NEW EDITION

New General Mathematics 4

'O' Level Course

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With Answers

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An 'O' Level Course

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Contents

Chapter 1

General arithmetic (4) Approximations, estimates, limits of accuracy

Approximation and estimation (revision)

Approximations are very useful when calculating. Numbers can be approximated to obtain rough estimates of calculations. For example, if someone pays \$93,86 tax each month, then the amount paid in a year is \$93,86 × 12. A rough estimate of the total amount paid in a year is

\$90 × 10 = \$900
(rounding to 1 significant figure)

or

\$95 × 10 = \$950
(rounding to the nearest 5)

or

\$90 × 12 = \$1 080
(rounding one number, but not the other)

Rough estimates are *not* accurate. (For example, the true amount in the above calculation is \$1 126,32.) However, they give some idea of the size or **order of magnitude** of the correct results of calculations. They are often used to check the correctness of answers to calculations. They are especially valuable when using calculators.

Example 1

Use a calculator to find the value of $\frac{4{,}386 \times 0{,}0894}{18{,}17}$.
Check your result by making a rough estimate.

$$\frac{4{,}386 \times 0{,}0894}{18{,}17} = \frac{0{,}392\,108\,4}{18{,}17} = 0{,}021\,579\,9$$

Rough estimate: $\frac{4 \times 0{,}09}{20} = \frac{0{,}36}{20} = 0{,}018$

The answer and the rough estimate are of the same order of magnitude (both are about 0,02). If the answer and the rough estimate are *not* of the same order of magnitude then the data should be re-entered into the calculator.

Rounding off numbers (revision)

Numbers can be rounded off to the nearest hundred, ten, whole number, etc., or to a given number of decimal places.

The digits 1, 2, 3 and 4 are rounded down and the digits 5, 6, 7, 8 and 9 are rounded up.

Example 2

Round off the number 163,864 (*a*) *to 2 decimal places;* (*b*) *to 1 decimal place;* (*c*) *to the nearest whole number;* (*d*) *to the nearest hundred.*

(a)	163,864 = 163,86	to 2 d.p.
(b)	163,864 = 163,9	to 1 d.p.
(c)	163,864 = 164	to the nearest whole number
(d)	163,864 = 200	to the nearest hundred

Significant figures (revision)

Numbers are sometimes rounded off to a given number of significant figures. The significance of a digit depends on its position in the number. Thus in the number 146,83; the 4 is more significant than the 6, the 6 is more significant than the 8 and the 1 is the most significant digit.

The first significant figure in a decimal fraction is the first non-zero digit in the fraction. For example, the first significant figure in the decimal 0,002 487 is 2.

Example 3

Round off 146,83 *(a) to* 1 *significant figure, (b) to* 2 *significant figures, (c) to* 3 *significant figures, (d) to* 4 *significant figures.*

(a) 146,83 = 100 to 1 s.f.
(b) 146,83 = 150 to 2 s.f.
(c) 146,83 = 147 to 3 s.f.
(d) 146,83 = 146,8 to 4 s.f.

Example 4

Round off 0,002 487 *to (a)* 1 s.f., *(b)* 2 s.f., *(c)* 3 s.f.

(a) 0,002 to 1 s.f.
(b) 0,002 5 to 2 s.f.
(c) 0,002 49 to 3 s.f.

Example 5

Round off (a) 8,026 *to* 3 s.f., *(b)* 50,95 *to* 2 s.f., (c) 18,057 *to* 1 *decimal place, (d)* $450 170 *to* 3 s.f.

(a) 8,026 = 8,03 to 3 s.f.
(b) 50,95 = 51 to 2 s.f.
(c) 18,057 = 18,1 to 1 d.p.
(d) $450 170 = $450 000 to 3 s.f.

Example 6

In 1987 *Zimbabwe's exports to Britain were* $240 million *to* 2 s.f. *What is the range of values of these exports?*

The range of values is between $235 million and $245 million.

Exercise 1a

1 Round off the following to the nearest whole number.

(a) 3,24 m (b) 0,97 m
(c) 86,02 ha (d) 341,77 cm
(e) 496 km (f) 164,90 mm
(g) 52,84 (h) $18,07
(i) 129,7 litres (j) $99,50

2 Round off the following to 1 decimal place.

(a) 0,078 6 (b) 18,04 (c) 18,954
(d) 0,786 (e) 7,926 (f) 20,08

3 Round off the following to 2 significant figures.

(a) 36,9 (b) 109,4
(c) 0,009 281 (d) 4,98
(e) 0,024 39 (f) 86 125
(g) 144 (h) 9,04
(i) 2,99 (j) 0,030 46

4 Round off the following to 3 significant figures.

(a) 7 579 (b) 52 069
(c) 352 289 (d) 1 789
(e) 0,089 21 (f) 170,08
(g) 83,352 (h) 0,906 4
(i) 827 502 (j) 8,007

5 In 1987 there was an estimated $389 400 000 in notes and coin in circulation in Zimbabwe. Give at least three ways in which a newspaper might have reported this amount.

6 In the 1982 census, the populations of Zimbabwe's three largest towns were given as shown in Table 1.1.

Table 1.1

town	population
Bulawayo	495 300
Chitungwiza	172 000
Harare	658 400

Round off these numbers to (a) 1 s.f., (b) 2 s.f.

7 A newspaper headline reads 'Government rejects a loan of $1,5 billion from IMF.' Between what two amounts does this figure lie?

8 The numbers of foreign visitors to Zimbabwe between 1984 and 1988 are given in Table 1.2.

Table 1.2

1984	1985	1986	1987	1988
339 598	389 465	433 372	487 716	491 721

(a) Round off the number to 1 s.f.
(b) Use your rounded numbers to estimate the total number of foreign visitors to Zimbabwe for the five years.

9 Find the order of magnitude of the outcomes of the following calculations. (Round the given numbers to 1 s.f.)

(a) $67{,}09 \times 4{,}38 \times 0{,}1178$

(b) $8\,956 \div 27{,}31$

(c) $\dfrac{55{,}73 \times 8\,607}{645{,}8}$

(d) $\dfrac{3{,}705}{22{,}73 \times 76{,}35}$

10 Use a calculator to find the true outcomes of the calculations in question 9. Give all your answers correct to 4 s.f.

Limits of accuracy

No measurement, however carefully made, is exact. The best measuring instruments are usually accurate to three figures only. Most measurements, therefore, are approximate. For example, if the length of a line is given as 23,8 cm, its true length will lie between a minimum, or **lower bound**, of 23,75 cm and a maximum, or **upper bound**, of 23,85 cm. This gives a possible **error** of ±0,05 cm in the given length. Figure 1.1 shows the end of the line (magnified) and the lower and upper bounds on its length. The difference between the upper and lower bound gives the range of measurement.

Fig. 1.1

Percentage error

$$\text{Percentage error} = \frac{\text{error}}{\text{measurement}} \times 100\%$$

In Fig. 1.1,

$$\begin{aligned}\text{percentage error} &= \pm \frac{0{,}05}{23{,}8} \times 100\% \\ &= \pm 0{,}210\,084\% \\ &= \pm 0{,}21\% \quad \text{(to 2 s.f.)}\end{aligned}$$

Example 7

The length of a pole is measured as 5 metres to the nearest metre. State (a) the upper and lower bounds of the length of the pole, (b) the greatest possible error, and calculate (c) the percentage error in the measurement.

(a) Since the measurement is given to the nearest metre, the true length of the pole will be between 4,5 m and 5,5 m.

upper bound = 5,5 m
lower bound = 4,5 m

(b) Greatest possible error = ± 0,5 m

(c) $\text{Percentage error} = \pm \dfrac{0{,}5\text{ m}}{5\text{ m}} \times 100\% = \pm 10\%$

Example 8

A girl and a boy estimate the length of a line to be 9 cm and 12 cm respectively. If the true length of the line is 9,6 cm, find (a) the absolute error, (b) the percentage error, for each student.

(a) Absolute error = approximate value − true value

For the girl,

$$\begin{aligned}\text{absolute error} &= 9\text{ cm} - 9{,}6\text{ cm} \\ &= -0{,}6\text{ cm}\end{aligned}$$

The girl has underestimated by 0,6 cm.

For the boy,

$$\begin{aligned}\text{absolute error} &= 12\text{ cm} - 9{,}6\text{ cm} \\ &= +2{,}4\text{ cm}\end{aligned}$$

The boy has overestimated by 2,4 cm.

(b) $\text{Percentage error} = \dfrac{\text{absolute error}}{\text{true value}} \times 100\%$

For the girl,

$$\begin{aligned}\text{percentage error} &= \frac{-0{,}6}{9{,}6} \times 100\% \\ &= -6{,}25\%\end{aligned}$$

The girl's estimate was 6,25% too low.

For the boy

$$\begin{aligned}\text{percentage error} &= \frac{+2{,}4}{9{,}6} \times 100\% \\ &= 25\%\end{aligned}$$

The boy's estimate was 25% too high.

Example 9

The length of a running track is measured and given as 400 m. Find the percentage error if the length is measured (a) to the nearest metre, (b) to the nearest 10 m, (c) to 1 significant figure.

(a) The range of actual measurement is between 399,5 m and 400,5 m.

Error = ± 0,5 m

$$\text{Percentage error} = \frac{\pm 0{,}5\text{ m}}{400\text{ m}} \times 100\% = \pm 0{,}125\%$$

(b) The range of actual measurement is between 395 m and 405 m.

Error = ± 5 m

$$\text{Percentage error} = \frac{\pm 5\text{ m}}{400\text{ m}} \times 100\% = \pm 1\tfrac{1}{4}\%$$

(c) The range of actual measurement is between 350 m and 450 m.

Error = ± 50 m

$$\text{Percentage error} = \frac{\pm 50\text{ m}}{400\text{ m}} \times 100\% = \pm 12\tfrac{1}{2}\%$$

Notice in Example 9 that the percentage error becomes greater as the method of estimation becomes rougher.

Example 10

Given the calculation $8,70 × 2,4, *a student gets a rough answer by rounding the given values to the nearest whole number. Calculate the resulting percentage error.*

Rough answer = $9 × 2
= $18

True answer = $8,70 × 2,4
= $20,88

Absolute error = $18 − $20,88
= − $2,88

$$\text{Percentage error} = \frac{-\$2{,}88}{\$20{,}88} \times 100\% = -13{,}8\% \text{ to 3 s.f.}$$

Example 11

The length and breadth of a rectangular plot are measured to the nearest metre as being 16 m *and* 11 m *respectively. Calculate the upper and lower bounds of the area of the plot.*

The upper bound of the area of the rectangle occurs when its length and breadth have their *greatest* possible values:

greatest possible length = 16,5 m
greatest possible breadth = 11,5 m
corresponding area (upper bound)
$= 16{,}5 \times 11{,}5\text{ m}^2$
$= 189{,}75\text{ m}^2$

The *lower bound* of the area of the rectangle occurs when its length and breadth have their *least* possible values:

least possible length = 15,5 m
least possible breadth = 10,5 m
corresponding area (lower bound)
$= 15{,}5 \times 10{,}5\text{ m}^2$
$= 162{,}75\text{ m}^2$

In Example 11, notice how errors become compounded when approximate values are used in computations. The upper and lower bounds of the area of the rectangle in the example differ by 27 m^2, quite a considerable amount.

Exercise 1b

1 What is the range of values of each of the following measurements?

(a) The length of a line segment is 9 cm to the nearest cm.

(b) A man is 1,75 m tall to the nearest cm.

(c) The radius of the earth is 6 400 km to 2 s.f.

(d) The distance from the school to the market is 9,8 km to 1 d.p.

(e) The population of Zimbabwe (1989) was 9 million to 1 s.f.

(f) A container holds 7,5 litres of petrol to the nearest 0,1 litre.

(g) A woman spent $20 000 to the nearest thousand dollars.

(h) The mass of a wrestler is 75,6 kg to the nearest 0,1 kg.

(i) The maximum temperature for a particular day in Kadoma was 30,2 °C to the nearest 0,1 °C.
(j) The volume of a sphere is 860 cm^3 to the nearest cm^3.

2 Calculate the percentage error in each of the following measurements.
(a) The capacity of a bucket is 7,5 litres to 1 d.p.
(b) The distance between two towns is 60 km to the nearest km.
(c) A pole is 125 cm high to the nearest cm.
(d) The volume of a box is 25 cm^3 to the nearest cm^3.
(e) The thickness of a book is 20 mm to the nearest mm.
(f) The mass of a girl is 62 kg to 2 s.f.
(g) The speed of an aircraft is 800 km/h to 1 s.f.
(h) The radius of a ball is 21 cm to the nearest cm.
(i) The area of a classroom is 400 m^2 to 2 s.f.
(j) The University of Zimbabwe has 9 000 students to 1 s.f.

3 There are 36 candles in a box. Each candle has a mass of 115 g.
(a) Round the given values to 1 s.f. and hence estimate the mass of the candles in the box.
(b) Calculate the true mass of the candles.
(c) Hence find, to 1 d.p., the percentage error of your estimation.

4 A French tourist has 980 Francs which she wishes to exchange for Dollars. The rate of exchange is \$0,347 to 1 Franc. She roughly estimates the value of her Francs by calculating $0{,}35 \times 1\,000$.
(a) Write down her rough estimate.
(b) Calculate the true value in Dollars and cents.
(c) Hence find, to 1 d.p., the percentage error in the estimation.

5 Given the calculation $783 \div 1{,}8$, a student gets a rough answer by rounding each number to 1 s.f. Calculate, to 1 d.p., the percentage error in his result.

6 13 g of salt is dissolved in 88 g of water to make a salt solution. If each amount is given to the nearest gram, calculate the upper and lower bounds of the masses of the solution.

7 A box contains 960 packets of coffee. Each packet is marked '250 g to the nearest 5 g'. Calculate, in kg, the upper and lower bounds of the masses of coffee in the box.

8 The length and breadth of a rectangular field are given as 100 m and 70 m to the nearest 5 m. Calculate the least and greatest possible areas of the field.

9 The radius of a circle is given as 6 cm to the nearest whole cm. Calculate, in terms of π, the lower and upper bounds of (a) the circumference, (b) the area, of the circle.

10 The edge of a cube is given as 8 cm to the nearest whole cm. Calculate the lower and upper bounds of its (a) surface area in cm^2, (b) volume in cm^3.

Degree of accuracy

Many calculations involve measurements. The degrees of accuracy of such measurements affect the degree of accuracy of the results of the calculations. Therefore, the degree of accuracy of measurements in a calculation must be taken into consideration when determining the answer to the calculation. The final result of a calculation should not be given to a number of significant figures more than the number of significant figures in any of the given data.

Rounded-off values are sometimes used in calculations. For example, π is often taken as 3,14 or 3,142.

Sometimes it seems that an answer found at an intermediate stage of a calculation may need to be rounded off before it is used in a subsequent stage. However, since such rounded-off values affect the degree of accuracy of the results of calculations, it is generally advisable *not* to round off intermediate values.

Example 12
Calculate the area of a circle of radius 3,5 cm.

$$\begin{aligned}\text{Area of circle} &= \pi\,(3{,}5)^2\ \text{cm}^2\\ &= 3{,}14 \times (3{,}5)^2\ \text{cm}^2\\ &= 38{,}465\ \text{cm}^2\\ &= 38\ \text{cm}^2 \text{ to 2 s.f.}\end{aligned}$$

Note that the radius, 3,5 cm, is given to 2 s.f. and 3,14 is a rounded-off value of π. In this case the answer should not be given to more than 2 significant figures.

Rough estimate: $3 \times 4 \times 4\,\text{cm}^2 = 48\,\text{cm}^2$

The rough estimate is of the same degree as the calculated answer. (To make rough estimates, first express each number to 1 s.f. before calculating.)

Exercise 1c

Give answers to reasonable degrees of accuracy unless otherwise stated.

1 A car travels a distance of 100 km for 1 h 38 min 45 s.
(a) Round off the time to the nearest $\frac{1}{4}$ hour.
(b) Use the rounded time to find the average speed for the journey in km/h. Give your answer to 1 significant figure.

2 In an experiment, the radius, r, of a spherical balloon is measured as 21 cm. Calculate the volume, V, of the balloon. Take π to be 3,14 and use $V = \frac{4}{3}\pi r^3$.

3 In the right-angled triangle, ABC, in Fig. 1.2, $\hat{B} = 90°$ and the lengths of AB and BC are given to the nearest centimetre. Calculate AC.

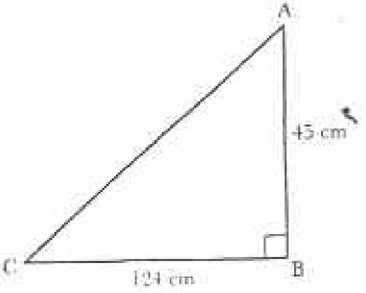


Fig. 1.2

4 Use four-figure tables or a calculator to calculate the following. Take π to be 3,142.
(a) $\pi\sqrt{3}$
(b) $2\pi \times 5{,}289$
(c) $(2{,}41)^3$
(d) $\dfrac{\pi}{28}$

5 Using square root and reciprocal tables, or a calculator, calculate the values of the following.
(a) $\dfrac{1}{\sqrt{110}}$
(b) $\dfrac{\sqrt{56}}{9}$
(c) $\dfrac{80}{\sqrt{12{,}5}}$

6 If a sheet of cardboard is 0,8 mm thick, calculate the range of values of the height of 328 sheets of the same cardboard. Also calculate the maximum and the minimum number of sheets which are in a pile 50 cm high.

7 A room is 4,6 m long, 3,7 m wide and 3,2 m high. Calculate
(a) the diagonal of the longer wall,
(b) the diagonal of the floor.

8 An aircraft travels 5 000 km (to 1 s.f.) in 6 hours (to the nearest hour). Calculate its speed in km/h.

9 Calculate the volume of a cylinder of radius 4 cm and height 20 cm. (Take π to be 3,14.)

10 Taking the value of π to be 3,14, calculate the area of a circle of radius 30 cm.

Chapter 2

Geometrical constructions (3) Locus

Basic constructions

Fig. 2.1 shows the basic constructions using set square, ruler and compasses.

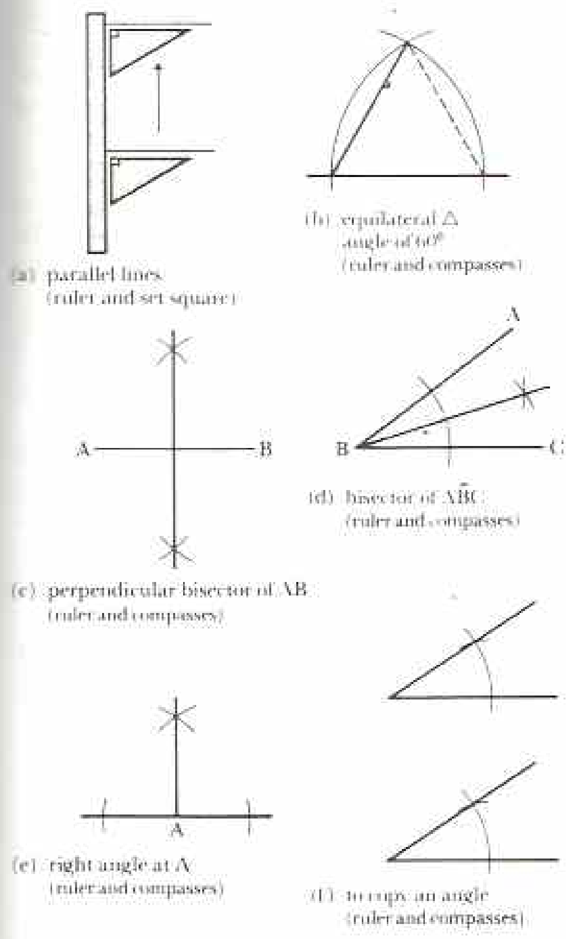


Fig. 2.1

To construct angles of 45° and 30°, bisect angles of 90° and 60° respectively (Fig. 2.2).

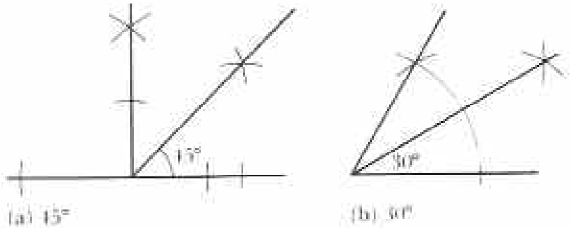


Fig. 2.2

Exercise 2a (Revision)

Make a sketch before constructing the figure.

1 (a) Construct △ABC such that AB = 8 cm, BC = 11 cm and AC = 10 cm.
(b) Mark a point X on AB such that BX = 5 cm. Use ruler and set square to construct a line through X parallel to BC to meet AC at Y.
(c) Measure XY and CY.

2 (a) Construct an equilateral triangle XYZ such that XY = 5 cm.
(b) With YZ as base, construct isosceles △AYZ such that AY = AZ = 8 cm.
(c) Similarly construct isosceles △BYZ such that B is on the opposite side of YZ to A with BY = BZ = 8 cm.
(d) What kind of quadrilateral is BYAZ?
(e) Measure AX and BX.

3 (a) Construct △ABC such that AB = 7 cm, BC = 6 cm and $A\hat{B}C$ = 60°.
(b) The bisector of $\hat{C}$ meets the perpendicular bisector of AC at X. Find the point X by construction.
(c) Measure BX.

4 (a) Use ruler and compasses to construct $\triangle PQR$ in which $\hat{Q} = 90°$, QR = 5 cm and PR = 10 cm.
(b) Measure PQ.
(c) By using Pythagoras' theorem now check the result.

5 (a) Use ruler and compasses to construct the parallelogram PQRS in which QR = 5 cm, RS = 11 cm and $Q\hat{R}S = 135°$.
(b) Measure the length of the shorter diagonal of PQRS.

6 (a) Construct quadrilateral ABCD such that AB = 5 cm, BD = DC = 8 cm, $A\hat{B}D = 30°$ and $B\hat{C}D = 45°$.
(b) Measure the diagonal AC.

To construct a perpendicular to a given straight line from a point outside the line.
Given a line AB and a point M outside the line.

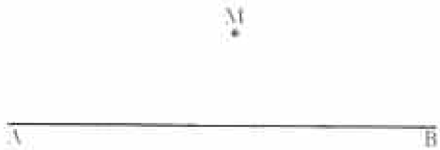


Fig. 2.3

It is required to construct a line through M perpendicular to AB.
(a) Place the sharp end of a pair of compasses on M. Open the compasses sufficiently to draw an arc which cuts AB at P and Q as in Fig. 2.4.

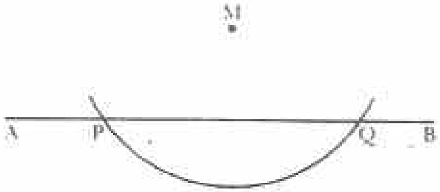


Fig. 2.4

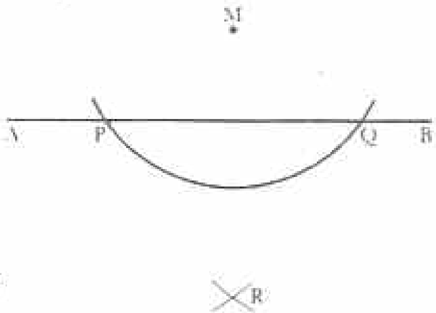


Fig. 2.5

(b) With centres A and B and equal radii, draw arcs to cut each other at R as in Fig. 2.5.
(c) Join MR, cutting AB at S (Fig. 2.6).
In Fig. 2.6, $M\hat{S}A = M\hat{S}B = 90°$. (Check by measurement.)

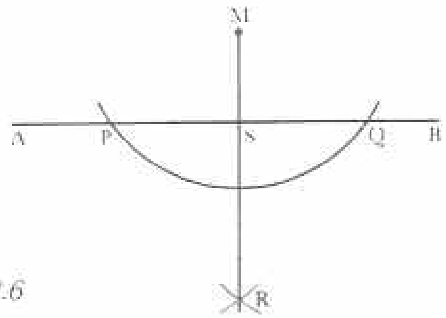


Fig. 2.6

Exercise 2b

Make a sketch before constructing the accurate figure.

1 (a) Construct a triangle with sides 6, 7, 9 cm. (b) Construct the three altitudes of the triangle. (c) What do you notice?

2 (a) Construct a triangle with sides 5, 6, 9 cm. (b) Construct the three altitudes as in question 1. (Two of the sides will have to be produced.) (c) Do the altitudes meet at one point?

3 (a) Construct a rectangle ABCD in which AB = 86 mm, BC = 58 mm.
(b) Construct the bisectors of $\hat{A}$ and $\hat{B}$. Let the bisectors meet at point E.
(c) Construct the perpendicular from E to AB and measure its length.

4 (a) Construct $\triangle LMN$ in which LM = 7,6 cm, MN = LN = 9,7 cm.
(b) Through N, construct a line parallel to LM.
(c) Construct MD perpendicular to the line constructed in (b), meeting it at D. Measure MD.

5 (a) Construct $\triangle ABC$ with sides 6, 8, 9 cm.
(b) Bisect all three angles of $\triangle ABC$. The bisectors should meet in one point. Call this I.
(c) Construct the perpendicular IX from I to BC.
(d) Draw a circle with radius IX and centre I. Does this circle do anything special?
(e) Measure the radius of the circle.

6 (a) Construct $\triangle XYZ$ with sides 5, 7, 9 cm.
(b) Use the method of question 5 to construct a circle which touches all three sides of the triangle.
(c) Measure the radius of the circle.

7 (a) Construct quadrilateral ABCD with $\hat{D} = 90°$, AD = 6 cm, DC = 9 cm, BC = 8,4 cm and AB = 5,4 cm.
(b) Bisect angles D and A and let the bisectors meet at X.
(c) Construct the perpendicular XP to DA. Draw the circle with centre X and radius XP.
(d) (i) Measure the radius of the circle.
(ii) Does the circle do anything special?

8 (a) Draw a circle of radius 5 cm. From any point P on the circumference draw chords PQ = 4 cm and PR = 6 cm and complete the $\triangle PQR$.
(b) Mark any point A on the circumference of the circle. Construct perpendiculars AX, AY, AZ from A to QR, RP, PQ respectively.
(c) Is there anything special about the points X, Y, Z?
(d) Is this always true? (Test by drawing another circle with chords of any convenient length.)

Fig. 2.7

Locus

Fig. 2.7 shows what happens to a raindrop which hits an umbrella on its way to the ground. The dotted line shows various positions of the raindrop.

The dotted line in Fig. 2.7 is called the **locus** of the raindrop. The locus can be thought of as the path traced out by the raindrop as it moves.

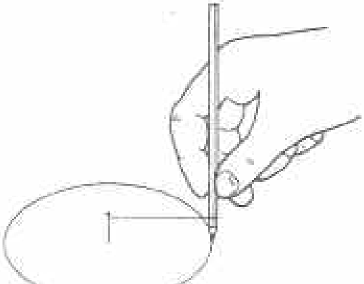


Fig. 2.8

In Fig. 2.8, a pencil point moves on a plane surface so that it is always a constant distance from a fixed point in the plane. The path that it traces out is a circle with the fixed point as centre. The locus of the pencil point is a circle.

The examples in Figs 2.7 and 2.8 demonstrate the simple definition of a locus: the path traced out by a moving point. However, this definition is not strictly correct. A point describes a *position* only, it has no size and cannot really be said to move. Hence, instead of describing a locus as the path traced out by a moving point, use the following definition:

A locus is the set of all possible positions occupied by an object which varies its position according to some given law.

In Fig. 2.7, the shape of the umbrella and the law of gravity determine the locus of the raindrop. In Fig. 2.8 the law which controls the pencil point is that it must be a constant distance from a fixed point in the same plane.

Common loci

Loci is the plural of locus.

Locus of points at a given distance from a fixed point.
In Fig. 2.9 overleaf, O is a fixed point. P_1, P_2, P_3, P_4 are points which are a constant distance

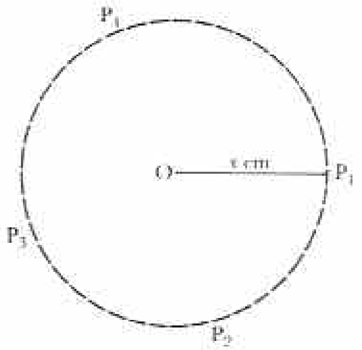


Fig. 2.9

x cm from O. The locus of the points is a **circle** of radius x cm (shown by the dotted line).

However, the locus need not necessarily lie in a plane. In 3 dimensions, the locus of P is a **spherical surface** of radius x cm and centre O (Fig. 2.10).

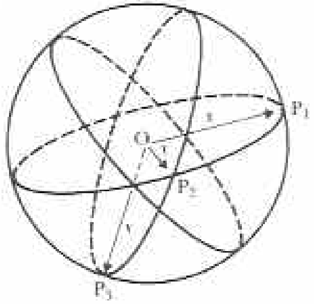


Fig. 2.10

Locus of points at a given distance from a straight line.
In Fig. 2.11, AB is a straight line which continues indefinitely in both directions. Points P_1, P_2, P_3, P_4 are each a distance x cm from AB. In 2 dimensions, the locus of the points consists of two straight lines parallel to AB, each a distance x cm from AB (shown by the dotted lines). Notice that this locus consists of **two separate lines**.

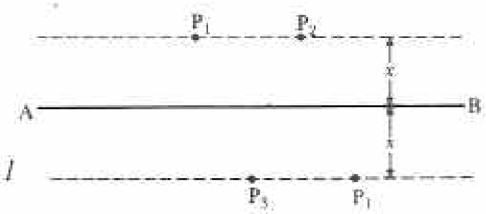


Fig. 2.11

In 3 dimensions, the locus is a **cylindrical surface** of radius x cm with AB as the central axis of the surface (Fig. 2.12).

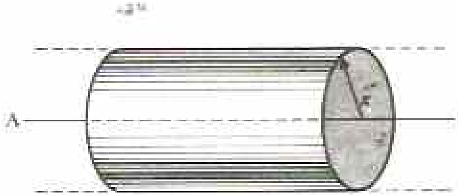


Fig. 2.12

Locus of points equidistant from two given points.
In Fig. 2.13, X and Y are two fixed points. Points P_1, P_2, P_3 are such that $P_1X = P_1Y$, $P_2X = P_2Y$ and $P_3X = P_3Y$. P_1, P_2, P_3 lie on the perpendicular bisector of XY. The locus of the points is the **perpendicular bisector** of XY (shown by the dotted line).

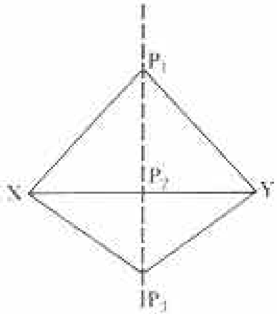


Fig. 2.13

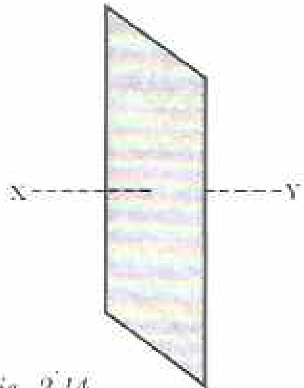


Fig. 2.14

In 3 dimensions, the locus is a **plane surface** which bisects XY perpendicularly (Fig. 2.14).

Exercise 2c

1 Place a cotton reel on a large sheet of paper. Unwind about 15 cm of thread and tie a loop at the free end. Put a pencil through the loop as in Fig. 2.15.
Keeping the thread tight, move the pencil so that the thread winds back onto the reel. What is the shape of the locus traced by the pencil point?

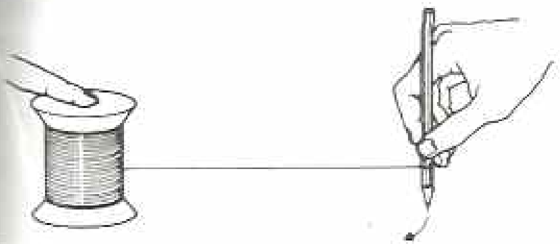


Fig. 2.15

2 Push two drawing pins through a sheet of paper so that they are about 5 cm apart. Make a loop of thread about 15 cm in length. Place the loop over the drawing pins and use a pencil to keep it tight as in Fig. 2.16.

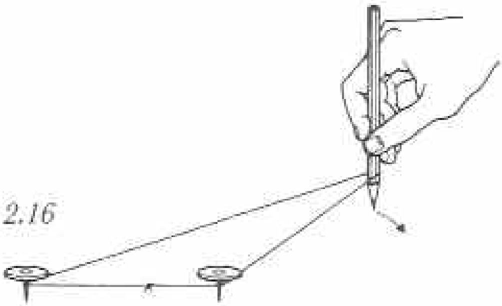


Fig. 2.16

Keeping the thread tight, move the pencil so that the point draws its locus on the paper. Describe the shape of the locus.

3 Draw two points A and B 4 cm apart. Use a 30° set square to plot a point P so that $A\hat{P}B = 30°$. See Fig. 2.17.

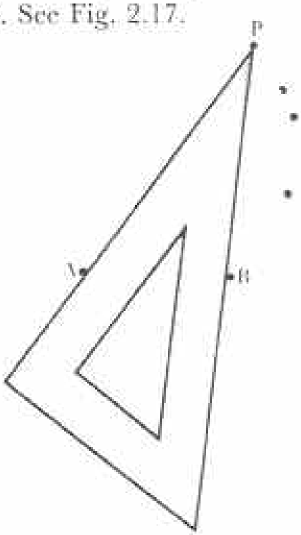


Fig. 2.17

Use this method to plot about 12 different positions of P. Hence draw the locus of P.

4 Repeat the method of question 3 so that $A\hat{P}B$ = (a) 45°, (b) 60°, (c) 90°.

5 Draw a circle of radius 4 cm and mark a fixed point A on its circumference. P can take any position on the circumference and Q is the mid-point of AP. For various positions of P, plot the corresponding positions of Q. Hence find the locus of Q.

6 Describe the following loci as accurately as possible.

(a) The locus of a door handle when the door opens through 180°.

(b) The locus of the tip of the minute hand of a clock during 1 hour.

(c) The locus of the mid-point of the hour hand of a clock during 1 hour.

(d) The locus of the centre of a coin which rolls in a straight line across a floor.

(e) The locus of a stone swinging on the end of a string.

(f) The locus of a table-tennis ball when served.

7 Describe the locus of a ball which rolls across the floor of a car at a constant speed, if the car is also travelling along a straight road at a constant speed.

8 A pencil slides inside a hemispherical bowl so that both ends are in contact with the bowl. Describe the locus of the mid-point of the pencil.

9 P is any position on a straight line AB. Q is a fixed point 3 cm from AB. Describe the locus of the mid-point of PQ.

10 A wire stretches from the top of a vertical pole to a point on the horizontal ground some distance from the foot of the pole. If the wire is tight, describe the locus of the lower end of the wire.

11 A set square lies flat on a table. It is rotated about its hypotenuse until it is flat on the table again. What is the path traced out by its right-angled corner?

12 A number of circles are drawn so that their circumferences pass through two fixed points. What is the locus of the centres of the circles?

13 A and B are fixed points. P is a variable point such that the area of $\triangle APB$ is constant. Find the locus of P.

14 In Fig. 2.18, the goat is tied by a rope $3\frac{1}{2}$ m long to a fixed wire 10 m long. A ring at the end of the rope can slide along the wire.

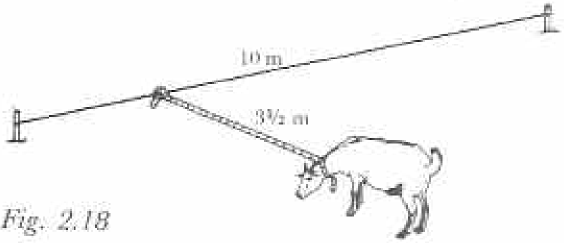


Fig. 2.18

Sketch the locus of the goat's head if the goat moves so as to keep as far as possible from the wire. Use the value $3\frac{1}{7}$ for π to calculate the approximate length of the locus.

15 In Fig. 2.19, a wheel of radius 14 cm starts with its centre in position A and rolls up two steps until its centre reaches position B. Sketch the locus of the centre of the wheel. Calculate the length of the locus using the value $\frac{22}{7}$ for π.

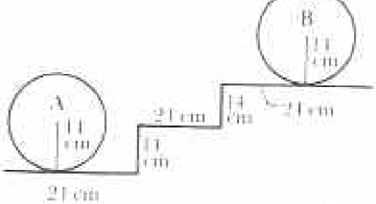


Fig. 2.19

Construction of loci in 2 dimensions

Example 1

$\triangle ABC$ *is such that* $\hat{B} = 70°$, AB = 5 cm *and* BC = 7,5 cm. *Find by construction the positions of a point* P *in the plane of* $\triangle ABC$ *which is equidistant from* B *and* C *and* 3,5 cm *from* A.

Fig. 2.20 is a scale drawing showing $\triangle ABC$ and the main details of the required construction. Since P is equidistant from B and C, it must lie on the perpendicular bisector of BC. P lies on locus l_1 in Fig. 2.20.

Since P is 3,5 cm from A, it must lie on a circle of radius 3,5 cm, centre A. P lies on locus l_2 in Fig. 2.20.

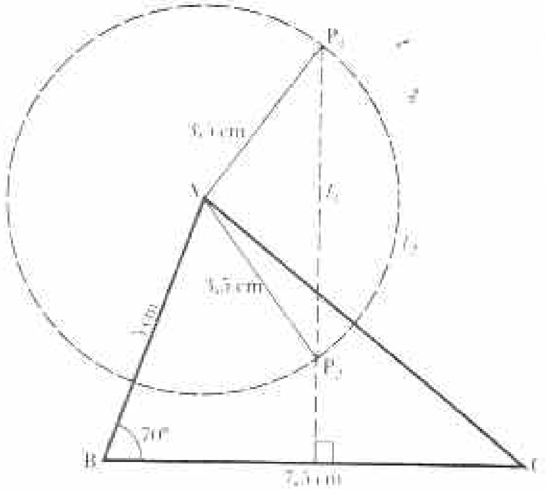


Fig. 2.20

To satisfy both conditions, P lies at the point(s) of intersection of these two loci, i.e. at P_1 and P_2 in Fig. 2.20.

Note: To make Fig. 2.20 clearer, some essential construction lines have been omitted. In practice these should be included. Also the two loci have been shown by dotted lines. Normally these would be solid lines.

Example 2

AB *is a straight line. A circle is drawn with centre* A *and radius* 2 cm. *Construct the points in the plane of the circle which are* 2,5 cm *away from* AB *and from the circumference of the circle.*

Fig. 2.21 is a scale drawing of the required construction.

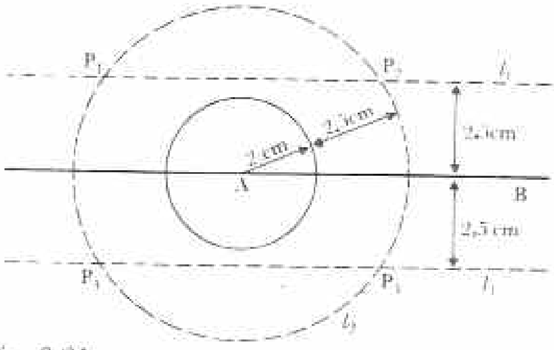


Fig. 2.21

Since the points are 2.5 cm away from AB, they must lie on lines which are parallel to AB and 2.5 cm from it, i.e. on locus l_1 in Fig. 2.21.

Since the points are 2,5 cm from the circumference of the circle, they must lie on the circumference of a circle of radius 4,5 cm, centre A, i.e. on locus l_2 in Fig. 2.21.

To satisfy both conditions, the required points are the points of intersection of these two loci, i.e. P_1, P_2, P_3, P_4 in Fig. 2.21.

Exercise 2d

In this exercise all loci are in 2 dimensions. Make a rough sketch first.

1 A and B are two points which are 7 cm apart. Construct the positions of a point P which is 4,2 cm from A and 5,6 cm from B. How many possible positions for P are there? Measure the distance between them.

2 AB and CD are two intersecting straight lines. Show how to construct the position of a point P which is 2 cm from AB and 3 cm from CD. How many possible positions for P are there?

3 △ABC is isosceles with AB = AC = 6 cm and BC = 4 cm. Construct points which are equidistant from B and C, and 3 cm from A. Measure their distances from BC.

4 △ABC has a right angle at C and AC = BC = 4 cm. Construct points which are equidistant from B and C, and 3 cm from A. Measure their distance from BC.

5 An aircraft flies at a height of 1 000 m on a straight, level course. The course takes the aircraft directly over two points A and B which are 1 500 m apart on the horizontal ground. On a scale drawing, construct two positions of the aircraft when its angle of elevation from A is 50°. In each case measure the angle of elevation from B.

6 A is a fixed point 3 cm from a straight line BC. Construct points which are 1 cm from BC and 3,5 cm from A. Measure the distance between them.

7 O is a fixed point on a straight line AB. P is a point which is 4 cm from AB and 5 cm from O. Construct 2 positions of P on the same side of AB. Measure the distance between them.

8 A boat is 20 m from a straight river bank and 29 m from a tree on the edge of the bank. By scale drawing, construct two possible positions of the boat. Measure the distance between them.

9 Show how to construct a quadrilateral PQRS in which QR = 4 cm, $\hat{R}$ = 110°, RS = 5 cm, $\hat{Q}$ = 90° and PQ = PS.

10 A and B are two fixed points 6 cm apart. A circle is drawn with centre B and radius 2 cm. Construct the positions of points which are equidistant from A and B, and 3 cm from the circumference of the circle. Measure the distance between them.

11 (a) Using ruler and compasses only, construct △ABC such that AC = 10 cm, BC = 8,5 cm and $A\hat{C}B$ = 135°.
(b) Using *any* geometrical instruments, find a point P within △ABC which is at a distance 2,8 cm from AC and 6 cm from B. Measure the length of AP.

12 Construct △ABC so that AB = 6 cm, BC = 9 cm, CA = 5 cm. Construct a point P, equidistant from A and C such that the area of △APB is 12 cm².

Further loci in 2 dimensions

Locus of points equidistant from two straight lines.

In Fig. 2.22, AB and CD are straight lines which intersect at O. P_1 is equidistant from AB and CD. Similarly P_2 is equidistant from the two lines. P_1 and P_2 lie on the bisector of the acute angle between the two lines.

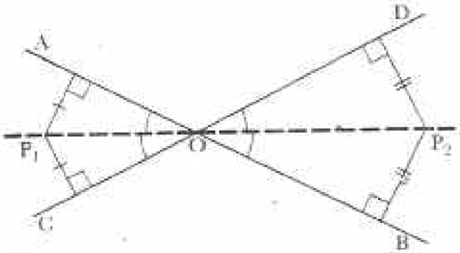


Fig. 2.22

In Fig. 2.23, P_3 is equidistant from AB and CD. P_3 lies on the bisector of the obtuse angle between the two lines.

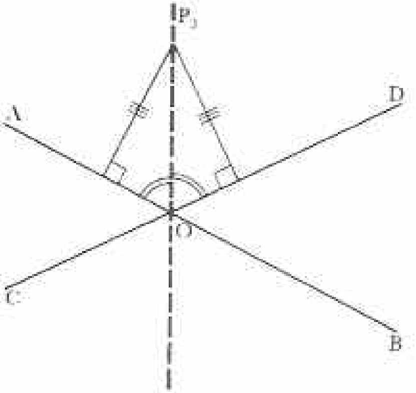


Fig. 2.23

The complete locus of points which are equidistant from two straight lines is the **pair of bisectors of the angles between the lines** (Fig. 2.24).

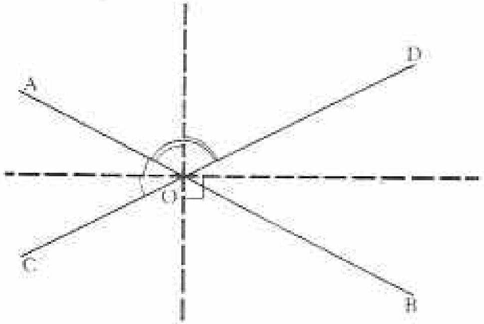


Fig. 2.24

Notice that the two parts of the locus intersect at right angles.

Locus of points which subtend a given angle on a given line segment.

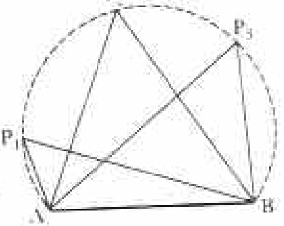


Fig. 2.25

In Fig. 2.25, AB is a given line segment. The arc $AP_1P_2P_3B$ is a major arc of a circle. It follows that $A\hat{P}_1B = A\hat{P}_2B = A\hat{P}_3B$ (angles in the same segment are equal). The arc is the locus of points which subtend a certain angle on a given line segment.

P can also be on the other side of AB. The complete locus of points which subtend a certain angle on AB is two circular arcs (Fig. 2.26).

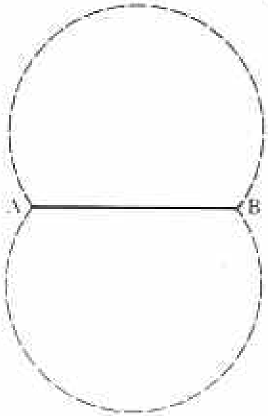


Fig. 2.26

Example 3

Using ruler and compasses only (a) construct $\triangle ABC$ *such that* AB = 6 cm, AC = 8,5 cm *and* $B\hat{A}C = 120°$. *(b) Construct the locus* l_1 *of points equidistant from* A *and* B. *(c) Construct the locus* l_2 *of points equidistant from* AB *and* AC. *(d) Find the points of intersection* P_1 *and* P_2, *of* l_1 *and* l_2 *and measure* P_1P_2.

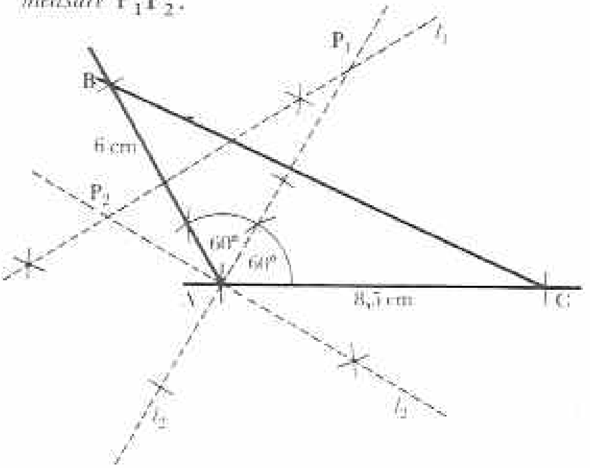


Fig. 2.27

Fig. 2.27 is a half-size scale drawing showing all construction lines and loci.

(a) Note the construction of $B\hat{A}C$ (= 120°).
(b) l_1 is the perpendicular bisector of AB.
(c) l_2 is in two parts. AP_1 is the bisector of $B\hat{A}C$. AP_2 is perpendicular to AP_1. Notice that points on AP_2 are equidistant from AB and CA *produced*.
(d) By measurement P_1P_2 = 6.9 cm.
(The loci are shown by dotted lines in Fig. 2.27. In your work use solid lines.)

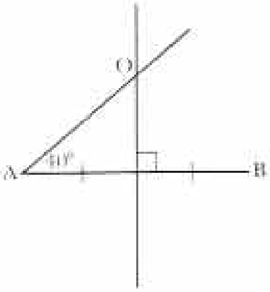


Fig. 2.29

Example 4
On a line AB, 3 cm *long, construct the locus of points which subtend an angle of* 50° *on* AB.

Method:
1 It is necessary to find the centre of the circular arc. Since $A\hat{P}B$ = 50°, $A\hat{O}B$ = 100° (angle at centre = 2 × angle at circumference). Hence the angles of △OAB are 100°, 40°, 40°.
2 Construct △OAB.
3 With centre O and radius OA draw the arc APB.

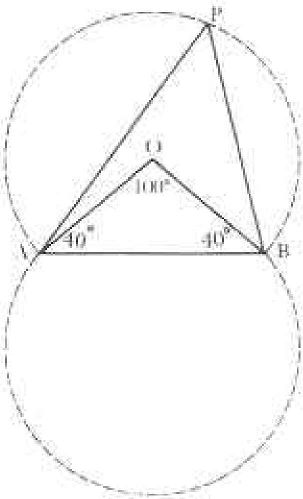


Fig. 2.28

4 Complete the second part of the locus on the other side of AB, using the same method.

An alternative method of finding O is to use the fact that the centre of the arc lies on the perpendicular bisector of AB. Since $O\hat{A}B$ = 40°, O can be found as in Fig. 2.29.

Circumcircle of a triangle

A circle which passes through the three vertices of a triangle is the **circumcircle** of the triangle. A circumcircle can be drawn for any triangle (Fig. 2.30).

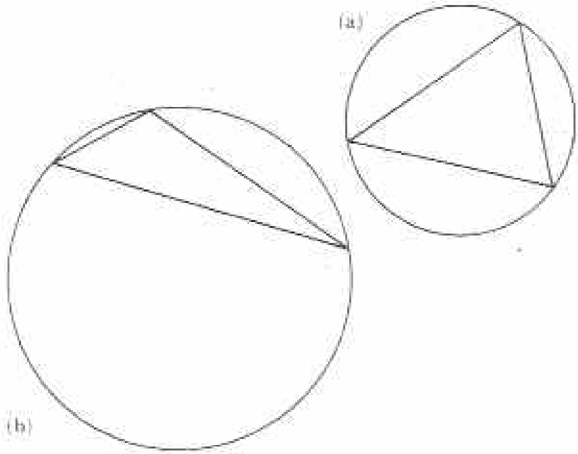


Fig. 2.30

The construction of the circumcircle of a triangle uses the fact that the perpendicular bisector of a chord of a circle passes through its centre. The three sides of the given triangle form three chords of its circumcircle.

If the perpendicular bisectors of the three sides of a triangle are constructed they will meet at a single point, the **circumcentre**, i.e. the centre of the circumcircle. The radius of the circumcircle is the distance of the circumcentre from any one of the vertices of the given triangle. The construction is shown in Fig. 2.31.

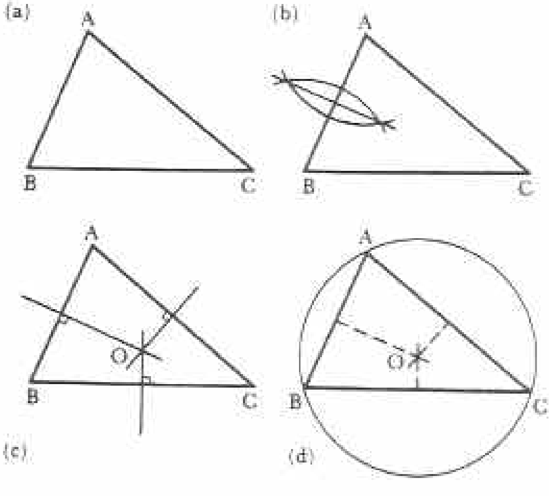


Fig. 2.31

Exercise 2e

In this exercise all loci are in 2 dimensions. Draw a rough sketch first.

1 Given any triangle XYZ, show how to construct a point P on YZ such that P is equidistant from XY and XZ.

2 Draw any triangle ABC. Construct the position of a point which is equidistant from AC and BC, and also equidistant from A and B. How many possible positions of the point are there?

3 On a line AB, 5 cm long, construct the locus of points which subtend an angle of 45° on AB.

4 On a line AB, 6 cm long, construct an arc of a circle which subtends an angle of 40° on AB. Find a point C on this arc such that AC = 8 cm. Measure BC

5 Draw △ABC in which AB = 5,2 cm, BC = 9,2 cm and CA = 8 cm. Construct the possible positions of a point which is equidistant from AB and AC, and 3,2 cm from BC.

6 Draw two straight lines AOB and COD intersecting at O. On OA mark a point Q such that OQ = 2,5 cm. Construct all the possible positions of a point equidistant from AB and CD, and 4 cm from Q.

7 On a line XY, 8 cm long, construct the locus of all points which subtend an angle of 100° on XY.

8 On a base BC, 6 cm long, construct △ABC of area 12 cm^2 such that $\hat{A}$ = 65°. Measure AB and AC.

9 Construct a trapezium ABCD in which AB || DC, AB = 4 cm, BC = 8 cm, CD = 11 cm, DA = 6 cm. (*Hint*: In a rough figure, divide the trapezium into parallelogram ABXD and △BCX. First construct △BCX.)

10 Using ruler, set square and compasses only, construct a quadrilateral ABCD, with AB = DA = 4 cm, BC = 6 cm, DC = 3 cm and BC || AD. (*Hint:* Draw a rough sketch. Divide the quadrilateral into △DCX and parallelogram ADXB. Draw △DCX first and produce CX to B.)

(a) Draw the locus, l_1, of points equidistant from the points C and D.

(b) Draw also the locus, l_2, of points equidistant from AD and AB.

(c) If P is the intersection of l_1 and l_2, measure AP.

11 Draw △ABC such that $\hat{A}$ = 75°, AB = 8 cm, AC = 7 cm. Construct two positions of a point P, equidistant from AC and BC, such that the area of △APB is 15,2 cm^2.

12 Construct the circumcircle of any

(a) acute-angled scalene triangle,

(b) obtuse-angled scalene triangle,

(c) right-angled scalene triangle.

Describe the positions of the circumcentres of the circumcircles that you have drawn.

13 Construct on a single diagram,

(a) triangle XYZ with base XY = 12 cm, XZ = 10 cm and YZ = 8,5 cm,

(b) the point P on the circumcircle of △XYZ such that P is equidistant from XY and XZ.

Measure PZ to the nearest mm.

14 (a) Construct the parallelogram ABCD in which AB = 7 cm, BC = 4 cm and $A\hat{B}C$ = 120°.

(b) Measure and write down AC.

(c) On the same diagram, construct (i) the circumcircle of △ABC, (ii) the locus of points equidistant from A and D.

(d) Mark the point P on the circumference of the circumcircle such that PA = PD and $A\hat{P}C$ = 60°.

Chapter 3

Circle geometry (2) Tangents

Tangent to a circle

In Fig. 3.1, a circle, centre O, is cut by a straight line MN at the two points X and Y. $\triangle OXY$ is isosceles and $O\hat{X}Y = O\hat{Y}X$. Hence $O\hat{X}M = O\hat{Y}N$.

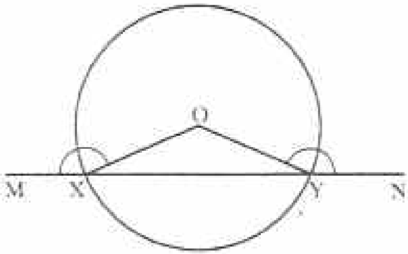


Fig. 3.1

Fig. 3.2 shows what happens to the positions of X and Y if MN moves downwards. X and Y occupy new positions such as X_1, Y_1 and X_2, Y_2, etc., becoming closer to each other.

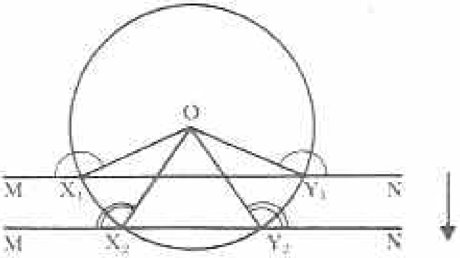


Fig. 3.2

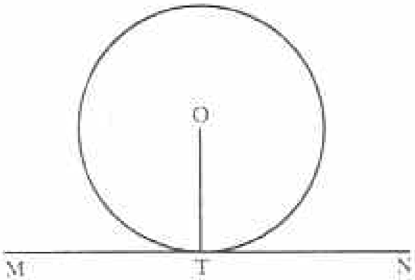


Fig. 3.3

Eventually the points X and Y will coincide at a single point T as in Fig. 3.3. Similarly the radii OX and OY will coincide to become one radius, OT.

Since $O\hat{X}M = O\hat{Y}N$ (Fig. 3.1), it follows that in Fig. 3.3,

$$O\hat{T}M = O\hat{T}N$$

and since MTN is a straight line,

$$O\hat{T}M = O\hat{T}N = 90°.$$

Hence $OT \perp MN$.

In Fig. 3.3, the line MN is said to be a **tangent** to the circle. The tangent *does not cut* the circle; it *touches* the circle.

Remember the following:

1 **A tangent to a circle is perpendicular to the radius drawn to its point of contact.**

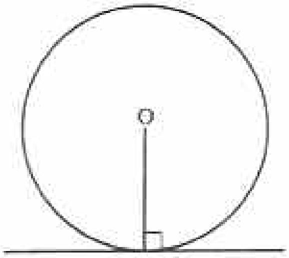


Fig. 3.4

2 **The perpendicular to a tangent at its point of contact passes through the centre of the circle.**

Example 1

TA *is a tangent at* A *to a circle, centre* O. AB *is a chord. If* $B\hat{A}T = x°$, *show that* $B\hat{O}A = 2x°$.

Fig. 3.5, overleaf, is a sketch of the circle.

If $B\hat{A}T = x°$

then $B\hat{A}O = (90 - x)°$ *(radius ⊥ tangent)*

and $A\hat{B}O = (90 - x)°$ *(isos.* $\triangle AOB$*)*

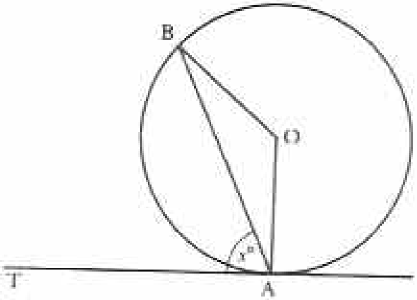


Fig. 3.5

$$\therefore \quad B\hat{O}A = 180° - 2(90 - x)° \quad (\textit{sum of angles of } \triangle ABO)$$
$$= 180° - 180° + 2x°$$
$$= 2x°$$

Exercise 3a

1 Calculate the size of angle α in each part of Fig. 3.6. In each figure, O is the centre of the circle.

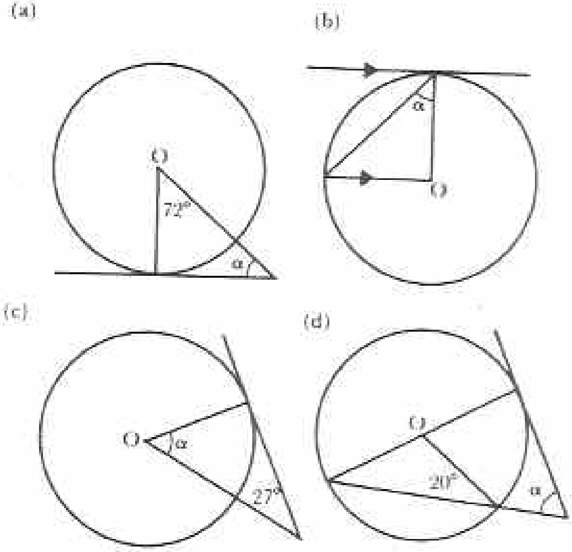
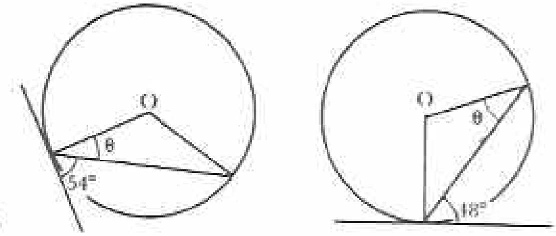


Fig. 3.6

2 Calculate the size of θ in each part of Fig. 3.7. In each figure, O is the centre of the circle.



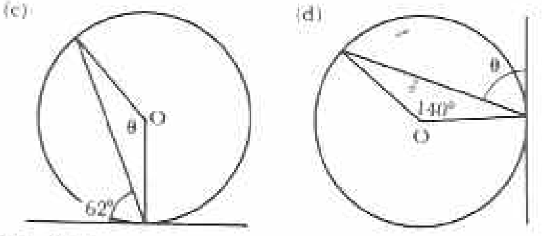


Fig. 3.7

3 Calculate OA in each part of Fig. 3.8. In each figure, O is the centre of the circle and the dimensions are in cm.

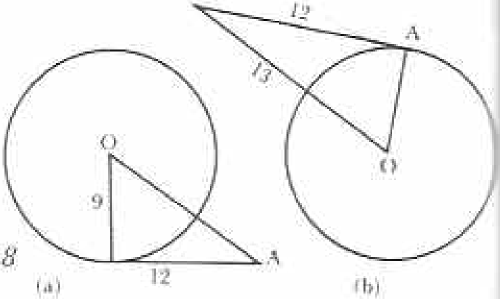


Fig. 3.8

4 In Fig. 3.9, O is the centre of the circle and TA is a tangent at A.

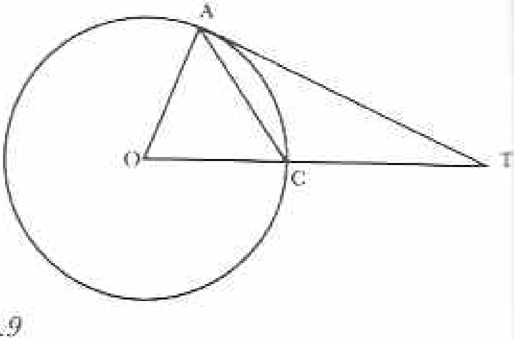


Fig. 3.9

Use Fig. 3.9 to answer the following.

(a) If $A\hat{O}C = 86°$, calculate $C\hat{A}T$.

(b) If $A\hat{T}O = 36°$, calculate $A\hat{C}O$.

(c) If $O\hat{A}C = 69°$, calculate $A\hat{T}C$.

(d) If $A\hat{C}T = 122°$, calculate $C\hat{A}T$.

5 The tangent from a point T touches a circle at R. If the radius of the circle is 2,8 cm and T is 5,3 cm from the centre, calculate TR.

6 A point P is 6,5 cm from the centre of a circle, and the length of the tangent from P is 5,6 cm. Calculate the radius of the circle.

7 AB is a chord and O is the centre of a circle. If $A\hat{O}B = 78°$, calculate the obtuse angle between AB and the tangent at B.

8 Two circles have the same centre and their radii are 15 cm and 17 cm. A tangent to the inner circle at P cuts the outer circle at Q. Calculate PQ.

9 In Fig. 3.10 AC is a tangent to the circle, centre O, and $B\hat{C}A = 90°$.

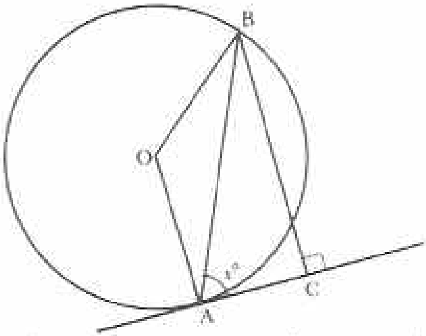


Fig. 3.10

(a) If $B\hat{A}C = x°$, find (i) $O\hat{B}A$, (ii) $A\hat{B}C$ in terms of x.
(b) Hence show that AB bisects $O\hat{B}C$.

10 AD is a diameter of a circle, AB is a chord and AT is a tangent. (a) State the size of $A\hat{B}D$. (b) If $B\hat{A}T$ is an acute angle of $x°$, find the size of $D\hat{A}B$ in terms of x. (c) Hence prove that $B\hat{A}T = A\hat{D}B$.

Tangents from an external point

Theorem

The tangents to a circle from an external point are equal.

Given: A point T outside a circle, centre O. TA and TB are tangents to the circle at A and B.
To prove: TA = TB.
Construction: Join OA, OB and OT.

Proof:
In $\triangle$s OAT and OBT (Fig. 3.11),

$\hat{A} = \hat{B} = 90°$	*(radius ⊥ tangent)*
OA = OB	*(radii)*
OT = OT	*(common side)*
∴ $\triangle OAT \equiv \triangle OBT$	*(RHS)*
∴ TA = TB	

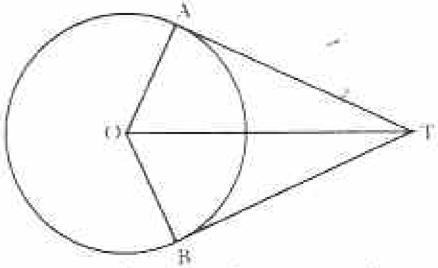


Fig. 3.11

Notice also that $A\hat{O}T = B\hat{O}T$ and $A\hat{T}O = B\hat{T}O$. Hence the line joining the external point to the centre of the circle bisects the angle between the tangents and the angle between the radii drawn to the points of contact of the tangents, i.e. OT, is on the line of symmetry of Fig. 3.11.

Example 2

In Fig. 3.12 O is the centre of the circle and TA and TB are tangents. If $A\hat{T}O = 39°$, calculate $T\hat{B}X$.

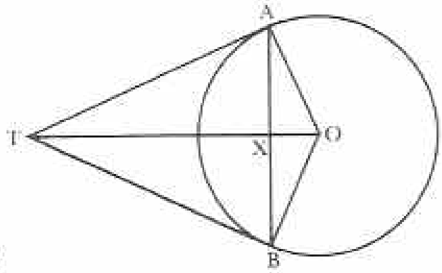


Fig. 3.12

In $\triangle TAX$,

$A\hat{X}T = 90°$	*(symmetry of Fig. 3.12)*
∴ $T\hat{A}X = 180° - (90° + 39°)$	*(sum of angles of △)*
$= 180° - 129°$	
$= 51°$	
∴ $T\hat{B}X = 51°$	*(symmetry)*

In Example 2 there are many ways of showing that $T\hat{B}X = 51°$, e.g. by noticing that $A\hat{T}X$ is the semi-vertical angle of isosceles triangle ATB.

Example 3

X, Y, Z are three points on a circle, centre O. The tangents at X and Y meet at T. If $X\hat{T}Y = 58°$, calculate $X\hat{Z}Y$.

See Fig. 3.13.

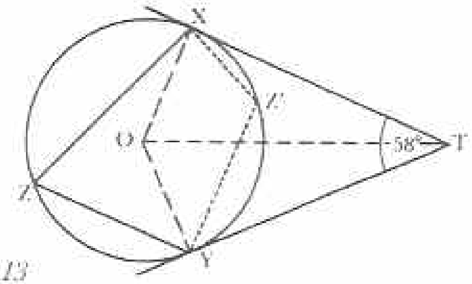


Fig. 3.13

Join OX and OY.
In quadrilateral TXOY,

$O\hat{X}T = O\hat{Y}T = 90°$ *(radius ⊥ tangent)*

$\therefore O\hat{X}T + O\hat{Y}T = 180°$

$\therefore X\hat{T}Y + X\hat{O}Y = 180°$ *(angles of quad.)*

$\therefore X\hat{O}Y = 180° - 58°$

$= 122°$

$\therefore X\hat{Y}Z = \frac{1}{2}$ of $122°$ *(angle at centre = 2 × angle at circumference)*

$= 61°$

If Z is on the minor arc XY (at Z') then

$X\hat{Z'}Y = 119°$ *(opp. angles of cyclic quad.)*

Hence $X\hat{Y}Z$ is 61° or 119°.

Exercise 3b

1 Use Fig. 3.12 to answer the following.
 (a) If $A\hat{T}O = 36°$, calculate $A\hat{B}O$.
 (b) If $A\hat{B}T = 57°$, calculate $A\hat{O}T$.
 (c) If $B\hat{T}O = 44°$, calculate $T\hat{A}X$.
 (d) If AB = 18 cm, and TB = 15 cm, calculate TX.

2 In Fig. 3.12, prove that TAOB is a cyclic quadrilateral.

3 In Fig. 3.12, if $A\hat{O}T = 47°$, calculate $A\hat{B}O$.

4 A, B, C are three points on a circle, centre O, such that $B\hat{A}C = 37°$. The tangents at B and C meet at T. Calculate $B\hat{T}C$. (*Hint:* Make a sketch and join BO and CO.)

5 P, Q, R are three points on a circle, centre O. The tangents at P and Q meet at T. If $P\hat{T}Q = 62°$, calculate $P\hat{R}Q$.

6 In Fig. 3.14 AB is a diameter of circle ABC centre O. TA and TC are tangents.

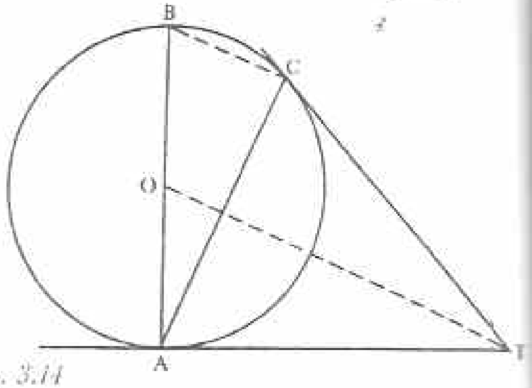


Fig. 3.14

If $A\hat{T}C = 2x$, show that $B\hat{A}C = x$. (*Hint:* Join OT.)

7 In Fig. 3.14 prove that BC is parallel to OT.

8 A quadrilateral PQRS is such that a circle can be drawn inside it to touch all four sides. Prove that PQ + RS = PS + QR.

9 A, B, C are three points on a circle. The tangents at A and B meet at T, and BC ∥ TA. Prove that AB bisects $T\hat{B}C$.

10 O is the centre of a circle, and two tangents from a point T touch the circle at A and B. BT is produced to C. If $A\hat{O}T = 67°$, calculate $A\hat{T}C$.

Contact of circles

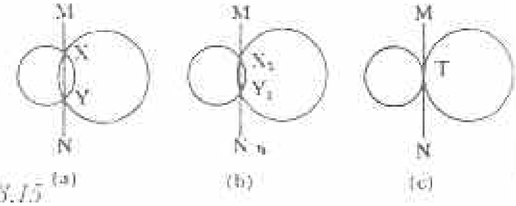


Fig. 3.15

Fig. 3.15(a) shows two circles with a common chord XY. If line MN is fixed and the circles move away from each other, the points X and Y become closer together, such as X_1, Y_1 in Fig. 3.15(b). Eventually the points will coincide at a single point as in Fig. 3.15(c). In Fig. 3.15(c), T is the point of contact of the circles and MN is a tangent to both circles at T.

Two circles touch each other if they both touch the same straight line at the same point.

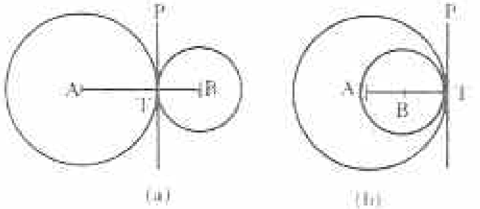


Fig. 3.16

In Fig. 3.16(a) the two circles touch each other **externally**, and in Fig. 3.16(b) they touch **internally**.

Let their centres be A and B and let the common tangent at T be PT.

In Fig. 3.16(a):

$\hat{ATP} = \hat{BTP} = 90°$ (*tangent* $\perp$ *radius*)

$\therefore$ $\hat{ATB} = 180°$

$\therefore$ ATB is a straight line.

In Fig. 3.16(b):

$\hat{ATP} = \hat{BTP} = 90°$

$\therefore$ AT and BT are both at right angles to TP.

$\therefore$ A, B and T lie in a straight line.

The straight line joining A and B in Fig. 3.16 is called the **line of centres**. In both parts of Fig. 3.16, **if two circles touch each other, the point of contact lies on the line of centres.**

The distance between their centres is the **sum** of their radii if the circles touch externally, and the **difference** of their radii if they touch internally.

Example 4

A and B are the centres of two circles which touch each other externally. Both circles touch a third circle, centre C, internally. If AB = 13 cm, BC = 14 cm, CA = 11 cm, calculate the radii of the circles.

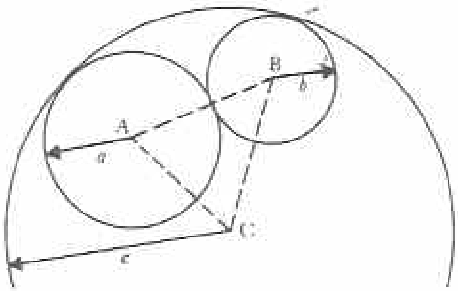


Fig. 3.17

Fig. 3.17 is a sketch of the three circles.
Let the radii of the circles centres A, B, C be a, b, c cm respectively. Then:

from AB, $a + b = 13$ (1)
from BC, $c - b = 14$ (2)
from CA, $c - a = 11$ (3)

Subtract (3) from (2)

$$-b + a = 3 \quad (4)$$

Add (1) to (4)

$$2a = 16$$
$$\therefore\ a = 8$$
$$\therefore\ b = 5$$
$$\therefore\ c = 19$$

The circles are of radii, 8 cm, 5 cm and 19 cm respectively.

Exercise 3c

1 Fig. 3.18 shows three circles which touch each other externally.

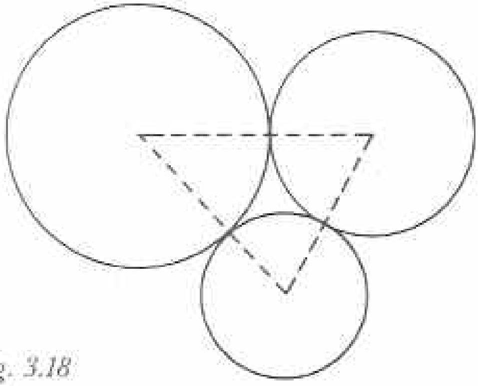


Fig. 3.18

If the centres of the circles form a triangle with sides of length 9 cm, 7 cm, 6 cm, calculate the radii of the circles.

2 In Fig. 3.18, find the radii of the circles if their centres form a triangle of sides 5 cm, 6 cm, 7 cm.

3 In Fig. 3.17, calculate the radii of the circles if AB = 11 cm, BC = 8 cm and CA = 9 cm.

4 Two circles, centres X and Y, touch externally at T. A is a point on their common tangent such that AT = 12 cm, AX = 13 cm and AY = 15 cm. (a) Calculate XY. (b) If the circles touch internally instead of externally, what is XY?

5 In Fig. 3.19, two circles touch at T and a line through T cuts the circles at A and B. OA and QB are radii.

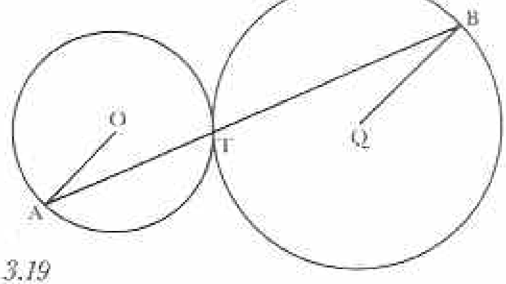


Fig. 3.19

Prove that OA ∥ QB. (*Hint:* Join OQ.)

6 Two circles touch at a point A. T is any point of their common tangent. Tangents from T touch one of the circles at P and the other at Q. Prove that TP = TQ.

7 Two circles touch each other at T. A straight line touches one circle at A and the other at B. Prove that $A\hat{T}B = 90°$.

8 Fig. 3.20 is a plan view of three equal cylindrical tins held together by an elastic band.

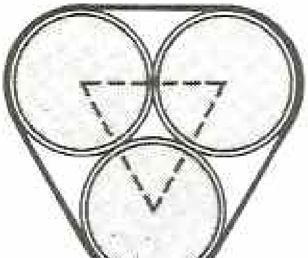


Fig. 3.20

If the centres of the tins form an equilate[ral] triangle of side 14 cm, use the value $3\frac{1}{7}$ f[or] π to calculate the length of the elastic ba[nd] in this position.

Alternate segment

In both parts of Fig. 3.21 SAT is a tangent [to] the circle at A. The chord AB divides the circ[le] into two segments APB and AQB.

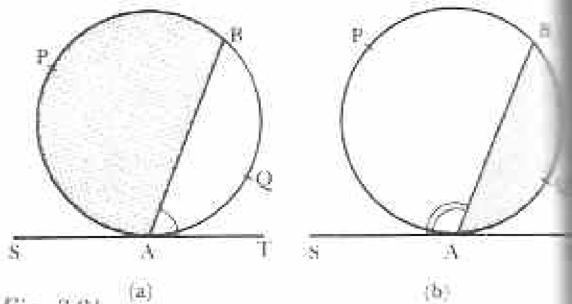


Fig. 3.21

In Fig. 3.21(a) the segment APB is th[e] **alternate segment** to $T\hat{A}B$, i.e. it is on th[e] other side of AB from $T\hat{A}B$. Similarly, in Fig 3.21(b), segment AQB is the alternate segmen[t] to $S\hat{A}B$.

Sometimes the word **opposite** is used instea[d] of alternate.

Theorem
If a straight line touches a circle, and from the point of contact a chord is drawn, the angles which the chord makes with the tangent are equal to the angles in the alternate segments.

Given: A circle, with SAT a tangent at A and chord AB dividing the circle into two segments APB and AQB. Segment APB is alternate to $T\hat{A}B$.
To prove: $T\hat{A}B = A\hat{P}B$ and $S\hat{A}B = A\hat{Q}B$.
Construction: Draw diameter AD. Join BD.

Proof:

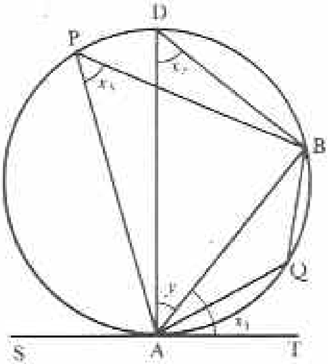


Fig. 3.22

With the lettering of Fig. 3.22,

$x_1 + y = 90°$ *(tangent ⊥ radius)*

also $\hat{ABD} = 90°$ *(angle in semicircle)*

$\therefore x_2 + y = 90°$ *(sum of angles of △)*

$\therefore x_1 = x_2$

$= x_3$ *(angles in same segment)*

$\therefore \hat{TAB} = \hat{APB}$

Also $\hat{SAB} = 180° - x_1$ *(angles on str. line)*

$= 180° - x_3$ *($x_1 = x_3$ proved)*

$= \hat{AQB}$ *(opp. angles of cyclic quad.)*

Example 5

In Fig. 3.23 PQX is a tangent to the circle QRS. Calculate $\hat{SQX}$.

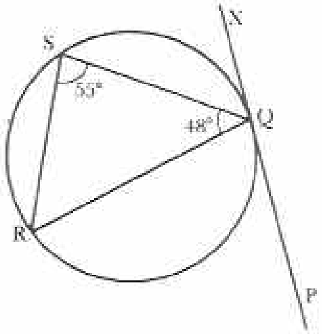


Fig. 3.23

In △QRS,

$\hat{SQX} = 180° - (55° + 48°)$ *(sum of angles of △)*

$= 180° - 103°$

$= 77°$

$\therefore \hat{SQX} = 77°$ *(alternate segment)*

Example 6

In Fig. 3.24 PT is a tangent to circle ABCT, BA = BT and $\hat{ATP} = 82°$. *Calculate* $\hat{BCT}$.

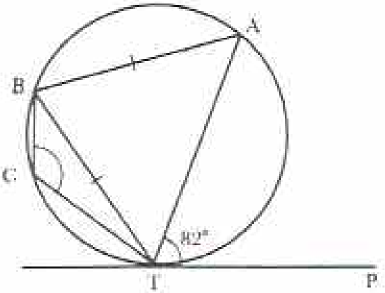


Fig. 3.24

$\hat{ABT} = 82°$ *(alternate segment)*

In △ABT,

$\hat{BAT} = \frac{1}{2}(180° - 82°)$ *(sum of angles of isos. △)*

$= \frac{1}{2} \times 98°$

$= 49°$

$\therefore \hat{BCT} = 180° - 49°$ *(opp. angles of cyclic quad.)*

$= 131°$

Exercise 3d

1 In Fig. 3.25 $\hat{XYZ}$ is a tangent to circle ABCY.

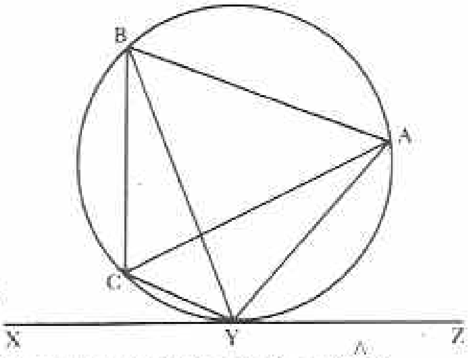


Fig. 3.25

(a) Name two angles equal to $\hat{AYZ}$.
(b) Name two angles equal to $\hat{CYX}$.
(c) Name an angle equal to $\hat{BYZ}$.
(d) Name an angle equal to $\hat{BYX}$.
(e) If $\hat{AYZ} = 58°$ what is $\hat{ACY}$?
(f) If $\hat{BCY} = 112°$ what is $\hat{BYZ}$?
(g) If $\hat{BCY} = 125°$ what is $\hat{BYX}$?
(h) If $\hat{BYZ} = 100°$ what is $\hat{BAY}$?

2 In Fig. 3.26 TAX and TBY are tangents to the circle and C is a point on the major arc AB.

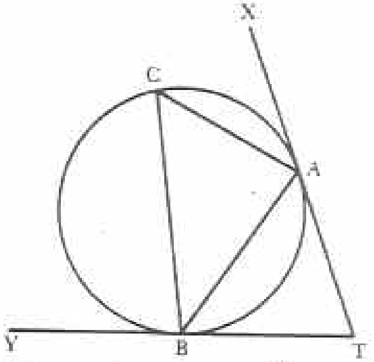


Fig. 3.26

(a) If $A\hat{T}B = 68°$, calculate $A\hat{C}B$.

(b) If $A\hat{B}C = 73°$, $B\hat{A}C = 83°$, calculate $A\hat{T}B$.

(c) If $A\hat{C}B = 59°$, $C\hat{B}Y = 78°$, calculate $C\hat{A}X$.

(d) If $C\hat{A}X = 65°$, $C\hat{B}Y = 76°$, calculate $A\hat{T}B$.

(e) If $A\hat{B}C = 48°$, $A\hat{T}B = 72°$, calculate $B\hat{A}C$.

3 In Fig. 3.27, XYZ is a tangent to the circle at Y. Name an angle equal to $Y\hat{Q}P$.

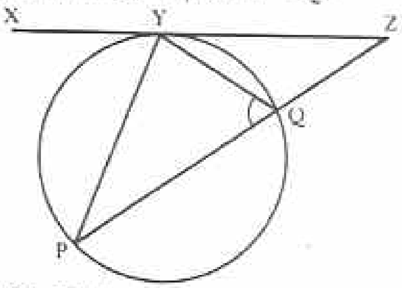


Fig. 3.27

4 In Fig. 3.28, TY is a tangent to the circle TVS. If $S\hat{V}T = 48°$ and VS = ST, what is $V\hat{T}Y$?

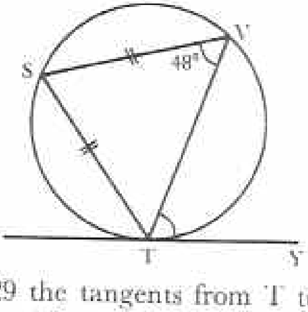


Fig. 3.28

5 In Fig. 3.29 the tangents from T touch a circle at A and B and BC is a chord parallel to TA. If $B\hat{A}T = 54°$, calculate $B\hat{A}C$.

6 In Fig. 3.29, if $A\hat{T}B = 82°$, calculate the angles of △ABC.

7 In Fig. 3.30, TS is tangent to circle PQRS. If PR = PS and $P\hat{Q}R = 117°$, calculate $R\hat{S}T$.

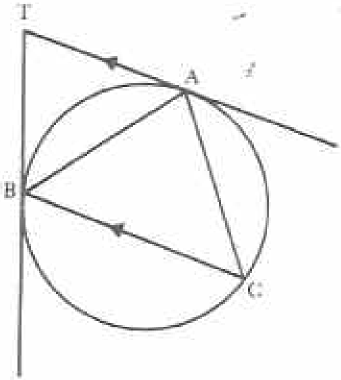


Fig. 3.29

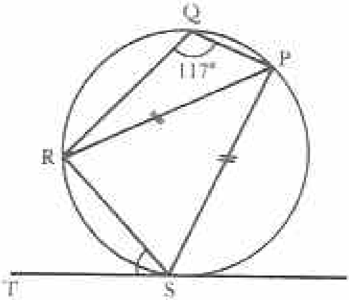


Fig. 3.30

8 AB is a chord of a circle and the tangents at A and B meet at T. C is a point on the minor arc AB. If $A\hat{T}B = 54°$ and $C\hat{B}T = 23°$, calculate $C\hat{A}T$.

9 In Fig. 3.31, if $A\hat{C}B = 37°$ and $A\hat{T}B = 42°$, calculate $A\hat{B}T$ and $A\hat{E}B$.

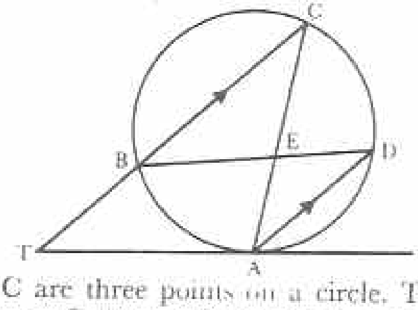


Fig. 3.31

10 A, B, C are three points on a circle. The tangent at C meets AB produced at T. If $A\hat{C}T = 103°$, $A\hat{T}C = 43°$, calculate the angles of △ABC.

11 The angles of a triangle are 40°, 60°, 80°, and a circle touches its sides at P, Q, R. Calculate the angles of △PQR.

12 AT is a tangent to the circle ABCD. $B\hat{A}C = 64°$ and $C\hat{A}T = 72°$. Calculate $B\hat{C}A$ and $C\hat{D}A$.

Chapter 4

The sine rule

Trigonometrical ratios of obtuse angles

The trigonometrical ratios of an acute angle in a right-angled triangle have already been defined.

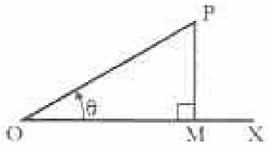


Fig. 4.1

In Fig. 4.1,

$$\sin\theta = \frac{MP}{OP} \quad \left(\frac{opp}{hyp}\right)$$

$$\cos\theta = \frac{OM}{OP} \quad \left(\frac{adj}{hyp}\right)$$

$$\tan\theta = \frac{MP}{OM} \quad \left(\frac{opp}{adj}\right)$$

If OX in Fig. 4.1 is kept fixed and OP allowed to rotate anticlockwise, there will come a stage when θ becomes obtuse (Fig. 4.2).

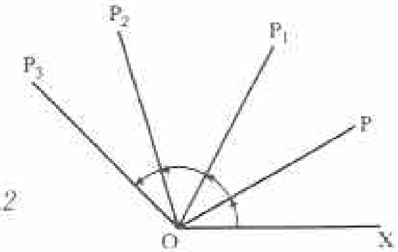


Fig. 4.2

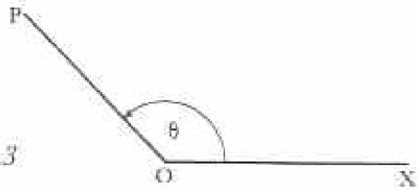


Fig. 4.3

When θ is obtuse, it is no longer in a right-angled triangle. It is therefore impossible to define $\sin\theta$, $\cos\theta$ and $\tan\theta$ in terms of the ratios of the hypotenuse, adjacent and opposite sides of a right-angled triangle (Fig. 4.3).

It is necessary, therefore, to define the trigonometrical ratios in such a way as to be suitable for obtuse angles as well as acute angles. Fig. 4.4 shows acute and obtuse angles θ within cartesian axes Ox, Oy.

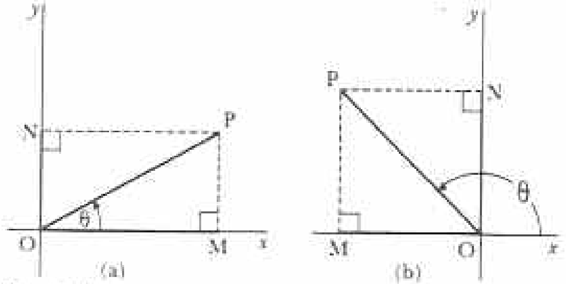


Fig. 4.4

Notice that the lettering of Fig. 4.4(a) is very much the same as that of Fig. 4.1.

In Fig. 4.4(a),

$$\sin\theta = \frac{MP}{OP} = \frac{ON}{OP}$$

$$\cos\theta = \frac{OM}{OP}$$

$$\tan\theta = \frac{MP}{OM} = \frac{ON}{OM}$$

In Fig. 4.4, OM is called the **projection of OP on Ox**. ON is the projection of OP on Oy. This makes it possible to define the trigonometrical ratios in a new way:

$$\sin\theta = \frac{\text{projection of OP on O}y}{\text{OP}}$$

$$\cos\theta = \frac{\text{projection of OP on O}x}{\text{OP}}$$

$$\tan\theta = \frac{\text{projection of OP on O}y}{\text{projection of OP on O}x}$$

With these definitions in Fig. 4.4(b),

$$\sin\theta = \frac{ON}{OP} \qquad \cos\theta = \frac{OM}{OP} \qquad \tan\theta = \frac{ON}{OM}$$

The ratios are now the same for both parts of Fig. 4.4. In Fig. 4.5, O is the centre of the circle and the origin of axes Ox, Oy. $P\hat{O}M = Q\hat{O}L = \theta$ (acute).

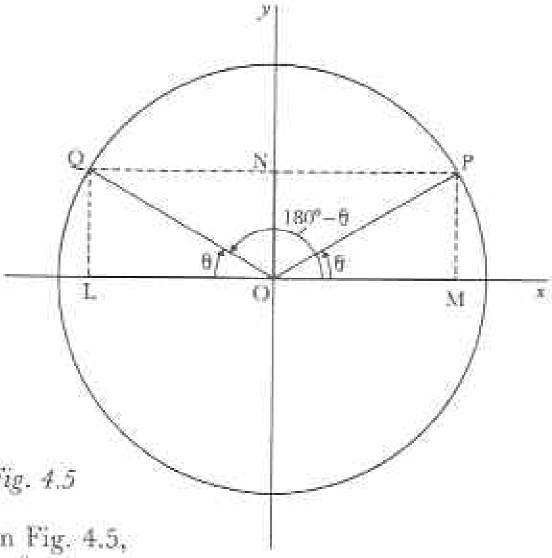


Fig. 4.5

In Fig. 4.5,

$M\hat{O}Q = 180° - \theta$ (obtuse)

OP = OQ (radii, both taken to be positive lengths)

ON is a positive length since it is on the positive part of Oy. OM is a positive length since it is on the positive part of Ox. OL is a negative length since it is on the negative part of Ox. From the symmetry of the figure,

OL = −OM.

Hence

$$\textbf{sin}\,(180° - \theta) = \frac{ON}{OQ} = \frac{ON}{OP} = \textbf{sin}\ \theta$$

$$\textbf{cos}\,(180° - \theta) = \frac{OL}{OQ} = \frac{-OM}{OP} = -\textbf{cos}\ \theta$$

$$\textbf{tan}\,(180° - \theta) = \frac{ON}{OL} = \frac{ON}{-OM} = -\textbf{tan}\ \theta$$

The following examples show how to use the above statements.

sin 160° = sin (180 − 20)° = sin 20° = 0,342 0
cos 160° = cos (180 − 20)° = −cos 20°
= −0,939 7
tan 160° = tan (180 − 20)° = −tan 20°
= −0,364 0

Exercise 4a

Use tables to find the values of the followin

1 sin 110°	**2** cos 110°
3 tan 110°	**4** sin 153°
5 sin 98°	**6** cos 106°
7 cos 142°	**8** tan 167°
9 tan 93°	**10** cos 128°
11 sin 156°30′	**12** tan 173,5°
13 cos 161,4°	**14** tan 131°42′
15 sin 93°12′	**16** cos 135,6°
17 cos 103°6′	**18** sin 118°42′
19 sin 178,35°	**20** tan 92°40′
21 sin 164°13′	**22** cos 118,83°
23 cos 121°31′	**24** sin 95,17°

Example 1

Find the values of θ lying between 0° and 180° each of the following.

(*a*) cos θ = 0,287 4 (*b*) sin θ = 0,936 1
(*c*) cos θ = −0,822 4 (*d*) tan θ = −2,164

(a) cos θ = 0,287 4
From tables, θ = 73,3°
Since cos θ is positive, θ is acute.

(b) sin θ = 0,936 1
From tables, θ = 69,4°
But sin 69,4° = sin (180 − 69,4)°
= sin 110,6°
θ = 69,4° or θ = 110,6°

(c) θ = −0,822 4
Since cos θ is negative, θ is obtuse.
First find the acute angle whose cosine is 0,822
From tables, 0,822 4 = cos 34,67°
⇒ −0,822 4 = cos (180 − 34,67)°
= cos 145,33°
⇒ θ = 145,33°

Or by scientific calculator (set in degree mode

Key	*Display*
AC	0
. 8 2 2 4 +/−	−0.8224
SHIFT	−0.8224
cos⁻¹	145.3257

θ = 145,33°

(d) $\tan\theta = -2{,}164$
Since $\tan\theta$ is negative, θ is obtuse.
From tables, $2{,}164 = \tan 65{,}2°$

$$\Rightarrow \quad \theta = 180° - 65{,}2° = 114{,}8°$$

Exercise 4b

Find the values of θ lying between 0° and 180° in each of the following. Give the answers in degrees to 1 or 2 d.p. where appropriate.

1 $\cos\theta = 0{,}809\,0$	**2** $\cos\theta = -0{,}809\,0$
3 $\tan\theta = 3{,}732$	**4** $\tan\theta = -3{,}732$
5 $\sin\theta = 0{,}920\,5$	**6** $\tan\theta = -1{,}963$
7 $\cos\theta = -0{,}939\,7$	**8** $\sin\theta = 0{,}422\,6$
9 $\cos\theta = -0{,}139\,2$	**10** $\tan\theta = -0{,}624\,9$
11 $\cos\theta = -0{,}727\,8$	**12** $\sin\theta = 0{,}608\,8$
13 $\sin\theta = 0{,}964\,6$	**14** $\tan\theta = -2{,}106$
15 $\tan\theta = -0{,}316\,6$	**16** $\cos\theta = -0{,}425\,3$
17 $\sin\theta = 0{,}832\,9$	**18** $\tan\theta = -0{,}723\,0$
19 $\sin\theta = 0{,}959\,4$	**20** $\cos\theta = -0{,}784\,6$
21 $\tan\theta = -1{,}678$	**22** $\sin\theta = 0{,}262\,8$
23 $\sin\theta = 0{,}744\,9$	**24** $\tan\theta = -12{,}43$

The sine rule

In any $\triangle$ABC, the angles are usually denoted by the capital letters A, B, C and the sides opposite these angles by a, b, c respectively.

Given: Any $\triangle$ABC (acute and obtuse-angled $\triangle$s are given in Fig. 4.6)

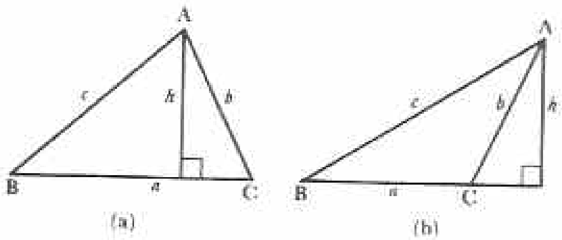


Fig. 4.6

To prove: $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C}$

Construction: Draw the perpendicular from A to BC (produced, if necessary)

Proof:
In Fig. 4.6(a) and (b),

$$\sin B = \frac{h}{c} \qquad (1)$$

In Fig. 4.6(a),

$$\sin C = \frac{h}{b} \qquad (2)$$

In Fig. 4.6(b),

$$\sin(180° - C) = \frac{h}{b}$$

$$\Rightarrow \sin C = \frac{h}{b} \qquad [\sin(180° - \theta) = \sin\theta] \quad (2)$$

From (1) $h = c\sin B$
From (2) $h = b\sin C$

$$\Rightarrow c\sin B = b\sin C$$

$$\Leftrightarrow \frac{b}{\sin B} = \frac{c}{\sin C}$$

Similarly, by drawing a perpendicular from C to AB,

$$\frac{a}{\sin A} = \frac{b}{\sin B}$$

$$\Rightarrow \frac{\mathbf{a}}{\mathbf{\sin A}} = \frac{\mathbf{b}}{\mathbf{\sin B}} = \frac{\mathbf{c}}{\mathbf{\sin C}}$$

This formula is used for solving triangles which are not right-angled and in which either **two angles and any side** are given or **two sides and the angle opposite one of them** are given.

Example 2

In $\triangle$ABC, B = 39°, C = 82°, a = 6,73 cm. *Find* c.

First draw a sketch of the information (Fig. 4.7).

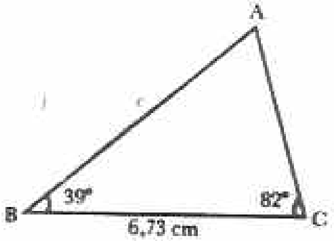


Fig. 4.7

In Fig. 4.7

$$A = 180° - (39° + 82°)$$
$$= 59°$$

$$\frac{c}{\sin C} = \frac{a}{\sin A}$$

$$\frac{c}{\sin 82°} = \frac{6,73}{\sin 59°}$$

$$\Leftrightarrow \quad c = \frac{6,73 \times \sin 82°}{\sin 59°} \text{ cm}$$

$$= 7,775 \text{ cm}$$
$$= 7,78 \text{ cm to 2 d.p.}$$

working:

No.	Log
6,73	0,8280
sin 82°	$\bar{1}$,9958
	0,8238
sin 59°	$\bar{1}$,9331
7,775	0,8907

Tables of logarithms of sines were used in the above working.

This calculation may also be done on a scientific calculator as follows:

Key	*Display*
AC	0
5 9 sin	0.8571673
M+ (or Min)	0.8571673
8 2 sin	0.990268
× 6 · 7 3	6.73
÷ MR	0.8571673
=	7.7750331

$c = 7,78$ cm

In the above calculator sequence, note the following:

1 the use of the memory keys,

2 the data are entered in virtually the reverse order to that given in the worked example.

Example 3

Find the remaining angles of △ABC in which $a = 12,5$ cm, $c = 17,7$ cm and $C = 116°$.

First make a sketch of the information.

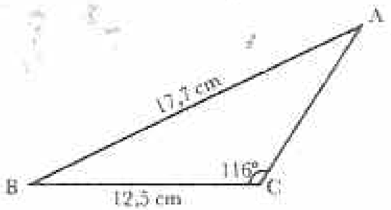


Fig. 4.8

In Fig. 4.8, c and C are known. Since a is also given, A can be found using the sine rule. The formula of the sine rule can be arranged so that the unknown comes first:

$$\frac{\sin A}{a} = \frac{\sin C}{c}$$

$$\frac{\sin A}{12,5} = \frac{\sin 116°}{17,7}$$

$$\Leftrightarrow \sin A = \frac{12,5 \times \sin 116°}{17,7}$$

$$= \frac{12,5 \times \sin 64°\,*}{17,7}$$

$$\Rightarrow A = 39,4° \text{ or } (180 - 39,4)°$$
$$= 39,4° \text{ or } 140,6°$$

working:

No.	Log
12,5	1,0969
sin 64°	$\bar{1}$,9537
	1,0506
17,7	1,2480
sin 39,4°	$\bar{1}$,8026

But C is obtuse, therefore A cannot be obtuse.
$\Rightarrow A = 39,4°$ (or 39°24′)

* *Sequence for scientific calculator:*

Key	*Display*
AC	0
6 4 sin	0.898794
× 1 2 · 5	12.5
÷ 1 7 · 7 =	0.6347415
SHIFT	0.6347415
$\sin^{-1}$	39.400819

$A = 39,4°$

$$B = 180° - (39,4 + 116)°$$
$$= 180° - 155,4°$$
$$= 24,6° \text{ (or } 24°36'\text{)}$$

The calculator sequences given in Examples 2 and 3 may be applied to the remaining examples and exercises in this chapter. See also Chapter 20 for further advice regarding the use of the scientific calculator to solve triangles.

Example 4

In △ABC, $a = 7{,}1$ cm, $b = 9{,}5$ cm and B = 63°18′. Solve the triangle completely.

To solve a triangle *completely* means to calculate all the unknown sides and angles.

Make a sketch of the given information.

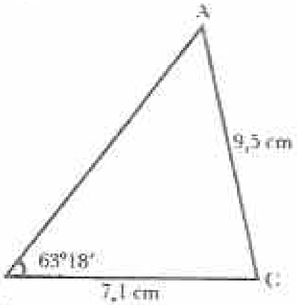


Fig. 4.9

$63°18' = 63\frac{18}{60}° = 63\frac{3}{10}° = 63{,}3°$

$$\frac{\sin A}{a} = \frac{\sin B}{b}$$

$$\frac{\sin A}{7{,}1} = \frac{\sin 63{,}3°}{9{,}5}$$

$$\Leftrightarrow \sin A = \frac{7{,}1 \times \sin 63{,}3°}{9{,}5}$$

working:

No.	Log
7,1	0,8513
sin 63,3°	$\bar{1}$,9510
	0,8023
9,5	0,9777
sin 41,89°	$\bar{1}$,8246

$$A = 41{,}89° \text{ or } (180 - 41{,}89)°$$
$$= 41{,}89° \text{ or } 138{,}11°$$

But $a < b$

$\Leftrightarrow$ A < B

$\Rightarrow$ A = 41,89°

$\Rightarrow$ C = 180° − (63,3 + 41,89)°

= 180° − 105,19°

= 74,81°

$$\frac{c}{\sin C} = \frac{b}{\sin B}$$

$$\frac{c}{\sin 74{,}81°} = \frac{9{,}5}{\sin 63{,}3°}$$

$$\Leftrightarrow c = \frac{9{,}5 \times \sin 74{,}81°}{\sin 63{,}3°}\text{ cm}$$

$$= 10{,}26 \text{ cm}$$

working:

No.	Log
9,5	0,9777
sin 74,81°	$\bar{1}$,9845
	0,9622
sin 63,3°	$\bar{1}$,9510
10,26	1,0112

Answers, correct to 1 d.p.

A = 41,9°, C = 74,8°, $c = 10{,}3$ cm

Answers, with angles correct to the nearest minute:

A = 41°53′, C = 74°49′, $c = 10{,}3$ cm

In general, four-figure tables give answers which are correct to 3 s.f. Answers should be rounded to 3 s.f. If a question gives angles in degrees and minutes, then answers should be given correct to the nearest minute. Note that 41,89° = 41° + (0,89 × 60)′ = 41° + 53,4′ = 41°53′ to the nearest minute.

Exercise 4c

1

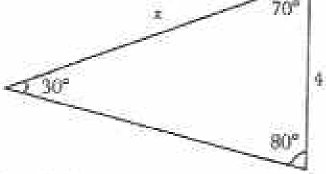


Fig. 4.10

Using the information given in Fig. 4.10, write down, but do not solve, an equation which can be used to find x.

2 In △ABC, A = 29°, B = 36°, $b = 15{,}8$ cm. Find a.

3 In △ABC, A = 54°12′, B = 71°30′, $a = 12{,}4$ cm. Find b.

4 In △ABC, B = 104,3°, C = 31,3°, $a = 29{,}0$ cm. Calculate c.

5 In △PQR, P = 83°, $p = 285$ m, $r = 216$ m. Calculate R.

6 In △ABC, C = 53°, $b = 3{,}56$ m, $c = 4{,}28$ m. Calculate B.

7 In △ABC, A = 115°, $a = 65$ m, $b = 32$ m. Solve the triangle completely.

8 In △ABC, B = 25°36′, C = 124°24′, $c = 39{,}2$ m. Solve the triangle completely.

9 In △XYZ, Y = 29,8°, Z = 51,4°, $x = 19{,}6$ cm. Solve the triangle completely.

10 In △ABC, A = 38°18′, $a = 252$ m, $b = 198$ m. Solve the triangle completely.

11 In △ABC, C = 96,2°, $b = 11{,}2$ cm, $c = 39{,}4$ cm. Solve the triangle completely.

12 Calculate the values of angles A and C of △ABC, where $b = 14{,}35$ cm, $a = 7{,}82$ cm and B = 115°36′.

Example 5

Two ships A and B leave a port P at the same time. A travels on a bearing of 159° and B travels on a bearing of 215°. After some time, A is 9 km from P, and the bearing of B from A is 256°. Calculate the distance of B from P.

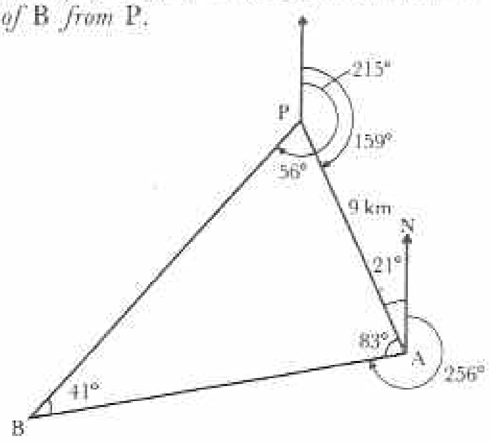


Fig. 4.11

Fig. 4.11 shows the positions of P, A and B. The arrows at P and A point northwards.

In $\triangle PAB$,

$$B\hat{P}A = 215° - 159° = 56°$$
$$N\hat{A}P = 180° - 159° = 21°$$
$$\Rightarrow P\hat{A}B = 360° - (256 + 21)° = 360° - 277° = 83°$$
$$\Rightarrow \quad \hat{B} = 180° - (56 + 83)° = 180° - 139° = 41°$$

$$\frac{PB}{\sin 83°} = \frac{9}{\sin 41°}$$

$$\Leftrightarrow \quad PB = \frac{9 \times \sin 83°}{\sin 41°}\text{ km}$$
$$= 13{,}61 \text{ km}$$
$$= 13{,}6 \text{ km to 3 s.f.}$$

working:

No.	Log
9	0,9542
sin 83°	$\bar{1}$,9968
	0,9510
sin 41°	$\bar{1}$,8169
13,61	1,1341

Ship B is 13,6 km from P.

In Example 5, the bearings 159°, 215°, 256° are examples of **three-figure true bearings**. Each bearing is measured clockwise from north and is given in the range 000° to 360°.

Example 6

A tree is on a bearing S 36° W *from a point* X *and* S 78°E *from a point* Y. *If* X *is* 200 m *due east of* Y, *calculate the distance of the tree from* Y *to the nearest metre.*

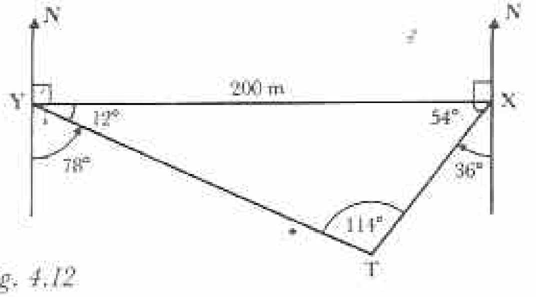


Fig. 4.12

In Fig. 4.12, T represents the position of the tree.

In $\triangle XYT$,

$$X\hat{Y}T = 90° - 78° = 12°$$
$$Y\hat{X}T = 90° - 36° = 54°$$
$$Y\hat{T}X = 180° - (12 + 54)° = 180° - 66° = 114°$$

$$\frac{YT}{\sin 54°} = \frac{200}{\sin 114°}$$

$$\Leftrightarrow \quad YT = \frac{200 \times \sin 54°}{\sin 114°}\text{ m}$$
$$= \frac{200 \sin 54°}{\sin 66°}\text{ m}$$
$$= 177{,}1 \text{ m}$$
$$= 177 \text{ m to the nearest m}$$

working:

No.	Log
200	2,3010
sin 54°	$\bar{1}$,9080
	2,2090
sin 66°	$\bar{1}$,9607
177,1	2,2483

The tree is 177 m from Y.

The bearing S 36° W is an example of a **compass bearing**. To find the direction S 36°W, first face south then turn through 36° in a westerly direction. In Fig. 4.12 it can be seen that S 36° W is the same as the three-figure bearing 216° (i.e. 180° + 36°). Similarly, the bearing S 78°E is equivalent to the three-figure bearing 102° (i.e. 180° − 78°).

Examples 5 and 6 show the importance of drawing a fully labelled diagram.

Exercise 4d

1 A point X is 34 m due east of a point Y. The bearings of a flagpole from X and Y are N 18° W and N 40°E respectively. Calculate the distance of the flagpole from Y.

2 A man walks due west for 4 km. He then changes direction and walks on a bearing of 197° until he is south-west of his starting point. How far is he then from his starting point?

3 A girl starts from a point A and walks 285 m to B on a bearing of 078°. She then walks due south to a point C which is 307 m from A. What is the bearing of A from C, and what is BC?

4 The bearing of a house from a point A is 319°. From a point B, 317 m due east of A, the bearing of the house is 288°. How far is the house from A?

5 A mass is hung from a horizontal beam by two strings as in Fig. 4.13. Calculate the length of the longer string.

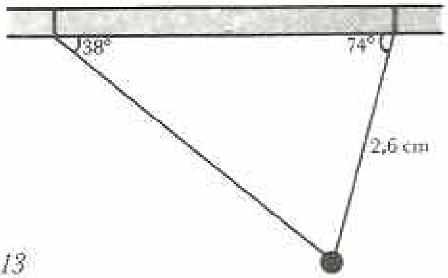


Fig. 4.13

6 An aircraft is timetabled to travel from A to B. Due to bad weather it flies from A to C then from C to B, where AC and CB make angles of 27° and 66° respectively with AB. If AC = 220 km, calculate AB.

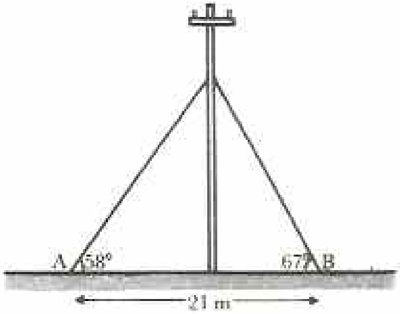


Fig. 4.14

7 Two wires support an electricity pole as shown in Fig. 14.14.
If the wires make angles of 58° and 67° with the ground and AB = 21 m, calculate the lengths of the wires.

8 From points A and B on level ground, the angles of elevation of the top of a building are 25° and 37° respectively. If AB = 57 m, calculate, to the nearest metre, the distances of the top of the building from A and B.

9 The bearings of ships A and B from a port P are 225° and 116° respectively. Ship A is 3,9 km from ship B on a bearing of 258°. Calculate the distance of ship A from P.

10 An aeroplane flies from a town X on a bearing 045° to another town Y, a distance of 200 km. It then changes course and flies to another town Z on a bearing 120°. If Z is due east of X, calculate (a) the distance from X to Z, (b) the distance from Y to XZ.

11 PQR is an acute-angled triangle in which $P\hat{Q}R = 42°14'$. S is the point on QR such that $P\hat{S}Q = 90°$. Given that PR = 5,23 cm and SR = 2,37 cm, calculate (a) $P\hat{R}Q$, (b) QR.

12 A ship sails directly from a port P to a point Q, which is 10 nautical miles due east of a lighthouse L.

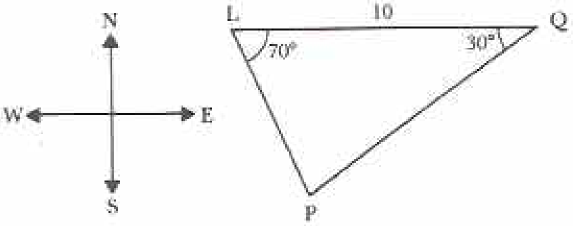


Fig. 4.15

Given that $P\hat{L}Q = 70°$ and $P\hat{Q}L = 30°$, calculate

(a) the bearing of Q from P,
(b) the bearing of L from P,
(c) the distance PQ,
(d) the shortest distance between the lighthouse and the ship during its journey.

Chapter 5

Graphs (3) Gradient

Gradient of a straight line

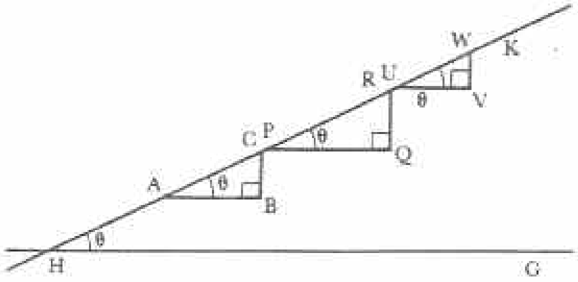


Fig. 5.1

In Fig. 5.1, HG is a horizontal line and HK is a line which makes an angle θ with HG. The $\triangle$s ABC, PQR, UVW are similar.

$$\text{Hence } \frac{BC}{AB} = \frac{QR}{PQ} = \frac{VW}{UV}.$$

Each of these fractions measures the **gradient** of the line HK. Hence the gradient of a straight line is the same at any point on it.

Also $\tan\theta = \frac{BC}{AB} = \frac{QR}{PQ} = \frac{VW}{UV}$, so $\tan\theta$ is also a measure of the gradient.

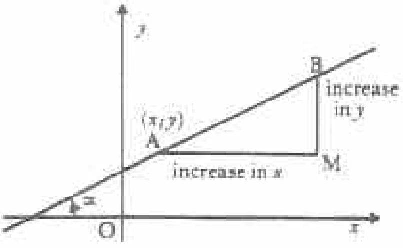


Fig. 5.2

In Fig. 5.2 the point A has coordinates $(x; y)$. In going from A to B the *increase* in x is AM. The corresponding *increase* in y is MB.

Gradient of AB

$$= \frac{\text{increase in } y \text{ from A to B}}{\text{increase in } x \text{ from A to B}} = \frac{MB}{AM}$$

Since y *increases* as x increases, the gradient is *positive*. AB makes an acute angle α with the positive direction of the x-axis and $\tan\alpha$ is positive.

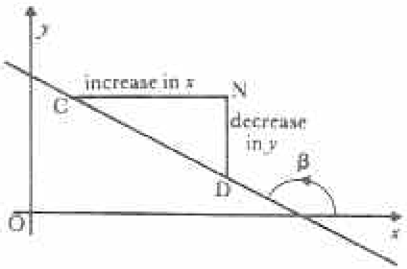


Fig. 5.3

In Fig. 5.3 the point C has coordinates $(x; y)$. In going from C to D the *increase* in x is CN. The corresponding *decrease* in y is ND. Consider a decrease to be a negative increase:

Gradient of CD

$$= \frac{\text{increase in } y \text{ from C to D}}{\text{increase in } x \text{ from C to D}} = \frac{-ND}{CN}$$

Since y *decreases* as x increases, the gradient is *negative*. CD makes an obtuse angle β with the positive direction of the x-axis and $\tan\beta$ is negative.

In algebraic graphs, the gradient of a straight line is the **rate of change of y compared with x**. For example, if the gradient is 3, then for any increase in x, y increases 3 times as much. Compare this with rates of change in distance–time and velocity–time graphs.

Example 1

Find the gradients of the lines joining (a) A(−1; 2) and B(3; −2), (b) C(0; −1) and D(4; 1).

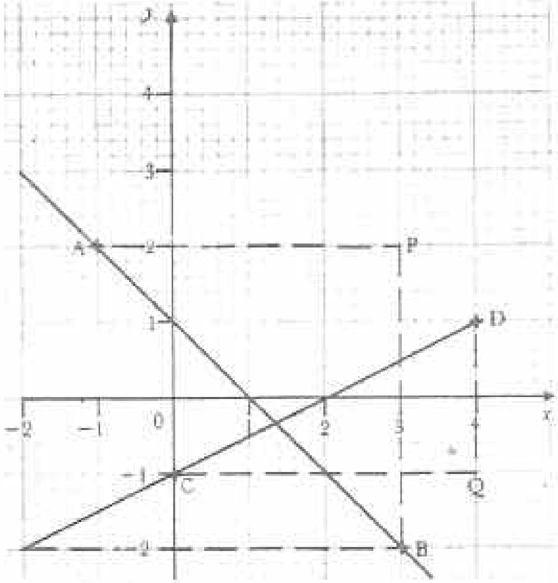


Fig. 5.4

Fig. 5.4 shows the points A, B, C, D and the lines AB and CD.

(a) $\text{Gradient of AB} = \dfrac{\text{increase in } y}{\text{increase in } x} = \dfrac{-PB}{AP}$

$$= \frac{-4}{4} = -1$$

Notice that the increase in y is negative (i.e. a decrease).

(b) $\text{Gradient of CD} = \dfrac{\text{increase in } y}{\text{increase in } x} = \dfrac{QD}{CQ}$

$$= \tfrac{2}{4} = \tfrac{1}{2}$$

The gradient can also be calculated without drawing the graph: consider a line which passes through the points $(x_1; y_1)$ and $(x_2; y_2)$.
Gradient of the line

$$= \frac{\text{increase in } y}{\text{increase in } x}$$

$$= \frac{\text{difference in the } y \text{ coordinates}}{\text{difference in the } x \text{ coordinates}} = \frac{y_2 - y_1}{x_2 - x_1}$$

For example, in Example 1,

$$\text{Gradient of AB} = \frac{(-2) - (2)}{(3) - (-1)} = \frac{-4}{4} = -1$$

$$\text{Gradient of CD} = \frac{(1) - (-1)}{(4) - (0)} = \frac{2}{4} = \tfrac{1}{2}$$

Exercise 5a

Find the gradients of the lines joining the following pairs of points.

1 (9; 7), (2; 5) **2** (2; 5), (4; 8)
3 (5; 3), (0; 0) **4** (6; 1), (1; 5)
5 (0; 4), (3; 0) **6** (−3; 2), (4; 4)
7 (2; 3), (6; −5) **8** (−4; 3), (8; −6)
9 (−4; −4), (−1; 5) **10** (7; −2), (−1; 2)

Example 2

(a) Draw a graph of the line represented by the equation $4x + 2y = 5$. (b) Find the gradient by taking measurements.

(a) First make a table of values.
When $x = 0$, $0 + 2y = 5 \Leftrightarrow y = 2\frac{1}{2}$;
when $x = 1$, $4 + 2y = 5 \Leftrightarrow y = \frac{1}{2}$;
when $x = 2$, $8 + 2y = 5 \Leftrightarrow y = -1\frac{1}{2}$.

x	0	1	2
y	$2\frac{1}{2}$	$\frac{1}{2}$	$-1\frac{1}{2}$

Fig. 5.5 shows the required graph.

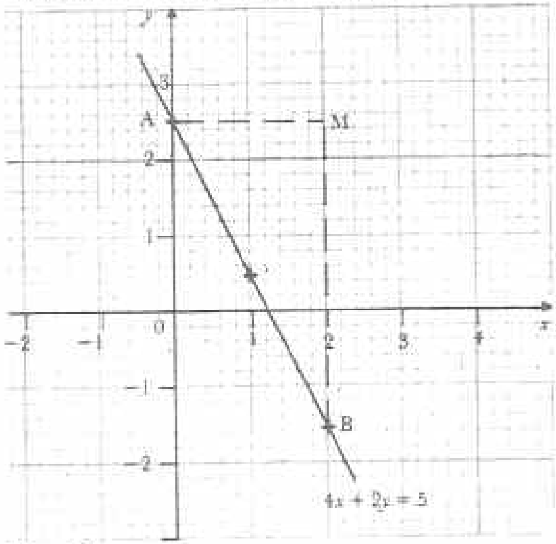


Fig. 5.5

(b) Choose two convenient points, such as A and B in Fig. 5.5.

$$\text{Gradient of AB} = \frac{\text{increase in } y}{\text{increase in } x} = \frac{-\text{MB}}{\text{AM}}$$

$$= \frac{-4}{2} = -2$$

Exercise 5b

Draw the graphs of the lines represented by the following equations. In each case find the gradient by taking measurements.

1 $y = 3x + 1$
2 $y = 3x - 2$
3 $y = -2x + 3$
4 $4x - 2y + 1 = 0$
5 $2x + 3y = 0$
6 $2x + 3y = 6$
7 $4x - 3y = 5$
8 $2x - 5y = 6$
9 $5x - 2y = 5$
10 $7x + 4y - 8 = 0$

Sketching graphs of straight lines

From Exercise 5b it can be seen that the gradient of a line depends only on the coefficients of x and y in its equation. For example, in questions 1 and 2 the following results were obtained:

$y = 3x + 1$, gradient 3
$y = 3x - 2$, gradient 3

The results of questions 5 and 6 were as follows:

$2x + 3y = 0$, gradient $-\frac{2}{3}$
$2x + 3y = 6$, gradient $-\frac{2}{3}$

Notice that the last two equations can be rearranged:

$2x + 3y = 0$
$\Leftrightarrow \quad 3y = -2x$
$\Leftrightarrow \quad y = -\frac{2}{3}x$
$2x + 3y = 6$
$\Leftrightarrow \quad 3y = -2x + 6$
$\Leftrightarrow \quad y = -\frac{2}{3}x + 2$

Hence, when y is the subject of the equation the coefficient of x gives the gradient. An equation in the form $\boldsymbol{y = mx + c}$ is that of a straight line with gradient m.

If the gradient and one point on the line are known, it is possible to make a rough sketch of the graph.

Example 3

Make a rough sketch of the line whose equation is $2x + 4y = 9$.

First: Rearrange the equation to make y the subject.

$2x + 4y = 9$
$\quad 4y = -2x + 9$
$\quad y = -\frac{1}{2}x + 2\frac{1}{4}$

Second: Find a point on the line. The simplest point is usually that where $x = 0$. When $x = 0$, $y = 2\frac{1}{4}$. $(0; 2\frac{1}{4})$ is a point on the line. Fig. 5.6 is a rough sketch of the line.

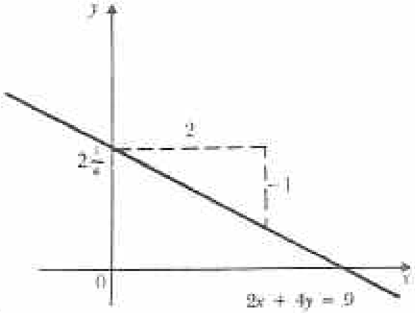


Fig. 5.6

Notice that the axes x and y should always be shown on a rough sketch.

An alternative method of sketching a straight line is to find the two points where the line crosses the axes.

Example 4

Sketch the graph of the line whose equation is $4x - 3y = 12$.

When $x = 0$, $-3y = 12$
$\Leftrightarrow \quad y = -4$

The line crosses the y-axis at $(0; -4)$.
When $y = 0$, $4x = 12$
$\Leftrightarrow \quad x = 3$

The line crosses the x-axis at $(3, 0)$.
Fig. 5.7 is a rough sketch of the line.

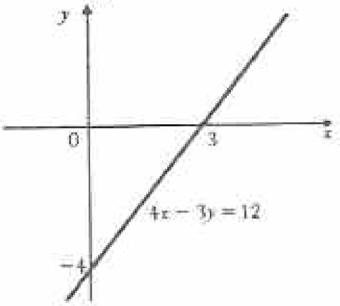


Fig. 5.7

Any line which is parallel to the x-axis has a **zero gradient**. The equations of such lines are always in the form $y = c$, where c may be any number. Fig. 5.8 shows the graphs of $y = 5$ and $y = -3$.

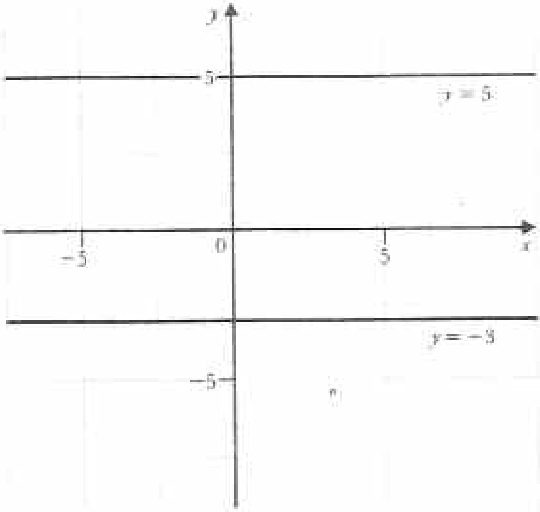


Fig. 5.8

Notice that the equation of the x-axis is $y = 0$.

The gradient of a line which is parallel to the y-axis is undefined (i.e. cannot be found). The equations of such lines are always in the form $x = a$, where a may be any number. Fig. 5.9 shows the graphs of the lines $x = 2$ and $x = -4$.

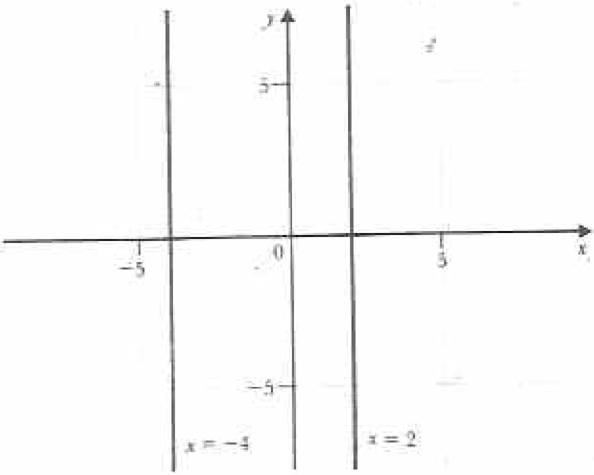


Fig. 5.9

Notice that the equation of the y-axis is $x = 0$.

Exercise 5c

1 Sketch the lines which pass through the following points with the given gradients:
(a) point $(2; 1)$, gradient 3
(b) point $(5; 0)$, gradient -2
(c) point $(1; -3)$, gradient -3
(d) point $(-4; -2)$, gradient $\frac{2}{3}$
(e) point $(5; -2)$, gradient $-\frac{4}{3}$

2 Write down the gradients of the lines represented by the following equations. Hence sketch the graphs of the lines.
(a) $y = 2x + 3$
(b) $y = \frac{1}{3}x$
(c) $y = \frac{5}{4}x - 2$
(d) $3x + 7y = 5$
(e) $4x - 7y = 7$

3 Find the coordinates of the points where the lines represented by the following equations cross the axes. Hence sketch the graphs of the lines.
(a) $y = 2x - 2$
(b) $y = \frac{1}{3}x + 1$
(c) $3x - 5y = 30$
(d) $4x + 3y = 2$
(e) $8x + 5y = 4$

4 Write down the gradients of the lines represented by the sketches in Fig. 5.10.

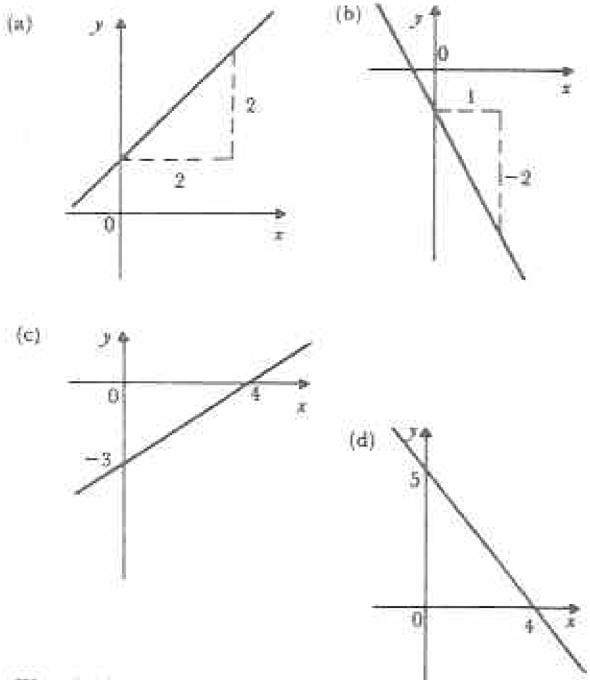


Fig. 5.10

5 Write down the gradients of the lines (a), (b), (c), (d), (e) in Fig. 5.11.

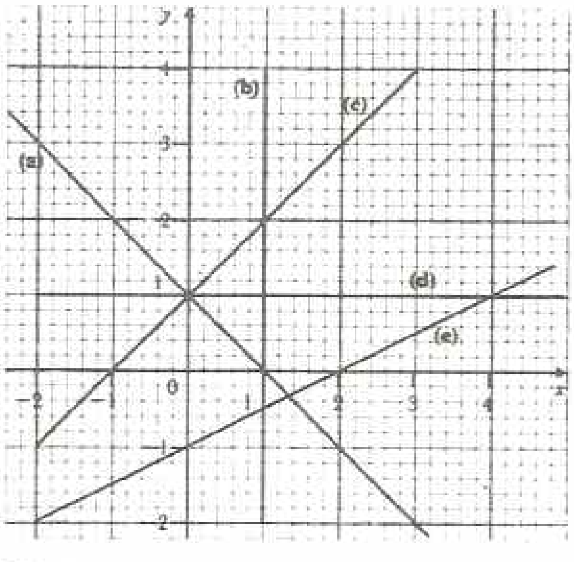


Fig. 5.11

Equation of a straight line

(a) Given its gradient and a point on the line

Example 5

A straight line of gradient 5 passes through the point B(3; −8). Find the equation of the line.

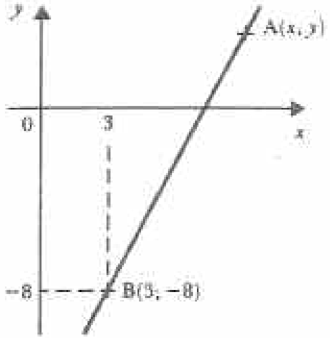


Fig. 5.12

Fig. 5.12 is a sketch of the line.
In Fig. 5.12 the point A(x; y) is any general point on the line.

$$\text{Gradient of AB} = \frac{y-(-8)}{x-3}$$

$$= \frac{y+8}{x-3}$$

Hence $\dfrac{y+8}{x-3} = 5$ since the gradient of AB is 5.

$$y + 8 = 5(x-3)$$
$$= 5x - 15$$
$$y = 5x - 23$$

The equation of the line is $y = 5x - 23$.

In general, the equation of a straight line of gradient m which passes through the point $(a; b)$ is given by

$$\frac{y-b}{x-a} = m$$

(b) Given two points on the line

Example 6

Find the equation of the straight line which passes through the points Q(−1; 7) *and* R(3; −2).

Figure 5.13 is a sketch of the line through Q and R.

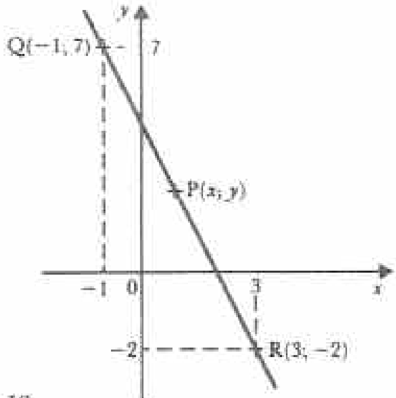


Fig. 5.13

In Fig. 5.13 the point $P(x;y)$ is any general point on the line.

$$\text{Gradient of QR} = \frac{7-(-2)}{(-1)-3} = \frac{9}{-4} = -2\tfrac{1}{4}$$

$$\text{Gradient of PR} = \frac{y-(-2)}{x-3} = \frac{y+2}{x-3}$$

But PQR is a straight line, hence

gradient of PR = gradient of QR

$$\frac{y+2}{x-3} = -2\tfrac{1}{4}$$

$$y+2 = -2\tfrac{1}{4}(x-3)$$

$$y+2 = -2\tfrac{1}{4}x + 6\tfrac{3}{4}$$

$$y = -2\tfrac{1}{4}x + 4\tfrac{3}{4}$$

The equation of the line is $y = -2\frac{1}{4}x + 4\frac{3}{4}$.

In general, the equation of a straight line which passes through the points $(a;b)$ and $(c;d)$ is given by:

$$\frac{y-b}{x-a} = \frac{d-b}{c-a}$$

Exercise 5d

1 Find the equation of the line which passes through the point
(a) (4; 9) and has a gradient of 3,
(b) (0; 0) and has a gradient of 3,
(c) (−2; 8) and has a gradient of −1,
(d) (6; 0) and has a gradient of $-\frac{3}{4}$,
(e) (0; −5) and has a gradient of −4,
(f) (−1; 2) and has a gradient of $2\frac{1}{2}$.

2 Find the equation of the line which passes through the points
(a) (0; 0) and (3; 7),
(b) (0; 0) and (3; −7),
(c) (−1; 4) and (5; −2),
(d) (−6; −6) and (4; −3),
(e) (7; 2) and (−9; 7),
(f) (2; −11) and (−4; 4).

3 A straight line is drawn through the points (7; 0) and (−2; 3). Find (a) its gradient, (b) its equation.

4 A straight line of gradient $4\frac{1}{2}$ passes through the point (4; −3). Write down (a) the equation of the line, (b) the equation of a parallel line which passes through the point $(0; \frac{1}{2})$.

5 Line l passes through the point (10; −1). Line m passes through the point $(-1\frac{1}{2}; -4\frac{1}{2})$. Find the equations of l and m if both lines pass through the point (1; 2).

6 (a) Find the gradient of the straight line through the points (2; −2) and (7; 8). (b) Find the equation of the straight line which passes through the point (3; 5) and has gradient −4. [Camb]

7 (a) Find the equation of the straight line which passes through the points (0; 5) and (5; 0).
(b) Show that the equation of the straight line which passes through $(0; a)$ and $(a; 0)$ is $x + y = a$.

8 (a) Write down an expression for the gradient of the line joining the points $(6; k)$ and (4; 1). Find the value of k if this gradient is $\frac{3}{5}$.
(b) Find the equation of the line through the point (−4; 5) with gradient −2. [Camb]

9 Find the equations of the sides of a triangle which has vertices A(0; 0), B(7; 0), C(5; 6).

10 Lines r and s both pass through the point $(k; 9)$. Line r has a gradient of $-\frac{4}{3}$ and passes through the point (5; −3).
(a) Find the value of k.
(b) Find the equation of line s given that it crosses the x-axis at (−14; 0).

Gradient of a curve

The gradient at any particular point on a curve is defined as being the gradient of the tangent to the curve at that point.

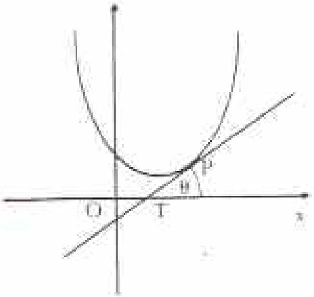


Fig. 5.14

In Fig. 5.14 the gradient of the curve at point P is the gradient of the tangent TP, i.e. $\tan\theta$. The tangent is drawn by placing a ruler against the curve at P and drawing a line, taking care that the 'angles' between the line and the curve appear equal.

Notice that the gradient of a straight line is the same at any point on the line, but that the gradient of a curve changes from point to point.

Example 7

Fig. 5.15 is the graph of the curve $y = 2 + x - x^2$ for values of x from −2 to 3. Use the given tangents to find the gradient of the curve at (a) P, (b) Q.

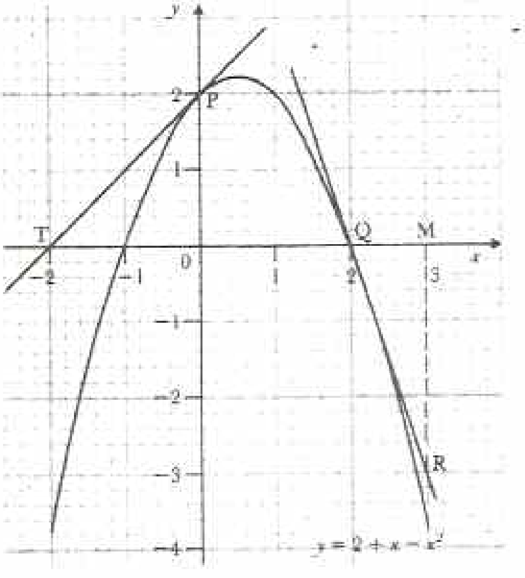


Fig. 5.15

(a) Gradient of the curve at P
= gradient of tangent TP

$$= \frac{OP}{TO} = \frac{2}{2} = 1$$

(b) Gradient of the curve at Q
= gradient of tangent QR

$$= \frac{-MR}{QM} = \frac{-3}{1} = -3$$

Notice that △s TOP and QMR were used find the gradients of the tangents. Any suital right-angled triangles could have been used. this case the intercepts TO and QM were convenient lengths.

Example 8

Draw the graph of $y = \frac{1}{4}x^2$ for values of x from to 3. Find the gradient of the curve at the point wh x has the value (a) 3, (b) −2.

Table 5.1 is the table of values.

Table 5.1

x	−2	−1	0	1	2	3
y	1	$\frac{1}{4}$	0	$\frac{1}{4}$	1	$2\frac{1}{4}$

Fig. 5.16 shows the required graph.

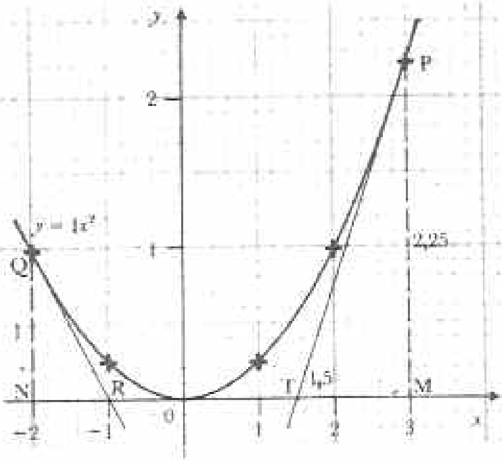


Fig. 5.16

Using the tangents drawn to the curve where $x = 3$ and $x = -2$:

(a) Gradient of the curve where $x = 3$

$= \text{gradient of tangent PT}$

$$= \frac{MP}{TM} = \frac{2{,}25}{1{,}5} = \frac{2\frac{1}{4}}{1\frac{1}{2}} = \frac{9}{6} = 1\tfrac{1}{2}$$

When $x = 3$, the gradient of the curve is $1\frac{1}{2}$ (i.e. at P, y is increasing $1\frac{1}{2}$ times as fast as x).

(b) Gradient of curve where $x = -2$

$= \text{gradient of tangent QR}$

$$= \frac{-QN}{NR} = \frac{-1}{1} = -1$$

When $x = -2$, the gradient of the curve is -1 (i.e. at Q, y is *decreasing* at the same rate as x is increasing).

Notice the following points:

1 In Example 8, the lengths MP, TM, QN, NR are measured according to the scales of the axes.

2 In Examples 7 and 8, the method of drawing tangents using a ruler can give inaccurate results. Gradients found by this method must only be taken as approximate.

Fig. 5.17 shows the tangents drawn at the **turning points** of two quadratic functions.

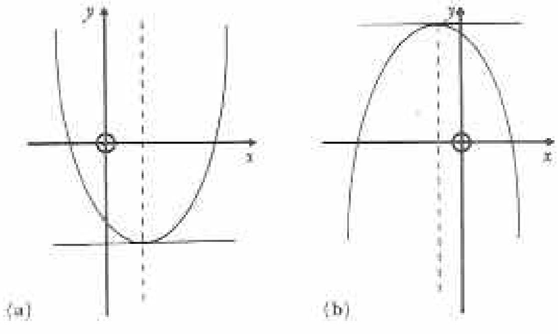


Fig. 5.17

In each case the tangent is parallel to the x-axis. Hence the gradient at a turning point is zero. In Fig. 5.17(a) the turning point corresponds to the **minimum** value of the function. In Fig. 5.17(b) the turning point corresponds to the **maximum** value of the function. In each figure the line of symmetry of the curve is shown by a broken line.

Exercise 5e

1 Write down the equation of the line of symmetry of the curve in (a) Fig. 5.15, (b) Fig. 5.16.

2 Fig. 5.18 is the graph of the function $x^2 - 6x + 4$.

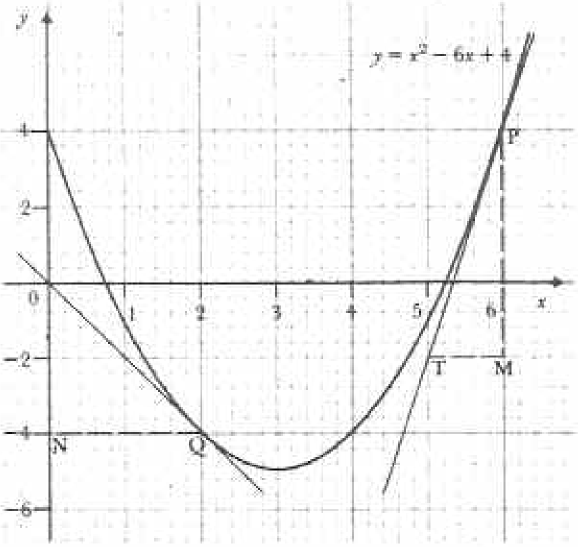


Fig. 5.18

(a) Use the given tangents to find the gradient of the curve (i) at P, (ii) at Q.

(b) Find the minimum value of the function.

(c) Write down the equation of the line of symmetry of the curve.

3 Fig. 5.19 is the graph of $y = 3 - 2x - x^2$.

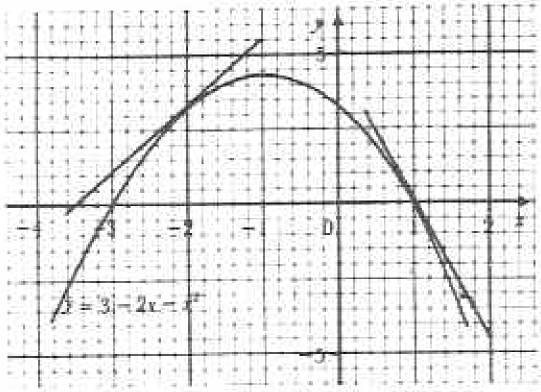


Fig. 5.19

(a) Use the given tangents to find the gradient of the curve (i) when $x = -2$, (ii) when $x = 1$.
(b) What is the maximum value of $3 - 2x - x^2$?
(c) Write down the equation of the line of symmetry of the curve.

4 (a) Copy and complete Table 5.2 for the relation $y = 3x - x^2$.

Table 5.2

x	-1	0	1	2	3	4
y	-4		2			

(b) Draw the graph of $y = 3x - x^2$ from $x = -2$ to $x = 4$, using a scale of 2 cm to 1 unit on both axes.
(c) Find the gradient of the curve at (i) $x = 0$, (ii) $x = 2$.
(d) Write down the equation of the line of symmetry of the curve
(e) Find the maximum value of $3x - x^2$.

5 Draw the graph of $y = x^2$ for values of x from -4 to 4. Use a scale of 1 cm to 1 unit on the x-axis and 2 cm to 1 unit on the y-axis. Find the gradient at the point where (a) $x = 3$, (b) $x = 1{,}5$, (c) $x = -2$.

6 Copy and complete Table 5.3 giving values for the function $y = 2x^2 - 4x + 3$ from $x = -2$ to $x = 4$.

Table 5.3

x	-2	-1	0	1	2	3	4
y	19		3	1			

Draw the graph of $y = 2x^2 - 4x + 3$, using a scale of 2 cm to 1 unit on the x-axis and 1 cm to 2 units on the y-axis.
From your graph, find
(a) the equation of the line of symmetry of the curve,
(b) the gradient of the curve at $x = 3$,
(c) the minimum value of y.

7 Draw the graph of $y = x^2 - 4x$ from $x = -1$ to $x = 5$. Use a scale of 2 cm to 1 unit on both axes. Find the gradient at the point where (a) $x = 4$, (b) $x = 2$, (c) $x = 0$.

8 Draw the graph of $y = 5x - 2x^2$ from $x = -1$ to $x = 4$. Use a scale of 2 cm to 1 unit on the x-axis and 1 cm to 1 unit on the y-axis. (a) Find the gradient of the curve at the point where $x =$ (i) 0, (ii) 1, (iii) 3. (b) Write down the equation of the line of symmetry of the curve.

9 Draw the graph of $y = x^2 - 3x + 2$ for values of x from -1 to 4. Find the gradient at the point where x has the value (a) $2\frac{1}{2}$, (b) $1\frac{1}{2}$, (c) 0, (d) $-\frac{1}{2}$.

10 Draw the graph of $y = 1 + x - x^2$ from $x = -2$ to $x = 3$. Find the gradient at the point where x has the value (a) $2\frac{1}{2}$, (b) $1\frac{1}{2}$, (c) $\frac{1}{2}$, (d) -1.

Chapter 6

Lengths and angles in solids

Angles between lines and planes

Fig. 6.1 shows a flagpole standing on horizontal ground. It is kept vertical by three straight wires attached to the pole at A and to the ground at K, L, M.

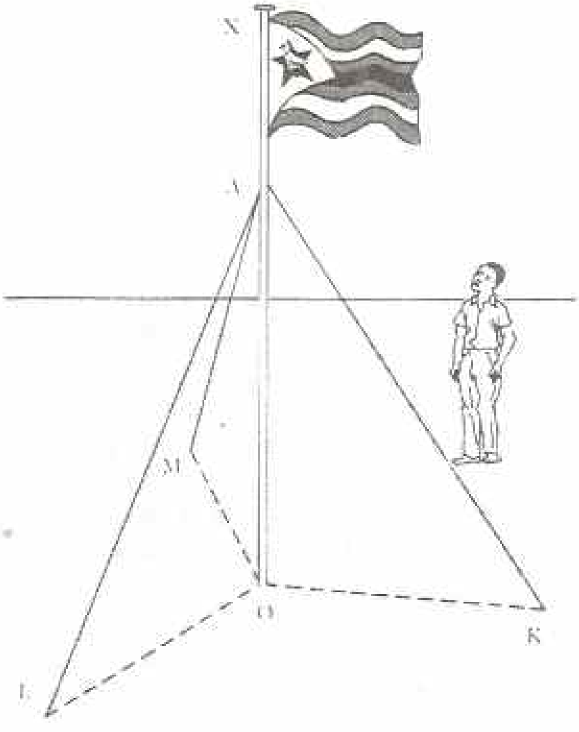


Fig. 6.1

The pole XO in Fig. 6.1 is perpendicular to the ground. The pole is said to meet the ground **normally** (i.e. perpendicularly).

The wires AK, AL, AM in Fig. 6.1 meet the ground **obliquely** (i.e. *not* perpendicularly). The angles between the wires and the ground are $A\hat{K}O$, $A\hat{L}O$, $A\hat{M}O$.

In Fig. 6.2 the line XY cuts the plane surface normally at O. Since XY is perpendicular to the plane, it is at right angles to every line drawn on the plane through O.

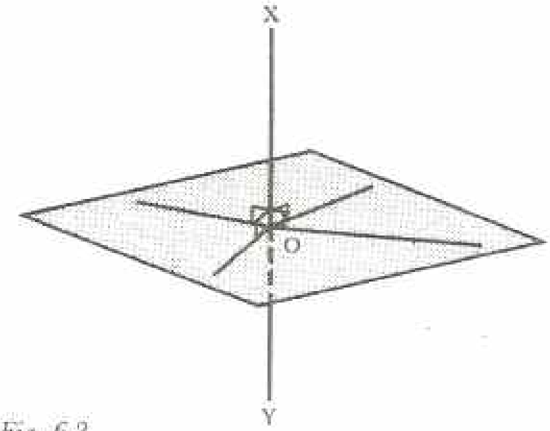


Fig. 6.2

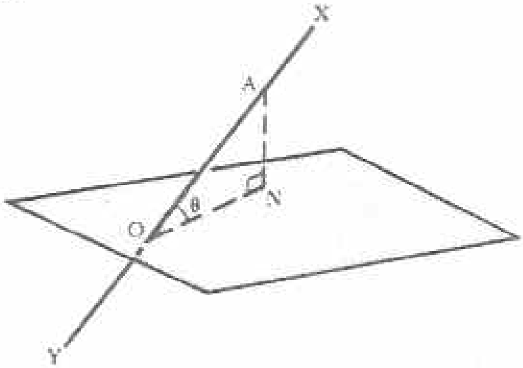


Fig. 6.3

In Fig. 6.3 the line XY cuts the plane obliquely at O. The angle between XY and the plane is found by drawing any perpendicular AN from XY to the plane. The angle $A\hat{O}N$ (θ) is the angle between the line and the plane. Figs. 6.4 and 6.5, overleaf, show another way of finding the angle between a line and a plane.

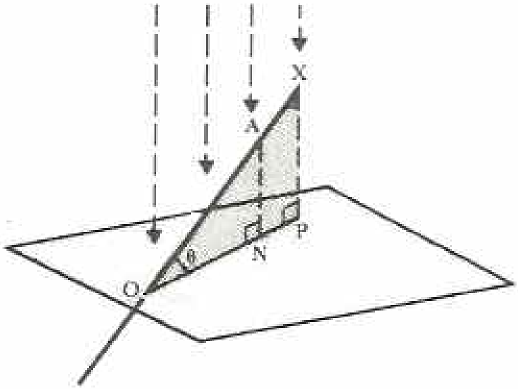


Fig. 6.4

In Fig. 6.4 OP is the **projection** of OX on the horizontal plane. Think of OP as the shadow of OX when the sun is vertically above the plane. $X\hat{O}P = A\hat{O}N = \theta$ is the angle between OX and the horizontal plane.

In many cases the plane is not horizontal. In Fig. 6.5 OQ is the projection of OX on the vertical plane (i.e. the shadow of OX when the plane is lit from the side). $X\hat{O}Q = A\hat{O}M = \alpha$ is the angle between OX and the vertical plane.

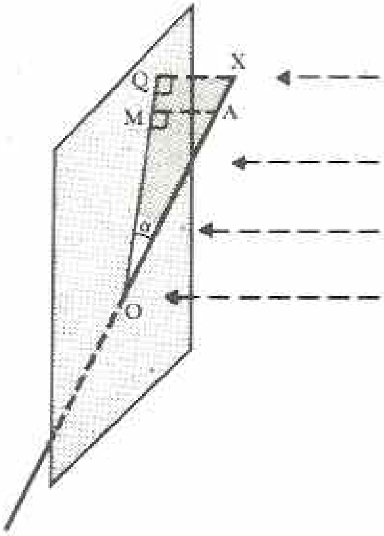


Fig. 6.5

Figs. 6.4 and 6.5 show that **the angle between a line and a plane is the angle between the line and its projection on the plane.**

Example 1

Fig. 6.6 shows a cuboid ABCDEFGH.

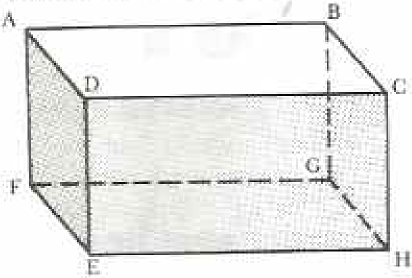


Fig. 6.6

(*a*) *Name four edges which are perpendicular to plane* BCHG.

(*b*) *Which one of the following is not a right angle?* $A\hat{F}G$, $A\hat{F}H$, $A\hat{F}E$, $C\hat{G}F$, $C\hat{B}E$, $B\hat{D}E$.

(*c*) *Name the projection of* DG *on* (*i*) *plane* EFGH, (*ii*) *plane* ADEF, (*iii*) *plane* ABGF.

(*d*) *Name the angle between* AH *and* (*i*) *plane* EFGH, (*ii*) *plane* BCHG, (*iii*) *plane* CDEH.

(a) AB, DC, EH, FG

(b) $C\hat{B}E$ (since BC⊥plane CDEH, $B\hat{C}E = 90°$. Hence △CBE is right-angled at C, *not* B. (See Fig. 6.7.)

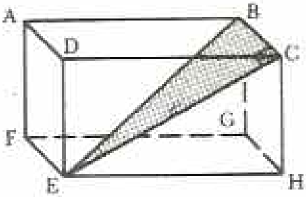


Fig. 6.7

(c) (i) EG, (ii) DF, (iii) AG.

(d) (i) $A\hat{H}F$, (ii) $A\hat{H}B$, (iii) $A\hat{H}D$.

Exercise 6a (discussion)

1 Name three right angles in Fig. 6.1.

2 In Fig. 6.8, O is the centre of the base of the right cone, vertex V.

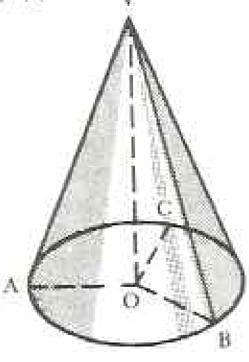


Fig. 6.8

(a) Name three right angles in Fig. 6.8.
(b) Name the angle between VB and the base of the cone.

3 In Fig. 6.9, VABCD is a square-based right pyramid. K, L, M, N are the mid-points of the edges shown in the figure.

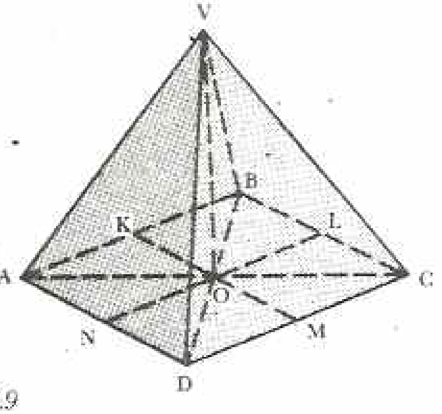


Fig. 6.9

(a) Name eight line segments which are perpendicular to VO.
(b) Which plane does AC meet perpendicularly?
(c) LN is a normal to which plane?
(d) Name the angle between VB and plane ABCD.
(e) Name three other angles which are equal in size to that in part (d).
(f) Name the angle between VK and plane ABCD.
(g) Name three other angles which are equal in size to that in part (f).
(h) Name the angle between VO and plane VCD.

4 In Fig. 6.10, PQRSTUVW is a cuboid.

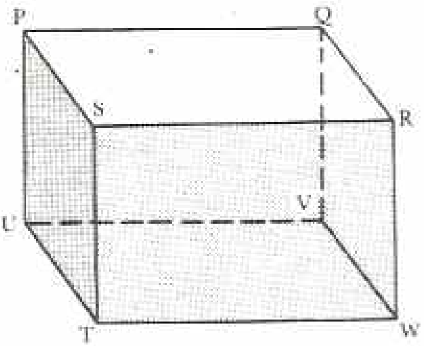


Fig. 6.10

(a) Name two planes which are perpendicula to edge RW.
(b) Name four normals to the plane PQVU
(c) Which of the following are right angles
$P\hat{U}T$, $Q\hat{R}T$, $Q\hat{V}S$, $R\hat{W}U$, $V\hat{T}P$, $P\hat{U}W$.
(d) Name the projection of QT on (i) plan TUVW, (ii) plane QRWY, (iii) plane RSTW
(e) Name the angle between RU an (i) plane PQRS, (ii) plane PSTU, (iii) plan PQVU.

5 In Fig. 6.11, ABCDEF is a prism whos cross-section is a right-angled triangle.

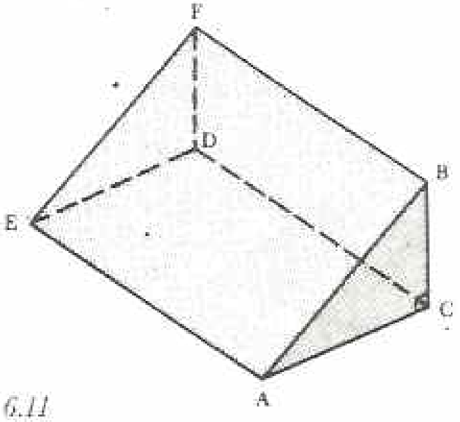


Fig. 6.11

(a) Name the plane to which ED is a norma
(b) Which one of the following is *not* a righ angle? $F\hat{D}C$, $F\hat{D}A$, $F\hat{D}E$, $E\hat{D}B$, $A\hat{F}D$.
(c) Name the projection of AF on (i) plan ACDE, (ii) plane BCDF.
(d) Name the angle that EB makes wit (i) plane ACDE, (ii) plane BCDF.

6 In Fig. 6.12, PQRSWXYZ is a cuboid.

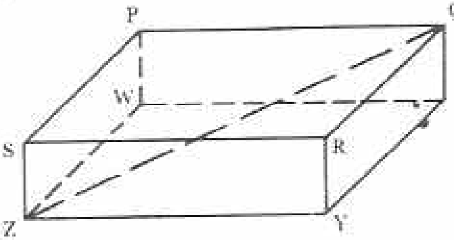


Fig. 6.12

Decide which of the following statemen about ZQ are true and which are false.
(a) The projection of ZQ on plane QXY is YQ.

(b) ZQ is the longest line segment that can be drawn in the cuboid.
(c) ZQ makes an angle of 45° with plane ZWPS.
(d) The angle between ZQ and plane QRYX is $Z\hat{Q}Y$.
(e) The projection of ZQ on the plane ZWPS is ZP.

Angles between planes

When one plane cuts another, they intersect along a straight line. Fig. 6.13 shows two examples of planes which meet. The dotted lines are the lines of intersection.

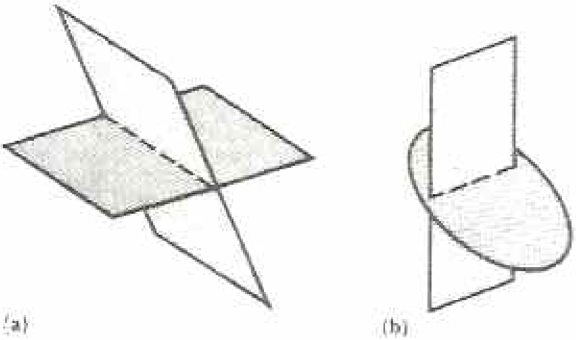


Fig. 6.13

The angle between two planes is found as follows. Choose a point on their line of intersection. From this point, draw a line on each plane at right angles to the line of intersection. The angle between these lines is the angle between the planes. This is shown in Fig. 6.14 where α and β are the angles between the planes.

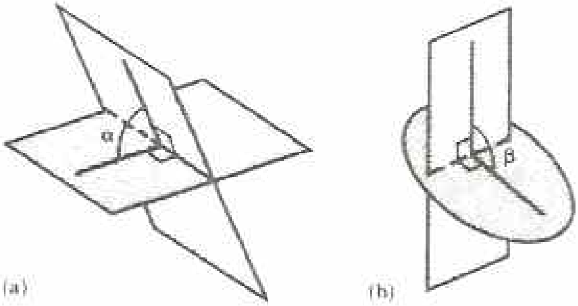


Fig. 6.14

Example 2

Name the angle between the shaded planes in each part of Fig. 6.15.

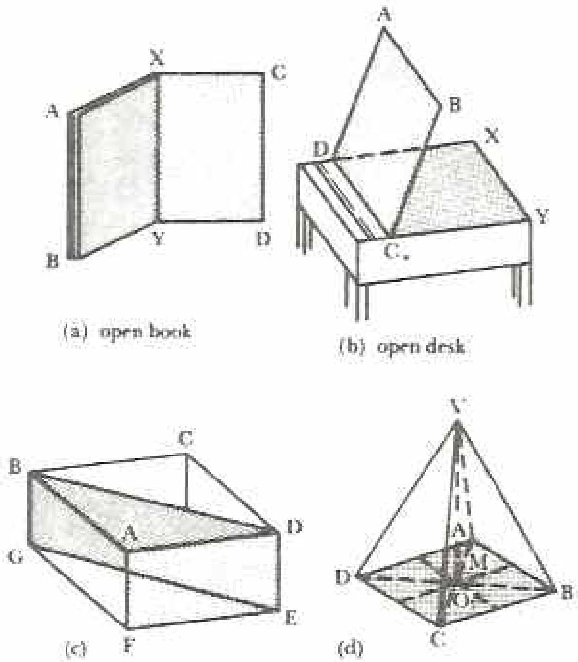


Fig. 6.15

(a) $A\hat{X}C$ or $B\hat{Y}D$
(b) $B\hat{C}Y$ or $A\hat{D}X$
(c) $B\hat{D}A$ or $G\hat{E}F$
(d) $V\hat{M}O$ (Note that MV is at right angles to AB; AV and BV are *not* at right angles to AB.)

Exercise 6b (discussion)

1 In Fig. 6.6 name the angle between the two shaded planes. What is the size of this angle?
2 In Fig. 6.7 name the angle between plane BCE (shaded) and plane BCHG.
3 In Fig. 6.8 name the angle between plane VOA and plane VOB.
4 In Fig. 6.9 name the angle between the following planes.
   (a) ABCD and VAC
   (b) ABCD and VNL
   (c) ABCD and VDC
   (d) ABCD and VAD
   (e) VAC and VNL
   (f) VAD and VBC

5 In Fig. 6.11 name the angle between the following planes.
(a) ACDE and ABC
(b) ACDE and ABFE
(c) ACDE and BCDF
(d) BCDF and ABFE

6 (a) Trace Fig. 6.12 into your exercise book.
(b) Draw and shade the triangular plane ZQW.
(c) Name the angle that plane ZQW makes with plane WXYZ.
(d) Name the angle that plane ZQW makes with plane PWZS.
(e) What is the size of the angle that plane ZQW makes with plane PQXW?

7 Fig. 6.16 shows a cuboid with a dividing plane.

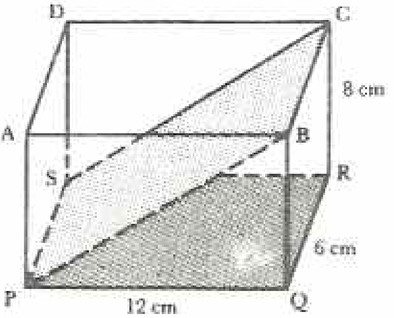


Fig. 6.16

(a) Name the angle between the shaded planes.
(b) Use the given dimensions to state the tangent of this angle in its simplest terms.

8 Fig. 6.17 is a view of a market stall.

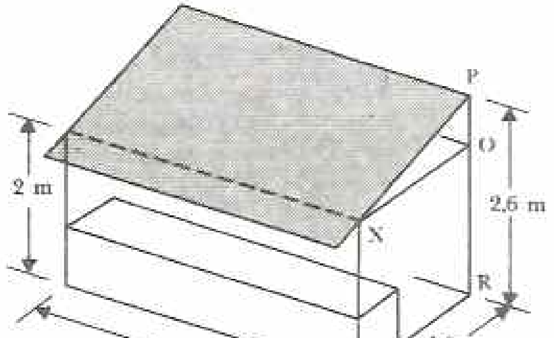


Fig. 6.17

(a) Name the angle that the roof makes with the horizontal.
(b) Use the given dimensions to state the tangent of this angle as a decimal fraction.

9 Fig. 6.18 represents a roof of a building. XOY and MON are lines of symmetry of the horizontal base of the roof.

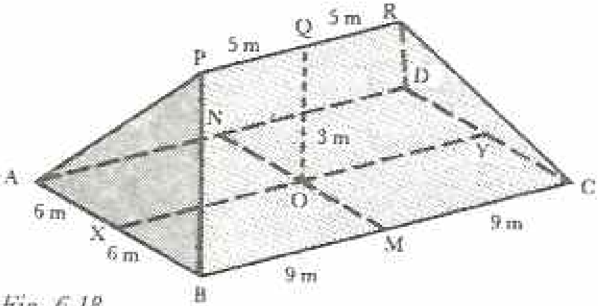


Fig. 6.18

(a) Name the angle between plane PBCR and the horizontal.
(b) Use the given dimensions to find the tangent of that angle.
(c) Name the angle between plane PAB and the horizontal.
(d) Find the tangent of that angle.

10 Fig. 6.19 is the net of a square-based pyramid.

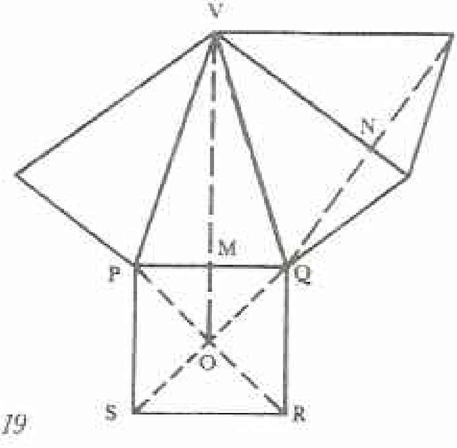


Fig. 6.19

(a) Sketch a view of the pyramid. Include all the given letters and lines on your sketch.
(b) Name the angle between plane VPQ and plane PQRS.
(c) Name the angle between planes VQR and VRS.

Calculating lengths and angles in solids

In most solids, unknown lengths and angles can be found by solving right-angled triangles.

Fig. 6.20 shows some of the right-angled triangles contained in a cuboid.

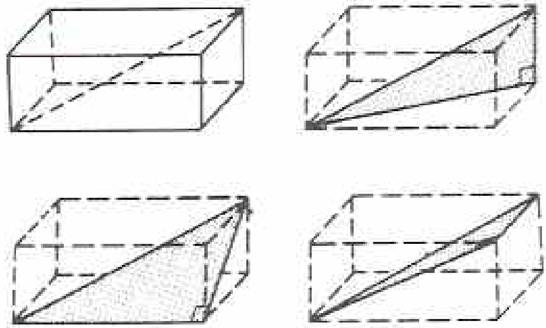


Fig. 6.20

Fig. 6.21 shows some of the right-angled triangles contained in a right pyramid.

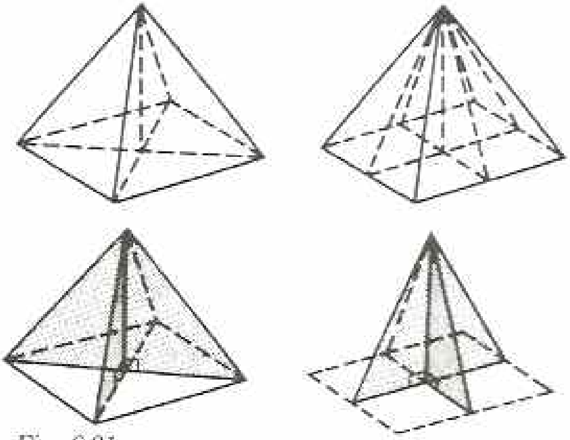


Fig. 6.21

The following examples show that when a right-angled triangle is used, it is advisable to sketch it separately from the solid.

Example 3

One end of a rectangular tank of length 6 m *is a square* ABCD *of side* 2 m. *If* AP *is a diagonal of the tank, calculate* (*a*) AP *correct to* 1 *decimal place,* (*b*) *the angle between* AP *and plane* ABCD, (*c*) *the shortest distance between plane* APD *and* BC.

Fig. 6.22 is a view of the tank.

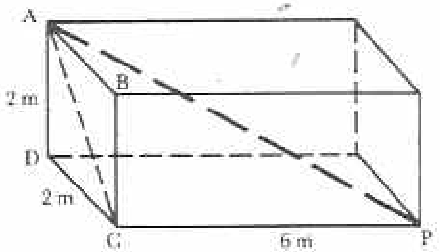


Fig. 6.22

(a) Fig. 6.23 shows the triangles used to calcula
AP.

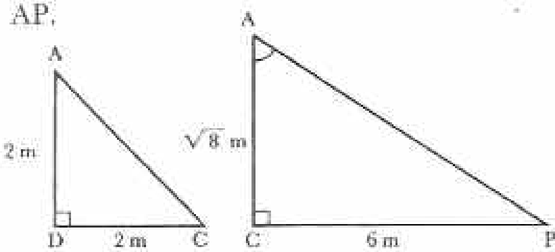


Fig. 6.23

In $\triangle$ADC,

$AC^2 = AD^2 + DC^2$ *(Pythagora*

$\quad = 2^2 + 2^2 = 4 + 4 = 8$

$AC = \sqrt{8}\,m$

In $\triangle$ACP,

$AP^2 = AC^2 + CP^2$ *(Pythagora*

$\quad = (\sqrt{8})^2 + 6^2 = 8 + 36 = 44$

$AP = \sqrt{44}\,m = 6{,}6\,m$ to 1 d.p.

(b) AC is the projection of AP on plane ABC
Hence $C\hat{A}P$ is the required angle. In $\triangle$AC

$$\tan C\hat{A}P = \frac{CP}{AC}$$

$$= \frac{6}{\sqrt{8}} = \frac{6}{2\sqrt{2}} = \frac{3}{\sqrt{2}} = \frac{3\sqrt{2}}{2}$$

$$= \frac{3 \times 1{,}414}{2} = 2{,}121$$

$C\hat{A}P = 64{,}75°$

AP meets plane ABCD at an angle of 65°
the nearest degree).

(c) In Fig. 6.24(a), CQ is the shortest distan
between BC and plane APD.

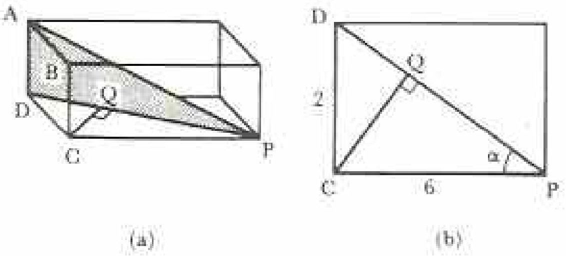


Fig. 6.24

In Fig. 6.24(b),

$DP^2 = DC^2 + CP^2$ *(Pythagoras)*

$= 2^2 + 6^2 = 4 + 36 = 40$

$DP = \sqrt{40}$ m

In $\triangle PQC$, $\sin\alpha = \dfrac{CQ}{CP} = \dfrac{CQ}{6}$

In $\triangle PCD$, $\sin\alpha = \dfrac{DC}{DP} = \dfrac{2}{\sqrt{40}}$

Hence $\dfrac{CQ}{6} = \dfrac{2}{\sqrt{40}}$

$$CQ = \frac{2 \times 6}{\sqrt{40}} = \frac{12\sqrt{40}}{40}\text{ m}$$

$$= \frac{3\sqrt{40}}{10} = \frac{3 \times 6{,}325}{10}\text{ m}$$

$= 1{,}897\,5$ m

$= 1{,}9$ m to 1 d.p.

BC is 1,9 m from plane APD.

Notice the value of sketching the various triangles (Figs. 6.23, 6.24). This makes it easy to place the right angle correctly and to solve the triangle using Pythagoras' theorem and trigonometry. Also notice in parts (b) and (c) how rationalising the denominators simplifies the arithmetic.

Example 4

A pyramid with vertex V *and edges* VA, VB, VC, VD *each* 13 cm *long has a rectangular base* ABCD *where* AB = CD = 8 cm *and* AD = BC = 6 cm. *Calculate* (a) *the height* VO *of the pyramid,* (b) *the angle between the base and an edge,* (c) *the angle between the base and* $\triangle$VBC, (d) *the angle between the base and* $\triangle$VCD.

Since the pyramid is symmetrical, the point O vertically below V is at the centre of the rectangle ABCD. This is shown in Fig. 6.25.

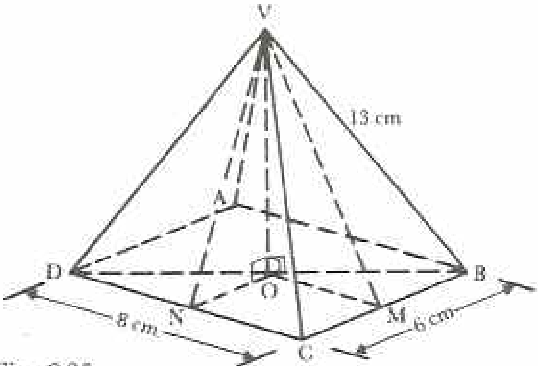


Fig. 6.25

The triangles in Fig. 6.26 are used to calculate VO.

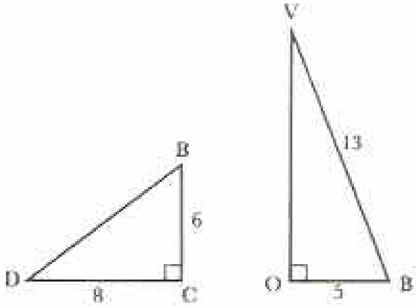


Fig. 6.26

(a) In $\triangle BCD$,

$BD^2 = 6^2 + 8^2$ *(Pythagoras)*

$= 36 + 64 = 100$

$BD = \sqrt{100}$ cm $= 10$ cm

$BO = \frac{1}{2}BD = 5$ cm

In $\triangle VOB$,

$VO^2 = 13^2 - 5^2$ *(Pythagoras)*

$= 169 - 25 = 144$

$VO = 12$ cm

(b) Since VO is perpendicular to the base, $V\hat{B}O$ is one of the angles between the base and an edge.

In $\triangle VBO$,

$$\tan V\hat{B}O = \frac{VO}{BO} = \frac{12}{5} = 2{,}4$$

$\therefore V\hat{B}O = 67{,}38°$

(c) Let M be the mid-point of BC. Since MV and MO are both perpendicular to edge BC, $V\hat{M}O$ is the angle between the base and face VBC. $V\hat{M}O$ can be found from $\triangle VMO$ (Fig. 6.27).

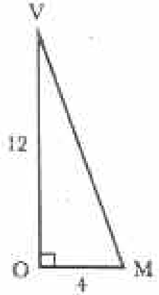


Fig. 6.27

In $\triangle VMO$, $\tan V\hat{M}O = \frac{12}{4} = 3$

$$V\hat{M}O = 71{,}57°$$

(d) Similarly, $V\hat{N}O$ is the angle between the base and face VCD.

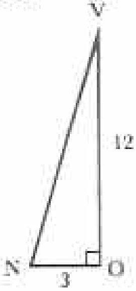


Fig. 6.28

In $\triangle VNO$, $\tan V\hat{N}O = \frac{12}{3} = 4$

$$V\hat{N}O = 75{,}97°$$

Example 5

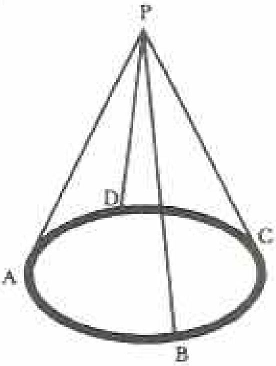


Fig. 6.29

In Fig. 6.29 a hoop of radius 80 cm *is suspended horizontally by four strings each* 160 cm *long and each attached to a nail vertically above the hoop at* P. *The strings are attached to points* A, B, C, D *which are equally spaced on the hoop. Calculate (a) the angle which* PA *makes with the horizontal, (b)* $B\hat{P}D$, *(c) the angle between* PA *and* PB.

Since the strings are of equal length, the centre of the hoop, O, is vertically below P. Fig. 6.30 shows, (a) the position of O and, (b) $\triangle PAC$.

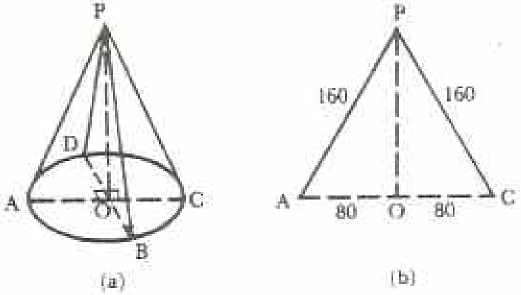


Fig. 6.30

(a) In $\triangle PAC$,
AC = 80 cm + 80 cm = 160 cm
Therefore $\triangle PAC$ is equilateral.
$\therefore\ P\hat{A}C = 60°$ *(angles of equilateral $\triangle$)*
PA makes an angle of 60° with the horizontal.

(b) Since the strings are equally spaced, $B\hat{P}D = A\hat{P}C$.
$A\hat{B}C = 60°$ *(angles of equilateral $\triangle$)*
$\Rightarrow B\hat{P}D = 60°$

(c) $A\hat{P}B$ is the angle between PA and PB. Fig 6.31 shows the triangles used to calculate $A\hat{P}B$

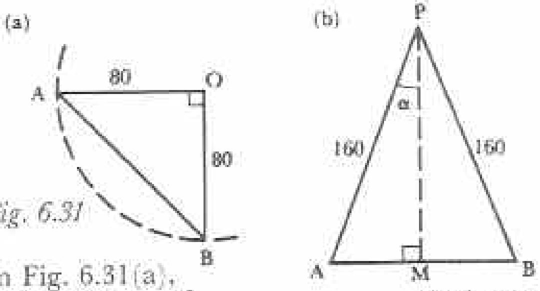


Fig. 6.31

In Fig. 6.31(a),
$AB^2 = 80^2 + 80^2$ *(Pythagoras*
$= 2 \times 80^2$
$AB = \sqrt{2 \times 80^2} = 80\sqrt{2}$ cm

In Fig. 6.31(b), M is the mid-point of AB.

$$\sin\alpha = \frac{AM}{AP} = \frac{\frac{1}{2}\times 80\sqrt{2}}{160} = \frac{\sqrt{2}}{4}$$

$$= \frac{1{,}414}{4} = 0{,}353\,5$$

$$\alpha = 20{,}7^\circ$$

$$A\hat{P}B = 2\alpha = 2\times 20{,}7^\circ = 41{,}4^\circ$$

PA and PB meet at 41,4°.

Exercise 6c

Draw as many sketches as are necessary.

1 The length, breadth and height of various cuboids are given below. Calculate the length of the long diagonal of each cuboid.
(a) 6 cm, 10 cm, 15 cm
(b) 4 cm, 4 cm, 2 cm
(c) 2 m, 5 m, 14 m
(d) 4 cm, 5 cm, 20 cm

2 Use the data of Fig. 6.16 to calculate (a) CQ, (b) PC, (c) the angle between the shaded faces.

3 Use the data of Fig. 6.17 to calculate the angle that the roof makes with the horizontal.

4 Fig. 6.32 shows a desk lid which is kept open with a 30 cm ruler.

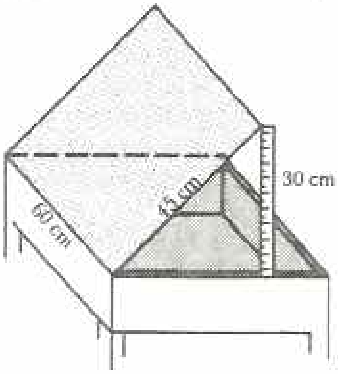


Fig. 6.32

Calculate the angle that the lid makes with the horizontal.

5 Fig. 6.33 shows another way of keeping a desk lid open with a ruler.
(a) Calculate the angle that the lid makes with the horizontal.
(b) Hence find the perpendicular distance between the top edge of the lid and the desk.

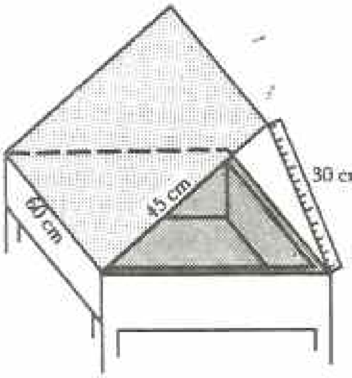


Fig. 6.33

6 In Fig. 6.34, PQRS is a horizontal rectangular assembly ground 80 m by 60 m and PT is a tower 47 m high.

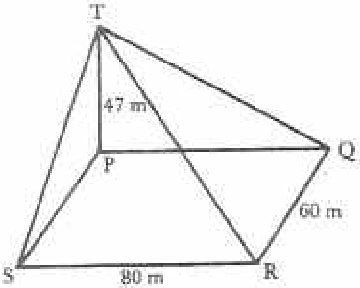


Fig. 6.34

Find, to the nearest degree, the angle of elevation of T from R.

7 Fig. 6.35 shows a stick, AB, leaning against the corner of a room. The foot of the stick, A, is 30 cm from one wall and 40 cm from the other. The top of the stick, B, is 120 cm above the floor.
Calculate (a) AO, (b) the angle that AB makes with the floor, (c) the length of the stick.

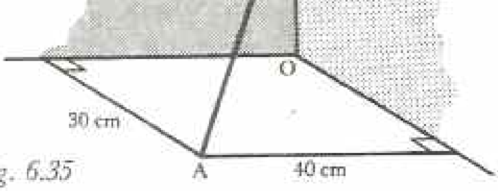


Fig. 6.35

8 A 50 cm stick leans, as in question 7, so that its foot is 10 cm from one wall and 20 cm from the other. Calculate (a) the height of the top of the stick above the floor, (b) the angle that the stick makes with the floor.

9 A pyramid has a square base of side 8 cm and sloping edges 9 cm long. Calculate (a) the height of the pyramid, (b) the angle, to the nearest $\frac{1}{10}$th of a degree, (i) between a sloping edge and the base, (ii) between a sloping face and the base.

10 A pyramid is 4 cm high and stands on a base which is a regular hexagon of side 3 cm. Calculate (a) the length of an edge of the pyramid, (b) the angle that the edge makes with the base, (c) the angle that a triangular face makes with the base.

11 The cuboid in Fig. 6.36 is 28 m long, 21 m wide and 12 m high. Calculate the angle between (a) PC and the plane CDSR, (b) the planes PBCS and QBCR.

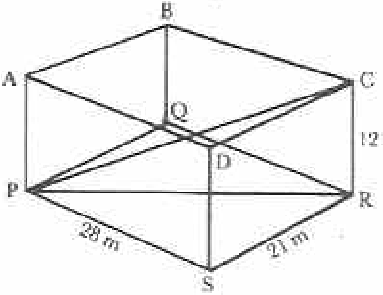


Fig. 6.36

12 Fig. 6.37 shows an 8 cm by 9 cm by 12 cm cuboid.

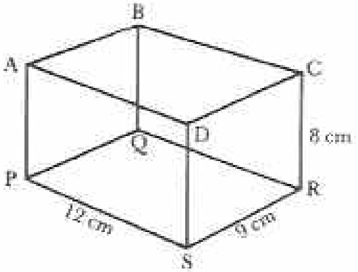


Fig. 6.37

Calculate (a) PR, (b) PC, (c) the angle between PC and (i) PQRS, (ii) DCRS, (iii) BCRQ, (d) the angle between ABCD and PBCS, (e) the angle between PQCD and ABQP.

13 A door 2 m high by 1,5 m wide is opened to an angle of 60° as shown in Fig. 6.38.

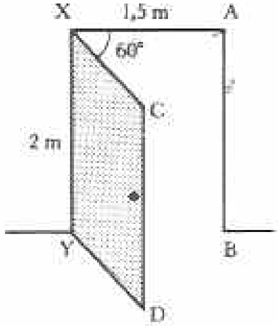


Fig. 6.38

Calculate (a) BD, (b) DX, (c) the angle that plane DXB makes with the horizontal.

14 O is the vertex of a right pyramid on a square base ABCD, the sides of the base being 10 cm long. The edges OA, OB, OC, OD are each 12,5 cm. Calculate (a) the height of the pyramid, (b) the angle between OAB and the base ABCD, (c) the volume of the pyramid in cm^3, correct to 3 s.f.

15 A right pyramid on a square base PQRS has a vertex T. Each of the sloping faces is an equilateral triangle of side 12 cm. Calculate (a) the base area of the pyramid, (b) the height of the pyramid, correct to 1 decimal place, (c) the volume of the pyramid, correct to 3 significant figures, (d) the angle which TS makes with the base PQRS.

16 The net of a pyramid consists of a square of side 16 cm and four isosceles triangles whose equal sides are each 17 cm. (a) Sketch the pyramid. (b) Calculate the height of the pyramid. (c) Calculate the angle between one of the 17 cm edges and the square base. (d) Calculate the angle between a triangular face and the square face.

17 Use the data of Fig. 6.18 to calculate the angle between the plane ABCD and (a) plane PRCB, (b) plane APB.

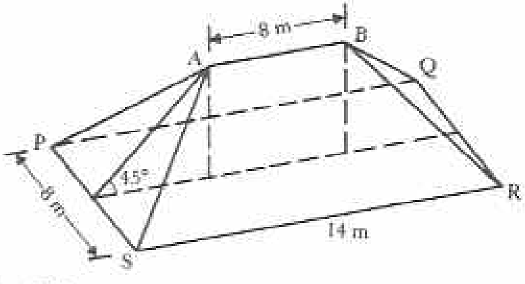


Fig. 6.39

18 Fig. 6.39 is a sketch of a symmetrical roof which has both triangular faces inclined at 45° to the horizontal.
Use the given dimensions to calculate (a) the height of AB above the plane PQRS, (b) the angle between the planes ABRS and PQRS.

19 In Fig. 6.40, which is not drawn to scale, TPQRSU is a solid which has a horizontal rectangular base PQRS in which PQ = 8 cm and PS = 10 cm. The triangular faces TPQ and URS are equilateral triangles and each makes an angle of 60° with the plane PQRS.

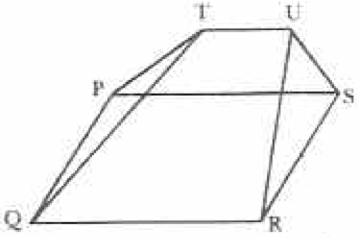


Fig. 6.40

Calculate (a) the height of TU above the plane PQRS, (b) TU, (c) the angle which TQ makes with the plane PQRS.

20 ABC is an equilateral triangle inscribed in a circle centre O, radius 80 cm, on a horizontal plane. A rod OE of length 60 cm is fixed vertically at O and stayed by wires from the top E to A, B and C. Calculate (a) the length of one of the wires, (b) the angle AEB correct to the nearest degree, (c) the angle between the planes BEC and BAC, correct to the nearest degree. (*Hint:* The property that the altitudes of an equilateral triangle trisect each other may be found useful.)

Inclined planes

Example 6

In Fig. 6.41, ABCD *and* XYCD *are rectangular planes such that* XYCD *is horizontal and* B *is* 10 cm *above* CY. EF || CB *and* EC = 16 cm, CP = 12 cm. *Calculate the angle that (a)* EF, *(b)* EB *makes with the horizontal.*

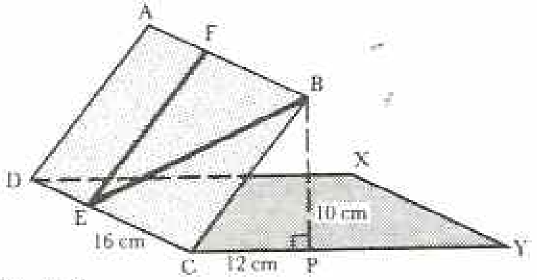


Fig. 6.41

(a) EF || CB so $\hat{BCP}$ equals the required angle.

In △BCP, $\tan \hat{BCP} = \frac{10}{12} = \frac{5}{6} = 0{,}833\,3$

$\Rightarrow \hat{BCP} = 39{,}8°$

EF makes an angle of 39,8° with the horizontal.

(b) PE is the projection of BE on plane XYCD. Hence $\hat{BEP}$ is the required angle. Fig. 6.42 shows the triangles used to find $\hat{BEP}$.

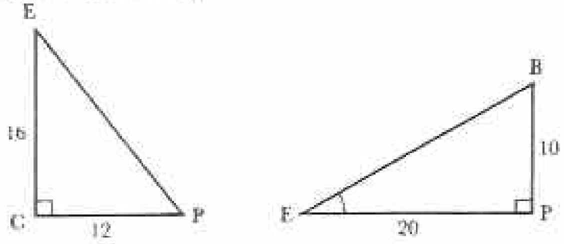


Fig. 6.42

In △ECP,

$EP^2 = 16^2 + 12^2$ (*Pythagoras*)

$= 256 + 144 = 400$

$EP = \sqrt{400} = 20$ cm

(*Note:* △ECP is a 3; 4; 5 △.)

In △BEP,

$\tan \hat{BEP} = \frac{BP}{EP} = \frac{10}{20} = 0{,}5$

$\Rightarrow \hat{BEP} = 26{,}57°$

EB makes an angle of 26,6° with the horizontal. The **slope** of a line or a plane is the angle tha it makes with the horizontal. In Fig. 6.41 it ha been shown that the slope of EF is 39,8° anc the slope of EB is 26,6°. The slope of EF is th same as the slope of plane ABCD. In this case EF is said to be a **line of greatest slope**. D and CB are also lines of greatest slope. EB i *not* a line of greatest slope; its slope (26,6°) i less than that of EF (39,8°).

Example 7

A rectangular lid 25 cm by 20 cm is kept open at an angle of 65° to the horizontal, the hinges being on one of the long edges. Calculate the slope of a diagonal of the lid.

Fig. 6.43 shows the lid ABCD. θ is the required angle.

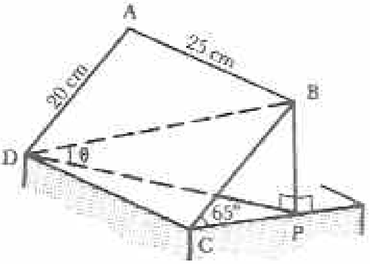


Fig. 6.43

Since θ is in the right-angled triangle BPD, it is necessary to find two sides of this triangle. The triangles in Fig. 6.44 each contain a side of $\triangle$BPD.

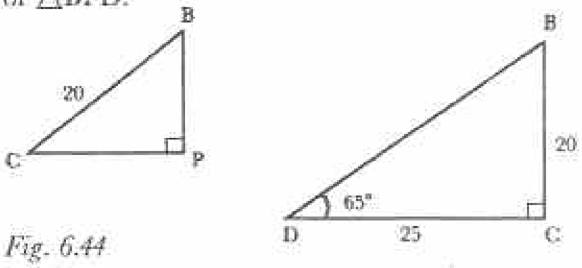


Fig. 6.44

In $\triangle$BPC, $\sin 65° = \dfrac{BP}{20}$

$\Rightarrow BP = 20 \sin 65°$ cm

In $\triangle$BCD,
$BD^2 = 20^2 + 25^2 = 400 + 625 = 1025$

$BD = \sqrt{1025} = 32{,}02$ cm

In $\triangle$BPD,

$\sin\theta = \dfrac{BP}{BD} = \dfrac{20 \sin 65°}{32{,}02}$

$\theta = 34{,}49°$

working:

No.	Log
20	1,3010
sin 65°	$\bar{1}$,9573
	1,2583
32,02	1,5054
sin 34,49°	$\bar{1}$,7529

The slope of the diagonal is 34,5°.

Notice in Example 7 that tables of log-sines were used. This is quicker than using natural sines first, then using logarithms.

Alternatively, the calculation can be done on a scientific calculator as follows:

Key	*Display*
AC	0
6 5 sin	0.9063077
× 2 0	20
÷ 3 2 . 0 2 =	0.5660885
SHIFT	0.5660885
$\sin^{-1}$	34.477917

$\theta = 34{,}48°$

Exercise 6d

Draw as many sketches as are necessary.

1 Name the lines of greatest slope in the following.
  (a) Fig. 6.11, plane ABFE
  (b) Fig. 6.16, plane CSPB
  (c) Fig. 6.18, (i) plane PAB, (ii) plane RCBP
  (d) Fig. 6.25, (i) plane VBC, (ii) plane VCD
  (e) Fig. 6.34, plane TSR
  (f) Fig. 6.43, plane ABCD

2 Name any lines which are at an angle to the line of greatest slope in the following.
  (a) Fig. 6.9, plane VAD
  (b) Fig. 6.18, plane PAB
  (c) Fig. 6.22, plane APC
  (d) Fig. 6.39, plane ABRS

3 In Fig. 6.41, calculate the slopes of EF and EB when EC = 20 cm and CP = BP = 15 cm.

4 Calculate the slope of a diagonal of the lid of the desk, (a) in Fig. 6.32, (b) in Fig. 6.33.

5 In Fig. 6.43, calculate the slope of DB when DC = 24 cm, CB = 7 cm and $B\hat{C}P = 30°$.

6 Fig. 6.45 shows a prism such that M is the mid-point of AD. Calculate the slopes of BM and BD.

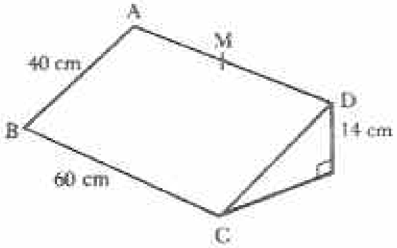


Fig. 6.45

Chapter 7

Fractions in algebra

Simplification of fractions

Lowest terms

When simplifying algebraic fractions, always fully factorise the numerators and denominators. It may then be possible to divide the numerator and denominator by any factors which they have in common.

Example 1

Reduce $\frac{6m^2u^2 - 4mu^3}{9m^3u - 4mu^3}$ *to its lowest terms.*

$\frac{6m^2u^2 - 4mu^3}{9m^3u - 4mu^3}$

$= \frac{2mu^2(3m - 2u)}{mu(9m^2 - 4u^2)}$ (*taking out common factors*)

$= \frac{2mu^2(3m - 2u)}{mu(3m + 2u)(3m - 2u)}$ (*difference of two squares*)

$= \frac{2u}{3m + 2u}$ [*dividing numerator and denominator by* $mu(3m - 2u)$]

Example 2

Simplify $\frac{a^2 + ax - 6x^2}{2x^2 + ax - a^2}$.

$$\frac{a^2 + ax - 6x^2}{2x^2 + ax - a^2} = \frac{(a - 2x)(a + 3x)}{(2x - a)(x + a)}$$

$$= -\frac{a + 3x}{x + a}$$

In Example 2, notice that

$$a - 2x = -(2x - a)$$

so that $\frac{a - 2x}{2x - a} = \frac{-(2x - a)}{(2x - a)} = -1$.

In general,

$$\frac{c}{-d} = -\frac{c}{d} \text{ and } \frac{-c}{d} = -\frac{c}{d}$$

so that $\frac{c}{-d} = \frac{-c}{d} = -\frac{c}{d}$.

Remember also that $\frac{-c}{-d} = \frac{c}{d}$ since two negative quantities, divided one by the other, give a positive result.

In the same way,

$$\frac{a - m}{2m - a} = \frac{a - m}{-(a - 2m)} = -\frac{a - m}{a - 2m}$$

$$\frac{a - m}{2m - a} = \frac{-(m - a)}{2m - a} = -\frac{m - a}{2m - a}$$

$$\frac{a - m}{2m - a} = \frac{-(m - a)}{-(a - 2m)} = \frac{m - a}{a - 2m}$$

Hence if the sign of the numerator *or* the denominator is changed, the sign of the fraction is changed. However, if the signs of *both* the numerator *and* the denominator are changed, the sign of the fraction is unchanged. Because of this, there will sometimes be alternative answers to those given in the examples and exercises in this chapter. (Notice that to change the signs of a term is equivalent to multiplying it by -1.)

Example 3

Simplify $\dfrac{a^2 - 5a + 6}{2 - 3a + a^2}$.

$$\frac{a^2 - 5a + 6}{2 - 3a + a^2} = \frac{(a-2)(a-3)}{(2-a)(1-a)}$$

$$= -\frac{a-3}{1-a} \text{ or } \frac{a-3}{a-1}$$

$$\text{or } \frac{3-a}{1-a} \text{ or } -\frac{3-a}{a-1}$$

Exercise 7a

Simplify the following fractions. If there is no simpler form, say so.

1 $\dfrac{mnu}{nuv}$ **2** $\dfrac{8x^2z}{10xyz}$ **3** $\dfrac{u+m}{u+n}$ **4** $\dfrac{ab+ac}{ad+ae}$

5 $\dfrac{a^2+ab}{a^2+ac}$ **6** $\dfrac{a^2+b^2}{a+b}$ **7** $\dfrac{u^2+uv}{uv+v^2}$ **8** $\dfrac{h^2-hk}{hk}$

9 $\dfrac{5d^2nv^3}{15d^3n^2v^4}$ **10** $\dfrac{c^2-cd}{d^2-cd}$ **11** $\dfrac{a^2-b^2}{b^2-ab}$

12 $\dfrac{x^2+xy}{x^2-y^2}$ **13** $\dfrac{28c^2d^2e^2}{35ce^4}$ **14** $\dfrac{x^2-4x}{x^2-4}$

15 $\dfrac{c^2-2cd+d^2}{c^2-cd}$ **16** $\dfrac{m^2+2mn+n^2}{m^2-n^2}$

17 $\dfrac{c^2-2c-15}{c^2-3c-10}$ **18** $\dfrac{d^2-9}{d^2-7d+12}$

19 $\dfrac{m^3n-2m^2n^2}{2mn^3-m^2n^2}$ **20** $\dfrac{xy-y^2}{(x-y)^2}$ **21** $\dfrac{xy-y^2}{(x+y)^2}$

22 $\dfrac{h^2+k^2}{(h+k)^2}$ **23** $\dfrac{h^2-k^2}{(h-k)^2}$

24 $\dfrac{u^2-5uv+6v^2}{u^2+uv-12v^2}$ **25** $\dfrac{x^2+xy-6y^2}{x^2-3xy+2y^2}$

26 $\dfrac{15+2x-x^2}{x^2-25}$ **27** $\dfrac{9a^2-m^2}{m^2-2am-3a^2}$

28 $\dfrac{8-2a-a^2}{2a^2-3a-2}$ **29** $\dfrac{a^2-am-an+mn}{a^2-am+an-mn}$

30 $\dfrac{a^2+am-an-mn}{a^2+am+an+mn}$ **31** $\dfrac{(u+w)^2-v^2}{(v+w)^2-u^2}$

32 $\dfrac{a^2-(b+c)^2}{c^2-(a-b)^2}$ **33** $\dfrac{2a^2m-3am^2+m^3}{am^2-a^2m-2a^3}$

34 $\dfrac{(2m-u)^2-(m-2u)^2}{5m^2-5u^2}$

35 $\dfrac{a(b+c)+(b+c)^2}{b^2-c^2+ab-ac}$ **36** $\dfrac{b^2+ac-ab-bc}{c^2-ac+ab-bc}$

Multiplication and division of fractions

Factorise fully first, then divide the numerator and denominator by any common factors.

Example 4

Simplify $\dfrac{a^2+2a-3}{a^2-16} \times \dfrac{a+4}{a^2+8a+15}$

Given expression

$$= \frac{\cancel{(a+3)}(a-1)}{(a-4)\cancel{(a+4)}} \times \frac{\cancel{a+4}}{(a+5)\cancel{(a+3)}}$$

$$= \frac{a-1}{(a-4)(a+5)}$$

The answer should be left in the form given. Do not multiply out the brackets.

Example 5

Simplify $\dfrac{m^2-a^2}{m^2+bm+am+ab} \div \dfrac{m^2-2am+a^2}{cm+bc}$

To divide by a fraction, multiply by its reciprocal.
Given expression

$$= \frac{m^2-a^2}{m^2+bm+am+ab} \times \frac{cm+bc}{m^2-2am+a^2}$$

$$= \frac{\cancel{(m-a)}\cancel{(m+a)}}{\cancel{(m+b)}\cancel{(m+a)}} \times \frac{c\cancel{(m+b)}}{\cancel{(m-a)}(m-a)}$$

$$= \frac{c}{m-a}$$

[dividing above and below by $(m-a)$, $(m+a)$, $(m+b)$*]*

Example 6

Simplify

$$\frac{a^2+ab}{a^2-2ab+b^2} \div \frac{a+3b}{a+2b} \times \frac{ab-a^2}{a^2+3ab+2b^2}.$$

Given expression

$$= \frac{a^2+ab}{a^2-2ab+b^2} \times \frac{a+2b}{a+3b} \times \frac{ab-a^2}{a^2+3ab+2b^2}$$

$$= \frac{a(a+b)}{(a-b)(a-b)} \times \frac{a+2b}{a+3b} \times \frac{a(b-a)}{(a+b)(a+2b)}$$

$$= -\frac{a^2}{(a-b)(a+3b)}$$ [*dividing above and below by* $(a+b)$, $(a+2b)$, $(a-b)$].

Notice that $(a-b)$ divides into $(b-a)$ to give -1. This is because $-1 \times (a-b) = (b-a)$.

Exercise 7b

Simplify the following.

1 $\frac{18ab}{15bc} \times \frac{20cd}{24de}$

2 $\frac{12dn^3}{15cd^3} \div \frac{9c^3n}{10c^2d^2}$

3 $\frac{m+n}{m} \times \frac{mn}{3m+3n}$

4 $\frac{uv}{3u-6v} \times \frac{4u-8v}{u^2v}$

5 $\frac{a^2-b^2}{a^2+ab} \div \frac{2a-2b}{ab}$

6 $\frac{a^2-4}{a^2-3a+2} \div \frac{a}{a-1}$

7 $\frac{m^2-9}{m^2-m-6} \times \frac{m^2+2m}{m^2}$

8 $\frac{18m^2u}{16n^3v^2} \div \frac{24m}{15nu^3} \times \frac{8n^2v^3}{30m^3u}$

9 $\frac{a^2-b^2}{ab+a^2} \times \frac{2a^3}{ab-a^2}$

10 $\frac{3d^2-12}{9d^2} \times \frac{6d^3}{4d+8}$

11 $\frac{2a-2b+2c}{8bc} \times \frac{10abc}{5a-5b+5c}$

12 $\frac{n^2-9}{n^2-n} \times \frac{n^2-3n+2}{n^2+n-6}$

13 $\frac{m^2-n^2}{m^2-2mn+n^2} \div \frac{m^2+mn}{n^2-mn}$

14 $\frac{a^2-ab-6b^2}{a^2+ab-6b^2} \times \frac{a^2-ab-2b^2}{a^2-2ab-3b^2}$

15 $\frac{5abc^2-10abcd}{3b^2c^2-6b^2cd} \times \frac{12bc^2d}{10acd}$

16 $\frac{c^2-cd}{d^2-de} \div \frac{d^2-cd}{cd-ce}$

17 $\frac{e^2-5e+6}{e^2+2e-3} \div \frac{3e-9}{2e^2+6e}$

18 $\frac{u^2+3u-10}{3u^2+12u} \div \frac{u^2-25}{u^2-u-20}$

19 $\frac{x^2-3x-4}{x^2-4x} \div \frac{x^2-4x+4}{x^2-4}$

20 $\frac{a^2+ab-2b^2}{a^2-2ab-3b^2} \times \frac{a^2-b^2}{ab+2b^2} \div \frac{a^2-2ab+b^2}{a^2-3ab}$

Addition and subtraction of fractions

Example 7

Simplify $2 + \frac{6a^2+2b^2}{3ab} - \frac{4a-b}{2b}$.

The denominators are $3ab$ and $2b$. The LCM of $3ab$ and $2b$ is $6ab$. Express each fraction in the expression with the denominator of $6ab$.

$$2 + \frac{6a^2+2b^2}{3ab} - \frac{4a-b}{2b}$$

$$= \frac{2\times 6ab}{6ab} + \frac{2(6a^2+2b^2)}{6ab} - \frac{3a(4a-b)}{6ab}$$

$$= \frac{12ab+2(6a^2+2b^2)-3a(4a-b)}{6ab}$$

$$= \frac{12ab+12a^2+4b^2-12a^2+3ab}{6ab}$$

$$= \frac{15ab+4b^2}{6ab}$$

$$= \frac{b(15a + 4b)}{6ab}$$

$$= \frac{15a + 4b}{6a}.$$

Example 8

Simplify $\dfrac{3}{m^2 + mn - 2n^2} - \dfrac{2}{m^2 - 4mn + 3n^2}$.

Factorise the denominators so that their LCM can be used as the common denominator. Given expression

$$= \frac{3}{(m - n)(m + 2n)} - \frac{2}{(m - n)(m - 3n)}$$

The LCM of the denominators is $(m - n)(m + 2n)(m - 3n)$.

Given expression

$$= \frac{3(m - 3n) - 2(m + 2n)}{(m - n)(m + 2n)(m - 3n)}$$

$$= \frac{3m - 9n - 2m - 4n}{(m - n)(m + 2n)(m - 3n)}$$

$$= \frac{m - 13n}{(m - n)(m + 2n)(m - 3n)}.$$

Example 9

Simplify $\dfrac{x + 4}{x^2 - 3x} - \dfrac{x - 1}{9 - x^2}$.

$$\frac{x + 4}{x^2 - 3x} - \frac{x - 1}{9 - x^2}$$

$$= \frac{x + 4}{x(x - 3)} - \frac{x - 1}{(3 - x)(3 + x)}$$

$$= \frac{x + 4}{x(x - 3)} + \frac{x - 1}{(x - 3)(3 + x)} *$$

$$= \frac{(x + 4)(x + 3) + x(x - 1)}{x(x - 3)(x + 3)}$$

$$= \frac{x^2 + 7x + 12 + x^2 - x}{x(x - 3)(x + 3)}$$

$$= \frac{2x^2 + 6x + 12}{x(x - 3)(x + 3)}$$

$$= \frac{2(x^2 + 3x + 6)}{x(x - 3)(x + 3)}$$

* Notice that the sign in front of the fraction changed since $(3 - x) = -(x - 3)$. This gi an LCM of $x(x - 3)(x + 3)$.

Example 10

Simplify

$$\frac{3a - 5m}{a^2 - 5am + 6m^2} + \frac{1}{a - 2m} - \frac{2}{a - 3m}.$$

Given expression

$$= \frac{3a - 5m}{(a - 2m)(a - 3m)} + \frac{1}{a - 2m} - \frac{2}{a - 3m}$$

$$= \frac{3a - 5m + (a - 3m) - 2(a - 2m)}{(a - 2m)(a - 3m)}$$

$$= \frac{3a - 5m + a - 3m - 2a + 4m}{(a - 2m)(a - 3m)}$$

$$= \frac{2a - 4m}{(a - 2m)(a - 3m)}$$

$$= \frac{2(a - 2m)}{(a - 2m)(a - 3m)}$$

$$= \frac{2}{a - 3m}$$

Exercise 7c

Simplify the following.

1 $\dfrac{3}{2ab} + \dfrac{4}{3bc}$ **2** $5 - \dfrac{a - b}{c}$

3 $\dfrac{3a - b}{5ab} - \dfrac{2b + 3c}{6bc} + \dfrac{3c - 2a}{15ac}$

4 $\dfrac{3}{2(x + y)} - \dfrac{1}{3(x + y)}$

5 $\dfrac{6}{a - 2b} + \dfrac{4}{2b - a}$

6 $3 + \dfrac{2b}{a - b}$ **7** $2 - \dfrac{x}{x + 2y}$

8 $\dfrac{1}{4(u-v)}-\dfrac{1}{5(v-u)}$

9 $\dfrac{7u}{2u+3v}-3$

10 $\dfrac{3a}{2a+b}-\dfrac{b}{4a+2b}$

11 $\dfrac{3mn}{2m^2+2n^2}+\dfrac{5mn}{3m^2+3n^2}$

12 $\dfrac{1}{4x-2y}-\dfrac{1}{y-2x}$

13 $\dfrac{a+b}{ab}-\dfrac{b+c}{bc}$

14 $\dfrac{u^2-v^2}{uv}+\dfrac{v}{u}-\dfrac{3uv-u^2}{v^2}$

15 $\dfrac{d+1}{2d-8}-\dfrac{d+2}{12-3d}$

16 $\dfrac{2}{a+1}+\dfrac{3}{a+2}$

17 $\dfrac{3x}{x-1}-\dfrac{4}{x+2}$

18 $\dfrac{e-2}{e+2}-\dfrac{e-1}{e+3}$

19 $\dfrac{3}{m-n}+\dfrac{m+3n}{(m-n)^2}$

20 $\dfrac{7c+2d}{(2c+d)^2}-\dfrac{3}{2c+d}$

21 $\dfrac{a+b}{(a-2b)^2}-\dfrac{1}{2b-a}$

22 $\dfrac{3c}{c^2-d^2}-\dfrac{3d}{d^2-c^2}$

23 $\dfrac{4m-9n}{16m^2-9n^2}+\dfrac{1}{4m-3n}$

24 $\dfrac{(m+n)^2}{m^2-n^2}+\dfrac{m^2+mn}{n^2-mn}$

25 $\dfrac{c(3-c)}{c^2+3c-10}+\dfrac{c-1}{c+5}$

26 $\dfrac{5}{d^2-2d-8}+\dfrac{2}{d^2-6d+8}$

27 $\dfrac{4}{3a+3b}-\dfrac{3}{2a-2b}+\dfrac{b}{a^2-b^2}$

28 $\dfrac{5}{d^2-16}+\dfrac{2}{(d+4)^2}$

29 $\dfrac{4}{x-3}-\dfrac{1}{x+2}-\dfrac{x+7}{x^2-x-6}$

30 $\dfrac{2a-1}{a^2-5a+6}-\dfrac{a+3}{a^2+2a-8}$

Example 11

Given that $x:y=9:4$, *evaluate* $\dfrac{8x-3y}{x-\frac{3}{4}y}$.

If $x:y=9:4$, then $\dfrac{x}{y}=\frac{9}{4}$.

Divide numerator and denominator of $\dfrac{8x-3y}{x-\frac{3}{4}y}$ by y.

$$\frac{8x-3y}{x-\frac{3}{4}y}=\frac{8\left(\frac{x}{y}\right)-3}{\frac{x}{y}-\frac{3}{4}}$$

Substitute $\frac{9}{4}$ for $\dfrac{x}{y}$ in the expression.

Value of expression

$$=\frac{8\times\frac{9}{4}-3}{\frac{9}{4}-\frac{3}{4}}=\frac{18-3}{1\frac{1}{2}}=\frac{15}{1\frac{1}{2}}$$

$$=15\div\tfrac{3}{2}=15\times\tfrac{2}{3}$$

$$=10$$

Example 12

If $x=\dfrac{2a+3}{3a-2}$, *express* $\dfrac{x-1}{2x+1}$ *in terms of* a.

Substitute $\dfrac{2a+3}{3a-2}$ for x in the given expression.

$$\frac{x-1}{2x+1}=\frac{\frac{2a+3}{3a-2}-1}{2\times\frac{2a+3}{3a-2}+1}$$

Multiply the numerator and denominator by $(3a-2)$.

$$\frac{x-1}{2x+1} = \frac{(2a+3)-(3a-2)}{2(2a+3)+(3a-2)}$$
$$= \frac{2a+3-3a+2}{4a+6+3a-2}$$
$$= \frac{-a+5}{7a+4}$$

Exercise 7d

1 If $\frac{x}{y} = \frac{3}{4}$, evaluate $\frac{2x-y}{2x+y}$.

2 Given $p:q = 9:5$, evaluate $\frac{15p-2q}{5p+16q}$.

3 If $a:b = 5:3$, evaluate $\frac{6a+b}{a-\frac{1}{3}b}$.

4 Given $\frac{x}{y} = \frac{2}{7}$, evaluate $\frac{7x+y}{x-\frac{1}{7}y}$.

5 If $a = \frac{d+1}{d-1}$, express $\frac{a+1}{a-1}$ in terms of d.

6 If $x = \frac{a+3}{2a-1}$, express $\frac{2x+1}{3x+1}$ in terms of a.

7 If $x = \frac{3m-5}{3m+5}$, express $\frac{x-1}{x+1}$ in terms of m.

8 If $x = \frac{3w-1}{w+2}$, express $\frac{2x-3}{3x-1}$ in terms of w.

9 If $X = \frac{2a+3}{3a-2}$, express $\frac{X-1}{2X+1}$ in terms of a.

10 If $h = \frac{m+1}{m-1}$, express $\frac{2h-1}{2h+1}$ in terms of m.

Equations with fractions

Example 13

Solve the equation $\frac{1}{3a-1} = \frac{2}{a+1} - \frac{3}{8}$.

The LCM of the denominators is $8(3a-1)(a+1)$. To clear fractions, multiply the terms on both sides of the equation by $8(3a-1)(a+1)$.

If $\frac{1}{3a-1} = \frac{2}{a+1} - \frac{3}{8}$

then $\frac{1}{3a-1} \times 8(3a-1)(a+1)$

$$= \frac{2}{a+1} \times 8(3a-1)(a+1)$$
$$- \frac{3}{8} \times 8(3a-1)(a+1)$$

i.e. $8(a+1) = 16(3a-1) - 3(3a-1)(a+1)$

$8a+8 = 48a - 16 - 3(3a^2+2a-1)$

$8a+8 = 48a - 16 - 9a^2 - 6a + 3$

$8a+8-48a+16+9a^2+6a-3 = 0$

$9a^2 - 34a + 21 = 0$

$(a-3)(9a-7) = 0$

$\therefore\ a = 3$ or $9a = 7$

$\therefore\ a = 3$ or $\frac{7}{9}$

Check: If $a = 3$, $\frac{1}{3a-1} = \frac{1}{9-1} = \frac{1}{8}$

and $\frac{2}{a+1} - \frac{3}{8} = \frac{2}{4} - \frac{3}{8} = \frac{1}{2} - \frac{3}{8} = \frac{1}{8}$

If $a = \frac{7}{9}$,

$$\frac{1}{3a-1} = \frac{1}{\frac{7}{3}-1} = \frac{1}{\frac{4}{3}} = \frac{3}{4}$$

and $\frac{2}{a+1} - \frac{3}{8} = \frac{2}{1\frac{7}{9}} - \frac{3}{8}$

$$= \frac{18}{16} - \frac{3}{8} = \frac{9}{8} - \frac{3}{8} = \frac{3}{4}$$

Example 14

Solve the equation $\frac{3}{x^2-5x+6} = \frac{2}{x^2-x-6}$.

Factorise the denominators of the fractions.

$$\frac{3}{(x-2)(x-3)} = \frac{2}{(x-3)(x+2)}$$

Multiply both sides by $(x-2)(x-3)(x+2)$

Then $3(x+2) = 2(x-2)$

$$\Leftrightarrow \quad 3x + 6 = 2x - 4$$
$$\Leftrightarrow \quad 3x - 2x = -4 - 6$$
$$\Leftrightarrow \quad x = -10$$

Check: If $x = -10$,

$$\frac{3}{x^2 - 5x + 6} = \frac{3}{100 + 50 + 6} = \frac{3}{156} = \frac{1}{52}$$

and $\dfrac{2}{x^2 - x - 6} = \dfrac{2}{100 + 10 - 6} = \dfrac{2}{104} = \dfrac{1}{52}$

Compare Examples 13 and 14 with Examples 7, 8, 9 and 10. In Examples 13 and 14, both sides of the equations are multiplied by the LCM of the denominators. Hence every denominator becomes 1 and the fractions are cleared. However, in Examples 7, 8, 9 and 10, the common denominators must stay in the given expressions, so that the expressions remain the same size. This is an important difference between **solving equations** with fractions and **simplifying expressions** with fractions.

Exercise 7e

Solve the following equations.

1 $\dfrac{3}{a} = a - 2$

2 $5 - 2d = \dfrac{2}{d}$

3 $\dfrac{7}{3} + \dfrac{2}{e} = e$

4 $c = \dfrac{3}{c + 2}$

5 $m = \dfrac{8}{3m + 2}$

6 $a + 3 = \dfrac{6}{a + 4}$

7 $3x - 2 = \dfrac{4}{x - 1}$

8 $n - 3 = \dfrac{4}{n - 3}$

9 $\dfrac{x - 2}{x + 4} = x$

10 $\dfrac{3n}{2n - 1} = n$

11 $\dfrac{a - 4}{7} = \dfrac{2}{3a - 1}$

12 $\dfrac{4}{w + 3} - \dfrac{3}{w + 2} = 0$

13 $\dfrac{3}{2b - 5} - \dfrac{4}{b - 3} = 0$

14 $\dfrac{c}{c - 2} = \dfrac{3}{2c - 5}$

15 $\dfrac{2}{d - 2} = \dfrac{3d}{4d + 12}$

16 $\dfrac{4n - 3}{6n + 1} = \dfrac{2n - 1}{3n + 4}$

17 $\dfrac{2m + 3}{2m + 5} - \dfrac{m - 1}{m - 2} = 0$

18 $\dfrac{3}{c + 2} - \dfrac{2}{2c - 3} = \dfrac{1}{7}$

19 $\dfrac{3}{x - 4} = \dfrac{2}{x - 1} - 4$

20 $\dfrac{1}{2a - 5} + \dfrac{7}{9} = \dfrac{2}{a + 5}$

21 $\dfrac{2}{d + 3} = \dfrac{3}{2d - 1} - \dfrac{4}{15}$

22 $\dfrac{11}{m + 3} = \dfrac{5}{2m} - \dfrac{1}{m - 4}$

23 $\dfrac{1}{2n - 3} + \dfrac{1}{2n + 1} - \dfrac{1}{n - 1} = 0$

24 $\dfrac{2}{u + 2} = \dfrac{2}{u + 1} - \dfrac{1}{u + 4}$

25 $\dfrac{3}{2d + 3} - \dfrac{1}{2d + 1} = \dfrac{1}{d + 1}$

26 $\dfrac{4a - 1}{a + 4} - 2 = \dfrac{2a - 1}{a + 2}$

27 $\dfrac{2}{x + 3} - \dfrac{x - 6}{x^2 - 9} = 0$

28 $\dfrac{4}{m^2 + 3m + 2} - \dfrac{3}{m^2 + 5m + 6} = 0$

29 $\dfrac{2}{u - 2} = \dfrac{2u - 1}{u^2 + u - 6} - \dfrac{3}{u + 3}$

30 $\dfrac{3}{v - 4} - \dfrac{v + 2}{v^2 - 3v - 4} = \dfrac{1}{2v + 2}$

Undefined fractions

Table 7.1 and Fig. 7.1 give a table of values and the corresponding graph of the function

$$\frac{1}{x-1}$$

Table 7.1

x	-1	0	$\frac{1}{2}$	$\frac{3}{4}$	$\frac{9}{10}$	$1\frac{1}{10}$	$1\frac{1}{4}$	$1\frac{1}{2}$	2
y	$-\frac{1}{2}$	-1	-2	-4	-10	10	4	2	1

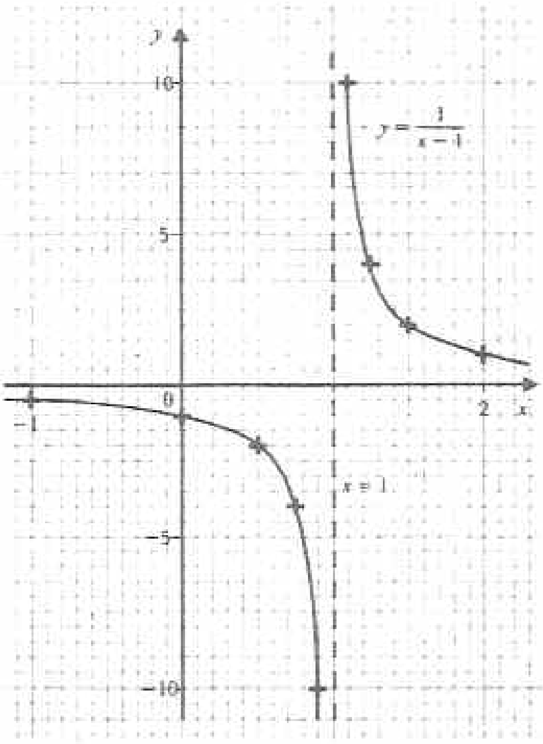


Fig. 7.1

Notice the following:

1 As the value of x approaches 1 from below, the value of $\frac{1}{x-1}$ *decreases* rapidly. For example, when $x = 0{,}999$,

$$\frac{1}{x-1} = \frac{1}{0{,}999-1} = \frac{1}{-0{,}001} = -1\,000$$

2 As the value of x approaches 1 from abov

the value of $\frac{1}{x-1}$ *increases* rapidly. F

example, when $x = 1{,}001$,

$$\frac{1}{x-1} = \frac{1}{1{,}001-1} = \frac{1}{0{,}001} = 1\,000$$

When $x = 1$, it is impossible to say what t

value of $\frac{1}{x-1}$ is.

Let $y = \frac{1}{x-1}$

When $x = 1$, $y = \frac{1}{1-1} = \frac{1}{0}$

Division by zero is impossible. The fracti

$\frac{1}{x-1}$ is said to be **undefined** when $x =$

Fig. 7.1 shows that $\frac{1}{x-1}$ is undefined wh

$x = 1$. If the denominator of a fraction has t value zero, the fraction will be undefined. If expression contains an undefined fraction, t whole expression is undefined.

Example 15

Find the values of x for which the following fracti are not defined.

(*a*) $\frac{3}{x+2}$ (*b*) $\frac{2x+13}{3x-12}$ (*c*) $\frac{5x}{x(5-}$

(*d*) $\frac{x^2-2x+3}{(x+3)(x-8)}$

(a) $\frac{3}{x+2}$ is undefined when $x + 2 = 0$.

If $x + 2 = 0$
then $x = -2$
The fraction is not defined when $x = -2$.

(b) $\frac{2x+13}{3x-12}$ is undefined when $3x - 12 =$
If $3x - 12 = 0$
then $3x = 12$
$x = 4$
The fraction is undefined when $x = 4$.

(c) $\dfrac{5x}{x(5-x)}$ is undefined when $x(5-x)=0$.

If $x(5-x)=0$
then either $x=0$ or $5-x=0$, i.e. $x=5$
The fraction is undefined when $x=0$ or when $x=5$.

(d) $\dfrac{x^2-2x+3}{(x+3)(x-8)}$ is undefined when $(x+3)(x-8)=0$
If $(x+3)(x-8)=0$
then either $x+3=0$ or $x-8=0$
i.e. either $x=-3$ or $x=8$
The fraction is undefined when $x=-3$ or when $x=8$.

If part of an expression is undefined, then the whole expression is not defined.

Example 16
Find the values of x for which the expression $\dfrac{a}{x}-\dfrac{b}{x^2+6x-7}$ *is not defined.*

$$\frac{a}{x}-\frac{b}{x^2+6x-7}=\frac{a}{x}-\frac{b}{(x-1)(x+7)}$$

The expression is not defined if any of its fractions has a denominator of 0.

$\dfrac{a}{x}$ is undefined when $x=0$.

$\dfrac{b}{(x-1)(x-7)}$ is undefined when

$(x-1)(x+7)=0$.

If $(x-1)(x+7)=0$
then either $(x-1)=0$ or $(x+7)=0$
i.e. either $x=1$ or $x=-7$
The expression is not defined when $x=0$, 1 or -7.

Exercise 7f
Find the values of x for which the following expressions are not defined.

1 $\dfrac{7}{x-3}$ **2** $\dfrac{2x}{4-x}$ **3** $\dfrac{3x+2}{x+7}$

4 $\dfrac{3+x}{x}$ **5** $\dfrac{6x}{2x-5}$ **6** $\dfrac{8}{15+3x}$

7 $\dfrac{y}{20-3x}$ **8** $\dfrac{2a}{x(x+2)}$ **9** $\dfrac{5b}{(1-2x)x}$

10 $\dfrac{3x+1}{(x+4)(x+3)}$ **11** $\dfrac{7x^2}{(x+1)(x-1)}$

12 $\dfrac{4}{(x-6)(x-6)}$ **13** $\dfrac{4x-3}{x(x+4)(x-9)}$

14 $\dfrac{1}{x^2-3x+2}$ **15** $\dfrac{6x-1}{x^2-8x-20}$

16 $\dfrac{x^2+12x+36}{x^2-3x-10}$ **17** $\dfrac{x^2-3x-10}{x^2+12x+36}$

18 $\dfrac{18}{x}+\dfrac{x^2+1}{x^2-9}$

19 $\dfrac{a}{x-2}+\dfrac{b}{x^2-2x}-\dfrac{c}{x+2}$

20 $\dfrac{x^2+10}{x^2+4x-5}-\dfrac{2x-1}{x^2+8x+15}$

Example 17
(a) For what value(s) of x is the expression $\dfrac{x^2+15x+50}{x-5}$ *not defined? (b) Find the value(s) of x for which the expression is zero.*

(a) The expression is not defined when its denominator is zero.
i.e. when $x-5=0$
$x=5$

(b) Let $\dfrac{x^2+15x+50}{x-5}=0$

Multiply both sides by $x-5$.
$x^2+15x+50=0$
$(x+5)(x+10)=0$
either $x+5=0$ or $x+10=0$
i.e. either $x=-5$ or $x=-10$
The expression is zero when $x=-5$ or $x=-10$.

Exercise 7g (miscellaneous practice)

1 Simplify $\dfrac{y^2+2yz+z^2}{y^2-z^2}$.

2 Simplify the expression $\dfrac{2m^2+m-15}{m^2-9}$ and state the value of m for which the simplified expression is not defined.

3 Simplify

$$\frac{3}{a^2-3a+2} \div \frac{2}{2a^2-5a+2}$$

and state the value of a for which the simplified expression is not defined.

4 Reduce $\dfrac{3}{x}-\dfrac{x}{2}+5$ to a single fraction.

5 Simplify $\dfrac{3}{2x-4}+\dfrac{2}{6-3x}$.

6 Simplify

(a) $\dfrac{2b}{a^2-b^2}+\dfrac{a}{b^2-ab}$,

(b) $\dfrac{2a}{a^2-b^2}+\dfrac{b}{b^2-ab}$.

7 Simplify $\dfrac{3x+2}{3}-\dfrac{x-1}{4}-\dfrac{5}{12}$.

8 Simplify $\dfrac{3y}{x^2-xy-2y^2}-\dfrac{2y}{x^2-2xy}+\dfrac{2x+y}{x^2+xy}$.

9 Simplify $\left(\dfrac{2}{x}-\dfrac{5}{y}\right)\div\dfrac{4}{xy}$.

10 If $a=\dfrac{2m+1}{2m-1}$, express $\dfrac{2a+1}{2a-1}$ in terms of m.

11 Solve $\dfrac{1}{2-m}+\dfrac{3m}{m^2-4}=0$.

12 Solve the equation

$$\frac{2}{2x-3}+\frac{3}{2x+3}=\frac{2}{4x^2-9}.$$

13 Solve $\dfrac{1}{3-a}-\dfrac{1}{3}+\dfrac{4}{2a-5}=0$.

14 If $x:y=9:1$, evaluate $\dfrac{11x+y}{x+y}$.

15 If $A=\dfrac{3x+2}{x+3}$, (a) for what value of x is A undefined; (b) for what range of values of x is $A<2$?

16 (a) For what value(s) of x is the expression $\dfrac{2x+11}{x^2+x-20}$ not defined?

(b) For what value(s) of x is the expression zero?

17 (a) Solve $\dfrac{5}{a+4}-\dfrac{2}{a-2}=0$.

(b) Simplify $\dfrac{5}{a+4}-\dfrac{2}{a-2}$.

18 (a) Simplify

$$\frac{5}{2b+2}-\frac{2b+1}{b^2-2b-3}-\frac{1}{3-b}$$

(b) Solve $\dfrac{5}{2b+2}-\dfrac{2b+1}{b^2-2b-3}=\dfrac{1}{3-b}$.

19 Simplify $\dfrac{2}{a+2}+\dfrac{1}{a+1}-\dfrac{2a+3}{a^2+3a+2}$ and find the value of a for which the simplified expression is not defined.

20 (a) Simplify

$$\frac{(x+1)(x-2)-(x-1)(x+4)}{2x-1}$$

(b) (i) If k is a constant not equal to zero find the value(s) of x for which the expression $\dfrac{k}{x}+\dfrac{b}{x-3}+\dfrac{c}{x(x-3)}$ is not defined.

(ii) If x is not equal to any of the values obtained in (i), find the value of k such that $\dfrac{k}{x}+\dfrac{2}{x-3}-\dfrac{6}{x(x-3)}=0$.

Chapter 8

Graphs (4) Velocity–time curves

Area under a curve

It is often necessary to estimate the area contained between a curve and the axes, or other lines. There are many ways of doing this. Example 1 gives two of the simplest methods.

Example 1

Estimate the area enclosed between the curve $y = 20 + x - x^2$ and the axes in the positive quadrant.

By counting squares:

Draw the curve on suitable graph paper (Fig. 8.1).

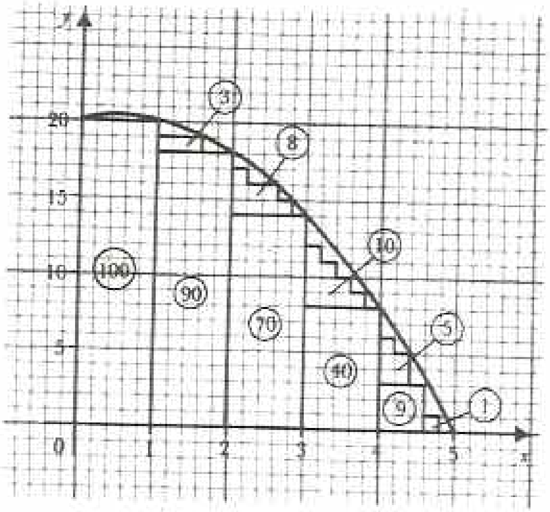


Fig. 8.1

Divide the area under the curve into convenient shapes, based on the squares of the graph paper. In each shape write the number of 'small squares' that it contains (in Fig. 8.1 the small squares have sides of 2 mm).

Estimate the remaining area by counting anything more than half a small square as one square, ignoring those bits which are less than half a small square.

Number of whole squares = 336
Estimated part squares = 19

Total area = 355 small squares

On the x-axis, 2 mm represents $\frac{1}{5}$ of a unit of length.

On the y-axis, 2 mm represents 1 unit of length.

Hence 1 small square represents $\frac{1}{5} \times 1$ unit of area $= \frac{1}{5}$ unit^2

$\Rightarrow$ 355 small squares represent $\frac{355}{5}$ unit^2
$= 71$ unit^2

Area under the graph $\simeq 71$ unit^2.

Notice the importance of the relationship between the small squares and the units of area.

By measuring trapeziums:

Draw the lines $x = 1$, $x = 2$, $x = 3$, $x = 4$ to divide the area under the curve into a rectangle (A), trapeziums (B), (C), (D) and triangle (E), Fig. 8.2.

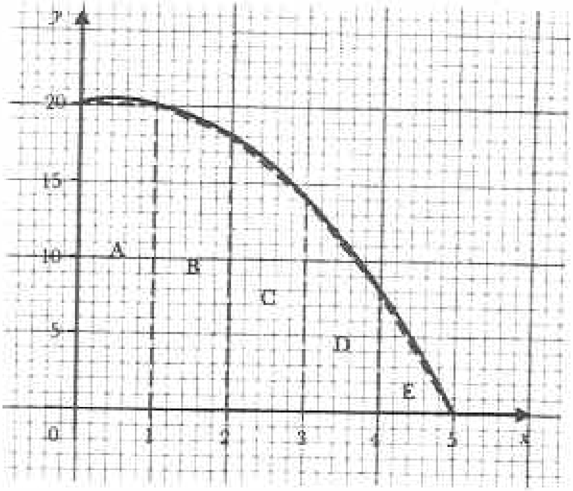


Fig. 8.2

Fig. 8.2 shows that the combined area of shapes (A), (B), (C), (D), (E) is approximately that contained between the curve and the axes.

$$\begin{aligned}\text{Total area} &\simeq 20 \times 1 + \tfrac{1}{2}(20 + 18)1 \\ &\quad + \tfrac{1}{2}(18 + 14)1 + \tfrac{1}{2}(14 + 8)1 \\ &\quad + \tfrac{1}{2} \times 8 \times 1 \text{ unit}^2 \\ &= 20 + 19 + 16 + 11 + 4 \text{ unit}^2 \\ &= 70 \text{ unit}^2\end{aligned}$$

Area under the graph $\simeq$ 70 unit^2.

The true area between the curve $y = 20 + x - x^2$ and the axes is 70,8 unit^2 to 1 d.p. Hence the method of counting squares is extremely accurate. Measuring trapeziums is slightly less accurate but is usually quicker.

Exercise 8a

In this exercise, give all answers to 2 s.f.

1 Use the method of counting squares to estimate the area bounded by the curve and the x-axis in Fig. 5.15 on page 38.

2 Look at Fig. 5.16 on page 38. Estimate the area bounded by the lines OM, MP and the curve OP.

3 Measure trapeziums to find the area below the curve but above the x-axis in Fig. 5.19 on page 39.

4 Use a suitable method to estimate the area above the curve but below the x-axis in Fig. 5.18 on page 39.

5 Draw the curve $y = 6x - x^2$ for values of x from 0 to 6, taking 2 cm for each x-unit and 1 cm for each y-unit. Find the area between the curve and the x-axis, both by counting squares and by measuring trapeziums.

6 Draw the curve $y = 2x(3 - x)$ for values of y corresponding to $x = 0, \frac{1}{2}, 1, 1\frac{1}{2}, 2, 2\frac{1}{2}, 3$. Use scales of 4 cm to 1 unit on the x-axis and 2 cm to 1 unit on the y-axis.
Estimate the area included between the x-axis and the curve.

Velocity–time curves

Curved graphs can be drawn to represent many physical situations. They are very commonly used to show the relationship between time and the velocity of moving objects.

Gradient/acceleration

In Chapter 5 it was shown that if a graph connects x and y, its gradient is the rate of change of y compared with x.

Hence if a graph connects velocity and time, its gradient is the rate of change of velocity compared with time. This rate of change is called **acceleration**.

Area under curve/distance

Fig. 8.3 is a simple graph of the motion of a car which travels at 15 m/s for 3 s.

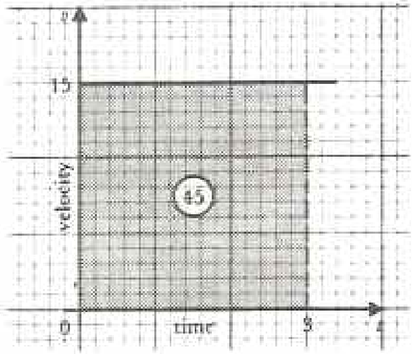


Fig. 8.3

In Fig. 8.3 the area under the graph is the area of the shaded rectangle

= length × breadth = 15 m/s × 3 s = 45 m

= distance travelled by the car

In general, the area under a velocity–time graph measures the distance travelled in a given time interval (Fig. 8.4).

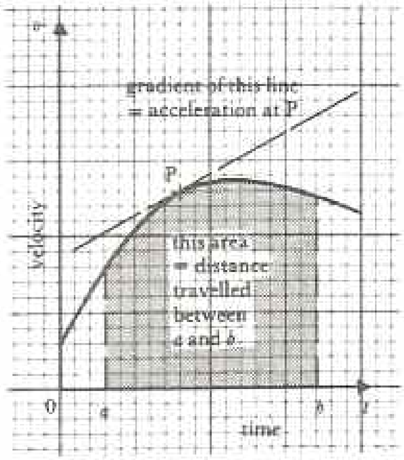


Fig. 8.4

Example 2, which follows, illustrates the use of gradient to represent acceleration and area to represent distance travelled.

Example 2

The speed of a car is noted at $\frac{1}{2}$-minute intervals and the results are as given in Table 8.1.

Table 8.1

time (minutes)	0	$\frac{1}{2}$	1	$1\frac{1}{2}$	2	$2\frac{1}{2}$	3
velocity (km/h)	20	$36\frac{1}{2}$	42	37	20	20	$26\frac{1}{2}$

(a) Estimate the distance travelled during the three minutes. (b) Estimate the acceleration in m/s² at $\frac{1}{2}$ min and 2 min from the start.

Fig. 8.5 is a graph of the journey.

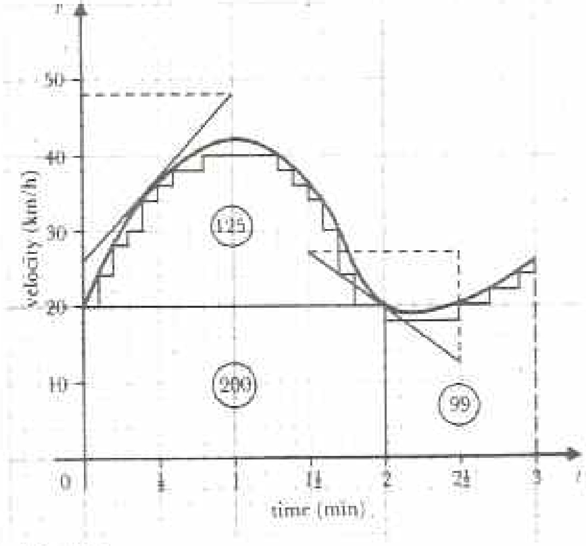


Fig. 8.5

(a) Estimated number of small squares = 444
1 small square represents 2 km/h × 0,1 min

$$= \frac{2 \times 1\,000}{60 \times 60} \text{ m/s} \times 6 \text{ s} = 3\tfrac{1}{3} \text{ m}$$

Distance travelled = $444 \times 3\frac{1}{3}$ m = 1 480 m
$\simeq$ 1 500 m

The car travels about 1 500 m.

(b) Draw the tangent to the curve at the point ($\frac{1}{2}$; $36\frac{1}{2}$). Its gradient shows an increase of velocity of 22 km/h in 1 min. (Compare the following sketch, Fig. 8.6, with the construction in Fig. 8.5.)

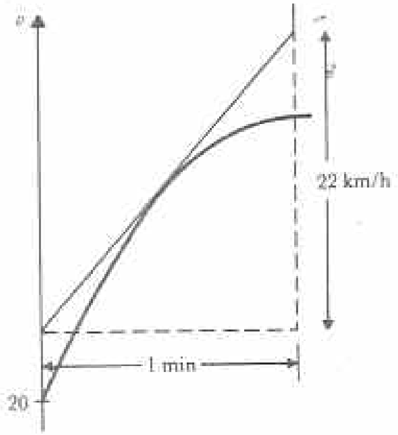


Fig. 8.6

$$\text{Acceleration} = \frac{22\,000}{60 \times 60} \text{ m/s in 60 s}$$

$$= \frac{22\,000}{60 \times 60 \times 60} \text{ m/s per second}$$

$$\simeq 0{,}102 \text{ m/s}^2$$

Similarly, at the point (2; 20), the rate of change of velocity is −15 km/h per minute.

$$\text{Deceleration}^* = \frac{15\,000}{60 \times 60 \times 60} \text{ m/s}^2$$

$$\simeq 0{,}069 \text{ m/s}^2$$

***Deceleration** is the rate at which velocity decreases; it may be thought of as negative acceleration. Fig. 8.7 shows that the acceleration of a moving body can be positive, negative or zero (at those points where the tangents to the velocity–time curve are parallel to the time-axis).

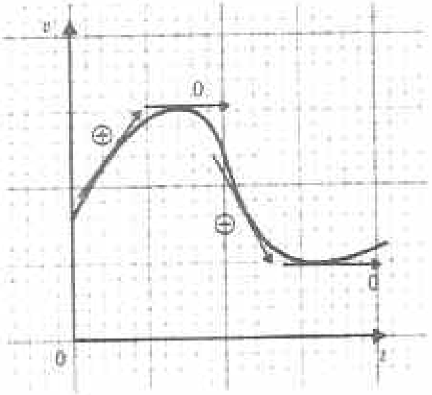


Fig. 8.7

Notice that just as the gradient on a velocity–time graph gives acceleration, so the gradient on a distance–time graph gives velocity, since velocity is the rate of change of distance with time. This definition is needed in questions 3 and 4 of the following exercise.

Exercise 8b

1 The velocity of a car accelerating from rest is taken at 10-second intervals. The results are given in Table 8.2.

Table 8.2

t (seconds)	0	10	20	30	40	50	60	70
v (m/s)	0	14,3	20,6	24,4	26,9	28,4	29,4	30

(a) Draw a v–t graph. (b) Estimate the accelerations 10 s and 45 s from the start. (c) Also estimate the total distance covered in the 70 seconds.

2 The depth of a river 60 m wide is found at 5-m intervals from one side to the other, straight across the river. The depths are 0; 11; 16,5; 18; 19,5; 22,5; 24,5; 25,5; 26; 24,5; 20; 11; 0 metres. (a) Draw the cross-section of the river. (b) Measure trapeziums to estimate the area of the cross-section in m^2. (c) If the river is flowing at 4 km/h, calculate its flow in litres/second giving the answer in standard form, correct to 2 s.f.

3 An object moves along a straight line so that its distance, s m, from a fixed point after t seconds is given by the formula $s = 5t - t^2$.

(a) Plot s against t for values of t from 0 to 5, taking a 2-cm scale on both axes.

(b) Estimate the gradients when $t = 1$ and $t = 3$.

(c) What do the gradients represent? What happens when the gradient is zero?

4 Water is poured at a steady rate into a pot of irregular shape. The depths of the water in the pot at 5-second intervals are 0; 2,2; 3,8; 5,0; 6,0; 6,8; 7,4; 7,9; 8,4; 8,9; 9,5; 10,6; 15,2 cm.

(a) Draw a graph showing depth against time.

(b) Estimate the rate, in cm/s, at which th level is rising in the pot 20 s and 35 s aft the start.

(c) Use your graph to guess the probab shape of the pot.

5 A particle moves along a straight line A so that, after t seconds, the velocity v m/s the direction AB is given by $v = 2t^2 - 9t +$ Corresponding values of t and v are give in Table 8.3 below.

Table 8.3

t	0	1	2	3	4	5	6	7
v	5		-5	-4	1	10	23	

Calculate the value of v when $t = 1$ and t value of v when $t = 7$.

Taking 2 cm to represent 1 second on t horizontal axis, and 2 cm to represent 5 m on the vertical axis, draw the graph $v = 2t^2 - 9t + 5$ for the range $0 \leqslant t \leqslant 7$.

Use your graph to estimate

(i) the values of t when the velocity is zer

(ii) the time at which the acceleration is zer

(iii) the acceleration after 6 seconds.

[Cam

6 A particle moves along a straight line that its velocity, v m/s, after t seconds is give by

$$v = 6 + 5t - t^2$$

(a) Copy and complete Table 8.4, givi corresponding values of t and v.

Table 8.4

t	0	1	2	3	4	5	6
v	6			12	10		

(b) Choose a suitable scale and draw graph to show the relationship between and t.

Use your graph to estimate

(c) the speed and time when the accelerati is zero,

(d) the acceleration after 5 seconds,

(e) the distance travelled in the first seconds.

7 A particle moves on a straight line so that its velocity v m/s at time t seconds from the start is given by $v = 0{,}1t^3$.
(a) Draw the graph of v against t for values of t from 0 to 4.
(b) Estimate the acceleration when $t = 2$ and when $t = 3{,}5$.
(c) Find the distance covered in the 4 seconds.

8 A car travels along a straight road and its speed v m/s when it is t seconds past a certain point is given in Table 8.5.

Table 8.5

t	0	5	10	15	20	25	30	35	40	45	50
v	19,7	23	25,7	27,7	29	29,7	29,6	29	28,2	27,2	25,8

(a) Find the maximum speed of the car in m/s and the time when it occurs.
(b) Calculate the acceleration when $t = 13$.
(c) Find the total distance travelled to the nearest 10 m.

9 A car changes its speed smoothly over 6 min of continuous running. Its speeds, in km/h, at 1-min intervals are successively 80; 65,4; 54,8; 47; 44,8; 52; 70.
By drawing a suitable graph, estimate, in m/s^2, the acceleration 5 min after the first observation and the deceleration after 2 min. Find also the total distance covered to the nearest 100 m.

10 The speed of a heavy lorry accelerating smoothly from rest is shown in Table 8.6.

Table 8.6

time (s)	0	10	20	30	40	50	60
speed (km/h	0	24,0	36,0	42,5	45,0	44,2	39,0

By drawing a suitable graph, estimate the distance covered during this time to the nearest 10 m. Find the acceleration 5 s after the start and the deceleration 55 s after the start (in m/s^2). What is the highest speed and when is it reached?

Chapter 9

Variation

Direct variation

Fig. 9.1 shows a new pencil cut into a number of pieces.

Fig. 9.1

The mass of each piece is proportional to its length. The ratio of mass to length is the same for all the pieces.

If a person walks at a steady speed, the distance travelled is proportional to the time taken.

These are both examples of **direct proportion**, or **direct variation.** In the first example the mass, M, **varies directly** with the length, L. In the second, the distance, D, varies directly with the time, T.

The symbol $\propto$ means 'varies with' or 'is proportional to'. The statements in the previous paragraph are written:

$$M \propto L$$
$$D \propto T$$

$D \propto T$ really means that the ratio $\frac{D}{T}$ is **constant** (i.e. stays the same).

Example 1
If $D \propto T$ and $D = 80$ when $T = 5$, find (a) the relationship between D and T, (b) the value of T when $D = 56$.

(a) If $D \propto T$
then $D = kT$, where k is a constant.
$D = 80$ when $T = 5$,
hence $80 = k \times 5$

$$\Leftrightarrow k = \frac{80}{5} = 16$$

The relationship between D and T is $D = 16T$.

(b) $D = 16T$
When $D = 56$,
$56 = 16T$

$$\Leftrightarrow T = \frac{56}{16} = \frac{7}{2} = 3\tfrac{1}{2}$$

Fig. 9.2 is a **sketch graph** of the relation $D \propto T$.

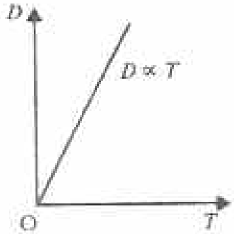


Fig. 9.2

Since $D = 16T$, the graph is a straight line of gradient 16 passing through the origin.

The graph of the relation between any two linear quantities which vary directly is always a straight line through the origin.

Example 2
Table 9.1 shows the extension E cm in an elastic string when it is pulled by a force of T newtons.

Table 9.1

T	5	8	11
E	7,5		16,5

(*a*) *Show that E is directly proportional to T.*
(*b*) *Find the value of E when $T = 8$.*

(a) By calculation:
if $E \propto T$
then $E = kT$, where k is a constant,

or $\frac{E}{T} = k$

i.e. If $E \propto T$, then $\frac{E}{T}$ should have a constant value for the results given in Table 9.1.
When $T = 5$, $E = 7{,}5$

$$\frac{E}{T} = \frac{7{,}5}{5} = 1{,}5$$

When $T = 11$, $E = 16{,}5$

$$\frac{E}{T} = \frac{16{,}5}{11} = 1{,}5$$

Since $\frac{E}{T} = 1{,}5$ in both cases, E is directly proportional to T.
Graphically: plot the given values (Fig. 9.3).

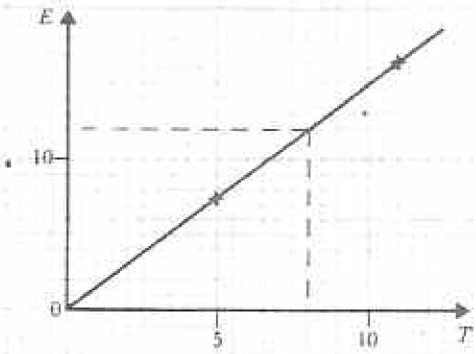


Fig. 9.3

Since the graph of E against T is a straight line passing through the origin, $E \propto T$.

(b) By calculation:
$E = 1{,}5T$
When $T = 8$,
$E = 1{,}5 \times 8 = 12$
or, from the graph:
when $T = 8$, $E = 12$

Exercise 9a

Questions 1–5 may be done orally.

1 If 1 m of wire has a mass of x g, what will be the mass of 25 m of the same wire?

2 If 1 jar of coffee costs \$$y$, what will be the cost of 4 jars of coffee?

3 If a man cycles 15 km in 1 hour, how far will he cycle in t hours if he keeps up the same rate?

4 If eggs cost n cents each, how much will 16 eggs costs?

5 If a cup holds d mℓ of water, how much water will 8 of these cups hold?

6 If $C \propto n$ and $C = 28$ when $n = 4$, find the formula connecting C and n.

7 If $D \propto t$ and $D = 32$ when $t = 2$, find the relationship between D and t.

8 If $x \propto y$ and $x = 3$ when $y = 12$, find the relationship between x and y.

9 If $d \propto s$ and $d = 120$ when $s = 30$, find the formula connecting d and s.

10 $a \propto b$ and $a = 2{,}4$ when $b = 3$. Find the relationship between a and b.

11 If $D \propto S$ and $D = 140$ when $S = 35$, find (a) the relationship between D and S, (b) the value of S when $D = 176$.

12 $x \propto y$ and $x = 30$ when $y = 12$. Find (a) the formula connecting x and y, (b) x when $y = 10$, (c) y when $x = 14$.

13 $P \propto Q$ and $P = 4{,}5$ when $Q = 12$. Find (a) the relationship between P and Q, (b) P when $Q = 16$, (c) Q when $P = 2{,}4$.

14 $A \propto B$ and $A = 1\frac{7}{8}$ when $B = \frac{5}{6}$. Find (a) A when $B = 0{,}4$, (b) B when $A = 7{,}5$.

15 $d \propto P$ and $d = 0{,}2$ when $P = 10$. Find (a) d when $P = 18$, (b) P when $d = 1{,}1$.

16 D is directly proportional to S.
(a) If $D = 120$ when $S = 30$, find the relationship between D and S.
(b) Find S when $D = 192$.

17 y varies directly with x.
(a) If $y = 9$ when $x = 45$ find the equation which connects x and y.
(b) Find y when $x = 40$.
(c) Find x when $y = 10$.

18 If $p \propto q$ and $p = 0{,}7$ when $q = 0{,}028$, find the relationship between p and q.

19 $x \propto y$ and $x = 17\frac{1}{2}$ when $y = 10\frac{1}{2}$.
(a) Find the equation which connects x and y.
(b) Find x when $y = 12$.

20 The mass of a plastic disc is proportional to its area. A disc of area 180 cm^2 has a mass of 200 g. If a similar disc has a mass of 250 g, what is its area?

21 What is the relation illustrated by the sketch graph in Fig. 9.4?

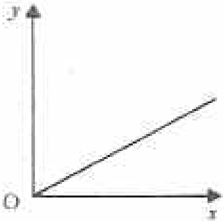


Fig. 9.4

22 Sketch the relation $N \propto s$.

23 Two variables A and B have corresponding values as shown in Table 9.2.

Table 9.2

A	6	10	12
B	34		68

(a) Either graphically, or by calculation, show that $B \propto A$.
(b) Find the value of B when $A = 10$.

24 The number of US Dollars (US\$) exchanged for a number of Pounds sterling (£) is given in Table 9.3.

Table 9.3

£	2	4	6	8	10
US\$	3,20	6,40	9,60	12,80	16,00

(a) Show that US\$ $\propto$ £.
(b) Find the law connecting US\$ and £.
(c) Find the value of £7 in US Dollars.
(d) Find the value of US\$8 in £.

25 The height (H cm) of liquid in a tube and the volume (V cm^3) of the liquid are as given in Table 9.4.

Table 9.4

V	3	6	12
H	2,6	5,2	10,4

(a) Show that $H \propto V$.
(b) Find the law of variation in the form $H = kV$.
(c) Find V when $H = 6,5$.
(d) Find H when $V = 4,5$.

Direct variation between non-linea quantities

Quantities which vary directly are not al in linear form. For example, the mass, m, cardboard square is directly proportional t area, A.

$m \propto A$

However, in Fig. 9.5, $A = x^2$
so it follows that $m \propto x^2$.

Fig. 9.5

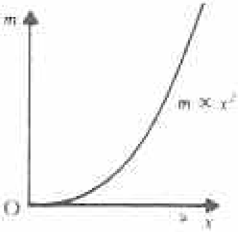


Fig. 9.6

$m \propto x^2$ is an example of direct proportio which one of the variables is in quadratic fo

Fig. 9.6 is a sketch graph of $m \propto x^2$. No that the curve is similar to the curve obtai in the graph of $y = x^2$.

Similarly, the volumes of spheres are dire proportional to the cubes of their radii:

$V = \frac{4}{3}\pi r^3$

or $V \propto r^3$, since $\frac{4}{3}\pi$ is a constant.

Fig. 9.7 is a sketch of the graph of $V \propto$

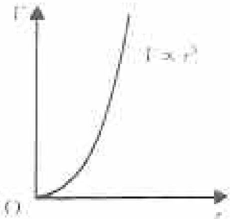


Fig. 9.7

Notice that the curve in Fig. 9.7 rises m steeply than that of Fig. 9.6 (also see Chapter

Example 3

$y \propto \sqrt{x}$ and $y = 4\frac{1}{2}$ when $x = 9$. (a) Find relationship between x and y. (b) Find y when $x =$ (c) Find x when $y = 6$. (d) Sketch the graph of $y \propto$

(a) If $y \propto \sqrt{x}$

then $y = k\sqrt{x}$, where k is a constant.

When $x = 9$, $y = 4\frac{1}{2}$

$$\Rightarrow 4\tfrac{1}{2} = k\sqrt{9}$$
$$= 3k$$
$$k = \frac{4\frac{1}{2}}{3} = \frac{3}{2}$$
$$\Rightarrow y = \tfrac{3}{2}\sqrt{x}$$

(b) When $x = 25$

$$y = \tfrac{3}{2}\sqrt{25} = 7\tfrac{1}{2}$$

(c) When $y = 6$

$$6 = \tfrac{3}{2}\sqrt{x}$$
$$\Rightarrow \sqrt{x} = \tfrac{12}{3} = 4$$
$$\Rightarrow x = 16$$

(d) Fig. 9.8 is a sketch graph of $y \propto \sqrt{x}$.

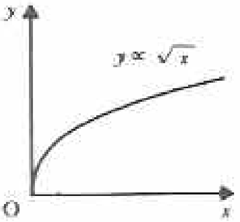


Fig. 9.8

The shape of the curve in part (d) is explained as follows:

If $y \propto \sqrt{x}$

then $y^2 \propto x$

or $x \propto y^2$

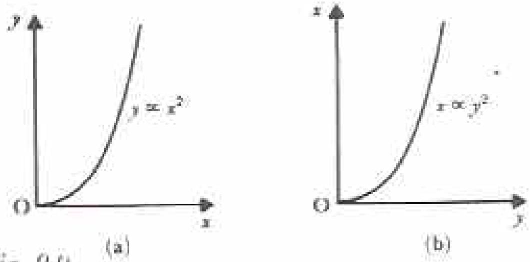


Fig. 9.9

Fig. 9.9(a) is a sketch of $y \propto x^2$. By interchanging the axes, Fig. 9.9(b) is a sketch of $x \propto y^2$. Fig. 9.8, $y \propto \sqrt{x}$, is equivalent to Fig. 9.9(b), $x \propto y^2$, with the x and y-axes in the standard positions.

The next example shows how a variation problem may be solved when the value of the constant k is not required.

Example 4

x is directly proportional to the square of y. What is the percentage change in x if y increases by 20%?

From the first sentence,

$$x \propto y^2$$

Let $x = ky^2$ (1)

Let x become X if y increases to $\frac{120}{100}y$.

Then $X = k\left(\frac{120}{100}y\right)^2$ (2)

Dividing (2) by (1),

$$\frac{X}{x} = \frac{k\left(\frac{120}{100}y\right)^2}{ky^2} = \left(\frac{120}{100}\right)^2 = \frac{144}{100}$$

$$\Leftrightarrow X = \frac{144}{100}x$$

Hence x increases by 44%.

Exercise 9b

1 A particle moves in such a way that its displacement, s metres, at time t seconds is given by the relation $s = at^2$, where a is a constant. Calculate a if $s = 32$ when $t = 4$.

2 y varies directly as the square of x. If $y = 98$ when $x = 7$, calculate y when $x = 5$.

3 $x \propto y^2$ and $x = 45$ when $y = 3$.
 (a) Find the relationship between x and y.
 (b) Find x when $y = 4$.
 (c) Find y when $x = 125$.

4 $A \propto B^3$ and $A = 32$ when $B = 4$.
 (a) Find the formula connecting A and B.
 (b) Find A when $B = 6$.
 (c) Find B when $A = 13.5$.

5 P varies directly as the square root of Q and $P = 10$ when $Q = 16$.
 (a) Find the equation in P and Q.
 (b) Find P when $Q = 9$.
 (c) Find Q when $P = 1\frac{7}{8}$.

6 $Z^2 \propto Y$ and $Z = 9$ when $Y = 27$.
 (a) Find the relation between Z and Y.
 (b) Find Z when $Y = 48$.
 (c) Find Y when $Z = 6$.

7 V varies directly as the cube of D and $V = 108$ when $D = 6$.
(a) Find the formula connecting V and D.
(b) Find V when $D = 3$.
(c) Find D when $V = 2{,}048$.

8 $D \propto \sqrt{H}$ and $D = 6$ when $H = 24$.
(a) Find the relation between D and H.
(b) Find D when $H = 150$.
(c) Find H when $D = 10\frac{1}{2}$.

9 State the relation which is illustrated by the sketch graph in Fig. 9.10.

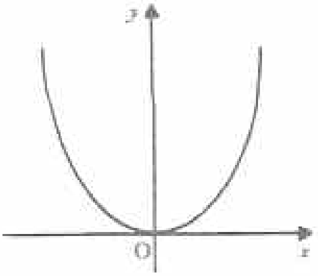


Fig. 9.10

10 Sketch the following curves, showing both positive and negative values of x.
(a) $y \propto x^2$ (b) $y \propto x^3$ (c) $y \propto \sqrt{x}$.

11 Draw a sketch graph of the relation y^2 is proportional to x.

12 y varies directly with x^2. Table 9.5 shows some corresponding values of x and y.

Table 9.5

x	-2	-1	$\frac{1}{2}$
y	16	4	

Find the relation between x and y and complete the table.

13 The power, P watts, used in an electric circuit is proportional to the square of the current, C amps. When the current is 4 amps the circuit uses 500 watts. Find the current when the circuit uses 2 420 watts.

14 The distance of the horizon from an observer varies directly with the square root of the height of the observer above ground level. At a height of 8 metres the horizon is 10 km away. Find the distance of the horizon from an observer at a height of 98 metres.

15 x varies directly with the square of y. Find the percentage change in x if y is
(a) increased by 10%,
(b) decreased by 10%.

16 x is directly proportional to the square root of y. What is the percentage change in x if y is increased by 44%?

17 If $V \propto R^3$, what is the percentage increase in V if R increases by 20%?

18 If $W \propto D^2$, what is the percentage decrease in W if D decreases by 15%?

Inverse variation

Fig. 9.11 shows a circle cut into (a) 5 equal sectors, (b) 12 equal sectors.

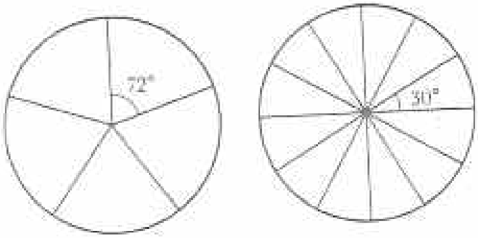


Fig. 9.11

The greater the number of sectors, the smaller the angle of each sector.

If a pot of tea is shared between some people, the *greater* the number of people, the *less* tea each will receive.

These are examples of **inverse proportion** or **inverse variation**. In the first example, the size of the angle, θ, **varies inversely** with the number of sectors, n. In the second, the volume of tea received, V, is inversely proportional to the number of people, n. These statements are written:

$$\theta \propto \frac{1}{n} \qquad V \propto \frac{1}{n}$$

Example 5

If V varies inversely with n and $V = 220$ when $n = 6$, find V when $n = 8$.

If $V \propto \frac{1}{n}$ then $V = \frac{k}{n}$, where k is a constant.
$V = 220$ when $n = 6$.

$220 = \frac{k}{6} \Leftrightarrow k = 6 \times 220$ and $V = \frac{6 \times 220}{n}$

When $n = 8$,

$$V = \frac{6 \times 220}{8}$$
$$= 165$$

Fig. 9.12 is a sketch graph of the relation $V \propto \frac{1}{n}$.

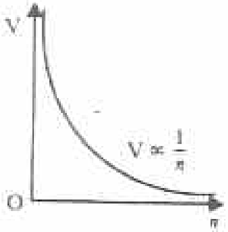


Fig. 9.12

Notice that the curve approaches both axes but does not reach them. This is because division by 0 is impossible. (See Fig. 7.1, Chapter 7; also see Chapter 15.)

Example 6

The number of spherical glass beads which can be made from a given volume of glass varies inversely with the cube of the diameter of the beads. When the diameter is 2 mm, *the number of beads is* 2 700. *How many beads of diameter* 3 mm *can be made from the glass?*

Let N = number of beads, d = diameter of the beads.

From the first sentence,

$N \propto \frac{1}{d^3}$ or $N = \frac{k}{d^3}$, where k is a constant.

From the second sentence,

$$2\,700 = \frac{k}{2^3}$$
$$\Leftrightarrow k = 8 \times 2\,700$$

So $N = \frac{2\,700 \times 8}{d^3}$

When $d = 3$,

$$N = \frac{2\,700 \times 8}{3^3}$$
$$= \frac{2\,700 \times 8}{27}$$
$$= 800$$

800 beads can be made.

Exercise 9c

1 If d varies inversely as t, use the symbol $\propto$ to show a connection between d and t.

2 A piece of string is cut into n pieces of equal length l.
(a) Does n vary directly or inversely with l?
(b) Use the symbol $\propto$ to show a connection between n and l.

3 A rectangle has a constant area, A. Its length is l and its breadth is b.
(a) Write a formula for l in terms of A and b.
(b) Write a formula for b in terms of A and l.
(c) Does l vary inversely or directly with b?

4 If $x \propto \frac{1}{y}$ and $x = 22$ when $y = 3$, find the relationship between x and y.

5 If $R \propto \frac{1}{T}$ and $T = 8$ when $R = 4$, find the relationship between R and T.

6 If y varies inversely as x, and $y = 2$ when $x = 3$, find y when $x = 6$.

7 P is inversely proportional to Q and $P = 5$ when $Q = 4$. What is the value of Q when $P = 25$?

8 If x varies inversely as the square of y, and $x = 4$ when $y = \frac{1}{2}$, what is y when x is 5?

9 Make x the subject of the relation $y \propto \frac{1}{\sqrt{x}}$.

10 A quantity $(y - k)$ varies inversely as the square of x. Make y the subject of an equation in x, k and h, where k and h are constants.

11 The electrical resistance R of a wire varies inversely as the square of the radius r. Use a constant k to show the relation between R and r.

12 P varies inversely as the square root of v and $P = 4.5$ when $v = 25$. Find v when $P = 15$.

13 Sketch the following curves, showing both positive and negative values of x.

(a) $y \propto \frac{1}{x}$ (b) $y \propto \frac{1}{x^2}$ (c) $y \propto \frac{1}{\sqrt{x}}$

(*Hint:* For sketching purposes, it may help to substitute = for $\propto$ and then plot some sample points.)

14 The variables X and Y are connected by the relation 'Y varies inversely as X'. Table 9.6 shows the values of Y for some selected values of X.

Table 9.6

X	10	20	30	40
Y	12	6	?	3

What is the missing value of Y?

15 The length of wire that can be made from a mass of copper is inversely proportional to the square of the diameter of the wire. When the diameter is 3 mm the length of the wire is 1,8 km. Find the length of the wire when its diameter is 1,2 mm.

Joint variation

The mass, M, of a coin of radius r and thickness h depends on the volume, V, of metal in the coin. i.e. $M \propto V$

or $M \propto \pi r^2 h$ (since $V = \pi r^2 h$)

or $M \propto r^2 h$ (since π is a constant)

$M \propto r^2 h$ means that the mass of the coin **varies jointly** with the square of the radius *and* the thickness. This is an example of **joint variation**.

Example 7

The mass of a wire varies jointly with its length and the square of its diameter. 500 m *of wire of diameter* 3 mm *has a mass of* 31,5 kg. *What is the mass of* 1 km *of wire of diameter* 2 mm?

Let M = mass in kg, d = diameter in mm and L = length in m.

Then, from the first sentence,

$M \propto Ld^2$

or $M = kLd^2$, where k is a constant.

From the second sentence,

$31{,}5 = k \times 500 \times 3^2$ (1)

From the third sentence,

$M = k \times 1\,000 \times 2^2$ (2)

Dividing (2) by (1),

$$\frac{M}{31{,}5} = \frac{1\,000 \times 2^2}{500 \times 3^2} = \frac{2 \times 4}{9} = \frac{8}{9}$$

$M = \frac{8}{9} \times 31{,}5 = 28$

Hence the mass is 28 kg.

Notice that it was not necessary to find k. Notice also that it is possible to mix dimensions, such as mm and m, so long as this is done consistently.

Example 8

If $X \propto YZ^{\frac{1}{3}}$ *and* $Y \propto Z^{-2}$ *show that* $X \propto Y^{\frac{5}{6}}$.

If $Y \propto Z^{-2}$

then $Y = \frac{k}{Z^2}$ where k is a constant.

$$\Rightarrow Z^2 = \frac{k}{Y}$$

$$\Rightarrow Z = \sqrt{\frac{k}{Y}}$$

Given that $X \propto YZ^{\frac{1}{3}}$ substitute $\sqrt{\frac{k}{Y}}$ for Z.

$$X \propto Y\left(\sqrt{\frac{k}{Y}}\right)^{\frac{1}{3}}$$

$$\Rightarrow X \propto Y\left(\frac{k}{Y}\right)^{\frac{1}{6}}$$

$$\Rightarrow X \propto Y \times \frac{k^{\frac{1}{6}}}{Y^{\frac{1}{6}}}$$

$$\Rightarrow X \propto k^{\frac{1}{6}}Y^{\frac{5}{6}}$$

$\Rightarrow X \propto Y^{\frac{5}{6}}$ since $k^{\frac{1}{6}}$ is a constant.

The method in Example 8 is to eliminate the variable which is not required.

Exercise 9d

1 $x \propto yz$. When $y = 2$ and $z = 3$, $x = 30$.
(a) Find the relation between x, y and z.
(b) Find x when $y = 4$ and $z = 6$.

2 $x \propto \frac{y}{z}$. $x = 27$ when $y = 9$ and $z = 2$.
(a) Find the relation between x, y and z.
(b) Find x when $y = 14$ and $z = 12$.

3 $p \propto \frac{q}{r^2}$ and $p = 3\frac{1}{3}$ when $q = 5$ and $r = 3$.
(a) Find the equation connecting p, q and r.
(b) Find p when $q = 9$ and $r = 1{,}2$.

4 The height (h) of a cone varies directly as its volume (V) and inversely as the square of its radius (r). Use the constant k to show the relationship between h, V and r.

5 $A \propto BC$ and $A = 6$ when $B = 4$ and $C = 9$.
(a) Find A when $B = 3$ and $C = 10$.
(b) Find C if $A = 20$ and $B = 15$.
(c) By what percentage does A change if B is increased by 10% and C is decreased by 10%?

6 x varies directly with the square of y and with z. When $y = 2$ and $z = 3$, $x = 4\frac{1}{2}$.
(a) Find x when $y = 5$ and $z = 4$.
(b) Find y when $x = 21$ and $z = 3\frac{1}{2}$.
(c) What happens to x if y is doubled and z halved?

7 x, y and z are related quantities such that x varies directly as y and inversely as the square root of z. When $x = 300$ and $y = 65$, $z = 25$. Calculate the value of x when $y = 468$ and $z = 144$.

8 If $P \propto \frac{1}{V}$ and $V \propto R^2$, how does P vary with R?

9 $x \propto y$ and $y \propto z^3$. How does x vary with z?

10 $x \propto y^2$ and $y \propto z^2$. How does x vary with z?

11 $A \propto BC$ and $B \propto \frac{1}{C^2}$. How does A vary with C?

12 y varies directly as x and inversely as z. x varies inversely as y^2. Prove that z^2 varies directly as x^3.

13 z varies directly as $\frac{x}{y^2}$ and y varies inversely as x. If $z = \frac{1}{3}$ when $x = 2$ and $y = \frac{1}{4}$, express
(a) y in terms of x,
(b) z in terms of x.
If the value of x is increased by 10%, find the corresponding increase in the value of z.

14 The mass, m, of a roller varies jointly with its length, l, and the square of its diameter, d. A roller of diameter 20 mm is 5 cm long and has a mass of 0,29 kg. Calculate the mass, in kg, of a roller 30 mm in diameter and 2 m long.

15 Given that the energy E varies directly as the resistance R and inversely as the square of the distance d, obtain an equation connecting E, R and d.
If $E = \frac{32}{25}$ when $R = 8$ and $d = 5$, calculate
(a) the value of R when $E = 16$ and $d = 3$;
(b) the value of d when $R = 5$ and $E = \frac{5}{6}$;
(c) the percentage increase in the value of R when each of E and d increases by 3%.

Partial variation

When feeding a large number of people, as in a hotel, the total cost depends on two separate factors: first the cost of the overheads (such as fuel and wages), secondly the cost of the food used.

The cost of the overheads generally remains constant, but the cost of the food is proportional to the number of people being fed. Hence the total cost is **partly constant and partly varies** as the number of people.

In algebraic terms, $C = a + kN$, where C is the total cost, N the number of people and a and k are constants. This is an example of **partial variation**.

Example 9

C is partly constant and partly varies as N. $C = 45$ when $N = 10$ and $C = 87$ when $N = 24$. (a) Find the formula connecting C and N. (b) Find C when $N = 18$.

(a) From the first sentence,

$C = a + kN$ where a and k are constants.

From the second sentence,

$45 = a + 10k$ (1)

and $87 = a + 24k$ (2)

Subtract (1) from (2),

$42 = 14k$

$k = \frac{42}{14} = 3$

Substitute 3 for k in (1),

$45 = a + 30$

$a = 15$

Hence $C = 15 + 3N$ is the required formula.

(b) When $N = 18$,

$C = 15 + 3 \times 18$

$= 15 + 54$

$= 69$

Example 10

The resistance to motion of a car is partly constant and partly varies as the square of the speed. At 40 km/h the resistance is 530 N, and at 60 km/h it is 730 N. What will be the resistance at 70 km/h?

Let R = resistance in newtons, V = speed in km/h, then, from the first sentence,

$R = a + kV^2$,

where a and k are constants.

From the second sentence,

$530 = a + 1\,600\,k$ (1)

and $730 = a + 3\,600\,k$ (2)

Subtract (1) from (2),

$200 = 2\,000\,k$

$k = \frac{1}{10}$

Substituting in (1),

$530 = a + 1\,600 \times \frac{1}{10}$

$= a + 160$

$a = 370$

Hence $R = 370 + \frac{1}{10}V^2$

When $V = 70$

$R = 370 + \frac{1}{10} \times 4\,900$

$= 370 + 490$

$= 860$

Hence the resistance is 860 N.

Notice in Examples 9 and 10, since there are two unknowns, two equations must be formed. These are then solved simultaneously.

Exercise 9e

1 x is partly constant and partly varies as y. When $y = 2$, $x = 30$, and when $y = 6$, $x = 50$. (a) Find the relationship between x and y. (b) Find x when $y = 3$.

2 x is partly constant and partly varies with y. When $y = 3$, $x = 11$, and when $y = 4$, $x = 14$. (a) Find the relationship between x and y. (b) Find x when $y = 10$.

3 x is partly constant and partly varies with y. When $y = 3$, $x = 7$, and when $y = 6$, $x = 9$.
(a) Find the relationship between x and y.
(b) Find x when $y = 4$.

4 D is partly constant and partly varies with V. When $V = 40$, $D = 150$, and when $V = 54$, $D = 192$.
(a) Find the formula connecting D and V.
(b) Find D when $V = 73$.

5 The cost of making a dress is partly constant and partly varies with the amount of time it takes to make. If it takes 3 hours to make, it costs \$40. If it takes 5 hours to make, it costs \$46. Find the cost if it takes $1\frac{1}{2}$ hours.

6 A varies partly as B and partly as the square root of B. When $B = 4$, $A = 22$ and when $B = 9$, $A = 42$. Find A when $B = 25$.

7 Table 9.7 is an incomplete table for the relation $y = \frac{1}{2}x^2 + k$, where k is a constant.

Table 9.7

x	−3	−2	1	2	3	4
y	6,5	4	2,5	4	6,5	

(a) What is the value of k?
(b) Find the value of y at $x = 4$.

8 Two quantities, P and Q, are connected by a linear relation of the form $P = kQ + c$ where k and c are constants.
(a) If $Q = 60$ when $P = 10$, and $Q = 240$ when $P = 100$, find the equation connecting P and Q.
(b) Sketch the graph of the relation, indicating where it cuts the two axes.

9 The fixed costs of a manufacturing business are \$30 000. The variable costs, which are proportional to the sales, are \$40 000 when the sales are \$80 000. Calculate the total costs and the profit when the sales are \$90 000.

10 The cost of feeding a number of students is partly constant and partly varies directly as the number of students. Feeding 75 students during a certain period costs \$875 and feeding 100 students during the same period of time costs \$1000. Find the cost of feeding 220 students over the same period of time.

11 The cost of making computers is partly constant and partly varies as the number of computers produced. The total cost of making 4 computers is \$18 700 and of making 10 computers is \$35 500. Find the total cost of making 20 computers.

12 The resistance R to the motion of a car is partly constant and partly proportional to the square of the speed v. When the speed is 30 km/h the resistance is 190 newtons and when the speed is 50 km/h the resistance is 350 newtons. Find for what speed the resistance is 302,5 newtons.

13 The diameter of a reel of sticky tape is partly constant and partly varies as the square root of the length of tape on the reel. When new, the reel contains 250 m of tape and is 8 cm in diameter. When all the tape has been used, the diameter of the empty reel is 2 cm. What length of tape is on the reel when its diameter is 5,6 cm?

14 The cost of running a hotel is partly constant and partly varies as the square of the number of people staying in the hotel. Accommodating 5 people costs \$200 and the cost for 8 people is \$395. Calculate the cost of accommodating 12 people in that hotel.

15 n is the algebraic sum of two terms, one of which varies directly as u and the other inversely as u^2. If $n = 11$ when $u = 2$ and $n = 25,16$ when $u = 5$, calculate n when $u = 3$.

Chapter 10

Mensuration of solid shapes

Surface area and volume of solids

Formulae for the areas and volumes of common solids already found in earlier books of this course are given below.

Prisms

In general,
volume = area of constant cross-section × perpendicular height
= area of base × height (Fig. 10.1)

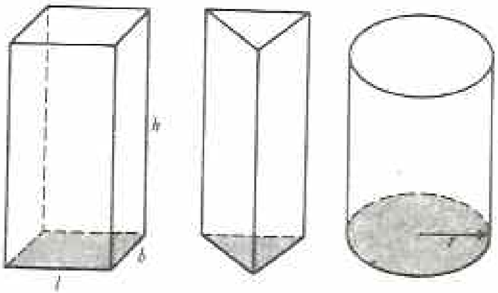


Fig. 10.1

cuboid
volume = lbh
surface area = $2(lb + lh + bh)$

cylinder
volume = $\pi r^2 h$
curved surface area = $2\pi rh$
total surface area = $2\pi rh + 2\pi r^2$
= $2\pi r(h + r)$

Pyramid and cone

In general
volume = $\frac{1}{3}$ × base area × height (Fig. 10.2)

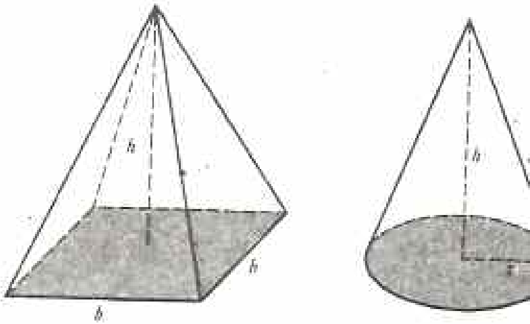


Fig. 10.2

square-based pyramid
volume = $\frac{1}{3}b^2h$

cone
volume = $\frac{1}{3}\pi r^2 h$
curved surface area = πrl
total surface area = $\pi rl + \pi r^2$
= $\pi r(l + r)$

Example 1

A car petrol tank is 0,8 m long, 25 cm wide and 20 cm deep. How many litres of petrol can it hold?

Working in cm,
volume of tank = $80 \times 25 \times 20\ \text{cm}^3$

1 litre = $1\,000\ \text{cm}^3$

capacity of tank = $\frac{80 \times 25 \times 20}{1\,000}$ litres
= 40 litres

The tank can hold 40 litres of petrol.

Example 2

A circular metal sheet 48 cm in diameter and 2 mm thick is melted and recast into a cylindrical bar 6 cm in diameter. How long is the bar?

Radius of sheet = $\frac{48}{2}$ cm = 24 cm

Radius of bar $= \frac{6}{2}$ cm $= 3$ cm

Let the bar be x cm long.

Then its volume $= \pi \times 3^2 \times x \text{ cm}^3$

Volume of circular sheet $= \pi \times 24^2 \times \frac{1}{5} \text{ cm}^3$

Hence $\pi \times 3^2 \times x = \pi \times 24^2 \times \frac{1}{5}$

$$\Rightarrow \quad x = \frac{\pi \times 24^2 \times \frac{1}{5}}{\pi \times 3^2}$$

$$= \frac{576}{9 \times 5}$$

$$= \frac{64}{5} = 12{,}8$$

The bar is 12,8 cm long.

Notice in Example 2 that no numerical value of π was needed. Never substitute a value for π unless it is necessary.

Example 3

A 216° sector of a circle of radius 5 cm is bent to form a cone. Find the radius of the base of the cone and its vertical angle.

In Fig. 10.3, the radius of the base of the cone is r cm and the vertical angle is 2α.

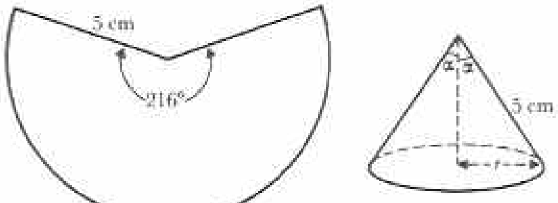


Fig. 10.3

circumference of base of cone

= length of arc of sector

$$\Rightarrow 2\pi r = \frac{216}{360} \times 2\pi \times 5$$

$$\Leftrightarrow \quad r = \frac{216}{360} \times 5 = 3$$

$\sin \alpha = \frac{3}{5} = 0{,}6000$

$\Rightarrow \quad \alpha = 36{,}87°$

$\Rightarrow \quad 2\alpha = 73{,}74°$

Radius of base = 3 cm

Vertical angle = 73,7° (to 0,1°)

Example 4

Fig. 10.4 shows a wooden block in the form of a prism. PQRS is a trapezium with PQ ∥ SR, PQ = 7 cm, PS = 5 cm *and* SR = 4 cm. *If the block is* 12 cm *long, calculate its volume.*

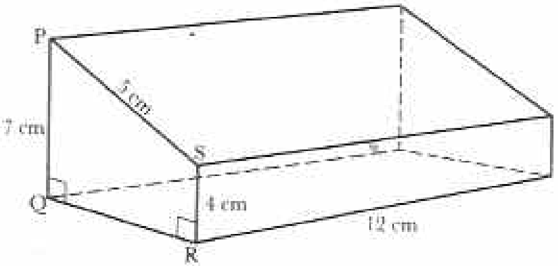


Fig. 10.4

Volume of block = area of PQRS × 12 cm³

area of PQRS $= \frac{1}{2}(7 + 4) \times$ QR cm²

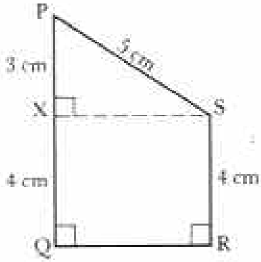


Fig. 10.5

With the construction of Fig. 10.5,

SX = QR

But SX = 4 cm (*the sides of* △PXS *form a* 3; 4; 5 *Pythagorean triple*)

Volume of block $= \frac{1}{2}(7 + 4) \times 4 \times 12 \text{ cm}^3$

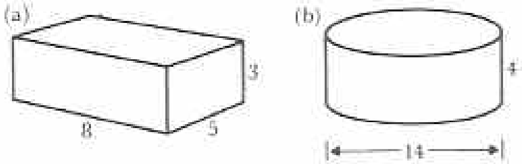
$= 11 \times 2 \times 12 \text{ cm}^3$

$= 264 \text{ cm}^3$

Exercise 10a

Use the value $3\frac{1}{7}$ for π where necessary.

1 Calculate the volumes of the solids in Fig. 10.6(a) – (f). All lengths are in cm.



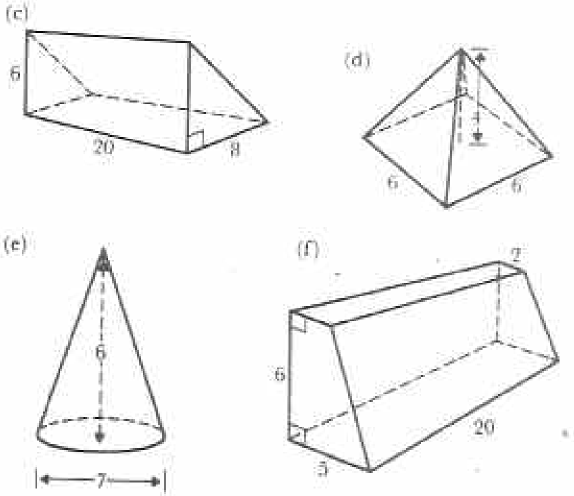


Fig. 10.6

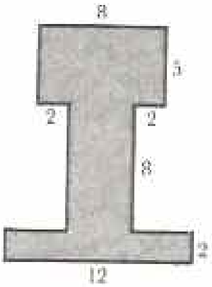


Fig. 10.8

2 Calculate the total surface areas of the solids in parts (a), (b), (c) of Fig. 10.6.

3 A rectangular tank is 76 cm long, 50 cm wide and 40 cm high. How many litres of water can it hold?

4 A water tank is 1,2 m square and 1,35 m deep. It is half full of water. How many times can a 9-litre bucket be filled from the tank?

5 $2\frac{1}{2}$ litres of oil are poured into a container whose cross-section is a square of side $12\frac{1}{2}$ cm. How deep is the oil in the container?

6 The diagrams in Fig. 10.7 show the cross-sections of steel beams. All dimensions are in cm. Calculate the volumes, in cm^3, of 5-metre lengths of the beams.

Fig. 10.7

7 Fig. 10.8 shows the cross-section of a steel rail, dimensions being given in cm. Calculate the mass, in tonnes, of a 20-metre length of the rail if the mass of 1 cm^3 of the steel is 7,5 g.

8 Fig. 10.9 shows the cross-section of a ruler.
(a) Calculate the volume of the ruler in cm^3 if it is 30 cm long.
(b) If the ruler is made of plastic and has a mass of 45 g, what is the density of the plastic in g/cm^3?
(c) Find the mass, to the nearest g, of the ruler if it is made of wood of density 0,7 g/cm^3.

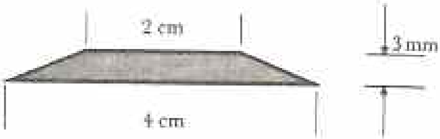


Fig. 10.9

9 Calculate, in terms of π, the total surface area of a solid cylinder of radius 3 cm and height 4 cm.

10 A paper label just covers the curved surface of a cylindrical tin of diameter 12 cm and height $10\frac{1}{2}$ cm. Calculate the area of the paper label.

11 A cylindrical tin full of engine oil has a diameter of 12 cm and a height of 14 cm. The oil is poured into a rectangular tin 16 cm long and 11 cm wide. What is the depth of the oil in the tin?

12 A cylindrical shoe polish tin is 10 cm in diameter and 3,5 cm deep.
(a) Calculate the capacity of the tin in cm^3.
(b) When full, the tin contains 300 g of polish. Calculate the density of the polish in g/cm^3 correct to 2 d.p.

13 A wire of circular cross-section has a diameter of 2 mm and a length of 350 m. If the mass of the wire is 6,82 kg, calculate its density in g/cm^3.

14 A measuring cylinder of radius 3 cm contains water to a height of 49 cm. If this water is poured into a similar cylinder of radius 7 cm, what will be the height of the water column?

15 A metal disc 12 cm in diameter and 5 cm thick is melted down and cast into a cylindrical bar of diameter 5 cm. How long is the bar?

16 A solid metal cylinder, 8 cm in diameter and 8 cm long, is to be made into discs 4 cm in diameter and 5 mm thick. Assuming no wastage, how many discs can be made?

17 Water flows through a 7-cm diameter pipe at the rate of 4 metres/second.

(a) How many cm^3 of water flow through the pipe in one second?

(b) Express the flow of water as a rate in litres/minute.

18 How many cylindrical glasses 6 cm in diameter and 10 cm deep can be filled from a cylindrical jug 10 cm in diameter and 18 cm deep?

19 A cylindrical container 30 cm in diameter holds approximately 30 litres of oil. How far does the oil level fall after 1 litre of oil has been used?

20 A pyramid 8 cm high stands on a rectangular base 6 cm by 4 cm. Calculate the volume of the pyramid.

21 A paper cone has a base diameter of 8 cm and a height of 3 cm.

(a) Calculate the volume of the cone in terms of π.

(b) Make a sketch of the cone and hence use Pythagoras' theorem to calculate its slant height.

(c) Calculate the curved surface area of the cone in terms of π.

(d) If the cone is cut and opened out into a sector of a circle, what is the angle of the sector?

22 Calculate the volume and curved surface area of a cone 14 cm in base diameter and 24 cm high.

23 If the cone in question 22 is made of paper, and the paper is flattened out into a sector of a circle, what is the angle of the sector?

24 Fig. 10.10 shows a cross-section of a dam wall. How many m^3 of concrete will it take to build a 100-m length of this wall?

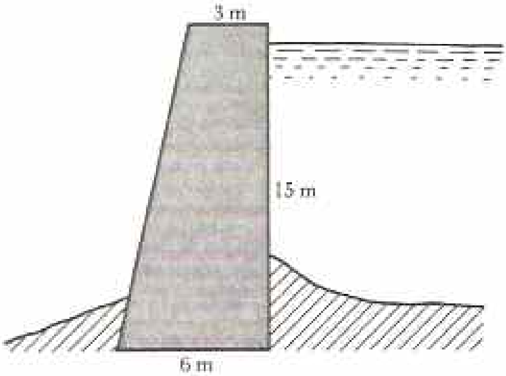


Fig. 10.10

25 A cone of height 9 cm has a volume of n cm^3 and a curved surface area of n cm^2. Find the vertical angle of the cone.

Sphere

Fig. 10.11 represents a solid sphere of radius r.

volume $= \frac{4}{3}\pi r^3$

curved surface area $= 4\pi r^2$

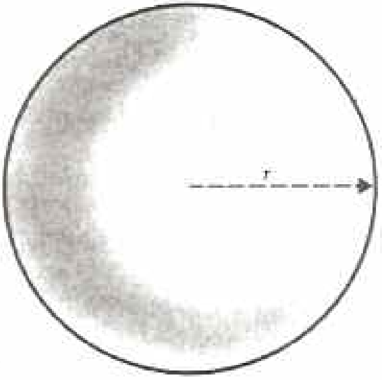


Fig. 10.11

(The proof of these formulae is beyond the scope of this course.)

Example 5

A solid sphere has a radius of 5 cm *and is made of metal of density* 7,2 g/cm^3. *Calculate the mass of the sphere in kg.*

$$\text{Volume of sphere} = \tfrac{4}{3}\pi \times 5^3\ \text{cm}^3$$
$$= \frac{4 \times \pi \times 125}{3}\ \text{cm}^3$$
$$= \frac{500\pi}{3}\ \text{cm}^3$$

$$\text{Mass of sphere} = \frac{500\pi}{3} \times 7{,}2\ \text{g}$$
$$= \frac{500\pi \times 7{,}2}{3 \times 1\,000}\ \text{kg} = \frac{7{,}2\pi}{3 \times 2}\ \text{kg}$$
$$= 1{,}2\pi = 1{,}2 \times 3{,}142$$
$$= 3{,}770\,4\ \text{kg}$$
$$= 3{,}77\ \text{kg to 3 s.f.}$$

Example 6

Calculate the total surface area of a solid hemisphere of radius 6,8 cm. *Use the value* 0,4971 *for* log π.

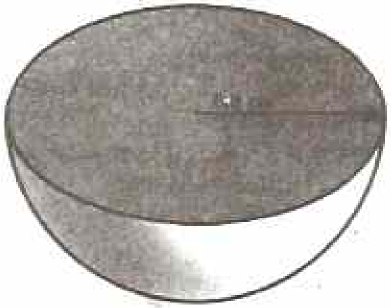


Fig. 10.12

In Fig. 10.12, total surface area
= curved surface area + plane surface area
$= 2\pi r^2 + \pi r^2$
$= 3\pi r^2$
When $r = 6{,}8$
total surface area
$= 3 \times \pi \times 6{,}8^2\ \text{cm}^2$
$= 435{,}7\ \text{cm}^2$
$= 436\ \text{cm}^2$ to 3 s.f.

working:

No	Log
$6{,}8^2$	$0{,}8325 \times 2$ = 1,6650
π	0,4971
3	0,4771
435,7	2,6392

Exercise 10b

Use the value 3,142 for π or 0,497 1 for log π, whichever is more convenient.

1 Calculate the volume and surface area to 3 s.f. of each of the following.
(a) A sphere, radius 10 cm
(b) A sphere, diameter 16 cm
(c) A hemisphere, radius 2 cm
(d) A hemisphere, diameter 9 cm

2 The diameter of an iron ball used in 'putting the shot' is 12 cm. If the density of iron is 7,8 g/cm^3, calculate the mass of the ball in kg to 3 s.f.

3 A cylinder and sphere both have the same diameter and the same volume. If the height of the cylinder is 36 cm, find their common radius.

4 A metal sphere 6 cm in diameter is melted and cast into balls of diameter $\frac{1}{2}$ cm. How many of the smaller balls will there be?

5 A sphere has a volume of 1 000 cm^3.
(a) Use tables to calculate its radius correct to 3 s.f.
(b) Hence calculate the surface area of the sphere.

Addition and subtraction of volumes

Many **composite solids** can be made by joining basic solids together.

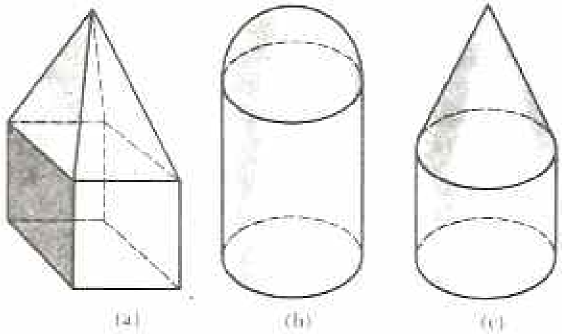


Fig. 10.13

In Fig. 10.13, the composite solids are made as follows:
(a) a cube and a square-based pyramid,
(b) a cylinder and a hemisphere,
(c) a cylinder and a cone.

Example 7

Fig. 10.14 represents a gas tank in the shape of a cylinder with a hemispherical top. The internal height and diameter are 1 m *and* 30 cm *respectively. Calculate the capacity of the tank to the nearest litre.*

Fig. 10.14

With the lettering of Fig. 10.14,
volume of tank

$= \text{volume of cylinder} + \text{volume of hemisphere}$

$= \pi r^2 h + \frac{2}{3}\pi r^3 = \pi r^2(h + \frac{2}{3}r)$

In Fig. 10.14

$r = 15$

$h = 100 - 15 = 85$

volume of tank

$= \pi 15^2(85 + \frac{2}{3} \times 15)\ \text{cm}^3$

$= 225\pi(85 + 10)\ \text{cm}^3$

$= 225\pi \times 95\ \text{cm}^3$

capacity in litres

$$= \frac{225 \times \pi \times 95}{1\,000}$$

$= 67{,}14 = 67$ to the nearest litre

working:

No.	Log
225	2,3522
π	0,4971
95	1,9777
	4,8270
1 000	3,0000
67,14	1,8270

Hollow shapes, such as boxes and pipes, have space inside.

The volume of material in a hollow object is found by subtracting the volume of the space inside from the volume of the shape as if it were solid.

Example 8

Fig. 10.15 represents an open rectangular box made of wood 1 cm *thick. If the external dimensions of the box are* 42 cm *long*, 32 cm *wide and* 15 cm *deep, calculate the volume of wood in the box.*

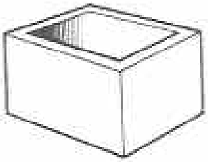


Fig. 10.15

The internal measurements of the box are 40 cm long, 30 cm, wide and 14 cm deep.

External volume $= 42 \times 32 \times 15\ \text{cm}^3$
$= 20\,160\ \text{cm}^3$

Internal volume $= 40 \times 30 \times 14\ \text{cm}^3$
$= 16\,800\ \text{cm}^3$

Volume of wood $= 20\,160\ \text{cm}^3 - 16\,800\ \text{cm}^3$
$= 3\,360\ \text{cm}^3$

Example 9

Find the mass of a cylindrical iron pipe 2,1 m *long and* 12 cm *in external diameter, if the metal is* 1 cm *thick and of density* 7,8 g/cm³. *Take* π *to be* $\frac{22}{7}$.

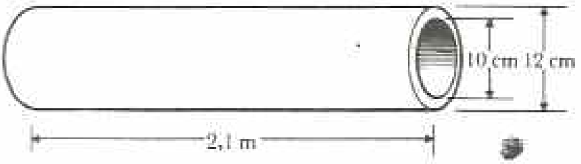


Fig. 10.16

Volume of outside cylinder $= \pi \times 6^2 \times 210\ \text{cm}^3$

Volume of inside cylinder $= \pi \times 5^2 \times 210\ \text{cm}^3$

Volume of iron

$= \pi \times 6^2 \times 210 - \pi \times 5^2 \times 210\ \text{cm}^3$

$= 210\pi(6^2 - 5^2)\ \text{cm}^3$

$= 210\pi(6 + 5)(6 - 5)\ \text{cm}^3$

$= 210\pi \times 11\ \text{cm}^3$

Mass of iron

$= 210\pi \times 11 \times 7{,}8\ \text{g}$

$$= \frac{210 \times 22 \times 11 \times 7{,}8}{7 \times 1\,000}\ \text{kg}$$

$= 56{,}6$ kg to 3 s.f.

(The working of the final line should be checked using tables.)

Example 10

Calculate, to one place of decimals, the volume in cm³ *of the metal in a hollow sphere* 10 cm *in external diameter, the metal being* 1 mm *thick.*

Outer radius = 5 cm

Inner radius = 4,9 cm

Volume of metal

$= \frac{4}{3}\pi \times 5^3 - \frac{4}{3}\pi \times 4{,}9^3\ \text{cm}^3$

$= \frac{4}{3}\pi(125 - 4{,}9^3)\ \text{cm}^3$ *

$= \frac{4}{3}\pi(125 - 117{,}7)\ \text{cm}^3$

working:

No.	Log
$4{,}9^3$	0,6902 × 3
117,7	2,0706

$= \dfrac{4 \times \pi \times 7,3}{3}\,\text{cm}^3$

$= \dfrac{29,2\pi}{3}\,\text{cm}^3$

$= 30,6\,\text{cm}^3$ to 1 d.p.

No.	Log
29,2	1,4654
π	0,4971
	1,9625
3	0,4771
30,58	1,4854

* This can be done on a calculator by treating it as $1,333 \times 3,142 \times (125 \div 4,9^3)$:

Key	*Display*
AC	0
4 · 9 × × = =	117.649
Min	117.649
1 2 5 ÷ MR =	7.351
× 3 · 1 4 2	3.142
× 1 · 3 3 3 =	30.78809

volume = $30,8\,\text{cm}^3$ (3)

(1) Notice how $4,9^3$ is done on the calculator.
(2) Many calculators have a $\boxed{\pi}$ button which automatically enters the value of π.
(3) The calculator value (30,8) is more accurate than the tables value (30,6). This is because the intermediate value in the tables calculation (7,3) was slightly inaccurate.

In Examples 7, 8, 9, 10, notice how factorisation simplifies the calculation.

Example 11

A right circular cylinder of height 12 cm *and radius* 4 cm *is filled with water. A heavy circular cone of height* 9 cm *and base-radius* 6 cm *is lowered, with vertex downwards and axis vertical, into the cylinder until the cone rests on the rim of the cylinder. Find (a) the volume of water which spills over from the cylinder, and (b) the height of the water in the cylinder after the cone has been removed.*

Fig. 10.17 shows the position of the cone and cylinder.

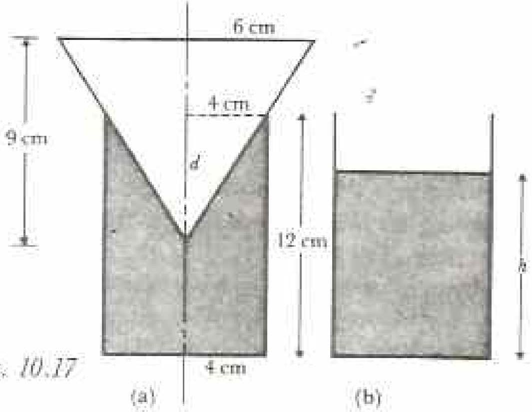


Fig. 10.17

(a) Let the cone be immersed to a depth d cm. By similar triangles,

$$\frac{d}{4} = \frac{9}{6}$$

$$d = \frac{9 \times 4}{6} = 6$$

Volume of water which spills over
= volume displaced by end of cone
$= \frac{1}{3}\pi \times 4^2 \times d\,\text{cm}^3 = \frac{1}{3} \times \frac{22}{7} \times 16 \times 6\,\text{cm}^3$
$= \frac{704}{7}\,\text{cm}^3 = 100\frac{4}{7}\,\text{cm}^3$

(b) Let the height of the water after the cone has been removed be h cm.
Volume of water in Fig. 10.17(a)
= volume of water in Fig. 10.17(b)

$$\pi \times 4^2 \times 12 - \tfrac{1}{3} \times \pi \times 4^2 \times 6 = \pi \times 4^2 \times h$$
$$\Leftrightarrow 12 - \tfrac{1}{3} \times 6 = h$$
$$\Leftrightarrow h = 12 - 2 = 10$$

The height of the water will be 10 cm.

Exercise 10c

Use the value $\frac{22}{7}$ for π or 0,4971 for log π, whichever is more convenient.

1 An open rectangular box has internal dimensions 2 m long, 20 cm wide and 22,5 cm deep. If the box is made of wood 2,5 cm thick, find the volume of the wood in cm^3.

2 An open concrete tank is internally 1 m wide, 2 m long and 1,5 m deep, the concrete being 10 cm thick. Calculate
(a) the capacity of the tank in litres, and
(b) the volume of concrete used in m^3.

3 Fig. 10.18 shows the plan of a foundation which is of uniform width 1 m.

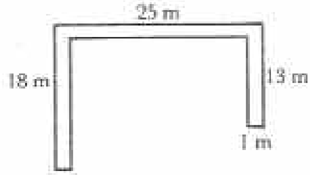


Fig. 10.18

If 1 m^3 of earth has a mass of $1\frac{1}{2}$ tonnes, what mass of earth will be removed when digging the foundation to a depth of $1\frac{1}{2}$ m?

4 A cast iron pipe has a cross-section as shown in Fig. 10.19, the iron being 1 cm thick. The mass of 1 cm^3 of cast iron is 7,2 g. Calculate the mass of a 2-metre length of the pipe.

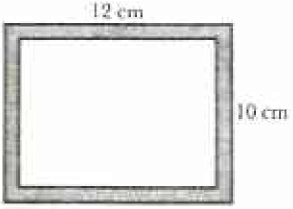


Fig. 10.19

5 A fish tank is in the shape of an open glass cuboid 30 cm deep with a base of 16 cm by 17 cm, these measurements being external. If the glass is 0,5 cm thick and its mass is 3 g/cm^3, find (a) the capacity of the tank in litres, and (b) the mass of the tank in kg.

6 Fig. 10.20 shows a storage tank made from a cylinder with a hemispherical end. Use the dimensions in Fig. 10.20 to calculate the volume of the tank in m^3.

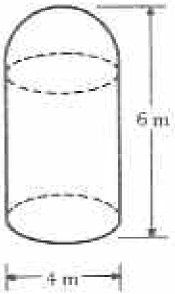


Fig. 10.20

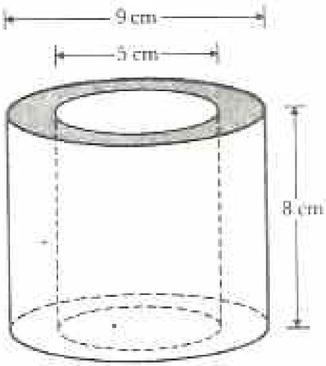


Fig. 10.21

7 Fig. 10.21 shows a cylindrical casting of height 8 cm and external and internal diameters 9 cm and 5 cm respectively. Calculate the volume of metal in the casting.

8 The outer radius of a cylindrical metal tube is R and t is the thickness of the metal.
(a) Show that the volume, V, of metal in a length, l units, of the tube is given by $V = \pi lt(2R - t)$.
(b) Hence calculate V when $R = 7{,}5$, $t = 1$ and $l = 20$.

9 A rectangular box, 18 cm by 12 cm by 6 cm, contains six tennis balls, each of diameter 6 cm. Calculate the percentage of the volume of the box occupied by the tennis balls.

10 Calculate the volume in cm^3 of the material in a cylindrical pipe 1,8 m long, the internal and external diameters being 16 cm and 18 cm respectively.

11 Calculate the approximate mass in kg of a 2-m length of cylindrical clay pipe of external and internal diameters 15 cm and 12 cm. The density of clay is 1,3 g/cm^3.

12 How far does the water level drop in a cylindrical tank of internal diameter 35 cm if 11 litres are drawn off?

13 A conical funnel 12 cm deep and 15 cm in diameter is full of liquid. It is emptied into a cylindrical tin 10 cm in diameter. Calculate the height of the liquid in the tin.

14 A spherical container 15 cm in diameter is half full of acid. The acid is poured into a tall cylindrical beaker of diameter 6 cm. How deep is the acid in the beaker?

15 A cylindrical tin of internal diameter 8 cm contains water to a depth of 6 cm. How far does the water level rise when a heavy ball of diameter 6 cm is placed in the tin?

16 A solid cube of side 8 cm is dropped into a cylindrical tank of radius 7 cm. Calculate the rise in the water level if the original depth of water was 9 cm.

17 An iron ball, 6 cm in diameter, is placed in a cylindrical tin 12 cm in diameter. Water is poured into the tin until its depth is 8 cm. If the ball is now removed, how far does the water level drop?

18 If, instead of the ball in question 17, an iron rod 8 cm long and 6 cm in diameter had

been placed flat in the tin, how far would the level have dropped when the rod was removed? (Assume that the other conditions remained the same.)

19 A wooden bowl is in the shape of a hollow hemisphere of external diameter 20 cm. The wood is 1 cm thick. Find the mass of the bowl if the wood has a density of 0,74 g/cm³.

20 A solid aluminium casting for a pulley consists of 3 discs, each $1\frac{1}{2}$ cm thick, of diameters 4 cm, 6 cm and 8 cm. A central hole 2 cm in diameter is drilled out as in Fig. 10.22. If the density of aluminium is 2,8 g/cm³, calculate the mass of the casting.

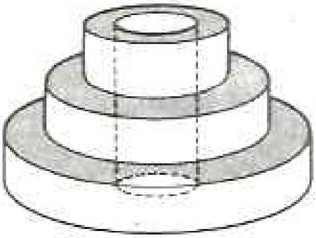


Fig. 10.22

Frustum of a cone or pyramid

If a cone or pyramid standing on a horizontal table is cut through parallel to the table, the top part is a smaller cone or pyramid. The other part is called a **frustum** (Fig. 10.23).

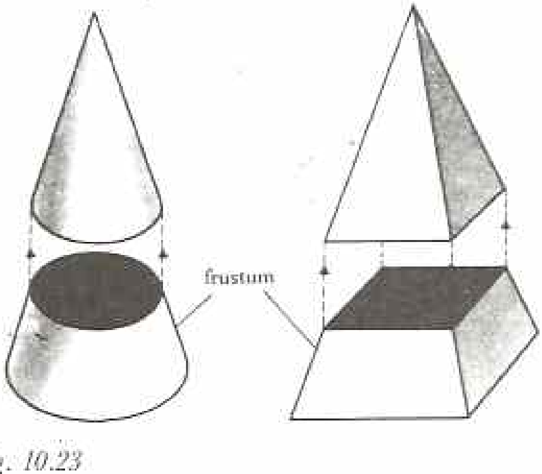


Fig. 10.23

To find the volume or surface area of a frustum, it is necessary to consider the frustum as a complete cone (or pyramid) with the smaller cone removed. Read Examples 12 and 13 carefully.

Example 12

Find the capacity in litres of a bucket 24 cm in diameter at the top, 16 cm in diameter at the bottom and 20 cm deep.

Complete the cone of which the bucket is a frustum, i.e. add a cone of height x cm and base diameter 16 cm as in Fig. 10.24.

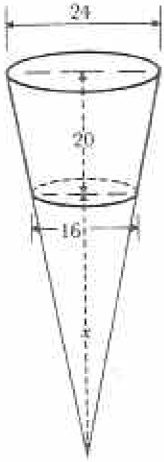


Fig. 10.24

By similar triangles,

$$\frac{x}{8} = \frac{x+20}{12}$$

$$12x = 8x + 160$$
$$4x = 160$$
$$x = 40$$

Volume of bucket

$$= \tfrac{1}{3}\pi 12^2 \times 60 - \tfrac{1}{3}\pi 8^2 \times 40 \text{ cm}^3$$
$$= \tfrac{1}{3}\pi(8\,640 - 2\,560)\text{ cm}^3$$
$$= \tfrac{1}{3}\pi \times 6\,080\text{ cm}^3 = 6\,366\text{ cm}^3$$

Capacity of bucket = 6,37 litres to 3 s.f.

Example 13

Find, in cm², the area of material required for a lampshade in the form of a frustum of a cone of which the top and bottom diameters are 20 cm and 30 cm respectively, and the vertical height is 12 cm.

Complete the cone of which the lampshade is a frustum as in Fig. 10.25.

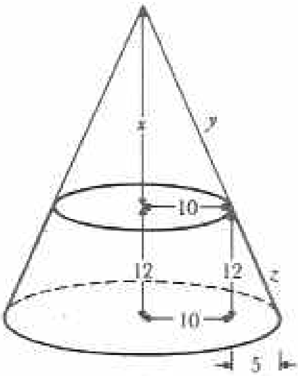


Fig. 10.25

With the lettering of Fig. 10.25, by similar triangles,

$$\frac{x}{10} = \frac{12}{5}$$

$x = 24$

By Pythagoras' theorem, $y = 26$ and $z = 13$

Surface area of frustum

$= \pi \times 15 \times 39 - \pi \times 10 \times 26\,\text{cm}^2$

$= 13\pi(45 - 20)\,\text{cm}^2$

$= 13\pi \times 25\,\text{cm}^2 = 1\,021\,\text{cm}^2$

Area of material required $= 1\,020\,\text{cm}^2$ to 3 s.f

Exercise 10d

1 A frustum of a cone has top and bottom diameters of 14 cm and 10 cm respectively and a depth of 6 cm. Find the volume of the frustum in terms of π.

2 A right pyramid on a base 10 m square is 15 m high.
(a) Find the volume of the pyramid.
(b) If the top 6 m of the pyramid are removed, what is the volume of the remaining frustum?

3 A frustum of a pyramid is 16 cm square at the bottom, 6 cm square at the top, and 12 cm high. Find the volume of the frustum.

4 A lampshade like that of Fig. 10.25 has a height of 12 cm and upper and lower diameters of 10 cm and 20 cm.
(a) What area of material is required to cover the curved surface of the frustum?
(b) What is the volume of the frustum?
(Give both answers in terms of π.)

5 The volume of a right circular cone is 5 litres. Calculate the volumes of the two parts into which the cone is divided by a plane parallel to the base, one-third of the way down from the vertex to the base. Give your answers to the nearest cm^3.

6 A storage container is in the form of a frustum of a right pyramid 4 m square at the top and 2,5 m square at the bottom. If the container is 3 m deep, what is its capacity in m^3?

7 The cone in Fig. 10.26 is exactly half full of water by volume. How deep is the water in the cone?

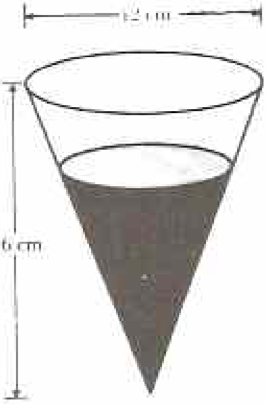


Fig. 10.26

8 A bucket is 20 cm in diameter at the open end, 12 cm in diameter at the bottom, and 16 cm deep. To what depth would the bucket fill a cylindrical tin 28 cm in diameter?

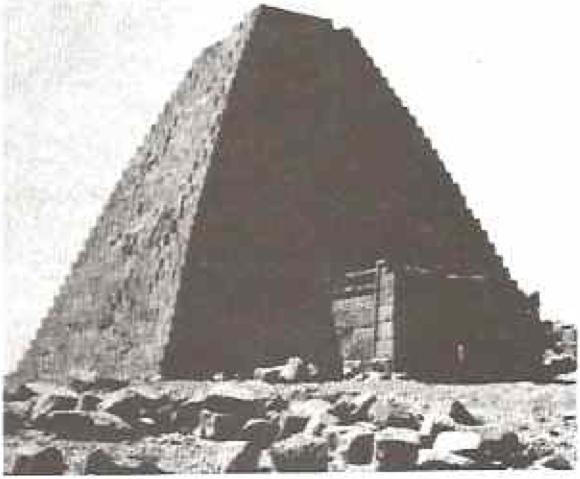


Fig. 10.27 This is one of the pyramids at Meroë in the Sudan. In what form is it?

Chapter 11

The cosine rule

The cosine rule

Given: Any $\triangle ABC$ (acute-angled and obtuse-angled triangles are given in Fig. 11.1).

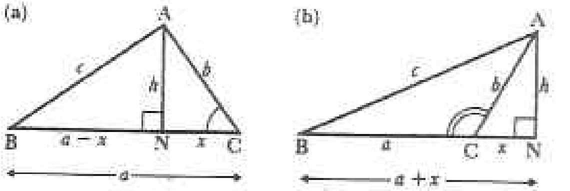


Fig. 11.1

To prove: $c^2 = a^2 + b^2 - 2ab\cos C$
Construction: Draw the perpendicular from A to BC (produced if necessary).

Proof:
In Fig. 11.1(a), with $\hat{C}$ acute,

$$c^2 = (a-x)^2 + h^2 \qquad \textit{(Pythagoras)}$$
$$= a^2 - 2ax + x^2 + h^2$$
$$= a^2 - 2ax + b^2 \quad (\textit{In } \triangle ACN,\ x^2 + h^2 = b^2)$$
$$= a^2 + b^2 - 2ab\cos C \quad \left(\textit{In } \triangle ACN,\ \frac{x}{b} = \cos C,\ x = b\cos C\right)$$

In Fig. 11.1(b), with $\hat{C}$ obtuse,

$$c^2 = (a+x)^2 + h^2 \qquad \textit{(Pythagoras)}$$
$$= a^2 + 2ax + x^2 + h^2$$
$$= a^2 + 2ax + b^2 \quad (\textit{In } \triangle ACN,\ x^2 + h^2 = b^2)$$
$$= a^2 + b^2 + 2a(-b\cos C)$$
$$\left(\textit{In } \triangle ACN,\ \frac{x}{b} = \cos A\hat{C}N = \cos(180^\circ - C) = -\cos C,\ x = -b\cos C\right)$$
$$= a^2 + b^2 - 2ab\cos C$$

In either case, $\boldsymbol{c^2 = a^2 + b^2 - 2ab\cos C}$
similarly, $\boldsymbol{b^2 = a^2 + c^2 - 2ac\cos B}$
and, $\boldsymbol{a^2 = b^2 + c^2 - 2bc\cos A}$

This formula is for solving triangles which are not right-angled in which **two sides and the included angle** are given.

Example 1
Find AB in Fig. 11.2.

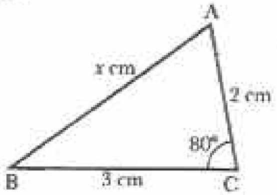


Fig. 11.2

By the cosine rule,

$$x^2 = 2^2 + 3^2 - 2 \times 2 \times 3 \times \cos 80^\circ$$
$$= 4 + 9 - 12 \times 0{,}173\,6$$
$$= 13 - 2{,}083\,2$$
$$= 10{,}916\,8 = 10{,}92 \text{ to 4 s.f.}$$
$$x = \sqrt{10{,}92}$$
$$x = 3{,}305 = 3{,}3 \text{ to 2 s.f.}$$
$$AB \simeq 3{,}3 \text{ cm}$$

Example 2
Find y in Fig. 11.3.

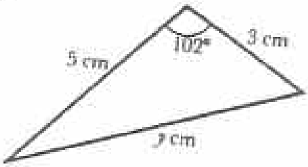


Fig. 11.3

$$y^2 = 5^2 + 3^2 - 2 \times 5 \times 3 \times \cos 102^\circ$$
$$= 25 + 9 - 30\ (-\cos 78^\circ)$$
$$= 34 + 30\cos 78^\circ$$
$$= 34 + 30 \times 0{,}207\,9$$
$$= 34 + 6{,}237$$
$$= 40{,}237$$
$$= 40{,}24 \text{ to 4 s.f.}$$
$$y = \sqrt{40{,}24} = 6{,}343$$
$$= 6{,}34 \text{ to 3 s.f.}$$

Notice that in Example 1, $x^2 < 2^2 + 3^2$ and in Example 2, $y^2 > 5^2 + 3^2$.

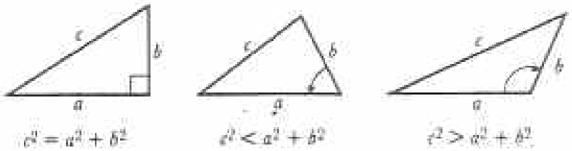


Fig. 11.4

Fig. 11.4 shows that

$c^2 = a^2 + b^2$ when c is opposite a *right* angle
$c^2 < a^2 + b^2$ when c is opposite an *acute* angle
$c^2 > a^2 + b^2$ when c is opposite an *obtuse* angle

Example 3

In $\triangle$ABC, $c = 8{,}44$ m, $a = 7{,}92$ m *and* B = 151,3°. *Calculate* AC.

First, make a sketch of the data (Fig. 11.5).

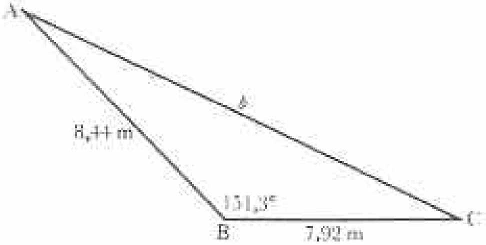


Fig. 11.5

By the cosine rule,

$b^2 = 8{,}44^2 + 7{,}92^2 - 2 \times 8{,}44 \times 7{,}92 \times \cos 151{,}3°$ *

$= 71{,}23 + 62{,}73 + 2 \times 8{,}44 \times 7{,}92 \times \cos 28{,}7°$

$= 133{,}96 + 16{,}88 \times 7{,}92 \times \cos 28{,}7°$

$= 133{,}96 + 117{,}3$

$= 251{,}26$

$= 251{,}3$ to 4 s.f.

$b = \sqrt{251{,}3}$

$= 15{,}85 = 15{,}9$ to 3 s.f.

AC = 15,9 m

working:

No	Log
16,88	1,2274
7,92	0,8987
cos 28,7°	$\bar{1}$,9431
117,3	2,0692

Notice the use of tables of squares, logarithms and square roots in Example 3.

* A scientific calculator may be used to evaluate this expression. Work *from right to left*, starting by evaluating the trigonometrical function:

Key	*Display*
1 5 1 · 3 cos	−0.8771461
× 7 · 9 2	7.92
× 8 · 4 4	8.44
× 2 +/− = M+	117.26532
7 · 9 2 × = M+	62.7264
8 · 4 4 × = M+	71.2336
MR √	15.850089

$b = 15{,}9$

Note the use of the calculator's memory to store the three products. The M+ key automatically adds the products.

Exercise 11a

Calculate the length of the side opposite the given angle in each of the $\triangle$s ABC. Give answers correct to 3 s.f.

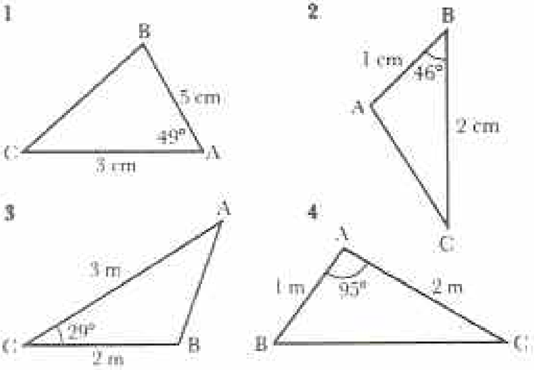


Fig. 11.6

5 A = 120°, b = 7 cm, c = 12 cm
6 B = 54°, c = 4 cm, a = 5 cm
7 C = 13°, a = 10 m, b = 15 m
8 B = 135,5°, c = 8 cm, a = 5 cm
9 A = 125,4°, b = 2,4 cm, c = 5 cm
10 C = 47,8°, a = 13,1 m, b = 24,2 m

Solving triangles using the sine and cosine rules

Example 4

In $\triangle ABC$, $a = 6{,}7$ cm, $c = 2{,}3$ cm *and* $B = 46{,}6°$. *Find* b, A *and* C.

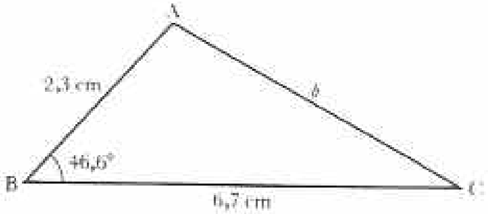


Fig. 11.7

In Fig. 11.7, using the cosine rule,

$$b^2 = 2{,}3^2 + 6{,}7^2 - 2 \times 2{,}3 \times 6{,}7 \cos 46{,}6°$$
$$= 5{,}29 + 44{,}89 - 4{,}6 \times 6{,}7 \times \cos 46{,}6°$$
$$= 50{,}18 - 21{,}17$$
$$= 29{,}01$$
$$b = \sqrt{29{,}01} = 5{,}386 \text{ cm}$$

working:

No	Log
4,6	0,6628
6,7	0,8261
cos 46,6°	$\bar{1}$,8370
21,17	1,3259

Using the sine rule,

$$\frac{\sin C}{2{,}3} = \frac{\sin 46{,}6°}{5{,}386}$$

$$\sin C = \frac{2{,}3 \sin 46{,}6°}{5{,}386}$$

$$C = 18{,}08°$$

working:

No	Log
2,3	0,3617
sin 46,6°	$\bar{1}$,8613
	0,2230
5,386	0,7313
sin 18,08°	$\bar{1}$,4917

$$= 180° - (46{,}6 + 18{,}08)°$$
$$= 180° - 64{,}68°$$
$$= 115{,}32°$$

To 1 d.p.
$b = 5{,}4$ cm, $C = 18{,}1°$, $A = 115{,}3°$

Notice the following points in Example 4:

1 The cosine rule is used to find b.
2 The sine rule is used to find one of the remaining angles.
3 The *smaller* of the two unknown angles is found first, since this must be an acute angle. If the sine rule had been used to find A, log sin A would have been $\bar{1}$,9561, which gives $A = 64{,}7°$ *or* $115{,}3°$. To avoid this ambiguity, always find the smaller angle first, since th angle must be acute.
4 Rounding off numbers only takes place a the *final* answer stage. Do *not* use rounde numbers at intermediate stages of the calcu lation.

Exercise 11b

Calculate the unknown side and angles in eac of the $\triangle$s ABC given. Give final answers correc to 3 s.f.

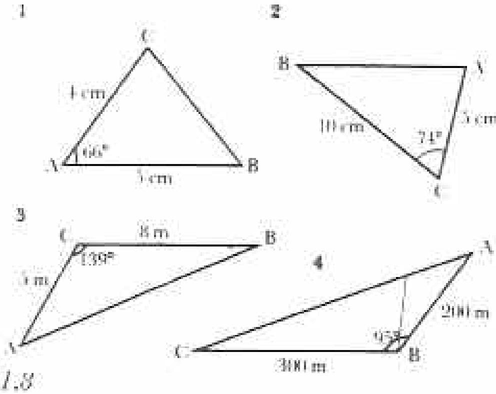


Fig. 11.8

5 $A = 58{,}1°$, $b = 10$ m, $c = 8{,}5$ m
6 $B = 126°$, $c = 5{,}6$ cm, $a = 5$ cm
7 $C = 25{,}7°$, $b = 3{,}5$ cm, $a = 6$ cm
8 $A = 140{,}15°$, $b = 45$ m, $c = 24$ m
9 $C = 143{,}3°$, $b = 3{,}8$ cm, $a = 2{,}3$ cm
10 $B = 34{,}5°$, $c = 2{,}8$ cm, $a = 5{,}1$ cm

Using the cosine rule to calculate angles

The formula $a^2 = b^2 + c^2 - 2bc \cos A$ can be rearranged with cos A as the subject of the formula:

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

$$\text{similarly, } \cos B = \frac{c^2 + a^2 - b^2}{2ca}$$

$$\text{and } \cos C = \frac{a^2 + b^2 - c^2}{2ab}$$

This formula can be used to calculate the angles of a triangle in which **all three sides** are given.

Example 5

Calculate the angles of a triangle with sides of length 4 m, 5 m and 7 m.

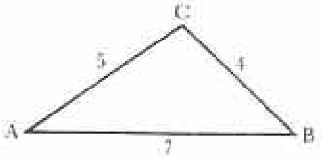


Fig. 11.9

With the lettering of Fig. 11.9,

$$\cos A = \frac{5^2 + 7^2 - 4^2}{2 \times 5 \times 7} = \frac{58}{70} = 0{,}828\,6$$

$$\Rightarrow A = 34{,}04°$$

$$\cos B = \frac{4^2 + 7^2 - 5^2}{2 \times 4 \times 7} = \frac{40}{56} = \frac{5}{7} = 0{,}714\,3$$

$$\Rightarrow B = 44{,}42°$$

$$\cos C = \frac{4^2 + 5^2 - 7^2}{2 \times 4 \times 5} = \frac{-8}{40} = -\frac{1}{5}$$

$$= -0{,}200\,0$$

$$\Rightarrow C = 180° - 78{,}46° = 101{,}54°$$

Check:

$$A + B + C = 34{,}04° + 44{,}42° + 101{,}54° = 180°$$

In questions such as Example 5, it is advisable to use the cosine formula to find every angle, then to check the results by addition.

Example 6

Calculate the angles of triangles which have sides (a) 400 m, 500 m, 700 m, (b) 2,8 cm, 4,2 cm, 5,6 cm.

In both cases, the calculation is simplified by considering similar triangles (i.e. equiangular) which have less complex numbers as sides.

(a) 400:500:700 = 4:5:7

Solve the triangle with sides of 4, 5 and 7 units, as in Example 5.

(b) 2,8:4,2:5,6 = 28:42:56
= 2:3:4

Solve the triangle with sides of 2, 3 and 4 units. This is left as an exercise.

Exercise 11c

Calculate the angles of the $\triangle$s ABC whose sides are given in cm. Give the final answers to the nearest 0,1°.

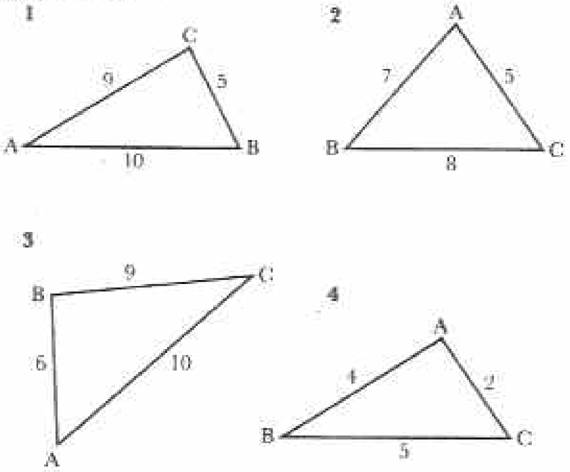


Fig. 11.10

5 $a = 5,\ b = 7,\ c = 9$
6 $a = 45,\ b = 33,\ c = 21$
7 $a = 5{,}2,\ b = 6{,}5,\ c = 7{,}8$
8 $a = 7{,}2,\ b = 6{,}3,\ c = 9{,}9$
9 $a = 14{,}4,\ b = 11{,}2,\ c = 7{,}6$
10 $a = 2{,}7,\ b = 3{,}7,\ c = 3{,}1$

Example 7

The sides of a parallelogram are 7 cm and 10 cm and one of its diagonals is 15 cm. Use the cosine formula to find the length of the other diagonal.

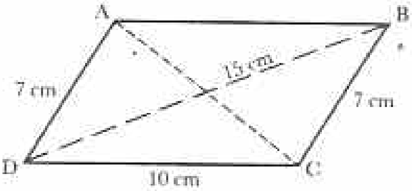


Fig. 11.11

In $\triangle$BDC,

$$\cos C = \frac{10^2 + 7^2 - 15^2}{2 \times 10 \times 7}$$

$$= \frac{149 - 225}{140}$$

$$= -\frac{76}{140}$$

In parallelogram ABCD,

$A\hat{D}C + D\hat{C}B = 180°$ (*adjacent angles of* $\|^{gm}$

$\Rightarrow \quad A\hat{D}C = 180° - D\hat{C}B$

$\Rightarrow \quad \cos A\hat{D}C = \cos(180° - D\hat{C}B)$

$\Rightarrow \quad \cos A\hat{D}C = -\cos D\hat{C}B = \frac{76}{140}$

In $\triangle$ ADC,

$$AC^2 = 7^2 + 10^2 - 2 \times 7 \times 10 \times \cos A\hat{D}C$$
$$= 49 + 100 - 140 \times \tfrac{76}{140}$$
$$= 149 - 76$$
$$= 73$$
$$AC = \sqrt{73}\text{ cm}$$
$$= 8{,}544\text{ cm}$$

The other diagonal is approximately 8,54 cm long.

Notice in Example 7 that it was not necessary to find the value of C in degrees.

Exercise 11d

1 In $\triangle$ABC in Fig. 11.12, M is the mid-point of BC.
(a) Calculate cos B in $\triangle$ABC.
(b) Hence calculate AM.
(*Note:* Do not find B in degrees.)

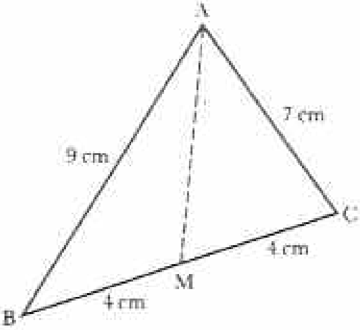


Fig. 11.12

2 With the data of Fig. 11.13, calculate
(a) $A\hat{B}C$, (b) AC, all lengths being given in cm.

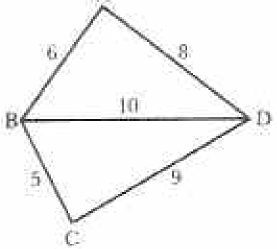


Fig. 11.13

3 In Fig. 11.14, find x and θ. All lengths a
given in cm.

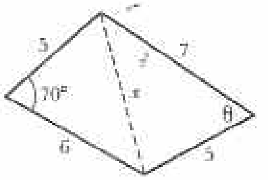


Fig. 11.14

4 In Fig. 11.15, PQRS is a cyclic quadrilatera
PQ = 7 cm, QR = 8 cm and PR = 7,5 cr

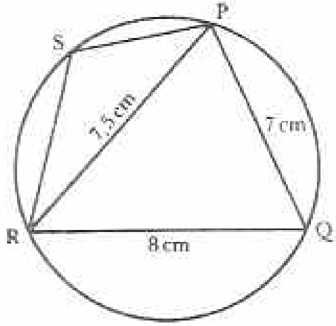


Fig. 11.15

(a) Calculate $P\hat{S}R$.
(b) Hence if SR = SP, calculate $S\hat{P}R$.
Give your answers correct to the near
tenth of a degree.

5 In a $\triangle$ABC, AB = 8 cm, BC = 4 c
CA = 5 cm and BC is produced to P so th
CP = 4 cm. Use the cosine rule to fi
$\cos A\hat{C}B$. Hence find AP.

6 Given Fig. 11.16, find $\cos \hat{P}$ in $\triangle$PQ
Hence find QX.

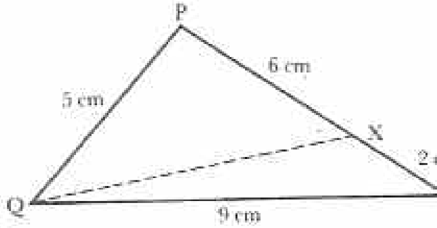


Fig. 11.16

7 The sides of a parallelogram are 3 cm a
5 cm and include an angle of 144°. Find
lengths of the diagonals of the parallelogra

8 Calculate y in Fig. 11.17. (First find co
but do not work out θ.)

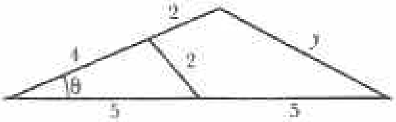


Fig. 11.17

9 In $\triangle PQR$ $p:q:r = \sqrt{3}:1:1$. Calculate the ratio $\hat{P}:\hat{Q}:\hat{R}$ in its simplest form.

10 In Fig. 11.18, ABCD is a trapezium with sides of lengths as shown.

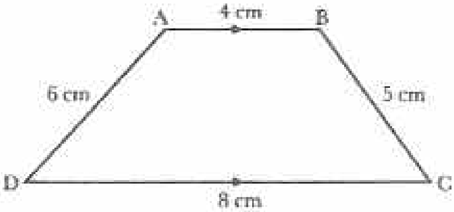


Fig. 11.18

Copy Fig. 11.18 and draw a line BX parallel to AD to cut CD in X. Hence calculate (a) $\hat{C}$, (b) BD.

Bearings and distances

Example 8

Three towns, A, B and C, are situated so that AB = 60 km and AC = 100 km. The bearing of B from A is 060° and the bearing of C from A is 290°. Calculate (a) the distance BC, (b) the bearing of B from C.

Fig. 11.19 is a sketch of the positions of A, B and C.

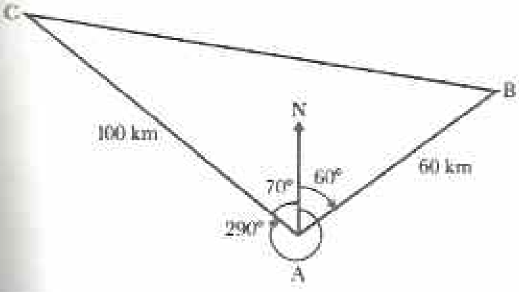


Fig. 11.19

In $\triangle ABC$,

$\hat{CAB} = 60° + (360 - 290)°$
$= 60° + 70°$
$= 130°$

By the cosine rule,

$BC^2 = 100^2 + 60^2 - 2 \times 100 \times 60 \times \cos 130°$
$= 10\,000 + 3\,600 + 12\,000 \cos 50°$
$= 13\,600 + 12\,000 \times 0{,}6428$
$= 13\,600 + 7\,713{,}6$
$= 21\,313{,}6 = 21\,310$ to 4 s.f.

$BC = \sqrt{21\,310}$ km
$= 146$ km

By the sine rule,

$$\frac{\sin C}{60} = \frac{\sin 130°}{146}$$

$$\sin C = \frac{60 \times \sin 130°}{146}$$

$$= \frac{60 \times \sin 50°}{146}$$

$C = 18{,}35°$

working:

No	Log
60	1,7782
sin 50°	$\bar{1}$,8843
	1,6625
146	2,1644
sin 18,35°	$\bar{1}$,4981

Fig. 11.20 shows the angles at C:

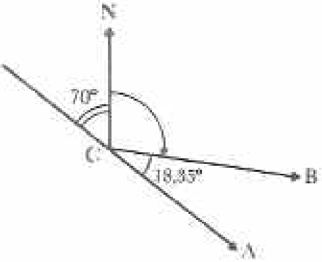


Fig. 11.20

$\hat{NCB}$ gives the bearing of B from C.

$\hat{NCB} = \hat{NCA} - 18{,}35°$
$= 110° - 18{,}35° = 91{,}65°$

To 3 s.f.,

(a) BC = 146 km,

(b) The bearing of B from C is 091,7°.

Exercise 11e

Give all distances correct to 3 s.f. Give all angles and bearings correct to 0,1°.

1 From a point on the edge of the sea, one ship is 5 km away on a bearing S 50° E and another is 2 km away on a bearing S 60° W. Find the distance between the ships.

2 A girl walks 50 m on a bearing 025° and then 200 m due east. How far is she from her starting point?

3 Two goal posts are 8 m apart. A footballer is 34 m from one post and 38 m from the other. Within what angle must he kick the ball if he is to score a goal?

4 City A is 300 km due east of city B. City C is 200 km on a bearing of 123° from city B. How far is it from C to A?

5 A triangular field has two sides 50 m and 60 m long, and the angle between these sides is 96°. How long is the third side?

6 Two boats A and B left a port C at the same time along different routes. B travelled on a bearing of 150° (S 30° E) and A travelled on the north side of B. When A had travelled 8 km and B had travelled 10 km, the distance between the two boats was found to be 12 km. Calculate the bearing of A's route from C.

7 A man prospecting for oil in the desert leaves his base camp and drives 42 km on a bearing of 032°. He then drives 28 km on a bearing of 154°. How far is he then from his base camp and what is his bearing from it?

8 Two ships leave port at the same time. One travels at 5 km/h on a bearing of 046°. The other travels at 9 km/h on a bearing of 127°. How far apart are the ships after 2 hours?

9 A boat sails 4 km on a bearing of 038° and then 5 km on a bearing of 067°.

(a) How far is the boat from its starting point?

(b) Calculate the bearing of the boat from its starting point.

10 A photographer is 350 m away from a lion and wants to get closer before she takes a photograph. There is a water-hole in the direct line between the lion and herself, so she moves at an angle of 8° to this line to a better position 200 m further on. Calculate her distance from the lion.

11 Two planks, of lengths 1 m and 1,2 m, lea against each other as shown in Fig. 11.21 If the angle between the planks is 36 (a) how far apart are the bottom edges o the planks, and (b) what angle does th longer one make with the floor?

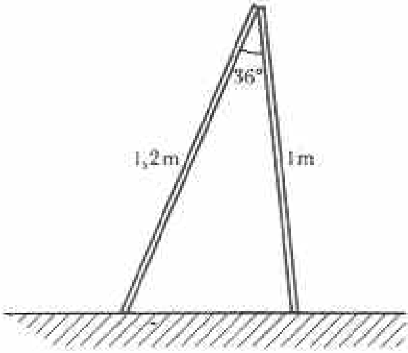


Fig. 11.21

12 An aeroplane flies due north from Harar Airport for 500 km. It then flies on a bearin of 060° for a further distance of 300 kn before overflying a road junction. Calculat (a) the distance of the aeroplane fron Harare Airport when it was directly abov the road junction, (b) the bearing of th aeroplane from Harare Airport at this instan

13 A ship leaves port and travels 21 km on a bearing of 032° and then 45 km on a bearin of 287°.

(a) Calculate its distance from the port.

(b) Calculate the bearing of the port fron the ship.

14 An aircraft flies along a triangular course The first leg is 200 km on a bearing of 115 and the second leg is 150 km on a bearin of 230°. How long is the third leg of th course and on what bearing must the aircraf fly?

15 Villages A, B, C, D are such that B is 4 kn due east of A, C is 3 km due south of B and D is 4 km S 50° W from C. Calculate th distance and bearing of A from D.

Chapter 12

Consumer arithmetic (2)

Taxation

A **tax** is a financial contribution which people are legally obliged to make to the State. The Government, acting on behalf of the State, decides the various rates of tax. It uses taxes to pay for services such as education, health, public transport and national defence.

Sales tax

A proportion of money paid for goods is given to the Government. The part which is given to the Government is called **sales tax**. At the time of going to press, sales tax was 12,5% (or one-eighth) of the selling price of consumable goods (such as clothing and petrol) and 20% (or one-fifth) of the selling price of durable goods (such as cars and furniture).

Example 1
A gold and diamond ring is advertised as

$1 495, excluding sales tax

What will be the total cost of the ring?

A ring is a durable item; sales tax is 20%.
True cost = 120% of $1 495

$$= \frac{6}{5} \times \$1\,495$$

$$= \$1\,794$$

Example 2
A coat costs $229,50 *including sales tax. How much tax does the Government receive?*

A coat is a consumable item; sales tax on it is 12,5%.
112,5% of basic cost = $229,50

$$1\% \text{ of basic cost} = \$\frac{229{,}50}{112{,}5}$$

$$12{,}5\% \text{ of basic cost} = \$\frac{229{,}50 \times 12{,}5}{112{,}5}$$

$$= \$\frac{229{,}50}{9}$$

$$= \$25{,}50$$

The Government receives $25,50 sales tax.

Income tax

Most people pay a part of their earnings to the Government. The part they pay is called **income tax**. In Zimbabwe the way of calculating income tax is roughly as follows.

1 Tax is paid on **taxable income**. The rates of taxable income for married and single people are given in Table 12.1.

Table 12.1

taxable income per annum	rate of tax (%)	
	married	single
first $1 000	10	14
second $1 000	12	16
third $1 000	14	18
fourth $1 000	16	20
fifth $1 000	18	22
and so on until ...	⋮	⋮
fifteenth $1 000	38	42,5
sixteenth $1 000	40	45
seventeenth $1 000	42,5	45
over $17 000	45	45

2 Abatements are given to help to meet the cost of personal and family commitments.

Here are some typical abatements:

single person	\$1 800
married person	\$3 000
children	\$600 per child
dependants	\$400 max

(Note: other abatements are available, but only the above will be used in this book.)

A taxpayer's total abatements are called the **abatable amount**. The maximum abatable amounts are \$6 600 and \$3 800 for married and single people respectively.

3 Annual tax is calculated as follows:

Chargeable income tax
= tax on gross income
− tax on abatable amount

Total tax payable
= chargeable income tax
+ 15% surcharge

(Note: in practice, the rate of the surcharge increases as total tax payable increases. However, for illustrative purposes, 15% will be used in this book.)

Example 3

A married teacher has an income of \$11 560. *He has 4 children and a dependent relative. How much income tax should he pay?*

First:
gross income = \$11 560
Second:
abatements:

married allowance	\$3 000
4 children	\$2 400
dependant	\$400
abatable amount =	\$5 800

Third:
tax on gross income of \$11 560
= \$100 + \$120 + \$140 + \$160
+ \$180 + \$200 + \$220 + \$240
+ \$260 + \$280 + \$300 + 32% of \$560
= \$2 379,20
tax on abatable amount of \$5 800
= \$100 + \$120 + \$140 + \$160
+ \$180 + 20% of \$800
= \$860

chargeable income tax
= \$2 379,20 − \$860
= \$1 519,20

Finally:

chargeable income tax	= \$1 519,20
surcharge of 15%	= \$227,88
Total tax payable	= \$1 747,08

Exercise 12a

Unless otherwise stated, use the rates given above in this exercise.

1 Find out how much customers pay for each item in the advertisement in Fig. 12.1.

Fig. 12.1

2 Each price in the advertisement in Fig. 12.2 includes a sales tax of 12,5%. How much money does the Government receive in each case?

Fig. 12.2

3 The 'Young Fashion' department of a large store advertises stone-washed denim tops at \$63,99 with skirts to match at \$72,99. A young woman buys one of each.
How much sales tax does she pay if the advertised prices are inclusive of sales tax at 12,5%?

4 A 3-band radio can be bought by paying '12 easy instalments of \$22,50'. How much sales tax is included in the total price? (*Note*: a radio is a durable item.)

5 Calculate the gross income tax on each of the following salaries when earned by (i) married, (ii) single people.
(a) \$6 000 (b) \$9 000 (c) \$14 000
(d) \$7 500 (e) \$8 600 (f) \$10 780

6 Calculate the abatable amounts for the following:
(a) a single woman with a dependent parent;
(b) a married man with 3 children;
(c) a married woman with 1 child and a dependent mother;
(d) a married man with 5 children and dependent parents.

7 A single taxpayer has an annual income of \$7 000 and total abatements of \$2 000. Calculate the amount of tax paid.

8 A graduate trainee accountant has a salary of \$13 320. If the trainee is single and has no dependants, find: (a) her abatable amount, (b) the tax on the gross income, (c) the tax on the abatable amount, (d) the chargeable income tax (i.e. the tax before surcharge), (e) the surcharge, (f) the total tax that she pays annually, (g) her remaining income after tax is paid.

9 An experienced trained accountant has a salary of \$27 800. He is married, with 4 children and a dependent mother.
(a) Calculate his abatable amount.
(b) Calculate the total tax that he pays.
(c) He is paid monthly and tax is taken from his pay in equal instalments. Find his monthly 'take home pay'.

10 Companies pay tax of 45% on all taxable income. In addition, their chargeable income tax is subject to a surcharge of 17,5%. Find the tax paid by a company which has a taxable income of \$88 724.

Household bills

Discount

Sometimes a trader will reduce the price of an item in order to sell it. The reduction in price is called a **discount**. Discounts are often given to customers who can pay in cash.

Example 4

A refrigerator costs \$899. A 10% discount is given for cash. What is the discount price?

Either:

$$\text{Discount} = 10\% \text{ of } \$899 = \frac{10}{100} \times \$899$$
$$= \$89{,}90$$
$$\text{Cash price} = \$899 - \$89{,}90$$
$$= \$809{,}10$$

Or:

$$\text{Cash price} = (100\% - 10\%) \text{ of } \$899$$
$$= 90\% \text{ of } \$899$$
$$= \frac{90}{100} \times \$899$$
$$= 0{,}9 \times \$899$$
$$= \$809{,}10$$

Example 5

A stationary shop sells notebooks at \$1,40 each or 5 for \$6. It gives a 35% discount to schools on orders of 100 notebooks or more. Calculate the unit costs of the notebooks at the three rates.

A 'unit cost' is the cost of a single item. Unit costs are used for comparing prices.

Non-discount rate:
Cost of 1 notebook = \$1,40

5 notebooks for \$6 rate:
Cost of 1 notebook = \$6 ÷ 5 = \$1,20

35% discount rate:

$$\text{Cost of 1 notebook} = 65\% \text{ of } \$1{,}40$$
$$= \frac{65}{100} \times \$1{,}40$$
$$= 91 \text{ cents}$$

Example 5 shows the savings that can be made when buying 'in bulk' (i.e. buying a lot of a particular item).

Hire purchase

Expensive items such as cars and television sets are too costly for most people to buy outright. People find it easier to buy such items by paying **instalments**. An instalment is a part payment. Paying by instalments is called **hire purchase**. The buyer hires the use of the item before paying for it completely. It costs money to hire an item. This is why hire purchase costs more than paying in cash.

Example 6

A motorbike costs \$2 676 *cash. Alternatively it can be bought for* 25% *deposit and* 24 *monthly instalments of* \$115. *How much more expensive is it to buy the motorbike by hire purchase?*

Hire purchase price

= deposit + instalments

= 25% of \$2 676 + 24 × \$115

= \$669 + \$2 760

= \$3 429

Price difference

= \$3 429 − \$2 676

= \$753

In Example 6:

1 The difference in price, \$753, represents the cost of hiring the motorbike during the time it was being paid for.

2 The deposit is given as a percentage of the cost price. This is commonly done.

Exercise 12b

1 Find the price if a discount of
(a) 10% is given on a cost price of \$59;
(b) $12\frac{1}{2}$% is given on a cost price of \$420;
(c) 15 cents in the dollar is given on a book marked at \$28;
(d) 20% is given on a \$45 shirt.

2 The selling price of an armchair is \$220. The shop gives a 25% discount for cash. What is the cash price?

3 During a sale a shop takes 35 cents in the \$ off all marked prices.
(a) What percentage discount does this represent?
(b) What is the sale price of a handbag marked at \$39?

4 A Co-operative sells eggs at the prices shown on the notice in Fig. 12.3.

Fig. 12.3

How much is saved by buying four dozen eggs in sixes instead of separately?

5 A 500 g packet of rice costs 98c. A 50 kg sack of the rice costs \$65. Calculate the two unit costs of the rice if 1 kg is taken as the unit. What is the saving per kg when buying in bulk?

6 The hire purchase price of a hi-fi set is \$678 spread over 24 equal fortnightly payments. How much is each payment?

7 The hire purchase price of a vehicle is a deposit of \$4 500 down and 36 monthly payments of \$950. What is the total paid for the hire purchase?

8 A carpet either costs \$699 cash or 30 weekly payments of \$27,50.
(a) Find the cost of the carpet when paying by hire purchase,
(b) Find the cost of the hire of the carpet for the 30 weeks.

9 A television set costs either
\$675 cash, or
52 weekly payments of \$16,50, or
104 weekly payments of \$9,50.
Find the total cost of each of the hire purchase methods of payment.
(Why should the two differ in cost?)

10 A new computer costs \$2 335. An 8% discount is given for cash. The hire purchase price of the computer is 15% down and 24 monthly payments of \$109,65.
Calculate the difference between paying cash and paying by hire purchase.

Electricity and water charges

People who receive public services such as water and electricity also receive bills to pay for them. The following give some typical charges. Remember, however, that rates and methods of payment for services vary from time to time and from place to place.

Electricity and water charges are sometimes shown on the same bill. Fig. 12.4 shows a typical bill for these charges issued by the City of Harare and Zimbabwe Electricity Supply Authority (ZESA).

City of Harare/ZESA

ELECTRICITY AND/OR WATER CHARGES

82,01	27 SEP	82,01	12 DEC	0,00

```
ZESA CHARGES
E A FIXED MONTHLY CHARGE                      7,30
E A      23705     22712      993            66,03
CITY OF HARARE CHARGES
W 1      6128      6071        57            28,15
10PC SUPERCHARGE ON ELEC ACC                  7,33
```

49 19 123	12 NOV	31 DEC 91	108,81

A RECEIPT ON THIS FORM IS NOT VALID UNLESS MACHINE PRINTED

Fig. 12.4

Although the electricity and water charges are shown on the same bill they are calculated differently as follows:

Electricity charges:

Fixed monthly charge:	\$7,30
Consumption rate:	6,65c per kWh
Surcharge:	10% of bill

The 10% surcharge is a handling charge and goes to the City Council. The rest goes to ZESA.

kWh is short for **kilowatt-hour**. 1 kWh is the amount of electricity that is used when 1 000 watts of electricity are consumed in 1 hour. The kWh is the basic 'unit' of electrical consumption.

Water charges:

Fig. 12.5 gives some typical water charges.

WATER TARIFF

Please note that the Water Tariffs have been amended in respect of water consumed after the normal reading date in October 1990. Details of the amended Tariffs are given below.

MUNICIPAL AREA

Type	Scale	Rate per month per cubic metre
Single Family Dwelling Units	W1	first 13 m³ at 36,5 c/m³ next 26 m³ at 48,5 c/m³ next 31 m³ at 60,0 c/m³ over 70 m³ at 71,5 c/m³
Commercial and Industrial	W2	48,5 c/m³

OUTSIDE MUNICIPAL AREA

All consumers	W3	57,5 c/m³

N.B. As water accounts are calculated in metric units all consumptions shown are cubic metres unless otherwise stated.

Fig. 12.5

The unit of water is the cubic metre (m^3). 1 m^3 is equivalent to 1 000 litres, or 1 kℓ, of water.

Example 7

Check the bill shown in Fig. 12.4.

Electricity charges:

No. of units used = 23 705 − 22 712
= 993

Charge for units = 993 × 6,65c
= \$66,034 5
= \$66,03 to nearest c

Fixed monthly charge = \$7,30

Total of electricity = \$73,33

10% surcharge = 10% of \$73,33
= \$7,33 to nearest c

Water charges:

No. of units used = 6 128 − 6 071
= 57

57 = 13 + 26 + 18

Charge for units
= (13 × 13,5c) + (26 × 48,5c) + (18 × 60c)
= \$4,745 + \$12,61 + \$10,80
= \$28,155 = \$28,15*

Grand total:
= \$73,33 + \$7,33 + \$28,15
= \$108,81

*Note that the 0,5c is rounded down.

Example 8

A household uses 716 *units of electricity. What will be the cost of the electricity?*

$$\begin{aligned}\text{Cost of units used} &= 716 \times 6{,}65\text{c}\\ &= 4\,761{,}4\text{c}\\ &= \$47{,}61\\ \text{Fixed monthly charge} &= \$7{,}30\\ \hline \text{Subtotal} &= \$54{,}91\\ 10\%\ \text{surcharge} &= 10\%\ \text{of}\ \$54{,}91\\ &= \$5{,}49\\ \text{Total cost} &= \$54{,}91 + \$5{,}49\\ &= \$60{,}40\end{aligned}$$

Example 9

Find the bill for 48 m³ *of water when it is calculated by (a) Scale W1, (b) Scale W2, (c) Scale W3.*

(a) Scale W1

$48 = 13 + 26 + 9$

Water bill

$= (13 \times 36{,}5\text{c}) + (26 \times 48{,}5\text{c}) + (9 \times 60\text{c})$

$= 474{,}5\text{c} + 1261\text{c} + 540\text{c} = 2\,275{,}5\text{c}$*

$= \$22{,}75$

(b) Scale W2

Water bill $= 48 \times 48{,}5 = 2\,328\text{c}$

$= \$23{,}28$

(c) Scale W3

Water bill $= 48 \times 57{,}5 = 2\,760\text{c}$

$= \$27{,}60$

*In practice the 0,5c is rounded down.

Examples 7, 8, 9 show that a calculator, if available, is a very useful aid for checking household bills.

Household rates (owners charges)

Property owners must pay for services which are supplied to their houses and families. Such services include road construction and maintenance, refuse removal and supply of public amenities such as civic centres, sports stadiums, street lighting and parks. Bills for this are called **owners charges**. They are also commonly called **household rates**, or simply just **rates**.

Fig. 12.6 shows a typical 6-monthly rates bill for a property in Harare.

Note that the bill contains three components.

1 Land

The plot of land (known as a *stand*) is evaluated and charged at a rate of 1,166 cents per $.

2 Improvements

Any improvements to the stand are evaluated and charged at a rate of 0,818 cents per $. Note that such improvements normally include the main house and outbuildings.

3 Refuse removal

A fixed charge of $25,25 for the regular disposal of rubbish.

City of Harare

P.O. Box 1680 HARARE

Business Hours

Monday, Tuesday 8.00–4.45
Wednesday 8.00–3.45
Thursday, Friday 8.00–4.45

IN RESPECT OF STAND 1903 MOUNT PLEASANT

Please quote Rates Code in communication

Rates Code 044968

Reflects payments to 19 MAR

ITEM	DATE FROM	DATE TO	VALUATION $	RATE cents per $	
LAND	1 JAN 91	30 JUN 91	6610	1,166	77,07
IMPROVEMENTS	1 JAN 91	30 JUN 91	15490	0,818	126,71
REFUSE REMOVAL	1 JAN 91	30 JUN 91			25,25
					229,03

Due date	Unpaid balance after	will attract interest
30 APR 91	31 MAY 91	10,75 PER CENT FROM DUE DATE

Fig. 12.6

Example 10
Check that the rates bill shown in Fig. 12.6 is correct.

Land charge:	6,610 × 1,166c =	$77,07
Improvements:	15 490 × 0,818c =	$126,71
Refuse removal (fixed charge)	=	$25,25
Total owner's charges	=	$229,03

Exercise 12c
Unless told otherwise, use the water charges, electricity charges and household rates as given above when doing this exercise.

1 The electricity bill for a household which uses 580 units is $50,46. This includes the fixed monthly charge and the 10% surcharge. Check that this bill is correct.

2 What would be the bill for a household which used half as many units as that in question 1?

3 The reading on an electricity meter changes from 18 091 to 19 172 in one month. Calculate the bill.

4 The water bill for a house charged on the W1 Scale was $24,55 when 51 m^3 of water was used. Check that this bill is correct.

5 What would be the bill for the house in question 4 if, next month, it used only two-thirds as much water?

6 At the beginning and end of a month the readings on a water meter are 2 391 m^3 and 2 448 m^3 respectively. Calculate the bill if charges are made on the W1 Scale.

7 70 m^3 of water are used in a month. Calculate the bill when charged on (a) Scale W1, (b) Scale W2, (c) Scale W3.
(Give possible reasons why the rates are different for different types of users.)

8 A property is valued as follows:

Land	$3 000
Improvements	$10 000

Calculate the 6-monthly owners charges for the property. Include charges for refuse removal.

9 The rates for the property in question 8 are increased to the following:

Land:	1,225c per $
Improvements:	0,922c per $
Refuse removal:	$27,85 (fixed)

Assuming that the valuations remain as before, calculate the new 6-monthly owner charges for the property.

10 A house in Harare has a land valuation of $6 500, an improvements valuation of $18 25 and is subject to Scale W1 water charges. Readings of the water and electricity meter show that in the month of June the household used 64 m^3 of water and 905 units of electricity.
Find the total amount that the bills will come to (i.e. the water and electricity bill for June and the owners charges for January to 30 June).

Insurance

Insurance is a financial arrangement for the payment of a sum of money to compensate for an unfortunate loss or injury. Those who wish to insure themselves make payments, or **premiums**, to insurance companies. For example, the Summit Insurance Company charges an annual premium of $66 to householders for insuring the contents of their houses against loss or theft of amounts up to $5 000. If a fire were to destroy or damage the house's contents, the householder would receive up to $5 000 in compensation.

It is possible to insure yourself for nearly any kind of loss. Specialist statisticians, called actuaries, are employed by insurance companies to calculate the premiums for the various risks involved. Table 12.2 shows the premiums to be paid by travellers to ensure that they obtain compensation in the form of benefits which are shown overleaf.

Table 12.2

Period of cover	Premium per person ($)
1–6 days	16,20
7–11 days	20,70
12–17 days	23,40
18–23 days	26,10
24–31 days	28,80
6 weeks	37,80
8 weeks	46,80
10 weeks	56,70

Benefits per person	
Benefit A Death by accident	$5 000
Benefit B Permanent total disablement following accident	$20 000
Benefit C Medical and other expenses. All necessary costs up to	$1 000 000
Benefit D Loss of luggage	up to $800
Benefit E Personal liability	$500 000
Benefit F Hijack of aircraft	£500
Benefit G Loss of money	up to £500

Example 11

Referring to Table 12.2, what would be the premium if a traveller wishes to insure himself for 2 weeks?

2 weeks = 14 days
This falls within the 12–17 days' range.
Premium = $23.40

Note: There is no reduction in premium if the time period is less than the upper limit of the range.

Example 12

The traveller in Example 11 has luggage valued at $450. If this, together with $210 worth of travellers cheques are stolen, what compensation could he expect?

Benefit D:
Compensation for lost luggage = $450

Benefit G
Compensation for stolen money = $210

Total compensation = $660

Note: Compensation is paid only on the value of the losses, not on the maximum amount insurable.

Mortgages

A **mortgage** is a loan given for the purpose of buying land and any property (buildings) which is on the land. Companies which give mortgages are called **building societies**. A building society owns the property until the loan on it is repaid.

Interest is charged on the loan. Typical rates of interest are 13.25% on residential property (e.g. houses) and 14.75% on commercial properties (e.g. offices, shops).

In practice, building societies seldom give a mortgage for the full value of a property. The maximum mortgage obtainable is usually 95% of the price of the property.

A mortgage is usually repaid over a long period of time: anything up to 25 years or more is quite common. During this time, both the interest and the loan are repaid.

Example 13

A house buyer borrows $50 000 *from the B&H Building Society to buy a* $70 000 *property. The mortgage is given over* 25 years *and the monthly repayments are* $576.39.

(*a*) *How much was the house buyer's deposit on the property?*

(*b*) *What is the total amount that is repaid to the B&H Building Society over the* 25 years?

(*c*) *How much does the house buyer have to pay for the property altogether?*

(a) Deposit = $70 000 − $50 000
= $20 000

(b) Total repayments per year
= $576.39 × 12
Total repayment over 25 years
= $576.39 × 12 × 25
= $172 917

(c) Total cost of property
= $172 917 + $20 000
= $192 917

Note that the monthly payment includes both the interest and some repayment of the capital borrowed.

Exercise 12d

1 Use the data in Table 12.2 to calculate the premiums for a husband and wife who insure themselves against travel losses during a 6-week trip to Europe.

2 The wife in question 1 falls ill during her holiday. She incurs hospital bills of $2 215 and is flown home to Zimbabwe at a further cost of $1 207. Which of the benefits will she be compensated from and how much should she receive?

3 A household insurance company charges an annual premium of $6 per $100 sum assured. What will be the total premium for a householder who insures his house contents for $14 300?

4 A car owner insures her car for 'third party, accident and theft', meaning that any losses *caused* by her or *incurred* by her will be paid under the insurance.
(a) Premiums are charged at the rate of $65 per $2 000 or part thereof, based on the value of the car. What will be the full premium if the car is worth $26 800?
(b) A 'no claims bonus' of 60% is given because the driver has made no claims against the insurance company within the past 5 years. What will be her actual premium?

5 Table 12.3 shows the premiums to be paid for insurance of home possessions in rural, urban and city areas for various amounts of cover.
(a) What would be the premium to insure home possessions for $4 000 in an urban area?
(b) What would be the premium to insure home possessions for $5 000 in the city of Bulawayo?
(c) Additional cover is given at a premium rate of $4 per $100 or part thereof in all areas. What would be the premium for a rural farmer who insures his possessions for $7 850?

Table 12.3

sum covered	Premiums: rural area	urban area	city area
$2 500	$34	$49	$54
$3 000	$37	$57	$68
$4 000	$43	$67	$82
$5 000	$48	$79	$99

6 A building society charges a monthly repayment of $11,54 for each $1 000 borrowed.
(a) Calculate the monthly repayment on a mortgage of $15 000.
(b) What is the yearly repayment?
(c) What will be the total repayment over 25 years?

7 If the Social Union Building Society's interest rate is 13,25% per annum, how much interest is paid on each $1 000 borrowed (a) per year, (b) per month?

8 The City of Harare sells a stand (plot of land) to Betterbuilder Brick Company for $56 750. The company obtains an 80% mortgage and makes monthly repayments of $758,13 over 10 years.
(a) Calculate the amount of the loan.
(b) Calculate the total amount repaid to the building society over the 10 years.

9 An 'interest only' mortgage is one on which only interest is paid. Payment of the capital takes place at the end of the period of the loan. A company obtained an 'interest only' mortgage of $90 000 at a rate of 14,75% per annum. After 5 years it repaid the capital. How much interest did the company pay?

10 A chicken farm is sold for $230 000. By selling her previous farm, the buyer can afford a deposit of $185 000.
(a) What size of mortgage does she need?
(b) The G&C Building Society offers her a 20-year mortgage at a monthly payment rate of $13,83 per $1 000 borrowed. Calculate the total monthly repayments.
(c) Calculate the total amount that is repaid over the 20 years.

Budgeting

In daily life it is important to keep accurate accounts of income and expenditure. This is true whether at a household, business, Co-operative or State level. Accurate accounts enable individuals, groups and countries to plan their spending. The planning of expenditure is called **budgeting**.

Budgeting is greatly assisted by keeping **cash accounts** of transactions. Fig. 12.7 is a simple household cash account which shows the income and expenditure for a household during a month.

Income		Expenditure	
Wages	1008,60	Food	226,11
Farm	247,17	Mortgage	354,25
		Insurance	26,44
		Elec/Water	94,67
		School costs	60,00
		Car running	156,80
		Entertainment	64,30
		Clothing	86,54
		Farm costs	58,00
		Savings	128,66
Total	1255,77	Total	1255,77

Fig. 12.7

Although Fig. 12.7 is a simplified account, it contains the main elements of all cash accounts:

1 A statement with details of income.
2 A statement with details of expenditure.
3 Total income and expenditure (which should balance).

All businesses and Co-operatives are required by law to keep cash accounts which show income and expenditure. Such accounts must show the cash which comes in and the cash which is paid out. Fig. 12.8 is a cash account showing one month's trading figures for a transport Co-operative.

Notice the following:

1 The cash account is in two parts:
   cash received (income) on the left.
   cash spent (expenditure) on the right.
2 The total on the left-hand side gives the total cash received.
3 The total on the right-hand side must balance the total on the left-hand side. To do this:
   (a) the actual expenditure is added,
   (b) the resulting sub-total is subtracted from the total cash received to give *cash carried forward*.
4 This cash is brought forward as cash received into the account for the next period.

Example 14

Use Fig. 12.7 to calculate the profit that the household made from farming.

Ace Public Transport Co-op Ltd.
Accounts for June 1991

		CASH RECEIVED			CASH SPENT
Date	Details	$	Date	Details	$
01/06/91	Cash brought fwd	8176,38	07/06/91	Wages	3412,80
08/06/91	Weekly takings	5038,63	08/06/91	Fuel bill	926,55
15/06/91	Weekly takings	4499,47	12/06/91	Bus repairs	417,50
20/06/91	Bus hire	550,00	14/06/91	Wages	3412,80
22/06/91	Weekly takings	3883,61	21/06/91	Wages	3412,80
29/06/91	Weekly takings	6171,62	22/06/91	Fuel bill	1514,83
30/06/91	School bus hire	430,00	25/06/91	6 new tyres	749,40
			28/06/91	Wages	3412,80
			30/06/91	Garage rent	1750,00
			30/06/91	Petty cash	76,16
			30/06/91	Cash carried fwd	9664,07
		28749,71			28749,71

Fig. 12.8

Profit from farming
= income from farming − farm costs
= \$247,17 − \$58 = \$189,17

Example 14 shows how keeping accounts can highlight the efficiency of a small project such as selling farm produce.

Example 15

Use Fig. 12.8 to find the operating profit or loss of Ace Public transport Co-op during the month of June.

Operating profit
= cash carried fwd (to July)
− cash brought fwd (from May)
= \$9 664,07 − \$8 176,38
= \$1 487,69

Note:

1 'fwd' is a common abbreviation of 'forward'.

2 If cash carried fwd < cash brought fwd, then a trading *loss* would have occurred.

Exercise 12e

1 Refer to Fig. 12.7.
(a) Which of the items under 'expenditure' is likely to be the same each month?
(b) Food, electricity and water are essential items. How much was spent on these essentials?
(c) Entertainment and clothing are non-essential items. How much was spent on non-essentials?

2 Refer to Fig. 12.8.
(a) The weekly takings come from bus fares. Find the total obtained from fares in June 1991.
(b) How much was obtained from hiring out buses?
(c) Find the total expenditure on fixed costs i.e. wages and garage rent.
(d) How much was spent on keeping the buses running during the month?
(e) How much did each new tyre cost?

3 Refer to Fig. 12.8. Next month the income was \$15 093,14 from bus fares and \$430,00 from school bus hire.
Expenditure amounted to 4-week's wages and garage rent as in June, \$2 674,37 for fuel and \$618,20 for bus maintenance. There was also \$112,74 in petty cash.
(a) Prepare a cash account for July 1991.
(b) What is the balance in hand at the end of July?
(c) Calculate the operating profit (or loss) for the month of July.

Fig. 12.9 is a summary account of the Central Government's budget for the financial years 1986/87 to 1988/89.

Government Budget (Z \$million) Expenditure and financing 1986/87 to 1988/89

	1986/87	1987/88	1988/89
REVENUE AND GRANTS	**3 056,5**	**3 784,9**	**4 356,4**
Income Tax	1 351,9	1 614,2	1 865,2
Sales Taxes	1 237,0	1 386,7	1 782,9
Other Revenues	467,6	784,0	708,3
EXPENDITURE AND NET LENDING	**4 053,3**	**4 680,9**	**5 467,4**
Current and Capital	3 822,1	4 295,7	4 960,2
Net Lending	231,2	385,2	507,2
DEFICIT	**−996,8**	**−896,0**	**−1 111,0**
FINANCING	**+996,8**	**+896,0**	**+1 111,0**
Foreign Loans	210,8	147,8	129,1
Domestic Loans	786,1	748,2	981,9

Fig. 12.9 (Source: Reserve Bank of Zimbabwe)

In Fig. 12.9:

The first row gives the total *revenue* or income. This is broken down to show revenue from income tax, sales tax and other sources.

The second row gives the main areas of Government *expenditure* on current and capital developments and on loans.

The third row gives the *budget deficit*, i.e. the excess of expenditure over revenue.

The fourth row shows how the deficit was *financed* or funded by foreign and domestic loans.

Refer to Fig. 12.9 and the above information when answering questions 4–10.

4 (a) How much revenue was obtained from sales tax in 1987/88?
(b) In which year did net lending amount to $507 200 000?
(c) Which item always produces the greatest source of revenue?
(d) Which year had the smallest budget deficit?

5 For the 3-year period, are the following statements true?
(a) The Government borrowed less and less foreign money to finance the budget deficit.
(b) Government expenditure rose steadily.
(c) The deficit each year was always between $900 million and $1 100 million.
(d) Each year income tax produced about twice as much revenue as sales tax.

6 In each of the main rows, the sub-totals usually add to give the main total. However, in the case of financing in 1986/87, there is a small discrepancy. Suggest how this arises.

7 What was the total revenue raised from income tax in the 3-year period?

8 What was the total spent on current and capital expenditure in the 3-year period?

9 (a) For 1986/87, express foreign loans as a percentage of the total amount needed to finance the deficit.
(b) Do likewise for 1988/89.
(c) What conclusion do you draw from your answers to parts (a) and (b)?

10 (a) Express the amount under revenue and grants in 1988/89 as a percentage of the 1986/87 figure.
(b) Do likewise for expenditure and net lending.
(c) What conclusion can you draw from your answers to parts (a) and (b)?

Chapter 13

Matrices (2)

Matrix arithmetic

Addition and subtraction

Example 1

If $\mathbf{A} = \begin{pmatrix} 3 & -2 \\ 1 & 0 \\ 0 & 4 \end{pmatrix}$ *and* $\mathbf{B} = \begin{pmatrix} -5 & 2 \\ 2 & 3 \\ -1 & 0 \end{pmatrix}$

find (a) $\mathbf{A} + \mathbf{B}$, *(b)* $\mathbf{A} - \mathbf{B}$, *(c)* $3\mathbf{A}$.

(a) $$\mathbf{A} + \mathbf{B} = \begin{pmatrix} 3 & -2 \\ 1 & 0 \\ 0 & 4 \end{pmatrix} + \begin{pmatrix} -5 & 2 \\ 2 & 3 \\ -1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 3 + (-5) & (-2) + 2 \\ 1 + 2 & 0 + 3 \\ 1 + (-1) & 4 + 0 \end{pmatrix}$$

$$= \begin{pmatrix} -2 & 0 \\ 3 & 3 \\ -1 & 4 \end{pmatrix}$$

(b) $$\mathbf{A} - \mathbf{B} = \begin{pmatrix} 3 & -2 \\ 1 & 0 \\ 0 & 4 \end{pmatrix} - \begin{pmatrix} -5 & 2 \\ 2 & 3 \\ -1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 3 - (-5) & (-2) - 2 \\ 1 - 2 & 0 - 3 \\ 0 - (-1) & 4 - 0 \end{pmatrix}$$

$$= \begin{pmatrix} 8 & -4 \\ -1 & -3 \\ 1 & 4 \end{pmatrix}$$

(c) $$3\mathbf{A} = 3\begin{pmatrix} 3 & -2 \\ 1 & 0 \\ 0 & 4 \end{pmatrix}$$

$$= \begin{pmatrix} 3 \times 3 & 3 \times (-2) \\ 3 \times 1 & 3 \times 0 \\ 3 \times 0 & 3 \times 4 \end{pmatrix}$$

$$= \begin{pmatrix} 9 & -6 \\ 3 & 0 \\ 0 & 12 \end{pmatrix}$$

Revision notes:

1 Matrices **A** and **B** are of the same *order*. They are both 3×2 matrices, i.e. 3 rows by 2 columns.

2 Matrices can be added or subtracted only if they are of the same order, in which case corresponding elements are added or subtracted as in parts (a) and (b) of Example 1.

3 When a matrix is multiplied by a *scalar*, as in part (c), the scalar multiplies every element of the matrix.

Multiplication

Example 2

If $\mathbf{M} = \begin{pmatrix} -2 & 4 \\ 3 & 5 \end{pmatrix}$ *and* $\mathbf{N} = \begin{pmatrix} 6 & 0 \\ -1 & 2 \end{pmatrix}$ *find the matrix products (a)* **MN**, *(b)* **NM**, *(c)* $\mathbf{M}^2$.

(a) $$\mathbf{MN} = \begin{pmatrix} -2 & 4 \\ 3 & 5 \end{pmatrix}\begin{pmatrix} 6 & 0 \\ -1 & 2 \end{pmatrix}$$

$$=\begin{pmatrix}(-2)\times 6+4\times(-1) & (-2)\times 0+4\times 2\\ 3\times 6+5\times(-1) & 3\times 0+5\times 2\end{pmatrix}$$

$$=\begin{pmatrix}-12+(-4) & 0+8\\ 18+(-5) & 0+10\end{pmatrix}$$

$$=\begin{pmatrix}-16 & 8\\ 13 & 10\end{pmatrix}$$

(b) $\mathbf{NM}=\begin{pmatrix}6 & 0\\ -1 & 2\end{pmatrix}\begin{pmatrix}-2 & 4\\ 3 & 5\end{pmatrix}$

$$=\begin{pmatrix}-12+0 & 24+0\\ 2+6 & -4+10\end{pmatrix}$$

$$=\begin{pmatrix}-12 & 24\\ 8 & 6\end{pmatrix}$$

(c) $\mathbf{M}^2=\begin{pmatrix}-2 & 4\\ 3 & 5\end{pmatrix}\begin{pmatrix}-2 & 4\\ 3 & 5\end{pmatrix}$

$$=\begin{pmatrix}4+12 & -8+20\\ -6+15 & 12+25\end{pmatrix}$$

$$=\begin{pmatrix}16 & 12\\ 9 & 37\end{pmatrix}$$

Revision notes:

4 In Example 2(a) **M** *pre-multiplies* **N**. In Example 2(b) **M** *post-multiplies* **N**.

5 Matrices can be multiplied only if there are as many columns in the first matrix as there are rows in the second matrix.

6 A $p\times q$ matrix will multiply a $q\times r$ matrix to give a $p\times r$ product:

$$(p\times\boxed{q\ \times\ q}\times r)\rightarrow(p\times r)$$

7 In general $\mathbf{AB}\neq\mathbf{BA}$ where **A** and **B** are matrices.

Exercises 13a (revision)

1 Simplify the following, giving each result as a single matrix.

(a) $\begin{pmatrix}3 & 9\\ 2 & 1\end{pmatrix}+\begin{pmatrix}4 & 0\\ -6 & 2\end{pmatrix}$

(b) $\begin{pmatrix}2\\ 6\end{pmatrix}+\begin{pmatrix}10\\ 3\end{pmatrix}$

(c) $\begin{pmatrix}5 & -1\\ 1 & -1\\ -2 & 7\end{pmatrix}-\begin{pmatrix}2 & 6\\ 1 & -7\\ 3 & 0\end{pmatrix}$

(d) $\begin{pmatrix}-4 & 2\\ 1 & -2\end{pmatrix}-\begin{pmatrix}6 & 9\\ -2 & -3\end{pmatrix}$

(e) $\begin{pmatrix}3\\ 9\end{pmatrix}-\begin{pmatrix}0\\ 5\end{pmatrix}+\begin{pmatrix}2\\ 6\end{pmatrix}$

(f) $\begin{pmatrix}3 & -2\\ 1 & 5\end{pmatrix}+\begin{pmatrix}-1 & 3\\ -2 & -1\end{pmatrix}-\begin{pmatrix}2 & 4\\ 7 & -2\end{pmatrix}$

2 If $\mathbf{A}=\begin{pmatrix}-1 & 5\\ 2 & 3\end{pmatrix}$ and $\mathbf{B}=\begin{pmatrix}6 & 0\\ 4 & -8\end{pmatrix}$ find

(a) $3\mathbf{A}$ (b) $2\mathbf{B}$ (c) $-2\mathbf{A}$
(d) $\frac{1}{2}\mathbf{B}$ (e) $2\mathbf{A}+\mathbf{B}$ (f) $\mathbf{A}-3\mathbf{B}$

3 Express each of the following products as a single element matrix.

(a) $(4;5)\begin{pmatrix}6\\ 1\end{pmatrix}$

(b) $(2;3;-1)\begin{pmatrix}\frac{1}{2}\\ -2\frac{1}{2}\\ 1\frac{1}{2}\end{pmatrix}$

(c) $(3;-1)\begin{pmatrix}x\\ y\end{pmatrix}$

(d) $(4;5;6)\begin{pmatrix}x\\ y\\ z\end{pmatrix}$

4 Find x and y if

$$4\begin{pmatrix}5 & x\\ -1 & 1\end{pmatrix}-3\begin{pmatrix}6 & x\\ 2 & 2\end{pmatrix}=2\begin{pmatrix}1 & 4\\ y & -1\end{pmatrix}$$

5 Express each of the following products as a single matrix.

(a) $\begin{pmatrix}5 & 1\\ 8 & 2\end{pmatrix}\begin{pmatrix}-2\\ 3\end{pmatrix}$

(b) $(2;5)\begin{pmatrix}6 & 2 & 3\\ 1 & 0 & -1\end{pmatrix}$

$$\begin{pmatrix} 4 & 9 \\ -2 & 5 \end{pmatrix}\begin{pmatrix} 1 & 3 \\ 0 & -1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 3 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 4 & 9 \\ -2 & 5 \end{pmatrix}$$

$$\begin{pmatrix} 6 & 3 \\ -1 & 2 \end{pmatrix}^2$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 2 & 3 & 0 \\ 5 & -1 & 6 \end{pmatrix}$$

Algebra of 2 × 2 matrices

Zero matrix (null matrix)

If any 2×2 matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is pre-multiplied or post-multiplied by the matrix $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, it is reduced to the matrix $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$.

Check that:

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

and

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ is called the **zero matrix** or **null matrix**.

Identity matrix

If any 2×2 matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is pre-multiplied or post-multiplied by the matrix $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ it remains unchanged.

Check that:

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

and

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ is called the **identity matrix**. It is given the symbol **I**.

Inverse of a 2 × 2 matrix

If $\begin{pmatrix} p & q \\ r & s \end{pmatrix}$ is a matrix such that

$$\begin{pmatrix} p & q \\ r & s \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \mathbf{I} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} p & q \\ r & s \end{pmatrix}$$

then $\begin{pmatrix} p & q \\ r & s \end{pmatrix}$ is the **inverse** of $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

If $\begin{pmatrix} p & q \\ r & s \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

then $pa + qc = 1$ (1)
$pb + qd = 0$ (2)
$ra + sc = 0$ (3)
$rb + sd = 1$ (4)

(1) × d: $pad + qcd = d$ (5)
(2) × c: $pbc + qcd = 0$ (6)
(5) − (6): $pad - pbc = d$
$\Leftrightarrow$ $p(ad - bc) = d$

$$p = \frac{d}{ad - bc}$$

Similarly $q = \dfrac{-b}{ad - bc}$

and $r = \dfrac{-c}{ad - bc}$

and $s = \dfrac{a}{ad - bc}$

Hence the inverse of any 2 × 2 matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is

$$\frac{1}{ad - bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

$ad - bc$ is called the **determinant** of the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

To find the inverse of a 2 × 2 matrix:

1 interchange the top left-hand and bottom right-hand elements;

2 multiply the other two elements by −1;

3 divide the resulting matrix by the determinant of the original matrix.

Example 3

Find the inverse of

(*a*) $\begin{pmatrix} 3 & -2 \\ 4 & 1 \end{pmatrix}$; (*b*) $\begin{pmatrix} 2 & 5 \\ 3 & 8 \end{pmatrix}$, (*c*) $\begin{pmatrix} 6 & 3 \\ 2 & 1 \end{pmatrix}$

(a) The determinant of $\begin{pmatrix} 3 & -2 \\ 4 & 1 \end{pmatrix}$ is

$$3 \times 1 - 4 \times (-2) = 3 + 8 = 11$$

Its inverse is $\frac{1}{11}\begin{pmatrix} 1 & 2 \\ -4 & 3 \end{pmatrix}$

Check: $\frac{1}{11}\begin{pmatrix} 1 & 2 \\ -4 & 3 \end{pmatrix}\begin{pmatrix} 3 & -2 \\ 4 & 1 \end{pmatrix}$

$$= \frac{1}{11}\begin{pmatrix} 11 & 0 \\ 0 & 11 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

(b) The determinant of $\begin{pmatrix} 2 & 5 \\ 3 & 8 \end{pmatrix}$ is

$$2 \times 8 - 3 \times 5 = 16 - 15 = 1$$

Its inverse is $\frac{1}{1}\begin{pmatrix} 8 & -5 \\ -3 & 2 \end{pmatrix} = \begin{pmatrix} 8 & -5 \\ -3 & 2 \end{pmatrix}$

Check: $\begin{pmatrix} 8 & -5 \\ -3 & 2 \end{pmatrix}\begin{pmatrix} 2 & 5 \\ 3 & 8 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

(c) The determinant of $\begin{pmatrix} 6 & 3 \\ 2 & 1 \end{pmatrix}$ is

$$6 \times 1 - 2 \times 3 = 0$$

The inverse of the given matrix would contain the fraction $\frac{1}{0}$. Since division by 0 is impossible it follows that $\begin{pmatrix} 6 & 3 \\ 2 & 1 \end{pmatrix}$ has *no* inverse.

Notice in Example 3 that a matrix whose determinant is zero has no inverse. Such matrices are called **singular** matrices.

Exercise 13b

Find the inverses of the following matrices where possible. Use multiplication to check each result.

1 $\begin{pmatrix} 6 & 3 \\ 1 & 2 \end{pmatrix}$ **2** $\begin{pmatrix} 5 & 3 \\ 2 & 3 \end{pmatrix}$

3 $\begin{pmatrix} 4 & 3 \\ 2 & 1 \end{pmatrix}$ **4** $\begin{pmatrix} 4\frac{1}{2} & 3 \\ 5 & 3 \end{pmatrix}$

5 $\begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix}$ **6** $\begin{pmatrix} 3 & 2 \\ 2 & 1 \end{pmatrix}$

7 $\begin{pmatrix} 6 & 14 \\ 3 & 7 \end{pmatrix}$ **8** $\begin{pmatrix} -5 & 3 \\ -1 & 4 \end{pmatrix}$

9 $\begin{pmatrix} 2 & -9 \\ 8 & -6 \end{pmatrix}$ **10** $\begin{pmatrix} -5 & -4 \\ -2 & 0 \end{pmatrix}$

11 $\frac{1}{2}\begin{pmatrix} 4 & 1 \\ 16 & 3 \end{pmatrix}$ **12** $\frac{1}{6}\begin{pmatrix} 0 & -3 \\ 2 & 0 \end{pmatrix}$

Matrices as operators

Simultaneous linear equations

If $\begin{pmatrix} 9 & 4 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 17 \\ 4 \end{pmatrix}$

then $\begin{pmatrix} 9x + 4y \\ 2x + y \end{pmatrix} = \begin{pmatrix} 17 \\ 4 \end{pmatrix}$

or $9x + 4y = 17$ (1)

and $2x + y = 4$ (2)

Hence the simultaneous equations (1) and (2) can be written as a single matrix equation:

$$\begin{pmatrix} 9 & 4 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 17 \\ 4 \end{pmatrix} \qquad (3)$$

In (3) the matrix $\begin{pmatrix} 9 & 4 \\ 2 & 1 \end{pmatrix}$ multiplies $\begin{pmatrix} x \\ y \end{pmatrix}$. Multiplication is an arithmetical operation; we say that the matrix acts as an **operator**.

The inverse of $\begin{pmatrix} 9 & 4 \\ 2 & 1 \end{pmatrix}$ is $\begin{pmatrix} 1 & -4 \\ -2 & 9 \end{pmatrix}$. Pre-multiply both sides of (3) by this inverse matrix:

$$\begin{pmatrix} 1 & -4 \\ -2 & 9 \end{pmatrix}\begin{pmatrix} 9 & 4 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & -4 \\ -2 & 9 \end{pmatrix}\begin{pmatrix} 17 \\ 4 \end{pmatrix}$$

$$\Leftrightarrow \quad \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$\Leftrightarrow \quad \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

or $x = 1$
and $y = 2$

Example 4

Solve the equations $3x - 4y = 1$ and $7x + y = 23$.

$3x - 4y = 1$
$7x + y = 23$

$$\Leftrightarrow \quad \begin{pmatrix} 3 & -4 \\ 7 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 23 \end{pmatrix} \qquad (1)$$

The determinant of $\begin{pmatrix} 3 & -4 \\ 7 & 1 \end{pmatrix}$ is:

$3 \times 1 - 7 \times (-4) = 31$

Its inverse is $\frac{1}{31}\begin{pmatrix} 1 & 4 \\ -7 & 3 \end{pmatrix}$.

Pre-multiply both sides of (1) by this inverse:

$$\frac{1}{31}\begin{pmatrix} 1 & 4 \\ -7 & 3 \end{pmatrix}\begin{pmatrix} 3 & -4 \\ 7 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{31}\begin{pmatrix} 1 & 4 \\ -7 & 3 \end{pmatrix}\begin{pmatrix} 1 \\ 23 \end{pmatrix}$$

$$\frac{1}{31}\begin{pmatrix} 31 & 0 \\ 0 & 31 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{31}\begin{pmatrix} 93 \\ 62 \end{pmatrix}$$

$$\Leftrightarrow \quad \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$$

$$\Leftrightarrow \quad \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$$

or $x = 3$
and $y = 2$

Check: $3 \times 3 - 4 \times 2 = 1$ and $7 \times 3 + 2 = 23$.

Exercise 13c

Use the matrix method to solve the following pairs of simultaneous equations.

1 $6x + 11y = 29$, $x + 2y = 5$
2 $7x + 4y = 29$, $4x + 2y = 16$
3 $2x + y = 3$, $x + 2y = 9$
4 $21x - 2y = 15$, $13x - y = 10$
5 $4x - 2y = 9$, $x + y = 3$
6 $x - 3y = 2$, $2x + 4y = -1$
7 $2a - 3b = 3$, $a + b = 4$
8 $\frac{1}{2}u - \frac{1}{3}v = 4$, $\frac{1}{3}u + \frac{1}{2}v = 7$
9 $15s - 6t = 4$, $6s + 18t = 5$
10 $6c - d = -2$, $10c - 3d = -10$

Further examples of the use of matrices as operators are given in the next chapter.

Exercise 13d (miscellaneous practice)

1 Evaluate as a single matrix

$$\begin{pmatrix} -3 & -8 \\ 6 & 2 \end{pmatrix} + 2\begin{pmatrix} -2 & 1 \\ -3 & 5 \end{pmatrix}.$$

2 If $\mathbf{M} = \begin{pmatrix} 2 & -6 \\ -1 & 4 \end{pmatrix}$, (a) find the value of the determinant of $\mathbf{M}$, (b) hence write down the inverse of $\mathbf{M}$.

3 Find the value of the determinant of the matrix $\begin{pmatrix} -2 & -4 \\ 5 & 3 \end{pmatrix}$. Hence write down the inverse of the matrix.

4 The value of the determinant of the matrix $\begin{pmatrix} 5 & -2 \\ -4 & x \end{pmatrix}$ is 7.

(a) Find the value of x.
(b) Hence write down the inverse of the matrix.

5 Find the value of k for which the matrix $\begin{pmatrix} 4 & k-2 \\ 8 & 6 \end{pmatrix}$ does not have an inverse.

6 If $\mathbf{A} = \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix}$ and $\mathbf{B} = (-2 \quad 5 \quad 0)$, evaluate

(a) **AB**,

(b) **BA**.

7 Express each of the following as a single matrix.

(a) $\begin{pmatrix} 1 & 5 \\ 4 & 0 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} -1 \\ 3 \end{pmatrix}$

(b) $\begin{pmatrix} 1 & 4 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 3 & 3 \\ 0 & 3 & 3 & 0 \end{pmatrix}$

8 Given that

$$\begin{pmatrix} 3 & 2 & 4 \\ 6 & 0 & 1 \end{pmatrix} \begin{pmatrix} 4 \\ m \\ -3 \end{pmatrix} = \begin{pmatrix} 10 \\ 7n \end{pmatrix}$$

find the values of m and n.

9 **P** is a 2×2 matrix such that $\begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix} \mathbf{P} - \mathbf{P} = \begin{pmatrix} -2 & 0 \\ 2 & 4 \end{pmatrix}$

Find the matrix **P**.

10 $\mathbf{M} = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}$ and $\mathbf{N} = \begin{pmatrix} -1 & 1 \\ 0 & 3 \end{pmatrix}$.

(a) Find $\mathbf{M} - 2\mathbf{N}$.

(b) Find the values of p and q if

$$\mathbf{M}\begin{pmatrix} 4 \\ p \end{pmatrix} = \mathbf{N}\begin{pmatrix} q \\ 6 \end{pmatrix}$$

11 Given that

$$\begin{pmatrix} 0 & 1 \\ -2 & 0 \end{pmatrix} \begin{pmatrix} q & -7 \\ p & 0 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 0 & -1 \end{pmatrix} + \begin{pmatrix} 0 & -2 \\ 6 & 3r \end{pmatrix}$$

find the value of p, of q and of r. [Camb

12 Find a and b if

$$\begin{pmatrix} 3 & 7 \\ b & a \end{pmatrix} \begin{pmatrix} a & -7 \\ -2 & 3 \end{pmatrix} = \mathbf{I}$$

where **I** is the identity matrix.

13 $\mathbf{A} = \begin{pmatrix} 4 & 2 \\ 0 & 3 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} \frac{1}{4} & k \\ 0 & \frac{1}{3} \end{pmatrix}$ and

$\mathbf{C} = \begin{pmatrix} 12 & 4 \\ -9 & m \end{pmatrix}$.

(a) Evaluate $\mathbf{A}^2$.

(b) Find the value of k which makes **AB** the identity matrix.

(c) Find the value of m which makes the determinant of **A** equal to the determinant of **C**. [Camb]

14 (a) Write down the inverse of the matrix $\begin{pmatrix} 3 & -4 \\ 5 & 7 \end{pmatrix}$

(b) Hence find x and y if

$$\begin{pmatrix} 3 & -4 \\ 5 & 7 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 10 \\ 3 \end{pmatrix}$$

15 Express the simultaneous equations

$3y = -5x + 3$

$2y = -3x - 1$

as a single matrix equation in the form

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} p \\ q \end{pmatrix}$$

where a, b, c, d, p, q are integers.
Hence find the values of x and y.

Chapter 14

Geometrical transformations (3)

Transformations and matrices

Translation

In Fig. 14.1 A is any point $(p; q)$ on the cartesian plane. OA or **a** is the **position vector** of A:

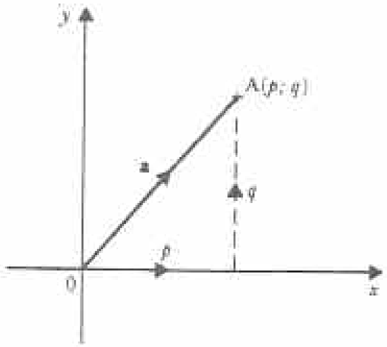


Fig. 14.1

$\vec{OA} = \mathbf{a} = \begin{pmatrix} p \\ q \end{pmatrix}$ = the displacement of A from the origin.

If A is the point (1; 2) and it is translated by vector $\begin{pmatrix} 4 \\ 2 \end{pmatrix}$, its final position B is shown in Fig. 14.2.

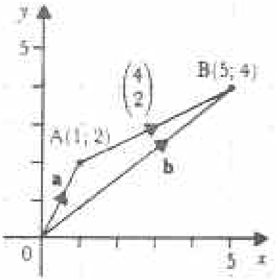


Fig. 14.2

The position of B is calculated by adding the translation vector to the position vector of A:

$$\vec{OB} = \mathbf{b} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} + \begin{pmatrix} 4 \\ 2 \end{pmatrix} = \begin{pmatrix} 5 \\ 4 \end{pmatrix}$$

Translation is a geometrical transformation often represented by the letter T. Hence if T represents the translation $\begin{pmatrix} 4 \\ 2 \end{pmatrix}$ then $T(\mathbf{a}) = \mathbf{b}$.

Similarly,

$$T(\mathbf{b}) = \mathbf{b} + \begin{pmatrix} 4 \\ 2 \end{pmatrix} = \begin{pmatrix} 5 \\ 4 \end{pmatrix} + \begin{pmatrix} 4 \\ 2 \end{pmatrix}$$
$$= \begin{pmatrix} 9 \\ 6 \end{pmatrix} = \mathbf{c}$$

Hence $T(T(\mathbf{a})) = \mathbf{c}$.
This is usually written as $T^2(\mathbf{a}) = \mathbf{c}$.
Notice that if the cartesian plane is given a translation T then every point on the plane is translated through T.

Rotation

Fig. 14.3 shows successive rotations of a **unit square** through 90° clockwise about the origin. [*Note:* a unit square has vertices at (0; 0), (1; 0), (1; 1), (0; 1)]

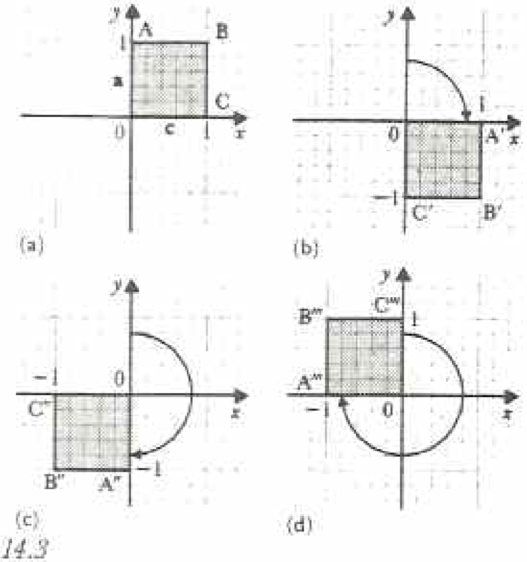


Fig. 14.3

In Fig. 14.3(a) the position vectors of the vertices A, B, C are **a**, **b**, **c** where

$$\mathbf{a}=\begin{pmatrix}0\\1\end{pmatrix},\quad \mathbf{b}=\begin{pmatrix}1\\1\end{pmatrix},\quad \mathbf{c}=\begin{pmatrix}1\\0\end{pmatrix}$$

If R represents a rotation through 90° clockwise about the origin, then from Fig. 14.3, parts (a) and (b):

$$\mathrm{R}(\mathbf{a})=\begin{pmatrix}1\\0\end{pmatrix} \qquad (1)$$

$$\mathrm{R}(\mathbf{c})=\begin{pmatrix}0\\-1\end{pmatrix} \qquad (2)$$

It is possible to represent R by a matrix. Let this matrix be $\begin{pmatrix}p & q\\r & s\end{pmatrix}$.

From (1) and (2) above:

$$\begin{pmatrix}p & q\\r & s\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix}=\begin{pmatrix}1\\0\end{pmatrix}$$

and $\begin{pmatrix}p & q\\r & s\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}=\begin{pmatrix}0\\-1\end{pmatrix}$

By multiplying out the matrices,

$0p+1q=1$

$\Leftrightarrow \quad q=1$

Similarly $s=0$, $p=0$, $r=-1$.

Hence $\mathrm{R}=\begin{pmatrix}0 & 1\\-1 & 0\end{pmatrix}$.

In Fig. 14.3(c), R^2 represents R followed by R, i.e. a rotation of 180° about 0.

$$\mathrm{R}^2=\begin{pmatrix}0 & 1\\-1 & 0\end{pmatrix}\begin{pmatrix}0 & 1\\-1 & 0\end{pmatrix}=\begin{pmatrix}-1 & 0\\0 & -1\end{pmatrix}$$

In Fig. 14.3(d), R^3 represents R followed by R followed by R, i.e. a clockwise rotation of 270° about 0.

$\mathrm{R}^3=\mathrm{R}\times\mathrm{R}^2$

$$=\begin{pmatrix}0 & 1\\-1 & 0\end{pmatrix}\begin{pmatrix}-1 & 0\\0 & -1\end{pmatrix}=\begin{pmatrix}0 & -1\\1 & 0\end{pmatrix}$$

R^4 represents four successive clockwise rotations of 90°, i.e. a rotation of 360°.

$\mathrm{R}^4=\mathrm{R}\times\mathrm{R}^3$ (or $\mathrm{R}^2\times\mathrm{R}^2$)

$$=\begin{pmatrix}0 & 1\\-1 & 0\end{pmatrix}\begin{pmatrix}0 & -1\\1 & 0\end{pmatrix} \text{ or }$$

$$\begin{pmatrix}-1 & 0\\0 & -1\end{pmatrix}\begin{pmatrix}-1 & 0\\0 & -1\end{pmatrix}$$

$$=\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix} \text{ in both cases.}$$

Hence R^4 is equivalent to the identity matrix **I**, as might be expected.

The matrices $\begin{pmatrix}0 & 1\\-1 & 0\end{pmatrix}$, $\begin{pmatrix}-1 & 0\\0 & -1\end{pmatrix}$, $\begin{pmatrix}0 & -1\\1 & 0\end{pmatrix}$, $\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}$ represent clockwise rotations of 90°, 180°, 270°, 360° about the origin.

Notice that if the cartesian plane is given a rotation R then every point, except the centre of rotation, is rotated through R. The centre of rotation is said to be an **invariant** point. *Invariant* means 'unchanging'.

Reflection

In Fig. 14.4, the unit square in part (a) is shown (b) reflected in the x-axis, and (c) reflected in the y-axis.

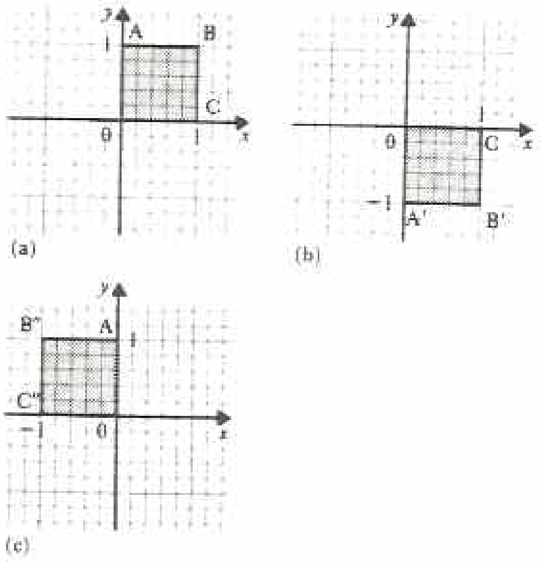


Fig. 14.4

If M represents the reflection in the x-axis, then by considering what happens to **OA** and **OC**,

$M(\mathbf{OA}) = \mathbf{OA'}$
and $M(\mathbf{OC}) = \mathbf{OC}$

If M is the matrix $\begin{pmatrix} p & q \\ r & s \end{pmatrix}$, then

$$\begin{pmatrix} p & q \\ r & s \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \end{pmatrix} \quad (1)$$

and $\begin{pmatrix} p & q \\ r & s \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad (2)$

By multiplying out the matrices,
$q = 0$, $s = -1$, $p = 1$ and $r = 0$.

Hence $M = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

Similarly, by considering Fig. 10.4(c), if N represents reflection in the y-axis it can be shown that

$$N = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

Notice that

$$M^2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$N^2 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Hence $M^2 = N^2 = \mathbf{I}$ as might be expected.

If the cartesian plane is given a reflection **M** then every point on the plane, except those on the line of reflection is transformed. The line of reflection is an **invariant line**.

Example 1

The vertices of a triangle are A(1; 2), B(3; 1) *and* C(−2; 1). *If* △ABC *is reflected in the x-axis, calculate the coordinates of the vertices of its image.*

The matrix M represents reflection in the x-axis where

$$M = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

If the image of △ABC is △A'B'C', then

$M(\mathbf{OA}) = \mathbf{OA'}$ (1)
$M(\mathbf{OB}) = \mathbf{OB'}$ (2)
$M(\mathbf{OC}) = \mathbf{OC'}$ (3)

From (1), $\mathbf{OA'} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$

From (2), $\mathbf{OB'} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ -1 \end{pmatrix}$

From (3), $\mathbf{OC'} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} -2 \\ 1 \end{pmatrix} = \begin{pmatrix} -2 \\ -1 \end{pmatrix}$

The vertices of the image of △ABC are A'(1; −2), B'(3; −1), C'(−2; −1).

Note: The working in Example 1 can be written more neatly by representing the vertices of △ABC by a single matrix,

$$\begin{pmatrix} 1 & 3 & -2 \\ 2 & 1 & 1 \end{pmatrix}.$$

Then, by matrix multiplication,

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\overset{\quad A \quad B \quad C}{\begin{pmatrix} 1 & 3 & -2 \\ 2 & 1 & 1 \end{pmatrix}} = \overset{\quad A' \quad B' \quad C'}{\begin{pmatrix} 1 & 3 & -2 \\ -2 & -1 & -1 \end{pmatrix}}$$

Example 2

Triangle XYZ *has vertices at* X(3; 4), Y(5; −1), Z(−2; 2). *Find the coordinates of the image of* X, Y, Z *if the triangle is first translated by vector* $\begin{pmatrix} -2 \\ 7 \end{pmatrix}$ *and then rotated through* 180° *about the origin.*

Let △X'Y'Z' be the image of △XYZ after translation $\begin{pmatrix} -2 \\ 7 \end{pmatrix}$.

$$\mathbf{OX'} = \begin{pmatrix} 3 \\ 4 \end{pmatrix} + \begin{pmatrix} -2 \\ 7 \end{pmatrix} = \begin{pmatrix} 1 \\ 11 \end{pmatrix}$$

$$\mathbf{OY'} = \begin{pmatrix} 5 \\ -1 \end{pmatrix} + \begin{pmatrix} -2 \\ 7 \end{pmatrix} = \begin{pmatrix} 3 \\ 6 \end{pmatrix}$$

$$\mathbf{OZ'} = \begin{pmatrix} -2 \\ 2 \end{pmatrix} + \begin{pmatrix} -2 \\ 7 \end{pmatrix} = \begin{pmatrix} -4 \\ 9 \end{pmatrix}$$

Let △X"Y"Z" be the image of △X'Y'Z' after rotation through 180° about the origin.

$\begin{pmatrix}-1 & 0\\ 0 & -1\end{pmatrix}$ is the matrix of rotation:

$$\begin{matrix} & & X' & Y' & Z' \end{matrix}$$

$$\begin{pmatrix}-1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}1 & 3 & -4\\ 11 & 6 & 9\end{pmatrix} =$$

$$\begin{matrix}X'' & Y'' & Z''\end{matrix}$$
$$\begin{pmatrix}-1 & -3 & 4\\ -11 & -6 & -9\end{pmatrix}$$

The vertices of the final image of $\triangle XYZ$ are at $X''(-1;-11)$, $Y''(-3;-6)$ and $Z''(4;-9)$.

Exercise 14a

1 Copy and complete Table 14.1.

Table 14.1

transformation	**matrix**
identity	$\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$
reflection in x-axis	
reflection in y-axis	
rotation of 180° about origin	

2 Find the matrices which are equivalent to anticlockwise rotations of (a) 90°, (b) 270° about the origin.

3 A triangle has vertices (1; 1), (2; 4) and (3; 7). Find the coordinates of the images of its vertices if it is rotated through 90° clockwise about the origin.

4 P′Q′ is the image of a line PQ after a translation of $\begin{pmatrix}7\\ -4\end{pmatrix}$. If the coordinates of P′ are (6; 1), calculate the coordinates of P.

5 A triangle has vertices A(0; 0), B(1; −1) and C(1; 1). Calculate the position of its vertices after a rotation of 180° about the origin.

6 Reflect the triangle of question 5 about the y-axis. Compare your answer with that of question 5.

7 A rhombus has vertices (2; 2), (1; −1), (−2; −2) and (−1; 1). Find the coordinates of its vertices when it is reflected in (a) the x-axis, (b) the y-axis.

8 In Fig. 14.5 there are three triangles, ABC, $A_1B_1C_1$ and $A_2B_2C_2$.

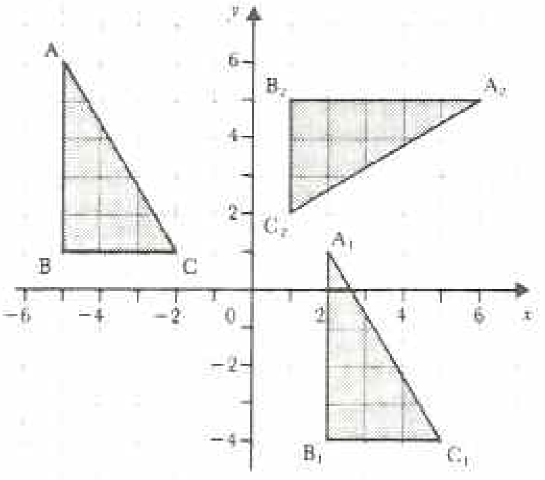


Fig. 14.5

(a) $A_1B_1C_1$ is the image of ABC under a single translation. Write down the column vector of this translation.

(b) $A_2B_2C_2$ is the image of ABC under an anticlockwise rotation about the origin. Write down (i) the angle of the rotation, (ii) the matrix of the rotation.

(c) $A_3B_3C_3$ (not shown in the diagram) is the image of ABC under a reflection represented by the matrix $\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$. Write down the equation of the straight line through B_3 and C_3.

9 $\triangle XYZ$ has vertices at X(1; 1), Y(4; 2) and Z(2; 3). Find the coordinates of the images of X, Y, Z if $\triangle XYZ$ is first rotated through 180° about the origin and then translated through $\begin{pmatrix}3\\ 5\end{pmatrix}$.

10 T is the translation $\begin{pmatrix}3\\ -1\end{pmatrix}$ and R is a clockwise rotation of 90° about the origin.

A is the point $(-5;2)$ and B is the point $(4;-3)$. Find the coordinates of
(a) T(A) (b) R(B)
(c) the point C which when given translation T followed by rotation R has its image at $(6;-2)$.

Enlargement

In Fig. 14.6 △OAB is enlarged to △OPQ by scale factor 2 with the origin as the centre of enlargement.

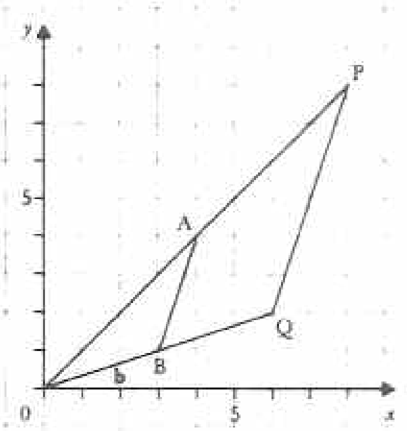


Fig. 14.6

From Fig. 14.6, $\mathbf{a} = \begin{pmatrix}4\\4\end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix}3\\1\end{pmatrix}$,

$\mathbf{OP} = \begin{pmatrix}8\\8\end{pmatrix}$ and $\mathbf{OQ} = \begin{pmatrix}6\\2\end{pmatrix}$

If operator E represents the enlargement, then $E(\mathbf{a}) = \mathbf{OP}$ and $E(\mathbf{b}) = \mathbf{OQ}$.

Let E be the matrix $\begin{pmatrix}p & q\\r & s\end{pmatrix}$, then

$$\begin{pmatrix}p & q\\r & s\end{pmatrix}\begin{pmatrix}4\\4\end{pmatrix} = \begin{pmatrix}8\\8\end{pmatrix}$$

and $$\begin{pmatrix}p & q\\r & s\end{pmatrix}\begin{pmatrix}3\\1\end{pmatrix} = \begin{pmatrix}6\\2\end{pmatrix}$$

from which $p = 2$, $q = 0$, $r = 0$, $s = 2$,

and $$E = \begin{pmatrix}2 & 0\\0 & 2\end{pmatrix} = 2\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix} = 2\mathbf{I}$$

In general, for any enlargement E with scale factor k and centre $(0;0)$,

$$E = \begin{pmatrix}k & 0\\0 & k\end{pmatrix} = k\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix} = k\mathbf{I}$$

If the cartesian plane is given an enlargement, there is only one invariant point: the centre of enlargement.

Example 3

Quadrilateral OABC *has vertices* O$(0;0)$, A$(6;-1)$, B$(-4;2)$, C$(9;9)$. OABC *is enlarged with scale factor* $-\frac{1}{3}$ *with the origin as centre. Find the coordinates of its enlargement* O′A′B′C′.

Enlargement matrix

$$= -\tfrac{1}{3}\mathbf{I} = -\tfrac{1}{3}\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix} = \begin{pmatrix}-\frac{1}{3} & 0\\0 & -\frac{1}{3}\end{pmatrix}$$

$$\begin{pmatrix}-\frac{1}{3} & 0\\0 & -\frac{1}{3}\end{pmatrix}\overset{\quad O \quad A \quad B \quad C}{\begin{pmatrix}0 & 6 & -4 & 9\\0 & -1 & 2 & 9\end{pmatrix}}$$

$$= \overset{\quad O' \quad A' \quad B' \quad C'}{\begin{pmatrix}0 & -2 & 1\frac{1}{3} & -3\\0 & \frac{1}{3} & -\frac{2}{3} & -3\end{pmatrix}}$$

The enlargement has coordinates O′$(0;0)$, A′$(-2;\frac{1}{3})$, B′$(1\frac{1}{3};-\frac{2}{3})$ and C′$(-3;-3)$.

Shear

In Fig 14.7 the unit square OABC is mapped onto the parallelogram ODEC by a shear parallel to the x-axis.

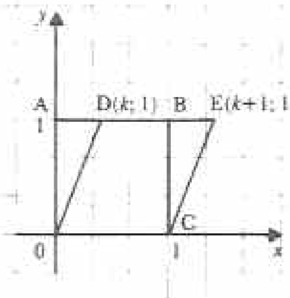


Fig. 14.7

Points on the x-axis remain where they are, while points above move by an amount proportional to their distance from the x-axis. If D is the point $(k; 1)$, the shearing matrix, H, can be found as before:

$$\begin{pmatrix} p & q \\ r & s \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

and $\begin{pmatrix} p & q \\ r & s \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} k \\ 1 \end{pmatrix}$

from which $p = 1$, $r = 0$, $q = k$ and $s = 1$.

H is the matrix $\begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}$.

This can be checked by considering the effect of the shear on point B:

$\begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1+k \\ 1 \end{pmatrix}$, as shown in Fig. 14.7.

Just as $\begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}$ is a shear parallel to the x-axis with the x-axis invariant, so the matrix $\begin{pmatrix} 1 & 0 \\ k & 1 \end{pmatrix}$ is a shear parallel to the y-axis with the y-axis invariant.

Example 4

G is a transformation represented by the matrix

$$\begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix}.$$

(a) Find the image of (4; 3) under G.
(b) Find the image of (4; −3) under G.
(c) Describe, completely, the transformation G.

(a) $\begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 4 \\ 3 \end{pmatrix} = \begin{pmatrix} -2 \\ 3 \end{pmatrix}$

The image of $(4; 3)$ is $(-2; 3)$.

(b) $\begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 4 \\ -3 \end{pmatrix} = \begin{pmatrix} 10 \\ -3 \end{pmatrix}$

The image of $(4; -3)$ is $(10; -3)$.

(c) Fig. 14.8 is a sketch showing how the transformation changes the line joining the given points.

From Fig. 14.8, G is a shear with the x-axis invariant. The shear factor is -2 which produces shearing to the left above the x-axis.

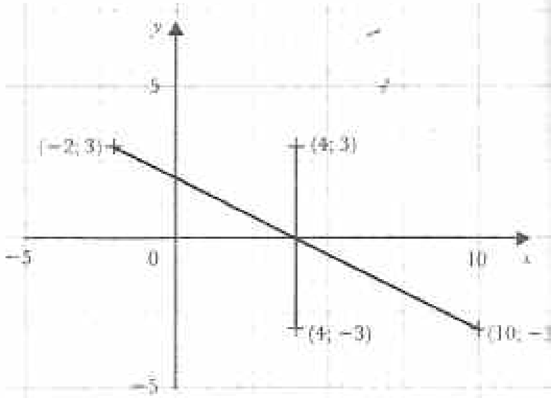


Fig. 14.8

Stretch

Consider the effect on the unit square of the operator $\begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}$.

$$\begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$$

$$\begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \end{pmatrix}$$

$$\begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

The effect is to stretch the square in two directions to form a rectangle as shown in Fig. 14.9.

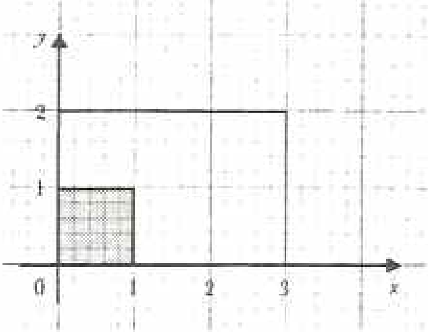


Fig. 14.9

Notice that the square is stretched by a factor of 3 in the direction of increase in x and a factor of 2 in the direction of increase in y. In general, a stretch, S, is given by

$$S = \begin{pmatrix} h & 0 \\ 0 & k \end{pmatrix}$$

which is a two-way stretch by factor h in the x-direction and k in the y-direction.

Notice that the origin is always mapped onto itself when multiplied by a 2×2 matrix.

Example 5

A′(0; 0), B′(−5; 6), C′(−2; −9) are the images of A(0; 0), B(5; 2), C(2; −3) under a transformation represented by a matrix of the form $\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$.

(a) Find the transformation matrix. (b) Find the matrix which will transform △A′B′C′ back to △ABC. (c) Complete the two matrices.

(a) $\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}\begin{pmatrix} 5 \\ 2 \end{pmatrix} = \begin{pmatrix} -5 \\ 6 \end{pmatrix}$ (1)

and $\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}\begin{pmatrix} 2 \\ -3 \end{pmatrix} = \begin{pmatrix} -2 \\ -9 \end{pmatrix}$ (2)

From (1), $5a = -5$

$\Leftrightarrow \quad a = -1$

and $2b = 6$

$\Leftrightarrow \quad b = 3$

The transformation matrix is $\begin{pmatrix} -1 & 0 \\ 0 & 3 \end{pmatrix}$, a stretch. Check this result in equation (2).

(b) Let the matrix $\begin{pmatrix} p & q \\ r & s \end{pmatrix}$ transform △A′B′C′ into △ABC. Hence:

$\begin{pmatrix} p & q \\ r & s \end{pmatrix}\begin{pmatrix} -5 \\ 6 \end{pmatrix} = \begin{pmatrix} 5 \\ 2 \end{pmatrix}$ (3)

and $\begin{pmatrix} p & q \\ r & s \end{pmatrix}\begin{pmatrix} -2 \\ -9 \end{pmatrix} = \begin{pmatrix} 2 \\ -3 \end{pmatrix}$ (4)

From (3): $-5p + 6q = 5$

$-5r + 6s = 2$

From (4): $-2p - 9q = 2$

$-2r - 9s = -3$

The solution of these equations gives $p = -1$, $q = 0$, $r = 0$, $s = \frac{1}{3}$

The matrix $\begin{pmatrix} -1 & 0 \\ 0 & \frac{1}{3} \end{pmatrix}$ will transform △A′B′C′ to △ABC.

(c) Let the matrices in parts (a) and (b) be S and T respectively.

$$S = \begin{pmatrix} -1 & 0 \\ 0 & 3 \end{pmatrix}$$

$$T = \begin{pmatrix} -1 & 0 \\ 0 & \frac{1}{3} \end{pmatrix} = -\tfrac{1}{3}\begin{pmatrix} 3 & 0 \\ 0 & -1 \end{pmatrix}$$

$= \text{inverse of S}.$

Each matrix is the inverse of the other.

Notice in Example 5 that if a shape is mapped onto an image by a matrix A, then the image can be mapped back onto the original shape by the inverse of A, sometimes written as A^{-1}. Remember that $AA^{-1} = \mathbf{I}$.

Exercise 14b

1 Use the matrix $\begin{pmatrix} -3 & 0 \\ 0 & -3 \end{pmatrix}$ to enlarge △ABC in Fig. 14.10.

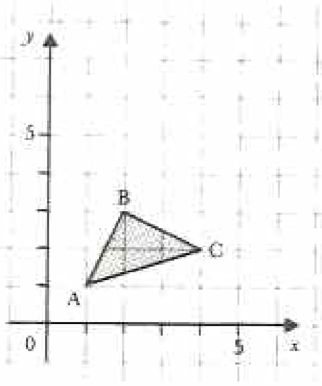


Fig. 14.10

2 Use the matrix $\begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}$ to shear △ABC in Fig. 14.10. Find the matrix which will map the image back onto △ABC.

3 Use the matrix $\begin{pmatrix} 2 & 0 \\ 0 & 5 \end{pmatrix}$ to transform $\triangle$ABC in Fig. 14.10. Describe the transformation as fully as possible.

4 Find the matrix E which has the effect of enlarging plane shapes by a scale factor $1\frac{1}{2}$ with the origin as centre of enlargement.

5 Use E of question 4 to enlarge the rectangle whose vertices are (0; 0), (3; 0), (0; 2), (3; 2).

6 The matrix $\begin{pmatrix} 1 & 0 \\ 5 & 1 \end{pmatrix}$ represents a transformation H.

(a) Find the image of (2; 5) under H.
(b) Find the image of (−2; 5) under H.
(c) Describe H as fully as possible.

7 In Fig. 14.11 $\triangle$KLM is mapped onto $\triangle$K′L′M′ by a two-way stretch, S.

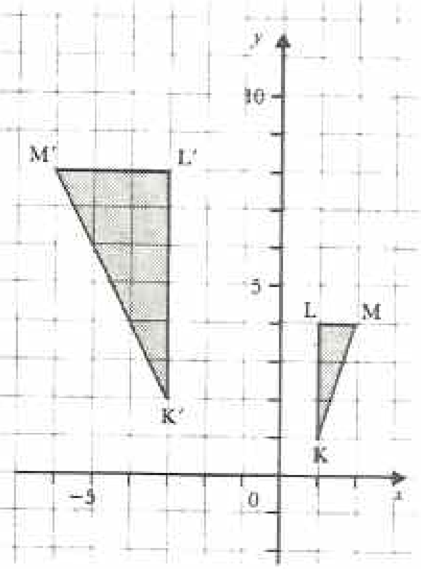


Fig. 14.11

(a) What is the stretch factor in the x-direction?
(b) What is the stretch factor in the y-direction?
(c) Hence, or otherwise, find the matrix which represents S.

8 (a) A single transformation U maps $\triangle$PQR of Fig. 14.12 onto $\triangle P_1Q_1R_1$ which has vertices P_1(0; 16), Q_1(12; 20), R_1(8; 2).
Describe U, writing down the matrix which represents the transformation.

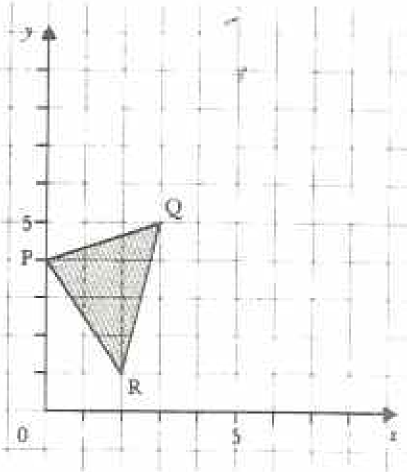


Fig. 14.12

(b) $\triangle P_2Q_2R_2$ is the image of $\triangle$PQR under a shear with the x-axis as the invariant line. If R_2 is the point (6; 1) find the coordinates of P_2 and Q_2.

9 A shear S is represented by the matrix

$$\begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}.$$

(a) Calculate the coordinates of the image of the point (2; −2) under S.
(b) Calculate the coordinates of the point which will be mapped onto (7; 4) by S.
(c) Write down the equation of the invariant line. [Camb]

10 $\triangle$PQR has vertices P(0; 0), Q(2; 1), R(−1; 3). Find the coordinates of the images of P, Q, R if the triangle is enlarged by scale factor 2 with origin as centre. The enlargement is rotated through 90° clockwise about the origin. Calculate the coordinates of the final image of $\triangle$PQR.

Combined transformations

Example 6

Triangle OAB *has vertices at* O(0; 0), A(2; 1), B(−1; 3). *Find the vertices of the triangle if it is first enlarged by* E, *then translated through* T *where*

$$E = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \text{ and } T = \begin{pmatrix} -3 \\ -5 \end{pmatrix}.$$

First, enlargment by E:

$$\begin{matrix} & & O & A & B & & O' & A' & B' \\ \end{matrix}$$
$$\begin{pmatrix}2 & 0\\0 & 2\end{pmatrix}\begin{pmatrix}0 & 2 & -1\\0 & 1 & 3\end{pmatrix}=\begin{pmatrix}0 & 4 & -2\\0 & 2 & 6\end{pmatrix}$$

Second, translation of points O′, A′, B′ through T:

$$\begin{pmatrix}0\\0\end{pmatrix}+\begin{pmatrix}-3\\-5\end{pmatrix}=\begin{pmatrix}-3\\-5\end{pmatrix}$$
$$\begin{pmatrix}4\\2\end{pmatrix}+\begin{pmatrix}-3\\-5\end{pmatrix}=\begin{pmatrix}1\\-3\end{pmatrix}$$
$$\begin{pmatrix}-2\\6\end{pmatrix}+\begin{pmatrix}-3\\-5\end{pmatrix}=\begin{pmatrix}-5\\1\end{pmatrix}$$

The vertices of the transformed triangle are O″(−3; −5), A″(1; −3) and B″(−5; 1). The combined transformation is shown in Fig. 14.13.

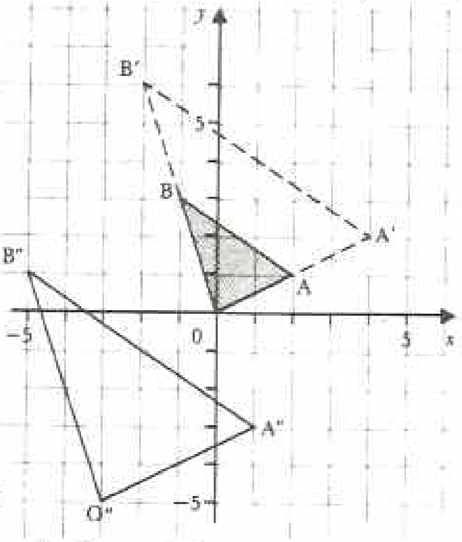


Fig. 14.13

The process in Example 6 can be written as

TE(**a**) = **b**

where TE means that E is done *before* T.

TE(**a**) = **b**

⇔ T[E(**a**)] = **b**

The order in which the operations are carried out is usually important. In general, TE ≠ ET. For example, with the data of Example 6:

$$TE\begin{pmatrix}2\\1\end{pmatrix}=\begin{pmatrix}1\\-3\end{pmatrix} \text{ as shown above}$$

$$\text{but } ET\begin{pmatrix}2\\1\end{pmatrix}=E\left[\begin{pmatrix}2\\1\end{pmatrix}+\begin{pmatrix}-3\\-5\end{pmatrix}\right]$$
$$=E\begin{pmatrix}-1\\-4\end{pmatrix}$$
$$=\begin{pmatrix}2 & 0\\0 & 2\end{pmatrix}\begin{pmatrix}-1\\-4\end{pmatrix}=\begin{pmatrix}-2\\-8\end{pmatrix}$$

Example 7

The rhombus whose vertices are at (0; 0), (1; 2), (3; 3) and (2; 1) is first reflected in the x-axis and then sheared by the operator $\begin{pmatrix}1 & 2\\0 & 1\end{pmatrix}$. *Find the vertices of the resulting figure.*

The matrix of reflection is M where

$M=\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}$ for reflection in the x-axis.

If H is the shearing matrix, then HM represents the combined transformation (i.e. M before H).

$$HM=\begin{pmatrix}1 & 2\\0 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}=\begin{pmatrix}1 & -2\\0 & -1\end{pmatrix}$$

$\begin{pmatrix}1 & -2\\0 & -1\end{pmatrix}$ is the combined matrix of transformation:

$$\begin{pmatrix}1 & -2\\0 & -1\end{pmatrix}\begin{pmatrix}0 & 1 & 3 & 2\\0 & 2 & 3 & 1\end{pmatrix}$$
$$=\begin{pmatrix}0 & -3 & -3 & 0\\0 & -2 & -3 & -1\end{pmatrix}$$

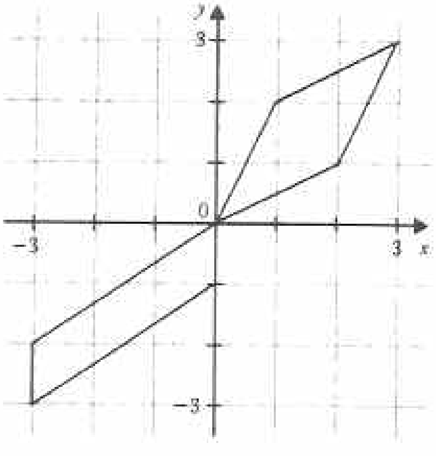


Fig. 14.14

The resulting figure has vertices at (0; 0), (−3; −2), (−3; −3) and (0; −1). The effect of the combined transformation on the original rhombus is shown in Fig. 14.14.

Example 8

(a′; b′) is the image of a point (a; b) after a transformation given by

$$\begin{pmatrix} a' \\ b' \end{pmatrix} = \begin{pmatrix} 4 & 0 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} + \begin{pmatrix} -7 \\ -2 \end{pmatrix}$$

(a) *Describe the transformation in words.*
(b) A′ *is the image of point* A(−1; 2). *Find the coordinates of* A′.
(c) *Find the coordinates of a point* B *which has an image at* B′(5; −5).

(a) First, the stretching matrix $\begin{pmatrix} 4 & 0 \\ 0 & 3 \end{pmatrix}$ acts on $(a; b)$. Second, the result is translated by vector $\begin{pmatrix} -7 \\ -2 \end{pmatrix}$.

(b) Substituting −1 for a and 2 for b in the given equation:

$$\begin{pmatrix} a' \\ b' \end{pmatrix} = \begin{pmatrix} 4 & 0 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} -1 \\ 2 \end{pmatrix} + \begin{pmatrix} -7 \\ -2 \end{pmatrix}$$
$$= \begin{pmatrix} -4 \\ 6 \end{pmatrix} + \begin{pmatrix} -7 \\ -2 \end{pmatrix}$$
$$= \begin{pmatrix} -11 \\ 4 \end{pmatrix}$$

A′ is the point (−11; 4).

(c) Let B have coordinates $(h; k)$, then:

$$\begin{pmatrix} 5 \\ -5 \end{pmatrix} = \begin{pmatrix} 4 & 0 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} h \\ k \end{pmatrix} + \begin{pmatrix} -7 \\ -2 \end{pmatrix}$$
$$\Rightarrow \begin{pmatrix} 5 \\ -5 \end{pmatrix} = \begin{pmatrix} 4h \\ 3k \end{pmatrix} + \begin{pmatrix} -7 \\ -2 \end{pmatrix}$$
$$\Rightarrow \begin{pmatrix} 5 \\ -5 \end{pmatrix} = \begin{pmatrix} 4h - 7 \\ 3k - 2 \end{pmatrix}$$
$$\Rightarrow \quad 5 = 4h - 7$$
$$\Leftrightarrow \quad h = 3$$
$$\text{and} \quad -5 = 3k - 2$$
$$\Leftrightarrow \quad k = -1$$

B is the point (3; 1).

Exercise 14c

1 Triangle XYZ in Fig. 14.15 is first rotated through 90° anticlockwise about the origin and then sheared by the operator $\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$. Calculate the coordinates of the vertices of its image.

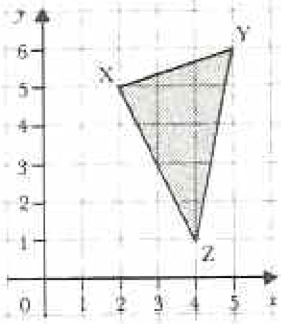


Fig. 14.15

2 Use the matrix $\begin{pmatrix} 1 & 0 \\ 5 & 1 \end{pmatrix}$ to shear △XYZ in Fig. 14.15. Then reflect the result in the y-axis. Calculate the coordinates of its final image.

3 Each of the following equations represents a transformation.

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} -3 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

(a) Describe the transformations in words.
(b) For each transformation find the image of the point (−1; 4).

4 T is a translation $\begin{pmatrix} 2 \\ 8 \end{pmatrix}$ and S is a stretch represented by the matrix $\begin{pmatrix} -2 & 0 \\ 0 & 3 \end{pmatrix}$.
Calculate the image of A(3; 2) under the following transformations:
(a) ST(A)
(b) TS(A)
(c) S^{-1}(A)

5 R is a clockwise rotation of 270° about the origin and H is a shear represented by the matrix $\begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix}$. Calculate the coordinates of the point that P(−3; 5) is mapped onto by the following combined transformations:
(a) RH(P) (b) HR(P) (c) $H^2(P)$

6 In Fig. 14.16, semicircle A can be mapped onto semicircle B by an anticlockwise rotation about the origin followed by a translation.
(a) State the angle of rotation.
(b) Find the matrix which represents the rotation.
(c) Find the column vector of the translation.
(d) Given that A can be mapped onto B by a single reflection in a line *m*, find the equation of *m*.

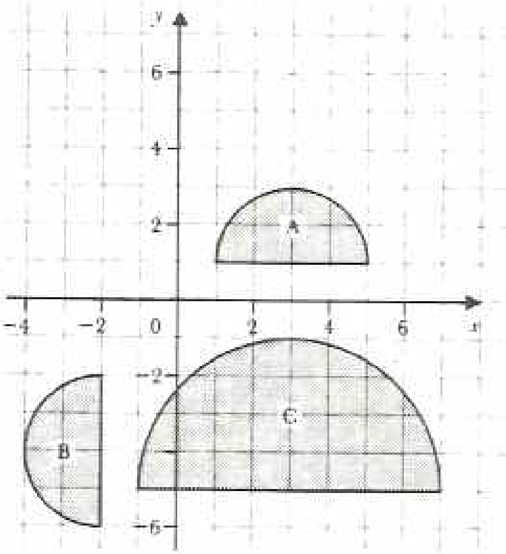


Fig. 14.16

7 In Fig. 14.16, semicircle C is the image of semicircle A under a transformation given by TE(A) = C, where E is an enlargement with the origin as centre and T is a translation.
(a) State the scale factor of E.
(b) Write down the matrix representing E.
(c) Express T as a column vector.
(d) The transformation can also be given by ET′(A) = C where T′ is a different translation and E is the same as before. Express T′ as a column vector.
(e) A can be mapped onto C by a single enlargement with centre (*h*; *k*). State the values of *h* and *k*.

8 The cartesian plane is first translated by vector $\begin{pmatrix} -1 \\ 3 \end{pmatrix}$ and then sheared by matrix $\begin{pmatrix} 1 & 0 \\ 3 & 0 \end{pmatrix}$. Show that the image of a point (*a*; *b*) on the plane is (*a* − 1; 3*a* + *b*). Hence write down the coordinates of the image of the origin under such a transformation.

9 (*x*′; *y*′) is the image of a point (*x*; *y*) after a combination of transformations given by

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 1 & 4 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} -1 \\ 5 \end{pmatrix}$$

(a) Find the coordinates of O′, the image of the point O(0; 0).
(b) Find the coordinates of A′, the image of the points A(3; 2).
(c) If B′(7; 8) is the image of B(*m*; *n*), form two equations which can be solved for *m* and *n*. Hence or otherwise find the values of *m* and *n*.

10 Answer this question on graph paper.
(a) Using a scale of 1 cm to 1 unit on each axis, draw *x* and *y* axes, taking values of *x* from −8 to 12 and values of *y* from −6 to 14. Draw and label the triangle X, with vertices (2; 4), (4; 4) and (4; 1).
(b) The single transformation U maps the triangle X onto the triangle U(X) which has vertices (6; 12), (12; 12) and (12; 3). Draw and label the triangle U(X) and describe fully the transformation U.
(c) The transformation R is a clockwise rotation of 90° about the origin. Draw and label the triangle R(X).
(d) The transformation T is the translation $\begin{pmatrix} -8 \\ 4 \end{pmatrix}$. Draw and label the triangle T(X) and the triangle RT(X).
(e) The single transformation V is represented by the matrix $\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$. Draw and label the triangle V(X) and describe fully the transformation V. [Camb]

Chapter 15

Graphs (5) Cubic and inverse functions, sketch graphs

Cubic functions

A **cubic function** of x is an expression in x in which 3 is the highest power of x. For example, $2x^3 + 5x^2 - x - 8$ is a cubic function of x.

Example 1
(a) Draw the graph of $y = x^3$ for values of x from -3 to $+3$. (b) Hence solve the equations $x^3 + 20 = 0$.

(a) Table 15.1 gives the necessary table of values. Notice that additional values of y for $x = \pm\frac{1}{2}$ have been calculated. These will be helpful when drawing the graph.

Table 15.1

x	-3	-2	-1	0	1	2	3	$\pm\frac{1}{2}$
y	-27	-8	-1	0	1	8	27	$\pm\frac{1}{8}$

Fig. 15.1 is the graph of $y = x^3$.

(b) If $x^3 + 20 = 0$, then $x^3 = -20$.
$x^3 = -20$ when, in Fig. 15.1, $y = -20$.
From Fig. 15.1, $y = -20$ at $x \simeq -2{,}7$.
$x = -2{,}7$ is the approximate solution of $x^3 + 20 = 0$.

Example 2
(a) Draw the graph of $y = x(x-2)(x+2)$ for values of x from -3 to $+3$. (b) Find the values of x at the points where the graph cuts the line $y = x + 2$.

Note: $x(x-2)(x+2) = x(x^2 - 4)$
$= x^3 - 4x$

Hence $x(x-2)(x+2)$ is a cubic function in x.

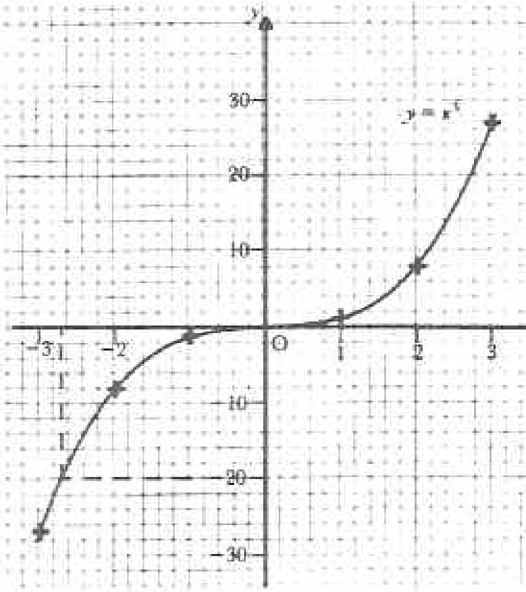


Fig. 15.1

(a) In Table 15.2, the values of y for integral values of x are first calculated. It then be seen that the values of y for $x =$ [illegible] and $x = \pm\frac{1}{2}$ will be helpful when drawing graph. These values are also calculated.

Table 15.2

x	-3	-2	-1	0	1	2	3	$2\frac{1}{2}$
$x-2$	-5	-4	-3	-2	-1	0	1	[illegible]
$x+2$	-1	0	1	2	3	4	5	$4\frac{1}{2}$
$y = x(x-2)(x+2)$	-15	0	3	0	-3	0	15	[illegible]

The curve in Fig. 15.2 is the graph of $y = x(x - 2)(x + 2)$.

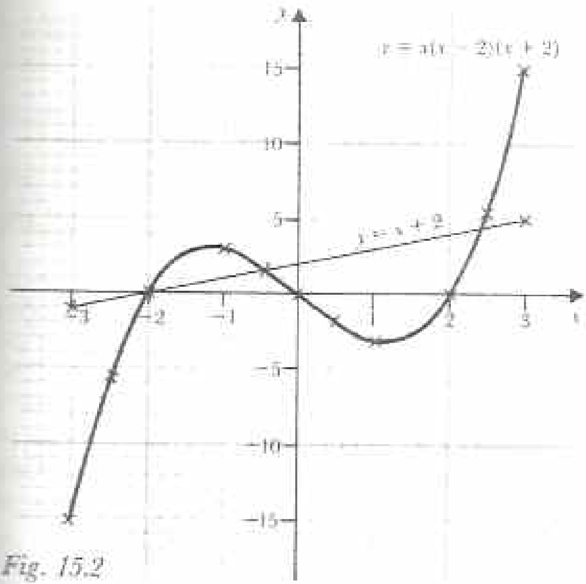


Fig. 15.2

(b) Draw the line $y = x + 2$ by plotting the values in Table 15.3.

Table 15.3

x	-3	0	3
y	-1	2	5

From Fig. 15.2, the values of x at the intersections of the curve and line are $x = -2; -0{,}6$ and $2{,}4$.

In part (b), since $y = x(x - 2)(x + 2)$ and $y = x + 2$, the values -2; $-0{,}6$ and $2{,}4$ are the roots of the equation $x(x - 2)(x + 2) = x + 2$.

Remember that readings from graphs usually give approximate results only. A bigger scale in Fig. 15.2 would give more accurate results.

Example 3

(a) Draw the graph of $y = 2x^2(5 - x)$ in the range $x = 0$ to $x = 5$. Hence find (b) the maximum value of y in the given range, (c) the value of x when y is a maximum.

(a) The graph is drawn by plotting the values in Table 15.4.

Fig. 15.3 is the graph of $y = 2x^2(5 - x)$ within the given range.

Table 15.4

x	0	1	2	3	4	5
$2x^2$	0	2	8	18	32	50
$(5 - x)$	5	4	3	2	1	0
y	0	8	24	36	32	0

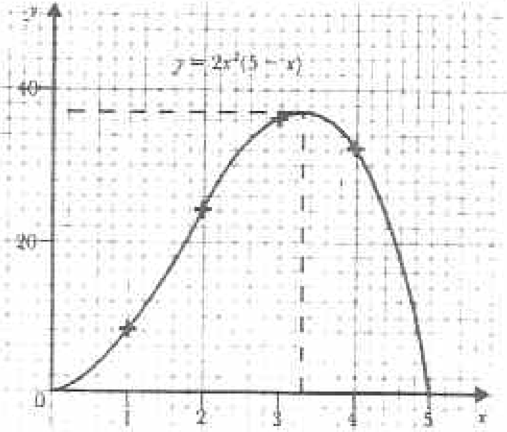


Fig. 15.3

(b) From Fig. 15.3, the maximum value of y is 37 (approximately).

(c) When $y = 37$, $x \simeq 3{,}3$.

Notice the following:

1 $y = 2x^2(5 - x) = 10x^2 - 2x^3$. Hence the curve in Fig. 15.3 is that of a cubic function in x. It does *not* have a line of symmetry.

2 Fig. 15.4 is a sketch graph of $y = 2x^2(5 - x)$ extended to include values outside the given range.

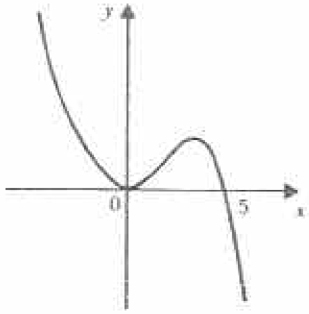


Fig. 15.4

It can be seen that y can have values greater than 37. The value of y found in part (b) is a maximum *for the given range* only.

Exercise 15a

1 (a) Draw the graph of $y = x^3$ for values of x from -4 to $+4$. (b) On the same axes, draw the graph of $y = -x^3$. (c) Use either graph to solve the equation $x^3 + 50 = 0$.

2 Solve the equation $x^3 = 5x + 2$ by drawing graphs of $y = x^3$ and $y = 5x + 2$ for values of x between -3 and $+3$.

3 (a) Given that $y = x^3 + 2x - 1$, copy and complete Table 15.5.

Table 15.5

x	-3	-2	-1	0	1	2	3
x^3	-27	-8	-1	0	1		
$+2x$	-6	-4	-2	0			
-1	-1	-1	-1				
y	-34	-13	-4				

(b) Use scales of 2 cm to 1 unit on the x-axis and 2 cm to 10 units on the y-axis and draw the graph of $y = x^3 + 2x - 1$. (c) Hence solve the equation $x^3 + 2x - 1 = 0$.

4 Solve graphically the equation $x^3 = 5x - 3$. *Hint: Either* use the method of question 2, drawing the graphs of $y = x^3$ and $y = 5x - 3$, *or* use the method of question 3, drawing the graph of $y = x^3 - 5x + 3$. In either case, use values of x in the range -3 to $+3$.

5 Solve the equation $x^3 + 3x - 7 = 0$ graphically. Use values of x in the range 0 to 4.

6 Given that $y = x(x - 3)(x + 3)$, copy and complete Table 15.6.

Table 15.6

x	-4	-3	-2	-1	0	1	2	3	$3\frac{1}{2}$
$x - 3$	-7	-6	-5	-4	-3				$\frac{1}{2}$
$x + 3$	-1	0	1	2	3				$6\frac{1}{2}$
y	-28	0	10	8					$11\frac{3}{8}$

(a) Using scales of 2 cm to 1 unit on the x-axis and 2 cm to 5 units on the y-axis, draw the graph of $y = x(x - 3)(x + 3)$.
(b) Use the graph to find the roots of the equation $x(x - 3)(x + 3) = 2x + 4$.

7 Given that $v = \frac{\pi x^2}{3}(30 - x)$, and taking the value of $\frac{\pi}{3}$ as approximately 1,05, complete Table 15.7, giving values correct to t nearest whole number.

Table 15.7

x	1	2	3	4	5
v	30		255		656

Draw the graph showing the variation of as x increases from 1 to 5. Hence find th value of x for which $v = 400$.

8 Draw on the same axes the graphs of $y =$ and $y = x(4x - 3)$ for values of x betwee 0 and 4. Use a scale of 2 cm to 1 unit o the x-axis and 2 cm to 10 units on the y-axi From your graph, (a) find the values of satisfying the equation $x^3 - 4x^2 + 3x =$ (b) find the range of values of x for whic $x^3 < x(4x - 3)$; (c) finds the gradient o the curve (i) $y = x^3$ at the point $x =$ (ii) $y = x(4x - 3)$ at $x = 2$.

9 A particle moves such that its distance, from its starting point after t seconds is give by $d = 12t - \frac{1}{4}t^3$. Draw the graph o $d = 12t - \frac{1}{4}t^3$ for values of t from 0 to 5. Us the graph to find (a) the greatest distanc that the particle is from its starting poin during the first 5 seconds, (b) the time tha it takes to return to the starting point.

10 A skeleton box on a square base of side x cm is made from 36 cm of wire (Fig. 15.5).

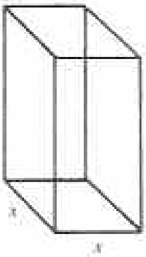


Fig. 15.5

(a) Find the height of the box in terms of x and hence show that its volume, v, is given by $v = x^2(9 - 2x)$.
(b) Draw the graph of $v = x^2(9 - 2x)$ for values of x from 0 to 4.
(c) Hence find (i) the maximum volume of the box, (ii) the dimensions of the box when the volume is a maximum.

Inverse functions

An **inverse function** of x is an expression in which x appears in the denominator of a fraction. For example, $\frac{6}{x}$ and $\frac{2x^2}{1-3x}$ are inverse functions of x.

Example 4

(a) Draw the graph of $y = \frac{6}{x}$ for values of x from -4 to $+2$. (b) Find the values of x at the point where the line $y = 2x + 3$ cuts the graph. (c) Of what equation in x are these values the roots?

(a) In Table 15.8, the values of y for integral values of x are first calculated. The extra values of y for $x = \pm 1\frac{1}{2}$ are then added.

Table 15.8

x	-4	-3	-2	-1	0	1	2	$\pm 1\frac{1}{2}$
$y = \frac{6}{x}$	$-1\frac{1}{2}$	-2	-3	-6		6	3	± 4

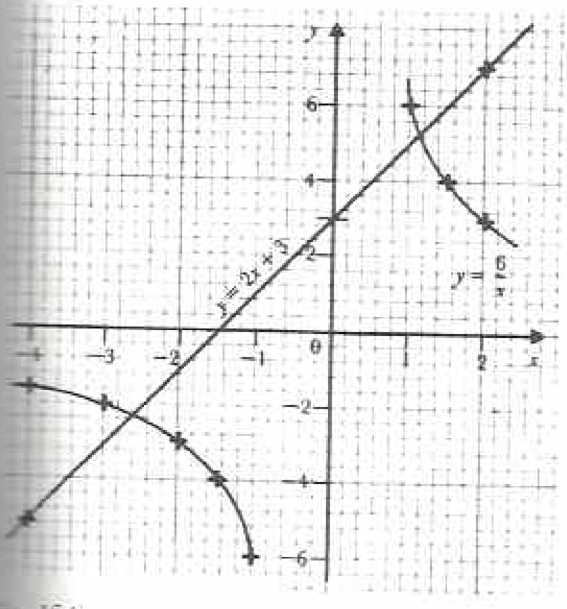


Fig. 15.6

Fig. 15.6 is the graph of $y = \frac{6}{x}$.

Notice that when $x = 0$, $\frac{6}{x}$ is undefined.

Notice that there is a break in continuity of the graph. It is in two branches which are separated by the axes. As x increases towards 0, y decreases in value; as x decreases towards 0, y increases in value.

This kind of curve is called a **hyperbola**.

(b) Draw the line $y = 2x + 3$ by plotting the points in Table 15.9.

Table 15.9

x	-4	0	2
y	-5	3	7

The line cuts the curve where $x \simeq -2{,}6$ and $x \simeq 1{,}1$.

(c) The curve and the line intersect where y simultaneously equals $\frac{6}{x}$ and $2x + 3$. Hence the values of x above are the roots of the equation: $\frac{6}{x} = 2x + 3$, i.e. $2x^2 + 3x - 6 = 0$.

Example 5

(a) Draw the graph of $y = 6x + \frac{20}{x}$ for values of x equal to $\frac{1}{2}$, 1, 2, 3, 4, 5. (b) Find the minimum value of y in the given range. (c) Find the corresponding value of x.

(a) In Table 15.10, values of y are calculated to the nearest whole number.

Table 15.10

x	$\frac{1}{2}$	1	2	3	4	5
$6x$	3	6	12	18	24	30
$\frac{20}{x}$	40	20	10	7	5	4
y	43	26	22	25	29	34

Fig. 15.7 is the required graph. (*Note:* to save space, the origin has not been included.)

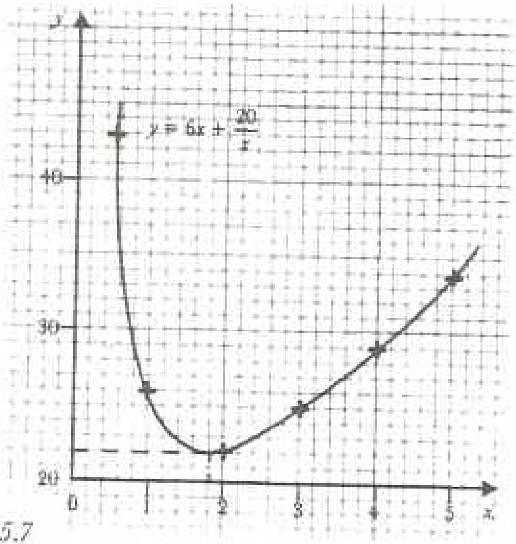


Fig. 15.7

(b) The minimum value of y is 21,9.
(c) The corresponding value of x is 1,8.

Fig. 15.8 is a sketch graph of $y = 6x + \frac{20}{x}$, showing values outside the range of Example 5.

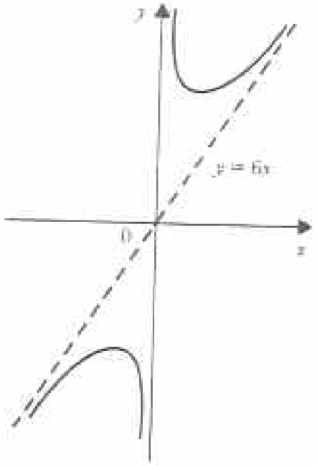


Fig. 15.8

It can be seen that y has values lower than 21,9 when $x < 0$.

Example 6

Solve the equation $2x^2 - x - 4 = 0$ *by drawing appropriate inverse and linear graphs within the same axes.*

Divide the given equation throughout by x.

$$2x - 1 - \frac{4}{x} = 0$$

Rearrange the resulting equation:

$$2x - 1 = \frac{4}{x}$$

Draw the graphs of $y = 2x - 1$ and $y = \frac{4}{x}$ on the same axes. See Fig. 15.9.

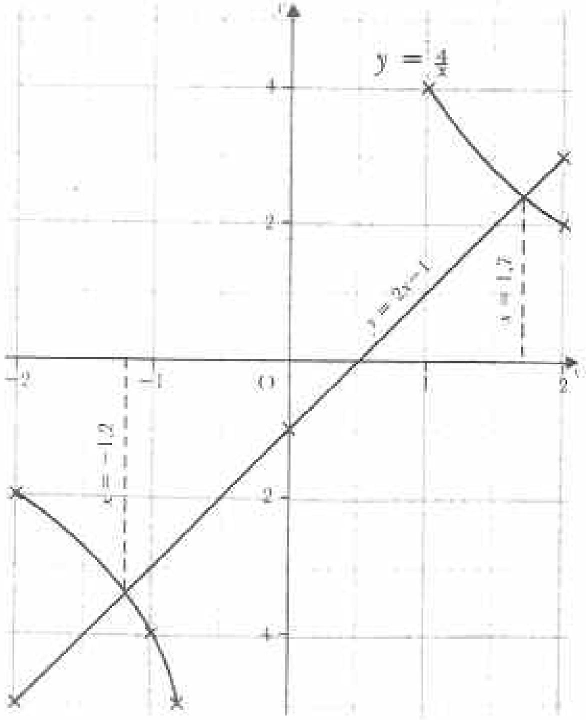


Fig. 15.9

The solution of the given quadratic equation is found by reading the values of x at the points of intersection of the graphs. From Fig. 15.9, the solutions of the equation are

$x = -1,2$ and $x = +1,7$

Exercise 15b

1 Draw the graphs of (a) $y = \frac{1}{x}$, (b) $y = \frac{1}{x^2}$ for values of x equal to ± 4, ± 2, ± 1, $\pm \frac{1}{2}$, $\pm \frac{1}{4}$.

2 Draw the graphs of $y = \frac{5}{x}$ and $y = x(x-1)$ from $x = \frac{1}{2}$ to $x = 5$. (a) Read off the value of x at the intersection of the curves. (b) Of what equation in x is this value a root?

3 (a) Given that $y = x^2 + \frac{3}{x}$, copy and complete Table 15.11.

Table 15.11

x	$\frac{1}{4}$	$\frac{1}{2}$	1	2	3	4
x^2	$\frac{1}{16}$	$\frac{1}{4}$	1	4	9	16
$\frac{3}{x}$	12	6	3	$1\frac{1}{2}$	1	
y	$12\frac{1}{16}$	$6\frac{1}{4}$	4	$4\frac{1}{2}$		

(b) Draw the graph of $y = x^2 + \frac{3}{x}$.

(c) Hence find the minimum value of y within the given range.

4 (a) Draw the graph of $y = x - \frac{5}{x}$ for values of x from -2 to $+4$.

(b) Hence solve the equation $x - \frac{5}{x} = 2$.

5 On the same axes, draw the graphs of $y = \frac{1}{x}$ and $y = 2x + 5$. Use the points of intersection of the graphs to solve the equation $2x^2 + 5x - 1 = 0$.

6 Solve the equation $x^2 + 7x + 4 = 0$ by drawing appropriate inverse and linear graphs within the same axes.

7 (a) Draw the graph of $y = x^2 + \frac{7}{x}$ for values of x from -4 to $+2$.

(b) Hence solve the equation $7 - x^2 = \frac{7}{x}$.

8 (a) Draw the graphs of $y = x + \frac{4}{x}$ and $y = x^2 - x$ for x equal to 1, $1\frac{1}{2}$, 2, $2\frac{1}{2}$, 3.

(b) Show that the intersections of these graphs satisfy the equation $x^3 - 2x^2 - 4 = 0$, and use the graph to find a root of this equation.

9 (a) Draw the graph of $y = x + \frac{1}{x}$ for values of x from 0,1 to 10.

(b) Use the graph to find approximately the roots of the equation $2x^2 - 9x + 2 = 0$. (*Hint:* Divide each term in the equation by $2x$.)

10 Copy and complete the following table of values for $y = x + \frac{25}{x}$ for $2 \geqslant x \geqslant 13$.

Table 15.12

(Values are rounded to one place of decimals.)

x	2	3	4	5	6	7	8	9	10	11	12	13
y	14,5			10,3		10,6	11,1		12,5		14,1	

Draw the graphs of $y = x + \frac{25}{x}$ and $2y = x + 15$, using the same axes, and a scale of 2 cm to 2 units on each axis. From the graphs, find:

(a) the values of x for which

$$2x + \frac{50}{x} = x + 15$$

(b) the gradient of the curve at $x = 3$.

11 Complete Table 15.13, which gives corresponding values of x and y for which

$$y = 30 - 3x - \frac{60}{x}$$

Table 15.13

x	2	$2\frac{1}{2}$	3	4	5	6	7	8	9	10
y	-6	$-1\frac{1}{2}$		3			0,4		$-3,7$	

Hence draw the graph of $y = 30 - 3x - \frac{60}{x}$.

(a) Read off the greatest value of y.

(b) To what value of x does this value of y correspond?

12 Copy and complete Table 15.14, overleaf, and use it to draw the graph of $y = \frac{3x}{x^2 + 1}$. Use a scale of 2 cm to 1 unit along the x-axis and 2 cm to $\frac{1}{2}$ unit along the y-axis.

Table 15.14

x	-3	$-2\frac{1}{2}$	-2	$-1\frac{1}{2}$	-1	$-\frac{1}{2}$
y	$-0,9$		$-1,2$		$-1,5$	

x	0	$\frac{1}{2}$	1	$1\frac{1}{2}$	2	$2\frac{1}{2}$	3
y	0	$1,2$		$1,38$		$1,03$	

Using the same axes, draw the graph of $y = \frac{2x}{3} + \frac{1}{3}$. From your graph find (a) the three roots of the equation $\frac{3x}{x^2+1} = \frac{2x}{3} + \frac{1}{3}$, (b) the maximum value of $\frac{3x}{x^2+1}$ within the given range, (c) the gradient of the curve at $x = -\frac{1}{2}$.

Sketch graphs

A **sketch graph** is a simple freehand drawing which shows the main features of a line or curve. Some examples of sketch graphs were given in Chapter 5.

Linear functions

Example 7

Sketch the graph of $2x - 3y = 24$.

Method: Find the **intercepts** on the axes; i.e. the positions where $x = 0$ and $y = 0$.

$2x - 3y = 24$.

When $x = 0$, $-3y = 24$

$y = -8$

When $y = 0$, $2x = 24$

$x = 12$

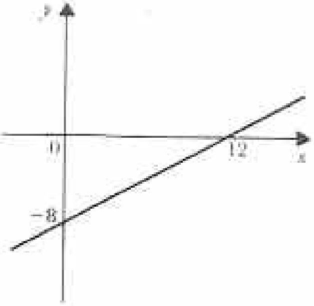


Fig. 15.10

The sketch graph can now be drawn (Fig. 15.1(

Always label the axes and origin. If possib

show where the line crosses the axes.

Example 8

Find the equation of the line represented by the ske graph in Fig. 15.11.

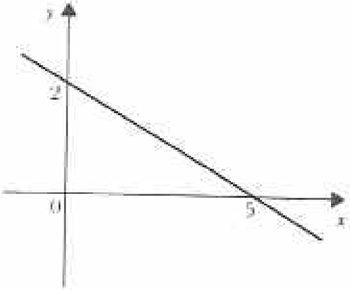


Fig. 15.11

Since the graph is a straight line, its equatic is of the form $y = mx + c$, where m is tl gradient of the line and c is the intercept on tl y-axis. From Fig. 15.11, $m = -\frac{2}{5}$*

$c = +2$

The equation is $y = -\frac{2}{5}x + 2$

i.e. $5y = -2x + 2$

or $2x + 5y = 2$

* As x increases by 5 units, y decreases by 2 unit

Quadratic functions

A quadratic function has an equation in th form $y = ax^2 + bx + c$, where a, b and c ar positive or negative constants. When a, th coefficient of x^2, is positive, the graph is cup-shaped parabola. When a is negative, th graph is a cap-shaped parabola (Fig. 15.12).

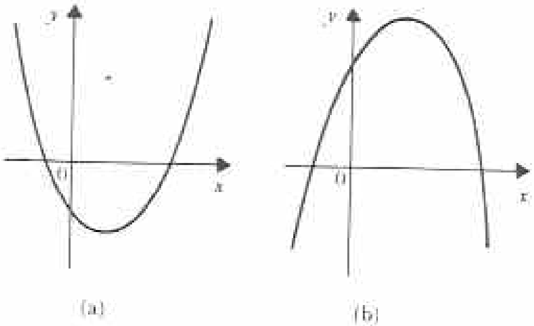


Fig. 15.12

Example 9

Sketch the graph of $y = x^2 - x - 12$, showing where the curve cuts the axes.

$y = x^2 - x - 12$

1 The curve cuts the y-axis when $x = 0$.
When $x = 0$, $y = -12$

2 The curve cuts the x-axis when $y = 0$.
When $y = 0$, $x^2 - x - 12 = 0$

$$\Leftrightarrow (x-4)(x+3) = 0$$
$$\Rightarrow x = 4 \text{ or } -3$$

3 The coefficient of x^2 is positive, hence the curve is a cup-shaped parabola.

The sketch in Fig. 15.13 is drawn using the data in *1*, *2* and *3* above.

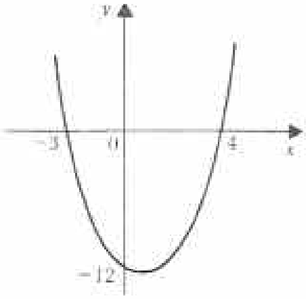


Fig. 15.13

Example 10

Find the equation of the curve represented by the sketch in Fig. 15.14.

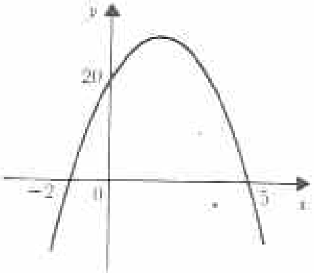


Fig. 15.14

Let the curve have the equation $y = ax^2 + bx + c$.

1 c is the intercept on the y-axis.
Hence, from Fig. 15.14, $c = +20$.

2 When $y = 0$, $ax^2 + bx + c = 0$. Hence from the intercepts on the x-axis in Fig. 15.14, the roots of the equation $ax^2 + bx + c = 0$ are -2 and $+5$.

$(x+2)(x-5) = 0$

$x^2 - 3x - 10 = 0 \qquad (1)$

Since $c = +20$, multiply each term in (1) by -2.

$-2x^2 + 6x + 20 = 0$

Hence $y = -2x^2 + 6x + 20$ is the required equation.

Notice that the coefficient of x^2 is negative an
that Fig. 15.14 shows a cap-shaped parabola

Inverse functions

Fig. 15.15 shows the graphs of (a) $y = \frac{1}{x+}$

and (b) $y = \frac{1}{x-3}$.

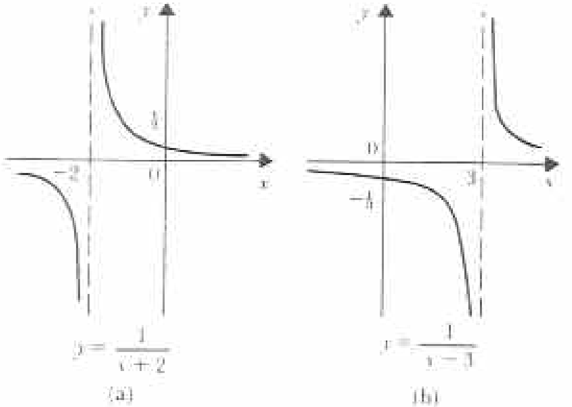


Fig. 15.15

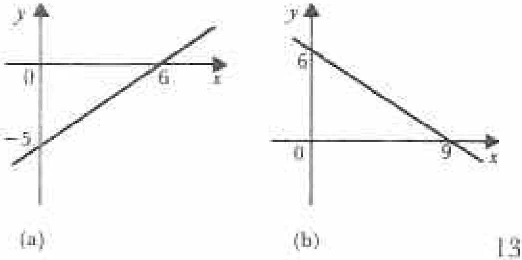
In each graph, the continuity of the curve
broken at the line $x = k$, where k is the valu
of x for which the fraction is undefined. Th
intercept on the y-axis is found by substitutin
$x = 0$ in the given equation. Neither curve cu
the x-axis.

Exercise 15c

Draw freehand graphs throughout this exercis
Whenever possible, show where the graph cu
the axes.

1 Sketch the graphs of the following.
(a) $3x - 2y = 12$ (b) $6x + 3y = 18$
(c) $y - 2x = 7$ (d) $y = 3x - 1$

2 Find the equations of the lines shown b
the sketch graphs in Fig. 15.16(a)–(d).



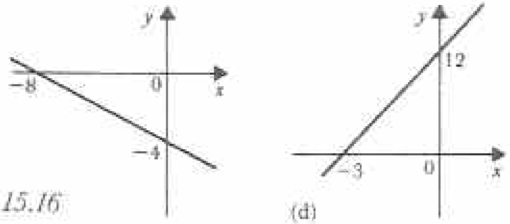


Fig. 15.16

3 What is the value of y at the point where the curve $y = x^2 + 3x - 11$ cuts the y-axis?

4 Sketch the graphs of the following, showing where the curve cuts the axes:

(a) $y = x^2 - 5x + 4$
(b) $y = 15 - 2x - x^2$
(c) $y = x^2 - 12x + 36$
(d) $y = 16 - x^2$
(e) $y = 3x^2 + 3x - 6$
(f) $y = 10 - 8x - 2x^2$

5 Fig. 15.17 is a sketch graph of $y = 6 + x - x^2$.

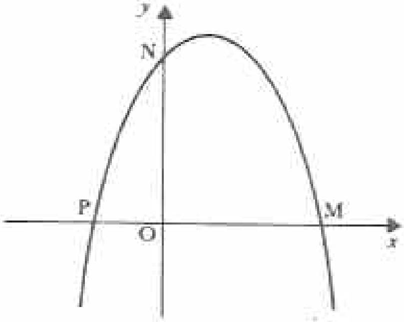


Fig. 15.17

(a) What is the value of x at M?
(b) Find the tangent of the angle OMN.

6 What equation is represented by the sketch graph in Fig. 15.18?

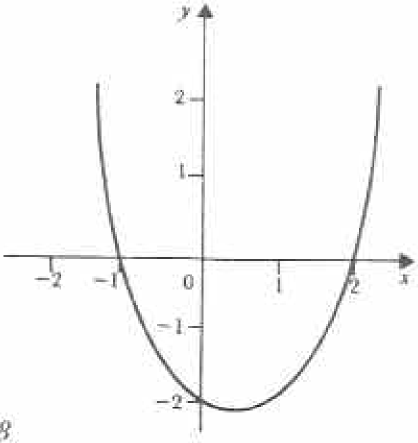


Fig. 15.18

7 What equations are represented by the sketch graphs in Fig. 15.19?

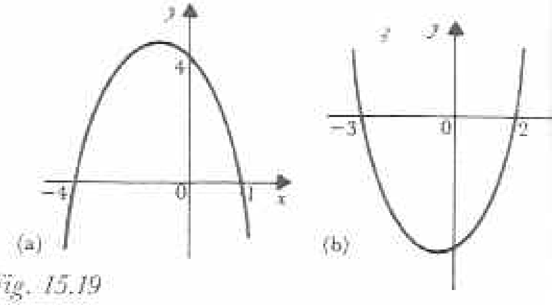


Fig. 15.19

8 Sketch the graphs of the following.

(a) $y = \dfrac{1}{x+1}$ (b) $y = \dfrac{1}{x-5}$

9 Find the equations of the curves shown by the sketch graphs in Fig. 15.20.

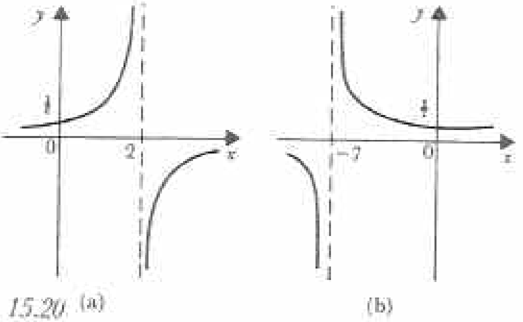


Fig. 15.20 (a) (b)

10 Find the equations of the line, l, the parabola, p, and the hyperbola, h, shown in the sketch graph in Fig. 15.21.

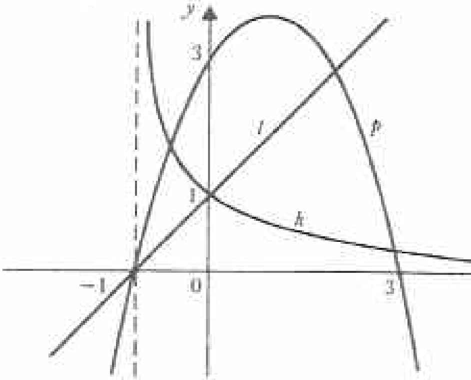


Fig. 15.21

Chapter 16

Statistics (5) Frequency distributions, histograms, cumulative frequency

Bar charts (revision)

A **bar chart** is a statistical graph in which bars are drawn such that their lengths or heights are proportional to the quantities they represent.

Example 1

Table 16.1 shows the number of schools in twelve towns.

Table 16.1

no. of schools	4	5	6	7	8
no. of towns	1	4	2	3	2

(a) *Draw a bar chart to illustrate the information.*
(b) *State the mode and median of the distribution.*
(c) *Calculate the mean of the distribution to the nearest whole number.*

(a) Fig. 16.1 is a bar chart showing the information in Table 16.1.

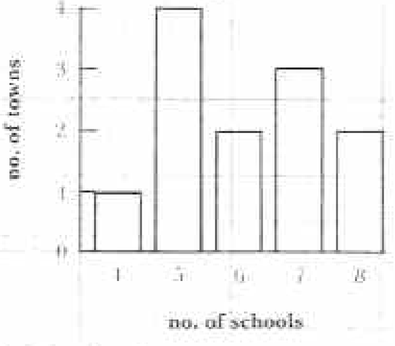


Fig. 16.1

(b) The mode is 5 schools (i.e. the most frequent number of schools; corresponding to the highest bar in Fig. 16.1). The median is 6 schools (i.e. the central value when the number of schools are arranged in order: 4 5 5 5 5 <u>6 6</u> 7 7 7 8 8).

(c) Mean number of schools per town

$$= \frac{\text{total number of schools}}{\text{total number of towns}}$$

$$= \frac{1 \times 4 + 4 \times 5 + 2 \times 6 + 3 \times 7 + 2 \times 8}{12}$$

$$= \frac{4 + 20 + 12 + 21 + 16}{12}$$

$$= \frac{63}{12} = 5\tfrac{1}{4}$$

= 5 to the nearest whole number.

The number of times any particular number occurs is called its **frequency**. In Example 1 3 towns have 7 schools; 3 is the frequency of 7 schools.

Exercise 16a (revision)

1 Fig. 16.2 is a bar chart showing the numbers of hours of each kind of programme broadcast by a radio station on a certain day.
(a) Which kind of programme was given most time?

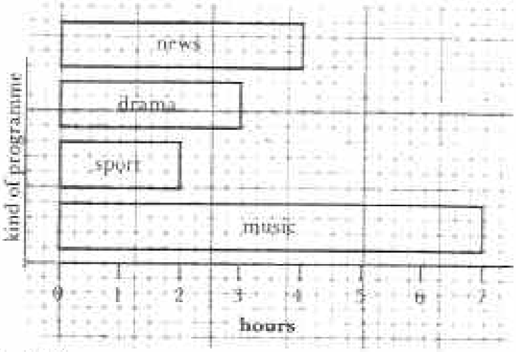


Fig. 16.2

(b) How many hours were given to drama?

(c) For how many hours did the radio station broadcast?

(d) What fraction of the programme time was given to news?

2 Fig. 16.3 is a bar chart showing the numbers of kilometres a group of 75 students walked in a sponsored march.

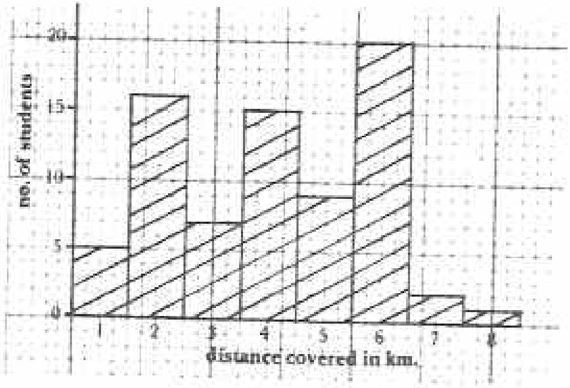


Fig. 16.3

(a) What is the mode of the distances covered?

(b) How many students covered less than 4 km?

(c) What is the total distance covered by all the students?

(d) If the students were listed in order according to the distances covered, how far would the student in the median position have walked?

3 Table 16.2 shows the numbers of seeds in 40 groundnuts.

Table 16.2

number of seeds	1	2	3	4
frequency	8	21	10	1

(a) Draw a bar chart to illustrate Table 16.2.

(b) State the mode and median number of seeds.

(c) Calculate the mean number of seeds per groundnut.

4 Table 16.3 shows the distribution of marks in a test.

Table 16.3

marks	40	41	42	43	44	45	46
frequency	7	4	6	2	4	2	6

(a) Draw a bar chart to show the distribution.

(b) How many people took the test?

(c) Find the median mark.

5 Table 16.4 shows the grades, in percentages, of 200 students in a test.

Table 16.4

grades (%)	10	20	30	40	50	60	70	80	90	100
no. of students	12	16	20	25	28	32	31	24	8	4

(a) Draw a bar chart for this distribution of grades.

(b) Find the mean of the distribution.

(c) Find the mode and the median of these grades.

Grouped data

Frequency distributions

When statistical data contain a large number of values, it is impractical to draw a bar chart and often difficult to calculate averages. For example, Table 16.5 shows the weekly pay of 50 people.

Table 16.5

weekly pay of 50 people ($)				
82	132	199	248	300
89	145	200	249	324
94	152	206	255	334
96	156	206	263	348
98	158	214	265	369
108	163	220	270	381
114	176	221	270	401
120	178	232	280	440
125	185	235	288	477
128	189	247	294	485

A bar chart of this data, if drawn, would contain 46 bars of height 1 unit and 2 bars of height 2 units (corresponding to the incomes \$206 and \$270). Such a graph would be difficult to draw and the result would show no pattern.

To overcome this problem, the data can be reduced to a **frequency distribution**. A frequency distribution is a table in which the given values are divided into **class intervals**. The number of values in each class interval is given as the **frequency** of the values in that interval. For example, the data in Table 16.5 can be grouped in equal class intervals of \$100 to give the frequency distribution shown in Table 16.6.

Table 16.6

class interval (weekly pay \$)	frequency (number of people)
0– 99	5
100–199	16
200–299	19
300–399	6
400–499	4

It is necessary to define the limits of the class intervals very clearly. Otherwise it may be difficult to decide the class in which to include borderline values, such as \$199 and \$200.

Exercise 16b

1 Make a frequency distribution of the data in Table 16.5 taking equal class intervals \$1–\$100, \$101–\$200, \$201–\$300, etc. Compare this distribution with that of Table 16.6.

2 Make a frequency distribution of the data in Table 16.5 by taking ten equal class intervals \$0–\$49, \$50–\$99, ..., \$450–\$499.

3 Fifty students were asked to estimate the size of an angle to the nearest degree. Their results, arranged in order of size, are given in Table 16.7.

Make a frequency distribution table, taking six equal intervals, 55–59, 60–64, 65–69, ..., 80–84.

Table 16.7

estimations (degrees)									
56	58	58	60	60	61	62	63	64	64
65	65	65	66	66	66	66	67	67	68
68	68	68	69	69	69	70	70	70	70
70	72	72	72	72	73	73	74	74	74
75	75	75	76	78	79	80	80	81	83

Histogram

A frequency distribution can be represented by a block graph called a **histogram**. Figure 16.4 is the histogram of the frequency distribution in Table 16.6.

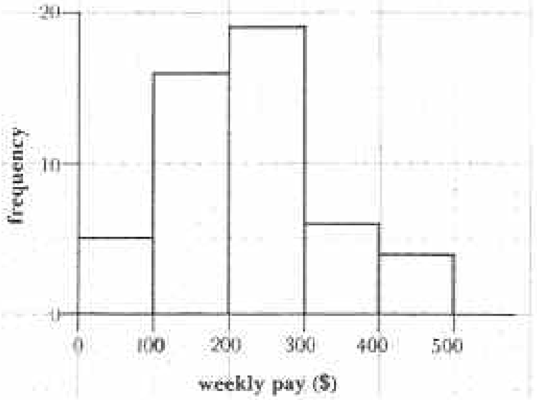


Fig. 16.4

A histogram consists of a number of rectangles. The horizontal width of each rectangle is given by the class interval.[1] The height is such that the area of the rectangle is proportional to the frequency in that interval.[2] Hence the areas of the rectangles show the frequency distribution.

Notes:

1 Since the true sizes of the class intervals are 0–99, 100–199, etc., there should be very small gaps between the rectangles. In practice these gaps are closed to give a continuous horizontal axis.

2 For a histogram with class intervals of equal widths, the vertical scale is proportional to the frequency of the distribution.

In Fig. 16.4, the **modal class** is 200–299, since this interval corresponds to the rectangle with the greatest area.

Frequency polygon

Grouped data can also be shown on a **frequency polygon**. Fig. 16.5 is a frequency polygon of the data shown in the histogram in Fig. 16.4.

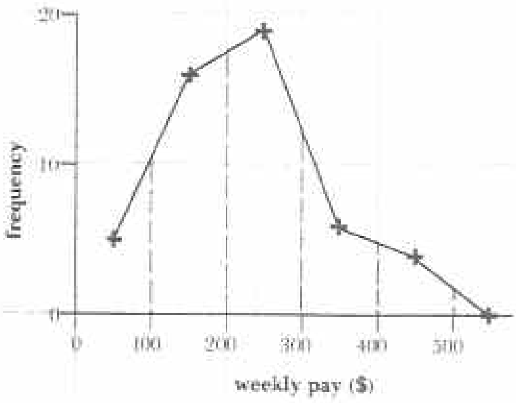


Fig. 16.5

In a frequency polygon, the frequencies are plotted at the *mid-points* of each class interval. The points are joined by straight lines.

Example 2

(*a*) *Draw a histogram and frequency polygon for the frequency distribution given in Table 16.8.*

Table 16.8

class interval	1–5	6–10	11–15	16–20	21–25
frequency	3	5	7	6	4

(*b*) *Calculate the mean of the distribution.*

(a) Fig. 16.6(a) and (b) shows the histogram and frequency polygon.

Note: The gaps between the intervals are 1 unit. These are too large to be ignored when drawing this histogram. To close the gaps, the widths of the rectangles are increased by $\frac{1}{2}$ unit on both

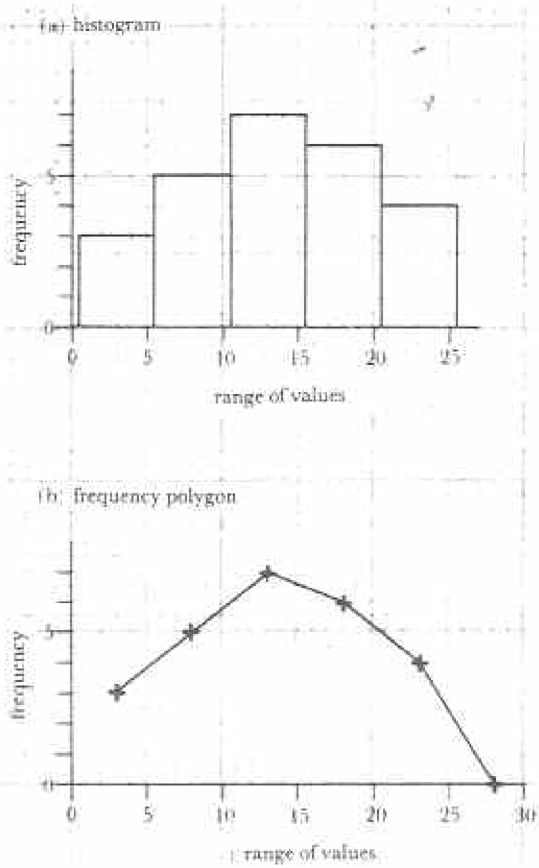


Fig. 16.6

sides. Compare this with Fig. 16.4 where the gaps were so small that they could be ignored.

(b) Let the values in each class interval be represented by the mid-value of that class. Hence, all values in the class 1–5 are counted as 3, all values in the 6–10 class are counted as 8, and so on.

Mean value

$$= \frac{3 \times 3 + 5 \times 8 + 7 \times 13 + 6 \times 18 + 4 \times 23}{3 + 5 + 7 + 6 + 4}$$

$$= \frac{9 + 40 + 91 + 108 + 92}{25}$$

$$= \frac{340}{25}$$

$$= 13.6$$

The assumption in part (b) of Example 2 leads to some inaccuracy. However, when there is a large number of values, the error is likely to be very small and can be ignored.

Example 3
Table 16.9 gives the marks of 50 students in a test.

Table 16.9

35	51	83	60	61	73	44	90	70	93
56	34	52	61	43	57	40	58	88	64
52	71	25	86	79	35	73	44	71	95
63	53	48	78	65	98	28	72	67	82
46	54	62	35	70	41	63	73	50	68

(*a*) *Construct the histogram, taking class intervals 21–30, 31–40, ..., 91–100.* (*b*) *What is the modal class?* (*c*) *Find the mean mark.*

(a) When the data are not in numerical order, use a tally system to count the frequencies. Take each mark in turn and enter a tally stroke against the proper class interval. The frequency total, 50, gives a check on the accuracy of working. The resulting frequency distribution is given in Table 16.10.

Table 16.10

class	frequency	
21–30	||	2
31–40	卌	5
41–50	卌 ||	7
51–60	卌 ||||	9
61–70	卌 卌 |	11
71–80	卌 |||	8
81–90	卌	5
91–100	|||	3
	total	50

Fig. 16.7 is the histogram of the distribution. Notice that class intervals of $20\frac{1}{2}$–$30\frac{1}{2}$, $30\frac{1}{2}$–$40\frac{1}{2}$, ..., have been drawn in Fig. 16.7. This removes the gaps between the bars.

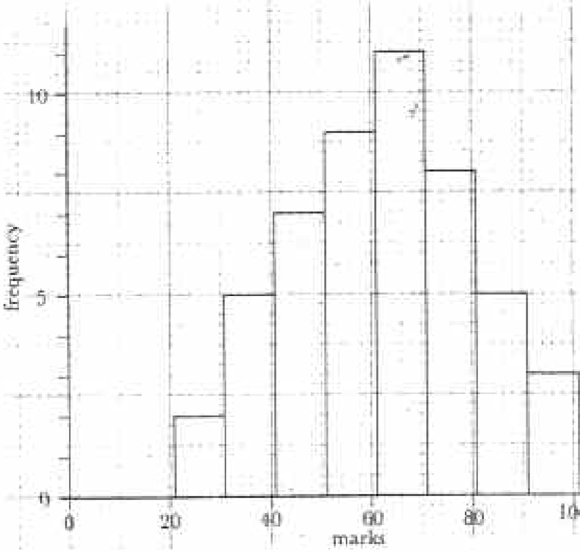


Fig. 16.7

(b) The modal class is 61–70. This can be seen in both the frequency distribution table and in the histogram.

The mode of the data is taken to be the central value of the modal class, 61–70, i.e. $65\frac{1}{2}$.

(c) To find the mean mark, choose a working mean and find the deviations from it.

In this example, $65\frac{1}{2}$ is taken as the working mean and the marks in each class interval are represented by the mid-mark of that class. For example, marks in the class interval 21–30 are counted as $25\frac{1}{2}$, and so on. The working is set out in columns as in Table 16.11.

Table 16.11

class interval	class centre	frequency (f)	deviation (d)	($f \times d$)
21–30	$25\frac{1}{2}$	2	−40	−80
31–40	$35\frac{1}{2}$	5	−30	−150
41–50	$45\frac{1}{2}$	7	−20	−140
51–60	$55\frac{1}{2}$	9	−10	−90
61–70	$65\frac{1}{2}$	11	0	0
71–80	$75\frac{1}{2}$	8	+10	+80
81–90	$85\frac{1}{2}$	5	+20	+100
91–100	$95\frac{1}{2}$	3	+30	+90
			total deviation	−190

From Table 16.11,
total deviation from working mean = −190

$$\text{mean deviation} = \frac{-190}{50}$$

$$= -3{,}8$$

$$\text{mean mark} = 65{,}5 - 3{,}8$$

$$= 61{,}7$$

As in Example 2, the final result in part (c) is likely to be slightly inaccurate. (In fact, the true mean of the marks in Table 16.9 is 61,2.) Nevertheless, this method should be used when a large number of values is given.

Exercise 16c

1 Draw a histogram and a frequency polygon for the frequency distribution in Table 16.12.

Table 16.12

class	1–5	6–10	11–15	16–20	21–25
frequency	2	4	6	5	3

State the modal class of the distribution.

2 Draw a histogram of the frequency distribution in Table 16.13.

Table 16.13

class	1–5	6–10	11–15	16–20	21–25
frequency	4	6	11	6	3

Calculate the mean of the data.

3 Draw a histogram of the data in Table 16.14.

Table 16.14

class	1–10	11–20	21–30	31–40	41–50
frequency	5	12	17	10	6

Estimate the mode of the data.

4 Draw a histogram and frequency polygon of the frequency distribution in Table 16.15.

Table 16.15

class	8–14	15–21	22–28	29–35	36–42	43–49
frequency	3	5	8	18	9	7

Find the mode and the mean of the data.

5 Table 16.16 shows the heights of plants to the nearest 5 cm together with the corresponding numbers of plants.

Table 16.16

height in cm	20	25	30	35	40	45
no. of plants	5	3	1	1	3	2

Draw a histogram to illustrate this informa-tion. How many plants have heights greate than the mode by more than 10 cm?

6 Use the frequency distribution constructe for question 2, Exercise 16b, to estimate th mode of the data in Table 16.5.

7 Use the frequency distribution table con-structed for question 3, Exercise 16b, t calculate the mean of the data in Table 16.7

8 Students taking a teacher-training cours are grouped by age as in Table 16.17.

Table 16.17

age group	19–20	20–21	21–22	22–23	23–24	24–25
number in group	4	5	10	16	12	3

Calculate the average age of the students.

9 Table 16.18 shows the numbers of absentees recorded each day of a school term.

Table 16.18

number absent	0–9	10–19	20–29	30–39	40–49	50–59
frequency	5	18	25	17	14	1

Calculate the average number of absentees per day.

10 The percentage marks of 100 students in a School Certificate examination are grouped as in Table 16.19.

Table 16.19

percentage	0–9	10–19	20–29	30–39	40–49
frequency	1	2	5	17	23

percentage	50–59	60–69	70–79	80–89	90–99
frequency	25	18	5	3	1

(a) Estimate the number of students who scored 15% less than the modal mark.
(b) Find the average percentage for the examination.

11 Small nails are sold in packets which have printed on them, 'Average contents 200 nails'. The contents of 100 packets, picked out at random, are counted and the results are given in Table 16.20.

Table 16.20

nails per packet	185–189	190–194	195–199	200–204	205–209	210–214
frequency	4	14	32	28	17	5

Is the statement on the packet true or not?

12 Table 16.21 gives the masses, in kg, of 30 students.

Table 16.21

43	45	50	47	51	58	52	47	42	54
61	50	45	55	57	41	46	49	51	50
59	44	53	57	49	40	48	52	51	48

(a) Taking class intervals 40–44, 45–49, ..., construct the frequency distribution of the data.
(b) Draw a histogram of the data.
(c) Calculate the mean mass of the students.

13 Table 16.22 gives the heights, in cm, of the 30 students in question 12.

Table 16.22

145	163	149	152	166	156	159	139	145	141
150	158	150	149	143	159	154	167	146	147
152	162	144	169	162	150	175	160	167	171

(a) Take class intervals 135–144, 145–154, ..., and construct the table of frequencies.
(b) Calculate the mean height of the students.

14 Table 16.23 gives the masses, in kg, of 50 international athletes.

Table 16.23

67	75	79	56	59	60	64	76	58	80
54	65	78	66	65	65	70	62	70	62
70	61	83	51	74	69	59	73	71	74
73	81	69	82	71	53	67	72	66	70
85	63	58	69	75	61	62	68	52	68

Taking class intervals of 51–55, 56–60, ..., 81–85, construct (a) the frequency distribution, (b) a histogram, to show this information. (c) Choose a suitable working mean and hence find the average mass of the athletes.

15 Table 16.24 gives the diameters, to the nearest millimetre, of 90 tins.

Table 16.24

diameter (mm)	<70	<80	<90	<95	<100	<105	<110	<120	<130
no. of tins	0	4	13	25	41	63	78	86	90

(a) State the limits between which the diameters of the 4 tins whose diameters are given as less than 80 mm must lie.
(b) Construct a frequency table showing the number of tins in each group.
(c) Draw a histogram to illustrate these frequencies.
(d) State which is the modal group and use your histogram to obtain an estimate of the actual mode.

Example 4

Find the median of the marks given in Table 16.9 on page 137.

Since there are 50 marks, the median is the mean of the 25th and 26th marks when all the marks are arranged in order of size. Table 16.25 gives the marks arranged in order.

Table 16.25

1	25	*11*	44	*21*	57	*31*	67	*41*	78
2	28	*12*	46	*22*	58	*32*	68	*42*	79
3	34	*13*	48	*23*	60	*33*	70	*43*	82
4	35	*14*	50	*24*	61	*34*	70	*44*	83
5	35	*15*	51	*25*	61	*35*	71	*45*	86
6	35	*16*	52	*26*	62	*36*	71	*46*	88
7	40	*17*	52	*27*	63	*37*	72	*47*	90
8	41	*18*	53	*28*	63	*38*	73	*48*	93
9	43	*19*	54	*29*	64	*39*	73	*49*	95
10	44	*20*	56	*30*	65	*40*	73	*50*	98

$$\text{Median} = \frac{61 + 62}{2} = 61.5$$

Cumulative frequency

In Example 4 it was very time-consuming to arrange so many marks in order of size.

To save time and to avoid making errors it is more usual to make a **cumulative frequency table** and to draw a **cumulative frequency curve**.

Table 16.26 is a cumulative frequency table of the data in Table 16.9.

Table 16.26

class interval	frequency	cumulative frequency
21– 30	2	2
31– 40	5	5 + 2 = 7
41– 50	7	7 + 7 = 14
51– 60	9	9 + 14 = 23
61– 70	11	11 + 23 = 34
71– 80	8	8 + 34 = 42
81– 90	5	5 + 42 = 47
91–100	3	3 + 47 = 50

Table 16.26 shows that
2 students scored 30 marks or less,
7 students (=5 + 2) scored 40 marks or less,
and so on.

Each number in the third column is found by adding the number in the second column to the previous total. This progressive increase in the total is what is meant by the word *cumulative*.

Cumulative frequency curve (ogive)

The data in Table 16.26 can be illustrated by plotting the cumulative frequencies against the corresponding upper limits of the class intervals. The points are joined by a smooth curve called an **ogive**.

Fig. 16.8 shows the ogive, or cumulative frequency curve, for the data which were given in Table 16.26.

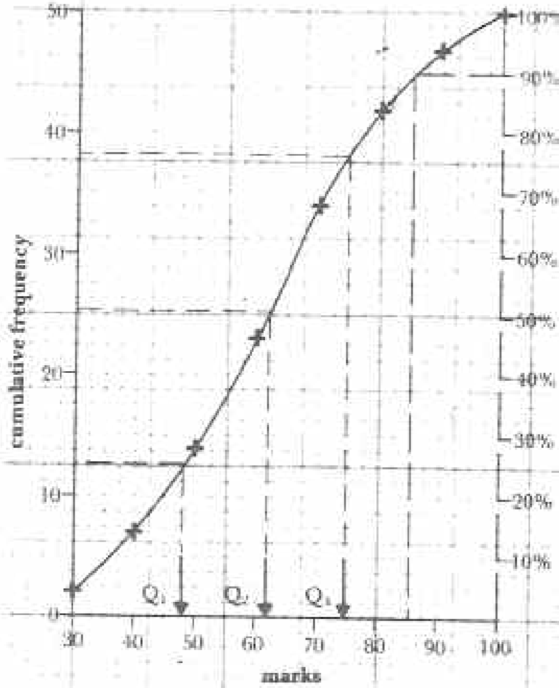


Fig. 16.8

Median and quartiles

The median is the mark that corresponds to the middle student. In Fig. 16.8 it can be seen that the $25\frac{1}{2}$th student gets a mark of 62. This estimate is reasonably close to the result (61.5) obtained in Example 4.

Just as the median is half-way up the distribution, the **lower quartile** is one-quarter of the way up and the **upper quartile** is three-quarters of the way up. If the total frequency is n, then the lower quartile is the value of the $\frac{1}{4}(n+1)$th item and the upper quartile is the value of the $\frac{3}{4}(n+1)$th item. Hence, in Fig. 16.8, the quartiles are at $12\frac{3}{4}$ and $38\frac{1}{4}$ on the cumulative frequency axis. These correspond to marks of 48 and 75 respectively.

The lower quartile is usually called Q_1.
The second quartile is the median, Q_2.
The upper quartile is usually called Q_3.
The **semi-interquartile range**, Q, is defined as

$$Q = \frac{Q_3 - Q_1}{2}$$

In Fig. 16.8, $Q_3 = 75$ and $Q_1 = 48$, hence

$$Q = \frac{75 - 48}{2} = \frac{27}{2} = 13\tfrac{1}{2}$$

This shows that about half of the students scored within $13\frac{1}{2}$ marks of the median. Hence the semi-interquartile range gives a measure of the spread of the distribution.

Percentiles

Using the right-hand vertical axis in Fig. 16.8, it can be seen that:
the lower quartile is about* 25% of the way up;
the median is about 50% of the way up;
the upper quartile is about 75% of the way up.
These are sometimes called the 25th, 50th and 75th **percentiles**. For example, the 90th percentile corresponds to a mark of 85. This means that 90% of the students scored 85 marks or less, or that 10% of the students scored over 85 marks.

* *Note:* In this example the lower quartile is actually about $25\frac{1}{2}$% of the way up. However, when the total frequency is high, the first, second and third quartiles coincide with the 25th, 50th and 75th percentiles.

Example 5

In Example 3 it is given that students who score over 45 marks pass the test. Use Fig. 16.8 to estimate the percentage of students that passed.

In Fig. 16.8, 45 marks is at the 21st percentile. This means that 21% of the students scored 45 marks or less.
Percentage scoring over 45 marks
= 100% − 21% = 79%
Percentage of students passing = 79%

Note: From Table 16.25 it can be seen that 39 students, or 78%, actually scored over 45 marks. Hence the estimate of 79% taken from the cumulative frequency curve is reasonably accurate.

Exercise 16d

1 In the test in Example 3, the students are *graded according to the marks scored as in* Table 16.27.

Table 16.27

marks scored	0–50	51–70	71–90	91–100
grade	re-sit	pass	credit	distinction

Use Fig. 16.8 to estimate,
(a) the percentage of students required to re-sit,
(b) the number of students that obtained a pass grade,
(c) the percentage of students awarded a distinction.

2 Draw a cumulative frequency curve of the data in Table 16.24 on page 139. Hence estimate
(a) the median diameter of the tins,
(b) the semi-interquartile range.

3 Table 16.28 shows the numbers of students who scored marks within 10-mark class intervals in a test.

Table 16.28

marks (class intervals)	1–10	11–20	21–30	31–40	41–50
number of students	2	7	9	11	13

marks (class intervals)	51–60	61–70	71–80	81–90	91–100
number of students	16	16	15	8	3

(a) Make a cumulative frequency table and hence draw an ogive showing the mark distribution.
(b) Estimate the median and upper and lower quartiles.
(c) Calculate the semi-interquartile range for the test.
(d) If any mark over 45 is a pass, estimate the percentage of students that passed.

4 Table 16.29 gives the mark distribution in a test.

Table 16.29

class interval	11–20	21–30	31–40	41–50	51–60
frequency	3	17	60	48	27

class interval	61–70	71–80	81–90	91–100
frequency	20	13	8	4

(a) Draw a cumulative frequency curve for the test.
(b) Find its median and semi-interquartile range.
(c) Estimate the percentage of the candidates that obtained more than 56 marks.
(d) Which mark is at the 70th percentile?

5 Table 16.30 shows the lives in hours, to the nearest hour, of 50 electric light bulbs.

Table 16.30

563	608	607	632	590	621	614	576	602	582
599	624	580	595	582	581	605	584	596	562
599	598	596	626	596	617	615	589	556	603
594	589	617	560	610	630	571	592	610	597
616	594	622	597	576	595	601	600	592	638

(a) Make a frequency distribution by grouping the values in Table 16.30 in 10-hour class intervals. Draw a histogram of the distribution and hence estimate the mode.
(b) By drawing a cumulative frequency curve, estimate the median and 80th percentile of the distribution.

6 36 girls were given a test in which the maximum mark available was 100. Table 16.31 shows the cumulative frequency of the results obtained.

Table 16.31

mark	10	20	30	40	50	60	70	80	90	100
number of girls scoring this mark or less	1	4	8	16	24	29	32	34	35	36

(a) Calculate how many girls scored a mark between 61 and 70 inclusive.
(b) Using a vertical scale of 2 cm to represent 5 girls and a horizontal scale of 1 cm to represent 10 marks, plot these values on graph paper and draw a smooth curve through your points.
(c) Showing your method clearly, use your graph to estimate the median mark.
[Camb]

Chapter 17

Inequalities (3) Linear programming

Solution of inequalities

Example 1

$y - x \leqslant 1$; $2x < 5$; $5y > -4x$ are simultaneous inequalities. (a) Show on a graph the region which contains the solution set of the inequalities. (b) If the solution set contains integral values of x and y only, list its members.

(a) The boundary lines of the region are
p: $y - x = 1$
q: $2x = 5$
r: $5y = -4x$

The unshaded region in Fig. 17.1 gives the solution set of all points $(x; y)$ which satisfy the three inequalities.

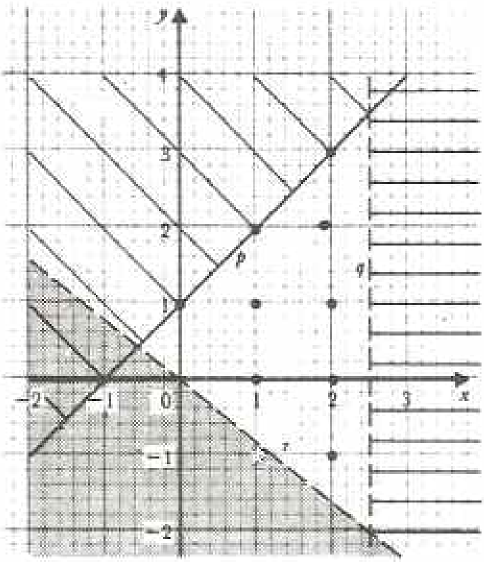


Fig. 17.1

(b) In Fig. 17.1 the solution set is shown by heavy points, indicating that the values of x and y are integral. The solution set is as follows: $\{(0; 1), (1; 0), (1; 1), (1; 2), (2; -1), (2; 0), (2; 1), (2; 2), (2; 3)\}$

Notes:
1. Chapter 5 explains how to draw graphs of straight lines.
2. The boundary line p is drawn solid to show that the set of points on the line is included in the required region.
3. The boundary lines q and r are drawn using broken lines to show that the points on those lines are *not* included in the required region.
4. Regions outside the boundaries are shaded to show that they are not required.

Example 2

Write down the three inequalities which define the unshaded area labelled A in Fig. 17.2.

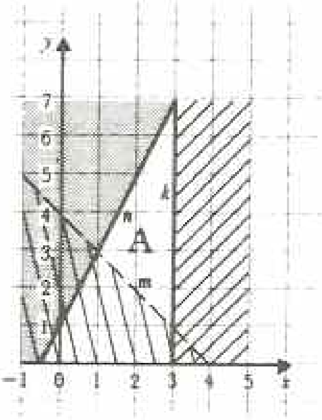


Fig. 17.2

The lines are labelled k, m, n for convenience.
Line k:
k is the line $x = 3$, k is solid. Points to the right of k are not required. Hence the corresponding inequality is $x \leqslant 3$.
Line m:
m has a gradient of -1 and cuts the y-axis at $(0; 4)$. Its equation is $y = -x + 4$, m is a broken line. Points below m are not required. Hence $y > 4 - x$ is the corresponding inequality.

Line n:
n has a gradient of 2 and cuts the y-axis at (0; 1). Its equation is $y = 2x + 1$. n is solid. Points above n are not required. Hence $y \leqslant 2x + 1$ is the corresponding inequality.
The inequalities which define the region A are $x \leqslant 3$; $y > 4 - x$ and $y \leqslant 2x + 1$.

Exercise 17a

1 Using graph paper, show the regions defined by each of the following. (Use solid and broken lines where appropriate and leave each required region unshaded.)
(a) $4y - x < 4$; $x - y < 3$; $x \geqslant -2$
(b) $x - y > -2$; $x + y < 4$; $x \geqslant -1$; $y > 0$
(c) $x < 4$; $y - 2x \leqslant 2$; $2 < y < 4$
(d) $6 \leqslant 2x + 3y \leqslant 12$; $x - 2y < 8$; $y < 3$

2 Solve each of the following graphically for integral values of x and y.
(a) $y > x$; $y \leqslant 3x$; $y + 2x < 8$
(b) $3x + 4y \leqslant 12$; $y - x \leqslant 2$; $y > 1$

3 Write down the three inequalities which define the unshaded area labelled A in Fig. 17.3.

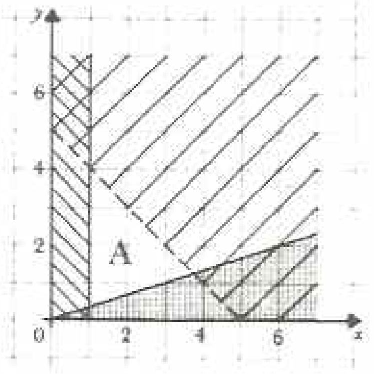


Fig. 17.3

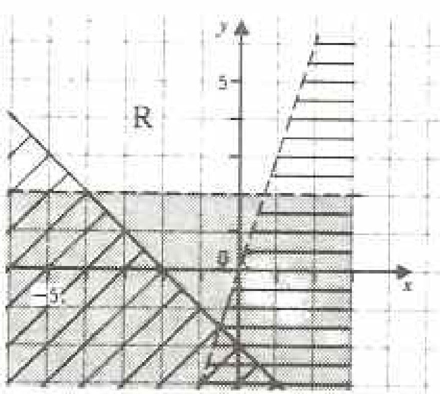


Fig. 17.4

4 What are the three inequalities which define the unshaded region R in Fig. 17.4?

5

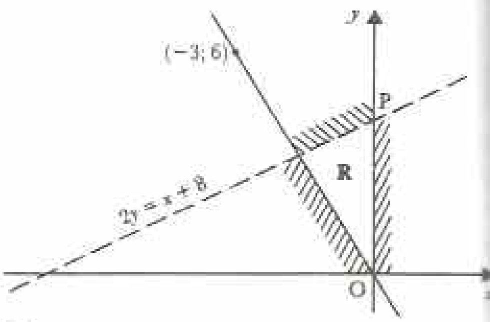


Fig. 17.5

In Fig. 17.5 find
(a) the coordinates of the point P where the line $2y = x + 8$ crosses the y-axis,
(b) the equation of the line which passes through the origin O and the point (−3; 6),
(c) the three inequalities which define the triangular region R in the diagram.
[Camb

Linear programming

Example 3

A student has \$2,50. She buys ballpens at 25c each and pencils at 10c each. She gets at least five of each and the money spent on ballpens is over 50c more than that spent on pencils.
Find (a) how many ways the money can be spent (b) the greatest number of ballpens that can be bought (c) the greatest number of pencils that can be bought

Let the student buy x ballpens at 25c and y pencils at 10c.
Then, from the first two sentences,
$25x + 10y \leqslant 250$
Since she gets at least 5 of each,
$x \geqslant 5$
and $y \geqslant 5$
Also, from the third sentence,
$25x - 10y > 50$
These inequalities are shown in Fig. 17.6.

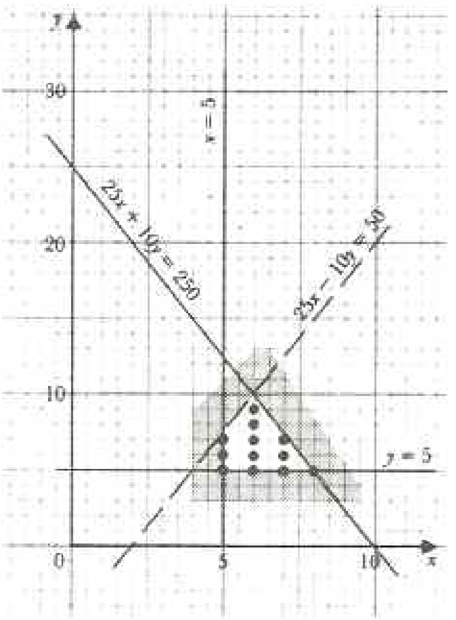


Fig. 17.6

(a) The solution set of the four inequalities is shown by the twelve points marked inside the unshaded region. For example the point (6; 8) shows that the student can buy 6 ballpens and 8 pencils. Hence there are 12 ways of spending the money.
(b) The greatest number of ballpens that can be bought is 8, corresponding to the point (8; 5).
(c) The greatest number of pencils is 9, corresponding to the point (6; 9).

Example 4
The student in Example 3 wants to buy as many items as possible. How many can she get and how much change will there be from the $2,50?

The number of items bought is $x + y$.
If the total is n, then $x + y = n$.
The general equation $x + y = n$ can be represented graphically by a family of parallel lines. Fig. 17.7 shows some members of the family when n has the values 5; 10; 12.

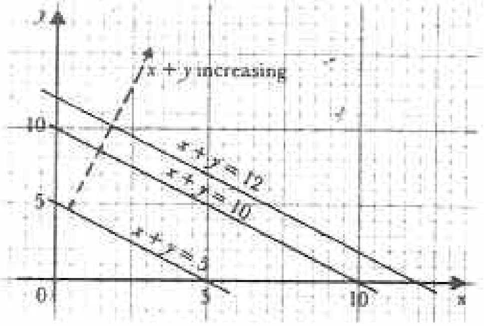


Fig. 17.7

Fig. 17.7 shows that as n increases, the lines appear to move to the right. Considering Fig. 17.6, the greatest possible value of n corresponds to the line which is parallel to $x + y = n$ and as far as possible to the right, but which also passes through the unshaded region.
Fig. 17.8 is a repeat of Fig. 17.6 with some of the family $x + y = n$ added.

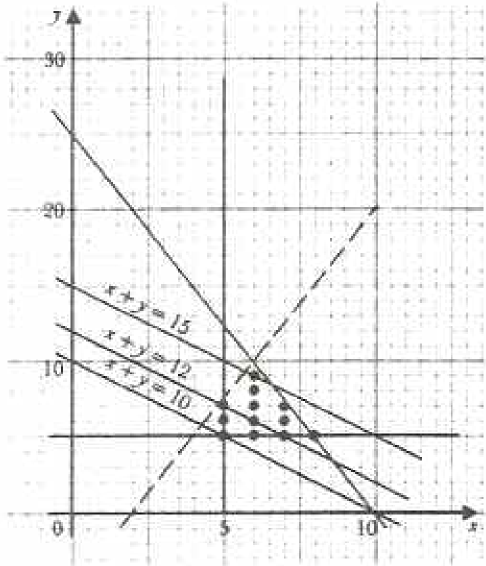


Fig. 17.8

From Fig. 17.8, the greatest value of n is 15, where the line $x + y = 15$ passes through the point (6; 9). The 15 items are made up as follows:

6 ballpens at 25c:	$1.50
9 pencils at 10c:	$0,90
total cost:	$2,40

There will be 10c change from $2,50.

The kind of problem solved in Examples 3 and 4 involves making decisions in a situation in which there are **restrictions**. Each restriction, such as the limit on the amount of money available, can be represented by a linear inequality. Hence the solution to the problem can be found graphically. This method is called **linear programming**. Linear programming can be used to solve a variety of realistic problems.

Example 5

To start a new bus company, a businessman needs at least 5 buses and 10 minibuses. He does not want to have more than 30 vehicles altogether. A bus takes up 3 units of garage space, a minibus takes up 1 unit of garage space and there are only 54 units of garage space available.

If x and y are the numbers of buses and minibuses respectively, (a) write down four inequalities which represent the restrictions on the businessman and (b) draw a graph which shows a region representing possible values of x and y.

Running costs are \$90 a day for a bus and \$48 a day for a minibus. (c) Write down an expression for the total cost per day, \$C. (d) Find the maximum daily cost and the corresponding numbers of buses and minibuses.

(a) From the first sentence,

$x \geqslant 5$

$y \geqslant 10$

From the second sentence,

$x + y \leqslant 30$

From the third sentence,

$3x + y \leqslant 54$

(b) In Fig. 17.9, R is the region which contains the possible values of x and y.

(c) $C = 90x + 48y$

(d) In Fig. 17.9, m is the line $90x + 48y = 720$. As m moves to the right, the cost increases. When m reaches m' the line passes through R at the point (12; 18), the point of maximum cost.

Maximum cost = 12 × \$90 + 18 × \$48
= \$1 080 + \$864
= \$1 944

It costs \$1 944 to run 12 buses and 18 minibuses.

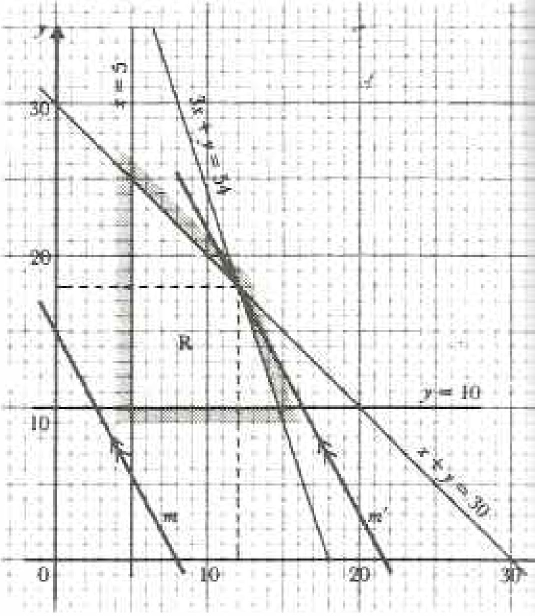


Fig. 17.9

Notes:

1 In part (d), C has been chosen as 720, t LCM of 90 and 48. This value gives convenient line m through the points (8, and (0, 15). m' is then drawn using a s square and ruler.

2 The working would be clearer and mo accurate if a larger scale had been used Fig. 17.9.

Exercise 17b

1 Redraw Fig. 17.9 using a scale of 2 cm to units on both axes. Use your graph answer the following.
When the bus company is running at fu *efficiency, the daily profit on a bus is fou* times that on a minibus. Find the numbe of bus and minibuses the businessman shoul buy to maximise his profit.

2 Notebooks cost 60c and pencils 36c. A gi has \$3,60 to spend and needs at least notebooks and 3 pencils. She decides t spend as much as possible of her \$3,60 (a) How many ways can she spend he money? (b) Do any of the ways give he change? If so, how much?

3 A car repair workshop uses large numbers of two types of spare part, one costing \$3 and the other \$4. The workshop owner allows \$300 to buy spare parts and he needs twice as many cheap ones as dear ones. There must be at least 50 cheap and 20 expensive parts.

(a) What is the largest number of spare parts he can buy, and in what way?

(b) If he decides to get as many of the expensive parts as conditions allow, how many of each type can he get?

4 A storeman fills a new warehouse with two types of goods, A and B. They both come in tall boxes which cannot be stacked. A box of A takes up $\frac{1}{2}$ m^2 of floor space and costs \$50. A box of B takes up $1\frac{1}{2}$ m^2 of floor space and costs \$300. The storeman has up to 100 m^2 of floor space available and can spend up to \$15 000 altogether. He wants to buy at least 50 boxes of A and 20 boxes of B.

(a) How many boxes of each should he buy in order to (i) spend all the money available and also to use as much space as possible? (ii) use all the space for the least cost?

(b) What is the cost in the second case?

5 Following an illness, a patient is required to take pills containing minerals and vitamins. The contents and costs of two types of pill, Feelgood and Getbetta, together with the patient's daily requirement, are shown in Table 17.1.

Table 17.1

	mineral	vitamin	cost
Feelgood	160 mg	4 mg	20c
Getbetta	40 mg	3 mg	10c
Daily requirement	800 mg	30 mg	

A daily prescription contains x Feelgood pills and y Getbetta pills.

(a) State the inequalities to be satisfied by x and y.

(b) Use a graphical method to show the solution set of x and y.

(c) Find the cheapest way of prescribing the pills and the cost.

6 While exploring for oil, it was necessary to carry at least 18 tonnes of supplies and 80 people into a desert region. There were two types of lorry available, Landmasters and Sandrovers. Each Landmaster could carry 900 kg of supplies and 6 people; each Sandrover could carry 1 350 kg of supplies and 5 people.

If there were only 12 of each type in good running order, find the smallest number of lorries necessary for the journey.

7 A shopkeeper orders packets of soap powder. The cost price of a large packet is \$2,70 and that of a small packet is \$1,20. She is prepared to spend up to \$60 altogether and needs twice as many small packets as large packets with a minimum of 10 large and 20 small packets.

(a) What is the greatest number of packets she can buy?

The profit is 30c on a large packet and 15c on a small packet.

(b) Which arrangement gives the greatest profit?

(c) What is that profit?

8 A dressmaker plans to buy new machines for her factory. Table 17.2 shows the cost, the necessary floor space and the output of each machine.

Table 17.2

machine	cost	floor space	output in components/hour
Machine A	\$300	3 m^2	10 per hour
Machine B	\$400	$2\frac{1}{2}$ m^2	15 per hour

She can spend \$3 600 altogether and she has 27 m^2 of floor space. Trade restrictions are such that she has to buy at least 3 of Machine A and 4 of Machine B.

(a) What is the maximum number of machines she can buy?

(b) What arrangement gives the biggest output?

9 A builder has \$960 000 and 8 ha of land available for building houses. Large houses cost \$24 000 each to build and need 0,25 ha;

small houses cost $15 000 each and occupy 0,1 ha. Permission to build is given so long as there are at least 16 large houses and 30 small houses.

(a) Find the greatest number of large houses that can be built.

(b) Find the distribution that (i) gives the greatest number of houses altogether, (ii) uses up all the land available.

10 A shopkeeper stocks two brands of drinks called Kula and Sundown, both of which are produced in cans of the same size. He wishes to order fresh supplies and finds that he has room for up to 1 000 cans. He knows that Sundown is more popular and so proposes to order at least twice as many cans of Sundown as Kula. He wishes, however, to have at least 100 cans of Ku and not more than 800 cans of Sundow Taking x to be the number of cans of Ku and y to be the number of cans of Sundow which he orders, write down the fo inequalities involving x and/or y whi satisfy these conditions.

The point $(x; y)$ represents x cans of Ku and y cans of Sundown. Using a scale 1 cm to represent 100 cans on each ax construct and indicate clearly, by shadir the unwanted regions, the region in whic $(x; y)$ must lie. The profit on a can of Ku is 3c and on a can of Sundown is 2c. U your graph to estimate the number of ca of each that the shopkeeper should order give the maximum profit. [Caml

Chapter 18

Vectors (2)

Vectors (revision)

Naming vectors

A **vector** is any quantity which has direction as well as size. Displacement (or translation), velocity, force, acceleration are all examples of vectors. Fig. 18.1 shows a vector which moves a point from position A to position B.

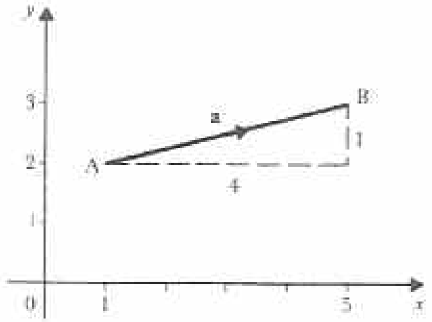


Fig. 18.1

The vector in Fig. 18.1 can be written in many ways:

either $\overline{AB}$, $\overrightarrow{AB}$, **AB**, $\underset{\sim}{AB}$

or $\bar{a}$, $\vec{a}$, **a**, $\underset{\sim}{a}$

Since the points are on a cartesian plane, **AB** can also be written as a column matrix, or **column vector**:

$$\mathbf{AB} = \mathbf{a} = \begin{pmatrix} 4 \\ 1 \end{pmatrix}$$

Direction is important. **BA** is in the opposite direction to **AB**, although they are both parallel and have the same size:

$$\mathbf{BA} = -\mathbf{AB} = -\begin{pmatrix} 4 \\ 1 \end{pmatrix} = \begin{pmatrix} -4 \\ -1 \end{pmatrix}$$

Magnitude

The **magnitude** or size of **AB** is represented by the length of the line segment AB. This is written as |**AB**| and is called the **modulus** of **AB**. In Fig. 18.1,

$$\mathbf{AB} = \begin{pmatrix} 4 \\ 1 \end{pmatrix}$$

$$|\mathbf{AB}| = \sqrt{4^2 + 1^2} \qquad \textit{(Pythagoras)}$$

$$= \sqrt{17}$$

The modulus of a vector is always positive.

Scalar multiplication

If a vector **AB** is multiplied by a **scalar** k, where k is any number, the result is a vector k times as big as **AB**:

$$\text{If } \mathbf{AB} = \begin{pmatrix} 4 \\ 1 \end{pmatrix}$$

$$\text{then } 3\mathbf{AB} = 3\begin{pmatrix} 4 \\ 1 \end{pmatrix} = \begin{pmatrix} 12 \\ 3 \end{pmatrix}$$

$$\text{and } -\tfrac{1}{2}\mathbf{AB} = -\tfrac{1}{2}\begin{pmatrix} 4 \\ 1 \end{pmatrix} = \begin{pmatrix} -2 \\ -\frac{1}{2} \end{pmatrix}$$

The effect of scalars can be summarised as follows:

1 If $\mathbf{a} = k\mathbf{b}$ then **a** is k times as big as **b** and parallel to it.

2 If $h\mathbf{a} = k\mathbf{b}$ then $\mathbf{a} \parallel \mathbf{b}$ or $h = 0$ and $k = 0$.

Addition and subtraction

Vectors may be added and subtracted. For example,

$$\text{If } \mathbf{p} = \begin{pmatrix} -9 \\ 0 \end{pmatrix} \text{ and } \mathbf{q} = \begin{pmatrix} 6 \\ -2 \end{pmatrix}$$

$$\text{then } \mathbf{p}+\mathbf{q} = \begin{pmatrix}-9\\0\end{pmatrix} + \begin{pmatrix}6\\-2\end{pmatrix}$$

$$= \begin{pmatrix}-9+6\\0+(-2)\end{pmatrix} = \begin{pmatrix}-3\\-2\end{pmatrix}$$

$$\text{and } \mathbf{p}-\mathbf{q} = \begin{pmatrix}-9\\0\end{pmatrix} - \begin{pmatrix}6\\-2\end{pmatrix}$$

$$= \begin{pmatrix}-9\\0\end{pmatrix} + -\begin{pmatrix}6\\-2\end{pmatrix}$$

$$= \begin{pmatrix}-9+(-6)\\0+(+2)\end{pmatrix} = \begin{pmatrix}-15\\2\end{pmatrix}$$

Example 1

In Fig. 18.2, below, **AB**, **BC**, **CD**, **DE** *are vectors as shown.*

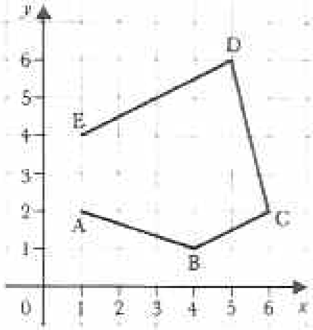


Fig. 18.2

(*a*) *Express each vector in the form* $\begin{pmatrix}a\\b\end{pmatrix}$.

(*b*) *Find* |**DE**|. (*c*) *Show that* **DE** = −2**BC**.

(*d*) *Express* **BC** + **CD** *as a single column vector.*

(*e*) *Express* **BC** − **CD** *as a single column vector.*

(a) $\overrightarrow{AB} = \begin{pmatrix}3\\-1\end{pmatrix}$, $\overrightarrow{BC} = \begin{pmatrix}2\\1\end{pmatrix}$,

$\overrightarrow{CD} = \begin{pmatrix}-1\\4\end{pmatrix}$, $\overrightarrow{DE} = \begin{pmatrix}-4\\-2\end{pmatrix}$

(b) $|DE| = \sqrt{(-4)^2 + (-2)^2}$

$= \sqrt{16+4} = \sqrt{20}$

(c) $\overrightarrow{DE} = \begin{pmatrix}-4\\-2\end{pmatrix} = -2\begin{pmatrix}2\\1\end{pmatrix} = -2\overrightarrow{BC}$

(d) $\overrightarrow{BC} + \overrightarrow{CD} = \begin{pmatrix}2\\1\end{pmatrix} + \begin{pmatrix}-1\\4\end{pmatrix} = \begin{pmatrix}1\\5\end{pmatrix}$

(e) $\overrightarrow{BC} - \overrightarrow{CD} = \begin{pmatrix}2\\1\end{pmatrix} - \begin{pmatrix}-1\\4\end{pmatrix}$

$= \begin{pmatrix}2\\1\end{pmatrix} + \begin{pmatrix}1\\-4\end{pmatrix} = \begin{pmatrix}3\\-3\end{pmatrix}$

Exercise 18a (revision)

1 In Fig. 18.2 state the column vector whi would displace E to A.

2 Express the following as positive vectors

(a) $-\begin{pmatrix}-3\\1\end{pmatrix}$ (b) $-\begin{pmatrix}4\\-5\end{pmatrix}$

(c) $-\begin{pmatrix}-8\\-6\end{pmatrix}$ (d) $-\begin{pmatrix}2\\7\end{pmatrix}$

3 If **XY** = $\begin{pmatrix}-8\\5\end{pmatrix}$ what is (a) |**XY**|, (b) **YX**

4 A shape is translated through $\begin{pmatrix}5\\-3\end{pmatrix}$. It then translated through $\begin{pmatrix}2\\8\end{pmatrix}$.

(a) What single translation is this equivale to? (b) How far is the shape from its startin position?

5 △PQR is such that **PQ** = $\begin{pmatrix}6\\-2\end{pmatrix}$ an **QR** = $\begin{pmatrix}-9\\9\end{pmatrix}$. Sketch △PQR and hence otherwise express **PR** and **RP** as colum vectors.

6 If $\mathbf{p} = \begin{pmatrix}3\\4\end{pmatrix}$, $\mathbf{q} = \begin{pmatrix}3\\-1\end{pmatrix}$, $\mathbf{r} = \begin{pmatrix}-4\\0\end{pmatrix}$ express each of the following as a sing column vector.

(a) $5\mathbf{p}$ (b) $-3\mathbf{q}$ (c) $\frac{1}{2}\mathbf{r}$

(d) $\mathbf{p}+\mathbf{q}$ (e) $\mathbf{r}-\mathbf{p}$ (f) $\mathbf{p}-\mathbf{r}$

(g) $3\mathbf{p}+\mathbf{r}$ (h) $\mathbf{p}-2\mathbf{q}$

(i) $5\mathbf{p}-4\mathbf{q}+\mathbf{r}$ (j) $3\mathbf{p}+\mathbf{q}-6\mathbf{r}$

7 Given **p**, **q**, **r** of question 6, evaluate th following, leaving the answers in surd for where necessary.

(a) $|\mathbf{p}|$ (b) $|\mathbf{q}|$ (c) $|\mathbf{r}|$

(d) $|\mathbf{p}+\mathbf{r}|$ (e) $|\mathbf{q}+\mathbf{r}|$ (f) $|\mathbf{p}-\mathbf{q}|$

8 Triangle ABC has coordinates A(1; 0), B(0; 2), C(1; 3). X is the point (4; 4). $\triangle$ABC is displaced by vector **AX**. Find
(a) the coordinates of the image of $\triangle$ABC,
(b) the modulus of **AX**.

9 (a) Find the vector **q** such that
$$\begin{pmatrix}1\\3\end{pmatrix} - \mathbf{q} = \begin{pmatrix}9\\-3\end{pmatrix}.$$
(b) Hence find $|\mathbf{q}|$.

10 Copy Fig. 18.3 to show points A, B, C and the origin O.

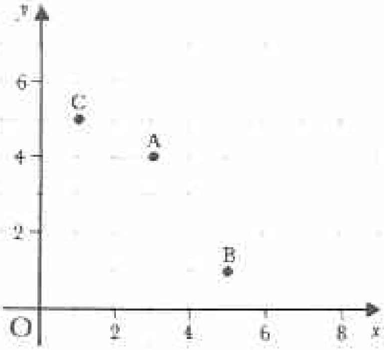


Fig. 18.3

Given that $\mathbf{AP} = \begin{pmatrix}3\\2\end{pmatrix}$, mark on the diagram and label clearly the point P. Given that **BQ** = 2**AC**, mark on the diagram and label clearly the point Q. Calculate |**OA**|. [Camb]

Position vectors

In Fig. 18.4, P is a point $(x; y)$ on the cartesian plane, origin O.

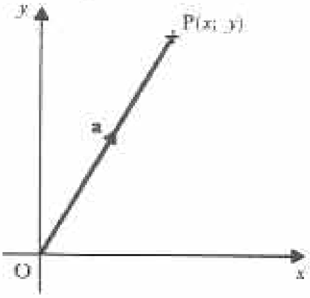


Fig. 18.4

Vector **a** is the displacement of P from O. Since this displacement gives the position of P relative to the origin, **a** is called the **position vector** of P.

In Fig. 18.4, $\mathbf{a} = \mathbf{OP} = \begin{pmatrix}x\\y\end{pmatrix}$.

Hence if a point has coordinates $(x; y)$, its position vector is $\begin{pmatrix}x\\y\end{pmatrix}$.

Position vectors can be used to find displacements between points.

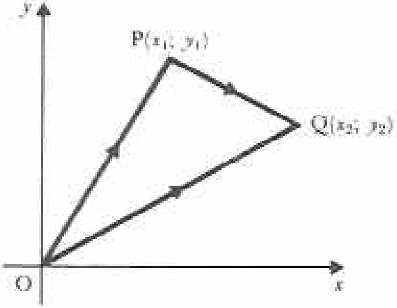


Fig. 18.5

In Fig. 18.5, by adding vectors,
$$\mathbf{OP} + \mathbf{PQ} = \mathbf{OQ}$$
$$\mathbf{PQ} = \mathbf{OQ} - \mathbf{OP}$$
$$\mathbf{PQ} = \begin{pmatrix}x_2\\y_2\end{pmatrix} - \begin{pmatrix}x_1\\y_1\end{pmatrix}$$
$$= \begin{pmatrix}x_2 - x_1\\y_2 - y_1\end{pmatrix}$$

Also, by Pythagoras' theorem,
$$|\mathbf{PQ}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$
The above results hold for any two general points $P(x_1; y_1)$ and $Q(x_2; y_2)$.

Example 2
If P and Q are the points (3; 7) and (11; 13) respectively, find **PQ** *and* |**PQ**|.

In Fig. 18.6 overleaf,
$$\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP}$$
$$= \begin{pmatrix}11\\13\end{pmatrix} - \begin{pmatrix}3\\7\end{pmatrix}$$
$$= \begin{pmatrix}11-3\\13-7\end{pmatrix} = \begin{pmatrix}8\\6\end{pmatrix}$$

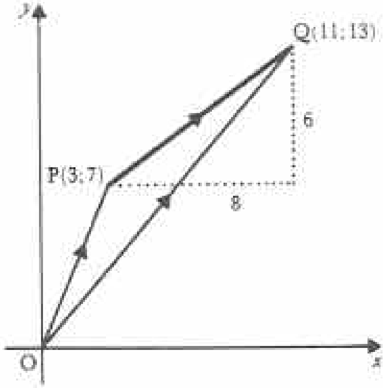


Fig. 18.6

$$|PQ| = \sqrt{(11-3)^2 + (13-7)^2}$$
$$= \sqrt{8^2 + 6^2} = \sqrt{100} = 10$$

Example 3

Quadrilateral OPQR *is as shown in Fig. 18.7.*

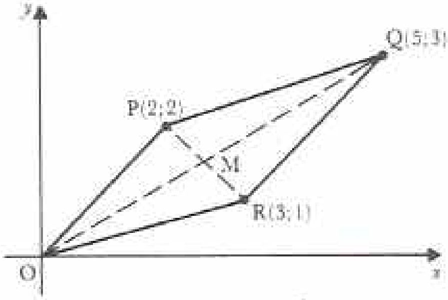


Fig. 18.7

(a) Show that OPQR *is a parallelogram, and*
(b) find the coordinates of the point of intersection of its diagonals.

(a) Using the position vectors of O, P, Q and R,

$$\vec{OP} = \begin{pmatrix}2\\2\end{pmatrix} - \begin{pmatrix}0\\0\end{pmatrix} = \begin{pmatrix}2\\2\end{pmatrix},$$

$$\vec{PQ} = \begin{pmatrix}5\\3\end{pmatrix} - \begin{pmatrix}2\\2\end{pmatrix} = \begin{pmatrix}3\\1\end{pmatrix},$$

$$\vec{RQ} = \begin{pmatrix}5\\3\end{pmatrix} - \begin{pmatrix}3\\1\end{pmatrix} = \begin{pmatrix}2\\2\end{pmatrix},$$

$$\vec{OR} = \begin{pmatrix}3\\1\end{pmatrix} - \begin{pmatrix}0\\0\end{pmatrix} = \begin{pmatrix}3\\1\end{pmatrix}$$

Hence $\vec{OR} = \vec{RQ}$

and $\vec{PQ} = \vec{OR}$

Considering the sides OP and RQ:

If $\vec{OP} = \vec{RQ}$

then $|\vec{OP}| = |\vec{RQ}|$ and $\vec{OP} \| \vec{RQ}$ since equ[al] vectors have the same magnitude and directio[n].

Hence OPQR is a parallelogram since it ha[s] a pair of opposite sides equal and parallel.

(b) Let the diagonals intersect at M.

$\vec{OM} = \frac{1}{2}\vec{OQ}$ (the diagonals of a parallelo[-]gram bisect each other)

$$\vec{OM} = \tfrac{1}{2}\begin{pmatrix}5\\3\end{pmatrix} = \begin{pmatrix}2\frac{1}{2}\\1\frac{1}{2}\end{pmatrix}$$

The diagonals intersect at the point $(2\frac{1}{2}; 1\frac{1}{2})$

Example 3 makes use of the following importa[nt] result:

If $\mathbf{a} = \mathbf{b}$
then $|\mathbf{a}| = |\mathbf{b}|$ and $\mathbf{a} \| \mathbf{b}$.

Example 4

In Fig. 18.8, A(5; 6), B(1; 8), C(*p*; 4), D *are t[he] vertices of a rhombus in the positive quadrant of t[he] cartesian plane.*

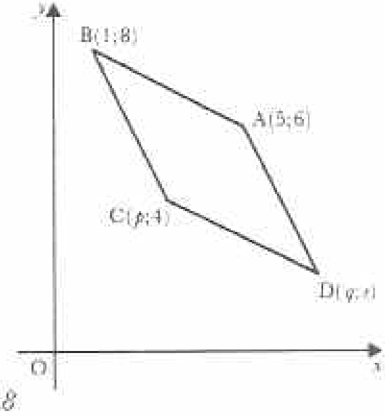


Fig. 18.8

Find p and hence find the coordinates of D.

If ABCD is a rhombus then adjacent sides a[re] equal:

$|\vec{AB}| = |\vec{BC}|$ (1)

and opposite sides are equal and parallel:

$\vec{AB} = \vec{DC}$ (2)

From (1),

$$\sqrt{(1-5)^2+(8-6)^2} = \sqrt{(p-1)^2+(4-8)^2}$$
$$\Rightarrow \quad (-4)^2+(2)^2 = (p-1)^2+(-4)^2$$
$$\Rightarrow \quad (p-1)^2 = (2)^2$$
$$\Rightarrow \quad p-1 = 2$$
$$\Rightarrow \quad p = 3$$

Let D have coordinates $(q; r)$.

From (2),

$$\begin{pmatrix} 1-5 \\ 8-6 \end{pmatrix} = \begin{pmatrix} p-q \\ 4-r \end{pmatrix}$$

Hence $$\begin{pmatrix} -4 \\ 2 \end{pmatrix} = \begin{pmatrix} 3-q \\ 4-r \end{pmatrix}$$

$$\Rightarrow \quad q = 7$$
and $$r = 2$$

$p = 3$ and D is the point $(7; 2)$.

Exercise 18b

1 Given points $A(7;8)$ and $B(2;-1)$, find (a) **AB**, (b) **BA**.

2 The points O, P, Q, R, S have coordinates $(0;0)$, $(1;5)$, $(3;8)$, $(7;10)$, $(10;3)$ respectively. Express each of the following as a column vector.
(a) **OQ** (b) **OS** (c) **PQ**
(d) **QR** (e) **QS** (f) **RP**

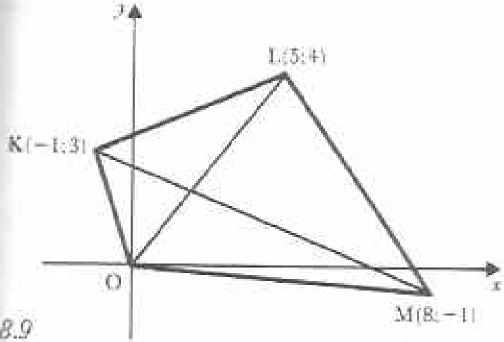


Fig. 18.9

3 Given Fig. 18.9, express each of the following as a single column vector.
(a) **OK** (b) **OM** (c) **KL**
(d) **LM** (e) **OL** (f) **KM**
(g) **OK** + **KL** (h) **OL** + **LM**
(i) **MK** + **KL** (j) **ML** + **LO**

4 OABC is a parallelogram where O is the origin, $\mathbf{OA} = \begin{pmatrix} 1 \\ 4 \end{pmatrix}$, $\mathbf{OC} = \begin{pmatrix} 5 \\ 2 \end{pmatrix}$.
(a) On graph paper mark and clearly label the points A, B and C.
(b) Express as a column vector (i) **OB**, (ii) **CA**. [Camb]

5 Use vectors to show that the quadrilateral $P(-3;0)$, $Q(-1;6)$, $R(3;5)$, $S(5;-2)$ is a trapezium.

6 Use vectors to show that the quadrilateral $A(3;-5)$, $B(8;5)$, $C(6;16)$, $D(1;6)$ is a rhombus.

7 Prove that the quadrilateral $O(0;0)$, $A(4;0)$, $B(7;5)$, $C(3;5)$ is a parallelogram.

8 Show that $P(3;2)$, $Q(9;4)$, $R(11;8)$, $S(5;6)$ is a parallelogram. Use a vector method to find the coordinates of the point of intersection of its diagonals.

9 $O(0;0)$, $P(4;6)$, Q, $R(8;2)$ are vertices of a quadrilateral. Find the coordinates of Q such that OPQR is a parallelogram. Find the coordinates of the point of intersection of its diagonals.

10 Points M and N have position vectors **m** and **n** respectively relative to the origin O. If $\mathbf{m} = \begin{pmatrix} 1 \\ -4 \end{pmatrix}$ and $\mathbf{MN} = \begin{pmatrix} 7 \\ 10 \end{pmatrix}$, find
(a) **n** (b) $|\mathbf{n}|$
(c) the coordinates of a point P such that OM is the short diagonal of parallelogram MNOP.

Properties of shapes

In the previous section, position vectors were restricted to the cartesian plane. However, vector methods can be used in any geometrical situation. They are often used to discover and prove properties of shapes.

In Fig. 18.10, PQRS is a parallelogram as shown.

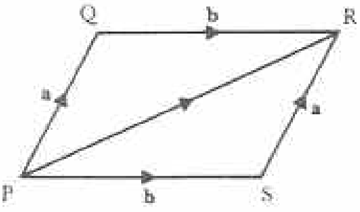


Fig. 18.10

PR = PQ + QR or **PS + SR**
= **a + b** or **b + a**

Hence **a + b = b + a**

This result shows that the addition of vectors is not affected by the order in which they are taken.

In Fig. 18.11 ABCD is any quadrilateral with vectors **a**, **b**, **c**, **d** as shown.

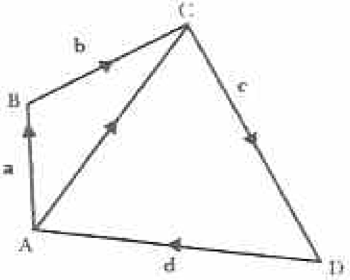


Fig. 18.11

a + b = AC
c + d = CA

adding,

a + b + c + d = AC + CA
but **AC + CA = 0**

so **a + b + c + d = 0**

If the vectors in Fig. 18.11 are taken to be displacements and the + sign is thought of as meaning 'followed by', the above result is hardly surprising. The total final displacement from the starting point, A, is zero when the vectors form the sides of a closed polygon.

Notice how the above results are used in the following examples.

Example 5

PQRS *is any quadrilateral.* A, B, C, D *are the mid-points of* PQ, QR, RS, SP *respectively. Prove that* ABCD *is a parallelogram.*

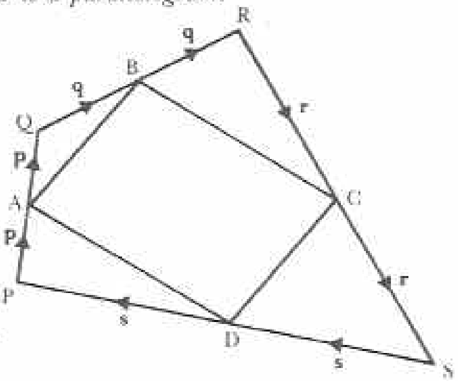


Fig. 18.12

Let $\overline{PQ} = 2\bar{p}$, $\overline{QR} = 2\bar{q}$, $\overline{RS} = 2\bar{r}$, $\overline{PS} = 2\bar{s}$, as shown in Fig. 18.12.

Considering the opposite sides AB and CD of quadrilateral ABCD:

$\overline{AB} = \bar{p} + \bar{q}$

$\overline{CD} = \bar{r} + \bar{s}$

But $2\bar{p} + 2\bar{q} + 2\bar{r} + 2\bar{s} = \mathbf{0}$

$\Leftrightarrow \quad \bar{p} + \bar{q} + \bar{r} + \bar{s} = \mathbf{0}$

$\Leftrightarrow \quad \bar{p} + \bar{q} = -\bar{r} - \bar{s}$

$\Leftrightarrow \quad \bar{p} + \bar{q} = -(\bar{r} + \bar{s})$

Hence $\overline{AB} = \bar{p} + \bar{q} = -(\bar{r} + \bar{s}) = -\overline{CD}$

i.e. $\overline{AB} = \overline{DC}$

If $\overline{AB} = \overline{DC}$, then AB || DC and AB = DC.

ABCD is a parallelogram since it has a pair of opposite sides which are parallel and equal.

Example 6

In Fig. 18.13, P *divides the line* AB *in the ratio* AP:PB = 7:3. *If* **OA = a** *and* **OB = b**, *express* **OP** *in terms of* **a** *and* **b**.

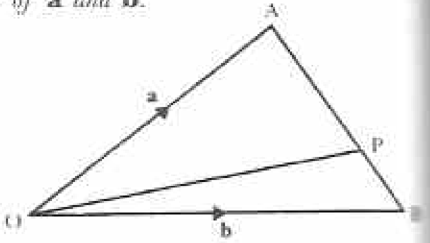


Fig. 18.13

In $\triangle$OAB,

$\overrightarrow{OA} + \overrightarrow{AB} = \overrightarrow{OB}$

$\bar{a} + \overrightarrow{AB} = \bar{b}$

$\overrightarrow{AB} = \bar{b} - \bar{a}$

Along $\overrightarrow{AB}$,

$\overrightarrow{AP} = \frac{7}{10}\overrightarrow{AB}$

$= \frac{7}{10}(\bar{b} - \bar{a})$

In $\triangle$OAP

$\overrightarrow{OP} = \overrightarrow{OA} + \overrightarrow{AP}$

$= \bar{a} + \frac{7}{10}(\bar{b} = - \bar{a})$

$= \bar{a} + \frac{7}{10}\bar{b} - \frac{7}{10}\bar{a}$

$= \frac{3}{10}\bar{a} + \frac{7}{10}\bar{b}$

Example 7
In Fig. 18.14, **OA** = **a** *and* **OB** = **b**.

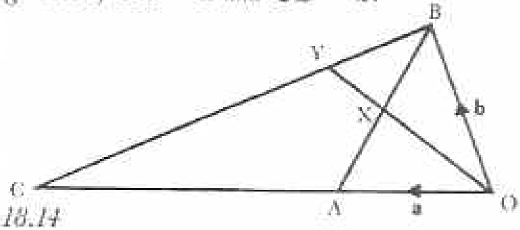


Fig. 18.14

(a) Express **BA** *in terms of* **a** *and* **b**.
(b) If X is the midpoint of BA, show that **OX** $= \frac{1}{2}(\mathbf{a} + \mathbf{b})$.
(c) Given that **OC** = 3**a**, *express* **BC** *in terms of* **a** *and* **b**.
(d) Given that **BY** = *m***BC**, *express* **OY** *in terms of* **a**, **b**, *m*.
(e) If **OY** = *n***OX** *use the results of (b) and (d) to evaluate m and n.*

(a) In △OPA,

$$\overrightarrow{OB} + \overrightarrow{BA} = \overrightarrow{OA}$$

$$\bar{b} + \overrightarrow{BA} = \bar{a}$$

$$\overrightarrow{BA} = \bar{a} - \bar{b}$$

(b) In △OBX,

$$\overrightarrow{BX} = \tfrac{1}{2}\overrightarrow{BA} = \tfrac{1}{2}(\bar{a} - \bar{b})$$

$$\overrightarrow{OX} = \overrightarrow{OB} + \overrightarrow{BA}$$

$$= \bar{b} + \tfrac{1}{2}(\bar{a} - \bar{b}) = \tfrac{1}{2}(\bar{a} + \bar{b})$$

(c) In △OBC,

$$\overrightarrow{BC} = \overrightarrow{BO} + \overrightarrow{OC}$$

$$= -\bar{b} + 3\bar{a} = 3\bar{a} - \bar{b}$$

(d) In △OBY,

$$\overrightarrow{OY} = \overrightarrow{OB} + \overrightarrow{BY}$$

$$= \bar{b} + m\overrightarrow{BC} = \bar{b} + m(3\bar{a} - \bar{b})$$

$$\overrightarrow{OY} = 3m\bar{a} + (1 - m)\bar{b} \qquad (1)$$

(e) $\overrightarrow{OY} = n\overrightarrow{OX}$

$$= n\tfrac{1}{2}(\bar{a} + \bar{b})$$

$$\overrightarrow{OY} = \tfrac{1}{2}n\bar{a} + \tfrac{1}{2}n\bar{b} \qquad (2)$$

also $\overrightarrow{OY} = 3m\bar{a} + (1 - m)\bar{b}$ (1)

Since the vectors are identical, the scalars multiplying **a** and **b** can be equated:

$\tfrac{1}{2}n = 3m$ (scalars of $\bar{a}$)
$\tfrac{1}{2}n = 1 - m$ (scalars of $\bar{b}$)
$3m = 1 - m \Leftrightarrow 4m = 1$
$\Leftrightarrow \quad m = \tfrac{1}{4}$
If $m = \tfrac{1}{4}$, then $\tfrac{1}{2}n = 3 \times \tfrac{1}{4}$
$\Leftrightarrow n = 1\tfrac{1}{2}$

Notice the last stage of Example 7. In general
if $h\mathbf{a} + k\mathbf{b} = n\mathbf{a} + m\mathbf{b}$
then $h = n$ and $k = m$
(or **a** = 0 and **b** = 0)

Exercise 18c

Make sketches where necessary.

1 Using Fig. 18.15, represent each of the following by a single vector.
(a) **PQ** + **QR** (b) **PR** + **RS**
(c) **PS** + **ST** (d) **PR** + **RT**
(e) **PQ** + **QR** + **RS** (f) **PQ** + **QT** + **TS**
(g) **PQ** + **QR** + **RS** + **ST**
(h) **PQ** + **QT** + **TR** + **RS**

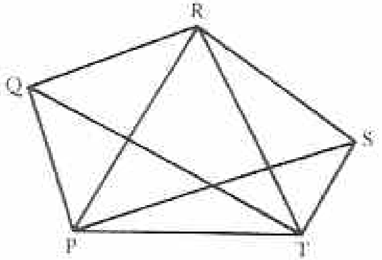


Fig. 18.15

2 In Fig. 18.16 **OA** = **a**, **OB** = **b** and M is the mid-point of AB.

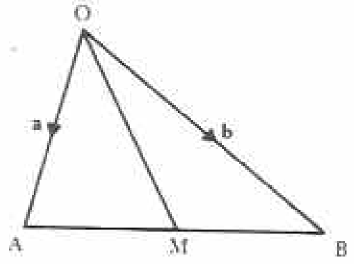


Fig. 18.16

Find **OM** in terms of **a** and **b**.

3 Given Fig. 18.17, overleaf, express **XY**, **YZ** and **ZX** in terms of **a**, **b** and **c**.

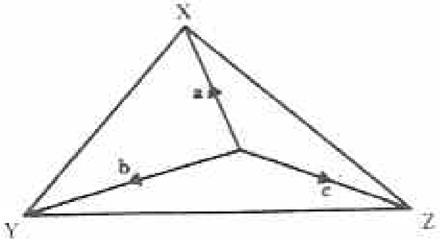


Fig. 18.17

4 In Fig. 18.18, **OP** = 2**a**, **PQ** = 2**b** − 3**a**, **OR** = 3**b**.

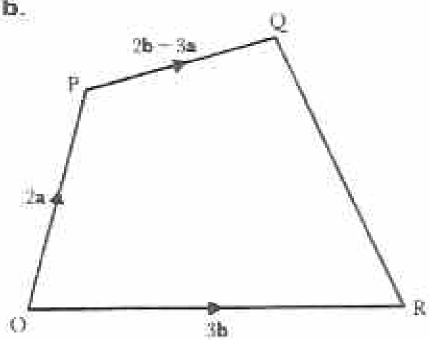


Fig. 18.18

Express (a) **OQ**, (b) **QR** in terms of **a** and **b** as simply as possible.

5 In Fig. 18.19, ORST is a parallelogram, **OR** = **r** and **OT** = **t**.

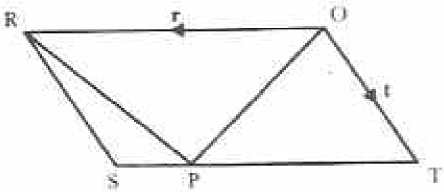


Fig. 18.19

If **ST** = 4**SP**, express the following in terms of **r** and/or **t**.

(a) **RS** (b) **ST** (c) **SP** (d) **RP** (e) **OP**

6 In △PQR, $\overrightarrow{PQ} = \mathbf{a}$, $\overrightarrow{PR} = \mathbf{b}$ and S is the mid-point of PR. Express the following in terms of **a** and/or **b**.

(a) $\overrightarrow{QR}$ (b) $\overrightarrow{PS}$ (c) $\overrightarrow{QS}$

7 PQRS is a trapezium in which PQ ∥ SR, **PQ** = **a** and **QR** = **b**. E is the mid-point of PS and PQ is half as long as SR. Express the following in terms of **a** and/or **b**.

8 ABCDEF is a regular hexagon. If $\overrightarrow{AB} = \mathbf{x}$ and $\overrightarrow{AF} = \mathbf{y}$, express the following in terms of **x** and **y**.

(a) $\overrightarrow{FC}$ (b) $\overrightarrow{BC}$ (c) $\overrightarrow{FE}$

(d) $\overrightarrow{AE}$ (e) $\overrightarrow{AD}$ (f) $\overrightarrow{AC}$

9 In △PQR, A is a point on PR such tha **PA** = $\frac{4}{5}$**PR** and B is the mid-point of QR Point C lies on PQ produced so tha **PC** = $\frac{3}{2}$**PQ**. If **PR** = **x** and **PQ** = **y**, expre the following in terms of **x** and **y**.

(a) **PA** (b) **PB** (c) **PC** (d) **AB** (e) **BC**

10 In Fig. 18.20, M and N are the mid-poin of AD and DC respectively.

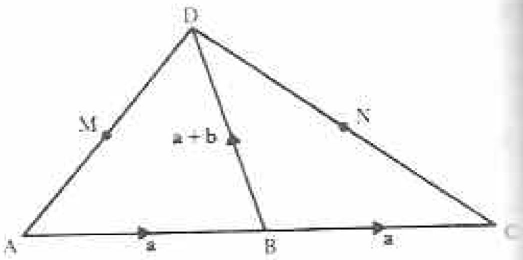


Fig. 18.20

It is given that **AB** = **BC** = **a** and th **BD** = **a** + **b**. Write down as simply a possible in terms of **a** and/or **b** expressions fo (a) **AD**, (b) **DC**, (c) **MN**. [Camb

11 Use vectors to show that if the diagonals o a quadrilateral bisect each other the quad rilateral is a parallelogram.

12 Use vectors to show that the diagonals of a parallelogram bisect each other.

13 In Fig. 18.21, OAB is any triangle, M an N are the mid-points of OA and OB respectively.

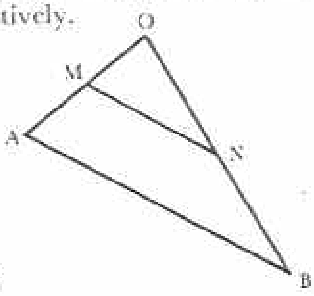


Fig. 18.21

If **OA** = **a** and **OB** = **b**,

(a) express **AB**, **OM**, **ON**, **MN** in terms o **a** and/or **b**, (b) hence describe any relation ship between line segments MN and AB.

14 ABCD is a quadrilateral whose diagonal are equal in length. The mid-points of AB

BC, CD, DA are joined in order to form a quadrilateral. Use a vector method to show that the quadrilateral so formed is a rhombus.

15 ABCD is a kite. The mid-points of AB, BC, CD, DA are joined to form a quadrilateral. Show that the quadrilateral so formed is a rectangle.

16 In trapezium PQRS, $\overrightarrow{QP} = \mathbf{a}$, $\overrightarrow{RQ} = \mathbf{b}$, RS = 3**a** and the diagonals interesect at X.

(a) Express $\overrightarrow{RP}$ and $\overrightarrow{QS}$ in terms of **a** and **b**.
(b) Show that PX:PR = QX:QS = 1:4.

17 In Fig. 18.22, P is a point on AB such that **BA** = 4**BP** and Q is the mid-point of OA. OP and BQ intersect at X.

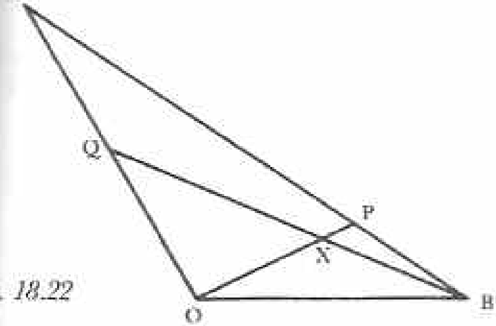


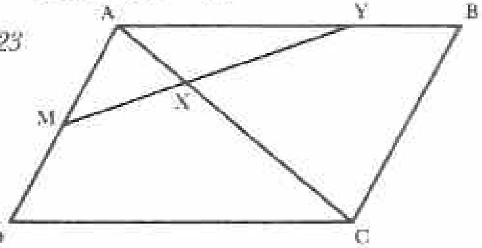
Fig. 18.22

Given **OA** = **a** and **OB** = **b**:

(a) Express the following in terms of **a** and **b**.
(i) **AB** (ii) **OP** (iii) **BQ**
(b) If **BX** = h**BQ**, express **OX** in terms of **a**, **b** and h.
(c) If **OX** = k**OP**, use the previous result to find h and k.
(d) Hence express **OX** in terms of **a** and **b** only.

18 In Fig. 18.23, OABC is a parallelogram, M is the mid-point of $\overrightarrow{OA}$ and $\overrightarrow{AX} = \frac{2}{7}\overrightarrow{AC}$. $\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{OC} = \mathbf{c}$.

Fig. 18.23



(a) Express the following in terms of **a** and **c**.
(i) $\overrightarrow{MA}$ (ii) $\overrightarrow{AB}$ (iii) $\overrightarrow{AC}$ (iv) $\overrightarrow{AX}$
(b) Using △MAX, express $\overrightarrow{MX}$ in terms of **a** and **c**.
(c) If $\overrightarrow{AY} = p\overrightarrow{AB}$, use △MAY to express $\overrightarrow{MY}$ in terms of **a**, **c** and p.
(d) Also if $\overrightarrow{MY} = q\overrightarrow{MX}$, use the result in (b) to express $\overrightarrow{MY}$ in terms of **a**, **c** and q.
(e) Hence find p and q and the ratio AY:YB.

19 In Fig. 18.24 **OP** = **a** and **OS** = **b**.

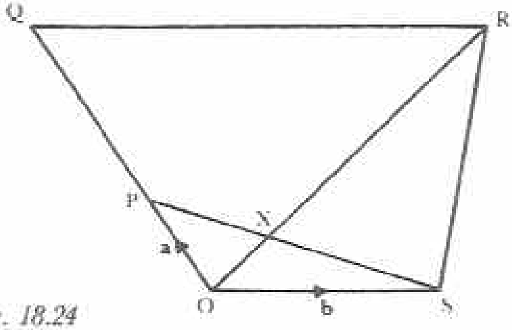


Fig. 18.24

(a) Express **SP** in terms of **a** and **b**.
(b) Given that **SX** = h**SP**, show that **OX** = h**a** + $(1 - h)$**b**.
(c) Given that **OQ** = 3**a** and **QR** = 2**b**, write down an expression for **OR** in terms of **a** and **b**.
(d) Given that **OX** = k**OR** use the results of parts (b) and (c) to find the values of h and k.
(e) Find the numerical value of the ratio $\frac{PX}{XS}$. [Camb]

20 ABC is any triangle. M and N are the mid-points of BC and AC respectively and AM and BN intersect at G. **AB** = **x** and **AC** = **y**.

(a) Express **AM** in terms of **x** and **y**.
(b) If **AG** = h**AM**, express **AG** in terms of **x**, **y** and h.
(c) Express **BN** in terms of **x** and **y**.
(d) If **BG** = k**BN**, express **AG** in terms of **x**, **y** and k.
(e) Use the results of (b) and (d) to find h and k.
(f) What can you deduce about the three lines joining the mid-points of the sides of a triangle to the opposite vertices?

Chapter 19

Probability (2) Combined probabilities

Probability

Probability is a numerical measure of the likelihood of an event happening or not happening. For example, if it has rained in Gwanda in 9 out of the last 12 Septembers, then, statistically, the probability of rain falling in Gwanda next September is $\frac{9}{12}$ (or $\frac{3}{4}$ or 0,75). This is an example of **experimental probability**. Since experimental probability uses numerical records of past events to predict the future, its predictions cannot be taken to be absolutely accurate.

Alternatively, the probability of throwing a five on a fair six-sided die is $\frac{1}{6}$, since any one of the six faces is equally likely. This is an example of **theoretical probability**. Theoretical probabilities are exact values which can be calculated by considering the physical nature of the given situations.

Example 1

Tendai and Samuel have played each other at tennis 15 times this season. Tendai has won 12 of the matches. They play each other in a championship. What is the probability that (a) the match is drawn, (b) Tendai wins, (c) either Tendai or Samuel wins?

(a) Tennis matches are either won or lost. They are never drawn.
Probability of a draw = 0

(b) Tendai has won 12 of the last 15 matches.
Experimental probability of Tendai winning the match
$= \frac{12}{15} = \frac{4}{5} = 0{,}8$

(c) Since one or other of Tendai or Samuel must win, probability of either person winning = 1.

If p is the probability of an event happening then p lies in the range $0 \geqslant p \geqslant 1$. The probability of an event *not* happening is p' where $p' = 1 -$
For instance, in Example 1, the probability o
Tendai *not* winning is $1 - 0{,}8$; i.e. 0,2.

Probability can also be described in se
language. If p(R) is the probability of a require
outcome happening, then

$$p(R) = \frac{n(R)}{n(\mathscr{E})}$$

where R = (required outcomes)
$\mathscr{E}$ = (all possible outcomes)

Example 2

A letter is chosen at random from the alphabet. Fi the probability that it is (a) F, *(b)*, F *or* T, *(c) o of the letters of the word* FREQUENCY, *(d) n one of the letters of the word* TABLE.

In every case,
$\mathscr{E} = \{A; B; C; \ldots; Z\}$

(a) Let A = {F}

then $p(A) = \frac{n(A)}{n(\mathscr{E})} = \frac{1}{26}$

The probability that F is chosen is $\frac{1}{26}$.

(b) Let B = {F; T}

then $p(B) = \frac{n(B)}{n(\mathscr{E})} = \frac{2}{26} = \frac{1}{13}$.

There is a $\frac{1}{13}$ probability that F or T is chosen

(c) Let C = {F; R; E; Q; U; N; C; Y}

then $p(C) = \frac{n(C)}{n(\mathscr{E})} = \frac{8}{26} = \frac{4}{13}$

The probability of choosing one of the letters of *frequency* is $\frac{4}{13}$.

(d) Let D = {T; A; B; L; E}

then $p(D) = \frac{n(D)}{n(\mathscr{E})} = \frac{5}{26}$

and $p(D') = 1 - p(D) = 1 - \frac{5}{26} = \frac{21}{26}$

There is a $\frac{21}{26}$ chance that none of the letters is in TABLE.

Notes:

1 'At random' means 'in a free irregular way'.
2 The letter E in part (c) of Example 2 was *not* counted twice.
3 Part (d) of Example 2 is most conveniently solved using the method of subtraction as shown. p(D') means 'the probability of not being in D'.

Exercise 19a (revision)

1 A statistical survey shows that 28% of all men take size 9 shoes. What is the probability that your friend's father takes size 9 shoes?

2 A school contains 357 boys and 323 girls. If a student is chosen at random, what is the probability that a girl is chosen?

3 A State Lottery sells $1\frac{1}{2}$ million tickets of which 300 are prizewinners. What is the probability of getting a prize by buying just one ticket?

4 Statistics show that 92 out of every 100 adults are at least 150 cm tall. What is the probability that a person chosen at random from a large crowd is less than 150 cm tall?

5 Fig. 19.1 is a magic square.

16	2	3	13
5	11	10	8
9	7	6	12
4	14	15	1

Fig. 19.1

If a number is picked at random from Fig. 19.1 what is the probability that it is
(a) odd, (b) prime,
(c) less than 10,
(d) exactly divisible by 3,
(e) a perfect square, (f) a perfect cube?

6 A bag contains 2 black balls, 3 green ball[s] 4 red balls. A ball is picked from the ba[g] at random. What is the probability that it i[s]
(a) black,
(b) green,
(c) red,
(d) yellow,
(e) not black,
(f) either black or red?

7 A fair six-sided die is thrown. Find th[e] probability of getting
(a) a 3
(b) a 4
(c) a 9
(d) either 1, 2 or 3
(e) a number divisible by 3
(f) a number less than 5

8 A letter is chosen at random from th[e] alphabet. Find the probability that it is
(a) M
(b) not A or Z
(c) either P, Q, R or S
(d) one of the letters ZIMBABWE

9 Table 19.1 gives the numbers of students i[n] age groups in a school.

Table 19.1

age	12	13	14	15	16	17	18
number	42	130	125	131	110	84	53

Find the probability that a student chose[n] at random is (a) 14, (b) 14 or less.

10 A card is picked at random from a pack o[f] playing cards*. Find the probability o[f] picking
(a) the 5 of ♡ (b) the K of ♠
(c) a 9 (d) a black Queen
(e) a diamond
(f) either a Jack, a 2 or an Ace
(g) a red card (h) a red club

* A packet of playing cards contains 5[2] cards in 4 suits: clubs (♣), diamonds (◇), hearts (♡), spades (♠). There are 1[3] cards in each suit: Ace (A), 2, 3, 4, 5, 6, 7, 8, 9, 10, Jack (J), Queen (Q), Kin[g] (K). Clubs and spades are black, diamond[s] and hearts are red.

Mutually exclusive events

Example 3

Find the probability that a letter chosen at random from the alphabet is either a vowel or one of the letters X, Y, Z.

If D = {desired outcomes}
then $D = V \cup L$
where V = {vowels} = {A; E; I; O; U}
and L = {X; Y; Z}
Hence D = {A; E; I; O; U} ∪ {X; Y; Z}
= {A; E; I; O; U; X; Y; Z}

$$p(D) = \frac{n(D)}{n(\mathscr{E})} = \frac{8}{26} = \frac{4}{13}$$

In Example 3, if p(V) and p(L) are the probabilities of choosing a vowel and one of X, Y, Z respectively, then

$p(V) = \frac{5}{26}$
$p(L) = \frac{3}{26}$

and, by inspection,

$p(D) = p(V) + p(L)$
$= \frac{5}{26} + \frac{3}{26}$
$= \frac{8}{26} = \frac{4}{13}$ as before

The task of choosing a letter which is either a member of V or a member of L involves separate events which cannot happen together, i.e. one event excludes the other. They are said to be **mutually exclusive** events. In such cases the separate probabilities are added to give the combined probability.

The Venn diagram in Fig. 19.2 represents the situation in which sets V and L are mutually exclusive, or disjoint:

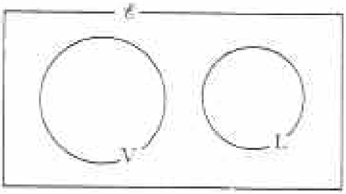


Fig. 19.2

In this case $p(V) + p(L) = p(V \cup L)$

Addition law

If events A, B, C, ... are mutually exclusive, the probability of A *or* B *or* C *or* ... happening is the sum of their individual probabilities: $p(A) + p(B) + p(C) + \ldots$.

Example 4

A number is chosen at random from the set {2; 4; 6; ...; 18; 20}. *Find the probability that it is either a factor of* 18 *or a multiple of* 5.

ℰ = {2; 4; 6; ...; 18; 20}
Let F = {factors of 18} = {2; 6; 18}
and M = {multiples of 5} = {10; 20}
It follows that F and M are mutually exclusive.

$$p(F) = \frac{n(F)}{n(\mathscr{E})} = \frac{3}{10}$$

$$p(M) = \frac{n(M)}{n(\mathscr{E})} = \frac{2}{10}$$

The probability that either F or M happens
$= p(F) + p(M)$
$= \frac{3}{10} + \frac{2}{10} = \frac{5}{10}$
$= \frac{1}{2}$

Notice that the addition law is used to solve problems which contain the words *or* or *either/or*.

Exercise 19b

1 A card is chosen at random from a pack of playing cards. What is the probability that it is either a heart or the Queen of spades?

2 F = {2; 3; 7} and T = {10; 20; 30; 40}.
(a) If one element is selected at random from F, write down the probability that it is odd.
(b) If one element is selected at random from T, write down the probability that it is a multiple of 5.
(c) If one element is selected at random from $F \cup T$ write down the probability that it is either a prime factor of 42 or a multiple of 4.

3 In a game of chance, an arrow spins at random. When it stops it points to one of eight sectors numbered as shown in Fig. 19.3.

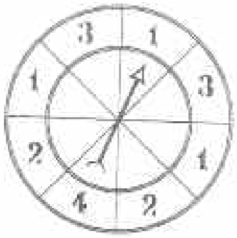


Fig. 19.3

Find the probability that the arrow points at (a) a 3 or a 4, (b) a 1 or a 4, (c) a 1 or a 2.

4 A bag contains 3 red balls, 4 blue balls, 5 white balls and 6 black balls. A ball is picked at random. What is the probability that it is either
(a) red or blue,
(b) red or white,
(c) blue or white,
(d) blue or black,
(e) red, white or blue,
(f) blue, white or black?

5 A letter is chosen at random from the word COMPUTER. What is the probability that it is
(a) either in the word CUT or in the word ROPE,
(b) neither in the word MET nor in the word UP?

Independent events

Example 5

A die is thrown and a coin is tossed. What is the probability of getting both a six and a tail?

Table 19.2 contains all the possible outcomes of throwing the die and the coin at the same time.

Table 19.2

	die 1	2	3	4	5	6
head (h)	h1	h2	h3	h4	h5	h6
tail (t)	t1	t2	t3	t4	t5	(t6)

Table 19.2 shows that there are 12 possible outcomes of which one, ringed, gives a *six* and a *tail*.

Required probability $= \frac{1}{12}$

In Example 5 if p(S) and p(T) are the probabilities of getting a six and a tail then

$p(S) = \frac{1}{6}$

$p(T) = \frac{1}{2}$

Probability of getting both

$= p(S) \times p(T)$

$= \frac{1}{6} \times \frac{1}{2} = \frac{1}{12}$ as before.

The task of getting both a six and a tail involves two events which have no effect on each other. They are said to be **independent events**. In such cases, the separate probabilities are multiplied to give the combined probability.

Product law

If events A, B, C, ... are independent, the probability of A *and* B *and* C *and* ... happening is the product of their individual probabilities: $p(A) \times p(B) \times p(C) \times \ldots$

The Venn diagram in Fig. 19.4 shows the intersecting sets S and T.

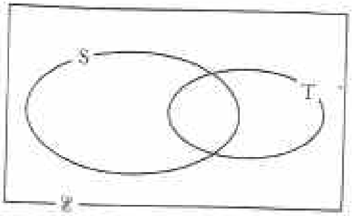


Fig. 19.4

Check that for any pair of intersecting sets,

$n(S \cup T) = n(S) + n(T) - n(S \cap T)$

Dividing each term by $n(\mathscr{E})$,

$$\frac{n(S \cup T)}{n(\mathscr{E})} = \frac{n(S)}{n(\mathscr{E})} + \frac{n(T)}{n(\mathscr{E})} - \frac{n(S \cap T)}{n(\mathscr{E})}$$

$\Rightarrow p(S \cup T) = p(S) + p(T) - p(S \cap T)$

The above probability equation may be used to simplify situations in which events are combined. If S and T had been mutually exclusive, i.e. disjoint, then $S \cap T = \varnothing$ and $p(S \cap T) = 0$, giving the addition law discussed earlier.

Example 6

Five girls and three boys put their names in a box. One name is picked out at random. Without replacing the first name, a second name is picked out at random. What is the probability that both are names of girls?

1st pick:
There are 5 girls and 8 names.
Probability that a girl's name is picked $= \frac{5}{8}$

2nd pick:
If a girl's name was picked, there now remain 4 girls and 7 names.
Probability that a girl's name is picked $= \frac{4}{7}$
Combined probability that both names are those of girls $= \frac{5}{8} \times \frac{4}{7} = \frac{5}{14}$

Notice that the product law is used to solve problems which contain the words *and* or *both/and*.

Exercise 19c

1 A card is chosen from a pack of playing cards then returned to the pack. A second card is chosen. What is the probability that both cards are black?

2 A coin is tossed and a die is thrown. What is the probability of getting a head and a perfect square?

3 If the arrow in Fig. 19.3 is spun twice, what is the probability of getting
(a) two 3s, (b) two 4s,
(c) a 1 followed by a 2,
(d) a 4 followed by a 3?

4 Five cards are lettered A, B, C, D, E. Three cards are chosen at random, one after the other, without replacement, and are placed in the order shown in Fig. 19.5.

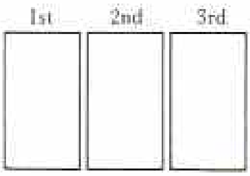


Fig. 19.5

What is the probability that the cards spell the word BED?

5 In a primary school 70% of the boys and 55% of the girls can ride a bicycle. If a boy and a girl are chosen at random what is the probability that (a) both of them can ride a bicycle, (b) neither of them can ride a bicycle?

Outcome tables, tree diagrams

Example 7

Two dice are thrown at the same time. Find the probability of getting (a) at least one 5, (b) a total score divisible by 5.

Table 19.3 shows all the possible outcomes when two dice are thrown.

Table 19.3

second die					P ↓		
6	7	8	9	⑩	11	12	
5	6	7	8	9	⑩	11	←Q
4	⑤	6	7	8	9	⑩	
3	4	⑤	6	7	8	9	
2	3	4	⑤	6	7	8	
1	2	3	4	⑤	6	7	
	1	2	3	4	5	6	
	first die						

The number of possible outcomes in Table 19.3 is $n(\mathscr{E})$ where $n(\mathscr{E}) = 36$
(a) Referring to the shaded column and row,
P = {outcomes with 5 on the first die}
Q = {outcomes with 5 on the second die}
$n(P \cup Q) = 11$
Probability of getting at least one five

$$= \frac{n(P \cup Q)}{n(\mathscr{E})} = \frac{11}{36}$$

(b) In Table 19.3, all of the total scores which are exactly divisible by 5 have been ringed.
Number of outcomes divisible by five $= 7$
Probability of getting a total score divisible by five $= \frac{7}{36}$

In Example 7, notice how the table helps to overcome the problem of finding the various numbers of outcomes. A similar method was used in Example 5 on page 161.

Example 8

A bag contains 3 black balls and 2 white balls.
(a) A ball is taken from the bag and then replaced. A second ball is chosen. What is the probability that (i) they are both black, (ii) one is black and one is white?
(b) Find out how those probabilities are affected if two balls are chosen without any replacement.

(a) The various possible ways of selecting the balls are shown in Fig. 19.6 on a **tree diagram**. In the diagram the branches of the tree show the different ways of choosing, with the related probabilities given as fractions.

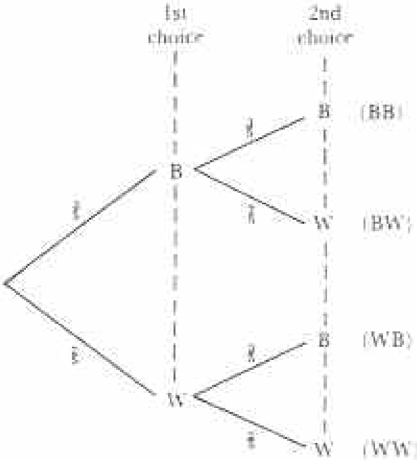


Fig. 19.6

By following the branches of the tree diagram,
(i) Probability that the first two balls are black:

$p(BB) = \frac{3}{5} \times \frac{3}{5} = \frac{9}{25}$

(ii) Probability that the first is black and the second is white:

$p(BW) = \frac{3}{5} \times \frac{2}{5} = \frac{6}{25}$

Probability that the first is white and the second is black:

$p(WB) = \frac{2}{5} \times \frac{3}{5} = \frac{6}{25}$

p(BW) and p(WB) are probabilities of mutually exclusive events. Hence the probability of getting a black ball and a white ball when the order does not matter.

$= p(BW) + p(WB)$

$= \frac{6}{25} + \frac{6}{25} = \frac{12}{25}$

(b) If there is no replacement, then there are only 4 balls left after the first is taken. Compare the probability fractions in Fig. 19.6 with those of Fig. 19.7.

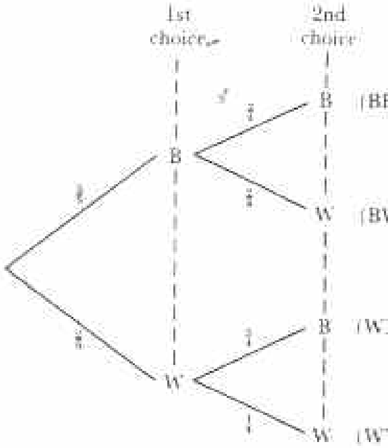


Fig. 19.7

From the tree diagram,

(i) $p(BB) = \frac{3}{5} \times \frac{2}{4} = \frac{3}{10}$

(ii) $p(BW) = \frac{3}{5} \times \frac{2}{4} = \frac{3}{10}$

$p(WB) = \frac{2}{5} \times \frac{3}{4} = \frac{3}{10}$

Probability of getting a black and white l regardless of order

$= \frac{3}{10} + \frac{3}{10} = \frac{3}{5}$

Notice in Example 8 that there are four pos outcomes: BB, BW, WB, WW and that in c case the sum of the probabilities of the outco is 1:

	(a)	(b)
BB	$\frac{9}{25}$	$\frac{3}{10}$
BW	$\frac{6}{25}$	$\frac{3}{10}$
WB	$\frac{6}{25}$	$\frac{3}{10}$
WW	$\frac{4}{25}$	$\frac{1}{10}$
Sum	1	1

This provides a useful check on calculatio

Example 9

If three cards are chosen from a pack without replacen what is the probability of getting at least two spa

Fig. 19.8, overleaf, shows the various way choosing the three cards.

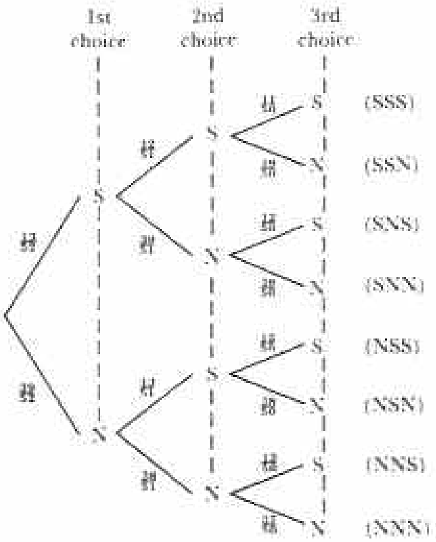


Fig. 19.8

Probability of choosing 3 spades
$= \frac{13}{52} \times \frac{12}{51} \times \frac{11}{50} = \frac{11}{850}$

Probability of choosing 2 spades
$= \frac{13}{52} \times \frac{12}{51} \times \frac{39}{50} + \frac{13}{52} \times \frac{39}{51} \times \frac{12}{50} + \frac{39}{52} \times \frac{13}{51} \times \frac{12}{50}$
$= \frac{3 \times 12 \times 13 \times 39}{52 \times 51 \times 50} = \frac{117}{850}$

Probability of getting at least 2 spades (i.e. 2 spades or 3 spades)
$= \frac{11}{850} + \frac{117}{850}$
$= \frac{128}{850} = \frac{64}{425}$

Exercise 19d

1 A pair of dice are thrown. What is the probability of getting (a) at least one six, (b) a total score of seven?

2 A game is played with a pentagonal spinner with sides marked 1 to 5. The score is on the side which comes to rest on the table. For example, in Fig. 19.9 the score is 3.

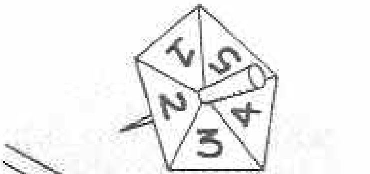


Fig. 19.9

In two spins, what is the probability of getting
(a) two 5s, (b) at least one 5,
(c) a total score of 5,
(d) a total score greater than 5?

3 When two dice are thrown what is the probability of the total score being a prime number?

4 In a large crowd, there are three times as many men as women. Three people are chosen at random. Assuming that there are so many people that choosing three has a negligible effect on the proportion of men to women, find the probability that they are (a) all men, (b) 2 women and 1 man.

5 In a school, 4 out of 5 students have pens. If 2 students are picked at random, what is the probability that (a) both will have a pen, (b) one has a pen and the other has not?

6 M and N are the mid-points of opposite sides of square ABCD (Fig. 19.10).

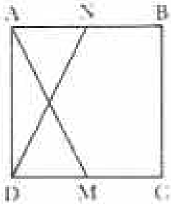


Fig. 19.10

A point is selected at random in the square. Find the probability that it lies
(a) in △ADM,
(b) in △ADM but not in △ADN,
(c) neither in △ADM nor in △ADN.

7 A ball is dropped at random into one of eight holes, numbered as shown in Fig. 19.11.

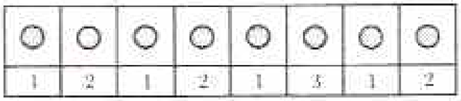


Fig. 19.11

The number under each hole gives the score obtained when the ball drops into that hole.
(a) State the probability of scoring 1.
(b) If the ball is dropped twice, find the probability of scoring (i) a total of 6, (ii) a total of 4. [Camb]

8 In order to choose an athletics team, 100 students were each timed over 1 500 metres. The data obtained is given in Table 19.4.

Table 19.4

time (min:sec)	4:00	4:30	5:00	5:30	6:00
number inside this time	7	28	65	88	100

(a) A student is chosen at random. What is the probability that the student's time was inside 5 min?

(b) A student is chosen at random from those whose time was inside 5 min 30 s. Find the probability that this student took less than 4 min 30 s.

(c) Two students are chosen at random. Find the probability that both took 5 min 30 s or more.

9 The probability of a seed germinating is $\frac{1}{3}$. If 3 of the seeds are planted, what is the probability that
(a) none germinate,
(b) at least one will germinate,
(c) only one will germinate?

10 When three dice are thrown together what is the probability of getting a total score of 10?

11 A coin is tossed 3 times. What is the probability of getting
(a) 2 heads and 1 tail,
(b) at least 1 head?

12 If two cards are drawn from a pack without replacement what is the probability of getting (a) an Ace and a King, (b) two Aces?

13 It is assumed that when children are born they are equally likely to be boys or girls. What is the probability that a family of 4 children contains
(a) 3 boys and 1 girl,
(b) 2 boys and 2 girls?

14 A bag contains 3 black balls, 4 white balls and 5 red balls. Three balls are removed without replacement. What is the probability of obtaining
(a) one of each colour,
(b) at least two red balls?

15 A committee consists of 6 men and 4 women. A randomly chosen subcommittee is made up of 3 of the committee members. What is the probability that
(a) they are all women,
(b) 2 of them are men?

Revision course

The remainder of this book is a revision course for students taking School Certificate/'O' level in mathematics.

It is assumed that readers have carefully worked through the previous chapters and books of the 'New General Mathematics' course. For this reason, explanations have been kept to a minimum. The revision course relies mainly on worked examples to show ways of solving problems and on exercises for further practice.

In order to make the revision course easy to use, each chapter covers both the elementary and more advanced parts of a topic.

At this stage the order in which the topics are studied is relatively unimportant. Therefore the revision chapters may be arranged to suit the needs of the reader. The certificate-level practice papers on pages 267 to 281 may be attempted after revision has been completed.

Contents of revision course

Chapter 21

General arithmetic

Fractions, decimals, percentages

Example 1

Which is greater, $\frac{5}{6}$ or $\frac{7}{8}$?

24 is the LCM of the denominators, 6 and 8. Express each fraction with a common denominator of 24.

$$\frac{5}{6}=\frac{5\times 4}{6\times 4}=\frac{20}{24}$$

$$\frac{7}{8}=\frac{7\times 3}{8\times 3}=\frac{21}{24}$$

Since $\frac{21}{24}>\frac{20}{24}$, $\frac{7}{8}$ is greater than $\frac{5}{6}$.

Example 2

Evaluate $2\frac{1}{6}+(3\frac{3}{5}\div 1\frac{1}{8})$.

$$2\tfrac{1}{6}+(3\tfrac{3}{5}\div 1\tfrac{1}{8})=\frac{13}{6}+\left(\frac{18}{5}\div\frac{9}{8}\right)$$

$$=\frac{13}{6}+\left(\frac{18}{5}\times\frac{8}{9}\right)$$

$$=\frac{13}{6}+\frac{16}{5}$$

$$=\frac{65+96}{30}$$

$$=\frac{161}{30}$$

$$=5\tfrac{11}{30}$$

Example 3

Evaluate 6,297 2 × 0,251 correct to 3 d.p.

$$6{,}297\,2\times 0{,}251$$

$$=\frac{62\,972}{10\,000}\times\frac{251}{1\,000}$$

$$=\frac{62\,972\times 251}{10\,000\,000}$$

$$=\frac{15\,805\,972}{10\,000\,000}$$

$$=1{,}580\,597\,2=1{,}581 \text{ to 3 d.p.}$$

working:
62 972
251
62 972
3 148 60
12 594 4
15 805 972

Example 4

Find the value of $\dfrac{2{,}25\times 7{,}5}{4{,}5}$

$$\frac{2{,}25\times 7{,}5}{4{,}5}$$

$$=\frac{0{,}25\times 1{,}5}{0{,}1}\quad \textit{(after equal divisions by 9 and 5)}$$

$= 0{,}25\times 15$ *(multiplying num. and den. by 10)*
$= 3{,}75$

Example 5

Convert 550 Deutchmarks into Francs if £1 = 4,40 DM and £1 = 10,80 F.

4,40 DM = 10,80 F (= £1)

$$\Leftrightarrow\quad 1\text{ DM}=\frac{10{,}80}{4{,}40}\text{F}$$

$$\Leftrightarrow\quad 550\text{ DM}=\frac{10{,}80}{4{,}40}\times 550\text{ F}=1\,350\text{ F}$$

Example 6

When the length of a spring is increased by 8% it becomes 351 mm long. What is its original length?

108% of the original length = 351 mm

$$1\% \text{ of the original length} = \frac{351}{108}\text{ mm}$$

$$\text{Original length } (100\%) = \frac{351}{108} \times 100\text{ mm}$$

$$= 325\text{ mm}$$

Example 7

When a refrigerator is sold for \$558 the profit is 24%. What should be the selling price to make a profit of 28%?

$$124\% \text{ of cost price} = \$558$$
$$\Rightarrow 128\% \text{ of cost price} = \$558 \times \tfrac{128}{124}$$
$$= \$558 \times \tfrac{32}{31}$$
$$= \$18 \times 32$$
$$= \$576$$

Notice that it was not necessary to find the cost price.

Example 8

Chido sells a second-hand car to Thabo and makes a profit of 10%. Thabo then sells the car to Anna for \$4 180 making a loss of 5%. How much did Chido pay for the car?

Chido paid $\frac{100}{110}$ of what Thabo paid.

Thabo paid $\frac{100}{95}$ of what Anna paid.

Hence Chido paid

$\frac{100}{110}$ of $\frac{100}{95}$ of what Anna paid

$$= \frac{100}{110} \times \frac{100}{95} \times 4\,180$$

$$= \frac{10}{11} \times \frac{20}{19} \times 4\,180$$

$$= 10 \times 20 \times 20$$

$$= \$4\,000$$

Example 9

Find the rate of simple interest in per cent per annum at which \$142 will amount to \$295,36 in 12 years.

$$\text{Amount} = \text{principal} + \text{interest}$$
$$\$295{,}36 = \$142 + \text{interest}$$
$$\Leftrightarrow \quad \text{interest} = \$295{,}36 - \$142$$
$$= \$153{,}36$$

But $I = \frac{PRT}{100}$, where I is the interest, P is the principal, R is the rate per cent per annum and T is the time in years.

$$\text{Hence } \$153{,}36 = \frac{\$142 \times R \times 12}{100}$$

$$\Leftrightarrow \quad R = \frac{100 \times 153{,}36}{142 \times 12}\%$$

$$= \frac{1\,278}{142}\% = \frac{639}{71}\%$$

$$= 9\%$$

Exercise 21a

1 Arrange the following fractions in order of size from smallest to largest. $\frac{2}{3}, \frac{11}{15}, \frac{7}{10}, \frac{5}{6}, \frac{4}{5}$

2 Evaluate the following.

(a) $2\frac{2}{3} + 1\frac{1}{2}$ (b) $5\frac{1}{3} - 2\frac{7}{8}$

(c) $1\frac{5}{8} + \frac{1}{2}$ (d) $1\frac{5}{6} + 2\frac{2}{3} - 3\frac{1}{2}$

(e) $\frac{3}{4} \times (1\frac{3}{5} \div \frac{1}{6})$ (f) $1\frac{1}{2} \times 6\frac{2}{5} \div \frac{2}{5}$

(g) $\dfrac{1\frac{2}{3} \times 7}{3\frac{1}{2}}$ (h) $2\frac{1}{2} \div (1\frac{1}{4} \div 3\frac{1}{3})$

3 (a) $\frac{2}{5}$ of the students in a class do history. If 14 students do history, how many students are there in the class?

(b) In an election there were three candidates; $\frac{2}{3}$ of the electors voted for the first candidate, $\frac{1}{4}$ for the second candidate and the rest for the third. If the third got 3 290 votes, how many votes did the winner get?

4 Calculate 129×54 and use the result to write down the value of

(a) $1{,}29 \times 5{,}4$ (b) $12{,}9 \times 0{,}54$

(c) $12{,}9 \times 0{,}005\,4$ (d) $0{,}129 \times 0{,}054$

5 Calculate the following without using tables.

(a) $22{,}7 \times 0{,}38$ (b) $8{,}848 \div 0{,}28$

(c) $86{,}13 \div 2{,}7$ (d) $0{,}991\,6 \div 5{,}36$

6 Express the following quantities in terms of the units shown in brackets.

(a) 405 cm (km)
(b) 158,7 m (km)
(c) 905 g (kg)
(d) 2,4 kg (g)
(e) 7,03 km (m)
(f) 1 305 mℓ (ℓ)

7 Round off the following numbers to the degrees of accuracy shown in brackets.

(a) 3,7846 (3 d.p.)
(b) 75,0794 (2 d.p.)
(c) 144 (2 s.f.)
(d) 9,84 (1 s.f.)
(e) 34,625 (2 s.f.)
(f) 34,625 (2 d.p.)

8 If £1 = Sh21,30 and £1 = 2,10 Roubles, convert Sh100 to Roubles. Give your answer correct to 2 d.p.

9 A woman changes $62 for Lire. If £1 = $1,55 and £1 = 2 200 lr, find the amount of Lire that she obtained.

10 (a) What percentage of 2 is 5?
(b) The original area of a farm is 250 ha. The farmer sells 15% of his land. What area is left?

11 The length round a running track should be 400 m. The actual length is found to be 401,2 m. Calculate the percentage error in the length of the track.

12 A bicycle manufacturer requires wheel spokes to measure 260 mm, with a tolerance (acceptable error) of $\pm 0{,}5\%$. Calculate the acceptable range of length for the spokes.

13 A trader loses 12% by selling a watch for $34,10. Find the cost price of the watch.

14 A trader makes a profit of $12\frac{1}{2}\%$ by selling some goods for $94,50. Find her cash profit.

15 A boy spends 57% of his pocket money. If the amount that he spends is $1,40 more than he has left, how much money had he originally?

16 When a car is sold for $1 870 the profit is 10%. What should be the selling price to make a profit of 18%?

17 A woman's salary was $12 600 in 1992 and was 15% more in 1993. If she paid a tax of $12\frac{1}{2}\%$ of her salary, how much tax did she pay in 1993?

18 A factory produced 3 456 radios in 1991 and 2 880 in 1992. (a) Calculate the percentage decrease in production from 1991 to 1992. (b) How many radios were produced in 1993 if there was a 15% increase over 1992?

19 A man's body-mass increases by 15%. He then goes on a diet and reduces his new body-mass by 15%. Is his final mass greater or less than the original, and by how much?

20 The total cost of a car service consists of a basic price plus a tax of 15%. Given that the total cost is $690, calculate the basic price of the service. [Camb]

21 Find the simple interest on $126 for 6 years at 7%.

22 Find the time in which $168,40 will earn $29,47 of interest at 5% per annum.

23 If $206,40 amounts to $237,36 in 2 years, find the rate of simple interest per annum.

24 Find the time in which $108,33 will amount to $123,50 at 8% per annum.

25 A salesman receives a basic monthly salary of $580. He also gets $\frac{1}{2}\%$ commission on sales. Out of his monthly income he pays 1% as union fees and 3% as income tax. What was his net income for a month in which his sales were $25 950?

Ratio, rate

A **ratio** is a numerical way of comparing quantities of the same kind. The quantities should be expressed in the same units. For example, the ratio of $2 to 40c is 5:1, whether working in cents: 200:40 = 5:1, or in Dollars: $2:\frac{2}{5} = 5:1$.

Example 10

Which ratio is greater, 3:4 or 6:11?

Express each ratio in the form n:1.

$3:4 = \frac{3}{4}:1 = 0{,}75:1$

$6:11 = \frac{6}{11}:1 = 0{,}545\ldots:1$

The ratio 3:4 is greater.

Example 11

Decrease $73,35 *in the ratio* 5:9.

Express the ratio 5:9 as the fraction $\frac{5}{9}$.
Since the money is decreased, multiply $73,35 by $\frac{5}{9}$.

$$\text{Required amount} = \$73{,}35 \times \tfrac{5}{9}$$
$$= \$8{,}15 \times 5$$
$$= \$40{,}75$$

Example 12

If 9 *men paint a building in* 21 *days, how long would* 7 *men take?*

7 men take more time than 9 men.
The number of men is *decreased* in the ratio 7:9.
Hence the time taken is *increased* in the ratio 9:7.
Time is to be found, so time comes last in each line of working.

9 men take 21 days

$\Rightarrow$ 7 men take $21 \times \frac{9}{7}$ days
= 27 days

This example illustrates **inverse** ratio.

Example 13

9 *notebooks cost* $12,15. *How many can be bought for* $17,55?

Money is *increased* in the ratio 17,55:12,15.
The number of books will be *increased* in the same ratio.
The number of books is to be found, so this comes last in each line of working.

$12,15 is the cost of 9 notebooks

$\Rightarrow$ $17,55 is the cost of $9 \times \dfrac{17{,}55}{12{,}15}$ notebooks

$= \dfrac{1\,755}{35}$ notebooks

= 13 notebooks

This is an example of **direct** ratio.

Quantities of different kinds may be connected in the form of a **rate**. For example, km/h, g/cm^3 and number of people/km^2 are all examples of rates.

Example 14

A car travels 132 km *in* 1 h 15 min. *Calculate the speed of the car.*

1 h 15 min = $1\frac{1}{4}$ h.
In $1\frac{1}{4}$ h the car travels 132 km.

$$\text{In 1 h the car travels } \frac{132}{1\frac{1}{4}}\text{ km} = \frac{132 \times 4}{5}\text{ km}$$
$$= \frac{528}{5}\text{ km}$$
$$= 105{,}6\text{ km}$$

The speed of the car is 105,6 km/h.

Example 15

A wooden cube has a mass of 50,48 g. *If one edge of the cube measures* 4 cm, *calculate the density of the wood correct to* 2 s.f.

$$\text{Volume of cube} = 4 \times 4 \times 4\text{ cm}^3 = 64\text{ cm}^3$$

$$\text{Density of wood} = \frac{50{,}48}{64}\text{ g/cm}^3 = \frac{6{,}31}{8}\text{ g/cm}^3$$
$$= 0{,}788\,75\text{ g/cm}^3$$
$$= 0{,}79\text{ g/cm}^3 \text{ to 2 s.f.}$$

Example 16

The population density of a village is 520 people/km^2. *If the village has an area of about* 3,3 km^2, *find its population to the nearest* 100 *people.*

The population density is 520 people/km^2,
i.e. 1 km^2 contains 520 people
Hence 3,3 km^2 contain 520 × 3,3 people
= 1 716 people
The village contains 1 700 people (to the nearest 100 people).

Exercise 21b

1 Write the following as ratios in their simplest form.

(a) 16:20	(b) 150c to $1
(c) $3\frac{1}{3}$:8	(d) 40 cm:1 m 20 cm

2 Express each of the following ratios in the form 1:*n*.

(a) 6:9	(b) 3:7
(c) 16:12	(d) 3,5:0,7

3 In each case, find out which one of the two ratios is greater.
(a) 17:8 or 15:6
(b) \$1,70:\$2 or \$3:\$4,80
(c) 1,5 g:2 kg or 0,5 kg:600 kg

4 The cost price, \$350, of a bicycle is reduced by \$105.
(a) Find the ratio by which the price is reduced.
(b) Express the price reduction as a rate of cents in the Dollar.

5 A car goes 60 km in 48 min. Find the speed of the car in km/h.

6 A shop sells oranges at 6 for \$1. A trader sells the same kind of oranges at 8 for \$1,20. Which price is cheaper, and by how much per orange?

7 A map is drawn on a scale of 1 cm to 5 km.
(a) Write the scale as a ratio in the form $1:n$.
(b) What distance does 2,8 cm on the map represent?

8 A town has an area of 80 hectares and a population of 2 500 people. Calculate the population density of the town in people/ha correct to the nearest whole person.

9 The density of aluminium is 2,7 g/cm^3. Calculate the volume of a piece of aluminium of mass 13,5 g.

10 The telegraph poles along a road are 20 m apart and a car travels from the first to the fifteenth in $10\frac{1}{2}$ seconds. Calculate the speed of the car in km/h.

11 A train 180 m long travels through a station at 60 km/h. Find how many seconds it takes for the train to pass a man who is standing on the station platform.

12 The ages of a mother and daughter are in the ratio 8:3. If the daughter's age now is 12, what will be the ratio of their ages in 4 year's time?

13 A person's income is increased in the ratio 47:40. Find the increase per cent.

14 A tankful of water lasts 15 weeks if 3 litres a day are used. How long will the tankful last if 10 litres a day are used?

15 A train normally travels between two stations at v km/h. If its average speed is increased in the ratio $m:n$, will it take more or less time? In what ratio is the time changed?

Proportion, mixtures

Example 17

Three people share 30 eggs in the ratio 1:2:3. *H*
many eggs does each get?

$1 + 2 + 3 = 6$

1st person gets $\frac{1}{6}$ of 30 eggs = 5 eggs
2nd person gets $\frac{2}{6}$ of 30 eggs = 10 eggs
3rd person gets $\frac{3}{6}$ of 30 eggs = 15 eggs
Check: $5 + 10 + 15 = 30$

Example 18

Mafa, Betty and Giya share \$180 *so that for ev*
\$1 *that Mafa gets, Betty gets* 50c, *and for every*
that Betty gets, Giya gets \$3. *Find Betty's share.*

Assuming that Mafa has 1 share, then Be
gets a $\frac{1}{2}$ share. Giya gets $1\frac{1}{2}$ times as much as Be
Hence Giya's share $= \frac{1}{2} \times 1\frac{1}{2} = \frac{3}{4}$.
They share the money in the ratio $1:\frac{1}{2}:\frac{3}{4}$
$= 4:2:3$
$4 + 2 + 3 = 9$
Betty's share $= \frac{2}{9}$ of \$180
$= \$40$

Example 19

Three numbers d, m, n are in the ratio 3:6:4. *Fi*
the value of $\dfrac{4d - m}{m + 2n}$.

Since $d:m:n = 3:6:4$,
let $d = 3k$, $m = 6k$, $n = 4k$.

Hence $\dfrac{4d - m}{m + 2n} = \dfrac{12k - 6k}{6k + 8k} = \dfrac{6k}{14k} = \dfrac{3}{7}$

Example 20

A and B are partners in a business. A invests \$10 0
for one year and B invests \$25 000 *for nine mont*
How should they share the first year's profit of \$4 60

A's investment = \$10 000 for 12 months
= 120 000 Dollar-months
B's investment = \$25 000 for 9 months
= 225 000 Dollar-months
Ratio of investments, A:B = 120 000:225 0
= 8:15

+ 15 = 23

...hould get $\frac{8}{23}$ of \$4 600 = \$1 600

...hould get $\frac{15}{23}$ of \$4 600 = \$3 000

...mple 21

...r sells rice in bags. 30 bags of rice costing \$50 ...g are mixed with 40 bags of another rice costing ...5 per bag. If the mixture is sold at \$66 per bag, ...e percentage gain.

...t of first 30 bags = \$50 × 30 = \$1 500
...of other 40 bags = \$58,75 × 40 = \$2 350
...l cost of 70 bags
= \$1 500 + \$2 350 = \$3 850

...age cost of 1 bag $= \dfrac{\$3\,850}{70} = \$55$

...n cash on 1 bag = \$66 − \$55 = \$11

Gain % $= \frac{11}{55} \times 100 = 20\%$

...cise 21c

...children divide \$9,45 between them in the ...tio 6:7:8. What is the size of the largest ...are?

...n a State lottery \$28 845 is divided between ...e 1st, 2nd and 3rd prizewinners in the ...tio 4:3:2. How much does the 3rd ...rizewinner get?

...0 is divided between Manasa, Bob and ...ophie so that Sophie's share is $\frac{3}{8}$ of Bob's ...d Bob's share is twice that of Manasa. ...ow much does Sophie receive?

...f $a:b:c = 5:2:3$, evaluate

...) $\dfrac{a-2b}{3b-c}$ (b) $\dfrac{a+b+c}{5a}$

...) $a-b:b+c$ (d) $4a-b:a+2b-c$

...ind the value of the ratio $m:n$ if

...) $4m = 7n$ (b) $5m - n = 2m + 5n$

...0 kg of coffee costing \$24 per kg is mixed ...ith 5 kg costing \$32 per kg. What should ...e the cost of the mixture per kg?

...shopkeeper has 5 shirts priced at \$23,20 ...ach, 3 at \$22,00 each and 4 at \$16,00 each. ...e mixes the shirts up and sells them all at ...20 each. How much does he gain or lose ...together by doing this? What is his average ...ain or loss per shirt?

8 Tea costing \$12,90 and \$11,55 per kg are mixed together in the ratio 2:1. What is the value of the mixture per kg?

9 Copper sulphate is made from
32 parts of copper
16 parts of sulphur
32 parts of oxygen
45 parts of water.
Find the mass of copper, sulphur, oxygen and water in 4,5 kg of copper sulphate.

10 Concentrated acid costing 90c per litre is diluted with water in the ratio 1:5 by volume. The diluted acid is sold at 24c per litre. What is the percentage profit?

11 Farai invests \$15 000 in a business. Four months later Denga invests \$36 000 in the business. At the end of the first year the profit is \$7 865. How much should Farai and Denga get if the profit is shared according to the amount and duration of investment?

12 Wines at \$4,50 and \$5 per litre are mixed with a third wine in the ratio 2:3:4. If the mixture costs \$5,20 a litre, what is the cost of the third wine?

13 Sacks of rice costing \$62 and \$55 per sack are mixed together in the ratio 5:2. Find the selling price of the mixture per sack in order to make a profit of 35%.

14 A motorist averages 80 km/h for the first 60 km of a journey and 96 km/h for the next 120 km. What is her average speed for the whole journey?

15 Sugar costing \$1 215/tonne is mixed with sugar costing \$1 380/tonne in the ratio 3:8. If the mixture is sold at \$1 735,50/tonne calculate the percentage profit.

Indices

The following **laws of indices** are true for all non-zero values of a, b and x.

1 $x^a \times x^b = x^{a+b}$

2 $x^a \div x^b = x^{a-b}$

3 $x^0 = 1$

4 $x^{-a} = \dfrac{1}{x^a}$

5 $(x^a)^b = x^{ab}$

6 $x^{\frac{1}{a}} = \sqrt[a]{x}$

7 $x^{\frac{a}{b}} = \sqrt[b]{x^a}$ or $(\sqrt[b]{x})^a$

Example 22 (Table 21.1)

Table 21.1

simplify	working	result
(a) $25^{\frac{1}{2}}$	$= \sqrt{25}$	$= \pm 5$
(b) $9^{\frac{1}{6}} \times 9^{\frac{1}{3}}$	$= 9^{\frac{1}{6}+\frac{1}{3}} = 9^{\frac{1}{2}} = \sqrt{9}$	$= \pm 3$
(c) $4^3 \div 4^5$	$= 4^{3-5} = 4^{-2} = \dfrac{1}{4^2}$	$= \frac{1}{16}$
(d) $2^6 \times 2^{-6}$	$= 2^{6+(-6)} = 2^{6-6} = 2^0$	$= 1$
(e) $(2^3)^{-2}$	$= 2^{-6} = \dfrac{1}{2^6}$	$= \frac{1}{64}$
(f) $27^{-\frac{2}{3}}$	$= \dfrac{1}{27^{\frac{2}{3}}} = \dfrac{1}{(\sqrt[3]{27})^2} = \dfrac{1}{3^2}$	$= \frac{1}{9}$
(g) $\left(\dfrac{81}{16}\right)^{-\frac{3}{4}}$	$= \sqrt[4]{\left(\dfrac{16}{81}\right)^3} = \left(\dfrac{\pm 2}{+3}\right)^3$	$= \pm \frac{8}{27}$

Example 23

Simplify (a) $9x^{-3} \times 2x^5$, *(b)* $(2d^3)^2$, *(c)* $2a^3 \div a^4$, *(d)* $(32n)^{\frac{1}{5}}$.

(a) $9x^{-3} \times 2x^5 = 9 \times 2 \times x^{-3} \times x^5$
$= 18 \times x^{-3+5}$
$= 18x^2$

(b) $(2d^3)^2 = 2^2 \times (d^3)^2$
$= 4d^6$

(c) $(2a^3 \div a^4 = 2(a^3 \div a^4) = 2a^{3-4}$

$= 2a^{-1} = \dfrac{2}{a}$

(d) $(32n)^{\frac{1}{5}} = 32^{\frac{1}{5}} \times n^{\frac{1}{5}}$
$= \sqrt[5]{32} \times \sqrt[5]{n}$
$= 2 \times \sqrt[5]{n}$

Exercise 21d

Simplify the expressions in 1–28.

1 $3^8 \times 3^3$ **2** $2^{-3} \div 2^5$ **3** $5^3 \times$

4 $3^6 \div 3^2$ **5** $(2^3)^4$ **6** (5^2)

7 $7^3 \times 7^{-3}$ **8** $(32)^{\frac{2}{5}}$ **9** 5^{-2}

10 $3^{-5} \div 3^{-3}$ **11** $2a^2 \times 5a$ **12** $(2a)^2$

13 $2a^2 \times (5a)^3$ **14** $2a^{-2} \times 5a$

15 $(2a)^{-2} \times 5a$ **16** $2a^{-2} \times (5a)^3$

17 2^{-4} **18** $8^{\frac{2}{3}}$ **19** $4^{-\frac{3}{2}}$

20 $\sqrt{1\frac{9}{16}}$ **21** $(\frac{1}{9})^{-\frac{1}{2}}$ **22** $2^{\frac{2}{3}} \times$

23 $0{,}09^{\frac{1}{2}}$ **24** $5^x \times 5^{-x}$ **25** $\sqrt[3]{4\ldots}$

26 $(\frac{27}{48})^{-\frac{3}{2}}$ **27** $0{,}216^{-\frac{2}{3}}$ **28** $\sqrt[3]{8a\ldots}$

29 Rewrite the following using positive in… only $\left(\text{e.g. } ab^{-2} = \dfrac{a}{b^2}\right)$.

(a) x^{-3} (b) xy^{-1} (c) $(xy)^{-}$

(d) $a^{-2}b^3$ (e) $(ab^{-3})^2$ (f) $3x^{-}$

30 Solve the following equations.

(a) $x^{\frac{1}{2}} = 3$ (b) $x^{\frac{1}{3}} = 2$ (c) $x^{-2} =$

(d) $3x^3 = 24$ (e) $x^{-\frac{2}{3}} = 9$ (f) $5x =$

Standard form

The number $A \times 10^n$ is said to be in **stan… form** or in **scientific notation** if n is a po… or negative integer and $1 \leqslant A < 10$. For exa… $5{,}3 \times 10^6$ and 9×10^{-2} are in standard fo…

Example 24

Find the value of $\dfrac{1{,}26 \times 10^3}{7 \times 10^{-1}}$, *expressing your … in standard form.*

$\dfrac{1{,}26 \times 10^3}{7 \times 10^{-1}} = \dfrac{1{,}26}{7} \times \dfrac{10^3}{10^{-1}}$

$= 0{,}18 \times 10^{3-(-1)} = 0{,}18 \times 10^4$
$= (1{,}8 \times 10^{-1}) \times 10^4$
$= 1{,}8 \times 10^{-1+4} = 1{,}8 \times 10^3$

Exercise 25e

1. Express the following numbers in standard form.

(a) [illegible] (b) 9 500 (c) 0,95

(d) [illegible]5 (e) 23 (f) 0,000 23

(g) [illegible] (h) 23 000

2. Express the following numbers in ordinary form.

(a) [illegible] $\times 10^4$ (b) $7{,}01 \times 10^2$

(c) [illegible] $\times 10^{-2}$ (d) 8×10^{-5}

(e) [illegible] $\times 10^6$ (f) $6{,}02 \times 10^3$

(g) [illegible] $\times 10^{-3}$ (h) $8{,}7 \times 10^{-1}$

3. Express $8 \times 10^{-3} + 5 \times 10^{-2} + 2 \times 10^{-1}$ (a) as a decimal fraction, (b) in standard form.

4. Find the value of each of the following, expressing your answers in standard form.

(a) ([illegible] $\times 10^{-3}) + (5{,}08 \times 10^{-2})$

(b) ([illegible] $\times 10^5) - (7{,}9 \times 10^4)$

(c) ([illegible] $\times 10^5) \times (5 \times 10^2)$

(d) ([illegible] $\times 10^{-2}) \div (9 \times 10^{-6})$

(e) $\dfrac{[illegible] \times 10^4}{[illegible] \times 10^{-4}}$

5. (a) Express in standard form (i) 480, (ii) 0,016.

(b) Evaluate $480 \div 0{,}016$, giving your answer in standard form. [Camb]

6. Given $m = 9{,}7 \times 10^4$ and $n = 8{,}3 \times 10^3$, evaluate (a) $m + n$, (b) $m - n$, giving your answers in standard form.

7. Given $p = 2{,}4 \times 10^3$ and $q = 6 \times 10^{-2}$ calculate (a) pq, (b) $\dfrac{p}{q}$, expressing each of your answers in standard form.

8. If $r = 3{,}6 \times 10^3$ express (a) r^2, (b) $\sqrt{r}$ as numbers in standard form.

9. Given that $a = 5 \times 10^4$ and $b = 3 \times 10^2$ find the value of (a) ab, (b) $a + b$, (c) $\dfrac{b}{a}$, expressing each of your answers in standard form. [Camb]

10. The pages of a dictionary are numbered from 1 to 1 632. The dictionary is 6,4 cm thick (neglecting the covers).

(a) How many thicknesses of paper make 1 632 numbered pages?

(b) Estimate the thickness of 1 page. Give your answer in metres in standard form correct to 1 significant figure.

Surds

Example 25

Simplify the following, making the number under the square root sign as small as possible.

(a) $2\sqrt{50}$ (b) $\dfrac{\sqrt{56}}{\sqrt{2}}$ (c) $\sqrt{72} \times \sqrt{75}$

(a) $2\sqrt{50} = 2 \times \sqrt{25 \times 2} = 2 \times \sqrt{25} \times \sqrt{2}$
$= 2 \times 5 \times \sqrt{2} = 10\sqrt{2}$

(b) $\dfrac{\sqrt{56}}{\sqrt{2}} = \sqrt{\dfrac{56}{2}} = \sqrt{28} = \sqrt{4 \times 7}$
$= 2 \times \sqrt{7} = 2\sqrt{7}$

(c) $\sqrt{72} \times \sqrt{75} = \sqrt{36 \times 2} \times \sqrt{25 \times 3}$
$= 6\sqrt{2} \times 5\sqrt{3}$
$= 6 \times 5 \times \sqrt{2} \times \sqrt{3}$
$= 30\sqrt{6}$

Example 26

Express each fraction with a rational denominator.

(a) $\dfrac{2}{\sqrt{5}}$ (b) $\dfrac{\sqrt{7}}{\sqrt{3}}$ (c) $\dfrac{\sqrt{8}}{2\sqrt{3}}$

(a) $\dfrac{2}{\sqrt{5}} = \dfrac{2 \times \sqrt{5}}{\sqrt{5} \times \sqrt{5}} = \dfrac{2\sqrt{5}}{5}$

(b) $\dfrac{\sqrt{7}}{\sqrt{3}} = \dfrac{\sqrt{7} \times \sqrt{3}}{\sqrt{3} \times \sqrt{3}} = \dfrac{\sqrt{7} \times \sqrt{3}}{3} = \dfrac{\sqrt{21}}{3}$

(c) $\dfrac{\sqrt{8}}{2\sqrt{3}} = \dfrac{\sqrt{8} \times \sqrt{3}}{2\sqrt{3} \times \sqrt{3}} = \dfrac{\sqrt{24}}{6} = \dfrac{2\sqrt{6}}{6}$
$= \dfrac{\sqrt{6}}{3}$

Example 27

Simplify $5\sqrt{18} - 3\sqrt{72} + 4\sqrt{50}$.

$$\begin{aligned}5\sqrt{18} - 3\sqrt{72} + 4\sqrt{50} &= 5\sqrt{9 \times 2} - 3\sqrt{36 \times 2} + 4\sqrt{25 \times 2}\\ &= 5 \times 3\sqrt{2} - 3 \times 6\sqrt{2} + 4 \times 5\sqrt{2}\\ &= 15\sqrt{2} - 18\sqrt{2} + 20\sqrt{2}\\ &= 17\sqrt{2}\end{aligned}$$

Exercise 21f

1 Simplify the following by making the number under the square root sign as small as possible.

(a) $\sqrt{18}$ (b) $\sqrt{28}$ (c) $\sqrt{108}$ (d) $\sqrt{44}$

2 Express each of the following as the square root of a single number.

(a) $2\sqrt{6}$ (b) $3\sqrt{7}$ (c) $5\sqrt{6}$ (d) $12\sqrt{2}$

3 Simplify the following by rationalising the denominators.

(a) $\dfrac{2}{\sqrt{3}}$ (b) $\dfrac{8}{\sqrt{2}}$ (c) $\dfrac{21}{\sqrt{7}}$ (d) $\dfrac{10\sqrt{2}}{\sqrt{12}}$

(e) $\dfrac{2\sqrt{5}}{\sqrt{10}}$ (f) $\dfrac{20}{\sqrt{45}}$ (g) $\sqrt{\dfrac{72}{75}}$

(h) $\dfrac{4\sqrt{5}}{3\sqrt{10}}$ (i) $\dfrac{12}{\sqrt{162}}$ (j) $\dfrac{2\sqrt{18}}{3\sqrt{12}}$

(k) $2\sqrt{3} - \dfrac{6}{\sqrt{3}} + \dfrac{3}{\sqrt{27}}$

(l) $\dfrac{3\sqrt{15} \times 2\sqrt{22}}{7\sqrt{2} \times \sqrt{165}}$

(m) $\dfrac{3\sqrt{8} \times 5\sqrt{3} \times \sqrt{7}}{\sqrt{42} \times 2\sqrt{3} \times \sqrt{15}}$

4 Simplify the following as far as possible.

(a) $\sqrt{20} + \sqrt{5}$ (b) $4\sqrt{3} - \sqrt{12}$

(c) $5\sqrt{7} - \sqrt{28}$

(d) $\sqrt{11} + \sqrt{44} - \sqrt{99}$

(e) $\sqrt{18} - \sqrt{32} + \sqrt{50}$

(f) $3\sqrt{27} - \sqrt{48} - 2\sqrt{75} + \sqrt{108}$

(g) $\sqrt{3} \times \sqrt{27}$ (h) $\sqrt{18} \times \sqrt{}$

(i) $\sqrt{27} \times \sqrt{15}$

(j) $\sqrt{5} \times \sqrt{6} \times \sqrt{10} \times \sqrt{12}$

(k) $(\sqrt{3})^3$ (l) $(\sqrt{2})^6$ (m) $(3\sqrt{}$

Number bases

Numbers are nearly always expressed in ten. However, theoretically, they m expressed in any base. In each case, the p of the digits indicates their value:

Base ten (denary)

$693_{ten} = \mathbf{6} \times 10^2 + \mathbf{9} \times 10^1 + \mathbf{3}$

Base five

$4\,302_{five} = \mathbf{4} \times 5^3 + \mathbf{3} \times 5^2 + \mathbf{0} \times 5^1 +$

Base two (binary)

$10\,110_{two}$
$= \mathbf{1} \times 2^4 + \mathbf{0} \times 2^3 + \mathbf{1} \times 2^2 + \mathbf{1} \times 2^1 +$

Notice that the greatest digit in any num 1 less than the base number.

Example 28

Express 271_{ten} *as a base five number.*

Use the method of repeated division.

5	271	
5	54 + **1**	i.e. $54 \times 5^1 + \mathbf{1} \times 1$
5	10 + **4**	i.e. $10 \times 5^2 + \mathbf{4} \times 5^1$
5	2 + **0**	i.e. $2 \times 5^3 + \mathbf{0} \times 5^2$
	0 + **2**	i.e. $0 \times 5^4 + \mathbf{2} \times 5^3$
	↑	↑

Reading the remainders upwards gives

$271_{ten} = 2\,041_{five}$

Check:

$$\begin{aligned}2\,041_{five} &= 2 \times 5^3 + 0 \times 5^2 + 4 \times 5^1 +\\ &= 250 + 0 + 20 + 1\\ &= 271_{ten}\end{aligned}$$

Example 29

Convert $1\,010\,110_{two}$ *to base five.*

First convert $1\,010\,110_{two}$ to base ten.

$$1\,010\,110_{two} = 2^6 + 0 + 2^4 + 0 + 2^2 + 2^1 + 0$$
$$= 64 + 0 + 16 + 0 + 4 + 2 + 0$$
$$= 86_{ten}$$

Then convert 86_{ten} to base five.

$$86_{ten} = 75 + 10 + 1$$
$$= 3 \times 25 + 2 \times 5 + 1$$
$$= 3 \times 5^2 + 2 \times 5^1 + 1$$
$$= 321_{five}$$

Example 30

Given $P = 242_{five}$ *and* $Q = 14_{five}$ *calculate*
(*a*) P + Q, (*b*) P − Q, (*c*) P × Q, (*d*) P ÷ Q

(a) *working:*

```
 242
+ 14
 311
```

method:
From the right,
$2 + 4 = 6 = 1 \times 5 + 1$
Write 1, carry 1,
$4 + 1 + 1 = 6 = 1 \times 5 + 1$
Write 1, carry 1
$2 + 1 = 3$

(b) *working:*

```
 242
− 14
 223
```

method:
$1 \times 5 + 2 - 4 = 3$
$3 - 2 = 2$

(c) *working:*

```
   242
  × 14
  2123
  242
 10043
```

method:
$4 \times 2 = 8 = 1 \times 5 + 3$
Write 3, carry 1.
$4 \times 4 + 1 = 17 = 3 \times 5 + 2$
Write 2, carry 3.
... and so on

(d) *working:*

```
     13
 14)242
    14
    102
    102
```

method:
As in normal long division with subcalculations in base five.

Exercise 21g

1 What is the value of the 3 in 4 312 if the number is in (a) base ten, (b) base five?

2 Arrange the following numbers in ascending order (i.e. from smallest to largest).
$10\,111_{two}$, 41_{five}, 22_{ten}

3 If $ab_{ten} = 32_{five}$, what is $a + b$?

4 What is the value, in base ten, of the digit 1 in each of the following?
(a) $10\,000_{ten}$ (b) 214_{five} (c) $1\,203_{five}$

5 Convert the following numbers to base ten.
(a) $11\,011_{two}$ (b) 230_{five}
(c) 413_{five} (d) $1\,011_{two}$

6 Express the following numbers in base two.
(a) 33_{ten} (b) 31_{ten}
(c) 97_{ten} (d) 111_{ten}

7 Express the following numbers in base five.
(a) 62_{ten} (b) 312_{ten}
(c) 626_{ten} (d) 555_{ten}

8 Express the following numbers in base two.
(a) 31_{five} (b) 104_{five}
(c) 202_{five} (d) 111_{five}

9 Express each of the following numbers in base five.
(a) $101\,010_{two}$ (b) $1\,110\,110_{two}$
(c) $1\,011\,111_{two}$ (d) $1\,111\,001_{two}$

10 Do the following additions in the given bases.
(a) 124 + 012 + 42 + 1 124 (*base five*)
(b) 11 011 + 1 011 + 11 110 (*base two*)

11 Subtract
(a) $2\,144_{five}$ from $10\,023_{five}$,
(b) $110\,111_{two}$ from $1\,110\,100_{two}$.

12 Find the missing number if this addition is in base five:

```
  1230
   232
  ****
 + 123
  4011
```

13 Simplify the following. Express each answer in the given base.
(a) $302_{five} \times 13_{five}$
(b) $11\,011_{two} \times 111_{two}$
(c) $143_{five} \div 22_{five}$
(d) $110\,110_{two} \div 1\,001_{two}$

14 Evaluate the following binary multiplications.
(a) 1 101 × 11 (b) 10 111 × 1 011
(c) 1 011 101 × 1 110 (d) 101 010 × 110

15 Evaluate the following binary divisions.
(a) 110 111 ÷ 1 011
(b) 110 010 ÷ 1 010
(c) 110 110 ÷ 1 001
(d) 10 000 101 ÷ 10 011

16 Fig. 21.9 shows how a strip of graph paper may be used as a punch tape. The first two rows show how to space the 'holes'.

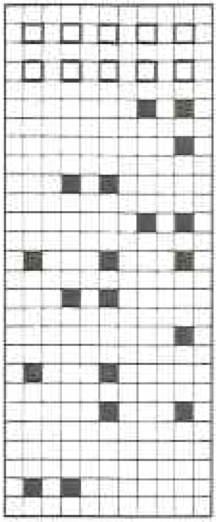


Fig. 21.1

The next row represents the first letter of the message. It shows the binary number 00 011, equivalent to 3. This represents C, the 3rd letter of the alphabet.
(a) Find out what the message is in Fig. 21.1.
(b) Use a strip of graph paper to write the following messages in binary code.
(i) APPROXIMATE
(ii) COLLECT DATA

Ready reckoners and tabulated data

A **ready reckoner** is a book which contains tables of numerical data which assist calculation. Sometimes ready reckoners are more convenient to use than calculators. Numerical information can be presented in many other ways, e.g. in simple lists and diagrams.

Exercise 21h contains extracts from a ... reckoner, conversion tables and other ... information. Read the titles and column ... very carefully to find out what each is ...

Exercise 21h

Table 21.2 is a page from a ready reckon... gives the cost of from 1 to 140 articles ... cents each.

Table 21.2

69 cents

1	0,69	36	24,84	71	48,99	106	73,14
2	1,38	37	25,53	72	49,68	107	73,83
3	2,07	38	26,22	73	50,37	108	74,52
4	2,76	39	26,91	74	51,06	109	75,21
5	3,45	40	27,60	75	51,75	110	75,90
6	4,14	41	28,29	76	52,44	111	76,59
7	4,83	42	28,98	77	53,13	112	77,28
8	5,52	43	29,67	78	53,82	113	77,97
9	6,21	44	30,36	79	54,51	114	78,66
10	6,90	45	31,05	80	55,20	115	79,35
11	7,59	46	31,74	81	55,89	116	80,04
12	8,28	47	32,43	82	56,58	117	80,73
13	8,97	48	33,12	83	57,27	118	81,42
14	9,66	49	33,81	84	57,96	119	82,11
15	10,35	50	34,50	85	58,65	120	82,80
16	11,04	51	35,19	86	59,34	121	83,49
17	11,73	52	35,88	87	60,03	122	84,18
18	12,42	53	36,57	88	60,72	123	84,87
19	13,11	54	37,26	89	61,41	124	85,56
20	13,80	55	37,95	90	62,10	125	86,25
21	14,49	36	38,64	91	62,79	126	86,94
22	15,18	57	39,33	92	63,48	127	87,63
23	15,87	58	40,02	93	64,17	128	88,32
24	16,56	59	40,71	94	64,86	129	89,01
25	17,25	60	41,40	95	65,55	130	89,70
26	17,94	61	42,09	96	66,24	131	90,39
27	18,63	62	42,78	97	66,93	132	91,08
28	19,32	63	43,47	98	67,62	133	91,77
29	20,01	64	44,16	99	68,31	134	92,46
30	20,70	65	44,85	100	69,00	135	93,15
31	21,39	66	45,54	101	69,69	136	93,84
32	22,08	67	46,23	102	70,38	137	94,53
33	22,77	68	46,92	103	71,07	138	95,22
34	23,46	69	47,61	104	71,76	139	95,91
35	24,15	70	48,30	105	72,45	140	96,60

Use Table 21.2 to answer questions 1-5.

1 Find the cost of the following.
(a) 8 articles at 69c each
(b) 22 articles at 69c each
(c) 49 articles at 69c each
(d) 87 articles at 69c each
(e) 133 articles at 69c each
(f) 10 articles at 69c each

2 Find the cost of the following.
   (a) 14 watches at $69 each
   (b) 29 cups at $6,90 each
   (c) 5 motorcycles at $6 900 each
   (d) 69 eggs at 25c each
   (e) 690 pencils at 32c each
   (f) 6,9 metres of cloth at $7,90 per metre

3 A trader bought 69 notebooks for 95c each and sold them all for $1,25 each. Find the profit.

4 A shop bought two dozen books for $122,50 altogether. The books were sold for $6,90 each. Calculate the profit.

5 Calculate the cost of sending a telegram if it contains 69 words, each word costing 36c.

Table 21.3 gives salaries from 320 units per annum to 9 000 units per annum. Each salary is broken down into monthly and weekly rates.

Table 21.3
Annual salary **320 – 9 000 units**

	Month	Week		Month	Week
320	26,667	6,154	1 600	133,333	30,769
325	27,083	6,250	1 700	141,667	32,692
330	27,500	6,346	1 750	145,833	33,654
340	28,333	6,538	1 800	150,000	34,615
350	29,167	6,731	1 900	158,333	36,538
360	30,000	6,923	2 000	166,667	38,462
370	30,833	7,115	2 100	175,000	40,385
375	31,250	7,212	2 200	183,333	42,308
380	31,667	7,308	2 250	187,500	43,269
390	32,500	7,500	2 300	191,667	44,231
400	33,333	7,692	2 400	200,000	46,154
425	35,417	8,173	2 500	208,333	48,077
450	37,500	8,654	2 600	216,667	50,000
475	39,583	9,135	2 700	225,000	51,923
500	41,667	9,615	2 750	229,167	52,885
525	43,750	10,096	2 800	233,333	53,846
550	45,833	10,577	2 900	241,667	55,769
575	47,917	11,058	3 000	250,000	57,692
600	50,000	11,538	3 100	258,333	59,615
650	54,167	12,500	3 200	266,667	61,538
700	58,333	13,462	3 300	275,000	63,462
750	62,500	14,423	3 400	283,333	65,385
800	66,667	15,385	3 500	291,667	67,308
850	70,833	16,346	4 000	333,333	76,923
900	75,000	17,308	4 500	375,000	86,538
950	79,167	18,269	5 000	416,667	96,154
1 000	83,333	19,231	5 500	458,333	105,769
1 100	91,667	21,154	6 000	500,000	115,385
1 200	100,000	23,077	6 500	541,667	125,000
1 250	104,167	24,038	7 000	583,333	134,615
1 300	108,333	25,000	7 500	625,000	144,231
1 400	116,667	26,923	8 000	666,667	153,846
1 500	125,000	28,846	9 000	750,000	173,077

Use Table 21.3 to answer questions 6–10. G[ive]
all answers to the nearest cent.

6 Find the monthly pay equivalent to a[n]
annual salary of
   (a) $950 (b) $3 100 (c) $8 000
   (d) $8 400 (e) $18 000 (f) $17 360.

7 Find the weekly wage equivalent to a[n]
annual pay of
   (a) $4 500 (b) $7 500 (c) $9 000
   (d) $12 000 (e) $17 500 (f) $8 575.

8 In 1992 the starting salary for a cleric[al]
officer was $11 340. How much was th[is]
per month?

9 A company employs 1 caretaker at $9 50[0]
per annum, 1 receptionist at $14 000 p[er]
annum and 3 clerks at $16 500 per annum.
Find the total wages paid to the worke[rs]
each week. (Discuss possible reasons wh[y]
the workers receive different wages.)

10 A Co-operative employs 69 workers who ar[e]
each paid $13 300 per annum. Use Table[s]
21.2 and 21.3 to find the total wage bill fo[r]
the Co-operative each week.

Table 21.4 is a *double conversion table* for convertin[g]
between litres and gallons. The central bol[d]
numbers refer to either of the columns on eac[h]
side. For example, 1 litre = 0,22 gallons an[d]
1 gallon = 4,546 litres.

Table 21.4

litres		gallons
4,546	**1**	0,220
9,092	**2**	0,440
13,638	**3**	0,660
18,184	**4**	0,880
22,730	**5**	1,100
27,276	**6**	1,320
31,822	**7**	1,540
36,368	**8**	1,760
40,914	**9**	1,980

Use Table 21.4 to answer questions 11–13.

11 Express the following in litres, correct to
one decimal place.
   (a) 4 gallons (b) 7 gallons
   (c) 9 gallons (d) 10 gallons
   (e) 60 gallons (f) 0,8 gallons

12 Express the following in gallons, correct to one decimal place.
(a) 3 litres (b) 5 litres (c) 8 litres
(d) 10 litres (e) 90 litres (f) 6,5 litres

13 The petrol tank of a car contains 42 litres. How many gallons does it hold?

Table 21.5 converts speeds of km/h into either m/s (metres per second) or mph (miles per hour).

Table 21.5

km/h	m/s	mph
1	0,28	0,62
2	0,56	1,24
3	0,83	1,86
4	1,11	2,49
5	1,39	3,11
6	1,67	3,73
7	1,94	4,35
8	2,22	4,97
9	2,5	5,59

Use Table 21.5 to answer questions 14–16.

14 Express the following speeds in metres per second.
(a) 5 km/h (b) 8 km/h
(c) 60 km/h (d) 70 km/h
(e) 89 km/h (f) 134 km/h

15 Express the following speeds in miles per hour.
(a) 4 km/h (b) 7 km/h
(c) 10 km/h (d) 80 km/h
(e) 8,5 km/h (f) 44 km/h

16 Many years ago, when imperial units were used, the speed limit in built up areas was 30 mph. This was changed to 50 km/h. Which speed is slower (and therefore safer)?

Figure 21.2 may be used to convert between temperatures in degrees Celsius and Fahrenheit.

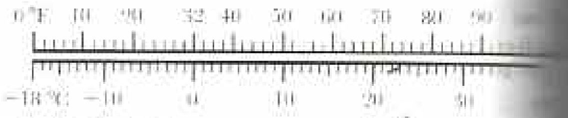


Fig. 21.2 Thermometer conversion

Use Fig. 21.2 to answer questions 17–19.

17 Convert the following temperatures to Celsius. Estimate your answers to the nea[rest] °C.
(a) 32 °F (b) 50 °F (c) 8[…]
(d) 68 °F (e) 23 °F (f) 9[…]

18 Convert the following temperatures to Fahrenheit. Estimate your answers to the nearest °F.
(a) −10 °C (b) 20 °C (c) 3[…]
(d) 5 °C (e) 34 °C (f) 1[…]

19 (a) If the temperature rises by 10° on the Celsius scale, how much does it rise on the Fahrenheit scale?
(b) Hence estimate what 50 °C is equiva[lent] to on the Fahrenheit scale.

20 Table 21.6 is a table for converting Z$ to U[S$].

Table 21.6

Z$	US$
1	0,47
2	0,93
5	2,34
10	4,67
20	9,35
50	23,37
100	46,73

Use Table 21.6 to change the follow[ing] amounts to US$.
(a) Z$15 (b) Z$40 (c) Z$9 (d) Z$[…]
(e) Z$135 (f) Z$26 (g) Z$38 (h) Z$[…]

Chapter 22

Algebraic processes

Sets

Refer to the list on page 285 for the symbols of set language and an explanation of their meanings.

Example 1

If $\mathscr{E} = \{1; 2; 3; 6; 9; 18\}$, $X = \{2; 6; 18\}$ *and* $Y = \{1; 3; 6\}$, *list the elements of* (*a*) $X' \cap Y$, (*b*) $X \cup Y'$, (*c*) $(X \cup Y)'$.

X' is the **complement** of X, i.e. those elements in $\mathscr{E}$ which are *not* in X.

$X' = \{1; 3; 9\}$
$Y' = \{2; 9; 18\}$

(a) $X' \cap Y = \{1; 3; 9\} \cap \{1; 3; 6\}$
$= \{1; 3\}$

(b) $X \cup Y' = \{2; 6; 18\} \cup \{2; 9; 18\}$
$= \{2; 6; 9; 18\}$

(c) $(X \cup Y)'$ is the complement of $X \cup Y$.
$X \cup Y = \{1; 2; 3; 6; 18\}$
$(X \cup Y)' = \{9\}$

Example 2

Two sets A *and* B *are such that* $n(A) = 11$ *and* $n(B) = 6$. *Given that* $n(\mathscr{E}) = 15$ *find* (*a*) *the smallest possible value of* $n(A \cap B)$, (*b*) *the largest possible value of* $n(A \cup B)'$. [*Camb*]

Let $n(A \cap B) = x$
and $n(A \cup B)' = y$.

Fig. 22.1 is a **Venn diagram** showing the numbers of elements in each subset.

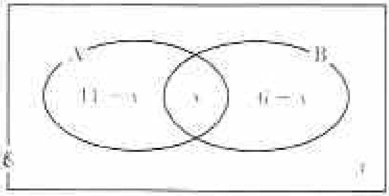


Fig. 22.1

Since $n(\mathscr{E}) = 15$,

$(11 - x) + x + (6 - x) + y = 15$
$17 - x + y = 15 \quad (1)$

(a) From (1), $x = 2 + y$

The smallest value of x occurs when y is lea
i.e. when y is 0.

The smallest possible value of $n(A \cap B)$ is 2.

(b) From (1), $y = x - 2$.

The largest value of y occurs when x is greate
i.e. when x is 6.

Note: 6 is the greatest value of x which will gi non-negative values for the numbers of elemen in the subsets.

The largest value of $n(A \cup B)' = 6 - 2 = 4$

Example 3

A number of travellers were questioned about transport they used on a particular day. Each of th used one or more of the methods shown in the Ve diagram in Fig. 22.2.

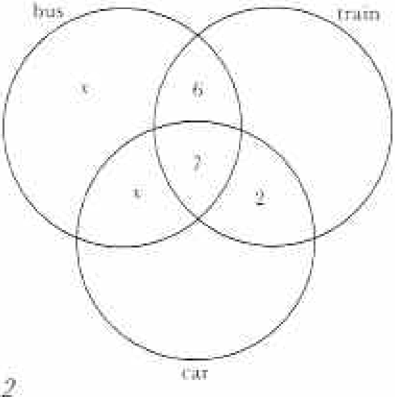


Fig. 22.2

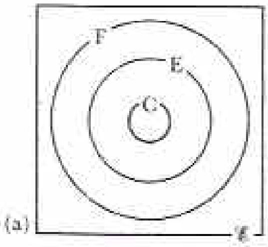
Of those questioned, 6 said that they travelled by b and train only, 2 by train and car only and 7 by bu train and car. The number x who travelled by bus on was equal to the number who travelled by bus and car on. Given that 35 people used buses and 25 people us trains find (*a*) *the value of* x, (*b*) *the number w*

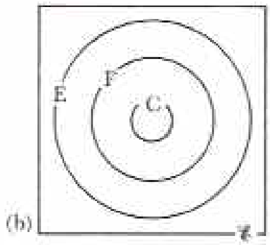
travelled by train only, (c) the number who travelled by at least two methods of transport.
Given also that 85 people were questioned altogether, calculate (d) the number who travelled by car only.
[*Camb*]

(a) If 35 people used buses, then
$x + x + 6 + 7 = 35$
$\Leftrightarrow \quad 2x + 13 = 35$
$\Leftrightarrow \quad 2x = 22$
$\Leftrightarrow \quad x = 11$
(b) Let y people travel by train only. If 25 people used trains, then
$6 + 7 + 2 + y = 25$
$\Leftrightarrow \quad 15 + y = 25$
$\Leftrightarrow \quad y = 10$
10 people travelled by train only.
(c) The number who travelled by at least two methods (i.e. those using two or three methods)
$= x + 6 + 2 + 7$
$= 11 + 15 = 26$
(d) Let z people travel by car only, then
$x + x + 6 + 7 + 2 + y + z = 85$
$11 + 11 + 15 + 10 + z = 85$
$\Leftrightarrow \quad 47 + z = 85$
$\Leftrightarrow \quad z = 38$
38 people travel by car only.

Example 4
Given $\mathscr{E}$ = {all vehicles}, F = {four-wheeled vehicles}, E = {vehicles with engines}, C = {cars} and (i) all cars have four wheels, (ii) all cars have engines. Rewrite statements (i) and (ii) in set notation and draw as many Venn diagrams as possible which correctly illustrate them. Which, if any, of the following deductions follow from the given statements? (a) All four-wheeled vehicles have engines. (b) All vehicles with engines are either cars or have four wheels.

Fig. 22.3 shows all the possible Venn diagrams.





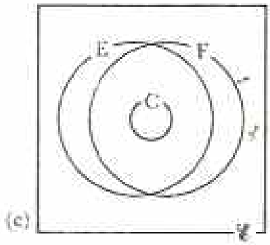


Fig. 22.3

(i): $C \subseteq F$ (ii) $C \subseteq E$

(a) *FALSE* The deduction is equivalent $F \subseteq E$ which is not true in either Fig. 22.3(or Fig. 22.3(c).
(b) *FALSE* The deduction is equivalent $E \subseteq C \cup F$ which is not true in either Fi 22.3(b) or Fig. 22.3(c).

Exercise 22a

1 Given $\mathscr{E}$ = {r;e;v;o;l;t;i;n;g}, A = {l;i;o; and B = {t;i;g;e;r}, list the members of th following sets.
(a) $A \cup B$ (b) $A \cap B$ (c) A'
(d) B' (e) $A' \cup B'$ (f) $A' \cap B'$
(g) $(A \cup B)'$ (h) $(A \cap B)'$ (i) $A' \cap B$

2 For each of the following, make a copy Fig. 22.4 then shade the given set.
(a) $(P \cup Q)'$ (b) $P' \cup Q'$ (c) $(P \cap Q)'$
(d) $P' \cap Q'$ (e) $P' \cup Q$ (f) $P \cap Q'$

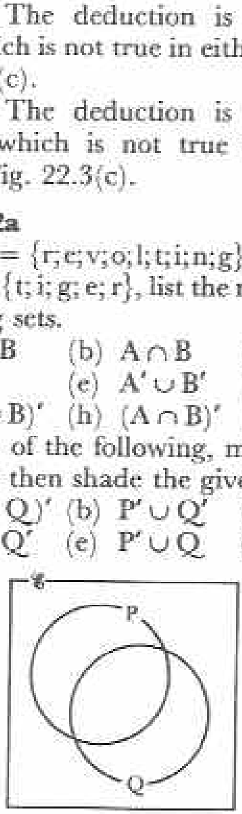


Fig. 22.4

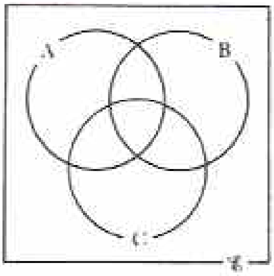


Fig. 22.5

3 For each of the following, make a copy of Fig. 22.5 then shade the given set.

(a) $(A \cup B) \cap C$ (b) $(A \cap B) \cup C$
(c) $A \cup (B \cap C')$ (d) $A \cap (B \cup C')$
(e) $A \cap (B \cup C)'$ (f) $(A \cap C') \cup (B \cap C)$

4 If $\mathscr{E} = \{1;2;3;...;10\}$, list the members of the following subsets of $\mathscr{E}$.

(a) $\{x:x \quad 2\}$
(b) $\{x:x$ is a multiple of $5\}$
(c) $\{x:3 < x < 8\}$
(d) $\{x:3x - 3 = 3\}$
(e) $\{x:x$ is a factor of $360\}$
(f) $\{x:(x-2)(x-4) = 0\}$

5 Given that $\mathscr{E} = \{x:x$ is an integer, $2 \geqslant x \geqslant 10\}$, $A = \{x:x$ is a prime number$\}$, $B = \{x:x$ is a multiple of $3\}$, (a) find $n(A \cup B)$, (b) list the elements of the set $A' \cap B'$. [Camb]

6 Given that $P = \{a;b;c;d\}$, $Q = \{b;c;d;e;f\}$ and $R = \{a;c;f;g\}$, mark the members of these sets clearly on the Venn diagram. (Make a copy of Fig. 22.6 in your exercise book.)

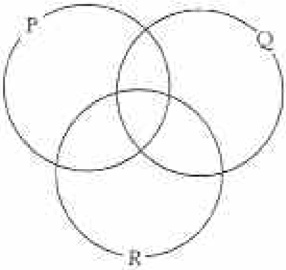


Fig. 22.6

Using your diagram, or otherwise, list the members of $(P \cup Q) \cap R$. [Camb]

7 $\mathscr{E} = \{$books$\}$, $A = \{$algebra books$\}$, $B = \{$books with brown covers$\}$. Show, by shading on a Venn diagram, the set of books with brown covers which are not algebra books.

8 A company employs 79 people, 52 of whom are men, 38 people, including all the women, are clerical staff. Draw a suitable Venn diagram to show this information. Hence or otherwise find the number of men that are clerical staff.

9 The numbers of elements of the subsets of the Venn diagram in Fig. 22.7 are as shown.

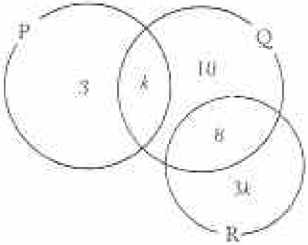


Fig. 22.7

If $\mathscr{E} = P \cup Q \cup R$ and $n(\mathscr{E}) = 33$, find

(a) k (b) $n(P \cup R)$
(c) $n(P \cap R)$ (d) $n(R' \cap Q)$

10 Sets $\mathscr{E}$, P and Q are such that $n(\mathscr{E}) = 30$, $n(P) = 12$ and $n(Q) = 25$. Find (a) the smallest possible value of $n(P \cap Q)$, (b) the range of possible values of $n(P' \cap Q')$.

11 In the Venn diagram of Fig. 22.8, $\mathscr{E}$ is the set of all children in a certain chosen group, $A = \{$children in Youth Club A$\}$ and $B = \{$children in Youth Club B$\}$. The letters p, q, x and y in the diagram represent the number of children in each subset.

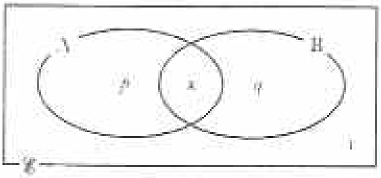


Fig. 22.8

Given that $n(\mathscr{E}) = 200$, $n(A) = 75$ and $n(B) = 35$,

(a) express p in terms of x,
(b) find the smallest possible value of y,
(c) find the largest possible value of x,
(d) find the value of q if $p = 45$. [Camb]

12 In a team of 30 athletes, some have a body-mass less than 70 kg, twice as many are over 60 kg and 12 have a body-mass between 60 kg and 70 kg. Show this information in a labelled Venn diagram. Find the number of athletes whose body-mass is (a) 60 kg or less, (b) 70 kg or more.

13 There are 150 people at an International Medical Conference. Forty are Africans, 70 are women and 110 are doctors. Twelve of the women are Africans, 46 of the doctors are women and 31 of the Africans are doctors. If 5 of the African

men are not doctors, (a) how many of the African women are doctors, and (b) how many of the men are neither African nor doctors?

14 The 90 members of a sports club play at least one of the games tennis, football, volleyball. 10 play tennis and football, 19 play football and volleyball and 29 play tennis and volleyball. n people play all three games. $2n$ people each play only one game. How many play football altogether?

15 Use a Venn diagram to show that if $G \subseteq H$, then $H' \subseteq G'$ and $G \cap H' = \varnothing$. Hence state two of the conclusions that can be deduced from the following statements.
$\mathscr{E}$ = {students}, G = {girls}, H = {happy students}, $H' \neq \varnothing$ and *all girls are happy students*.

Simplification of algebraic expressions

Collecting like terms

Expressions such as $2x$, $3x$, $-7x$ are called **terms in** x. Since they are all terms in x they are called **like terms**. $2x$ and $3y$ are **unlike terms**. When simplifying algebraic expressions like terms are grouped together.

Example 5

Simplify $2x^2 - 3ax + 4x^2 - 6xa - x^2$.

$2x^2 - 3ax + 4x^2 - 6xa - x^2$
$= 2x^2 + 4x^2 - x^2 - 3ax - 6ax$
$= 5x^2 - 9ax$

Notice that $6xa = 6ax$; the order of the letters is not important.

Exercise 22b

Simplify the following expressions.

1 $6x - 9y + 10x$ **2** $2p^2 + 3q - 8q$
3 $7a - 2b - 3a - 8b - a$
4 $8 - 3x^2 + 21 + 18x^2 - 6$
5 $x^2 - xy - 9xy + 9y^2$
6 $5u^2v - 3vu^2 + uv^2 - 5v^2u$
7 $d^2 + 2ad - 2d^2 + 8ad + 3d^2 - 20ad$
8 $2a^2 - a + 6a^2 - 8a^2 - 6a + 5a^2 - + 10a$
9 $-3x^2 + 5x^2 + 6x^2 + 4x^3 + 2x^4 - x^2$
10 $m^2 - 2mn + 3n^2 + 5n^2 + 8mn - 4n^2 - 7m$

Removing brackets

Example 6

Remove brackets from the expression $-3(2x - 5y + 6z)$.

$-3(2x - 5y + 6z)$
$= (-3) \times (2x) + (-3) \times (-5y) + (-3) \times (+6z)$
$= -6x + 15y - 18z$

Notice that every term inside the bracket is multiplied by the quantity outside the bracket.

Example 7

Remove brackets and simplify $x(x - 2y) - 5y(3x - 6y)$.

$x(x - 2y) - 5y(3x - 6y)$
$= x^2 - 2xy - 15xy + 30y^2$
$= x^2 - 17xy + 30y^2$

The expression $(a + b)(c + d)$ can be expanded in two ways:

either: $(a + b)(c + d) = (a + b)c + (a + b)d$
$= ac + bc + ad + bd$
or: $(a + b)(c + d) = a(c + d) + b(c + d)$
$= ac + ad + bc + bd$
$= ac + bc + ad + bd$

The result is the same in both cases. Examples 8 and 9 show the two methods.

Example 8

Expand $(2n + 3)(3n - 7)$.

$(2n + 3)(3n - 7)$
$= (2n + 3)3n + (2n + 3)(-7)$
$= 6n^2 + 9n - 14n - 21$
$= 6n^2 - 5n - 21$

Example 9

Expand $(2x - 3y)(5x - 2y)$.

$(2x - 3y)(5x - 2y)$
$= 2x(5x - 2y) - 3y(5x - 2y)$
$= 10x^2 - 4xy - 15xy + 6y^2$
$= 10x^2 - 19xy + 6y^2$

By expanding brackets and collecting terms it can be shown that:

$$(\boldsymbol{a} + \boldsymbol{b})^2 = \boldsymbol{a}^2 + 2\boldsymbol{ab} + \boldsymbol{b}^2$$
$$(\boldsymbol{a} - \boldsymbol{b})^2 = \boldsymbol{a}^2 - 2\boldsymbol{ab} + \boldsymbol{b}^2$$

Example 10

Expand (a) $(4p + q)^2$, (b) $(5x - 3y)^2$.

(a) $(4p + q)^2 = (4p)^2 + 2(4p)(q) + (q)^2$
$= 16p^2 + 8pq + q^2$

(b) $(5x - 3y)^2 = (5x)^2 - 2(5x)(3y) + (3y)^2$
$= 25x^2 - 30xy + 9y^2$

Exercise 22c

Expand brackets and collect like terms where possible.

1	$4(9a + 1)$	**2**	$-5(6x - 2y + 1)$
3	$3x(14 - y)$	**4**	$-2a(5a - b + 3c)$
5	$8 - (x + 3)$	**6**	$11x^2 - x(3x + 4)$
7	$6x(x - 2) - 5(x - 2)$		
8	$5(6 - b^2) + 2(1 - b^2)$		
9	$(a + 2b)(3c + 4d)$	**10**	$(x + 1)(x + 8)$
11	$(2a + b)(5a + 8b)$	**12**	$(2a + b)(5a - 8b)$
13	$(2a - b)(5a + 8b)$	**14**	$(2a - b)(5a - 8b)$
15	$(2x - 5)(x - 3)$	**16**	$(3 - x)(4 + 5x)$
17	$(3a + 1)^2$	**18**	$(b - 4)^2$
19	$(2c - 5d)^2$	**20**	$(4m - 3n)^2$

Factorisation

Example 11

Factorise $(x - a)(3x + 2a) - (x - a)^2$.

$(x - a)$ is a **common factor** of the two parts of the expression.

$(x - a)(3x + 2a) - (x - a)^2$
$= (x - a)[(3x + 2a) - (x - a)]$
$= (x - a)(3x + 2a - x + a)$
$= (x - a)(2x + 3a)$.

Example 12

Factorise $am + 3bn - an - 3bm$.

The terms am and an have a in common.
The terms $3bm$ and $3bn$ have $3b$ in common.
Grouping pairs in this way,

$am + 3bn - an - 3bm = am - an - 3bm + 3bn$
$= a(m - n) - 3b(m - n)$
$= (m - n)(a - 3b)$

Example 13

Factorise $10x^2 + 5x - 2x - 1$.

$10x^2 + 5x - 2x - 1$
$= 5x(2x + 1) - 1(2x + 1)$
$= (2x + 1)(5x - 1)$

Example 14

Factorise $3x^2 - 13x - 10$.

1st step: Find the product of the first and last term
$3x^2 \times (-10) = -30x^2$

2nd step: Find two terms such that their produc is $-30x^2$ and their sum is $-13x$ (the middl term). Since the middle term is negative, onl consider those factors in which the negativ term is numerically greater than the positive term

	factors of $-30x^2$	*sum of factors*
(a)	$-30x$ and $+x$	$-29x$
(b)	$-15x$ and $+2x$	$-13x$
(c)	$-10x$ and $+3x$	$-7x$
(d)	$-6x$ and $+5x$	$-x$

Of these, only (b) gives the required result.

3rd step: Replace $-13x$ in the given expressio by $-15x + 2x$. Factorise by grouping.

$3x^2 - 13x - 10 = 3x^2 - 15x + 2x - 10$
$= 3x(x - 5) + 2(x - 5)$
$= (x - 5)(3x + 2)$

Notice in Example 14, that when the require result was found in line (b), it was not reall necessary to do lines (c) and (d).

Example 15

Factorise $6 - 15x + 9x^2$.

3 is a common factor of all the terms. Take thi out first.

$6 - 15x + 9x^2 = 3(2 - 5x + 3x^2)$
$= 3(2 - 2x - 3x + 3x^2)$*
$= 3[2(1 - x) - 3x(1 - x)]$
$= 3(1 - x)(2 - 3x)$

* The reason why $-2x - 3x$ was substituted fo $-5x$ is left as an exercise. See Example 14.

$$\boldsymbol{a}^2 + 2\boldsymbol{ab} + \boldsymbol{b}^2 = (\boldsymbol{a} + \boldsymbol{b})^2$$
$$\boldsymbol{a}^2 - 2\boldsymbol{ab} + \boldsymbol{b}^2 = (\boldsymbol{a} - \boldsymbol{b})^2$$
$$\boldsymbol{a}^2 - \boldsymbol{b}^2 = (\boldsymbol{a} + \boldsymbol{b})(\boldsymbol{a} - \boldsymbol{b})$$
$\boldsymbol{a}^2 + \boldsymbol{b}^2$ **has no factors**

Example 16
Factorise $d^2 - 10dm + 25m^2$.

d^2 is the square of d.
$25m^2$ is the square of $5m$.
$10dm$ is twice the product of d and $5m$.
$d^2 - 10dm + 25m^2 = (d - 5m)^2$.

Example 17
Factorise $16a^2 - 25b^2$.

$$16a^2 - 25b^2 = (4a)^2 - (5b)^2 = (4a + 5b)(4a - 5b)$$

Example 18
Factorise $5m^2 - 80$.

$$5m^2 - 80 = 5(m^2 - 16) = 5(m + 4)(m - 4)$$

Exercise 22d
Factorise:

1 $(3c - d)(m - n) + (3c - d)(2m - 3n)$
2 $ac + bc + 2ad + 2bd$
3 $x^2 - 16a^2$ **4** $4c^2 - 25d^2$
5 $a^2 - 3a - 10$ **6** $a^2 - 3ab - 10b^2$
7 $a^2b^2 - 3ab - 10$ **8** $5m^2 - 45n^2$
9 $hm - 2km - 2hn + 4kn$
10 $(2a + b)^2 - (2a + b)(a - 3b)$
11 $3m^2 - 10m + 3$ **12** $c^2d^2 - 81$
13 $16x^2 - 9a^2m^2$ **14** $6n^2 + 13n + 6$
15 $4a^2 + 20a + 25$ **16** $9h^2 - 36k^2$
17 $(5a - 3b)(a - 2b) - (4a + b)(a - 2b)$
18 $25a^2b^2c^2 - 9d^2$ **19** $\dfrac{m^2}{9} - \dfrac{n^2}{4}$
20 $3su + tu - 6sv - 2tv$
21 $9x^2 - 12xy + 4y^2$
22 $12d^2 + 5d - 2$ **23** $x^4 - y^2$
24 $10m^2n^2 - 7mn - 12$
25 $16 - n^4$ **26** $(a + b)^2 - c^2$
27 $x^2 - (m - n)^2$
28 $(m + 2n)(3a - b) - (a - 3b)(m + 2n)$
29 $2 - h - 15h^2$
30 $2am - bm + 3bn - 6an$
31 $a^2 - 15ab + 54b^2$ **32** $m^2 - 15mn - 54n^2$
33 $(c - 2d)^2 - 9c^2$
34 $12x^2 + 35xy + 18y^2$
35 $(h - k)(2h - 3k) + (h - k)^2$
36 $6acx - 8ady + 4acy - 12adx$
37 $6a^2 - 19ax - 36x^2$ **38** $25a^2 - 4(m + 2n)^2$
39 $25a^2 - 4(a - 2b)^2$ **40** $9a^2 - (3a - 2b)^2$

Factorisation can often be used to simplify calculations.

Example 19
Simplify $63 \times 29 + 37 \times 29$.

$$63 \times 29 + 37 \times 29 = 29(63 + 37) = 29 \times 100 = 2\,900$$

Example 20
Evaluate $17{,}9^2 - 12{,}1^2$ *by using factorisation.*

$$17{,}9^2 - 12{,}1^2 = (17{,}9 + 12{,}1)(17{,}9 - 12{,}1) = 30 \times 5{,}8 = 174$$

Exercise 22e
Use factorisation to simplify the following.

1 $18 \times 57 + 18 \times 43$
2 $23 \times 119 - 23 \times 19$
3 $243 \times 4 + 243 \times 6$
4 $28 \times 752 + 28 \times 248$
5 $63 \times 47 - 43 \times 47$
6 $61 \times 127 - 77 \times 61$
7 $106^2 - 94^2$ **8** $8{,}78^2 - 1{,}22^2$
9 $5 \times 9{,}2^2 - 5 \times 4{,}8^2$
10 $\pi R^2h - \pi r^2h$, where $\pi = 3\frac{1}{7}$, $R = 18$ cm, $r = 10$ cm and $h = 15$ cm.

Algebraic fractions

Example 21
Simplify $\dfrac{3x - 2}{5} + \dfrac{x + 1}{3}$.

$$\frac{3x - 2}{5} + \frac{x + 1}{3} = \frac{3(3x - 2) + 5(x + 1)}{5 \times 3}$$
$$= \frac{9x - 6 + 5x + 5}{15}$$
$$= \frac{9x + 5x - 6 + 5}{15}$$
$$= \frac{14x - 1}{15}$$

Example 22

Simplify $\dfrac{a-4}{2a} - \dfrac{9b-2}{6b} + 1.$

$$\frac{a-4}{2a} - \frac{9b-2}{6b} + 1$$

$$= \frac{3b(a-4) - a(9b-2) + 6ab(1)}{6ab}$$

$$= \frac{3ab - 12b - 9ab + 2a + 6ab}{6ab}$$

$$= \frac{2a-12b}{6ab} = \frac{2(a-6b)}{6ab} = \frac{a-6b}{3ab}$$

Example 23

Simplify $\dfrac{6-x-x^2}{x^2-4}.$

$$\frac{6-x-x^2}{x^2-4} = \frac{(3+x)(2-x)}{(x+2)(x-2)}$$

$$= -\frac{(3+x)(x-2)}{(x+2)(x-2)} = -\frac{3+x}{x+2}$$

Example 24

Simplify $\dfrac{3a^3}{3a^2-6ab} + \dfrac{4b^3}{2b^2-ab}.$

$$\frac{3a^3}{3a^2-6ab} + \frac{4b^3}{2b^2-ab}$$

$$= \frac{3a^3}{3a(a-2b)} + \frac{4b^3}{b(2b-a)}$$

$$= \frac{a^2}{a-2b} + \frac{4b^2}{2b-a}$$

$$= \frac{a^2}{a-2b} - \frac{4b^2}{a-2b}$$

$$= \frac{a^2-4b^2}{a-2b} = \frac{(a+2b)(a-2b)}{(a-2b)} = a+2b$$

Example 25

Simplify $\dfrac{x}{x-3} - \dfrac{x-1}{x+2}.$

$$\frac{x}{x-3} - \frac{x-1}{x+2} = \frac{x(x+2) - (x-1)(x-3)}{(x-3)(x+2)}$$

$$= \frac{x^2+2x-(x^2-4x+3)}{(x-3)(x+2)}$$

$$= \frac{x^2+2x-x^2+4x-3}{(x-3)(x+2)}$$

$$= \frac{6x-3}{(x-3)(x+2)}$$

$$= \frac{3(2x-1)}{(x-3)(x+2)}$$

Exercise 22f

Simplify the following.

1 $\dfrac{x-4}{3} + \dfrac{x+3}{2}$

2 $\dfrac{b+5}{7} - \dfrac{2-b}{5}$

3 $\dfrac{3d+6}{10} + \dfrac{d+3}{3}$

4 $\dfrac{2x-5}{7} - \dfrac{8x-8}{6}$

5 $\dfrac{2}{3ab} - \dfrac{3}{4bc}$

6 $5 - \dfrac{p-q}{q}$

7 $\dfrac{ab+ac}{ab-ac}$

8 $\dfrac{a^2-b^2}{a^2-ab}$

9 $\dfrac{a^2+ab}{ab+b^2}$

10 $\dfrac{m^2-2mn+n^2}{m^2-n^2}$

11 $\dfrac{2x^2-x-1}{x-1}$

12 $\dfrac{x^2-5x+6}{x^2-9}$

13 $\dfrac{15-2x-x^2}{x^2-9}$

14 $\dfrac{9-a^2}{a^2+6a+9}$

15 $\dfrac{5xy}{5x^2-10xy} + \dfrac{8xy}{2y^2-xy}$

16 $\dfrac{2a+b}{a^2-ab} - \dfrac{2b+a}{ab-b^2}$

17 $\dfrac{2}{x+2} - \dfrac{x-6}{x^2-4}$

18 $\dfrac{b}{a^2-ab} + \dfrac{a+b}{ab}$

19 $\dfrac{x-5}{6} - \dfrac{6}{x-5}$

20 $\dfrac{1}{m-1} + \dfrac{9}{2m+3} - \dfrac{8}{m+4}$

Substitution

Example 26

Find the value of $3(x + y)$ if $x = -2$ and $y = 7$.

When $x = -2$ and $y = 7$,

$$3(x + y) = 3(-2 + 7) = 3 \times 5 = 15$$

Example 27

Evaluate $ut - \frac{1}{2}ft^2$ when $t = 5$, $u = -20$ and $f = 10$.

$$ut - \tfrac{1}{2}ft^2 = (-20) \times 5 - \tfrac{1}{2}(10)(5)^2 = -100 - 125 = -225$$

Example 28

Evaluate $\dfrac{ab^2 - c^2}{2bc} + \dfrac{a^2}{2b + c}$ when $a = 2$, $b = -3$ and $c = -2$.

$$\frac{ab^2 - c^2}{2bc} + \frac{a^2}{2b + c}$$

$$= \frac{2(-3)^2 - (-2)^2}{2(-3)(-2)} + \frac{2^2}{2(-3) + (-2)}$$

$$= \frac{2 \times 9 - 4}{12} + \frac{4}{-6 - 2}$$

$$= \frac{14}{12} + \frac{4}{-8} = \frac{7}{6} - \frac{1}{2} = \frac{7}{6} - \frac{3}{6} = \frac{2}{3}$$

Exercise 22g

1 Evaluate $u + at$ if $a = 10$, $u = 4$ and $t = 2$.
2 If $p = 42$ and $r = 3$, find the value of $\dfrac{p - r^2}{r}$.
3 Find $\sqrt{u^2 + 2as}$ when $u = 11$, $a = 4$ and $s = 13$.
4 If $x = 3$, $y = -5$ and $z = 2$, what is the value of $\dfrac{x^2(y - z)}{6z + y}$?
5 Evaluate $x^2 - 3x + 8$ when $x =$ (a) 3, (b) 0, (c) -1, (d) -4.
6 If $y = 3x^2 - 5x - 2$, find y when $x =$ (a) 0, (b) 1, (c) 2, (d) 3.
7 If $V = k\dfrac{R}{T}$, find k when $V = 14$, $R =$ and $T = 45$.
8 Evaluate $x^2y + y^2x$ when $x = 5$ and $y = -$
9 If $a = -4$, $b = 6$ and $c = -3$, find t values of (a) $5a - 3c + 2b$, (b) $9a^2c^2 -$ (c) $\dfrac{3ac}{b^2 - c^2}$.
10 If $s = u + 980t$, (a) find s when $u = 0$ an $t = 5$, (b) find t when $s = 10\,000$ an $u = 3\,000$.

Variation

Direct variation

y varies directly as x is written as $y \propto x$. $y \propto$ means that $y = kx$, where k is a constant.

Example 29

If $P \propto R$ and $P = 10$ when $R = 6$, find the relationsh between P and R. Hence find R when $P = 12{,}5$.

Let $P = kR$

$10 = k6$ (from 1st sentence)

$$\Leftrightarrow \quad k = \frac{10}{6} = \frac{5}{3}$$

$\therefore P = \frac{5}{3}R$ is the required relationship between P and R.

When $P = 12{,}5$, $12{,}5 = \frac{5}{3}R$

$$\Leftrightarrow \quad R = \tfrac{3}{5} \times 12{,}5 = 7{,}5$$

Fig. 22.9 is a graph of the relationship between R and P.

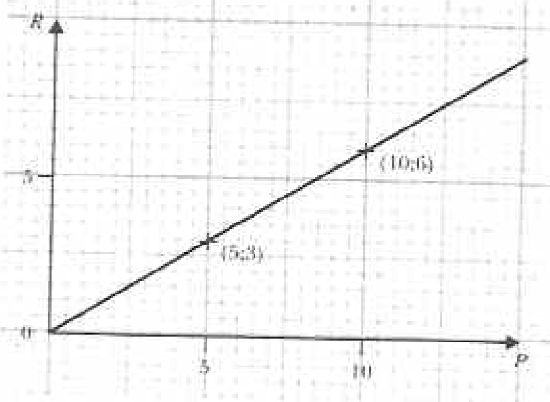


Fig. 22.9

Inverse variation

If y varies inversely as x then $y \propto \frac{1}{x}$ or $y = \frac{k}{x}$, where k is a constant. Notice that if $y \propto \frac{1}{x}$, then $x \propto \frac{1}{y}$. Similarly, if $x \propto y^3$, then $y \propto \sqrt[3]{x}$, and so on.

Example 30
Given that N varies as D^3 and that $N = 240$ when $D = 3$, find N when $D = 2$.

Let $N = \frac{k}{D^3}$

then $240 = \frac{k}{3^3}$

$\Leftrightarrow k = 240 \times 27$

When $D = 2$,

$$N = \frac{240 \times 27}{2^3}$$

$$= \frac{240 \times 27}{8}$$

$$= 810$$

Example 31
$R \propto \sqrt{M}$ and $R = 6$ when $M = 16$. Find the law connecting R and M. Find R when $M = 6\frac{1}{4}$ and find M when $R = 15$.

Let $R = k\sqrt{M}$

then $6 = k\sqrt{16}$

$= 4k$

$\Leftrightarrow k = \frac{3}{2}$

Hence $R = \frac{3}{2}\sqrt{M}$, which is the required law.

When $M = 6\frac{1}{4}$,

$$R = \tfrac{3}{2}\sqrt{6\tfrac{1}{4}} = \tfrac{3}{2}\sqrt{\tfrac{25}{4}} = \tfrac{3}{2} \times \tfrac{5}{2} = \tfrac{15}{4} = 3\tfrac{3}{4}$$

When $R = 15$,

$$15 = \tfrac{3}{2}\sqrt{M}$$

$$\Leftrightarrow \quad \sqrt{M} = 15 \times \tfrac{2}{3} = 10$$

$$\Rightarrow \quad M = 10^2 = 100$$

Notice that if $R = \frac{3}{2}\sqrt{M}$, then $R^2 = \frac{9}{4}M$ or $M = \frac{4}{9}R^2$. It is often convenient to use an alternative form of the original.

Joint variation

Joint variation involves three or more variables. The relationships between them can be in many forms:

$$M \propto R^2T, \qquad F \propto \frac{Mm}{d^2}, \qquad W = kDS,$$

$$t = k\frac{mV^2}{r}$$

Example 32
The mass of a solid metal ball varies jointly as its specific gravity and the cube of its diameter. When the diameter is 6 cm and the specific gravity 7,5, the mass is 850 g. Find the mass of a ball of specific gravity 10,5 and diameter 8 cm.

Let M = mass in grams
D = diameter in cm
S = specific gravity

Then $M = kSD^3$

$\therefore\ 850 = k \times 7{,}5 \times 6^3$ (1)

and $M = k \times 10{,}5 \times 8^3$ (2)

Dividing (2) by (1),

$$\frac{M}{850} = \frac{10{,}5}{7{,}5} \times \left(\frac{8}{6}\right)^3 = \frac{7 \times 64}{5 \times 27}$$

$$\Leftrightarrow \quad M = 850 \times \frac{7}{5} \times \frac{64}{27} \simeq 2\,821$$

The mass is approximately 2 821 g.

Example 32 shows a method that may be used when the value of k is not required.

Partial variation

Partial variation consists of two or more parts added together. For example:
$S = a + bT, \qquad R = a + bV^2,$
$E = aMH + bMV^2$

Notice that in partial variation there are at least two constants, such as a and b above. These constants have to be found separately. See Examples 9 and 10 on pages 190 and 191.

Exercise 22h

1 Express the following as relationships using the given letters and *either* the symbol $\propto$ *or* any constants which may be necessary.

(a) The length, l, of a rectangle of constant area varies inversely as its breadth, b.

(b) The resonance frequency, r, of a series circuit varies inversely as the square root of its capacitance, c.

(c) The gravitational attraction, G, between two particles of mass m_1 and m_2 varies jointly as the product of their masses and the inverse of the square of their distance apart, d.

(d) The energy, E, of a moving body varies partly as the height of the body above sea level, h, and partly as the square of its velocity, v.

2 If $y \propto x$ and $y = 10$ when $x = \frac{1}{2}$ find the law of the variation. Find x if $y = 35$.

3 $A \propto M$ and $A = 8$ when $M = 20$. Find A when $M = 15$ and M when $A = 7$.

4 $P \propto Q$ and $P = 14$ when $Q = 8$. Find P when $Q = 6$ and Q when $P = 28$.

5 $D \propto V$ and $D = 108$ when $V = 3$. Find D when $V = 3{,}75$ and V when $D = 189$.

6 $P \propto Q^2$. $P = 27$ when $Q = 6$. Find the law, P when $Q = 10$, Q when $P = 18\frac{3}{4}$.

7 $x \propto \frac{1}{y}$. $x = 7\frac{1}{2}$ when $y = 4$. Find the law, x when $y = 12$, y when $x = 20$.

8 $M \propto R^3$. $M = 40$ when $R = 4$. Find the law, M when $R = 10$, R when $M = 2{,}56$.

9 $\sqrt{Y} \propto Z$. $Y = 4$ when $Z = 3$. Find the law, Y when $Z = 15$, Z when $Y = 16$.

10 $A \propto BC$. When $B = 6$ and $C = 3$, $A = 7\frac{1}{2}$. Find A when $B = 8$ and $C = 9$; also B if $A = 25$ and $C = 8$.

11 $P \propto \frac{Q}{R^2}$. When $Q = 5$ and $R = 3$, $P = 20$. Find P when $Q = 6$ and $R = 4$; also R when $P = 21{,}6$ and $Q = 15$.

12 x is partly constant and partly varies as When $y = 5$, $x = 7$; and when $y = 7$, $x =$ Find the law of the variation and also when $y = 11$.

13 x varies partly as y and partly as y^2. Whe $y = 4$, $x = 52{,}8$; and when $y = 5$, $x = 8$ Find x when $y = 6$.

14 $x \propto y$ and $y \propto z^2$. How does x vary with

15 $x \propto y^2$ and $y \propto \frac{1}{z}$. How does x vary with

16 A car takes 6 hours to travel from X to at a constant speed. How long does the sam journey take for

(a) a lorry travelling at half the speed the car?

(b) a helicopter travelling at 3 times th speed of the car?

(c) an aeroplane travelling at 6 times th speed of the car?

17 If y is inversely proportional to x, comple Table 22.1.

Table 22.1

x	10		20	25	30	
y		$\frac{1}{3}$	$\frac{1}{4}$			$\frac{1}{7}$

18 The illumination of a small object by a lam varies directly as the candlepower of th lamp and inversely as the square of th distance between the lamp and the objec If a light bulb of 8 candlepower, fixe 150 cm above a table, is replaced by a candepower bulb, how far must the ne light be lowered to give the object the sam illumination as before?

19 The resistance to the motion of a vehicle partly constant and partly varies as th square of its speed. At 30 km h^{-1} th resistance is 496 N, and at 50 km h^{-1} it 656 N. Find the resistance at 60 km h^{-1}.

20 The time taken for a committee meeting partly constant and partly varies as th square of the number of members presen If there are twelve members present th meeting lasts only 56 minutes, but wit twenty it takes exactly two hours. How lon will it last if there are sixteen members?

Example 33

Draw the graph of $y = x^2 + 3x - 4$ for values of x from -5 to $+2$. Find (a) y when $x = 1{,}5$, (b) y when $x = -2{,}5$, (c) x when $y = 4$, (d) the least value of $x^2 + 3x - 4$, (e) the value of x for which y is least.

Obtain the values of y by adding the values of x^2, $3x$ and -4 for each value of x. Table 22.2 shows the method.

Table 22.2

x	-5	-4	-3	-2	-1	0	1	2
x^2	25	16	9	4	1	0	1	4
$+3x$	-15	-12	-9	-6	-3	0	3	6
-4	-4	-4	-4	-4	-4	-4	-4	-4
y	6	0	-4	-6	-6	-4	0	6

Fig. 22.10 is the graph of $y = x^2 + 3x - 4$. Note that scales of 2 cm to 1 unit on the x-axis and 1 cm to 1 unit on the y-axis would give a larger, clearer graph.

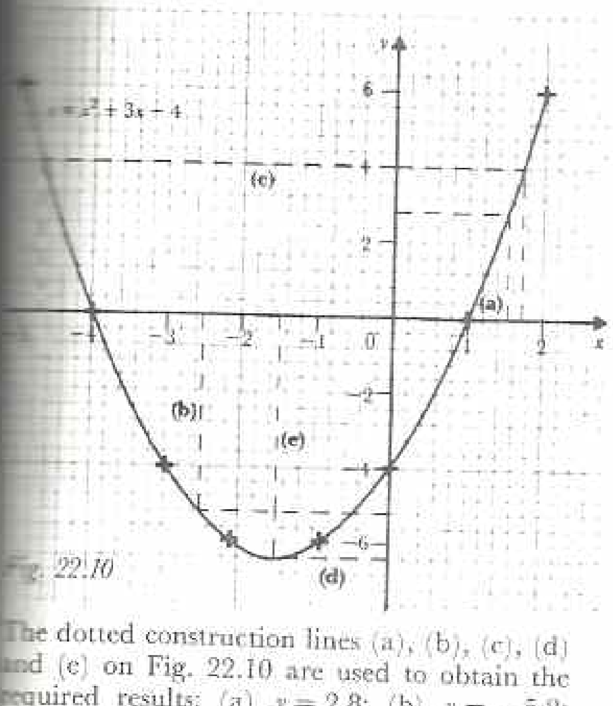


Fig. 22.10

The dotted construction lines (a), (b), (c), (d) and (e) on Fig. 22.10 are used to obtain the required results: (a) $y = 2{,}8$; (b) $y = -5{,}2$; (c) $x = 1{,}7$ or $-4{,}7$; (d) $-6{,}2$; (e) $x = -1{,}5$.

Example 34

On the same axes, draw the graphs of $y = \dfrac{3}{x-2}$ and $y = 2x - 3$ for values of x from -2 to $+4$. Find their points of intersection, if any.

Tables 22.3 and 22.4 give corresponding values of x and y for each graph.

Table 22.3 $y = \dfrac{3}{x-2}$

x	-2	-1	0	1	2	3	4	1,5	2,5
y	0,75	1	1,5	3	udf	3	1,5	6	6

Notice that no value of y is given for $x = 2$; y is undefined (udf) when $x = 2$. Values of y when $x = 1{,}5$ and $x = 2{,}5$ are found instead.

Table 22.4 $y = 2x - 3$

x	0	2	4
y	-3	1	5

Since $y = 2x - 3$ is a linear relation, three points are sufficient. Fig. 22.11 shows the graphs drawn on the same axes.

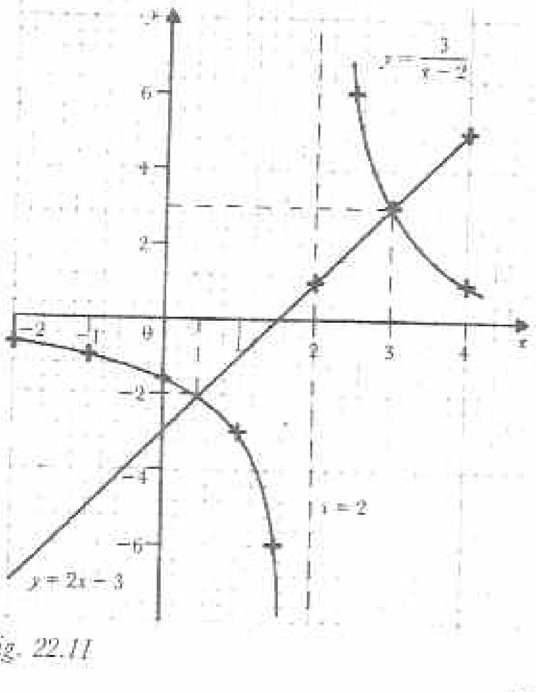


Fig. 22.11

The graphs intersect at the points (3; 3) and $(\frac{1}{2}; -2)$.

Exercise 22i

1 Copy and complete Table 22.5, giving values for the relation $y = 3x^2 - 5x + 3$.

Table 22.5

x	-2	-1	0	1	2	3	4
y	25					15	31

(a) Draw a graph of the relation using scales of 2 cm to 1 unit on the x-axis and 2 cm to 5 units on the y-axis.
(b) What is the minimum value of $3x^2 - 5x + 3$?
(c) Write down the equation of the line of symmetry of the curve.
(d) Find the values of x when $y = 5$.

2 Draw the graph of $y = x^2 - 3x - 4$ for values of x from -2 to $+5$. Use scales of 2 cm to 1 unit on the x-axis and 1 cm to 1 unit on the y-axis. (a) Read off the value of y when $x =$ (i) 4,5; (ii) 2,5. (b) Find the values of x when $y =$ (i) 4, (ii) 0. (c) For what value of x is y a minimum?

3 Draw the graph of $y = 4x - x^2$ for values of x from -1 to $+5$. (a) Read off the value of y at the point where the line $x = -0,5$ cuts the curve. (b) Find the points of intersection of the curve with the line $y = -3$. (c) For what value of x does $4x - x^2$ have its greatest value, and what is this greatest value?

4 Choose a suitable scale and draw the graph of $y = x^2 + 4x + 1$ for values of x from -5 to $+1$. Find (a) the range of values of x for which y increases as x increases, and (b) the coordinates of the point at which y has its least value.

5 Draw the graph of $y = (x + 1)(x - 2)$ fo[r] values of x from -3 to $+4$. Find the ra[nge] of values of x for which
(a) y decreases as x increases;
(b) y is negative.

6 Choose suitable scales and then draw the graph of

(a) $y = \dfrac{3}{x + 3}$ from -6 to 0.

(b) $y = \dfrac{12}{x - 2}$ from -1 to $+5$

(c) $y = x(x - 1)(x - 3)$ from -2 to $+$[…]

(d) $y = x^3 - 4x - 1$ from -3 to $+3$.

7 *Without drawing the graph* of the relati[on] $y = (4 - x)(2 + x)$, write down the coordinates of the point where the cur[ve] cuts (a) the x-axis, (b) the y-axis.
(c) State whether the curve has a maxim[um] or minimum value of y and (d) write down the value of x at this point.
(e) Hence calculate the corresponding val[ue] of y.

8 On the same axes, draw the graphs of $y = x^2 - 5x$ and $y = 5x - x^2$ for values of x from -1 to $+6$. (a) At what points do the curves intersect? (b) Write down the equations of *two* lines of symmetry of the completed curves.

9 On the same axes, draw the graphs of $y = \frac{2}{x}$ and $y = \frac{2}{3}x - 1$, for values of x from -3 to $+3$. Find the values of x and y at the points of intersection of the graphs.

10 On the same axes, draw the graphs of $y = \dfrac{-1}{x + 2}$ and $x + y + 2 = 0$ for values of x from -5 to $+2$. Read off the values of x and y at the points of intersection of the graphs.

Chapter 23

Equations and inequalities

Linear equations

Example 1

Solve the equation $2(x+3) = -11$.

$2(x+3) = -11$

Remove brackets:

$2x + 6 = -11$

Subtract 6 from both sides:

$2x = -17$

Divide both sides by 2:

$x = -8\frac{1}{2}$

Example 2

Solve $\frac{8}{t} - 1 = 3$ *for t.*

$\frac{8}{t} - 1 = 3$

Multiply both sides by t:

$8 - t = 3t$

Add t to both sides:

$8 = 4t$

Divide both sides by 4:

$2 = t$

or $t = 2$

Exercise 23a

Solve the following equations for x.

1 $3 - 2x = 7$ **2** $10 - 5x + 6 = 0$

3 $x + 13 = 5x - 7$ **4** $2x - 13 = 5 + 6x$

5 $-3(x-1) = 9$

6 $2(3x-1) - 10 = 0$

7 $6(3-x) = 5(4-x)$

8 $\frac{11}{x} - 1\frac{1}{2} = \frac{5}{x}$ **9** $\frac{6}{x} + \frac{2}{3x} = 2$

10 $5(3x + \frac{1}{2}) = 2x - 17$

Change of subject of a formula

Example 3

Make P the subject of the formula $R = \frac{Q^2 - PR}{Q + P}$.

$$R = \frac{Q^2 - PR}{Q + P}$$

Multiply both sides by $(Q + P)$ to clear fractions:

$R(Q + P) = Q^2 - PR$

Clear brackets:

$RQ + PR = Q^2 - PR$

Collect terms in P on the LHS of the equation:

$2PR = Q^2 - RQ$

Divide both sides by $2R$:

$$P = \frac{Q^2 - RQ}{2R} = \frac{Q(Q-R)}{2R}$$

Example 4

Make r the subject of the formula $b = \sqrt{\frac{4\pi r^3}{3h}}$.

$$b = \sqrt{\frac{4\pi r^3}{3h}}$$

Square both sides:

$$b^2 = \frac{4\pi r^3}{3h}$$

Multiply both sides by $\frac{3h}{4\pi}$:

$$\frac{3hb^2}{4\pi} = r^3 \qquad \text{or } r^3 = \frac{3hb^2}{4\pi}$$

Take the cube root of both sides:

$$r = \sqrt[3]{\frac{3hb^2}{4\pi}}$$

Chapter 23

Equations and inequalities

Syllabus references 6.5.4, 6.6.4 and 6.5.4

Linear equations

Example 1

Solve the equation $2(x+3) = -11$.

$2(x+3) = -11$

Remove brackets:

$2x + 6 = -11$

Subtract 6 from both sides:

$2x = -17$

Divide both sides by 2:

$x = -8\frac{1}{2}$

Example 2

Solve $\frac{8}{t} - 1 = 3$ *for* t.

$\frac{8}{t} - 1 = 3$

Multiply both sides by t:

$8 - t = 3t$

Add t to both sides:

$8 = 4t$

Divide both sides by 4:

$2 = t$

or $t = 2$

Exercise 23a

Solve the following equations for x.

1 $3 - 2x = 7$ **2** $10 - 5x + 6 = 0$

3 $x + 13 = 5x - 7$ **4** $2x - 13 = 5 + 6x$

5 $-3(x-1) = 9$

6 $2(3x-1) - 10 = 0$

7 $6(3-x) = 5(4-x)$

8 $\frac{11}{x} - 1\frac{1}{2} = \frac{5}{x}$ **9** $\frac{6}{x} + \frac{2}{3x} = 2$

10 $5(3x + \frac{1}{2}) = 2x - 17$

Change of subject of a formula

Example 3

Make P *the subject of the formula* $R = \frac{Q^2 - PR}{Q+P}$.

$$R = \frac{Q^2 - PR}{Q+P}$$

Multiply both sides by $(Q+P)$ to clear fractions:

$R(Q+P) = Q^2 - PR$

Clear brackets:

$RQ + PR = Q^2 - PR$

Collect terms in P on the LHS of the equation:

$2PR = Q^2 - RQ$

Divide both sides by $2R$:

$$P = \frac{Q^2 - RQ}{2R} = \frac{Q(Q-R)}{2R}$$

Example 4

Make r *the subject of the formula* $b = \sqrt{\frac{4\pi r^3}{3h}}$.

$$b = \sqrt{\frac{4\pi r^3}{3h}}$$

Square both sides:

$$b^2 = \frac{4\pi r^3}{3h}$$

Multiply both sides by $\frac{3h}{4\pi}$:

$$\frac{3hb^2}{4\pi} = r^3 \quad \text{or } r^3 = \frac{3hb^2}{4\pi}$$

Take the cube root of both sides:

$$r = \sqrt[3]{\frac{3hb^2}{4\pi}}$$

Substitute 3 for x in equation (3):

$$y = 11 - 9 = 2$$

The solution is $x = 3$, $y = 2$.

Or by elimination:

$2x - y = 4$ (1)

$3x + y = 11$ (2)

Add (1) and (2) to eliminate y:

$5x = 15$

$\Leftrightarrow \quad x = 3$

Substitute 3 for x in (1):

$6 - y = 4$

$\Leftrightarrow \quad y = 2$

The solution is $x = 3$, $y = 2$.

Or by graph:

Fig. 22.2 shows the graphs of $2x - y = 4$ and $3x + y = 11$ drawn on the same axes.

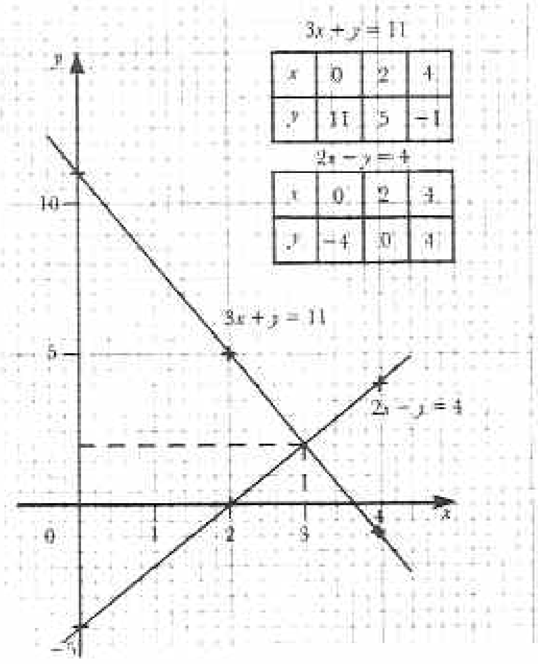


Fig. 23.2

The point of intersection of the two lines is (3;2). The solution of the equations is $x = 3$, $y = 2$.

Example 9

Solve the equations $\frac{2}{3}x - \frac{1}{2}y = 2$, $\frac{3}{4}x - \frac{1}{3}y = 3\frac{1}{6}$.

$\frac{2}{3}x - \frac{1}{2}y = 2$ (1)

$\frac{3}{4}x - \frac{1}{3}y = 3\frac{1}{6}$ (2)

Simplify the equations by clearing the fractions

(1) × 6: $4x - 3y = 12$ (3)

(2) × 12: $9x - 4y = 38$ (4)

Solve in the usual way:

(3) × 4: $16x - 12y = 48$ (5)

(4) × 3: $27x - 12y = 114$ (6)

(6) − (5): $11x = 66$

$\Leftrightarrow \quad x = 6$

Substituting 6 for x in (3):

$24 - 3y = 12$

$\Leftrightarrow -3y = -12$

$\Leftrightarrow \quad y = 4$

The solution is $x = 6$, $y = 4$.

Example 10

Solve the equations

$3x - 2y + 1 = 2x - 5y - 10 = 4x - 3y$.

Pair the three expressions in any two different ways:

$3x - 2y + 1 = 4x - 3y$

$\Leftrightarrow \quad x - y = 1$ (1)

$2x - 5y - 10 = 4x - 3y$

$\Leftrightarrow \quad 2x - 8y = -10$

$\Leftrightarrow \quad x - 4y = -5$ (2)

Solve equations (1) and (2) in the usual way. This gives $x = 3$ and $y = 2$ as the solution.

Exercise 23d

Solve the following pairs of equations. Use either the substitution, elimination or graphical method, whichever is most suitable.

1 $x + y = 6$, $2x + 3y = 14$

2 $2x - y = 11$, $x + 2y = -7$

3 $3x - 4y = 7$, $x - 2y = 5$

4 $3x + 2y = 7$, $7x - 3y = 1$

5 $5x - 2y = -23$, $3x + 4y = 7$

6 $3x - 2y = 4$, $2x - 7y = 31$

7 $x - 2y = 1$, $x + 2y = 9$

8 $a + 3b = -13$, $2a - 9b = 4$

9 $4c - 3d = 1$, $2c + 4d = 17$

10 $3x + 4y = -1$, $3x + 8y = 4$

11 $5m - 2n = 15$, $3m + 5n = 9$

12 $3x + 7y = -8$, $5y = 2x + 15$

13 $3a + 4m = 0$, $a = 2m - 5$

14 $5d = 4n + 8$, $5n + 10 = 4d$

15 $5x + 3y - 2 = 0$
$3x - 32 = 7y$

16 $\frac{3}{4}x + \frac{1}{5}y = 4$
$\frac{1}{2}x = \frac{3}{5}y - 1$

17 $1{,}2x - 1{,}1y = 7{,}9$
$1{,}8x + 0{,}7y = 0{,}1$

18 $\frac{d}{4} - \frac{n}{3} = 6$
$\frac{3d}{2} + \frac{5n}{6} = 2$

19 $3x - y + 12 = 5x + 2y + 4 = x + y$

20 $5(a + b) = 2(a + 3b) + 1$
$3(a + 2b) - 7 = a + 3b + 1$

21 $2{,}5d - 1{,}5g = -2$
$1{,}3d + g + 4{,}6 = 0$

22 $\frac{2}{3}(a - 6) = m - 3$
$\frac{3}{5}(a + 2) = 2m + 1$

23 $c + d + 6 = 2c - 5d - 3 = 3c - 4d + 2$

24 $3(2u - w) = 5u - w + 1$
$6u + 7w + 4 = 4(u + 2w)$

25 $\frac{3}{4}(4x + y) = 4$
$2x = \frac{3}{5}(2x - 3y)$

Simultaneous inequalities

Linear programming

This topic is fully covered in Chapter 17, pages 143 to 148.

Exercise 23e

1

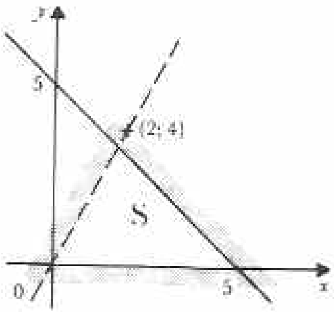


Fig. 23.3

Given Fig. 23.3, find
(a) the equation of the line through the origin and the point (2;4),
(b) the equation of the line through (0;5) and (5;0),
(c) the inequalities which define the triangular shape S.

2 Write down the three inequalities which define the unshaded triangular area A in Fig. 23.4.

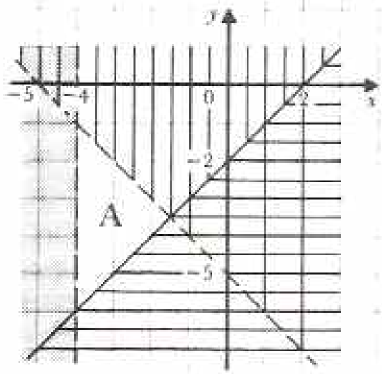


Fig. 23.4

3 List the three inequalities which define the unshaded area labelled A in Fig. 23.5.

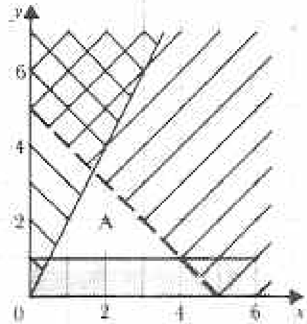


Fig. 23.5

4 Write down the four inequalities which define the unshaded region R in Fig. 23.6.

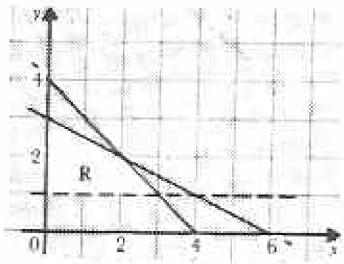


Fig. 23.6

5 K is the set of points $(x; y)$ which satisfies the four inequalities $y - x \geqslant 1$, $2x \geqslant 5$, $5y > -4x$, $y \geqslant 2$.
Show on a graph the region which represents K. Use your graph to find the greatest value of $(x + y)$.

6 (a) The set of points with coordinates $(x; y)$ satisfies the five inequalities

$x \leqslant 0,\ y \leqslant 0,\ y + 2x \leqslant 8,$
$4y + 3x \geqslant 24$ and $3y \geqslant 2x$.

(i) Using 2 cm to represent 1 unit on each axis, construct accurately on graph paper, and clearly indicate by shading the unwanted regions, the region in which the set of points $(x; y)$ must lie.
(ii) Using your graph, find the least value of $(x - y)$ for points in the region.
(b) The owner of a large piece of land plans to divide it into not more than 36 plots and to build either a house or a bungalow on each plot. He decides that he will build at least 20 houses and that there will be at least twice as many houses as bungalows.

Taking h to represent the number of houses and b the number of bungalows, write down three inequalities, other than $h \leqslant 0$ and $b \leqslant 0$, which satisfy the above conditions. [Camb]

7 It takes 2 m of cloth to make a shirt and 3 m to make a dress. A tailor has 36 m of cloth and he needs to make at least 6 of each. If the profit on a shirt is the same as that on a dress, what arrangement of shirts and dresses gives the greatest profit?

8 A shopkeeper orders two sizes of notebooks, large at $1,35 each and small at 60c each. She needs twice as many small ones as large ones, with minimum quantities of 10 large and 20 small. If she spends up to $30, what is the maximum number she can buy?
If the profit is 20c on a large notebook and 10c on a small one, what arrangement gives the greatest profit, and how much is that profit?

9 A company plans to spend $500 000 on new machines. Table 23.1 shows the cost and necessary floor space for the two types of machine to be bought.

Table 23.1

machine	cost	floor space
A	$20 000	$6\ m^2$
B	$25 000	$4\ m^2$

More of machine A than of machine B are needed and there are only $120\ m^2$ of factory floor space available.
Which purchase arrangement (a) uses all the space available, (b) is more expensive?

10 A new book is to be published in both a hardback and a paperback edition. A bookseller agrees to purchase
(i) 15 or more hardback copies,
(ii) more than 25 paperback copies,
(iii) at least 45, but fewer than 60, copies altogether.

Using h to represent the number of hardback copies and p to represent the number of paperback copies, write down the inequalities which represent these conditions. The point $(h;p)$ represents h hardback copies and p paperback copies. Using a scale of 2 cm to represent 10 books, on each axis, construct, and indicate clearly by shading the *unwanted* regions, the region in which $(h;p)$ must lie.

Given that each hardback copy costs $5 and that each paperback copy costs $2, calculate the number of each sort that the bookseller must buy if he is prepared to spend between $180 and $200 altogether and he has to buy each sort in packets of five. [Camb]

Quadratic equations

Example 11
Solve the equation $y^2 - 4y = 0$.

$$y^2 - 4y = 0$$
$$\Leftrightarrow \quad y(y - 4) = 0$$
$$\Leftrightarrow \quad \text{either } y = 0 \text{ or } y - 4 = 0$$
$$\Leftrightarrow \quad y = 0 \text{ or } y = 4$$

Example 12
Solve the equation $3x^2 + 5x - 28 = 0$.

$$3x^2 + 5x - 28 = 0$$

Factorise the quadratic expression (see Chapter 22).

$$(x + 4)(3x - 7) = 0$$
$$\Leftrightarrow \quad \text{either } x + 4 = 0 \text{ or } 3x - 7 = 0$$
$$\Leftrightarrow \quad x = -4 \text{ or } 3x = 7$$
$$\text{i.e. } x = -4 \text{ or } x = 2\tfrac{1}{3}$$

Exercise 23f

Use factorisation to solve the following equations.

1 $x^2 - 10x + 21 = 0$ **2** $m^2 + 3m + 2 = 0$
3 $a^2 + a - 6 = 0$ **4** $n^2 - 3n - 10 = 0$
5 $x^2 + x - 2 = 0$ **6** $y^2 + 3y = 0$
7 $2d^2 - 7d + 6 = 0$ **8** $4e^2 + 11e + 6 = 0$
9 $a^2 - 4 = 0$ **10** $a^2 - 4a = 0$
11 $2t^2 + 7t + 5 = 0$ **12** $3u^2 - 10u - 8 = 0$
13 $8w^2 - 18w + 9 = 0$
14 $12d^2 - 19d - 18 = 0$
15 $8n^2 + 2n - 21 = 0$

Graphical solution of quadratic equations

Example 13

Solve $x^2 - 2x - 4 = 0$ graphically.

Let $y = x^2 - 2x - 4$. Make a table of corresponding values of x and y (Table 23.2). Fig. 23.7 is the graph of $y = x^2 - 2x - 4$.

Table 23.2 $y = x^2 - 2x - 4$.

x	-2	-1	0	1	2	3	4
x^2 $-2x$ -4	4 4 -4	1 2 -4	0 0 -4	1 -2 -4	4 -4 -4	9 -6 -4	16 -8 -4
y	4	-1	-4	-5	-4	-1	4

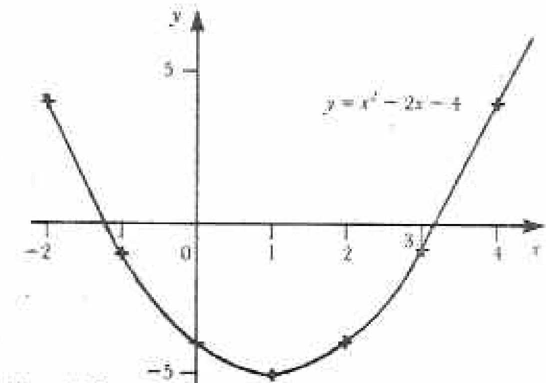


Fig. 23.7

$x^2 - 2x - 4 = 0$ when $y = 0$, i.e. where the curve intersects the x-axis at $x = 3{,}2$ and $x = -1{,}2$. These are the approximate solutions to the equation $x^2 - 2x - 4 = 0$.

Example 14

(a) Draw the graph of $y = 2x^2 + 3x - 6$ for values of x from -4 to 2. (b) Use the graph to solve the equations (i) $2x^2 + 3x - 6 = 0$, (ii) $2x^2 + 3x - 3 = 0$. (c) By drawing the line $y = 2x + 1$ on the same axes, solve the equation $2x^2 + x - 7 = 0$.

(a) The values in Table 23.3 are used to draw the curve in Fig. 23.8.

Table 23.3 $y = 2x^2 + 3x - 6$

x	-4	-3	-2	-1	0	1	2
$2x^2$ $+3x$ -6	32 -12 -6	18 -9 -6	8 -6 -6	2 -3 -6	0 0 -6	2 3 -6	8 6 -6
y	14	3	-4	-7	-6	-1	8

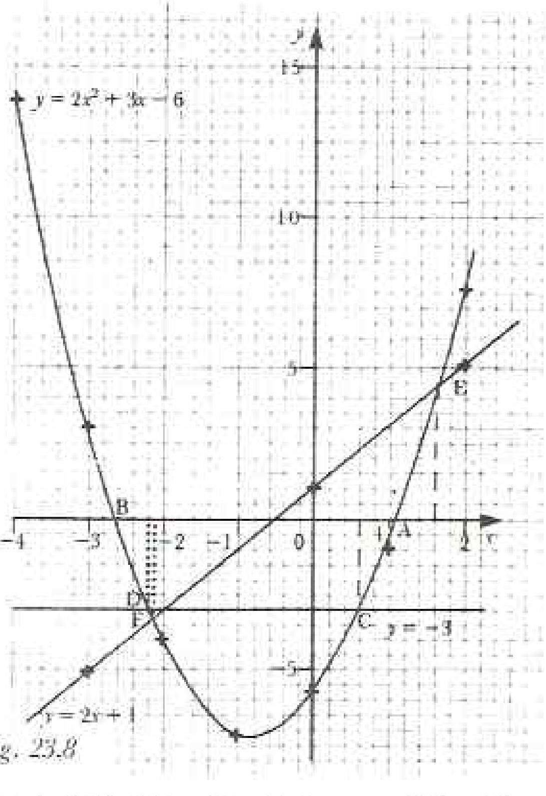


Fig. 23.8

(b) (i) $2x^2 + 3x - 6 = 0$ when $y = 0$, i.e. along the x-axis, at the points A and B in Fig. 23.8. Hence $x = 1{,}1$ and $x = -2{,}6$ are the approximate solutions.

(ii) $2x^2 + 3x - 3 = 0$

$$2x^2 + 3x = 3$$
$$2x^2 + 3x - 6 = -6$$

i.e. $2x^2 + 3x - 6 = -3$

This is true at the points where the line $y = -3$ cuts the curve; i.e. at the points C and D in Fig. 23.8.

Hence $x = 0{,}7$ and $x = -2{,}2$ are the approximate solutions.

(c) Table 23.4 is used to draw the straight line $y = 2x + 1$ in Fig. 23.8.

Table 23.4

x	-3	0	2
y	-5	1	5

The curve and the straight line intersect at the points E and F in Fig. 23.8. At these points $y = 2x^2 + 3x - 6$ and $y = 2x + 1$

$\Rightarrow\ 2x^2 + 3x - 6 = 2x + 1$

$\Leftrightarrow\ 2x^2 + x - 7 = 0$

Hence $x = 1{,}6$ and $x = -2{,}1$ are the approximate solutions of $2x^2 + x - 7 = 0$.

Exercise 23g

1 Copy and complete Table 23.5 for the relation $y = x^2 - 4x + 2$.

Table 23.5

x	-2	-1	0	1	2	3	4	5
y	14			-1	-2		2	

Use a scale of 1 cm to 1 unit on both axes and draw a graph of the relationship. Use your graph to find (a) the solutions of $x^2 - 4x + 2 = 0$, (b) the least value of $x^2 - 4x + 2$.

2 Copy and complete Table 23.6 for the relation $y = 3x^2 - 6x + 1$.

Table 23.6

x	-2	-1	0	1	2	3	4
y	25			-2		10	

Using scales of 2 cm to 1 unit on the x-axis and 2 cm to 5 units on the y-axis, draw a graph of the relationship. Use the graph to solve $3x^2 - 6x + 1 = 0$.

3 Draw the graph of $y = x^2 - 3x + 2$, taking values of x from -1 to 4. Use the graph to read off the roots of the equation
(a) $x^2 - 3x - 2 = 0$
(b) $x^2 - 3x + 1 = 0$
(c) $x^2 - 3x - 4 = 0$.

4 Draw the graph of $y = 3x^2 + 2x - 1$, taking values of x from -3 to 2. Use the same graph to read off the roots of the equations:
(a) $3x^2 + 2x = 0$
(b) $3x^2 + 2x - 7 = 0$
(c) $3x^2 + 2x = 3$
(d) $3x^2 + 2x - 12 = 0$.

5 Draw the graph of $y = 3x^2 - 5x + 3$ and use it to find the roots of (a) $3x^2 - 5x + 3 = 0$, (b) $3x^2 - 5x = 0$, (c) $3x^2 - 5x - 1 = 0$.

6 Draw the graph of $y = 2x$ to cut the curve drawn in question 5. Hence find the solution of the equation $3x^2 - 7x + 3 = 0$.

7 Draw the graph of $y = x(x - 1)$ for values of x from -2 to 3. Read off the values of x at the points where the curve cuts the line $y = 3 - x$. Of what equation in x are these values the roots?

8 Draw the graphs of $y = 2x^2 - 3x - 6$ and $y = 1 - 3x$ on the same axes for values of x from -2 to 3. Use the graph to find the roots of the equation $2x^2 - 7 = 0$.

9 Copy and complete Table 23.7 for the relation $y = 2 + x - x^2$.

Table 23.7

x	-2	$-1{,}5$	-1	$-0{,}5$	0	0,5	1	1,5	2	2,5	3
y			0	1,25	2		2	1,25			-4

(a) Draw the graph of the relation using a scale of 2 cm to 1 unit on each axis.
(b) From your graph, find the greatest value of y and the value of x at which this occurs.
(c) Using the same axes, draw the graph of $y = 1 - x$.
(d) From your graphs, determine the roots of the equation $1 + 2x - x^2 = 0$.

10 Solve the equation $x^2 = \frac{1}{2}x + 4$ by drawing the graphs of $y = x^2$ and $y = \frac{1}{2}x + 4$ on the same axes for values of x from -3 to 3. Check your result by drawing a separate graph of $y = x^2 - \frac{1}{2}x - 4$ for the same range of values of x.

Simultaneous linear and quadratic equations

Example 15

Solve the simultaneous equations
$2x - 5y = 1$; $4x^2 + 25y^2 = 41$.

$$2x - 5y = 1 \qquad (1)$$
$$4x^2 + 25y^2 = 41 \qquad (2)$$

From (1): $x = \dfrac{1 + 5y}{2} \qquad (3)$

Substitution $\frac{1}{2}(1 + 5y)$ for x in (2):

$$4 \times [\tfrac{1}{2}(1 + 5y)]^2 + 25y^2 = 41$$
$$4 \times \tfrac{1}{4}(1 + 10y + 25y^2) + 25y^2 = 41$$
$$\Leftrightarrow \quad 50y^2 + 10y - 40 = 0$$
$$\Leftrightarrow \quad 5y^2 + y - 4 = 0$$
$$\Leftrightarrow \quad (5y - 4)(y + 1) = 0$$
$$\Leftrightarrow \quad y = \tfrac{4}{5} \text{ or } -1$$

Substitute for y in (3):

When $y = \frac{4}{5}$: $\quad x = \dfrac{1 + 4}{2} = 2\frac{1}{2}$

When $y = -1$: $\quad x = \dfrac{1 - 5}{2} = -2$

The solutions may be given in the form of ordered pairs: $(2\frac{1}{2}; \frac{4}{5})$ and $(-2; -1)$.

Notice that part (c) of Example 14 gives a graphical method of solving simultaneous linear and quadratic equations. Questions 6–10 of Exercise 23g provide practice in the use of graphical methods.

Exercise 23h

Solve the following pairs of equations.

1 $2x + y = 5$; $x^2 + y^2 = 25$

2 $4x - y = 7$; $xy = 15$

3 $x + y = 3$; $x^2 - y^2 = -3$

4 $4x^2 + y^2 = 61$; $2x + y = 1$

5 $2x - y = 5$; $4x^2 - y^2 = 15$

6 $2x^2 - y^2 = -2$; $3x + y = 1$

7 $2x + 3y = 1$; $4x^2 - 9y^2 = -17$

8 $x + 2y = 2$; $x^2 + 2xy = 8$

9 $x^2 + 2y^2 = 3$; $x - 3y = 2$

10 $xy = 30$; $3x + y = 21$

11 $25x^2 - 4y^2 = 36$; $5x - 2y = 2$

12 $x - 3y = 1$; $x^2 - 2xy - y^2 = 7$

13 $3xy - y^2 = 2$; $2x - 3y = -4$

14 $25x^2 - 7y^2 = 29$; $5x + 7y + 1 = 0$

15 $9x^2 + 16y^2 = 52$; $3x - 4y = 2$

Word problems

Example 16

The ages of a man and his son add up to 39 years. In 3 years the man will be 4 times as old as his son. Find the ages of the man and his son.

Let the ages of the man and son be x and y years respectively. From the first sentence:

$$x + y = 39 \qquad (1)$$

From the second sentence:

$$x + 3 = 4(y + 3)$$
$$x + 3 = 4y + 12$$
$$x - 4y = 9 \qquad (2)$$

Solve equations (1) and (2) simultaneously.
Subtract (2) from (1):

$$5y = 30$$
$$y = 6$$

Substitute 6 for y in (1):

$$x = 33$$

The man is 33 years and the son is 6 years.

Check: $33 + 6 = 39$ *(1st sentence)*
$36 = 4 \times 9$ *(2nd sentence)*

Example 16 led to simultaneous linear equations. Examples 17 and 18 which follow, lead to quadratic equations.

Example 17

Find two numbers whose difference is 4 and whose product is 192.

Let the smaller number be x.
Then the larger number is $x + 4$.
Their product is $x(x + 4)$.

$$\text{Hence } x(x+4) = 192$$
$$\Leftrightarrow \quad x^2 + 4x - 192 = 0$$
$$\Leftrightarrow \quad (x-12)(x+16) = 0$$
$$\Leftrightarrow \quad x = 12 \text{ or } -16$$

The other number is 4 more, i.e. $12 + 4$ or $-16 + 4$, i.e. 16 or -12.

The two numbers are 12 and 16, or -16 and -12.

Check: $12 \times 16 = 192$ and $-16 \times -12 = 192$.

Compare Example 17 with Example 18 which follows. Notice the use of units and the elimination of one root because it is not a sensible result.

Example 18

The length of a rectangular compound is 5 m *more than the width. Its area is* 500 m². *Find the width and length of the compound.*

Let the width be x m. Then, from the 1st sentence, the length is $(x + 5)$ m. The area is $x(x + 5)$ m². From the second sentence:

$$x(x+5) = 500$$
$$\Leftrightarrow \quad x^2 + 5x - 500 = 0$$
$$\Leftrightarrow \quad (x-20)(x+25) = 0$$
$$\Rightarrow \quad x = 20 \text{ or } -25$$

An answer of -25 m is clearly not sensible for the width of a compound. Hence the width is 20 m and the length, 5 m more, is 25 m.

Check: $20\text{ m} \times 25\text{ m} = 500\text{ m}^2$.

Exercise 23i

1 A notebook and a pencil cost \$1,32. If the notebook costs 74c more than the pencil, find the cost of each.

2 Divide 27 into two parts such that their product is 180.

3 The area of a rectangle is 360 cm² and its length is 2 cm more than its width. Find the width.

4 A father is 28 years older than his daughter. In 2 years' time he will be 3 times as old as his daughter. Find their present ages.

5 The perimeter of a rectangle is 42 cm and its area is 68 cm². Find its length and breadth.

6 The ages of two sisters are 11 and 8 years. In how many years' time will the product of their ages be 378?

7 When a man cycles for 1 hour at x km/h and 2 hours at y km/h he travels 32 km. When he cycles for 2 hours at x km/h and 1 hour at y km/h he travels 34 km. Find x and y.

8 A rectangular garden measures 12 m by 5 m. A path of uniform width runs along one side and one end. If the total area of the garden and path is 98 m², find the width of the path.

9 A number is subtracted from 20 and from 17. The product of the numbers so obtained is 180. Find the original number.

10 If I subtract 1 from the numerator of a fraction, the fraction becomes $\frac{1}{2}$. If I add 1 to both the numerator and denominator of the fraction, it becomes $\frac{2}{3}$. What is the fraction?

$\left(\textit{Hint:} \quad \text{Let the fraction be } \dfrac{x}{y}.\right)$

11 A rectangular piece of cardboard measures 21 cm by 16 cm. When strips of equal width are cut off one side and one end, the area of the remaining piece is 234 cm². Find the width of the strips.

12 A woman is 35 years old and her son is 12 years old. How many years ago was the product of their ages 174?

13 A table costs \$$x$ and a chair costs \$$y$. If the price of each is raised by \$20, the ratio of their prices becomes 5:2 respectively. If the price of each is reduced by \$5, the ratio becomes 5:1. Express the ratio $x:y$ in its lowest terms.

14 Two rectangles have the same area of 24 cm². The second rectangle is 4 cm shorter and 1 cm wider than the first. What is the length and breadth of the first rectangle?

15 A girl bought some pencils for \$3,60. If she had paid 4c less for each pencil she could have bought 3 more pencils. How many pencils did she buy?

Chapter 24

Properties of plane shapes, constructions, locus

Syllabus reference 6.7

Angles

Angle is a measure of rotation or turning.

1 revolution = 360 degrees (1 rev = 360°)

1 degree = 60 minutes ($1° = 60'$)

The names of angles change with their size (Fig. 24.1).

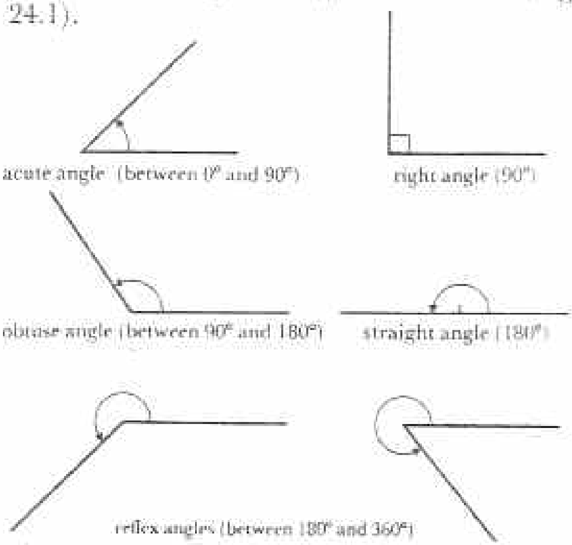


Fig. 24.1

Angles are formed when lines meet or intersect. Remember the following facts:

Angles at a point add up to 360°.

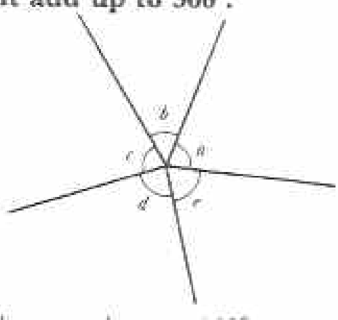


Fig. 24.2

In Fig. 24.2, $a + b + c + d + e = 360°$.

Adjacent angles on a straight line add up to 180°.

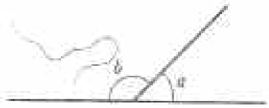


Fig. 24.3

In Fig. 24.3, $a + b = 180°$.

Vertically opposite angles are equal.

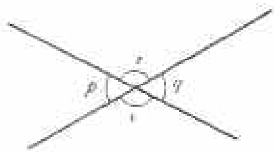


Fig. 24.4

In Fig. 24.4, $p = q$ and $r = s$.

Alternate angles are equal.

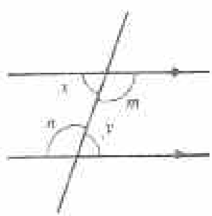


Fig. 24.5

In Fig. 24.5, $x = y$ and $m = n$.

Corresponding angles are equal.

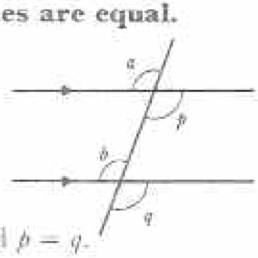


Fig. 24.6

In Fig. 24.6, $a = b$ and $p = q$.

Interior opposite angles add up to 180°.

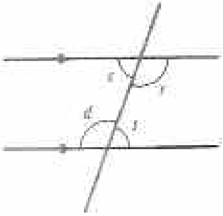


Fig. 24.7

In Fig. 24.7, $c + d = 180° = r + s$.

Example 1

Find the angles marked a, b and c in Fig. 24.8.

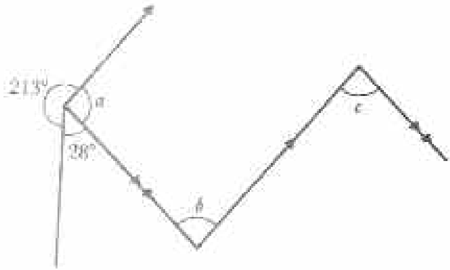


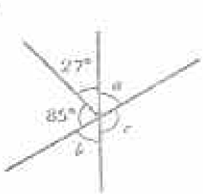
Fig. 24.8

$a = 360° - (213° + 28°)$ (*angles at a point*)
$\quad = 360° - 241° = 119°$
$b = 180° - a$ (*int. opp.*)
$\quad = 180° - 119° = 61°$
$c = b$ (*alt. angles*)
$\quad = 61°$

Exercise 24a (oral or written)

Find the lettered angles in Fig. 24.9.

1



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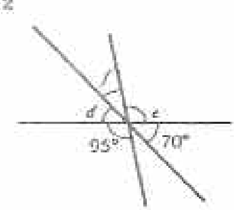

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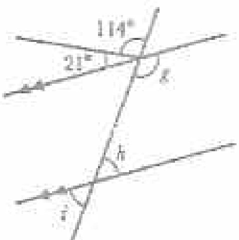

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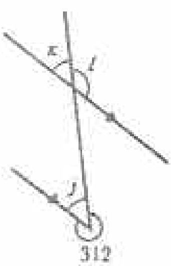

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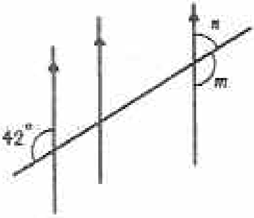

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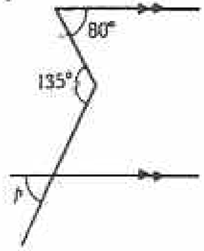

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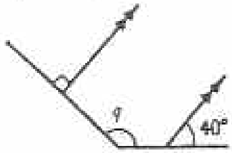

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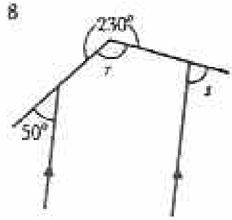


Fig. 24.9

Triangles

Angles in a triangle

The sum of the angles of a triangle is 2 right angles, or 180°.

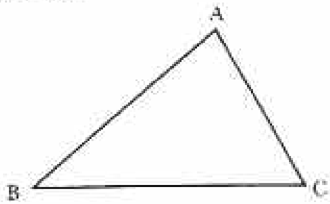


Fig. 24.10

In Fig. 24.10, $\hat{A} + \hat{B} + \hat{C} = 180°$.

The exterior angle of a triangle equals the sum of the two interior opposite angles.

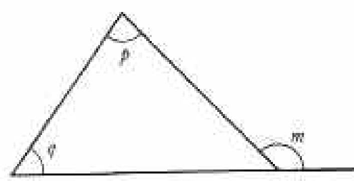


Fig. 24.11

In Fig. 24.11, $m = p + q$.

Types of triangle

Fig. 24.12 gives the names and properties of common types of triangle.

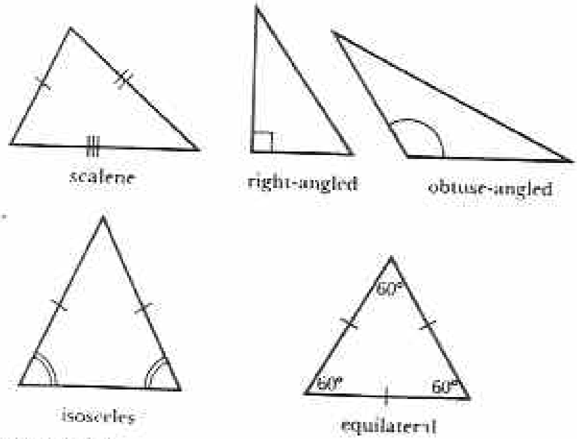


Fig. 24.12

Example 2

Find $\hat{P}$ *in Fig. 24.13.*

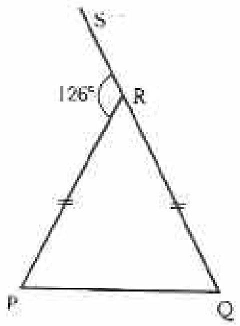


Fig. 24.13

$$\hat{P} = \hat{Q} \qquad \textit{(base } \angle\textit{s of isos. } \triangle\textit{)}$$
$$\hat{P} + \hat{Q} = 126° \qquad \textit{(ext. } \angle \textit{ of } \triangle\textit{)}$$
$$\therefore\ 2\hat{P} = 126° \qquad (\hat{P} = \hat{Q})$$
$$\therefore\ \hat{P} = 63°$$

Congruent triangles

Congruent means the same in all respects. Two triangles are congruent if:

1 **two sides and the included angle** of one are respectively equal to two sides and the included angle of the other (SAS), *or*

2 **two angles and a side** of one are respectively equal to two angles and a side of the other (ASA or AAS), *or*

3 **three sides** of one are respectively equal to three sides of the other (SSS), *or*

4 they are **right-angled** and have the **hypotenuse and another side** of one equal to the hypotenuse and side of the other (RHS).

Example 3

Given Fig. 24.14, name the triangle which is congruent to △XYZ, keeping the letters in the correct order.

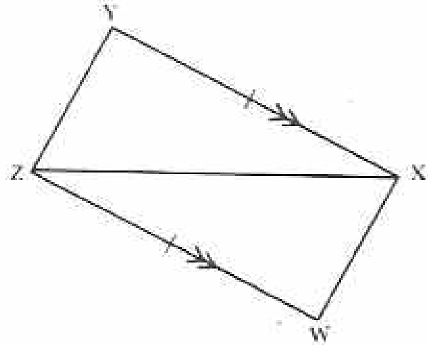


Fig. 24.14

$\triangle XYZ \equiv \triangle ZWX$ *(SAS)*

Reason: $XY = ZW$ *(given)*

$Y\hat{X}Z = W\hat{Z}Y$ *(alt.* $\angle$s, $YX \parallel ZW$*)*

$XZ = ZX$ *(common side)*

Exercise 24b

1 Name and calculate the sizes of the exterior angles shown in Fig. 24.15.

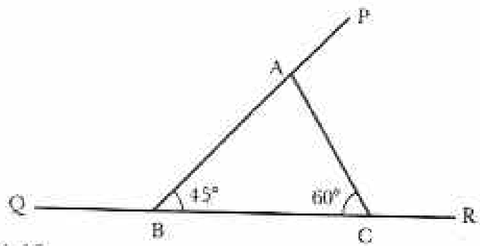


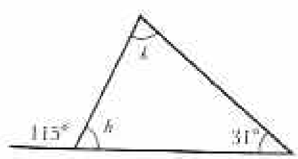
Fig. 24.15

2 In each of the following, two angles of a triangle are given. Find the third angle and name the type of triangle.

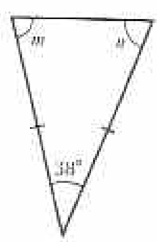
(a) 69°, 46° (b) 38°, 71°
(c) 43°, 94° (d) 60°, 60°
(e) 35°, 55° (f) 58°, 25°

3 Calculate the sizes of the lettered angles in Fig. 24.16(a)–(f).

(a)



(b)



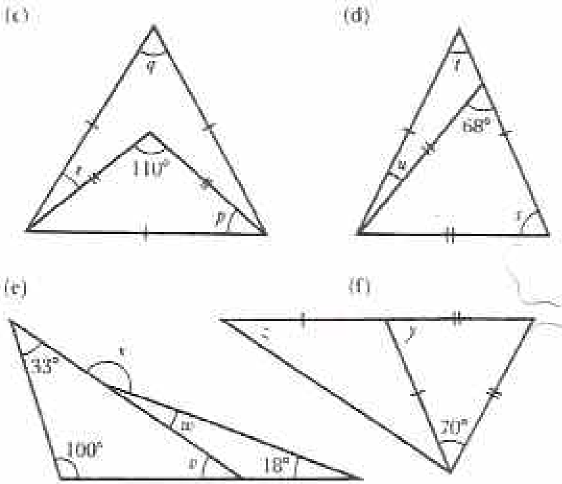


Fig. 24.16

4 Find the value of x in Fig. 24.17.

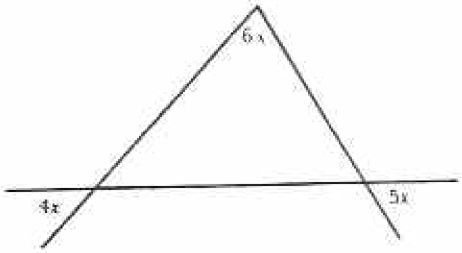


Fig. 24.17

5 Find the value of x in each part of Fig. 24.18. Hence state which type of triangle each is.

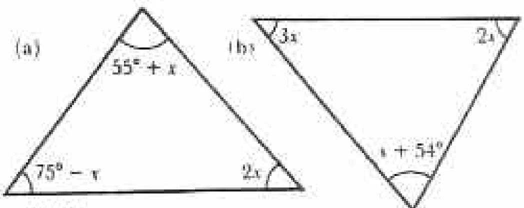


Fig. 24.18

6 In each part of Fig. 24.19, name the triangle which is congruent to $\triangle XYZ$. Keep the letters in the correct order and state the case of congruency using the abbreviations, *RHS*, *SSS*, *SAS*, *ASA* or *AAS* as appropriate.

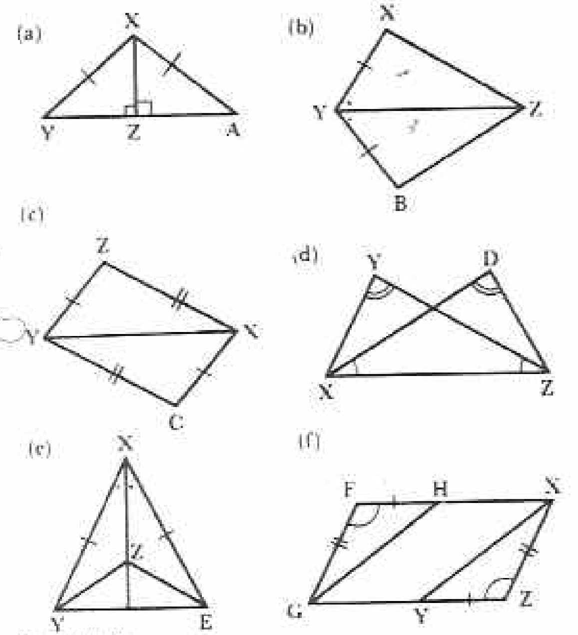


Fig. 24.19

7 In Fig. 24.20, CD ∥ AB and AD bisects $B\hat{A}C$. If $C\hat{D}E = 27°$ and $A\hat{C}B = 69°$, find $B\hat{E}D$.

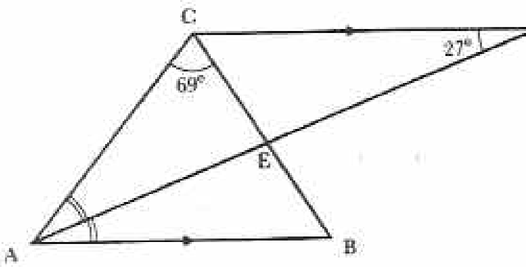
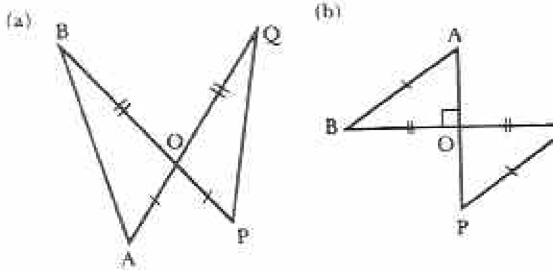


Fig. 24.20

8 In each part of Fig. 24.21, $\triangle AOB \equiv \triangle POQ$. State the case of congruency for each.



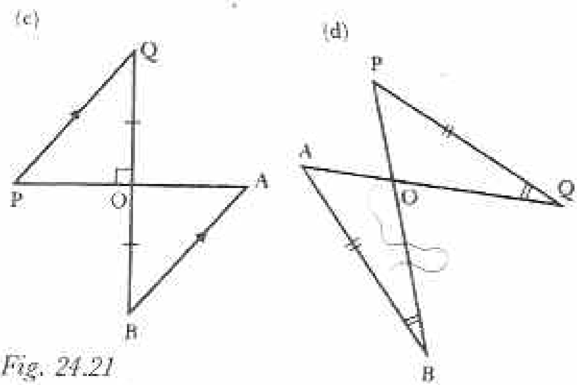


Fig. 24.21

9 In Fig. 24.22, BC ∥ XY, $B\hat{X}Y = 50°$, $B\hat{Y}X = 28°$ and AB = BY. Calculate $A\hat{C}B$.

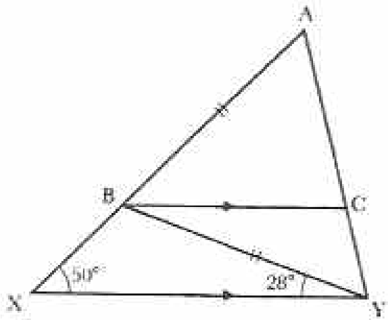


Fig. 24.22

10 Which of the triangles in Fig. 24.23 are congruent? State the case(s) of congruency.

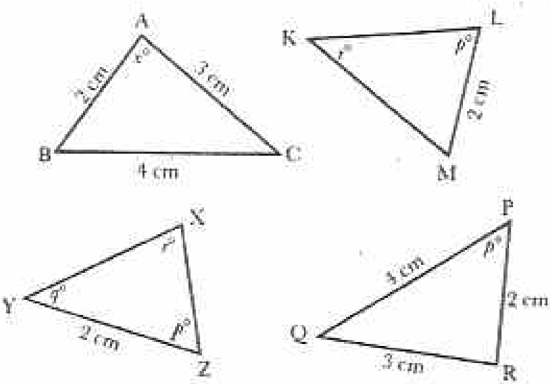


Fig. 24.23

Polygons

Any plane figure with straight sides is called a **polygon**. A **regular polygon** has all its sides of equal length and all its angles of equal size. Polygons are named after the number of sides they have. Fig. 24.24 gives the names of some common regular polygons.

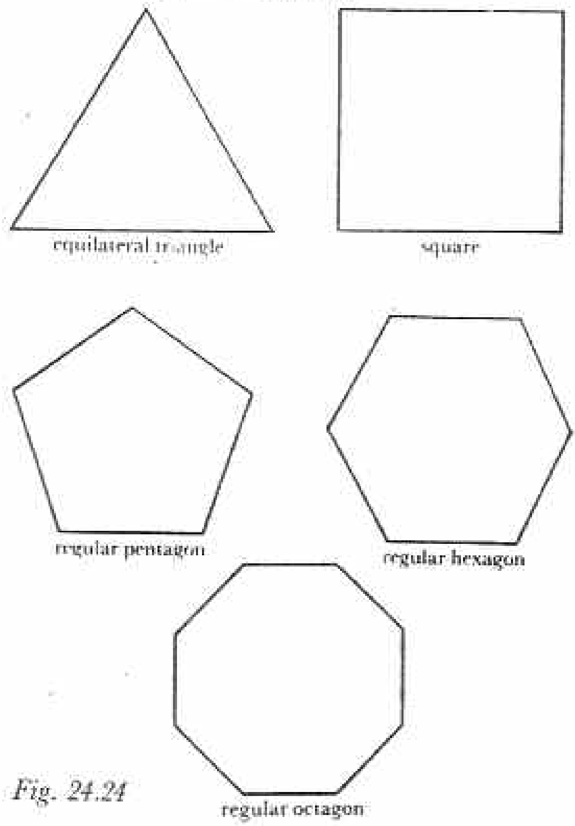


Fig. 24.24

Angles of a polygon

The sum of the angles of an *n*-sided polygon is $(n - 2) \times 180°$ or $(2n - 4)$ right angles. The sum of the exterior angles of a polygon is 4 right angles, or 360°.

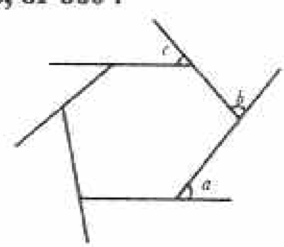


Fig. 24.25

In Fig. 24.25, $a + b + c + \ldots = 360°$.

Example 4

Calculate the interior angles of a regular pentagon.

A pentagon has 5 sides.
The five exterior angles add up to 360°.
Since the pentagon is regular, the exterior angles are equal.

$$\text{Each exterior angle} = \frac{360°}{5} = 72°$$

Hence each interior angle

$= 180° - 72°$ ($\angle$*s on a str. line*)
$= 108°$

Example 5

Each of the angles of a polygon is 140°. Find the number of sides that the polygon has.

Either:
Let the polygon have n sides.
Sum of angles of polygon $= n \times 140° = 140n°$
Also, sum of angles $= (n-2)180°$
So $(n-2)180° = 140n°$
$\Leftrightarrow \quad 180n - 360 = 140n$
$\Leftrightarrow \quad 40n = 360$
$\Leftrightarrow \quad n = \frac{360}{40} = 9$

or:
Each exterior angle $= 180° - 140° = 40°$
But the sum of the exterior angles $= 360°$

$$\text{Number of exterior angles} = \frac{360°}{40°} = 9$$

The polygon has 9 sides.

Properties of quadrilaterals

A **trapezium** is a quadrilateral which has one pair of opposite sides **parallel** (Fig. 24.26).

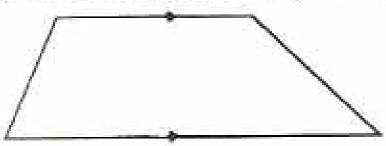


Fig. 24.26

A **parallelogram** is a quadrilateral which has both pairs of opposite sides parallel (Fig. 24.27).

In a paralellogram,
1 the opposite sides are parallel,
2 the opposite sides are equal,

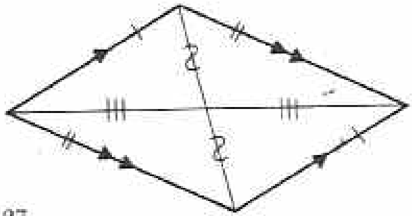


Fig. 24.27

3 the opposite angles are equal,
4 the diagonals bisect one another.

A **rhombus** is a quadrilateral which has all four sides equal (Fig. 24.28).

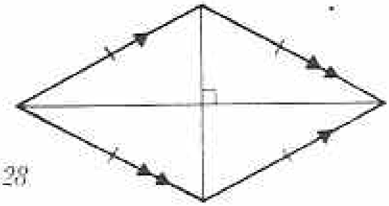


Fig. 24.28

In a rhombus,
1 all four sides are equal,
2 the opposite sides are parallel,
3 the opposite angles are equal,
4 the diagonals bisect one another at right angles,
5 the diagonals bisect the angles.

A **rectangle** is a quadrilateral in which every angle is a right angle (Fig. 24.29).

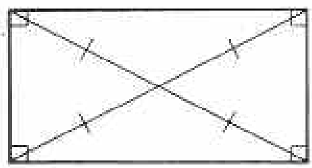


Fig. 24.29

In a rectangle, all of the facts given for a parallelogram are true. In addition:
1 all four angles are right angles,
2 the diagonals are of equal length.

A **square** is a rectangle which has all four sides equal (Fig. 24.30).

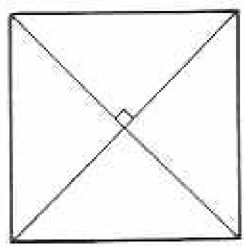


Fig. 24.30

In a square, all of the facts given for a rhombus are true. In addition:

1 all four angles are right angles,

2 the diagonals are of equal length,

3 the diagonals meet the sides at 45°.

Exercise 24c

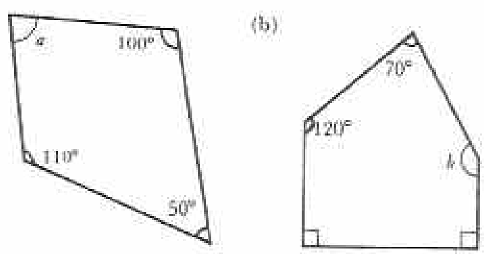
1 Use the $(2n - 4) \times 90°$ formula to complete Table 24.1.

Table 24.1

polygon	sum of interior angles
3 sides triangle	
4 sides quadrilateral	
5 sides pentagon	
6 sides hexagon	
7 sides heptagon	
8 sides octagon	
10 sides decagon	
12 sides dodecagon	

2 Calculate the sizes of the lettered angles in Fig. 24.31.

(a) (b)



(c) (d)

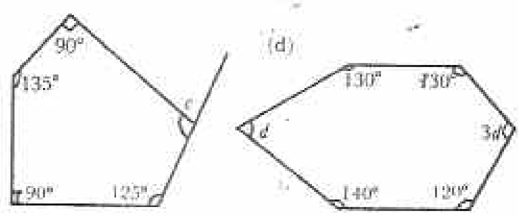


Fig. 24.31

3 Calculate the interior angles of a regular 15-sided polygon.

4 A regular polygon has interior angles of 160°. How many sides has it?

5 In Fig. 24.32, pentagon ABCDE is such that AB ∥ ED and BC ∥ AE. If $A\hat{B}C = 118°$ and $C\hat{D}E = 130°$, calculate $B\hat{C}D$, $E\hat{A}B$ and $A\hat{E}D$.

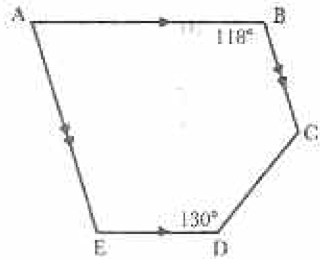


Fig. 24.32

6 In Fig. 24.33, QABS is a diagonal of rhombus PQRS and square PARB.

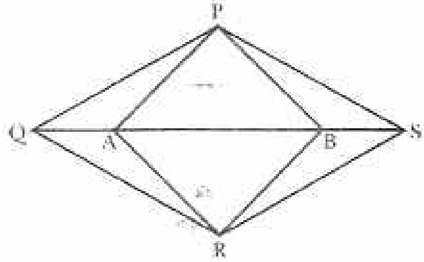


Fig. 24.33

If $P\hat{S}R = 68°$, what is the size of $Q\hat{P}A$?

7 In Fig. 24.34, ABDE and BCDE are parallelograms. Use the measurements given on the figure to calculate the perimeter of trapezium ABCDE.

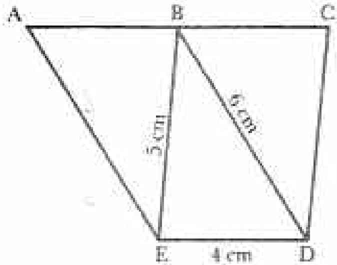


Fig. 24.34

8 ABCD is a trapezium such that AB || DC. X is a point on CD such that CX = BA. If $A\hat{B}C = 102°$ and $D\hat{A}X = 47°$, calculate $A\hat{D}X$.

9 Two of the exterior angles of a polygon are 63° each. The remaining exterior angles are each 26°. How many sides has the polygon?

10 The angles of a pentagon are $4x$, $5x$, $6x$, $7x$ and $8x$. Find x and hence state the sizes of the angles of the pentagon.

Circles

Fig. 24.35 gives the names of the lines and regions of a circle.

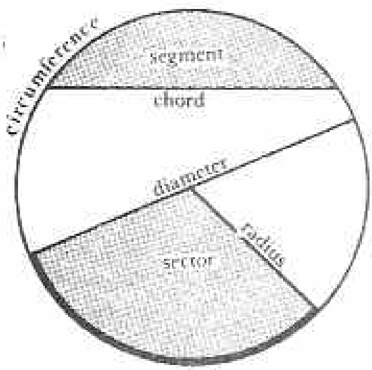


Fig. 24.35

Chords of a circle

1 The line joining the centre of a circle to the mid-point of a chord is perpendicular to the chord (Fig 24.36).

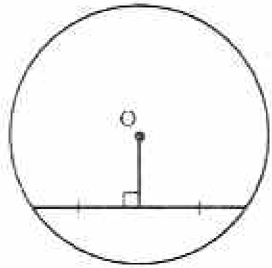


Fig. 24.36

2 Equal chords are equidistant from the centre of a circle, and conversely.

3 In equal circles equal chords are equidistant from the centres, and conversely.

Angle properties of circles

1 The angle which an arc of a circle subtends at the centre of a circle is twice that which it subtends at any point on the remaining part of the circumference.

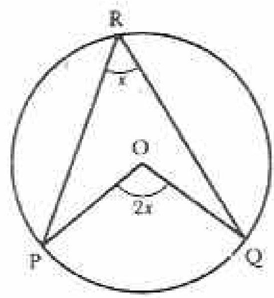


Fig. 24.37

In Fig. 24.37, $P\hat{O}Q = 2 \times P\hat{R}Q$.

2 The angle in a semicircle is a right angle.

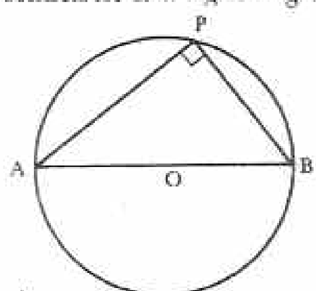


Fig. 24.38

In Fig. 24.38, $A\hat{P}B = 90°$.

3 Angles in the same segment are equal.

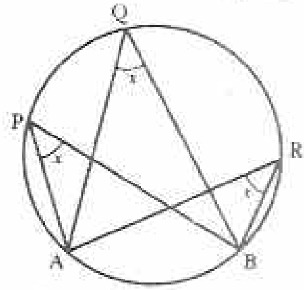


Fig. 24.39

In Fig. 24.39, $A\hat{P}B = A\hat{Q}B = A\hat{R}B = \ldots$

4 Angles in opposite segments are supplementary.

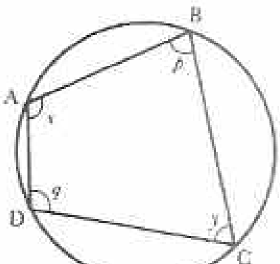


Fig. 24.40

In Fig. 24.40, $x + y = p + q = 180°$. ABCD is known as a **cyclic quadrilateral**. Hence, opposite angles of a cyclic quadrilateral are supplementary. It follows that the exterior angle of a cyclic quadrilateral is equal to the interior opposite angle (Fig. 24.41).

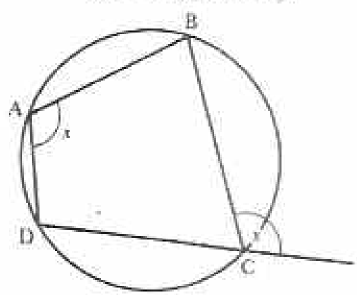


Fig. 24.41

Tangents to circles

A **tangent** to a circle is a line which meets the circle in one point only.

1 A tangent is perpendicular to the radius at the point of contact.

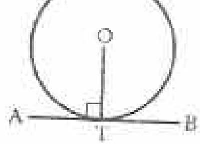


Fig. 24.42

In Fig. 24.42, OT ⊥ ATB.

2 The tangents to a circle from an external point are equal in length.

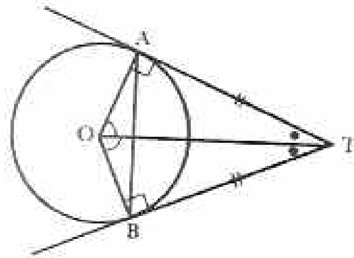


Fig. 24.43

In Fig. 24.43, TA = TB. Notice also that $A\hat{T}O = B\hat{T}O$, $A\hat{O}T = B\hat{O}T$ and OT bisects AB at right angles.

3 The angle between a tangent to a circle and a chord through the point of contact is equal to the angle in the alternate segment.

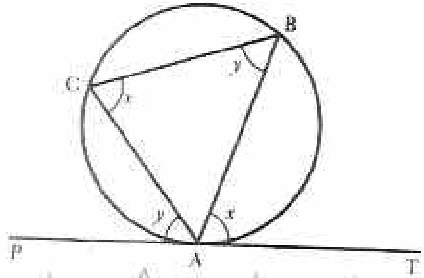
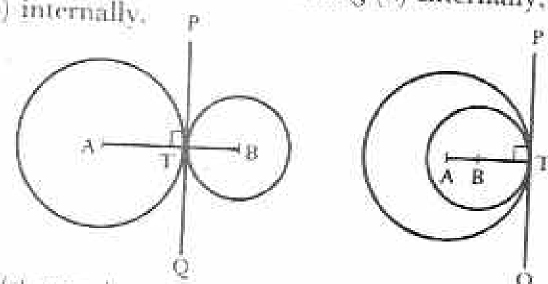


Fig. 24.44

In Fig. 24.44, $T\hat{A}B = A\hat{C}B$ and $P\hat{A}C = A\hat{B}C$.

Contact of circles

Fig. 24.45 shows circles touching (a) externally, (b) internally.



(a) external contact (b) internal contact

Fig. 24.45

In both cases the line joining the centres of the circles, AB, passes through their point of contact, T. The line of centres, AB, is at right angles to the common tangent, PTQ.

Exercise 24d

1 Redraw the figures in Fig. 24.46 and write in the sizes of the marked angles. (Some construction lines may be necessary.)

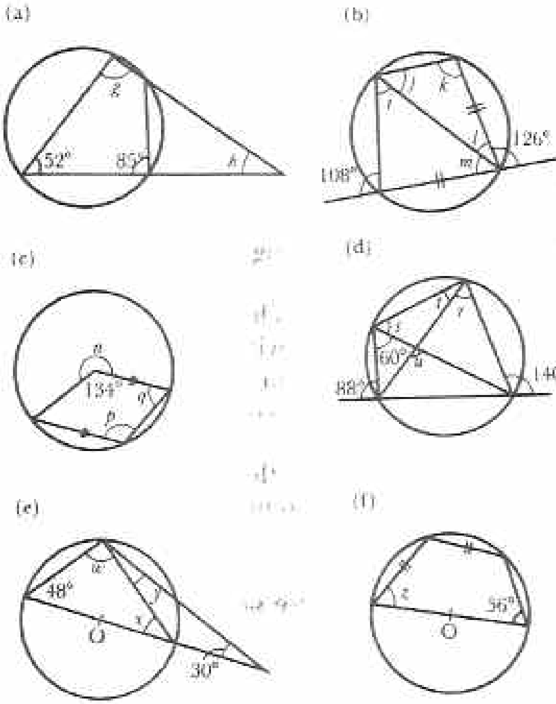


Fig. 24.46

2 Three circles, centres A, B and C with radii 4 cm, 3 cm and 2 cm respectively touch one another externally. Calculate the lengths of the sides of △ABC.

3 Three circles touch one another externally. Their centres form a triangle with sides 10 cm, 9 cm and 7 cm. Find the radii of the circles.

4 In Fig. 24.47, the circle touches the sides of △ABC at X, Y, Z. If BC = 11 cm, CA = 10 cm and AB = 9 cm, find AY and BX.

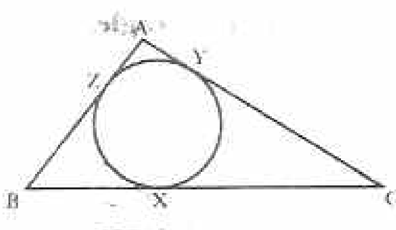


Fig. 24.47

5 In Fig. 24.48, lines drawn through T are tangents and O is the centre of any given circle. Find the lettered angles.

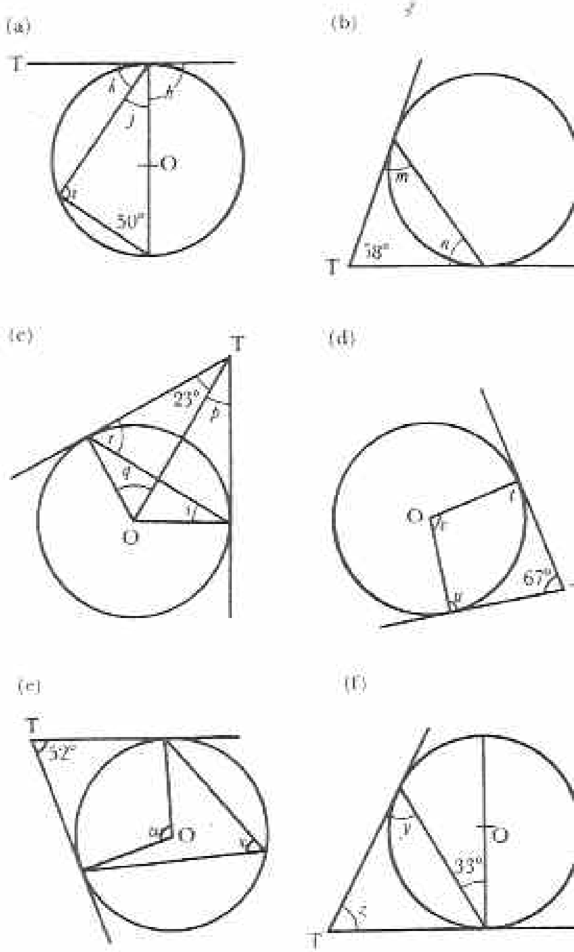
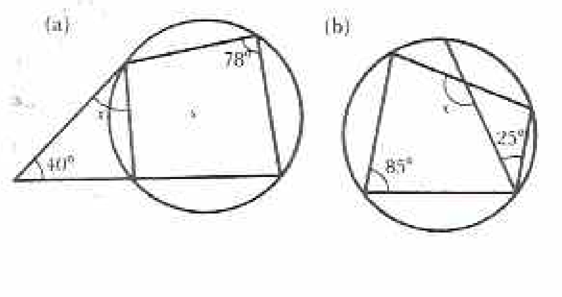


Fig. 24.48

6 A circle is drawn inside a triangle ABC to touch the sides BC, CA and AB at P, Q and R respectively. If $\hat{A} = 56°$ and $\hat{B} = 68°$, find the angles of △PQR.

7 In each part of Fig. 24.49, find the value of x. In part (d), O is the centre of the circle.



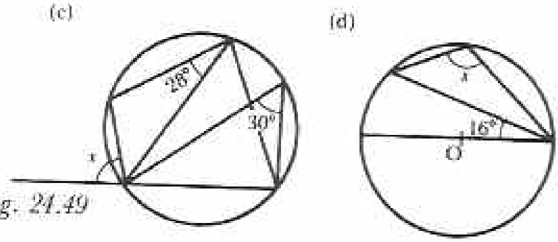


Fig. 24.49

8 In Fig. 24.50, find the sizes of the lettered angles.

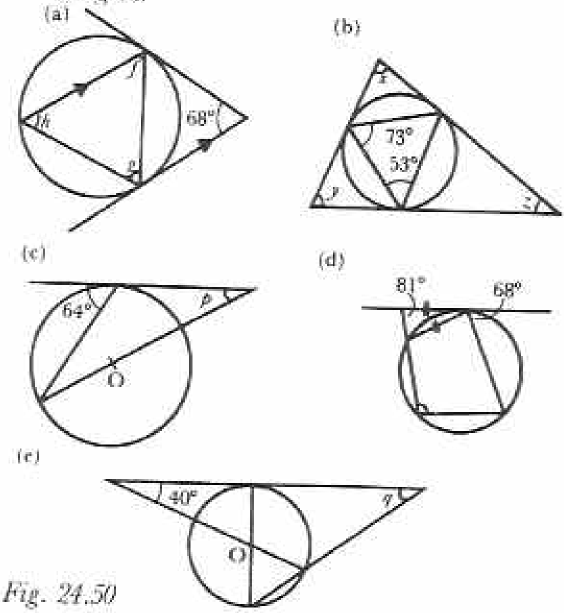


Fig. 24.50

9 In Fig. 24.51, express y in terms of x.

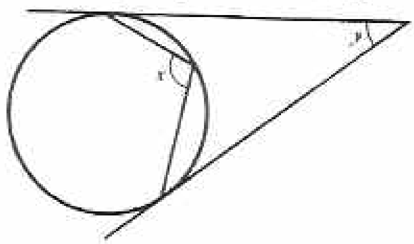
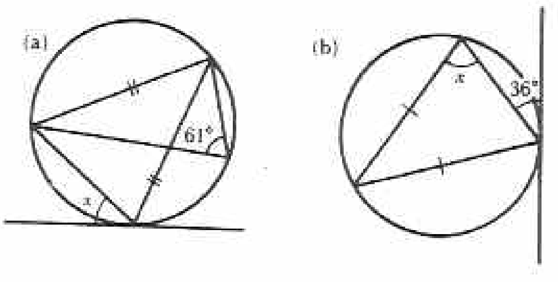


Fig. 24.51

10 In each part of Fig. 24.52, find the value of x.



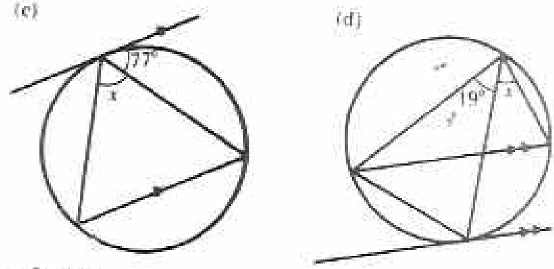


Fig. 24.52

Constructions

Remember the following when making constructions:

1 Make a rough sketch. This helps in anticipating problems associated with the construction.

2 Leave all construction lines visible. Do not rub off anything that contributes to the final result.

3 Use a hard pencil with a sharp point. This enables lines and points to be as fine and accurate as possible.

To bisect a given line segment, AB

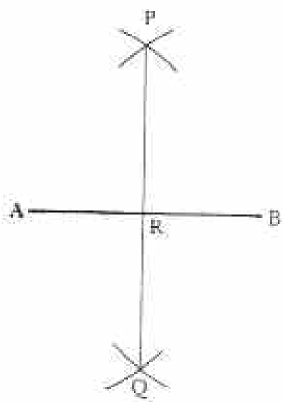


Fig. 24.53

With centres A and B and equal radii, draw arcs to cut each other at P and Q. Join PQ to cut AB at R (Fig. 24.53). R is the mid-point of AB.

To bisect a given angle, $A\hat{B}C$

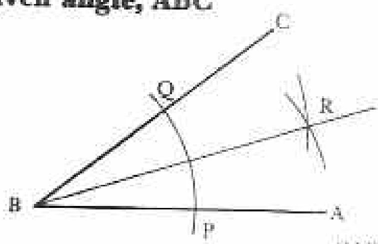


Fig. 24.54

With centre B and any radius, draw an arc to cut BA and BC at P and Q. With centres P and Q and equal radii, draw arcs to cut each other at R. Draw BR (Fig. 24.54). BR is the required bisector.

To construct a line perpendicular to a given straight line, AB, from a point, M, outside the line

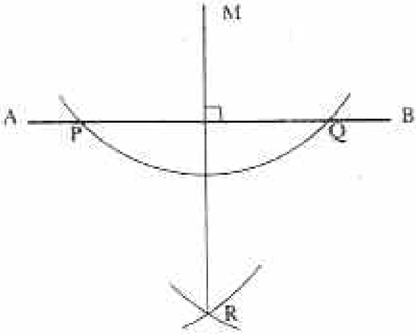


Fig. 24.55

With centre M and any radius, draw an arc to cut AB at P and Q. With centres P and Q and equal radii, draw arcs to cut each other at R (Fig. 24.55). MR is perpendicular to AB.

To construct parallel lines, using ruler and set square

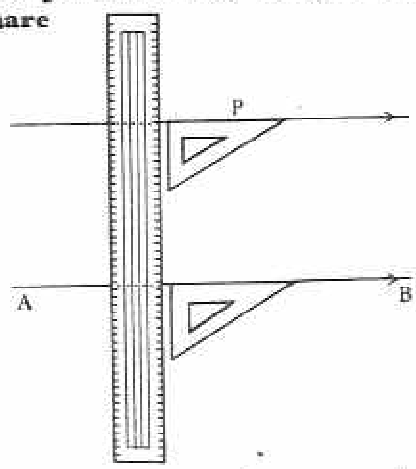


Fig. 24.56

Place a set square with one edge accurately along a given line AB. Place a ruler against one of the other edges of the set square. Holding the ruler firmly, slide the set square along its edge until the edge that was originally along AB passes through the required point P. Use that edge of the set square to draw a line parallel to AB (Fig. 24.56).

To construct an angle of 60°

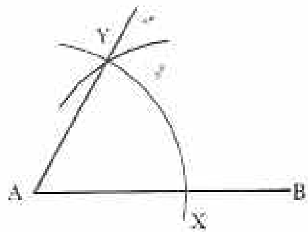


Fig. 24.57

With centre A and any convenient radius draw an arc cutting AB at X. With centre X and the *same* radius, draw an arc to cut the first arc at Y. Join AY to give $B\hat{A}Y = 60°$ (Fig. 24.57).

To construct an angle of 30°, first construct an angle of 60° as above and then bisect it. Further bisections will give **angles of 15°, $7\frac{1}{2}$°, etc.**

To construct an angle of 45°, first construct a right angle and then bisect it. Further bisections will give **angles of $22\frac{1}{2}$°, etc.**

To copy a given angle, $A\hat{B}C$

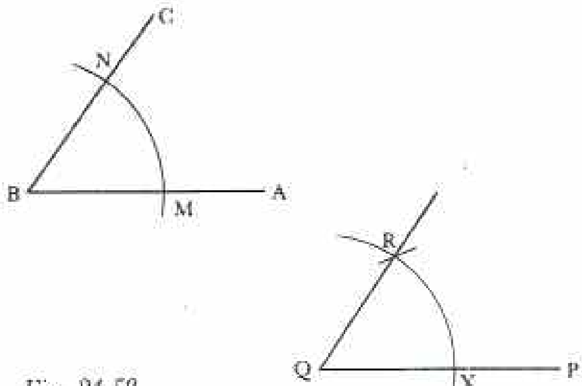


Fig. 24.58

Draw any line PQ. With centre B and any radius draw an arc to cut BA and BC at M and N. With centre Q and the *same* radius, draw an arc to cut QP at X. With centre X and radius MN, draw an arc to cut the arc through X at R (Fig. 24.58). Then $P\hat{Q}R = A\hat{B}C$.

Exercise 24e

1 Construct angles of 60°, 30°, 75°, 45°, 120°, $37\frac{1}{2}$°, 135°.

2 Construct an equilateral triangle with sides of length 7,2 cm. Construct the perpendicular from a vertex to the opposite side and measure its length.

3 Construct $\triangle PQR$ such that $\hat{Q} = 90°$, $\hat{P} = 60°$ and $PQ = 8$ cm. Draw the bisectors of $\hat{P}$ and $\hat{R}$ to intersect at O. Measure OP.

4 Construct an isosceles triangle with the equal sides 9 cm long and the angle between them 45°. Measure the third side.

5 Construct $\triangle ABC$ such that $AB = 10{,}8$ cm, $BC = 6{,}4$ cm and $AC = 7$ cm. Draw the perpendicular bisectors of the three sides to meet at O. Draw the circumcircle of the triangle, i.e. the circle, centre O, which passes through A, B and C.

6 Construct the parallelogram ABCD in which $BD = 104$ mm, $DC = 48$ mm and $B\hat{C}D = 30°$. Measure AC.

7 Construct a trapezium PQRS in which $PQ \parallel SR$, $PQ = 6$ cm, $PS = 5$ cm, $SR = 11$ cm and $QS = 9$ cm. Measure QR.

8 Construct a rhombus ABCD so that $AC = 6$ cm and $BD = 8$ cm. Measure a side of the rhombus.

9 Construct a triangle with sides of 5 cm, 6 cm, 7 cm. A rhombus with sides of length 4 cm has acute angles each equal to the smallest angle of the triangle. Copy the smallest angle of the triangle and hence construct the rhombus. Measure the longest diagonal of the rhombus.

10 Draw a circle of radius 3 cm. Draw a diameter RT and construct a tangent to the circle at T. Find a point P on the tangent such that $\triangle RTP$ is isosceles. Measure RP.

Locus

The simple idea of a locus is that it is the path traced out by a point which moves in accordance with a certain law. However, it is more correct to define **a locus as the set of all possible positions** occupied by an object which varies its position according to some given law.

Table 24.2 on page 222 gives some common **loci** (plural of locus) in 2 and 3 dimensions.

Example 6

$AB = 4$ cm *and a point C moves so that the area of* $\triangle ABC$ *is* 6 cm^2 *and* $A\hat{C}B = 60°$. *Construct two possible positions of C. Find AC and BC in each case.*

Let C be a distance h cm from AB, then

$$\tfrac{1}{2}AB \times h = 6$$
$$\tfrac{1}{2} \times 4 \times h = 6$$
$$\Leftrightarrow h = 3$$

C is a distance 3 cm from AB.
Fig. 24.59 shows (a) sketch and (b) the accurate drawing giving two positions of C.

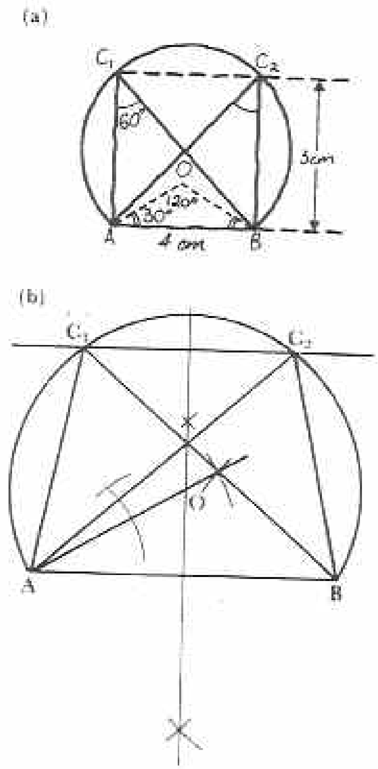
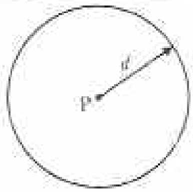
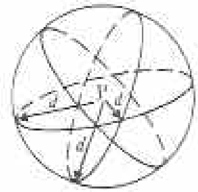
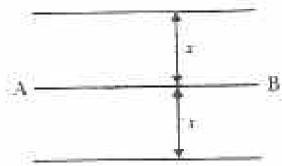
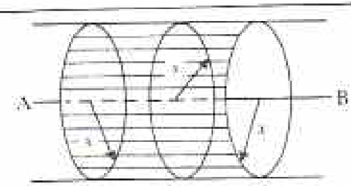
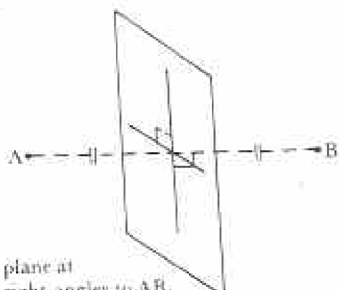
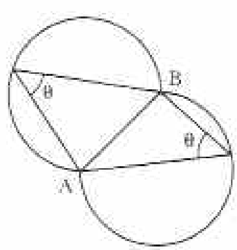


Fig. 24.59

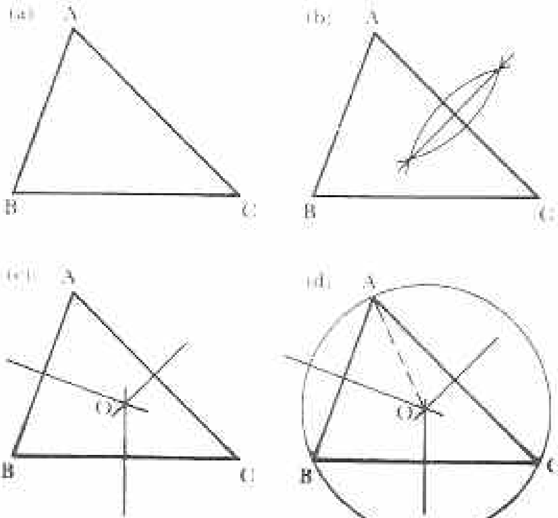
From Fig. 24.59,
either $AC = 3{,}1$ cm and $BC = 4{,}5$ cm,
or $AC = 4{,}5$ cm and $BC = 3{,}1$ cm.

Table 24.2

a set of points	locus	
	in 2 dimensions	in 3 dimensions
. . . which are a given distance, *d*, from a given point P	 circumference of a circle, centre P, radius *d*	 outer shell of a sphere, centre P, radius *d*
. . . which are a given distance, *x*, from a given straight line AB	 a pair of lines each parallel to AB	 a cylindrical surface with AB as its central axis
. . . which are equidistant from two given points AB	 the perpendicular bisector of AB	 plane at right-angles to AB, bisecting AB
. . . at which a given segment of a straight line AB subtends a given angle θ	 two segments of a circle on chord AB	non-applicable in 3 dimensions

Circumcircle

Fig. 24.60 shows how to construct the circumcircle of any triangle ABC.



(a) Given any △ ABC.

(b) Perpendicular bisector of AC.

(c) Perpendicular bisectors of all three sides meeting at O (lines of construction not shown here).

(d) Circumcircle, centre O, radius OA (or OB or OC).

Fig. 24.60

Exercise 24f

1 A pencil slides around inside a hemispherical bowl. Describe the locus of the mid-point of the pencil.

2 A pole stands vertically on horizontal ground. A wire is stretched tightly from the top of the pole to a point on the ground some distance from the foot of the pole. Describe the locus of the lower end of the wire.

3 Draw two intersecting lines. Construct the locus of points which are equidistant from the two lines.

4 A and B are fixed points. P can move so that $\hat{APB} = 45°$. Use a 45° set square to plot several positions of P, and hence draw its locus.

5 A circle is drawn so that it passes through two fixed points. Describe the locus of the centre of the circle as it varies in radius.

6 Draw two straight lines AXB and PXQ intersecting at X at an angle of 70°. Draw the locus of points which move so that they are (i) 2 cm from AB, (ii) 3 cm from PQ. How many points are common to both loci?

7 On the same diagram, draw the following loci: (a) a point P which moves so that $\hat{APB} = 90°$ and AB = 8 cm; (b) points which are 3 cm from AB. How many points are common to both loci?

8 If AB = 5 cm, construct the locus of the set of points C such that the area of △ABC is $10\,cm^2$. Use a suitable construction to find all the positions of C where $\hat{ACB} = 54°$. Hence find two possible values for AC.

9 (a) **Construct in a single diagram,**
   (i) **triangle ABC such that AB = 9 cm, AC = 7 cm and BC = 5 cm,**
   (ii) **the locus of points which are 4 cm** from A,
   (iii) the circumcircle of triangle ABC.
(b) If $\mathscr{E}$ = {P: P lies inside the circumcircle of △ABC}
   X = {P: P lies inside △ABC}
   Y = {P: AP < 4 cm}
   show the region $X' \cap Y'$ by shading in your diagram.

10 Construct the triangle XYZ in which XY = 5 cm, $\hat{X} = 60°$ and $\hat{Y} = 90°$. Measure and write down the length of YZ. On the same diagram,
(a) construct the circumcircle of △XYZ,
(b) construct, on the same side of XY as Z, the locus of the point P, such that the area of △XYP equals half the area of △XYZ,
(c) mark, and label clearly, a point Q such that $\hat{XQY} = 30°$ and the area of △XYQ is half the area of △XYZ.

Given that M is a point such that $\hat{XMY} = 30°$, find the largest possible area of △XMY. [Camb]

Chapter 25

Mensuration

Syllabus references 6.4, 6.7.5 and 6.8.2

Perimeter

The **perimeter** of a plane shape is the distance round the edge of the shape. The perimeters of shapes are found by measurement. In some cases there are formulae which enable perimeters to be calculated:

Perimeter of rectangle in Fig. 25.1 $= 2(l + b)$

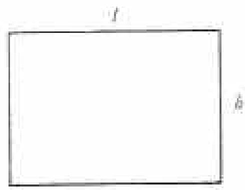


Fig. 25.1

Perimeter of circle in Fig. 25.2 $= 2\pi r$
The perimeter of a circle is often called the **circumference**.

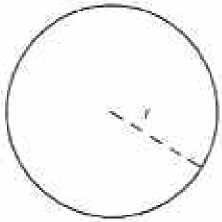


Fig. 25.2

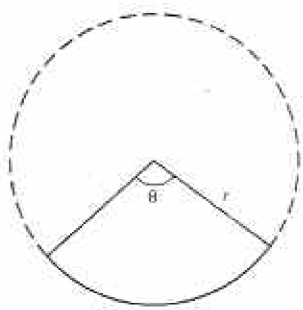


Fig. 25.3

Length of arc in Fig. 25.3 $= \dfrac{\theta}{360}$ of $2\pi r$

Perimeter of sector $= \dfrac{\theta}{360} \times 2\pi r + 2r$

Example 1

Find the perimeter of a sector of a circle of radius 3,5 cm, *the angle of the sector being* 144°.

Perimeter = length of arc + $2r$

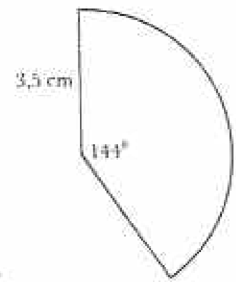


Fig. 25.4

$$\text{Length of arc} = \frac{144}{360} \times 2\pi \times 3{,}5 \text{ cm}$$

$$= \frac{144}{360} \times 2 \times \frac{22}{7} \times \frac{7}{2} \text{ cm}$$

$$= \frac{44}{5} \text{ cm} = 8{,}8 \text{ cm}$$

Perimeter $= (8{,}8 + 2 \times 3{,}5)$ cm $= 15{,}8$ cm

Exercise 25a

Take the value of π to be $3\frac{1}{7}$, unless told otherwise.

1 Use the value 3,14 for π to calculate the circumference of a circle of diameter 20 cm.

2 The minute hand of a wall-clock is 10,5 cm long. How far does its tip travel in 24 hours?

3 Through what angle does the minute hand of a clock move in 25 min? If the minute hand is 6,3 cm long, how far does its tip move in 25 min?

4 How many revolutions does a bicycle wheel of diameter 70 cm make in travelling 110 m?

5 An arc subtends an angle of 72° at the centre of a circle of radius 17,5 cm. Find the length of the arc.

6 Two circles have circumferences of 10π cm and 12π cm. What is the difference in their radii?

7 A chord of a circle subtends an angle of 60° at the centre of a circle of radius 7 cm. Find the perimeter of the minor segment of the circle.

8 A piece of thread was wound tightly round a cylinder for 20 complete turns. The thread was found to be 3,96 m long. Calculate the diameter of the cylinder in cm.

9 Calculate the perimeter of a sector of a circle of radius 27 cm, the angle of the sector being 140°.

10 Fig. 25.5 is a sketch of a doorway. The arc at the top subtends an angle of 60° at the centre of a circle of radius 75 cm.

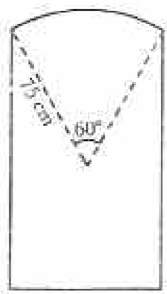
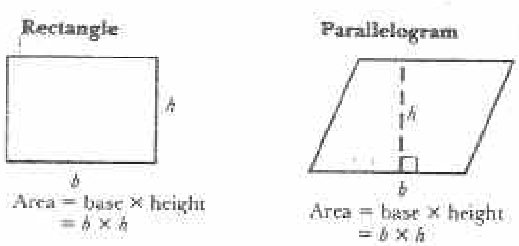


Fig. 25.5

Use the value 3,14 for π to calculate the perimeter of the doorway to the nearest $\frac{1}{2}$-centimetre.

Area of plane shapes

The common units of area are cm^2, m^2 and km^2. The **hectare** (ha) is often used for land measure. 1 ha = 10 000 m^2. Formulae for the areas of the common plane shapes are given in Fig. 25.6.



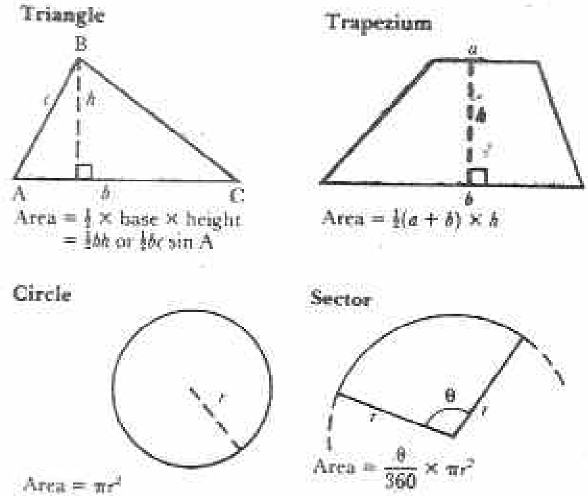


Fig. 25.6

Example 2

Calculate the area of $\triangle$ABC *in which* AB = 8 cm, AC = 4 cm *and* $\hat{BAC}$ = 58°.

Area of $\triangle ABC = \frac{1}{2} \times 8 \times 4 \times \sin 58°\ cm^2$
$= 16 \times 0{,}848\ cm^2$
$= 13{,}6\ cm^2$ to 3 s.f.

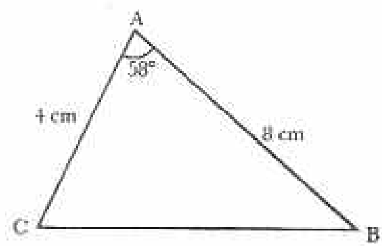


Fig. 25.7

Example 3

Calculate the area of the parallelogram in Fig. 25.8.

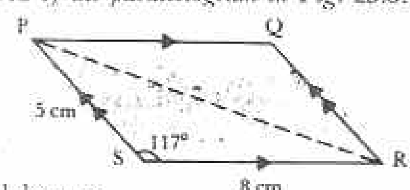


Fig. 25.8

Area of parallelogram
$= 2 \times$ area of $\triangle PRS$
$= 2 \times (\frac{1}{2} \times 5 \times 8 \times \sin 117°)\ cm^2$
$= 5 \times 8 \times \sin 63°\ cm^2$
$= 40 \times 0{,}891\,0\ cm^2 = 35{,}6\ cm^2$

Example 4

The trapezium in Fig. 25.9, overleaf, has an area of 456 cm^2. *Find the distance between its parallel sides.*

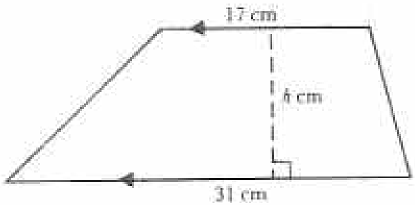


Fig. 25.9

Area of trapezium $= \frac{1}{2}(17 + 31)h \text{ cm}^2$

$\frac{1}{2}(17 + 31) \times h = 456$

$\Leftrightarrow \quad \frac{1}{2} \times 48 \times h = 456$

$\Leftrightarrow \quad 24h = 456$

$\Leftrightarrow \quad h = \frac{456}{24} = 19$

The parallel sides are 19 cm apart.

Example 5

In Fig. 25.10, the chord AB subtends an angle of 120° at the centre of the circle, radius 7 cm. Use the value $\frac{22}{7}$ for π to calculate the area of the minor segment of the circle.

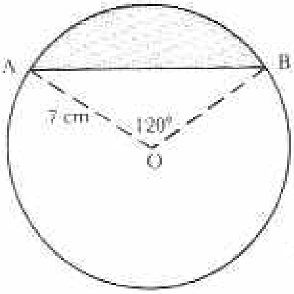


Fig. 25.10

Area of minor segment

$= \text{area of sector AOB} - \text{area of } \triangle\text{AOB}$

Area of sector AOB $= \frac{120}{360} \times \pi \times 7^2 \text{ cm}^2$

$= \frac{1}{3} \times 22 \times 7 \text{ cm}^2$

Area of $\triangle$AOB $= \frac{1}{2} \times 7 \times 7 \times \sin 120° \text{ cm}^2$

$= \frac{1}{2} \times 7 \times 7 \times 0{,}866\,0 \text{ cm}^2$

$= 7 \times 7 \times 0{,}433 \text{ cm}^2$

Area of segment

$= \frac{1}{3} \times 22 \times 7 - 7 \times 7 \times 0{,}433 \text{ cm}^2$*

$= 7(\frac{1}{3} \times 22 - 7 \times 0{,}433) \text{ cm}^2$

$= 7(7{,}333 - 3{,}031) \text{ cm}^2 = 7 \times 4{,}302 \text{ cm}^2$

$= 30{,}1 \text{ cm}^2$ to 3 s.f.

* Alternatively the calculation may be done on a scientific calculator as follows.

Area of segment

$= \frac{1}{3} \times 22 \times 7 - \frac{1}{2} \times 7 \times 7 \times \sin 120° \text{ cm}^2$

$= \frac{154}{3} - 24{,}5 \times \sin 60°$

$= 51{,}333\,333 - 24{,}5 \times \sin 60°$**

On the calculator:

Key	*Display*
60 sin × 24·5 = +/−	−21 217622
+ 51·333333 =	30 115711

Area of segment $= 30{,}1 \text{ cm}^2$

** Where possible, always simplify calculations before using a calculator. This reduces the possibility of making keystroke errors.

Exercise 25b

Use the value $\frac{22}{7}$ for π unless told otherwise.

1 Calculate the area shaded in each of the shapes in Fig. 25.11.

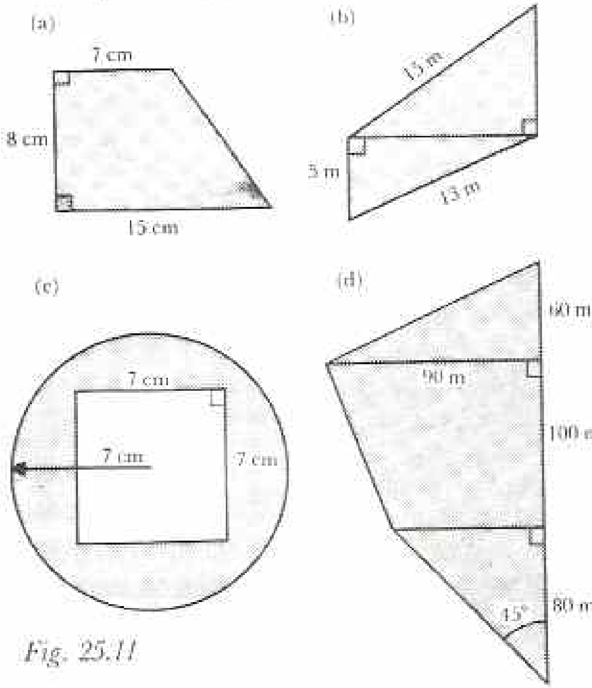


Fig. 25.11

2 The diagonals of a parallelogram are 6 cm and 8 cm long and they intersect at an angle of 55°. Find the area of the parallelogram.

3 Two sides of a triangular field are 120 m and 200 m. If the angle between the sides is 68°, find the area of the field in hectares.

4 Find the area of a circle of radius 35 cm. If a sector of angle 80° is removed from the circle, what area is left?

5 Circular discs of diameter 4 cm are punched out of a sheet of brass of mass 0,84 g/cm^2. What is the mass of 500 discs?

6 If 350 of the discs in question 5 are punched from a square sheet of brass, 80 cm by 80 cm, what percentage of the sheet is not used?

7 What is the diameter, to the nearest metre, of a circular sports ground of area exactly one hectare?

8 A washer is 4,5 cm in diameter with a central hole of diameter 1,5 cm. Use the value 3,142 for π to calculate the surface area of the washer correct to 3 s.f.

9 Fig. 25.12 shows a cross-section of a tunnel in the form of a major segment of a 6 m diameter circle. The path at the base of the tunnel is 3 m wide. Use the value 3,142 for π to find the cross-sectional area of the tunnel correct to 3 s.f.

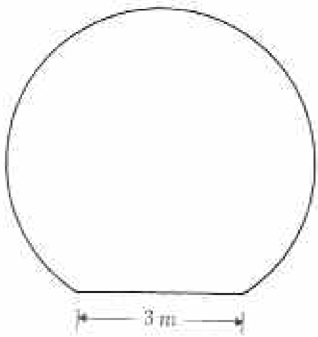


Fig. 25.12

10 Calculate the area of the doorway in Fig. 25.5 on page 225.

11 A chord subtends an angle of 140° at the centre of a circle of radius 10 cm. Calculate the area of the minor segment of the circle.

12 The floor of a classroom measures 6 m by 8 m. The walls are 3 m high and they contain four rectangular windows measuring 1,2 m by 75 cm. There is a door which measures 2 m by 80 cm. What percentage of the total wall area is taken up with windows and doors? (2 s.f.)

Surface area and volume of solids

Fig. 25.13 contains a summary of the formulae for surface area and volume of common solid shapes.

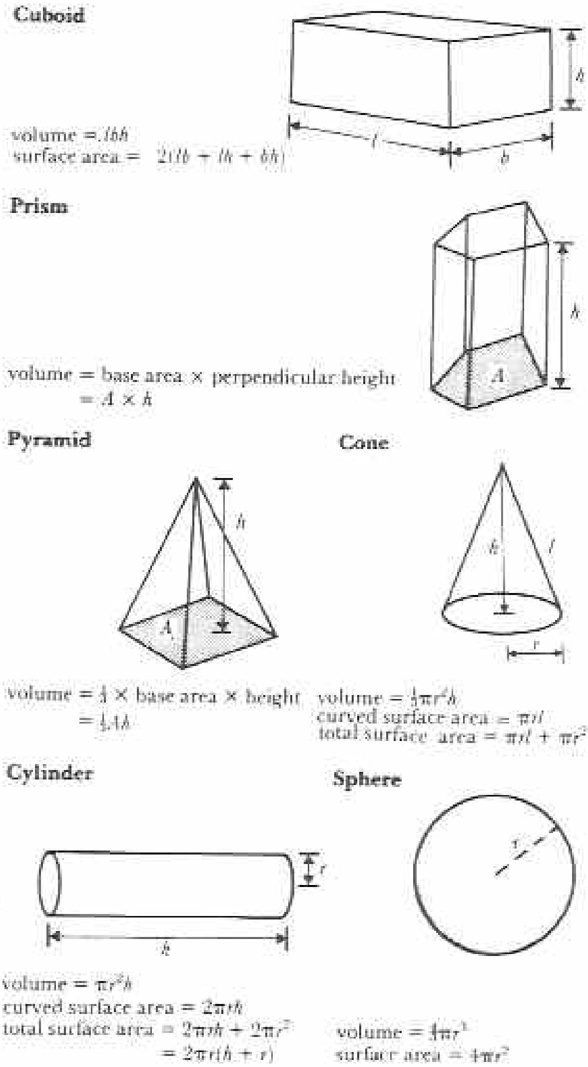


Fig. 25.13

Example 6

An open rectangular box (Fig. 25.14, overleaf) *measures externally* 32 cm *long,* 27 cm *wide and* 15 cm *deep. If the box is made of wood* 1 cm *thick, what volume of wood is used?*

The internal dimensions are 30 cm, 25 cm and 14 cm respectively.

External volume $= 32 \times 27 \times 15\ \text{cm}^3$
$= 12\,960\ \text{cm}^3$

Internal volume $= 30 \times 25 \times 14\ \text{cm}^3$
$= 10\,500\ \text{cm}^3$

Volume of wood $= 12\,960\ \text{cm}^3 - 10\,500\ \text{cm}^3$
$= 2\,460\ \text{cm}^3$

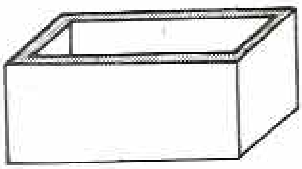


Fig. 25.14

Example 7

How many litres of oil does a cylindrical drum 28 cm *in diameter and* 50 cm *deep hold?*

Volume of drum $= \pi r^2 h = \frac{22}{7} \times 14^2 \times 50\ \text{cm}^3$

Capacity of drum $= \dfrac{22 \times 14^2 \times 50}{7 \times 1\,000}$ litres

$= 30{,}8$ litres

Example 8

A cylindrical metal bar 50 cm *long and* 6 cm *in diameter is pulled out to form a wire of diameter* 3 mm. *(a) What length is the wire? (b) How does the curved surface area of the wire compare with that of the bar?*

(a) Let the length of the wire be x cm.

Volume of wire $= \pi\left(\dfrac{3}{20}\right)^2 \times x\ \text{cm}^3$

Volume of bar $= \pi \times 3^2 \times 50\ \text{cm}^3$

$$\pi\frac{9x}{400} = \pi \times 9 \times 50$$

$$\Leftrightarrow \qquad x = \frac{9\pi \times 50 \times 400}{9\pi} = 20\,000$$

length of wire $= 20\,000$ cm $= 200$ m

(b) Considering curved surfaces only:
area of bar $= 2\pi 3 \times 50\ \text{cm}^2 = 300\pi\ \text{cm}^2$

area of wire $= 2\pi\dfrac{3}{20} \times 20\,000\ \text{cm}^2$

$= 6\,000\pi\ \text{cm}^2 = 20 \times 300\pi\ \text{cm}^2$
$= 20 \times$ area of bar

Notice in Example 8 that it was not necessary to use a numerical value for π.

Example 9

Find the capacity in litres of a bucket 24 cm *in diameter at the top,* 16 cm *in diameter at the bottom and* 18 cm *deep.*

The bucket is in the shape of a frustum of a cone. It is necessary to consider the whole cone. Complete the cone as in Fig. 25.15 and let the depth of the extension be x cm.

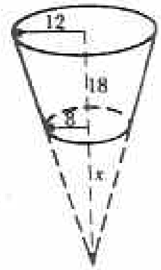


Fig. 25.15

In Fig. 25.15,

$$\frac{8}{x} = \frac{12}{x + 18} \qquad \textit{(similar } \triangle\text{s)}$$

$8x + 144 = 12x$
$\Leftrightarrow \quad 4x = 144$
$\Leftrightarrow \quad x = 36$

Volume of frustum
$= \frac{1}{3}\pi 12^2 \times 54\ \text{cm}^3 - \frac{1}{3}\pi 8^2 \times 36\ \text{cm}^3$
$= \frac{1}{3}\pi 4^2 \times 18(3^2 \times 3 - 2^2 \times 2)\ \text{cm}^3$
$= \pi \times 16 \times 6 \times 19\ \text{cm}^3 \simeq 5\,730\ \text{cm}^3$

Capacity of bucket $\simeq 5{,}73$ litres

Example 10

A solid is made up of a cylinder with a hemisphere on top as in Fig. 25.16. Calculate (a) the surface area (b) the volume of the solid.

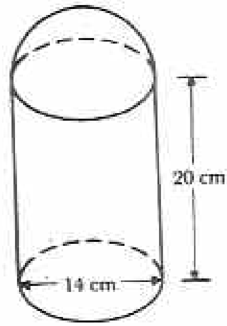


Fig. 25.16

(a) Total surface area
$= \pi r^2 + 2\pi rh + \frac{1}{2}(4\pi r^2)$
$= \pi 7^2 + 2\pi \times 7 \times 20 + 2\pi \times 7^2\ \text{cm}^2$
$= 7\pi(7 + 40 + 14)\ \text{cm}^2$
$= 7 \times \frac{22}{7} \times 61\ \text{cm}^2$
$= 1\,342\ \text{cm}^2$

(b) Volume $= \pi r^2 h + \frac{1}{2}(\frac{4}{3}\pi r^3)$
$= \pi \times 7^2 \times 20 + \frac{2}{3}\pi \times 7^3\ \text{cm}^3$
$= 49\pi(20 + \frac{2}{3} \times 7)\ \text{cm}^3$
$= 49 \times \frac{22}{7} \times \frac{74}{3}\ \text{cm}^3$
$\simeq 3\,800\ \text{cm}^3$

Exercise 25c

Use the value $\frac{22}{7}$ or 3,142 for π as appropriate.

1 Water in a 14 mm diameter pipe flows at 2 m/s. How many litres flow along the pipe in 1 min?

2 What is the mass in kg of a cylindrical metal bar 35 cm long and 3 cm in diameter if 1 cm^3 of the metal has a mass of 8 g?

3 154 litres of oil are poured into a cylindrical barrel of diameter 35 cm. To what depth is the drum filled?

4 Fig. 25.17 shows a casting with dimensions given in mm. What is the mass of the casting in kg if it is made of iron of density 7,2 g/cm^3?

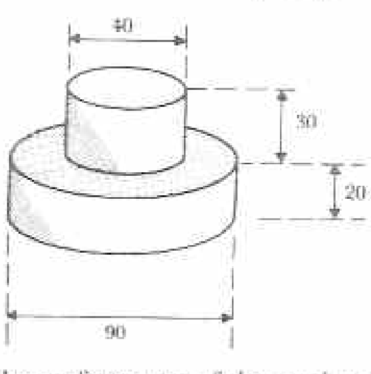


Fig. 25.17

5 Calculate the surface area of the casting in Fig. 25.17 in cm^2.

6 6,6 mm of rain fall onto a rectangular roof 8 m long and 6 m wide. The rainwater drains into a cylindrical barrel of diameter 60 cm. How far does the water level rise in the barrel?

7 An open rectangular box made of wood 1,5 cm thick measures externally 63 cm long, 48 cm wide and 50 cm deep. Calculate (a) the volume of wood in the box, (b) the mass of the box if the density of the wood is 0,8 g/cm^3.

8 The most economical shape for a cylindrical container is one in which the height and diameter are equal. Find the capacity in litres to 2 s.f. of such a tin which is 10 cm high.

9 A cylindrical tin 8 cm in diameter contains water to a depth of 4 cm. If a cylindrical wooden rod 4 cm in diameter and 6 cm long is placed in the tin, it floats exactly half submerged. What is the new depth of water?

10 Calculate (a) the slant height, (b) the curved surface area, (c) the volume, of a cone of height 8 cm and base diameter 12 cm. Leave the answers in terms of π.

11 If the cone in question 10 is made of paper which is cut and opened out into a sector of a circle, what is the angle of the sector?

12 A lampshade is in the shape of an open frustum of a cone. Its top and bottom diameters are 10 and 20 cm and its height is 12 cm. Find, in terms of π, the area of material required for the curved surface of the shade.

13 A pile of sand is in the form of a cone 20 m in diameter at the bottom and 6 m high. What is the mass of the sand in tonnes if 1 m^3 has a mass of 2,5 tonnes?

14 A lead ball of diameter 6 cm is melted down and cast in balls 5 mm in diameter. How many of the smaller balls are there?

15 A heavy 9 cm ball is placed in an empty cylindrical tin of diameter 12 cm. Enough water is poured into the tin to cover the ball. If the ball is then removed, how far does the water-level fall?

16 A pyramid 7 cm high stands on a base 12 cm square. Calculate the volume of the pyramid.

17 In Fig. 25.18, overleaf, the diameters and heights of the solids are all equal.
(a) What fraction of the volume of the cylinder is (i) the sphere, (ii) the cone?
(b) What is the ratio of the curved surface area of the sphere to that of the cylinder?

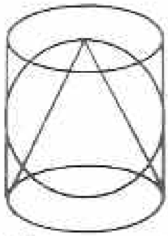


Fig. 25.18

Fig. 25.19

18 A solid consists of a cone attached to a hemisphere as in Fig. 25.19. Calculate the volume of the solid if the diameter of the hemisphere is 3 cm and the overall height of the object is 7 cm.

19 A spherical retort 15 cm in diameter is half full of acid. The acid is poured into a tall cylindrical beaker of diameter 6 cm. How deep is the acid in the beaker?

20 A hollow metal sphere is made of metal 4 mm thick and has an external diameter of 12 cm. If the density of the metal is 8,8 g/cm^3, what is the mass of the sphere?

Areas and volumes of similar shapes

If two similar shapes have corresponding lengths in the ratio $a:b$,

(i) the ratio of their areas is $a^2:b^2$,

(ii) the ratio of their volumes is $a^3:b^3$.

Example 11

A board 1 m long and 80 cm wide costs $4,80. What would be the cost of a similarly shaped board 75 cm long?

$$\text{Ratio of corresponding lengths} = \frac{75\text{ cm}}{100\text{ cm}} = \frac{3}{4}$$

Ratio of costs

$$= \text{ratio of corresponding areas} = \left(\frac{3}{4}\right)^2 = \frac{9}{16}$$

$$\text{Cost of smaller piece} = \frac{9}{16}\text{ of }\$4{,}80 = \$2{,}70$$

Example 12

Two similarly shaped cans hold 2 litres and 6,75 litres respectively. If the smaller can is 16 cm in diameter, what is the diameter of the larger?

$$\text{Ratio of volumes} = \frac{6{,}75\text{ litres}}{2\text{ litres}} = \frac{27}{8} = \left(\frac{3}{2}\right)^3$$

Ratio of corresponding lengths $= \frac{3}{2}$

Diameter of larger can $= \frac{3}{2}$ of 16 cm $= 24$ cm

Exercise 25d

1 A photograph is 20 cm long and 15 cm wide. The length of a small print of the photograph is 4 cm. Find (a) the width of the smaller print, (b) ratio of the areas of the two photographs.

2 8 kg of fertiliser are needed for a garden. How much fertiliser would be needed for a garden of double the linear dimensions?

3 1 kg of grass seed is needed for a rectangular plot 25 m long. How much seed would be needed for a similar plot which is 100 m long?

4 The diameter of a sphere is 3 times that of another sphere. How many times greater is its surface area?

5 A box of height 8 cm has a volume of 320 cm^3. What is the volume of a similar box of height 6 cm?

6 Two balls have diameters of 10 cm and 6 cm. Find (a) the ratio of their diameters in its simplest form, (b) the ratio of their surface areas, (c) the ratio of their volumes.

7 A cylinder is 8 cm high and its base diameter is 4 cm. The height of a similar cylinder is 12 cm. (a) Find the diameter of the base of the larger cylinder. (b) What is the ratio of the volume of the larger cylinder to that of the smaller one?

8 The area of a lake is 18 km^2. It is represented by an area of 2 cm^2 on the map. (a) What area in km^2 is represented by 1 cm^2 on the map? (b) What length does 1 cm on the map represent? (c) What is the ratio of lengths on the map to actual lengths?

9 A school has an area of 3 025 m^2 and it is represented on a plan by an area of 144 cm^2. Find the actual length of a wall which is shown on the plan by a line 8,4 cm long.

10 Two similar boxes have volumes of 250 cm^3 and 54 cm^3. What is the ratio of (a) their heights, (b) their surface areas?

Chapter 26

Solution of triangles

Syllabus reference 6.8.1

Solving right-angled triangles

Pythagoras' theorem

In a right-angled triangle the square on the hypotenuse is equal to the sum of the squares on the other two sides.

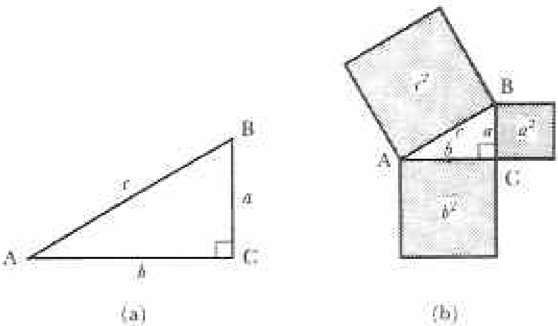


Fig. 26.1

In Fig. 26.1,

$AB^2 = BC^2 + AC^2$

or $c^2 = a^2 + b^2$

Notice also that

$a^2 = c^2 - b^2$

and $b^2 = c^2 - a^2$

Fig. 26.1(b) gives a geometrical interpretation of Pythagoras' theorem.

Example 1

In Fig. 26.2 if AC = 12 cm, BC = 5 cm, CD = 11 cm, *find* AD.

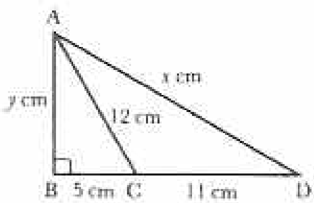


Fig. 26.2

Let AD = x cm and AB = y cm.

In △ABC,

$y^2 = 12^2 - 5^2 = 144 - 25 = 119$

In △ABD,

$x^2 = y^2 + 16^2 = 119 + 256 = 375$

$\therefore \quad x = \sqrt{375} \simeq 19{,}4$

AD = 19,4 cm to 3 s.f.

Notice that y represented an *intermediate* length. When y^2 was found to be 119 there was no need to find the value of y, since it was the value of y^2 that was needed in the subsequent working.

Exercise 26a

1 The dimensions in Fig. 26.3 are all cm. In each case x is a whole number of cm. Find the value of x in each part.

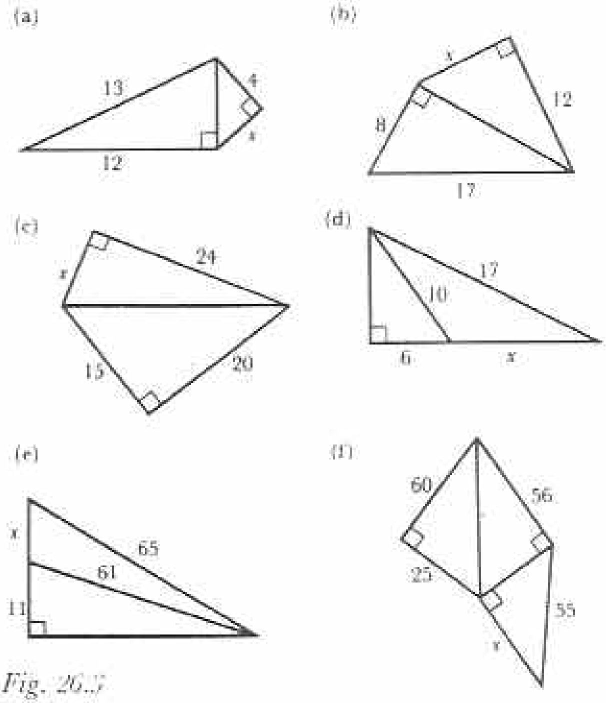


Fig. 26.3

2 A radio-mast 36 m high is supported by straight wires attached to its top and to points on the level ground 12 m from its base. Calculate the length of each wire.

3 A ladder 8,5 m long leans against a vertical wall so that its upper end is 7,5 m from the ground. How far is the foot of the ladder from the wall?

4 One side of a right-angled triangle is 24 cm long and its hypotenuse is 25 cm long. Calculate (a) the length of the third side of the triangle, (b) the area of the triangle.

5 A cone has a slant height of 29 cm and a circular base of diameter 42 cm. Calculate the vertical height of the cone.

6 In each part of Fig. 26.4, calculate the value of x correct to 3 s.f.

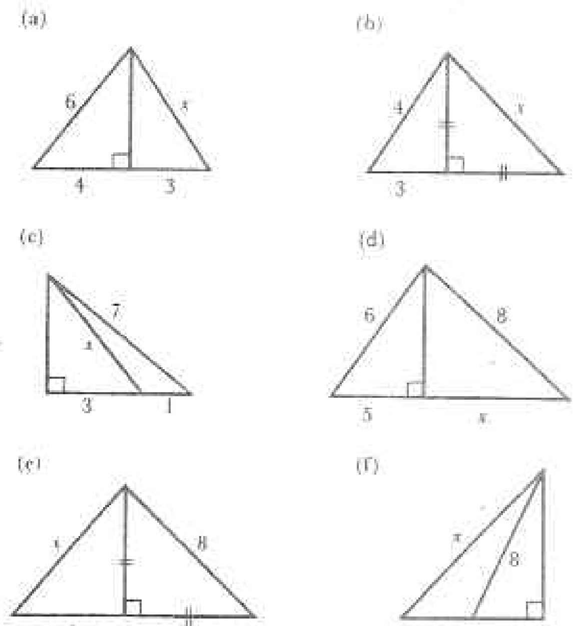


Fig. 26.4

7 A chord of a circle is 10 cm from the centre of the circle. Calculate the length of the chord given that the radius of the circle is 31 cm.

8 A rectangle is 4,3 cm long and the length of each diagonal is 5,1 cm. Calculate (a) the width, (b) the area of the rectangle, giving both answers correct to 3 s.f.

9 In △ABC, AC = 2 m, BC = 3 m and C is obtuse. The perpendicular from A to BC produced is AD. If CD = 1 m, calculate AB.

10 ABCD is a rectangle in which AB = 2,8 cm and AD = 3,3 cm. E is a point on DC produced such that △AED and rectangle ABCD are equal in area. Calculate (a) DE, (b) AE.

Sine, cosine, tangent

The trigonometrical ratios sine, cosine and tangent are defined in terms of the hypotenuse, opposite and adjacent sides of a right-angled triangle as follows:

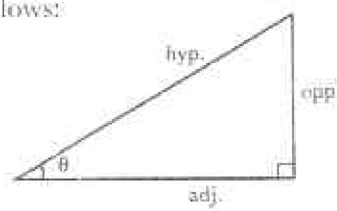


Fig. 26.5

$$\text{sine } \theta = \frac{\text{opp.}}{\text{hyp.}}$$

$$\text{cosine } \theta = \frac{\text{adj.}}{\text{hyp.}}$$

$$\text{tangent } \theta = \frac{\text{opp.}}{\text{adj.}}$$

Example 2

Calculate angle x in Fig. 26.6.

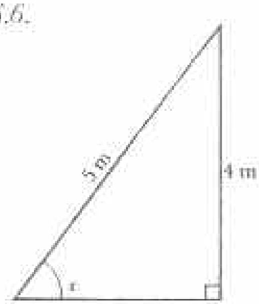


Fig. 26.6

$$\sin x = \frac{4}{5} = 0{,}800\,0$$

$$x = 53°8' \text{ or } 53{,}13° \qquad \textit{(from sine tables)}$$

Example 3

Calculate the lengths of the sides marked a and b in Fig. 26.7.

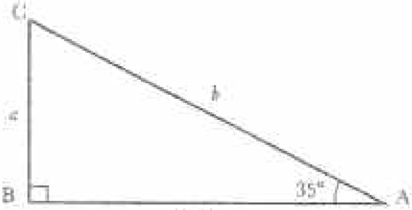


Fig. 26.7

$$\tan 35° = \frac{a}{5}$$

$$a = 5 \times \tan 35°$$
$$= 5 \times 0{,}700\,2 = 3{,}501$$

$$\cos 35° = \frac{5}{b}$$

$$b = \frac{5}{\cos 35°}$$
$$= 6{,}103$$

working:

No	Log
5	0,6990
cos 35°	$\bar{1}$,9134
6,103	0,7856

On a scientific calculator:

Key	*Display*
35 cos 1/x × 5 =	6.1038729

$b = 6{,}1$

Note the use of the reciprocal key 1/x on the calculator. This saved the need to use the memory key.

The sides are 3,5 cm and 6,1 cm long (2 s.f.).

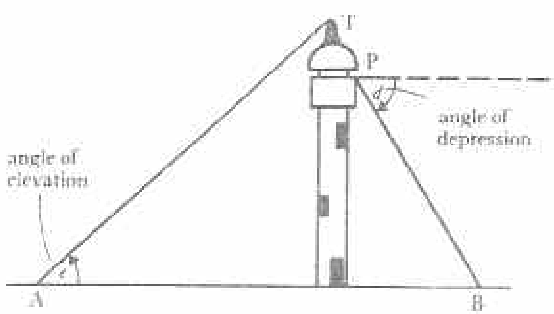


Fig. 26.8

Fig. 26.8 shows the **angle of elevation**, *e*, of the top of a tower, T, from a point A below.

Fig. 26.8 also shows the **angle of depression**, *d*, of a point B on the ground from a point, P, on the tower.

Example 4

From a window 10 m *above level ground, the angle of depression of an object on the ground is* 25,4°. *Calculate the distance of the object from the foot of the building.*

Note that the angle of depression is the angle between the horizontal and the line joining the window and the object (Fig. 26.9).

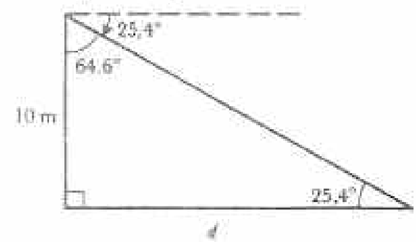


Fig. 26.9

Either, $\tan 25{,}4° = \frac{10}{d}$

$$d = \frac{10}{\tan 25{,}4°} = \frac{10}{0{,}474\,8}$$

$= 21{,}06$ (*from recip. tables*)

or, using the **complement** of 25,4°,

$$\tan 64{,}6° = \frac{d}{10}$$

$$d = 10 \times \tan 64{,}6°$$
$$= 10 \times 2{,}106$$
$$= 21{,}06$$

The object is 21,1 m from the foot of the building (3 s.f.).

Angles of 45°, 60°, 30°, 0°, 90°

From Fig. 26.10,

$$\sin 45° = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$$

$$\cos 45° = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$$

$$\tan 45° = \frac{1}{1} = 1$$

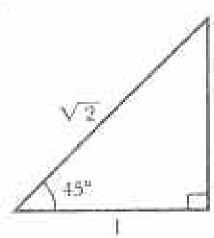


Fig. 26.10

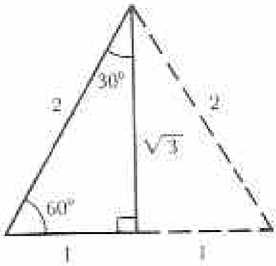


Fig. 26.11

From Fig. 26.11,

$\sin 60° = \frac{\sqrt{3}}{2}$ $\quad$ $\sin 30° = \frac{1}{2}$

$\cos 60° = \frac{1}{2}$ $\quad$ $\cos 30° = \frac{\sqrt{3}}{2}$

$\tan 60° = \frac{\sqrt{3}}{1} = \sqrt{3}$ $\quad$ $\tan 30° = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$

$\sin 0° = 0$ $\quad$ $\sin 90° = 1$

$\cos 0° = 1$ $\quad$ $\cos 90° = 0$

$\tan 0° = 0$ $\quad$ $\tan 90°$ is undefined

Example 5

Given Fig. 26.12, calculate RQ *and* RS.

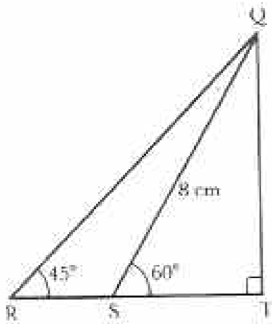


Fig. 26.12

In $\triangle$QST, the sides are in the ratio $1{:}2{:}\sqrt{3}$ (a 30°, 60°, 90° $\triangle$).

$ST = \frac{1}{2} \times SQ = \frac{1}{2} \times 8\text{ cm} = 4\text{ cm}$

and $TQ = \sqrt{3} \times ST = 4\sqrt{3}\text{ cm}$.

In $\triangle$QRT, the sides are in the ratio $1{:}1{:}\sqrt{2}$ (a 45°, 45°, 90° $\triangle$).

$$RQ = \sqrt{2} \times TQ = \sqrt{2} \times 4\sqrt{3}\text{ cm}$$
$$= 4\sqrt{6}\text{ cm}$$

and $RS = RT - ST$
$= TQ - ST$
$= 4\sqrt{3}\text{ cm} - 4\text{ cm}$
$= 4(\sqrt{3} - 1)\text{ cm}$

Exercise 26b

1 A plane takes off at an angle of 5°10′ to the ground. How high is it when it has moved 2 000 m horizontally from its take-off point?

2 The angle of elevation of the top of a vertical mast from a point on level ground 240 m from its foot is 31,5°. How high is the mast?

3 A town B is due north of A. A third town C is 10 km on a bearing 020° from A. If B is on a bearing of 290° from C, calculate (a) BC, (b) AB.

4 Calculate the values of *a*, *b* and *c* in Fig. 26.13.

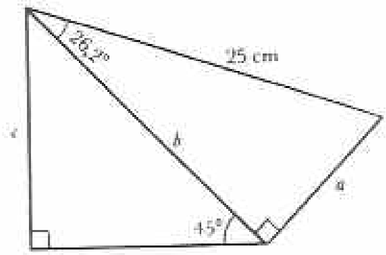
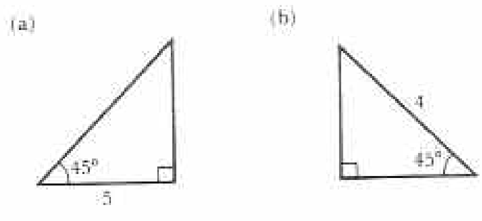


Fig. 26.13

5 A chord of a circle is 5 cm long and subtends an angle of 24,3° in the major segment. Calculate (a) the perpendicular distance of the chord from the centre, (b) the radius of the circle.

6 In each part of Fig. 26.14 the length of one side of a triangle is given in cm. Find the lengths of the other two sides, giving the answers in surd form with rational denominators where necessary.



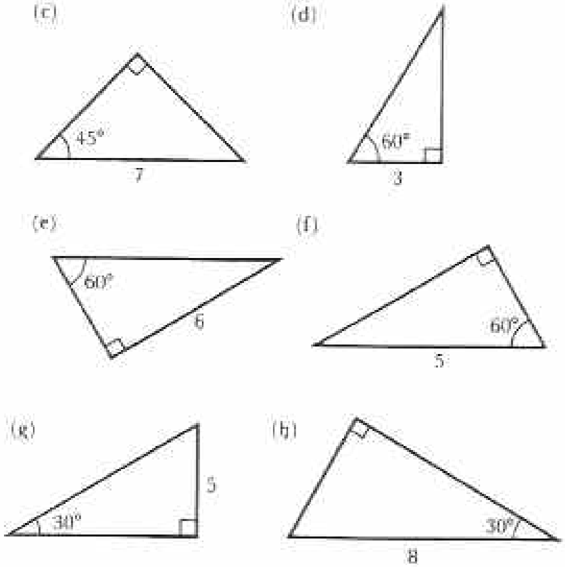


Fig. 26.14

7 In Fig. 26.15 the given lengths are in cm. In each part, find the length marked x, giving the answers in surd form with rational denominators where necessary.

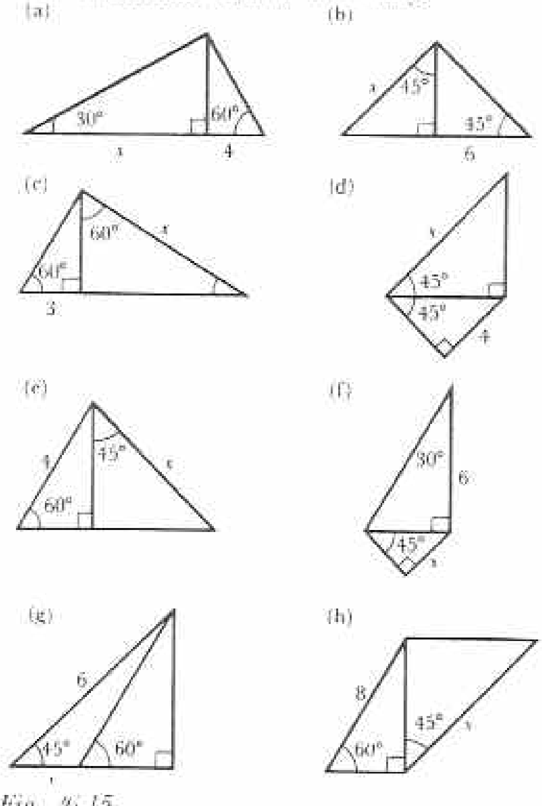


Fig. 26.15

8 The angle of elevation of the top of a flagpole from a point on level ground is 30°. From another point on the ground, 20 m nearer the flagpole, the angle of elevation is 60°. Calculate the height of the flagpole.

9 A tripod consists of three legs each 1,05 m long. The height of the top of the tripod above the ground is 90 cm. What is the inclination of each leg to the horizontal?

10 A man walks 11 km due north from A to B. He then walks 6,5 km due east from B to C. Calculate (a) the bearing of C from A, (b) AC.

11 A plank rests with one end on the ground and the other end on the back of a lorry 1,2 m above the ground. How long is the plank if it is inclined at 21° to the horizontal?

12 Farai walks 5 km from A to B on a bearing of 035°. She then walks 6 km from B to C on a bearing of 125°. Calculate (a) the distance, (b) the bearing of C from A.

13 A girl 160 cm tall stands 150 m from the foot of a building. She finds that the angle of elevation of the top of the building is 17°. Calculate the height of the building to the nearest $\frac{1}{2}$-metre.

14 The angle of depression of a point on the 225 m contour line is 10,2° from the top of a hill 915 m high. Calculate the horizontal distance between the two points. Find the difference between this distance and the actual distance between the two points.

15 A point Y is 400 m north-west of X. A tree is north-east of Y and on a bearing 015° from X. Find the distance of the tree from (a) X, (b) Y.

Lengths and angles in solids

Revise Chapter 6, pages 41 to 52.

Example 6

A pyramid with a vertex O *and edges* OA, OB, OC, OD *each* 10 cm *long stands on a square base* ABCD *of side* 8 cm. *Calculate* (*a*) *the height* OP *of the* *pyramid, (b) the angle between an edge and the base, (c) the angle between a sloping face and the base.*

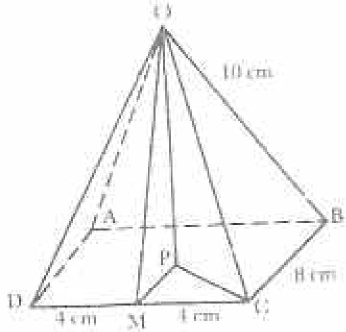


Fig. 26.16

In Fig. 26.16, M is the mid-point of the edge DC.

(a)

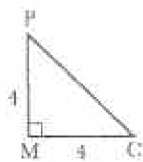


Fig. 26.17

In △PMC (Fig. 26.17),

$PC^2 = 4^2 + 4^2$ *(Pythagoras)*

$= 16 + 16 = 32$

Fig. 26.18

In △OPC (Fig. 26.18),

$OP^2 = OC^2 - PC^2$ *(Pythagoras)*

$= 100 - 32 = 68$

$OP = \sqrt{68}\text{ cm} = 8{,}246\text{ cm}$

(b) $O\hat{C}P$ is the angle between an edge OC and the base ABCD.

$$\text{In } \triangle OPC,\ \cos O\hat{C}P = \frac{\sqrt{32}}{10} = 0{,}565\,7$$

$$O\hat{C}P = 55{,}55°$$

(c) $O\hat{M}P$ is the angle between the sloping face ODC and the base. Notice that OM and MP are both perpendicular to the common edge DC.

Fig. 26.19

In △OMP (Fig. 26.19),

$$\tan O\hat{M}P = \frac{8{,}246}{4} = 2{,}061\,5$$

$$O\hat{M}P = 64{,}1°$$

Example 7

An area of sloping ground is 16 m wide, 12 m long and slopes at 25° to the horizontal as in Fig. 26.20. Find the angle of slope of the diagonal BD.

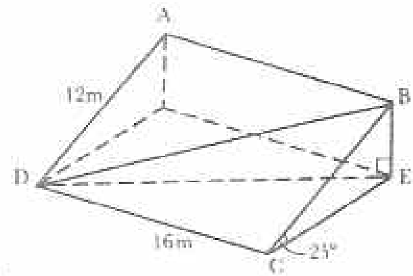


Fig. 26.20

In △BDE,

$BE = 12 \sin 25°$ *(from △BCE*

$BD^2 = 12^2 + 16^2$ *(Pythagoras in △BCD*

$= 400$

$BD = 20\text{ m}$

$$\sin B\hat{D}E = \frac{BE}{BD} = \frac{12 \sin 25°}{20}$$

$$= 0{,}6 \times 0{,}422\,6 = 0{,}253\,6$$

$$B\hat{D}E = 14°41'$$

Notice that Examples 6 and 7 were answere by solving the appropriate right-angled triangl Example 6 shows the value of sketching th various triangles used.

Exercise 26c

1 A cube has edges of 4 cm length. Calculate (a) the length of a diagonal of the cube, (b) the angle between the diagonal of the cube and its base.

2 The vertex of a pyramid on a square base of side 12 cm is 7 cm above the base. Calculate (a) the length of each sloping edge, (b) the angle between each sloping edge and the base, (c) the angle between each sloping face and the base.

3 A triangular prism like that of Fig. 26.20 has BE = 4 cm., AB = 20 cm and AD = 16 cm. Make a suitable sketch and calculate the slope of (a) BC, (b) BD.

4 A room in the shape of a cuboid has a floor which measures 5 m by 4 m. The longest diagonal of the room makes an angle of 35° with the floor. Find the height of the room.

5 A pyramid OABCD stands on a square base ABCD and OA = OB = OC = OD = 16 cm. $\hat{BOD} = 90°$. Calculate the angle between a sloping face and the base.

6 A prism like that of Fig. 26.20 has AB = 24 cm, AD = 7 cm and BE = 3 cm. X is a point on AB such that $\hat{ADX} = 45°$. Calculate the slope of BD and of DX.

7 A pole is resting in the corner of a room. The top of the pole is 2,5 m above the floor and the bottom is 1 m from each wall. Find the length of the pole and the angle which it makes with the floor.

8 A mast 45 m high is supported by 4 equal straight wires attached to the top of the mast and to the corners of a square of side 60 m on the level ground. Calculate the inclination of each wire to the horizontal.

9 Fig. 26.21 represents an open door. Assume that ABCD and ABEF are rectangles each measuring 2 m by 1,5 m.

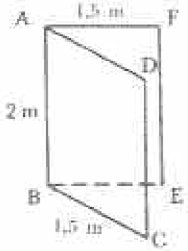


Fig. 26.21

If $\hat{DAF} = 60°$ calculate the slopes of (a) DE, (b) △ACE.

10 Fig. 26.22 shows an open rectangular box 7 cm long, 6 cm wide and 6 cm high. A rod 12 cm long rests with its lower end in one bottom corner and is supported by the opposite corner.

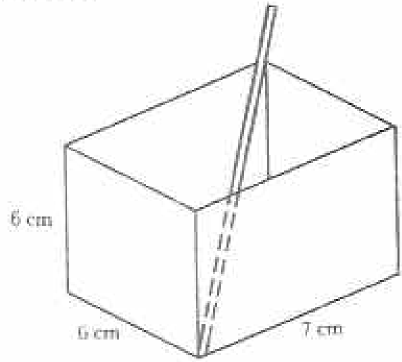


Fig. 26.22

Calculate (a) the inclination of the rod to the horizontal and (b) the height of its top end above the level of the base of the box.

Solving non-right-angled triangles

The sine rule

Revise Chapter 4, pages 25 to 31.

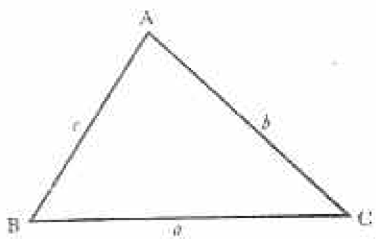


Fig. 26.23

The sine rule states that

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

or $$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

for any triangle ABC in which A, B, C are the angles of the triangle and a, b, c are the lengths of the sides opposite these angles.

In obtuse-angled triangles, $\sin\theta = \sin(180° - \theta)$.

Example 8

Solve completely the $\triangle$ABC *in which* $a = 12{,}4$ cm, $c = 14{,}7$ cm *and* C = 72°4′.

To 'solve completely' means to find all the unknown lengths and angles.

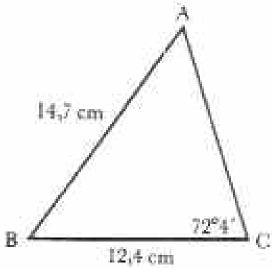


Fig. 26.24

In Fig. 26.24,

$$\frac{\sin A}{a} = \frac{\sin C}{c}$$

$$\frac{\sin A}{12{,}4} = \frac{\sin 72°4'}{14{,}7}$$

$$\Leftrightarrow \sin A = \frac{12{,}4 \sin 72°4'}{14{,}7}$$

$$A = 53°23' \text{ or } 126°37'$$

working:

No	Log
12,4 sin 72°4′	1,0934 $\bar{1}$,9784
14,7	1,0718 1,1673
sin 53°23′	$\bar{1}$,9045

But $a < c$, therefore A < C.

$$A = 53°23'$$

and B = 54°33′ (∠s of △)

$$\frac{b}{\sin B} = \frac{c}{\sin C}$$

$$\frac{b}{\sin 54°33'} = \frac{14{,}7}{\sin 72°4'}$$

$$b = \frac{14{,}7 \sin 54°33'}{\sin 72°4'}$$

$$\simeq 12{,}6 \text{ cm}$$

working:

No	Log
14,7 sin 54°33′	1,1673 $\bar{1}$,9110
sin 72°4′	1,0783 $\bar{1}$,9784
12,59	1,0999

This expression may be evaluated on a scientific calculator as follows:

Key	*Display*
4 ÷ 60 + 72 = sin M+	0.95141
33 ÷ 60 + 54 = sin	0.81462
× 14.7 =	11.9749
÷ MR =	12.5864

$b = 12{,}6$ cm to 3 s.f.

Hence A = 53°23′, B = 54°33′ and $b = 12{,}6$ cm

The cosine rule

Revise Chapter 11, pages 88 to 94.

The cosine rule is used for solving triangles in which two sides and the included angle are given.

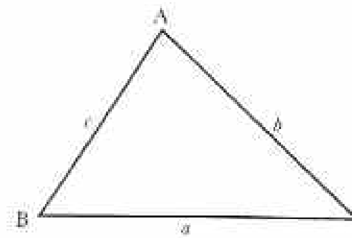


Fig. 26.25

In $\triangle$ABC,

$$\boldsymbol{a^2 = b^2 + c^2 - 2bc\cos A}$$
$$\boldsymbol{b^2 = a^2 + c^2 - 2ac\cos B}$$
$$\boldsymbol{c^2 = a^2 + b^2 - 2ab\cos C}$$

These formulae are true for both acute and obtuse angles. In obtuse-angled triangles $\cos\theta = -\cos(180° - \theta)$.

Example 9

Find x in Fig. 26.26.

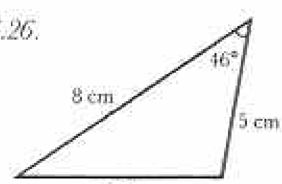


Fig. 26.26

$$x^2 = 8^2 + 5^2 - 2 \times 8 \times 5 \times \cos 46°$$
$$= 64 + 25 - 80 \times 0{,}6947$$
$$= 89 - 55{,}576 = 33{,}424$$
$$x = \sqrt{33{,}42} = 5{,}781 \simeq 5{,}78 \text{ cm}$$

Example 10
Find y in Fig. 26.27.

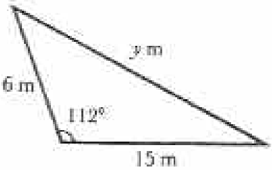


Fig. 26.27

$$\begin{aligned} y^2 &= 6^2 + 15^2 - 2 \times 6 \times 15 \times \cos 112° \\ &= 36 + 225 - 180 \times (-\cos 68°) \\ &= 261 + 180 \times 0{,}374\,6 \\ &= 261 + 67{,}428 = 328{,}428 \end{aligned}$$

$$y = \sqrt{328{,}4} = 18{,}12 \simeq 18{,}1\,\text{m}$$

When all three sides of a triangle are known, the angles can be calculated by rearranging the basic formulae as follows:

$$\mathbf{\cos A = \frac{\boldsymbol{b^2 + c^2 - a^2}}{\boldsymbol{2bc}}}$$

$$\mathbf{\cos B = \frac{\boldsymbol{a^2 + c^2 - b^2}}{\boldsymbol{2ac}}}$$

$$\mathbf{\cos C = \frac{\boldsymbol{a^2 + b^2 - c^2}}{\boldsymbol{2ab}}}$$

Example 11
Calculate the angles of a triangle which has sides 5 cm, 8 cm *and* 11 cm.

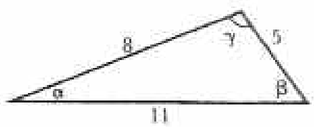


Fig. 26.28

Lettering the angles of the △ as in Fig. 26.28,

$$\cos\alpha = \frac{8^2 + 11^2 - 5^2}{2 \times 8 \times 11}$$

$$= \frac{160}{16 \times 11} = \frac{10}{11} = 0{,}909\ 1$$

$$\alpha = 24{,}6°$$

$$\cos\beta = \frac{5^2 + 11^2 - 8^2}{2 \times 5 \times 11}$$

$$= \frac{82}{110} = 0{,}745\,5$$

$$\beta = 41{,}8°$$

$$\cos\gamma = \frac{8^2 + 5^2 - 11^2}{2 \times 8 \times 5} = \frac{-32}{80} = -0{,}4$$

$$\therefore\ \gamma = 180° - 66{,}4° = 113{,}6°$$

Check: $\alpha + \beta + \gamma = 180°$

Examples which show how to use a scientific calculator with the sine and cosine rules are given in Chapter 20, pages 166 to 172.

Example 12
A village P *is* 10 km *from a point* X *on a bearing* 025° *from* X. *Another village,* Q, *is* 6 km *from* X *on a bearing of* 162°. *Calculate the distance and bearing of* P *from* Q.

Enter the details on a sketch such as Fig. 26.29.

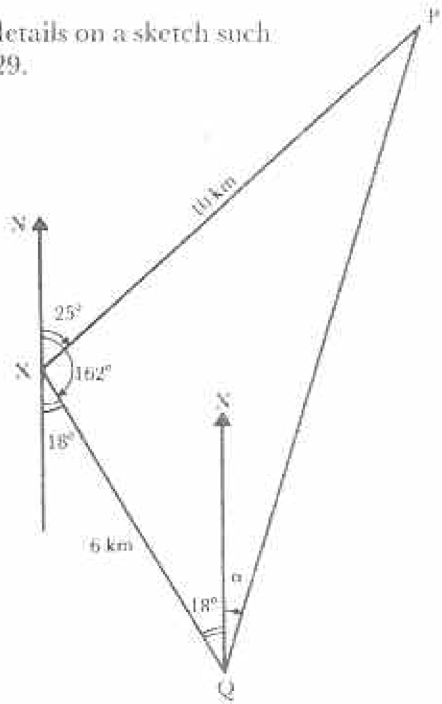


Fig. 26.29

$$P\hat{X}Q = 162° - 25° = 137°$$

$$\begin{aligned} PQ^2 &= 10^2 + 6^2 - 2 \times 10 \times 6 \times \cos 137° \\ &= 100 + 36 - 120 \times (-\cos 43°) \\ &= 136 + 120 \times 0{,}731\,4 \\ &= 136 + 87{,}768 = 223{,}768 \end{aligned}$$

$$\begin{aligned} PQ &= \sqrt{223{,}8}\,\text{km} = 14{,}96\,\text{km} \\ &= 15{,}0\,\text{km to 3 s.f.} \end{aligned}$$

$$\frac{\sin Q}{10} = \frac{\sin 137°}{14{,}96}$$

$\sin Q = \dfrac{10 \sin 43^\circ}{14{,}96}$ *

$\hat{Q} = 27°8'$

From Fig. 26.29,

α = bearing of P from Q

$= 27°8' - 18°$

$= 009°8'$

working:

No	Log
10	1,0000
sin 43°	$\bar{1}$,8338
	0,8338
14,96	1,1749
sin 27°8'	$\bar{1}$,6589

* This calculation may be done on a scientific calculator as follows.

Key	*Display*
43 sin ÷ 1·496 = SHIFT $\sin^{-1}$	27.12165

Exercise 26d

1 In $\triangle$ABC, $\hat{B} = 38°$, $\hat{C} = 48°$, $c = 18{,}8$ cm. Find b.

2 In $\triangle$ABC, $\hat{A} = 98°$, $\hat{C} = 36°$, $a = 34{,}4$ cm. Find c.

3 In $\triangle$ABC, $\hat{B} = 29°$, $b = 8{,}6$ cm, $c = 3{,}1$ cm. Find $\hat{C}$.

4 In $\triangle$ABC, $\hat{A} = 96°13'$, $a = 39{,}4$ cm, $b = 11{,}2$ cm. Find $\hat{B}$.

5 In $\triangle$ABC, $\hat{A} = 60°$, $b = 5$ cm, $c = 3$ cm. Find a.

6 In $\triangle$ABC, $\hat{B} = 22°$, $a = 4$ cm, $c = 5$ cm. Find b.

7 In $\triangle$ABC, $\hat{C} = 123°$, $a = 3$ m, $b = 2$ m. Find c.

8 In $\triangle$ABC, $\hat{B} = 143°$, $a = 25$ cm, $c = 40$ cm. Find b.

9 In $\triangle$ABC, $a = 5$ m, $b = 6$ m, $c = 7$ m. Find $\hat{B}$.

10 In $\triangle$ABC, $a:b:c = 15:21:24$. Find $\hat{B}$.

11 In $\triangle$ABC, $a = 8$ km, $b = 14$ km, $c = 18$ km. Find $\hat{C}$.

12 In $\triangle$ABC, $a = 9$ m, $b = 6$ m, $c = 4{,}8$ m. Find $\hat{A}$.

13 A boy starts from a point X and walks 220 m on a bearing 063°. He then walks to a point Y on a bearing 156°. If Y is due east of X, calculate XY.

14 Three villages X, Y and Z are connected by straight level roads. XY = 5 km, YZ = 4 km and $X\hat{Y}Z = 160°$. What distan[ce] is saved by walking from X to Z direct[ly] instead of through Y?

15 In Fig. 26.30 the lengths are in cm. Calcula[te] d and α.

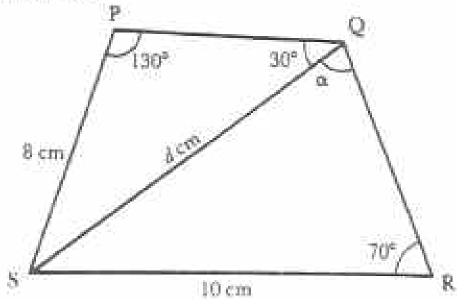


Fig. 26.30

16 In Fig. 26.31 the lengths are in m. Find [x] and θ.

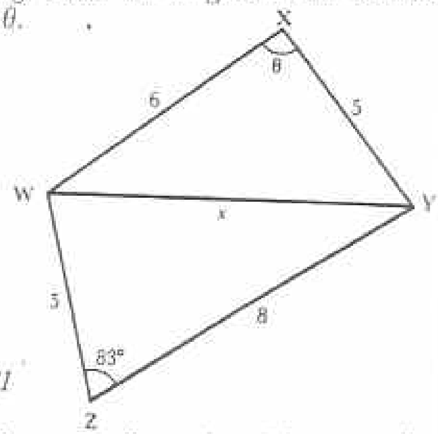


Fig. 26.31

17 In the quadrilateral ABCD, AD || BC, AB = 15 cm, DC = 14 cm, $D\hat{A}B = 60°$ an[d] $A\hat{D}B = 40°$. Calculate (a) BD, to 2 s.f. (b) $B\hat{C}D$, correct to the nearest degree.

18 ABCD is a cyclic quadrilateral in whic[h] AB = 5 cm, BC = 4 cm, CD = 7 cm, DA = 6 cm. Calculate $A\hat{B}C$.

19 An aeroplane leaves an airport and flies du[e] north for $1\frac{1}{2}$ hours at 500 km/h. It then flie[s] 400 km on a bearing 053°. Calculate its fina[l] distance and bearing from the airport.

20 Two explorers leave camp at the same tim[e]. One walks at 5 km/h on a bearing 039°. Th[e] other walks at 7,5 km/h on a bearing 265°. After two hours how far apart are they an[d] what is the bearing of the second from th[e] first?

Chapter 27

Matrices, transformations, vectors

Matrices

A **matrix** is a rectangular pattern of elements, usually numbers. A matrix can be used as a store or as an operator.

Addition and subtraction

Matrices can be added and subtracted only if they have the same number of rows and columns. Only corresponding elements in each matrix may be combined.

Example 1

Simplify

$$\begin{pmatrix}2 & 6\\1 & -4\end{pmatrix}-\begin{pmatrix}-3 & 2\\0 & -5\end{pmatrix}+\begin{pmatrix}-1 & 8\\-5 & 7\end{pmatrix}.$$

$$\begin{pmatrix}2 & 6\\1 & -4\end{pmatrix}-\begin{pmatrix}-3 & 2\\0 & -5\end{pmatrix}+\begin{pmatrix}-1 & 8\\-5 & 7\end{pmatrix}$$

$$=\begin{pmatrix}2-(-3)+(-1) & 6-2+8\\1-0+(-5) & -4-(-5)+7\end{pmatrix}$$

$$=\begin{pmatrix}4 & 12\\-4 & 8\end{pmatrix}=4\begin{pmatrix}1 & 3\\-1 & 2\end{pmatrix}$$

In Example 1, notice that 4 is a factor of each element in the resultant matrix. The 4 can be taken out as a **scalar** factor.

Multiplication

Matrices can be multiplied only if the number of columns in the first or **pre-multiplying** matrix is the same as the number of rows in the second or **post-multiplying** matrix. A $p \times q$ matrix will multiply a $q \times r$ matrix to give a $p \times r$ product. In general $\mathbf{AB} \neq \mathbf{BA}$, where $\mathbf{A}$ and $\mathbf{B}$ are matrices.

To multiply matrices, find the sum of the products of corresponding elements in each row of the pre-multiplying matrix and each column of the post-multiplying matrix. The following examples demonstrate the method of matrix multiplication.

Example 2

product	general case	example
$(1 \times 2) \times (2 \times 1)$	$(a \;\; b)\begin{pmatrix}p\\q\end{pmatrix}$ $= (ap+bq)$	$(-2 \;\; 3)\begin{pmatrix}5\\1\end{pmatrix}$ $=(-2 \times 5 + 3 \times 1)$ $=(-7)$
$(2 \times 1) \times (1 \times 2)$	$\begin{pmatrix}a\\b\end{pmatrix}(p \;\; q)$ $=\begin{pmatrix}ap & aq\\bp & bq\end{pmatrix}$	$\begin{pmatrix}-4\\7\end{pmatrix}(-1 \;\; 2)$ $=\begin{pmatrix}4 & -8\\-7 & 14\end{pmatrix}$
$(2 \times 2) \times (2 \times 2)$	$\begin{pmatrix}a & b\\c & d\end{pmatrix}\begin{pmatrix}p & q\\r & s\end{pmatrix}$ $=\begin{pmatrix}ap+br & aq+bs\\cp+dr & cq+ds\end{pmatrix}$	$\begin{pmatrix}5 & 1\\2 & 7\end{pmatrix}\begin{pmatrix}-2 & 0\\3 & 1\end{pmatrix}$ $=\begin{pmatrix}-7 & 1\\17 & 7\end{pmatrix}$

Identity and inverse

$\mathbf{I}$ is the 2×2 **identity matrix** where

$$\mathbf{I}=\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}.$$

Any 2×2 matrix is left unchanged when pre- or post-multiplied by $\mathbf{I}$.

In most cases a matrix $\mathbf{M}$ has an **inverse**, $\mathbf{M}^{-1}$, such that $\mathbf{M} \times \mathbf{M}^{-1} = \mathbf{M}^{-1} \times \mathbf{M} = \mathbf{I}$.

$$\text{If } \mathbf{M}=\begin{pmatrix}a & b\\c & d\end{pmatrix}$$

$$\text{then } \mathbf{M}^{-1}=\frac{1}{ad-bc}\begin{pmatrix}d & -b\\-c & a\end{pmatrix}.$$

The cross-product-difference, $ad - bc$, is the **determinant** of the given matrix.

Example 3
Find the inverse of the following where possible.

(*a*) $\begin{pmatrix} 5 & -2 \\ 9 & -3 \end{pmatrix}$ (*b*) $\begin{pmatrix} 10 & 3 \\ 7 & 2 \end{pmatrix}$ (*c*) $\begin{pmatrix} 6 & 9 \\ 4 & 6 \end{pmatrix}$

(a) The determinant of the given matrix

$$= 5 \times (-3) - 9 \times (-2)$$
$$= -15 + 18 = 3$$

Its inverse is $\frac{1}{3}\begin{pmatrix} -3 & 2 \\ -9 & 5 \end{pmatrix}$.

(b) The determinant of the given matrix

$$= 10 \times 2 - 7 \times 3$$
$$= 20 - 21 = -1$$

Its inverse is $-1\begin{pmatrix} 2 & -3 \\ -7 & 10 \end{pmatrix}$

or $\begin{pmatrix} -2 & 3 \\ 7 & -10 \end{pmatrix}$.

(c) The determinant of the given matrix

$$= 6 \times 6 - 9 \times 4$$
$$= 36 - 36 = 0$$

The inverse of the given matrix would contain the undefined fraction $\frac{1}{0}$. This is an example of a **singular** matrix; such a matrix has *no* inverse.

Further examples and information on matrices can be found in Chapter 13.

Exercise 27a

1 Find the determinant of the matrix $\begin{pmatrix} -5 & 2 \\ 4 & -3 \end{pmatrix}$. Hence write down the inverse of the matrix.

2 Find the value of x for which the matrix $\begin{pmatrix} 5x-3 & 7 \\ 2x+4 & 2 \end{pmatrix}$ has no inverse.

3 $\mathbf{A} = \begin{pmatrix} 2 & 0 \\ 3 & 1 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 1 & 2 \\ -1 & 3 \end{pmatrix}$.

(a) Find $\mathbf{A} + 2\mathbf{B}$.

(b) Given that $\mathbf{A}\begin{pmatrix} x \\ 2 \end{pmatrix} = \begin{pmatrix} 8 \\ 2y \end{pmatrix}$, find the value of x and y. [Camb]

4 Given that $\mathbf{A} = \begin{pmatrix} 3 & -1 \\ 2 & 0 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 0 & \frac{1}{2} \\ -1 & m \end{pmatrix}$, $\mathbf{C} = \begin{pmatrix} -9 & 4 \\ 4 & n \end{pmatrix}$, find (a) $\mathbf{A}^2$, (b) m if $\mathbf{B} = \mathbf{A}^{-1}$, (c) n if $\mathbf{A}$ and $\mathbf{C}$ have equal determinants.

5 Given that the value of the determinant of the matrix $\begin{pmatrix} x & -3 \\ -1 & 2 \end{pmatrix}$ is 5, find the value of x. Hence write down the inverse of the matrix. [Camb]

6 Find a, b, c such that

$$\begin{pmatrix} a & b \\ 0 & 2 \end{pmatrix}\begin{pmatrix} 0 & 3 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 9 \\ 5 & 0 \end{pmatrix} - \begin{pmatrix} 4 & -6 \\ 3 & 2c \end{pmatrix}.$$

7 (a) Given that $\mathbf{P} = \begin{pmatrix} 2 & 3 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & -4 \\ 3 & 0 \end{pmatrix}$, find the matrix $\mathbf{P}$.

(b) Find the inverse of the matrix $\begin{pmatrix} 3 & 5 \\ 2 & 4 \end{pmatrix}$.

(c) Given that $\mathbf{R}$ is a 2×2 matrix such that $\mathbf{R} + \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}\mathbf{R} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$, find $\mathbf{R}$.

8 Find two values of k such that $\begin{pmatrix} 2k+2 & k \\ 4k-3 & k+3 \end{pmatrix}$ is a singular matrix.

9 (a) Write down the inverse of the matrix $\begin{pmatrix} -1 & 4 \\ 1 & 3 \end{pmatrix}$.

(b) Hence or otherwise find x and y if $\begin{pmatrix} -1 & 4 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 13 \\ 1 \end{pmatrix}$.

10 Express the simultaneous equations

$$2x + 5y = 7$$
$$x - y = 7$$

as a single matrix equation

$$\mathbf{M}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 7 \\ 7 \end{pmatrix},$$

where $\mathbf{M}$ is a 2×2 matrix. Pre-multiply both sides of the matrix equation by $\mathbf{M}^{-1}$ to find the values of x and y.

Geometrical transformations

A figure is **transformed** when its position and/or shape changes. The **image** of a shape is the figure obtained after a transformation.

If the image has the same dimensions as the original shape, the transformation is called a **congruency** or **isometry**. Fig. 27.1 shows a shaded triangle ABC and its images after a typical **translation**, T, **rotation**, R, **reflection**, M.

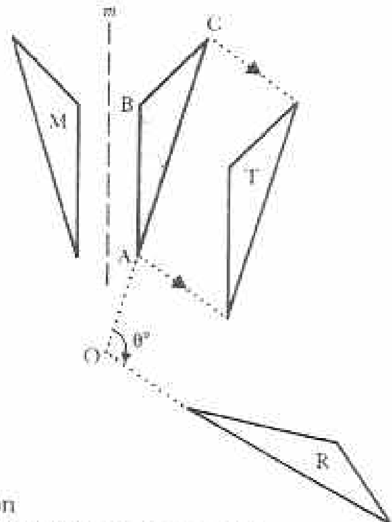


Fig. 27.1

T: translation

R: rotation of $\theta°$ clockwise about O

M: reflection in line m

Isometric (congruent) shapes have corresponding lengths and angles equal.

Fig. 27.2 shows triangle ABC and its images after **enlargements** E and E′.

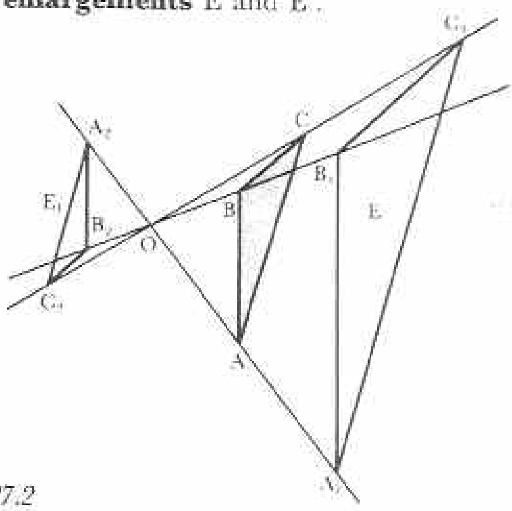


Fig. 27.2

E: enlargement of factor k where $k = \frac{OA_1}{OA} = \frac{OB_1}{OB}$

E′: enlargement of factor K where $K = \frac{OA_2}{OA} = \frac{OB_2}{OB}$

Notice the following:

(a) Enlarged shapes are geometrically similar and have corresponding angles equal.

(b) If the enlargement factor is k then the original area will be enlarged by factor k^2.

(c) In Fig. 27.2, K is negative and fractional.

Shear and **stress** are shown in Fig. 27.3.

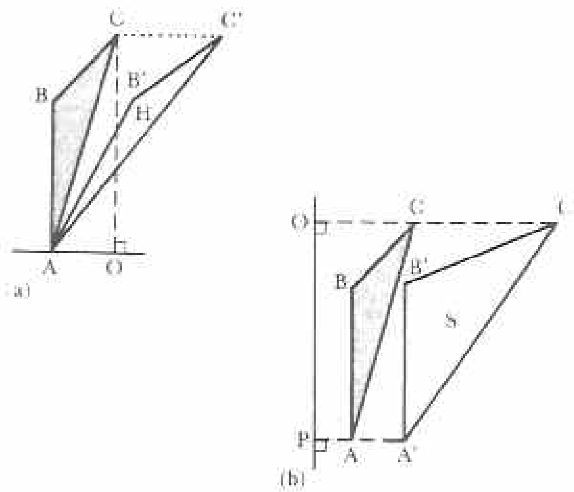


Fig. 27.3

H: shear of factor k where $k = \frac{CC'}{OC}$ and the line through AO is invariant.

S: stretch of factor K where $K = \frac{OC'}{OC} = \frac{OA'}{OA}$ and the line through OP is invariant.

Notice the following:

(a) Any shape has the same area as its image after shearing.

(b) If a shape is stretched with factor K, then its image has an area K times that of the shape.

Matrices can be used as operators which transform shapes drawn on the cartesian plane. Table 27.1, on page 244, gives a summary of the most common transformations of the cartesian plane and their related matrices.

Example 4
△PQR has coordinates P(−3; 1), Q(−2; 4), R(0; 4).
(a) If △PQR is given a shear of factor 1 with the x-axis invariant, find by drawing or calculation the coordinates of its image, △P′Q′R′.
(b) If △P′Q′R′ is given a transformation represented by the matrix $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ *find the coordinates of the new image △P″Q″R″.*

(a) *Either* by drawing as in Fig. 27.4:

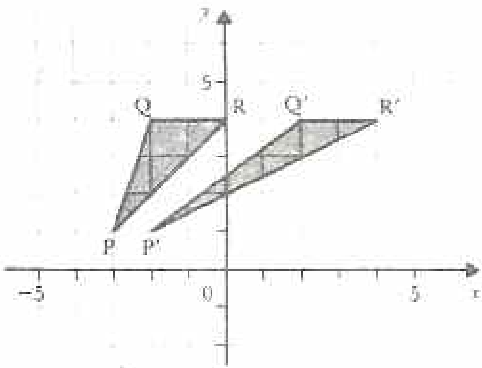


Fig. 27.4

△P′Q′R′ has coordinates P′(−2; 1), Q′(2; 4), R′(4; 4).
Or by calculation:

$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ is the matrix which represents the shear.

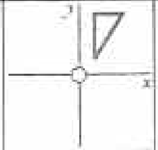
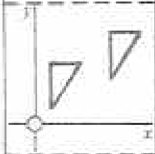
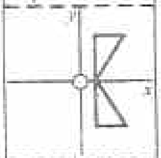
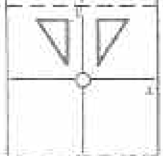
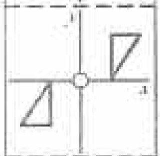
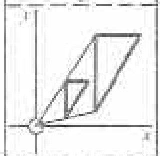
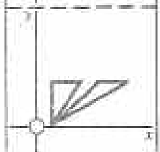
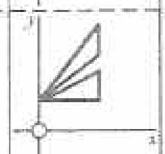
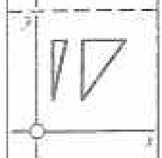
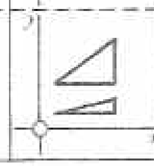
$$\begin{matrix} & \text{P} & \text{Q} & \text{R} & & \text{P}' & \text{Q}' & \text{R}' \end{matrix}$$
$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} -3 & -2 & 0 \\ 1 & 4 & 4 \end{pmatrix} = \begin{pmatrix} -2 & 2 & 4 \\ 1 & 4 & 4 \end{pmatrix}$$

△P′Q′R′ has coordinates P′(−2; 1), Q′(2; 4), R′(4; 4), as above
(b)
$$\begin{matrix} & \text{P}' & \text{Q}' & \text{R}' & & \text{P}'' & \text{Q}'' & \text{R}'' \end{matrix}$$
$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} -2 & 2 & 4 \\ 1 & 4 & 4 \end{pmatrix} = \begin{pmatrix} -2 & 2 & 4 \\ -1 & -4 & -4 \end{pmatrix}$$

△P″Q″R″ has coordinates P″(−2; −1), Q″(2; −4), R″(4; −4).

See Chapter 14, pages 113 to 123 for further coverage of matrices and transformations.

Table 27.1

Transformation	Sketch	Matrix
Identity		$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$
Translation		$\begin{pmatrix} a \\ b \end{pmatrix}$
Reflection in x-axis		$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$
Reflection in y-axis		$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$
Rotation of 180° about (0, 0)		$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$
Enlargement centre (0, 0)		$\begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}$
Shear x-axis invariant		$\begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}$
Shear y-axis invariant		$\begin{pmatrix} 1 & 0 \\ k & 1 \end{pmatrix}$
Stretch y-axis invariant		$\begin{pmatrix} k & 0 \\ 0 & 1 \end{pmatrix}$
Stretch x-axis invariant		$\begin{pmatrix} 1 & 0 \\ 0 & k \end{pmatrix}$

Exercise 27b

1 Describe completely the single transformation which maps $\triangle PQR$ onto $\triangle KLM$ in Fig. 27.5.

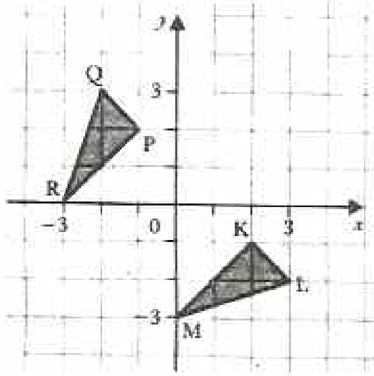


Fig. 27.5

2 In Fig. 27.6 $\triangle OPQ$ is enlarged to $\triangle OP'Q'$ by scale factor $-1{,}4$ with O as centre.

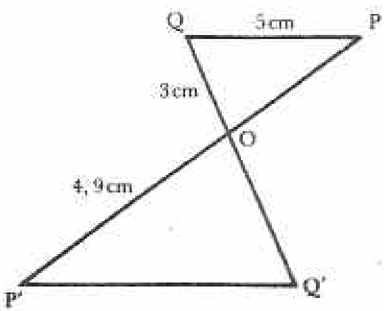


Fig. 27.6

If $PQ = 5$ cm, $OQ = 3$ cm and $OP' = 4{,}9$ cm calculate (a) $P'Q'$, (b) OQ' and (c) OP.

3 $\triangle ABC$ is as given in Fig. 27.7.

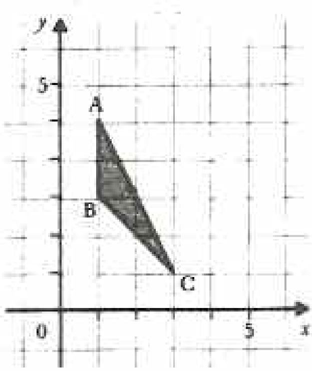


Fig. 27.7

S is a stretch of factor 3 in the x-direction with the y-axis invariant. If $\triangle PQR$ is the image of $\triangle ABC$ under S, find

(a) the coördinates of $\triangle PQR$,

(b) the numerical value of $\dfrac{\text{area of } \triangle ABC}{\text{area of } \triangle PQR}$.

4 The matrix $\begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix}$ represents a transformation Q.

(a) Find the image of $(5; -2)$ under Q.

(b) Find the image of $(5; 2)$ under Q.

(c) Describe completely the transformation Q.

5 T is the translation $\begin{pmatrix} 3 \\ 2 \end{pmatrix}$ and R is an anticlockwise rotation of 90° about the origin. A is the point $(2; -5)$, B is $(-1; 4)$ and C is $(-4; 4)$. Find the coordinates of (a) T(A), (b) R(B), (c) the point D if RT(D) = C. [Camb]

6 A shear H is represented by the matrix $\begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix}$. Line A'B' is the image of line AB under H.

(a) If the coordinates of B and A' are $(1; 4)$ and $(4; -3)$ respectively, find the coordinates of B' and A.

(b) Write down the coordinates of any two points which remain invariant under H.

7 F is a transformation of matrix $\begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}$.

The images of $A(2; 2)$, $B(-2; 4)$, $C(0; 8)$ under F are $A'(5; 5)$, $B'(-5; 10)$, $C'(0; 12)$.

(a) Using a suitable scale, draw triangles ABC and A'B'C' on the same graph.

(b) Describe fully the transformation F and write down the value of k.

(c) Find the coordinates of the vertices of the image of ABC after rotation of 270° clockwise about the point $(3; 2)$.

8 $P'(0; 0)$, $Q'(3; 13)$, $R'(-2; -11)$ are the images of $P(0; 0)$, $Q(3; 1)$, $R(-2; -3)$ under a transformation represented by a matrix of the form $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

(a) Find the transformation matrix.
(b) Find the matrix which will transform $\triangle P'Q'R'$ back to $\triangle PQR$.

9 P is the point $(-1; 7)$ on the shape given in Fig. 27.8.

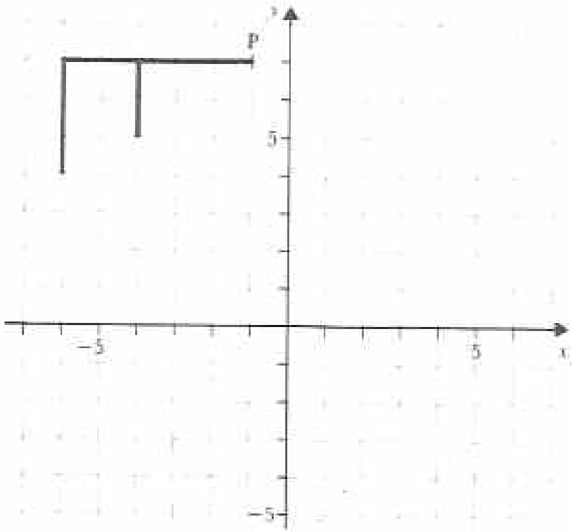


Fig. 27.8

M is a reflection in the line $2 - y = 0$.
N is a reflection in the line $x + y = 0$.
(a) Find the image of P under (i) MN(P), (ii) NM(P).
(b) Describe fully the single transformation K such that K[MN(P)] = P.

10 Answer the whole of this question on a sheet of graph paper.
The triangle ABC has vertices A(2; 0), B(4; 4), C(0; 1). The triangle PQR has vertices P(8; −2), Q(4; 0), R(7; −4). The triangle LMN has vertices L(−2; −7), M(−6; −9), N(−3; −5). Draw these triangles on graph paper, using a scale of 1 cm to 1 unit on each axis, and label the vertices. $\triangle ABC$ can be mapped onto $\triangle PQR$ by an anticlockwise rotation about the origin followed by a translation.
(a) State the angle of rotation.
(b) Find the matrix which represents this rotation.
(c) Find the column vector of the translation.
(d) Given that $\triangle ABC$ can be mapped onto $\triangle PQR$ by a single rotation, find the coordinates of the centre of this rotation.
(e) Given that $\triangle ABC$ can be mapped onto $\triangle LMN$ by a translation of $\begin{pmatrix} 0 \\ -3 \end{pmatrix}$ followed by a reflection in the mirror line m, draw the line m on your graph and label it clearly.
(f) Find the equation of m. [Camb]

Vectors

Chapter 18, pages 149 to 157, contains complete revision notes on vectors. Read the content and worked examples of Chapter 18 before attempting the following revision exercise.

Exercise 27c

1 Fig. 27.9 shows two vectors **a** and **b**.

Fig. 27.9

Copy Fig. 27.9 onto squared paper. Draw and clearly label the lines OP, OQ, OR, OS such that (a) $\mathbf{OP} = 4\mathbf{a}$, (b) $\mathbf{OQ} = -2\mathbf{b}$, (c) $\mathbf{OR} = 3\mathbf{a} + 2\mathbf{b}$, (d) $\mathbf{OS} = 5\mathbf{a} - 3\mathbf{b}$.

2 PQRS is a rhombus. **PQ** is shown in Fig. 27.10 and **QR** is the column vector $\begin{pmatrix} 4 \\ 1 \end{pmatrix}$.

Fig. 27.10

(a) On a copy of Fig. 27.10, mark and label the points R and S.
(b) Express the following as column vectors.
(i) **SR** (ii) **PR** (iii) **QS**

3 OABC is a trapezium such that O is the origin, $\mathbf{OA} = \begin{pmatrix} 2 \\ 5 \end{pmatrix}$, $\mathbf{OC} = 2\mathbf{AB} = \begin{pmatrix} 6 \\ 2 \end{pmatrix}$.
(a) On squared paper, mark and label the points O, A, B and C.
(b) Express the following as column vectors.
(i) **OB** (ii) **BC** (iii) **CA**

4 $\mathbf{OA} = \begin{pmatrix} -4 \\ 9 \end{pmatrix}$, $\mathbf{OB} = \begin{pmatrix} 6 \\ -3 \end{pmatrix}$ and M is the mid-point of AB. Express the following as column vectors.
(a) **AB** (b) **AM** (c) **OM**

5 In Fig. 27.11, C, D and F are the mid-points of BE, CE and AE respectively. $\mathbf{AB} = \mathbf{a}$ and $\mathbf{AC} = \mathbf{a} + \mathbf{b}$.

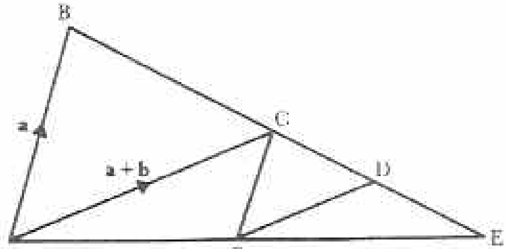


Fig. 27.11

Write down as simply as possible in terms of **a** and **b**, expressions for the following.
(a) **BC** (b) **CF** (c) **FD**
(d) **DE** (e) **FE** (f) **BF**

6 $\mathbf{OP} = \begin{pmatrix} -3 \\ 4 \end{pmatrix}$ and $\mathbf{OQ} = \begin{pmatrix} 1 \\ q \end{pmatrix}$. Find
(a) |**OP**|, (b) q if O, P and Q are vertices of a square OPQR.

7 It is given that $\mathbf{u} = 4\mathbf{a} + 3\mathbf{b}$, $\mathbf{v} = 5\mathbf{a} - \mathbf{b}$ and $\mathbf{w} = h\mathbf{a} + (h + k)\mathbf{b}$, where h and k are constants. If $\mathbf{w} = 3\mathbf{u} - 2\mathbf{v}$, calculate the value of h and k. [Camb]

8 If $\mathbf{OA} = 3\mathbf{p} - 2\mathbf{q}$, $\mathbf{OB} = \mathbf{p} + 7\mathbf{q}$ and $\mathbf{AB} = 2m\mathbf{p} + (m - n)\mathbf{q}$, find the values of m and n.

9 In Fig. 27.12, OABC is a square and X, Y, Z are the mid-points of OC, CB, BA respectively. $\mathbf{OA} = \mathbf{a}$ and $\mathbf{OC} = \mathbf{c}$.

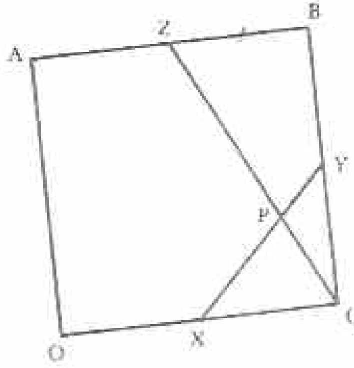


Fig. 27.12

(a) Express **XY** and **CZ** in terms of **a** and **c**.
(b) If $\mathbf{XP} = h\mathbf{XY}$, express **XP** in terms of **a**, **c** and h.
(c) If $\mathbf{CP} = k\mathbf{CZ}$, express **CP** in terms of **a**, **c** and k.
(d) Use the fact that $\mathbf{XP} = \mathbf{XC} + \mathbf{CP}$ to evaluate h and k.

10 (a) Given that $\mathbf{OK} = \begin{pmatrix} 16 \\ 2 \end{pmatrix}$, $\mathbf{OL} = \begin{pmatrix} 4 \\ -3 \end{pmatrix}$ and that M and N are the mid-points of OK and OL respectively, (i) express **MN** as a column vector, (ii) find the value of |**KL**|.
(b)

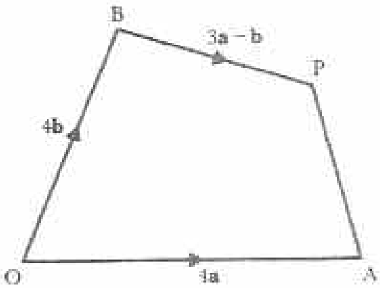


Fig. 27.13

Given that $\mathbf{OA} = 4\mathbf{a}$, $\mathbf{OB} = 4\mathbf{b}$ and $\mathbf{BP} = 3\mathbf{a} - \mathbf{b}$, express as simply as possible, in terms of **a** and **b**, (i) **OP**, (ii) **AP**.
The lines OA produced and BP produced meet at Q. Given that $\mathbf{BQ} = m\mathbf{BP}$ and $\mathbf{OQ} = n\mathbf{OA}$, form an equation connecting m, n, **a** and **b**. Hence deduce the values of m and n. [Camb]

Chapter 28

Travel graphs, statistics, probability

Syllabus references 6.5.2 and 6.11

Travel graphs

Distance–time graphs

Fig. 28.1 is a typical distance–time graph of a journey which is in three stages.

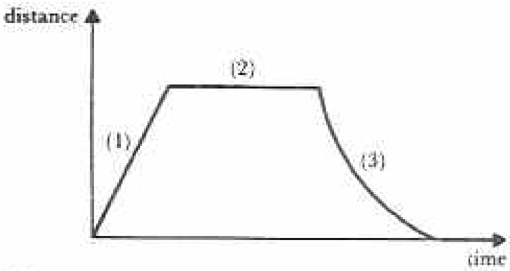


Fig. 28.1

In Fig. 28.1,

Stage 1: Distance is changing uniformly with time. The **gradient** of the line is a measure of the **speed**, or rate of change of distance with time.

Stage 2: Distance does not change with time. The object is at rest.

Stage 3: The gradient of the tangent at any point on the curve gives the speed at that point. During this stage the speeds are negative. This shows that the object is travelling in a direction opposite the original direction.

Example 1

At 0800 *a cyclist left home and cycled at* 13 km/h *to a village* 20 km *away. She stayed for* 2 hours *at the village before cycling home at* 9 km/h. *Use a graphical method to find the time when she arrived home.*

Fig. 28.2 is a graph of the cyclist's journey.

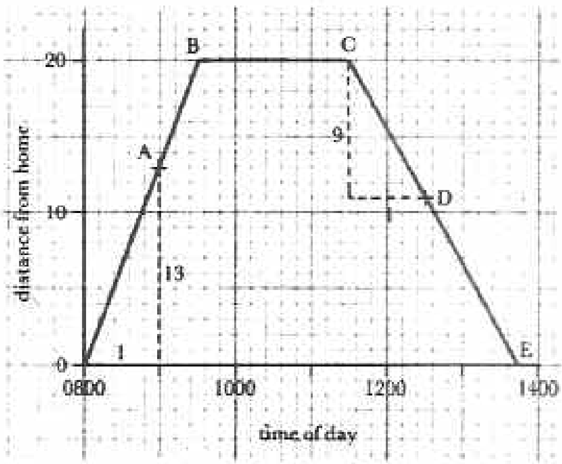


Fig. 28.2

From the graph, the time of arrival is found at E, approximately 1345.

Method

Choose suitable scales with time on the horizontal axis. Then:

(a) In 1 hour the cyclist covers 13 km. Plot the point A (0900; 13).

(b) Produce the line through OA to B, 20 km from the start.

(c) On the horizontal through B, mark C, 2 hours after B. C represents the starting point of the journey home.

(d) Going home the cyclist covers 9 km in 1 hour. Plot D 1 hour and 9 km from C.

(e) Produce the line through CD to cut the time axis at E.

Speed–time graphs

Fig. 28.3 is a typical speed–time graph.

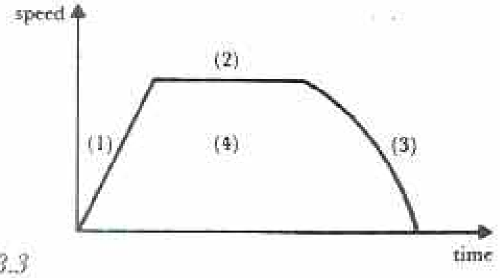


Fig. 28.3

In Fig. 28.3,
Stage 1: Speed is changing uniformly with time. The **gradient** of this line is a measure of the **acceleration**, or rate of change of speed with time.
Stage 2: Speed does not change with time. The horizontal line represents a period of constant speed.
Stage 3: Speed is decreasing with time. This gives a **negative acceleration** or **deceleration**. The gradient of the tangent at any point on the curve gives the deceleration at that point.
Region 4: The **area** under the graph is a measure of the **distance** travelled. The area can be estimated by counting squares or by considering the areas of trapeziums.

Chapter 8, pages 63 to 67, gives a full explanation of how to estimate the gradient at a point on a curve and how to use trapeziums to estimate the area under a curve. Also see Example 3 below.

Example 2

Fig. 28.4 is the speed–time graph of a train journey.

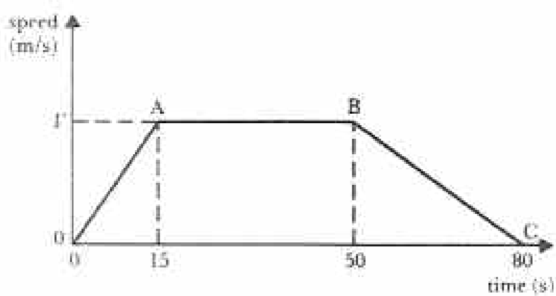


Fig. 28.4

If the total distance travelled in the 80 seconds is 920 m, *calculate (a), V, (b) the acceleration of the train during the first* 15 seconds, *(c) the distance travelled in the final* 40 seconds.

(a) Distance travelled

$= \text{area of trapezium OABC}$

$= \frac{1}{2}(AB + OC)V = \frac{1}{2}(35 + 80)V$

Hence, $920 = \frac{1}{2}(35 + 80)V$

$\Leftrightarrow \quad 1840 = 115V$

$\Leftrightarrow \quad V = \frac{1840}{115} = 16$

(b) Acceleration during first 15 seconds

$= \text{gradient of OA}$

$= \frac{V}{15} = \frac{16}{15}\,\text{m/s}^2$

$= 1\frac{1}{15}\,\text{m/s}^2$

(c)

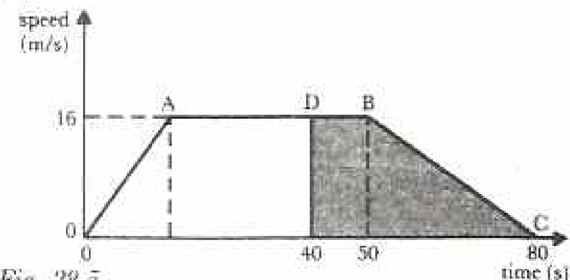


Fig. 28.5

Distance travelled during the last 40 seconds

$= \text{Area under DBC in Fig. 28.5}$

$= \frac{1}{2}(DB + 40)16\,\text{m}$

$= \frac{1}{2}(10 + 40)16\,\text{m}$

$= 400\,\text{m}$

Example 3

A particle moves in a straight line so that after t seconds its velocity v m/s is given by $v = 5t^2 - 12t + 8$. *Some corresponding values of v and t are given in Table 28.1.*

Table 28.1

t	0	$\frac{1}{2}$	1	$1\frac{1}{2}$	2	$2\frac{1}{2}$	3	$3\frac{1}{2}$	4
v	8	$3\frac{1}{4}$	p	$1\frac{1}{4}$	4	q	17	$27\frac{1}{4}$	40

(a) Calculate p and q.
(b) Draw the graph of $v = 5t^2 - 12t + 8$ for the range $0 \geqslant t \geqslant 4$.
(c) Use the graph to estimate (i) the time at which the acceleration is zero, (ii) the acceleration when $t = 3$. (d) By using trapeziums with $\frac{1}{2}$-second intervals, estimate the distance travelled during the 4 seconds.

(a) When $t = 1$,

$$p = 5(1)^2 - 12(1) + 8$$
$$= 5 - 12 + 8 = 1$$

When $t = 2\frac{1}{2}$,

$$q = 5(2\tfrac{1}{2})^2 - 12(2\tfrac{1}{2}) + 8$$
$$= 31\tfrac{1}{4} - 30 + 8 = 9\tfrac{1}{4}$$

(b) Fig. 28.6 is the required graph.

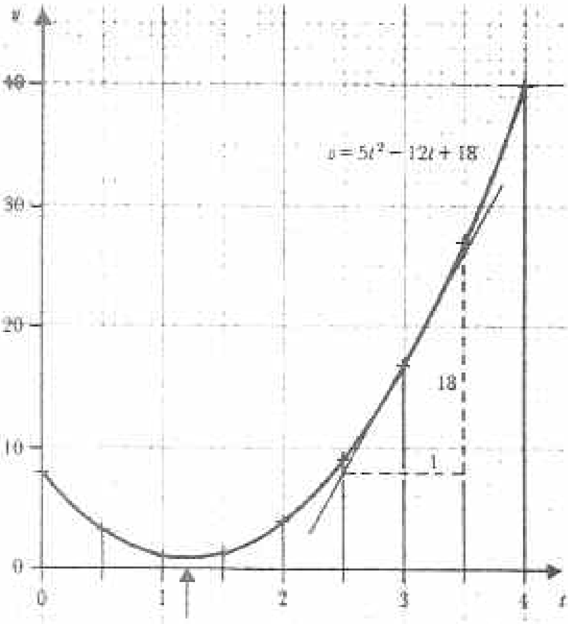


Fig. 28.6

(c) (i) The acceleration is zero when the tangent to the curve is horizontal, i.e. at the lowest point of the curve in Fig. 28.6. At this point, $t = 1,2$ seconds.
(ii) In Fig. 28.6 a tangent has been drawn at the point where $t = 3$.

$$\text{Gradient of tangent} = \frac{18\,\text{m/s}}{1\,\text{s}} = 18\,\text{m/s}^2$$

When $t = 3$, the acceleration is $18\,\text{m/s}^2$.

(d) Distance travelled
= area under curve
≃ sum of areas of trapeziums in Fig. 28
$= \frac{1}{2}(8 + 3\frac{1}{4})\frac{1}{2} + \frac{1}{2}(3\frac{1}{4} + 1)\frac{1}{2} + \frac{1}{2}(1 + 1\frac{1}{4})\frac{1}{2}$
$+ \ldots + \frac{1}{2}(17 + 27\frac{1}{4})\frac{1}{2} + \frac{1}{2}(27\frac{1}{4} + 40)\frac{1}{2}$
$= \frac{1}{2}[\frac{1}{2}(8 + 40) + 3\frac{1}{4} + 1 + 1\frac{1}{4} + 4 + 9\frac{1}{4}$ –
$17 + 27\frac{1}{4}]\,\text{m}$
$= \frac{1}{2}[24 + 63]\,\text{m}$
$= 43\frac{1}{2}\,\text{m}$

Exercise 28a

1 **Fig. 28.7 shows the journey of a car fro Harare to Kadoma and of a bus fro** Kadoma to Harare on the same road.

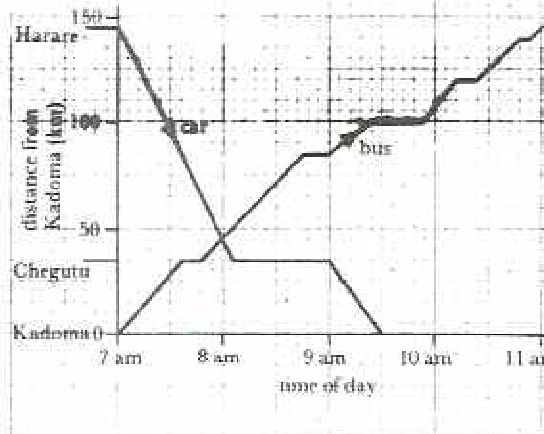


Fig. 28.7

(a) How many times did the bus st between Kadoma and Harare?
(b) What is the distance between Kador and Harare?
(c) What was the car's average speed f the whole journey?
(d) What was the bus's average speed the nearest km/h for the whole journey?
(e) What was the car's average spe between Harare and Chegutu?
(f) What was the car's average spe between Chegutu and Kadoma?
(g) How far were they both from Chegu when they passed each other on the roa

2 Fig. 28.8 shows how Sola walked fro school to the Post Office and how Go walked from the Post Office to the schoo

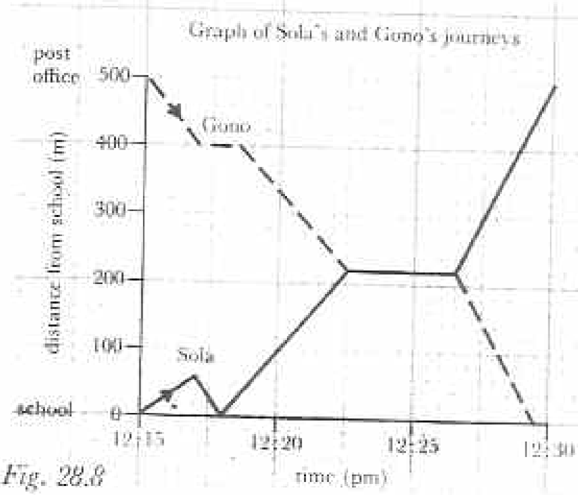


Fig. 28.8

(a) How far were they from the Post Office when they met?
(b) How long did they stand talking?
(c) After leaving school, Sola suddenly remembered a letter she had to post. She returned to school for the letter. At what time did she remember the letter?
(d) Gono stopped for $1\frac{1}{2}$ min to buy some bread. How far is the bakery from the school?
(e) After leaving Gono, what was Sola's walking speed in m/min?
(f) How much further did Sola walk than Gono?

3 Three cars, A, B, C, start one after the other in alphabetical order at 5-minute intervals. They travel at 80, 100, 120 km/h respectively. Given that A leaves at midday, draw a distance–time graph which enables you to find when (a) B passes A, (b) C passes A, (c) C passes B.

4 An aircraft travels at an average speed of 800 km/h. It left airport A at 1230 and arrived at airport B at 1345. After stopping at airport B for 45 min it left for airport C, arriving there at 1900.

Draw a distance–time graph of its journey using a scale of 2 cm to 1 hour on the time axis and 1 cm to 400 km on the distance axis. Use your graph to find (a) the distance from A to C, (b) the distance of the plane from B at 1500.

5 A cyclist sets out at 0705 to reach his office 13 km away at 0800. After 5 min he finds he has forgotten his briefcase. He rides home again at 16 km/h and then takes 2 min to find the briefcase. Use a graphical method to answer the following. (a) How fast must he ride to get to work on time? (b) If his top speed is 17 km/h, how many minutes will he be late for work?

6 During a journey a train accelerates uniformly for 6 min. Its speed, v km/h, is given in 1-min intervals in Table 28.2.

Table 28.2

time (min)	0	1	2	3	4	5	6
v (km/h)	7	16	25	34	43	52	61

By drawing a speed–time graph find, (a) the train's acceleration in km/h per min, (b) the distance travelled in km during the whole 6 min.

7 Fig. 28.9 is the speed–time graph of a sprinter.

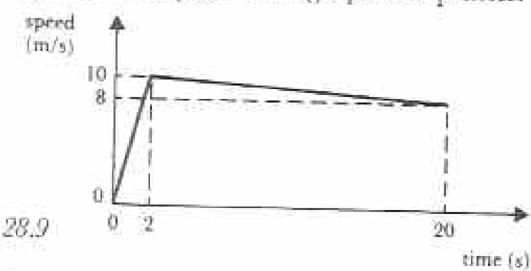


Fig. 28.9

Calculate (a) the acceleration of the sprinter over the first 2 seconds, (b) the deceleration of the sprinter over the next 18 seconds, (c) the distance covered in the 20 seconds.

8 Fig. 28.10 is the speed–time graph of a journey.

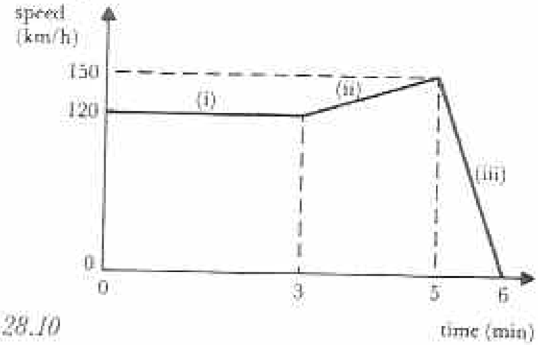


Fig. 28.10

(a) Describe the three parts of the journey in your own words.
(b) Calculate the accelerations represented by parts (ii) and (iii) of the graph (answers in km/h per minute).
(c) Calculate the total distance travelled (answer in km).

9 Fig. 28.11 is the speed–time graph of a car journey.

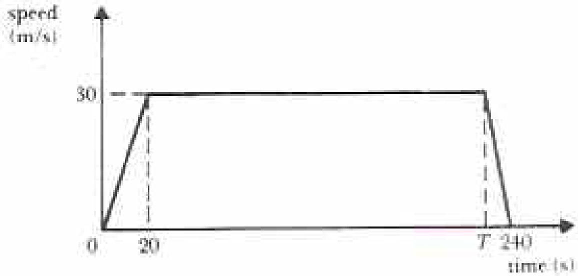


Fig. 28.11

(a) Calculate the acceleration of the car during the first 20 seconds.
(b) Given that the final deceleration is 0,5 m/s^2, calculate (i) T, (ii) the total distance travelled by the car.

10 Fig. 28.12 is the speed–time graph of the last ten seconds of a car journey. It travels at a constant speed of 31 m/s for 4 seconds and slows down uniformly first to 16 m/s then to rest after a further 5 seconds and 1 second respectively.

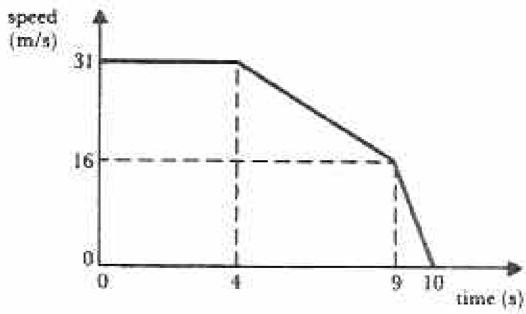


Fig. 28.12

Calculate (a) the speed of the car 2 seconds before it stopped, (b) the average speed of the car during the whole 10 seconds.

11 An aeroplane accelerates from rest at 40 km/h per minute until its speed is 840 km/h. I then travels at this speed for $2\frac{1}{2}$ h befor decelerating at an average of 15 km/h pe minute until it comes to rest.
(a) Sketch the journey on a speed (km/h)- time(min) graph.
(b) Calculate the total time taken for th journey in minutes.
(c) Calculate the distance travelled in km

12 Fig. 28.13 is a speed–time curve of part o a taxi journey.

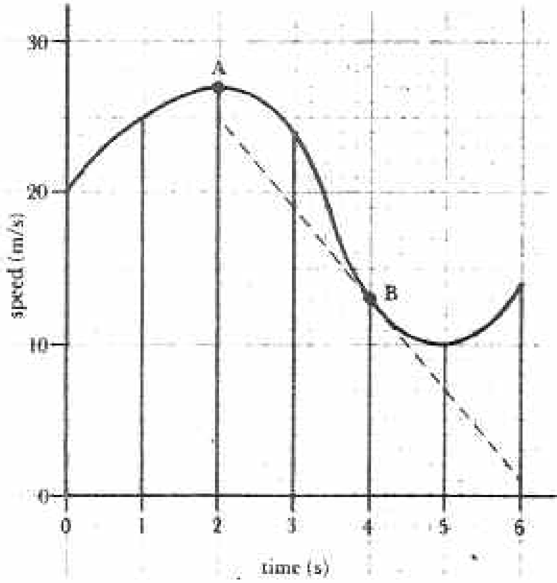


Fig. 28.13

(a) What is the acceleration of the taxi at A
(b) Use the tangent drawn at B to estimat the acceleration of the taxi at that point.
(c) By using suitable trapeziums or other wise, estimate the distance travelled by th taxi in the 6 seconds.

13 Table 28.3 gives the speed, v m/s, of a object taken at 1-second intervals.

Table 28.3

t(s)	0	1	2	3	4	5	6	7	8
v(m/s)	32	35	36	35	32	27	20	11	0

(a) Draw a v–t graph of the motion.
(b) By constructing a suitable tangent, estimate the acceleration of the object after 5 seconds.
(c) Estimate the total distance travelled during the 8 seconds.

14 The velocity, v m/s, of an object after t seconds is given by the equation
$v = 4t^2 - 12t + 11$.
Table 28.4 contains some corresponding values of v and t.

Table 28.4

t(s)	0	$\frac{1}{2}$	1	$1\frac{1}{2}$	2	$2\frac{1}{2}$	3	$3\frac{1}{2}$	4
v(m/s)	11	6	3	m	32	6	11	n	27

(a) Calculate m and n.
(b) Taking 2 cm to represent 1 second on the horizontal axis and 2 cm to represent 5 m/s on the vertical axis, draw the graph of $v = 4t^2 - 12t + 11$ for the range $0 \leqslant t \leqslant 4$.
(c) From your graph, find the times when the velocity is 8 m/s.
(d) By drawing a tangent, find the acceleration of the object after 3 seconds.
(e) Estimate the distance travelled during the 4 seconds.

15 The velocity, v m/s, of a car after t seconds is given by
$v = 7 + 6t - t^2$.
Table 28.5 contains some corresponding values of v and t.

Table 28.5

t	0	1	2	3	4	5	6	7
v	7	12	15	a	15	12	7	b

(a) Calculate a and b.
(b) Draw the graph of $v = 7 + 6t - t^2$ for the range $0 \leqslant t \leqslant 7$ using scales of 2 cm to 1 second horizontally and 1 cm to 1 m/s vertically.
(c) Use your graph to find the speed of the car after 4.3 seconds.
(d) By drawing suitable tangents, find the acceleration of the car after (i) 1 second, (ii) 6 seconds.
(e) Estimate the distance travelled during the 7 seconds.

Statistics

Bar charts and pie charts

Example 4
Table 28.6 shows how a wage of $150 is spent.

Table 28.6

item	amount
food	\$50
rent	\$25
clothing	\$20
savings	\$30
other expenses	\$25

Show this information on a bar chart.

Represent the items by bars of the same width. The height, or length, of each bar is proportional to the amount of money. Fig. 28.14 shows that the bars may be vertical or horizontal.

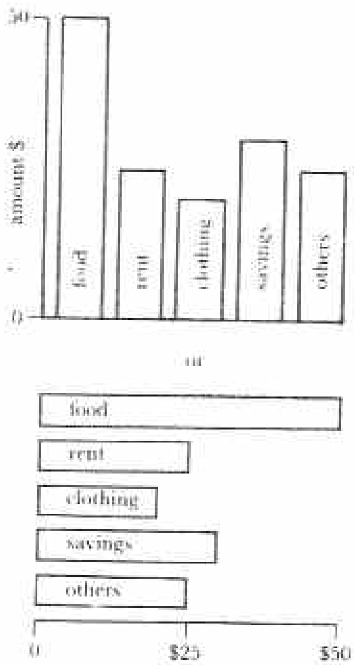


Fig. 28.14

Example 5

The number of students admitted to a university in a particular year is distributed among five faculties as follows:
Education, 350;
Medicine, 150;
Engineering, 200;
Law, 100;
Arts, 100.
Draw a pie chart to represent this information.

Table 28.7 shows how to calculate the angles of the sectors of the pie chart.

Table 28.7

faculty	number of students	angle of sector in pie chart
Education	350	$\frac{350}{900} \times 360° = 140°$
Medicine	150	$\frac{150}{900} \times 360° = 60°$
Engineering	200	$\frac{200}{900} \times 360° = 80°$
Law	100	$\frac{100}{900} \times 360° = 40°$
Arts	100	$\frac{100}{900} \times 360° = 40°$
totals	900	360°

Fig. 28.15 is the required pie chart.

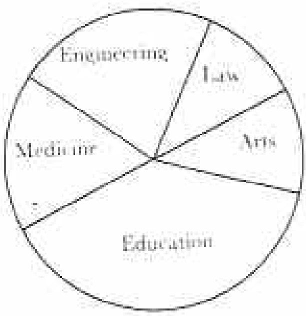


Fig. 28.15

Mean, median, mode

Given a set of values, the **mean** is the sum of all the values divided by the number of them.

If a set of numbers is arranged in order of size, the middle term is called the **median**. If there is an even number of terms the median is taken as the mean of the two middle terms.

The number of times any particular value occurs in a set is called its **frequency**. The number which occurs most often, i.e. the value that has the greatest frequency, is called the **mode**.

Example 6

Find (a) the mean, (b) the median, (c) the mode of the following set of numbers: 12; 16; 8; 11; 12; 8; 2; 8; 1; 14.

(a) Sum of the numbers
$= 12 + 16 + 8 + 11 + 12 + 8 + 2 + 8 + 1 + 14$
$= 92$
Number of numbers = 10

$$\text{Mean} = \frac{92}{10} = 9.2$$

(b) Arrange the numbers in ascending order:
1; 2; 8; 8; 8; 11; 12; 12; 14; 16

$$\text{Median} = \frac{8 + 11}{2} = 9.5$$

(c) The mode is 8.

Example 7

The ages of 15 *students in years and months are* 14.5; 15.2; 14.3; 13.9; 14.10; 14.11; 13.8; 15.3; 14.6; 15.6; 15.8; 16.1; 15.4; 14.4; 14.7. *Find the averag age of the students to the nearest month.*

The ages range from 13.8 to 16.1. Take 15.(as a working mean. Make two columns of numbers. The one marked (+) shows all the deviations in months for ages over 15.0; the other, (−), gives the deviations in months fo ages under 15.0.

(+)	(−)
2	7
3	9
6	15
8	2
13	1
4	16
	6
	8
	5
36	69

33 (i.e. 69 − 36)

The working shows that the total deviation for the 15 students is 33 months less than 15.0. Hence:

$$\text{mean age} = 15 \text{ yr } 0 \text{ mo} + \left(\frac{-33}{15}\right) \text{mo}$$
$$= 15 \text{ yr } 0 \text{ mo} - 2.2 \text{ mo}$$
$$= 14 \text{ yr } 10 \text{ mo to the nearest month}$$

The method in Example 7 is useful when given a large set of numbers of roughly the same size.

Exercise 28b

1 A student spent a full day as shown in Table 28.8.

Table 28.8

activity	time
at lectures	5 h
reading	6 h
sleeping	7 h
sports	2 h
others	4 h

Show this data on a bar chart and a pie chart.

2 Table 28.9 shows how a wage of \$200 was spent.

Table 28.9

item	amount
food	\$60
rent	\$40
clothing	\$20
savings	\$40
others	\$30

Show this data on a bar chart and a pie chart.

3 A university admitted the following numbers of students from 1988 to 1992.

Table 28.10

year	number of students
1988	800
1989	1 200
1990	1 350
1991	1 570
1992	2 250

(a) Represent this information on a bar chart.
(b) Calculate the mean number of students admitted per year.

4 Table 28.11 shows the numbers of different types of books in a school library.

Table 28.11

subject	number of books
mathematics	48
science	110
engineering	54
novels	496
others	372

Draw a pie chart to show this information.

5 Find the mean, median and mode of the following sets of numbers.
(a) 2; 4; 4; 6; 7
(b) 3; 5; 5; 7; 7; 7; 9; 9
(c) 11; 9; 6; 4; 3; 12; 1; 6; 5

6 Use a working mean of 115 to find the mean of the following:
110; 120; 113; 116; 119;
127; 117; 118; 118; 113

7 The ages of 16 students in years and months are as follows:

17.2	17.10	18.2	19.5
18.0	17.11	18.7	19.7
19.3	19.8	16.11	17.9
17.10	17.5	18.5	18.1

Choose a suitable working mean and use it to find the average age of the students.

8 The heights, in cm, of 10 boys are
145 163 159 162 167
149 150 160 170 155
Calculate the mean and median heights.

9 The masses, to the nearest kg, of 15 girls are
45 38 51 44 43
60 55 47 45 42
52 46 41 50 53
Find the mean and median masses.

10 Table 28.12 gives the age distribution of members of a school choir.

Table 28.12

age	12	13	14	15	16	17
frequency	2	1	3	6	5	3

(a) How many students are in the choir?
(b) What is the modal age?
(c) Draw a pie chart to show the age distribution.
(d) Calculate the mean age of the choir members.

Grouped data

Histogram, cumulative frequency curve

Example 8

Table 28.13 is the frequency distribution of the masses of 40 pupils in a class.
(a) Draw a histogram of the distribution.
(b) State the modal class.

Table 28.13

mass (kg)	**number of pupils**
41–45	3
46–50	7
51–55	12
56–60	10
61–65	6
66–70	2

(c) Calculate the mean mass of the pupils.
(d) Draw a cumulative frequency curve of the distribution.
(e) Hence estimate (i) the median, (ii) the semi-interquartile range.

(a) Fig. 24.16 is the required histogram. Notice that the gaps between the bars have been closed by increasing the width of each rectangle by $\frac{1}{2}$ unit on both sides.

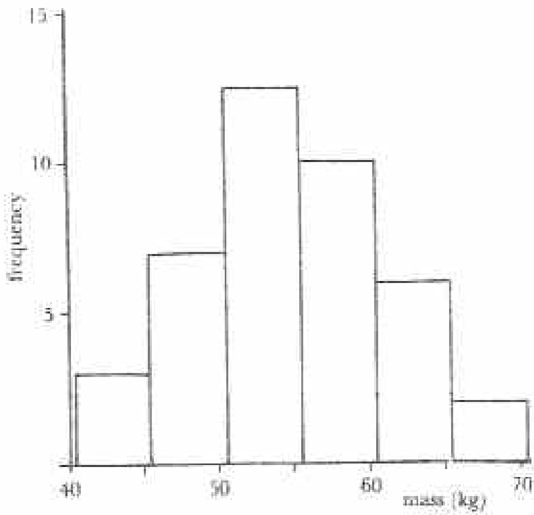


Fig. 28.16

(b) The modal class is 51–55. This class has the highest frequency.
(c) The mid-value of each class can be taken to be representative of that class as a whole. Table 28.14 shows the mid-values and the corresponding deviations from a working mean of 53 kg.

Table 28.14

mass (kg)	frequency f	mid-value	deviation d	$f \times d$
41–45	3	43	−10	−30
46–50	7	48	−5	−35
51–55	12	53	0	0
56–60	10	58	+5	+50
61–65	6	63	+10	+60
66–70	2	68	+15	+30
				total deviation = +75

Either, using the mid-values only:
mean mass

$$= \frac{3 \times 43 + 7 \times 48 + 12 \times 53 + 10 \times 58 + 6 \times 63 + 2 \times 68}{3 + 7 + 12 + 10 + 6 + 2} \text{ kg}$$

$$= \frac{2\,195}{40} \text{ kg} = 54{,}875 \text{ kg}$$

or using a working mean of 53 kg, from Table 28.14:

$$\text{mean deviation} = \frac{+75}{40} \text{ kg} = 1{,}875 \text{ kg}$$

$$\text{mean mass} = 53 + (+1{,}875) \text{ kg} = 54{,}875 \text{ kg}$$

(d) Table 28.15 is used to draw the cumulative frequency curve in Fig. 28.17.

Table 28.15

mass (kg)	frequency	cumulative frequency
41–45	3	3
46–50	7	10
51–55	12	22
56–60	10	32
61–65	6	38
66–70	2	40

(e) From Fig. 28.17, (i) median $= Q_2 = 54\frac{1}{2}$ kg
(ii) semi-interquartile range

$$= \frac{Q_3 - Q_1}{2} \text{ kg}$$

$$= \frac{59\frac{1}{4} - 50\frac{1}{4}}{2} \text{ kg}$$

$$= 4\frac{1}{2} \text{ kg}$$

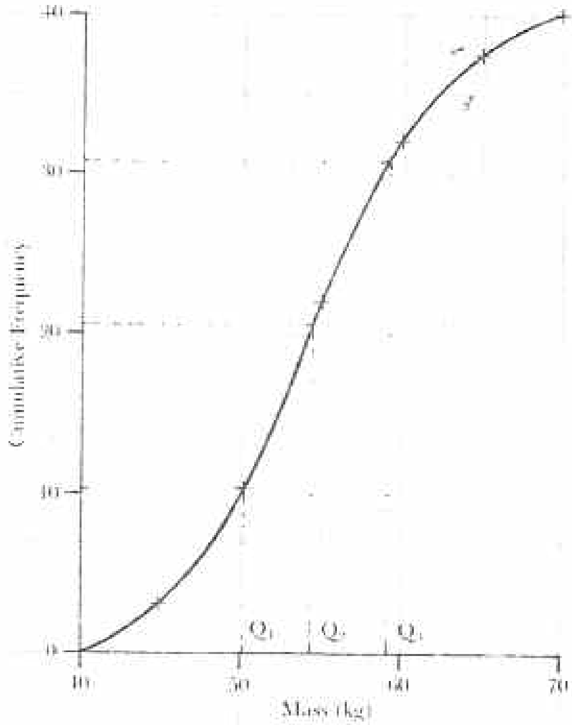


Fig. 28.17

Further information on frequency distributions, histograms and cumulative frequency curves can be found in Chapter 16, pages 133 to 142.

Exercise 28c

1 Table 28.16 shows the frequencies, f, of children of age x years in a hospital.

Table 28.16

x	1	2	3	4	5	6	7	8
f	3	4	5	6	7	6	5	4

(a) What is the modal class?
(b) How many children are in the hospital?
(c) Calculate the mean age of the children.

2 Table 28.17, overleaf, shows the length of life of 200 electric light bulbs.

Table 28.17

length of life (hours)	number of bulbs
201–300	10
301–400	16
401–500	32
501–600	54
601–700	88

(a) Draw a histogram of this distribution.
(b) Use a working mean of 550,5 hours to calculate the mean life of the light bulbs.

3 Table 28.18 shows the number of work-days lost through illness among 500 factory employees during a 1-year period.

Table 28.18

number of days	number of employees
0– 4	250
5– 9	158
10–14	33
15–19	29
20–24	15
25–29	10
30–34	5

(a) Draw a histogram of the distribution.
(b) State the modal class.
(c) Calculate the mean number of days lost.

4 The masses, in kg to the nearest kg, of 40 students are as follows:

59	54	51	56	59	61	60	61	59	58
62	61	63	64	58	57	56	60	62	60
61	65	58	57	54	52	62	67	69	49
56	58	60	60	62	58	51	57	70	63

(a) Take class intervals of 46–50, 51–55, ..., and make up a table of frequencies.
(b) Draw the corresponding histogram.
(c) Find the median mass.
(d) Calculate the mean mass.

5 The examination marks of 50 students are as follows:

65	58	51	36	23	40	53	59	70	51
46	59	50	67	46	39	61	62	73	60
71	51	47	32	48	40	40	51	58	67
60	69	43	52	37	26	38	50	59	40
44	54	42	47	68	74	45	39	48	55

(a) Make a frequency distribution using class intervals of 21–30, 31–40, ...
(b) Draw a cumulative frequency curve.
(c) Hence estimate (i) the median, (ii) the semi-interquartile range.
(d) Find the percentage of students that got more than 45 marks.

6 Table 28.19 is the frequency distribution of the heights of 40 pupils.

Table 28.19

height (cm)	number of pupils
131–140	2
141–150	11
151–160	14
161–170	10
171–180	3

(a) Draw a histogram of the distribution.
(b) State the modal class.
(c) Calculate the mean height of the pupils.
(d) Draw a cumulative frequency curve of the distribution.
(e) Hence estimate the median height of the pupils.

Probability

The **probability** of an event happening can be given a numerical value x where

$$x = \frac{\text{number of required outcomes}}{\text{number of possible outcomes}}$$

and $1 \geqslant x \geqslant 0$.

If $x = 1$, then the event is certain to happen. If $x = 0$, then the event cannot happen. $1 - x$ is the probability of the event *not* happening.

Example 9
Table 28.20 shows the numbers of students in each age group in a class.

Table 28.20

age (years)	16	17	18
number	7	22	13

What is the probability that a student chosen at random from the class is (a) 17 years old, (b) not 17 years old, (c) over 16 years old?

(a) Probability that the student is 17 years old

$$= \frac{\text{number of 17 year olds}}{\text{total number of students}}$$

$$= \frac{22}{7 + 22 + 13} = \frac{22}{42} = \frac{11}{21}$$

(b) Probability that the student is *not* 17 years old

$$= 1 - \frac{11}{21} = \frac{10}{21}$$

(c) Probability that the student is over 16 years old

$$= \frac{\text{number of students over 16}}{\text{total number of students}}$$

$$= \frac{22 + 13}{42} = \frac{35}{42} = \frac{5}{6}$$

Probabilities of mutually exclusive events and independent events together with the use of outcome tables and tree diagrams are explained in Chapter 19, pages 158 to 165.

Exercise 28d

1 A box contains 40 oranges, 12 of which are unripe. I pick one at random. What is the probability that it is (a) ripe, (b) unripe?

2 A bag contains 10 white balls, 4 red balls and 6 black balls. If a ball is selected at random, what is the probability that it is (a) white, (b) red, (c) black?

3 A card is picked at random from a pack of 52 playing cards. What is the probability that it is (a) the Queen of diamonds, (b) a black five, (c) a nine, (d) a black heart.

4 A class contains 15 boys and 21 girls. A student is chosen at random. What is the probability that a boy is chosen?

5 A fair 6-sided die is thrown. Find the probability of getting (a) a four, (b) an even number, (c) a number less than five.

6 Three coins are tossed at the same time. (a) Write down all the possible ways that they can fall, using H for head and T for tail. (b) Find the probability of getting 1 head and 2 tails.

7 A bag contains 24 tennis balls, some white and some green. If a ball is chosen at random, the probability of getting a green ball is $\frac{3}{8}$.
How many white balls are in the bag?

8 A worker is chosen at random from among the 500 employees in Table 28.18 on page 258. What is the probability that she had been absent for less than 10 days during the year in question?

9 A number has three digits formed by arranging 4, 5 and 6 in a random order. Write down all the possible numbers. Hence find the probability that the number is divisible by
(a) 3 (b) 4 (c) 5 (d) 6
(e) 7 (f) 8 (g) 9 (h) 13

10 Fig. 28.18 shows a target made from four concentric rings of radii, r, $2r$, $3r$, $4r$. A bullet hits the target at random.
Calculate the probability that it hits the shaded region.

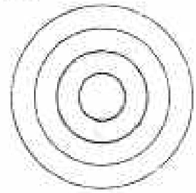


Fig. 28.18

11 Table 28.21 shows the numbers of students in each age group in a class.

Table 28.21

age (years)	16	17	18	19
frequency	9	11	11	5

(a) A student is chosen at random from the class. What is the probability that the age of the student is (i) 16 years, (ii) under 18 years, (iii) not 19 years?

(b) Calculate the average age of the students in years and months.

(c) Two students are chosen at random. Find the probability that they are both 19.

12 A = {2; 3; 6; 9; 11} and B = {13; 14; 15; 17; 19; 21}.

(a) If one element is selected at random from A, what is the probability that it is odd.

(b) If one element is selected from each set, calculate the probability that both elements are even, expressing your answer as a fraction in its lowest terms. [Camb]

13 A class of 35 students took a test which was marked out of 10. 27 of the students scored 7 marks or less.

(a) State the probability that a student chosen at random from the class scored more than 7?

A second class took the test and a quarter of them scored more than 7. If one student is chosen at random from each class, find the probability that

(b) both scored more than 7,

(c) only one scored more than 7.

14 From a group of five children, consisting of three girls and two boys, one child is chosen at random. Write down the probability that the child chosen is a girl.

A second child is then chosen at random from the remaining four children. Given that the first child chosen is a girl, write down the probability that the second child chosen is also a girl.

Two children on another occasion are chosen at random from this same group of three girls and two boys. Calculate the probability that (a) both are girls, (b) both are boys, (c) they are of different sexes. [Camb]

15 Table 28.22 shows the distribution of grades obtained by 30 students in an examination.

Table 28.22

grade	A	B	C	D	E
frequency	5	6	9	7	3

A pupil is selected at random from the group. Find the probability that the student got (a) a grade E, (b) either grade A or grade E.
Two students are selected at random from the group. Find the probability that (c) both obtained C, (d) one obtained grade C.

Chapter 29

Non-routine problems

In each section the questions have been categorised into problems, puzzles and investigations.

Problems can usually be solved in a fairly straightforward manner, using conventional methods.

Puzzles are less straightforward. You may have to consider unusual approaches. Do not be afraid to use the method of trial and error with puzzles. Allow time for thinking.

Investigations tend to be open-ended. Sometimes it is difficult to know when to stop. Investigations are best approached systematically. If a problem seems too complex, try a simpler example of the same problem. Find a helpful way of setting out your working and recording your results. This usually means making lists and/or tables. When enough results have been collected, it may then be possible to discover a rule.

Number, algebra and pattern

1 Arrange seven chairs in a row and ask three boys and three girls to sit on them as in Fig. 29.1.

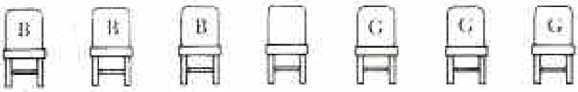


Fig. 29.1

The aim is to change over the boys and girls. Here are the rules:

(i) a student can move into an adjacent empty chair,

(ii) a student can jump over one adjacent student of the opposite sex into an empty chair,

(iii) no backward moves are allowed.

(a) What is the minimum number of moves needed? [Puzzle]

(b) What is the least number of moves needed if there are 5 chairs, 2 boys and 2 girls? [Puzzle]

(c) What if there were 9 chairs, 4 boys and 4 girls?

(d) Generalise for n boys and n girls. [Investigation]

(e) What if there were 2 empty chairs?

(f) What if there were unequal numbers of boys and girls?

(g) Investigate some ideas of your own. [Investigation]

2 (a) Choose any number less than 100 (e.g. 57).

(b) Form a new number by squaring each of the digits and adding.

$(57 \rightarrow 5^2 + 7^2 = 25 + 49 = 74)$

(c) Repeat.

$(74 \rightarrow 7^2 + 4^2 = 49 + 16 = 65)$

(d) Carry on doing this. You will know when to stop!

(e) Repeat the above with a different starting number.

(f) Investigate for all numbers less than 100.

3 Choose any four-digit number, e.g. 1 952. [Investigation]

Write digits in descending order:	9 521
Write digits in ascending order:	−1 259
Subtract:	8 262

Repeat the process on the answer (8 262 → 8 622 − 2 268)

Keep on repeating. You will know when to stop! [Problem]

Investigate with numbers which have three digits, five digits, etc. [Investigation]

4 Find a four-digit number which is exactly four times greater when its digits are reversed. [Puzzle]

Make up a similar puzzle for 3- and 5-digit numbers. [Puzzle]

5 123×45, 543×21, 421×53, $5 \times 41 \times 32$ are all products using the digits 1, 2, 3, 4,

5. Use a calculator to find the arrangement of these digits which gives the greatest product. [Problem]
Extend the problem to the digits 1, 2, 3, 4, 5, 6. Try to find a rule. [Investigation]

6 Make a copy of the grid in Fig. 29.2.

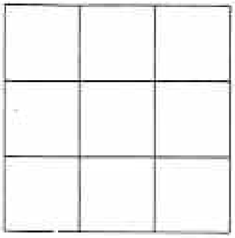


Fig. 29.2

Place the numbers 2, 2, 2, 3, 3, 3, 4, 4, 4 in the cells of the grid so that when any line of three numbers is added up in any direction the total is always 9. [Puzzle]

7 Make another copy of the grid in Fig. 29.2. Colour the squares either red, white or blue so that:

each red touches a white
each white touches a blue
each blue touches a red [Puzzle]

8 In a copy of Fig. 29.3 replace the asterisks by the numbers 1 to 12 in such a way that there are three numbers totalling 17 along each side. [Puzzle]

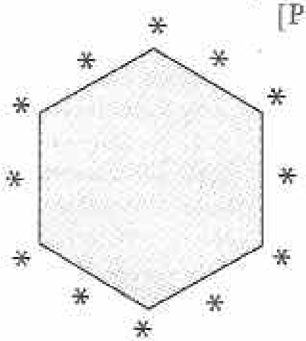


Fig. 29.3

9 x and y are numbers such that $xy = 1\,000\,000$. Find x and y if neither of them contain any 0's. [Problem]

10

6	9	15	24	39

To make the above row of numbers, start with 6 and 9 and add them to get 15. Then add 9 and 15 to get 24; finally add 15 and 24 to get 39.

The same rule has been used in the nex row, but the numbers in the middle arc missing.

5				43

Find the missing numbers. [Puzzle]
If the first number was m and the last number was n, what, in terms of m and n would be the middle number? [Problem]
Investigate for rows of various length. [Investigation]

11 Refer to the following set of whole numbers: {1, 2, 5, 9, 16}
(a) Which is the smallest prime number?
(b) Which is a multiple of 4?
(c) Write down all the square numbers.
(d) Write down three factors of 18.
(e) Find three numbers a, b, c such that $(a + b)^2 = c$.
(f) Find three numbers x, y, z such that $x^2 = y + z$.
(g) Find the numbers p, q, r such that $p^3 = q - r$. [Problem]

12 (a) Extend Table 29.1 as far as 33.

Table 29.1

denary	binary	number of ones in binary
1	1	1
2	10	1
3	11	2
4	100	1
5	101	2
6	110	2
7	111	3
8	1000	1
9	1001	2
⋮	⋮	⋮
33		

(b) Classify the denary numbers into sets, according to the number of 1's in their binary equivalents. For example:

$S_1 = \{1, 2, 4, 8, \ldots\}$
$S_2 = \{3, 5, 6, 9, \ldots\}$
$S_3 = \{7, 11, \ldots\}$
$S_4 = \{15, \ldots\}$ [Problem]

(c) Investigate the members of S_1, S_2, S_3, S_4, S_5, etc.
(d) What patterns can you find?
[Investigation]

13 A tin of cooking oil costs $10. If the oil is worth $9 more than the tin, what is the value of the tin? [Puzzle]

14 A company proposes a choice of two pay plans to a Union negotiator:
(a) Initial salary of $10 000, to be increased by $500 after each 12 months;
(b) Initial salary of $10 000, to be increased by $125 after each 6 months.
Which plan should the Union negotiator recommend? [Puzzle]

15 What is the value of
$$(x-a)(x-b)(x-c)\ldots(x-z)?$$
[Puzzle]

16 There are some goats and some hens on a farm. They have a total of 99 heads and legs. There are twice as many hens as goats. How many hens are there? [Problem]

17 A herd boy counts his cattle. When he counts them in threes there is one left over. When he counts them in fives there are two left over.
How many cattle might he have? [Puzzle]
Investigate the possible numbers of cattle he could have and suggest a rule.
[Investigation]

18 A student types some patterns using x's and m's. She makes the patterns according to a rule. Fig. 29.4 shows three of her patterns:

```
 x x x          x x x x x          x x x x x x
m m m m        m m m m m m        m m m m m m m
 x x x          x x x x x          x x x x x x
```

Fig. 29.4

(a) Draw another way she could have typed the pattern.
(b) How many x's would she need to type if she made a pattern which had 15 m's?
(c) How many m's would she need to type if she had a pattern which had 90 x's?
(d) If X stands for the number of x's and M stands for the number of m's, find the rule, or formula, that connects X and M.
$X = \ldots.$
(e) Another student types x's and m's according to the rule $X = \frac{1}{3}(2M+2)$. Find the number of x's if this student types 29 m's.
(f) Find the value of X in the second student's rule when $M = 10$. What does your result tell you? [Problem]

19 Start with any two single-digit numbers, e.g. 4 and 9. Here is a chain made by starting with these numbers:

$$4 \rightarrow 9 \rightarrow 3 \rightarrow 2 \rightarrow 5 \rightarrow 7 \rightarrow 2 \rightarrow$$

(a) How is the chain made? [Puzzle]
(b) Continue the chain. What happens? [Puzzle]
(c) Make similar chains with other starting numbers.
(d) Investigate what happens.
[Investigation]
(e) What happens if you use numbers in base five?

20 You have a large supply of 1c, 2c, 5c and 10c coins.
(a) How many ways can you make up a total of 10c? [Problem]
(b) How many ways can you make up a total of 15c? [Problem]
(c) Investigate for other sums of money.
[Investigation]

Spatial awareness and pattern

1 A farmer gives the field shown in Fig. 29.5 to his four children, provided they can divide it up according to his instructions.

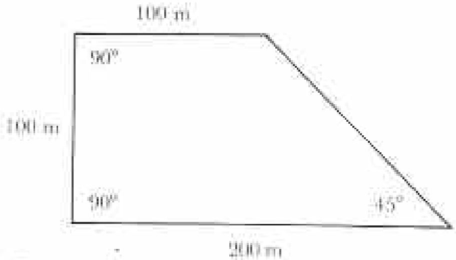


Fig. 29.5

All pieces should be equal in area. They must also be similar in shape to the original field. How can the children divide the field?
[Puzzle]

2 (a) In Fig. 29.6(a) find the ratio between the areas of the two circles.

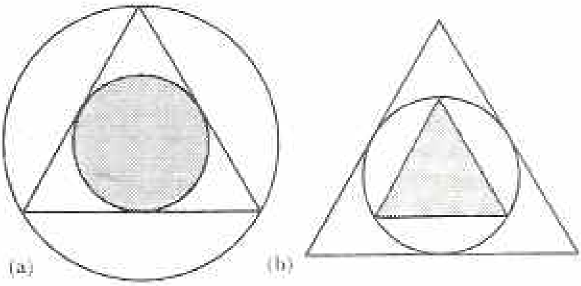


Fig. 29.6

(b) In Fig. 29.6(b) find the ratio between the areas of the two triangles. [Problem]

3 How many squares are there on an 8 × 8 chessboard? (64 is *not* the correct answer. Nor is it 65.) [Problem]

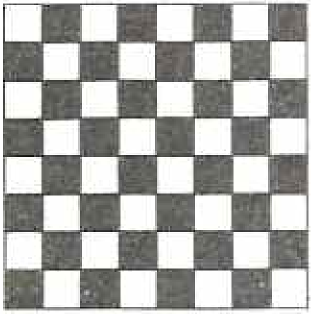


Fig. 29.7

It may help if you consider simpler cases, e.g. a 2 × 2 or a 3 × 3 chessboard (Fig. 29.8):

Fig. 29.8

4 Make a paper rectangle 8 cm by 3 cm like that in Fig. 29.9.

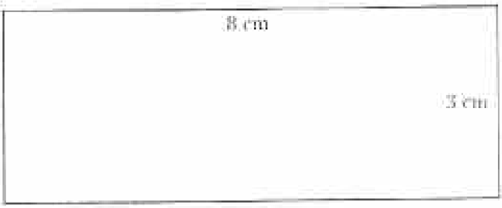


Fig. 29.9

Cut the paper into two pieces which can be rearranged to make a rectangle which measures 12 cm by 2 cm. [Puzzle]

5 Fig. 29.10 is a view of a crate which can hold 24 bottles of Cola.

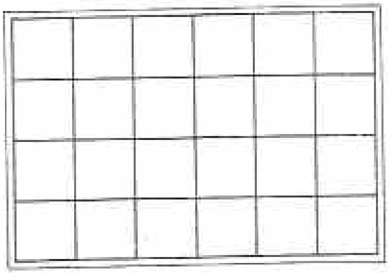


Fig. 29.10

(a) Place 18 bottles of Cola in the crate so that each row and each column has an *even* number of bottles in it.

(b) Find three different ways of doing this. [Puzzle]

(c) Is it possible to do the problem with 17 bottles?

(d) Investigate with different numbers of bottles.

(e) Investigate with crates of different dimensions. [Investigation]

6 How many acute angles can you see in Fig. 29.11? [Puzzle]

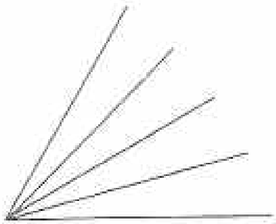


Fig. 29.11

7 Fig. 29.12 shows two dissections of a 4 × 4 square into rectangles (note that {squares} ⊂ {rectangles}).

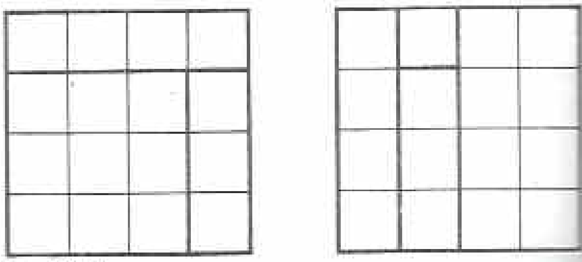


Fig. 29.12

The rules for dissection are:
(i) in each case all the rectangles must be different;
(ii) the edges of the rectangles are a whole number of units.
Find other dissections of a 4×4 rectangle.
Investigate for other starting squares.
[Investigation]

8 Arrange some matches to make five squares as shown in Fig. 29.13.

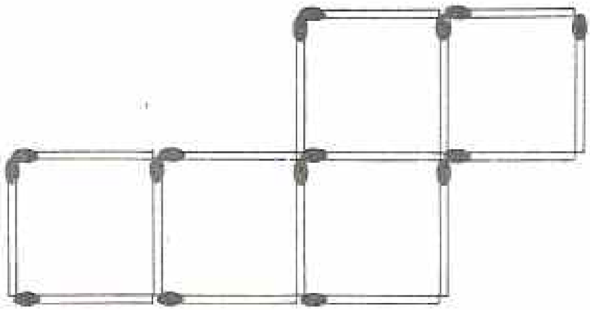


Fig. 29.13

Change the positions of just two of the matches to reduce the number of squares to four. (No 'loose ends' are allowed.)
[Puzzle]

9 In Fig. 29.14, three squares and three diagonals are arranged as shown.

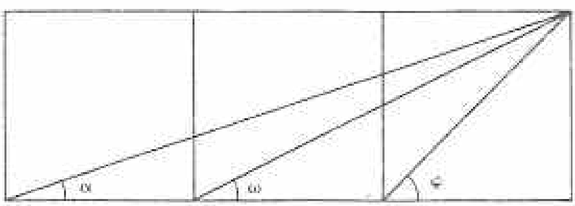


Fig. 29.14

There is a simple relationship between the three angles, α, ω and ϕ. Guess the relationship, then prove it. [Puzzle]

10 Fig. 29.15 shows a 5×3 grid with one diagonal drawn.
The diagonal cuts 7 of the grid's squares.
Investigate the numbers of squares cut by diagonals of grids of various dimensions.
How many squares would be cut by the diagonal of an $m \times n$ grid? [Investigation]

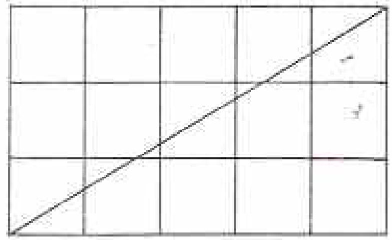


Fig. 29.15

Miscellaneous

1 A yellow ball, a blue ball and two red balls are placed in a bag. The bag is shaken. Someone takes two balls out of the bag, looks at them, and says, 'At least one of these balls is red.'
What is the probability that the other ball is also red? [Puzzle]

2 It is recommended that the water in a swimming pool should be a 1% chlorine solution. How would you decide how much chlorine to add to a swimming pool near your school?
You may have to ask your chemistry teacher what a '1% chlorine solution' is.
[Real problem]

3 Nine people arrive at a meeting. Each person shakes hands once with every other person present. How many handshakes are there? [Problem]
How many if only five people are at the meeting?
What if there were 500 people at the meeting?
[Investigation]

4 The maths teacher sets an exam paper for her class and left it in her desk overnight. Next morning it had disappeared. Four girls were in the classroom that morning: Aquilina, Bernadette, Cynthia and Dora. Of these students, only one of them tells the truth.
Aquilina said, 'Cynthia took it'.
Bernadette said, 'I didn't take it'.
Cynthia said, 'Bernadette is lying'.
Dora said, 'Cynthia is lying'.
Who took the exam paper? [Puzzle]

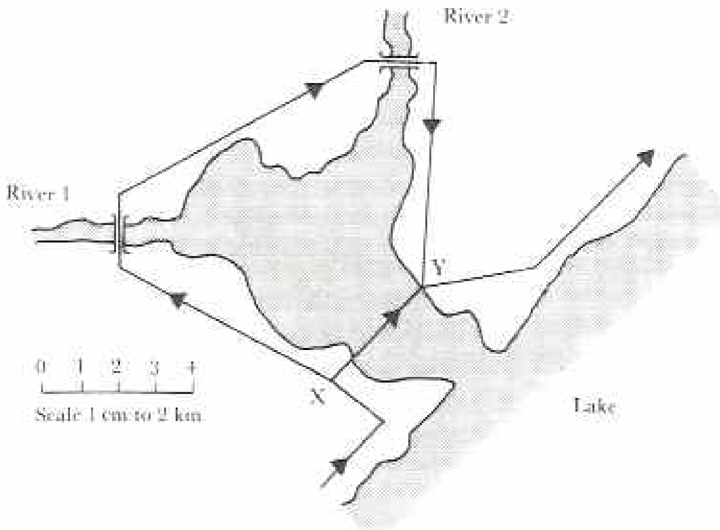


Fig. 29.6

5 Fig. 29.16 is a map showing two rivers flowing into a lake. A road runs northeast. To negotiate the estuary, a cyclist *either* has to make a detour, via two bridges, or can travel from X to Y with the help of a ferry.

(a) Measure the distance in cm from X to Y using the ferry.

(b) Measure the distance in cm from X to Y using the bridges.

(c) What are the actual distances in km for each route?

(d) What is the saving in km if the cyclist uses the ferry?

(e) The cyclist cycles at 20 km/h and the ferry travels at 8 km/h. How many minutes does the journey XY take by ferry? (Assume that there is no waiting for the ferry.) [Problem]

6 A car has travelled 30 000 km altogether. It has one spare tyre. The tyres were changed at intervals so that each tyre had been used for the same number of km. For how many km had each tyre been used? [Puzzle]

7 One day four university students sat together at lunchtime. Their names were Tembo, Festus, John and Henry and they studied geology, history, biology and law though not necessarily in that order.

John and Henry ate fruit for lunch. The law student ate sandwiches. Tembo and Henry often play tennis with the biology student and the law student. Henry sat between the biology student and the history student.

One of the students did not eat lunch. What was he studying? [Puzzle]

8 How many times will the long hand of a clock pass the short hand between midday today and midday tomorrow? (Since both hands are together at the starting time and finishing times, these do not count as passes.) [Puzzle]

9 Two students, X and Y, run a race of 100 metres. X beats Y by 5 metres. They decide to race again, but this time to introduce a handicap system. X starts the race 5 metres behind the starting line, therefore giving Y a 5 metre start. If both students run at the same rate as before, what will be the result of the second race? [Puzzle]

10 I am thinking of a whole number.

It is odd.
It is less than 1 000.
All the digits are different.
The sum of its digits is 12.
The difference between the first two digits is the same as the difference between the last two.
The hundreds digit is greater than the sum of the digits in the tens and units columns.

What is the number I am thinking of? [Puzzle]

Certificate-level practice examinations

This chapter contains two full-scale practice examinations in Mathematics at School Certificate/'O' level. They are included as a final revision of the senior secondary school course and to provide practice in examination technique. To be effective, each paper should be done under examination conditions, i.e. the times for each paper should be observed and neither the textbook nor notes should be referred to.

General advice

1 Be sure to read and understand the examination *rubric* (instructions). Typical rubric is given on the papers which follow. Note, however, that the Examinations Syndicate may change the rubric if it wishes. So, always check the rubric carefully.

2 Work out roughly how much time you can afford to spend on each question. Allow time for reading the questions and for checking your answers at the end. This advice is especially important in the short-answer paper where approximately 30 questions are to be answered in $2\frac{1}{2}$ hours (i.e. only about 5 minutes per question).

3 If any question involves drawing, make a rough sketch first. This helps you to position your final answer on the paper, whether it is a graph or a scale drawing.

4 Check your answers to see that they are sensible in terms of the given data. For example, a car costing $27 or walking speed of 5 600 km/h are sure signs that a slip has been made.

5 Show all your working in the body of the question you are answering. Do not be ashamed of your rough working. Examiners know what to look for. They may be able to give marks for such working, but only if they see it on the paper.

Examination 1

Paper 1 (2 h 30 min)

Instructions to candidates:
All *questions may be attempted.*
*Answers are to be written in the spaces provided.**
If working is needed for any question, it must be shown in the space below that question.
Omission of essential working will result in loss of marks.
Questions 1 to 19 carry 3 marks each.
Questions 20 to 27 carry 4 marks each.
Question 28 carries 5 marks.
Question 29 carries 6 marks.

*The examination paper contains spaces for working and answers. To save space in this book, these spaces have been omitted.

NEITHER MATHEMATICAL TABLES NOR SLIDE RULES NOR CALCULATORS MAY BE USED IN THIS PAPER.

1 Find the value of
(a) $1\frac{3}{4} - \frac{1}{3}$
(b) $2\frac{2}{5} \times 1\frac{2}{3}$
(c) $(\frac{1}{2} + \frac{1}{4}) \div \frac{1}{3}$

2 Find the value of
(a) $6,3 + 2,87$
(b) $0,8 \div 0,05$
(c) $0,056 \times 0,003$.

3 The universal set
$\mathscr{E} = \{u; n; i; v; e; r; s; a; l\}$,
$R = \{r; i; v; e; r\}$ and $V = \{\text{vowels}\}$.
(a) List the members of V.
(b) Find n(R).
(c) List the members of $R \cup V$.

4 50 students were asked, 'How did you travel to school today?' An incomplete list of results is given in Table R1 overleaf.

Table R1

method of travel	bus	car	bicycle	walking
no. of students	13	4	8	

(a) How many students walked to school?
(b) What percentage travelled by bus?

5 (a) Express 924_{ten} in base five.
(b) Find the sum of $10\,100_{two}$ and $1\,100_{two}$ as a binary number.
(c) Find $2\,033_{five} - 424_{five}$, giving the answer in base ten.

6 If $p = 9 \times 10^6$ and $q = 4 \times 10^7$, find the value of the following in standard form.

(a) $p + q$ (b) $\sqrt{p}$ (c) $\frac{1}{q}$

7

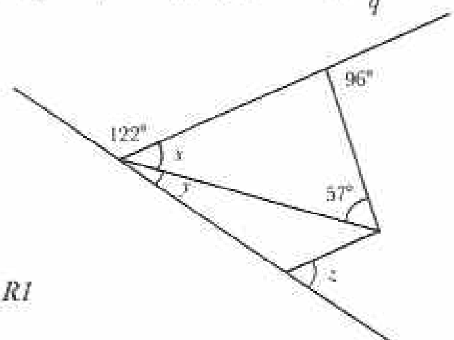


Fig. R1

Given Fig. R1, find x, y, z.

8 What is
(a) 20 mm as a fraction, in its lowest terms, of 9 m?
(b) 0,004 09 correct to 3 decimal places?
(c) 0,312 5 as a common fraction in its lowest terms?

9 (a) Three students measure their body-mass to the nearest kg as 39 kg, 42 kg and 51 kg. What is the greatest possible value for their combined body-mass?
(b) If x is an integer such that $3x < 40$ and $27 - 2x \leqslant 5$, list the possible values of x.

10 A straight line joins the points A(1; 3) and B(4; 7).
(a) What is the length of AB?
(b) What is the gradient of AB?
(c) A line parallel to AB passes through the origin and the point $(3; k)$. What is the value of k?

11 Fig. R2 shows points O, P and Q drawn on the cartesian plane.

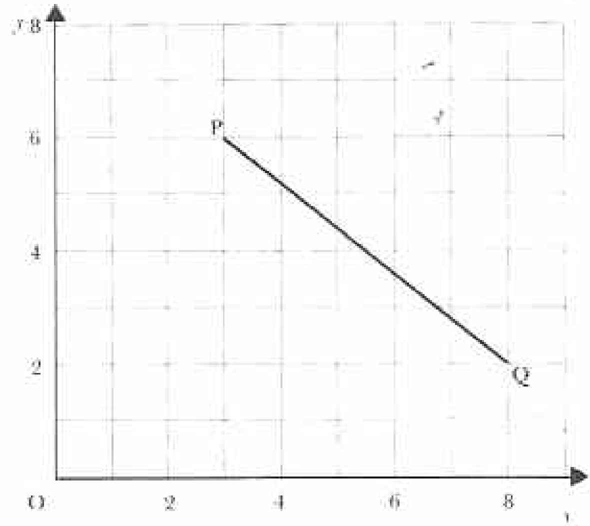


Fig. R2

Express each of the following as a single column vector.
(a) **PQ** (b) **OQ** (c) **OP** − **OQ**

12 Fig. R3 contains two regular pentagons A and B drawn on a common base-line as shown.

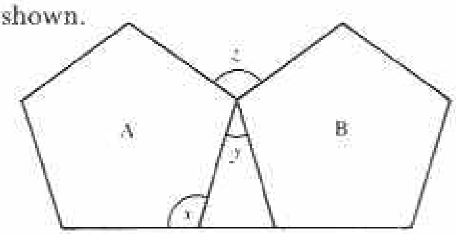


Fig. R3

Calculate the angles x, y and z.

13 Fig. R4 shows the minimum and maximum temperatures recorded on a particular day.

Fig. R4

(a) How many degrees did the temperature change?
(b) Calculate the mean of the two temperatures.

14 Factorise
(a) $x^2 - 14x + 49$,
(b) $6ab - 2b^2 - 4b + 12a$.

15 Solve the simultaneous equations:

$$2x + y - 2 = 0$$
$$4x - 3y - 19 = 0.$$

16 Fig. R5 is a graph showing the journey of a cyclist who left home at 11 am to visit a friend 5 km away, before returning home.

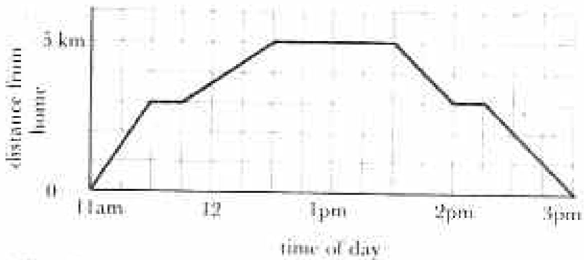


Fig. R5

(a) When did the cyclist arrive at his friend's house?
(b) On both the outward journey and the homeward journey, the cyclist called to greet his mother. How far does his mother live from his home?
(c) What was the cyclist's average speed for the whole journey?

17 A mattress is priced at \$200 plus sales tax of 20%.
(a) Find the total price of the mattress including sales tax.
(b) A shop offers the mattress for sale with a discount of 20% of its total price. What is the discount price?

18 A drinking glass is in the shape of a cylinder of internal diameter 7 cm and height 12 cm. If the glass is two-thirds full, how many mℓ of liquid does it contain? [Take π to be $3\frac{1}{7}$ and assume that 1 mℓ of liquid occupies 1 cm^3.]

19 A large cake is divided into n pieces, each of mass m grams.
(a) Which of the following describes the relation between m and n?

(i) $m \propto n$ (ii) $m \propto \frac{1}{n}$ (iii) $\frac{1}{m} \propto \frac{1}{n}$

(b) If $m = 360$ when $n = 15$, write down a formula connecting m and n.
(c) If the reciprocal of m is 0,002 5, calculate m.

20 A cinema can hold n people. If all the seats are taken, then 540 people are in the cinema.
(a) Represent this information by an inequality in n.
(b) If a school party visits the cinema, the manager insists that there must be one teacher for every 30 students. If a school party of 120 students visits the cinema, how many seats will be left for members of the general public?
(c) Seats normally cost \$3,00 each, but the manager gives a 15% discount for parties of 20 or over. What would be the total bill for the school party in part (b)? (Include all the teachers.)

21 Fig. R6 is a sketch showing the relative positions of Mutare (M) and Harare (H).

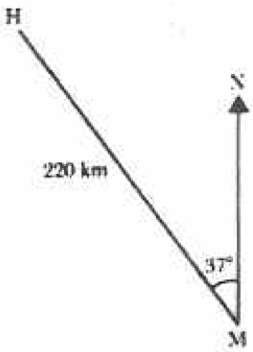


Fig. R6

(a) What is the bearing of Harare from Mutare? (b) What is the bearing of Mutare from Harare? (c) Use Fig. R6 and as much of the information below as is necessary to find how far Harare is north of Mutare (sin 37° = 0,60, cos 37° = 0,80, tan 37° = 0,75).

22 Fig. R7 shows a circle ABCT, centre O with PT a tangent at T.

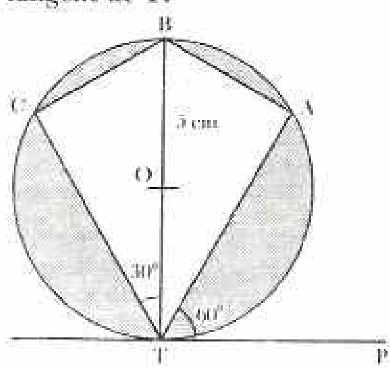


Fig. R7

(a) If $A\hat{T}P = 60°$ and $C\hat{T}B = 30°$, calculate $A\hat{B}C$.

(b) If the radius of the circle is 5 cm, show that the total area of the shaded regions is in the form $k(\pi - \sqrt{3})$. State the value of k.

23 In Fig. R8, $\triangle$ABC is similar to $\triangle$APQ.

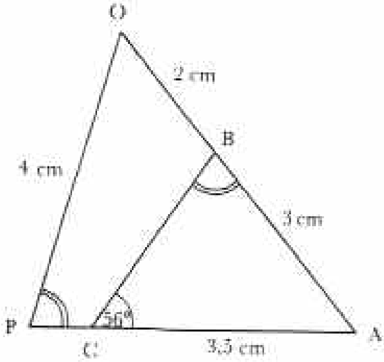


Fig. R8

Given AB = 3 cm, BQ = 2 cm, AC = 3,5 cm, PQ = 4 cm and $B\hat{C}A = 56°$, and as much of the information given below as is necessary, calculate (a) BC, (b) the area of $\triangle$APQ. ($\sin 56° = 0{,}829$, $\cos 56° = 0{,}559$)

24 Solve the equations

(a) $\dfrac{x-1}{12} + \dfrac{x+1}{3} = \dfrac{2x-1}{4}$

(b) $(x+2)^2 = (2x-5)^2$.

25 Table R2 shows the number of fish caught by a fisherman in one week.

Table R2

Sunday	0
Monday	0
Tuesday	8
Wednesday	19
Thursday	48
Friday	0
Saturday	2

For this data, calculate (a) the mode, (b) the median, (c) the mean.

26 A bag contains 5 red discs and 4 blue discs. Two discs are chosen at random, without replacement.

(a) Copy and complete the probability tree diagram in Fig. R9.

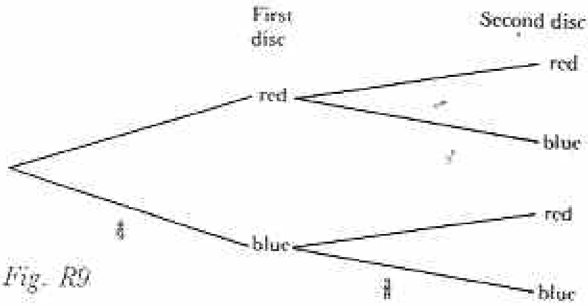


Fig. R9

(b) Find the probability that both discs are red.

27 Here is a number pattern:

$1^3 = 1$
$2^3 = 3 + 5$
$3^3 = 7 + 9 + 11$
$4^3 = 13 + 15 + 17 + 19$
$5^3 =$

(a) Write down the line of numbers for 5^3.

(b) How many numbers will be in the line for 100^3?

(c) Use the pattern to find the value of
$1^3 + 2^3 + 3^3 + 4^3 + 5^3 + 6^3$
without working out each cube separately.

28 The grid in Fig. R10 shows $\triangle$ABC. A′(3; 2) is the image of A(1; 2) after a one-way stretch with the y-axis invariant.

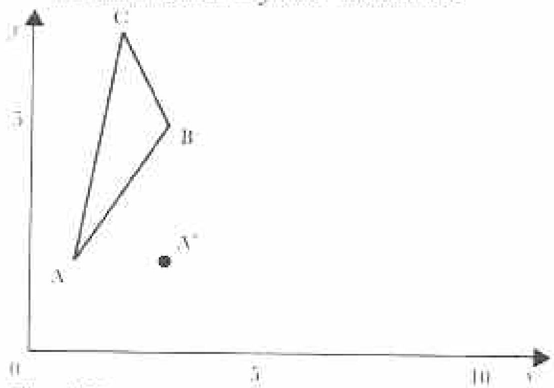


Fig. R10

(a) What is the stretch factor?

(b) What are the coordinates of B′ and C′, the images of B and C under the same stretch?

(c) What is the matrix which represents the stretch?

29 Fig. R11 (overleaf) is a conversion graph which gives the rate of exchange from Z$ to French Francs.

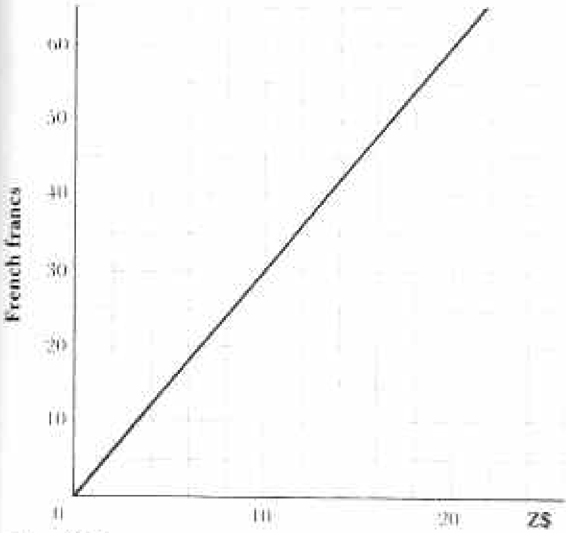


Fig. R11

(a) Use the graph to find
(i) The number of French Francs equivalent to Z$10.
(ii) The cost in Z$ of a watch bought in France for 55 Francs.
(b) A new exchange rate gives Z$1 = 3,25 Francs.
(i) On a copy of the graph, draw a *new line* to represent this.
(ii) Use your line to estimate the new cost in Z$ of the watch above.

Examination 1

Paper 2 (2 h 30 min)

Instructions to candidates:
Answer **all** *the questions in Section A and any* **three** *questions from Section B.*
The intended marks for questions or parts of questions are given in brackets [ ].
All working must be clearly shown. It should be done on the same sheet as the rest of the answer. Omission of essential working will result in loss of marks. If the degree of accuracy is not specified in the question and if the answer is not exact, three-figure accuracy is required.
Mathematical tables [or electronic calculators] may be used to evaluate* **explicit** *numerical expressions.*
Mathematical tables and graph paper are provided.

* The section within the square brackets is omitted in the non-calculator version of the paper.

Section A [64 marks]
Answer **all** *the questions in this section.*

1 (a) Ms Dube saved $5 200 to buy a house. She estimates that she will have to use $500 for legal costs and $800 for moving expenses. She will use the remainder as a deposit for the house.

Table R3

Use this Ready Reckoner to decide which plan is best for you				
	Plan A	**Plan B**	**Plan C**	**Plan D**
Minimum cover you get	$10 000	$20 000	$35 000	$50 000
Double cover if death is accidental	$20 000	$40 000	$70 000	$100 000
Age (nearest)	**Amount you pay monthly**			
20–30	$5,00	$6,00	$7,35	$10,50
31–35	$5,00	$6,20	$9,10	$13,00
36–40	$5,40	$8,40	$12,95	$18,50
41–45	$6,00	$12,00	$19,25	$27,50
46–50	$9,30	$18,60	$30,80	—
51–55	$14,70	$29,40	—	

(i) How much does she have for the deposit? [1]

(ii) A building society offers her a maximum loan of 95% of the cost of the house. The other 5% represents the deposit, which she must pay. What would be the most expensive house she could afford? [4]

(b) Table R3, on page 271, is a ready reckoner which can be used to work out the cost of life insurance premiums.

(i) Ms Dube, who is 29, chooses Plan B. How much is her monthly premium? [1]

(ii) A 46-year-old man wants a minimum cover of $35 000. What total premium will he pay in a year? [2]

(iii) What is the maximum age of a person who can be insured for an accidental death benefit of $100 000? [2]

(iv) A teacher pays an *annual* premium of $109,20. What is the age range of the teacher and what plan was chosen? [2]

2 (a) In Fig. R12, the rectangle on the left is twice the area of that on the right.

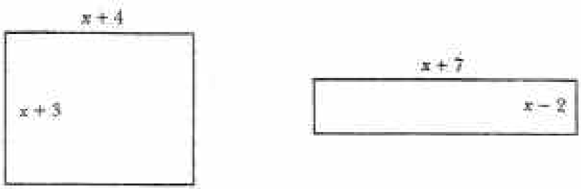


Fig. R12

(i) Form a quadratic equation in x and show that it reduces to $x^2 + 3x - 40 = 0$. [3]

(ii) Solve the equation, stating which solution is realistic in terms of the given data. [3]

(iii) Find the area of the larger rectangle. [1]

(b) When the weather is cold, a strong wind makes it feel colder. The following formula gives the *effective* temperature, e °C, when the *actual* temperature is a °C and the wind is blowing at w km/h:

$$e = \left(1 + \frac{w}{50}\right)a - (3 + \tfrac{3}{5}w)$$

For example, when the wind is 10 km/h the formula simplifies to

$e = 1{,}2a - 9$

(i) Find the simplified formula when the wind is 25 km/h. [2]

(ii) If the actual temperature is 7 °C when the wind speed is 30 km/h, find the effective temperature. [3]

3 In Fig. R13, ABCD is a parallelogram in which AB = 7 cm, BC = 10 cm and diagonal AC = 9 cm.

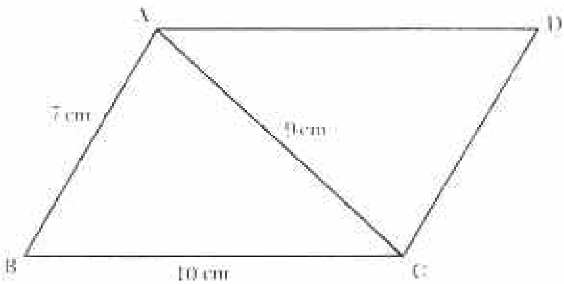


Fig. R13

Calculate:

(a) $B\hat{A}C$, [4]

(b) the perpendicular distance between AB and DC. [3]

4 (a) Ice lollipops are made of frozen fruit juice. They are in the shape of a triangular prism 11 cm long whose regular cross-section is an equilateral triangle of side 2,5 cm.

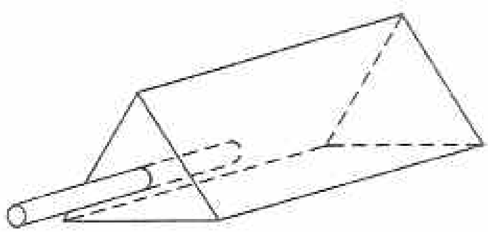


Fig. R14

Calculate

(i) the area of the triangular cross-section, [3]

(ii) the volume of the prism, [1]

(iii) the volume of frozen fruit juice if the stick takes up 4 cm^3 of the prism, [1]

(iv) the volume of *liquid* fruit juice required to make one lollipop if the fruit juice increases in volume by 6% when it is frozen. [3]

(b) A model ship is made to a scale of 1:200.

(i) The mast of the model is 9 cm high. What is the height of the mast on the ship in metres? [2]

(ii) The volume of the part of the ship below the water line is $3{,}2 \times 10^4$ m^3. Calculate the volume of the corresponding part on the model in cm^3. [3]

5 (a) In Fig. R15, tangent PQ touches circle ABT at T. $O\hat{A}B = 12°$ and BO ∥ TA.

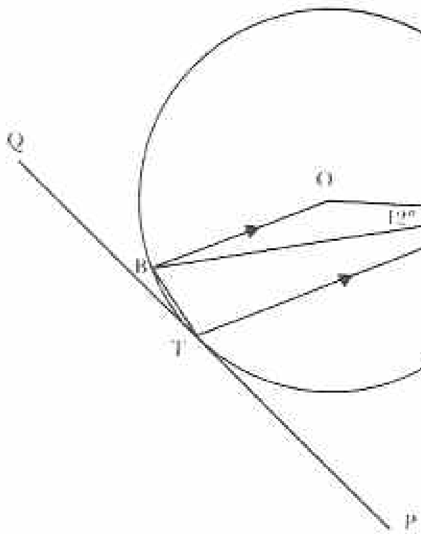


Fig. R15

Calculate, giving reasons,

(i) $O\hat{B}A$, [1]

(ii) reflex $B\hat{O}A$, [2]

(iii) $A\hat{T}B$, [1]

(iv) $Q\hat{T}B$, [2]

(v) $A\hat{T}P$. [1]

(b) In Fig. R16, **OA** = 3**a**, **OB** = 4**b**, OP:PA = 2:1 and OQ:QB = 3:1. M is the mid-point of PQ and N is the mid-point of AB.

(i) Express **OP** in terms of **a**. [1]

(ii) Express **QP** in terms of **a** and **b**. [2]

(iii) Find **MN** in terms of **a** and **b**. [3]

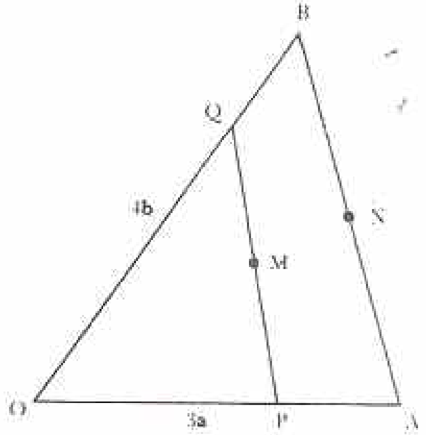


Fig. R16

6 The coordinates of the vertices of △ABC are A(0; 0), B(2; 0) and C(1; 2). $\triangle A_1B_1C_1$ is the image of △ABC under the transformation T_1 whose matrix is $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.

(a) Find the coordinates of A_1, B_1 and C_1. [2]

$\triangle A_2B_2C_2$ is the image of $\triangle A_1B_1C_1$ under transformation T_2 whose matrix is $\begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$.

(b) Find the coordinates of A_2, B_2, C_2. [2]

(c) Find the matrix of the single transformation under which $\triangle A_2B_2C_2$ is the image of △ABC. [3]

Section B [36 marks]

Answer **three** *questions in this section.*
Each question carries 12 marks.

7 A manager and a craftsman together form a small Co-operative making two models of chairs: Standards and Specials. The craftsman works for not more than 42 hours per week. Due to administration commitments, the manager cannot spend more than 34 hours a week in the workshop making chairs. The Standard chair requires 1 hour's work by the craftsman and 2 by the manager. The Special chair requires 3 hour's work by the craftsman and 1 by the manager. At least 6 of the Standard and 6 of the Special models are made each week.

(a) If, in a week, x Standard and y Special chairs are made, express the above information in the form of four inequalities in x and y. [4]

(b) Taking scales of 2 cm to 10 models on each axis, draw these inequalities. Show the region in which x and y are valid by leaving it clear. [4]

(c) If the profit on a Standard chair is $24 and on a Special chair is $40, find

(i) how many of each model should be made each week in order to obtain the greatest profit. [3]

(ii) the profit in this case. [1]

8 21 people work in an office. Fig. R17 gives some details about the people. In the Venn diagram,

A = {audio-typists}
S = {shorthand-typists}
W = {word-processing users}

Each person is a member of at least one of these sets.

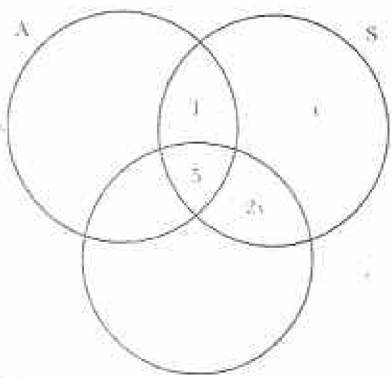


Fig. R17

(a) 15 people can do shorthand-typing. Find x. [2]

(b) 4 people cannot do either audio-typing or shorthand-typing. How many can do audio-typing? [4]

(c) No one does only audio-typing. Find the number of people who can

(i) use a word-processor, [2]

(ii) use a word-processor *and* do audio-typing. [2]

(d) Make a copy of the Venn diagram and shade the region $W \cap A' \cap S'$. [1]
Write a brief description of this set. [1]

9 VABCD is a right pyramid with a square base ABCD of side 5 cm. Each of the pyramid's four triangular faces is inclined at 75° to the base. Calculate

(a) the perpendicular height of the pyramid [2

(b) the length of the slant edge VA, [4

(c) the angle that VA makes with th horizontal, [2

(d) the total surface area of the pyramid. [4

10 The container of a petrol lorry is a cylinde with its axis horizontal. Its internal lengt is 7 m and its internal diameter is 3 m.
Fig. R18 shows a vertical section of th cylinder when the maximum depth of petro it contains is 0,75 m. AB indicates the uppe surface of the petrol.

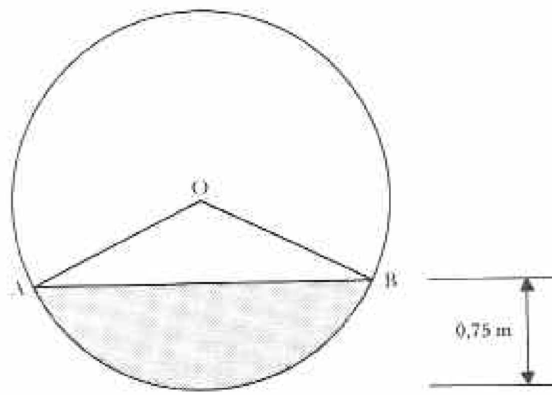


Fig. R18

Calculate

(a) the angle AOB, where O is the cent of the circular section, [

(b) the area of sector AOB, [

(c) the shaded area in Fig. R18, [

(d) the mass of petrol in the lorry, if 1 r of petrol has a mass of 700 kg. [

11 The chest measurements of 100 people we found to the nearest cm. Table R4 gives t result of the survey.

(a) Make a cumulative frequency tal which shows the numbers of people w chest measurements of 80 cm or less, 85 or less, 90 cm or less, ..., 115 cm or less.]

Table R4

chest (cm)	76–80	81–85	86–90	91–95	96–100	101–105	106–111	111–115
frequency	3	10	14	20	23	15	9	6

(b) Using a horizontal scale of 2 cm to represent 5 cm of chest measurement and a vertical scale of 2 cm to 10 people, draw a cumulative frequency graph of the survey measurements from 80 cm to 115 cm. [4]
(c) Use the graph to estimate the median chest measurement. [2]
(d) Use the graph to estimate the number of people with a chest measurement less than or equal to 103 cm. [1]
(e) Two people are chosen at random from those surveyed. What is the probability that both people have chest measurements greater than 100 cm? [3]

12 Given $y = \frac{9}{x}$, (a) copy and complete Table R5, giving values of y to 2 decimal places where necessary. [2]

Table R5

x	1	2	3	4	5	6	7	8	9
y	9	4,5		2,25				1,13	1

(b) Using scales of 1 cm to 1 unit on both axes, draw the graph of $y = \frac{9}{x}$ for $1 \leqslant x \leqslant 9$. [4]
(c) On the same axes draw the graph of $y = 8 - x$. [2]
(d) Find the values of x at the points where the straight line meets the curve. [2]
(e) Of which quadratic equation are the above values the solutions? Give your answer in the form $ax^2 + bx + c = 0$. [2]

Examination 2

Paper 1 (2 h 30 min)

Instructions to candidates:
All *questions may be attempted.*
*Answers are to be written in the spaces provided.**
If working is needed for any question, it must be shown in the space below that question.
Omission of essential working will result in loss of marks.
Questions 1 to 19 carry 3 marks each.
Questions 20 to 27 carry 4 marks each.
Question 28 carries 5 marks.
Question 29 carries 6 marks.

* The examination paper contains spaces for working and answers. To save space in this book, these spaces have been omitted.

NEITHER MATHEMATICAL TABLES NOR SLIDE RULES NOR CALCULATORS MAY BE USED IN THIS PAPER.

1 Express 60 300
(a) to two significant figures,
(b) in standard form,
(c) as a product of prime factors.

2 Evaluate
(a) $(\frac{3}{4} - \frac{2}{3}) \times 1\frac{1}{5}$
(b) $\frac{0,6 + 0,75}{0,6 \times 0,75}$.

3 If $x = 4$ and $y = -1$, evaluate
(a) $x + 4y$, (b) $x^2 - y^2$, (c) $(x - y)^2$.

4 Show the solution set of the inequality
$$3(2 + 3x) < 2x - 1$$
on a copy of the number line in Fig. R19.

Fig. R19

5 (a) factorise $a^2 + 5ax - 3ax - 15x^2$.
(b) Express m dollars and n cents in cents.

6 $\triangle$ABC is right-angled at B. X is a point on AB such that AX = XC. If $\hat{ACB} = 71°$ calculate (a) $\hat{BAC}$, (b) $\hat{BXC}$.

7 Each of the small circles in Fig. R20 is of radius r cm.

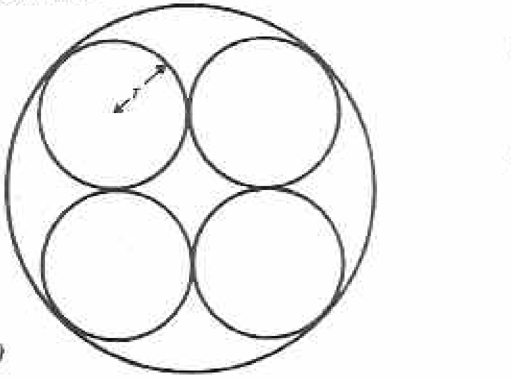


Fig. R20

(a) Express the radius of the large circle in terms of r.
(b) Express the ratio

$$\frac{\text{area of large circle}}{\text{area of one small circle}}$$

in its simplest form.

8 Write down the positive square roots of the following.
(a) 0,002 5 (b) $7\frac{1}{9}$ (c) $25d^{36}$

9 Two geometrically similar tins have heights of 7 cm and 21 cm. If the smaller tin holds 250 g of milk powder, how many kg does the larger one hold?

10 Find the value of
(a) $32^{\frac{3}{5}}$; (b) $0{,}04^{-1\frac{1}{2}}$; (c) $25^{\frac{1}{2}} \times 8^{-\frac{2}{3}}$.

11 In a family, the average mass of the five children is 34 kg, the mother is 59 kg and the father is 72 kg. Calculate the average mass of all 7 members of the family.

12 A jug of water has a mass of 3,9 kg when full and 1,5 kg when quarter full. What is the mass of the jug when empty?

13

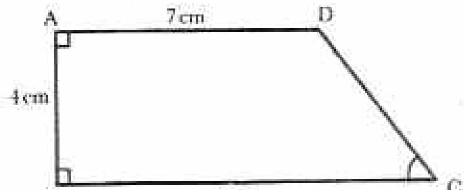


Fig. R21

Given Fig. R21, (a) calculate CD, (b) find the value of cos $B\hat{C}D$, expressing your answer as a decimal.

14 Given that x is an even integer, find the values of x which satisfy both $x \geqslant 4$ and $3x + 7 > 10$.

15 A hall contains 175 people. 12% of them are children and there are 56 men. How many women are in the hall?

16 If $c = ab - \frac{b}{a}$,
(a) find c when $a = \frac{2}{3}$ and $b = -6$,
(b) express b in terms of a and c.

17 Fig. R22 is a regular pentagram, i.e. a fully symmetrical five-pointed star as shown.

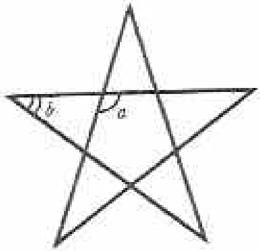


Fig. R22

Calculate the sizes of the angles marked a and b.

18 In Fig. R23, BC = 10 cm, $\hat{A} = 40°$, $\hat{C} = 50°$.

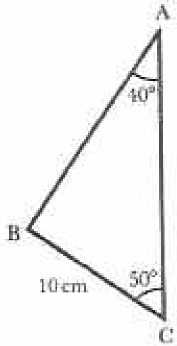


Fig. R23

Use as much of the given information as is necessary to calculate (a) AB, (b) AC.
(sin 40° = cos 50° = 0,64
cos 40° = sin 50° = 0,77
tan 40° = 0,84, tan 50° = 1,19)

19 A machine part is made of metal. The metal is a mixture of copper and zinc whose masses are in the ratio 13:7. If there are 147 g of zinc in the machine part, what is its total mass?

20 Given that $\mathscr{E} = \{1; 2; 3; \ldots; 9; 10\}$
S = {perfect squares}
T = {1; 3; 6; 10},
(a) Mark the members of these sets on a copy of the Venn diagram in Fig. R24.

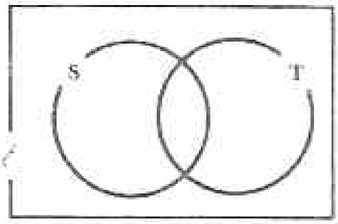


Fig. R24

(b) Using your diagram or otherwise (i) list the members of $S' \cap T'$, (ii) find $n(T' \cup S)$.

21 In Fig. R25, **AC** represents the column vector $\begin{pmatrix} 2 \\ -4 \end{pmatrix}$.

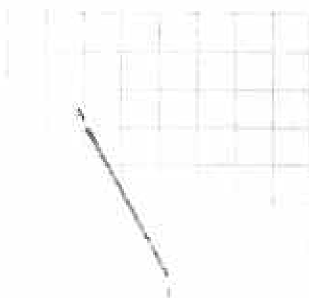


Fig. R25

AC is a diagonal of a square ABCD such that $\mathbf{AD} = \begin{pmatrix} p \\ q \end{pmatrix}$ where p and q are both negative.

(a) Sketch the square ABCD on a copy of Fig. R25.

(b) Write down the values of p and q.

(c) Write down **AB** in column vector form.

(d) Write down **BD** in column vector form.

22 $\begin{pmatrix} \frac{4x}{1-3x} & x-5 \\ & x \end{pmatrix}$ is the inverse matrix of $\begin{pmatrix} x & 5-x \\ 3x-1 & 4x \end{pmatrix}$.

Find two values of x for which this is true.

23 A straight line passes through the points (5; 0) and (−4; 3). Find (a) the gradient of the line, (b) its equation, (c) the coordinates of the point where it cuts the line $x = 7$.

24 In Fig. R26, ABCD is a trapezium such that $A\hat{D}B = B\hat{C}D$, AB = 4 cm and BD = 6 cm.

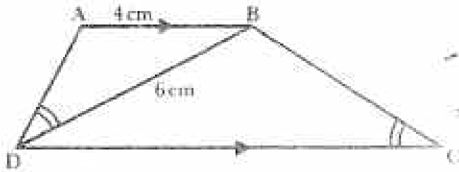


Fig. R26

(a) Calculate the length of CD.

(b) Find the ratio $\frac{\text{area of } \triangle ABD}{\text{area of } \triangle BCD}$.

25 In Fig. R27, O is the centre of the circle APB and $O\hat{A}B = 34°$.

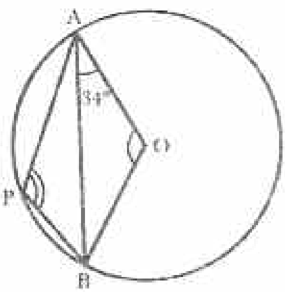


Fig. R27

Find the size of (a) $A\hat{O}B$, (b) $A\hat{P}B$.

26 Fig. R28 shows a shape S drawn on a cartesian plane.

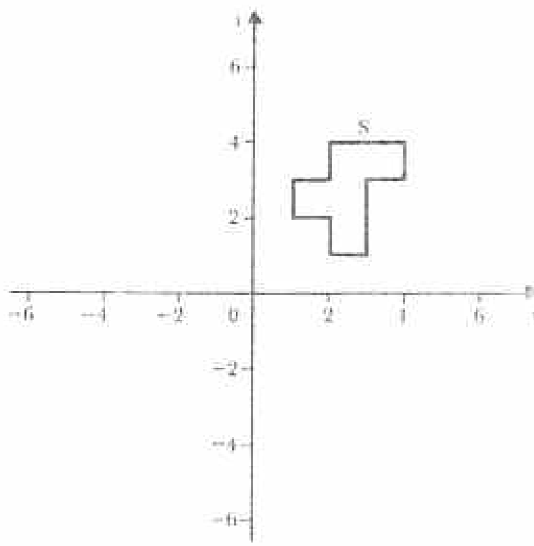


Fig. R28

On a copy of Fig. R28, draw the image of S under a

(a) translation by vector $\begin{pmatrix} -7 \\ 1 \end{pmatrix}$ (label the image T),

(b) reflection in the line $y = -x$ (label the image M),

(c) clockwise rotation of 90° about the point (0; −2) (label the image R).

27 y varies inversely as the positive square root of x and $y = 12$ when $x = \frac{4}{9}$.

(a) Express y in terms of x.

(b) Find y when $x = 12\frac{1}{4}$.

(c) Find n such that $x = y = n$.

28 The curve in Fig. R29 is the graph of $y = 7 - x - x^2$ for values of x from −4 to 3. The region above the x-axis between $x = -3$ and $x = 2$ has been shaded.

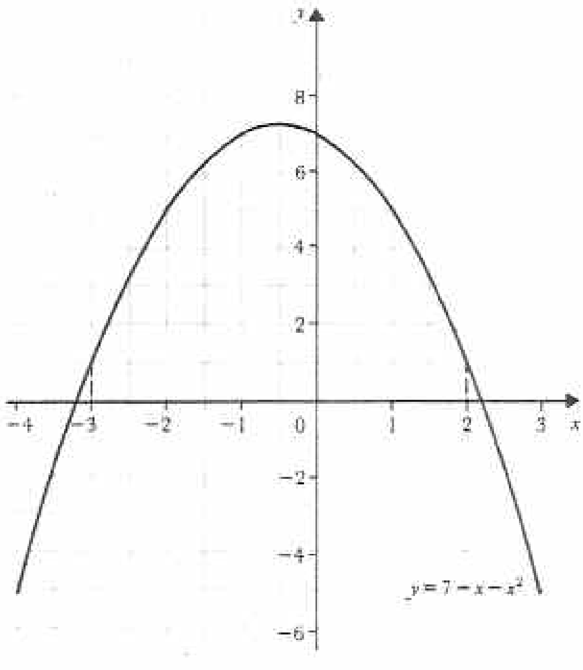


Fig. R29

Estimate

(a) the greatest value of $7 - x - x^2$,

(b) the solution of the equation $7 - x - x^2 = 0$,

(c) the area of the shaded region.

29 Fig. R30 is the speed–time graph of a car journey.

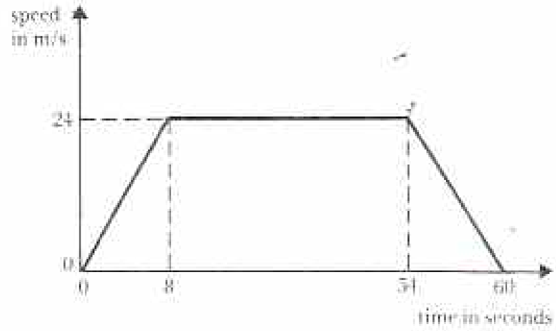


Fig. R30

Calculate

(a) the total distance travelled during the 60 seconds,

(b) the acceleration of the car during the first 8 s,

(c) the speed of the car at the end of the 56th second.

Examination 2

Paper 2 (2 h 30 min)

Instructions to candidates:

Answer **all** *the questions in Section A and any* **three** *questions from Section B.*

The intended marks for questions or parts of questions are given in brackets [ ].

All working must be clearly shown. It should be done on the same sheet as the rest of the answer. Omission of essential working will result in loss of marks. If the degree of accuracy is not specified in the question and if the answer is not exact, three figure accuracy is required.

Mathematical tables [or electronic calculators] may be used to evaluate* **explicit** *numerical expressions.*

Mathematical tables and graph paper are provided.

* The section within the square brackets is omitted in the non-calculator version of the paper.

Section A [64 marks]

Answer **all** *the questions in this section.*

1 (a) A farmer has 350 ha of land. After selling some land he is left with 329 ha. What percentage of his land did he sell? [2]

(b) $M = \begin{pmatrix} 4 & 0 \\ -2 & 2 \end{pmatrix}$ and $N = \begin{pmatrix} 1 & -5 \\ 1 & -2 \end{pmatrix}$.

Find

(i) MN [2]

(ii) NM [2]

(iii) x and y if

$$M\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -2 \\ 7 \end{pmatrix}.$$ [4]

2 (a) In Fig. R31, O is the centre of circle PQRS, PQ is a diameter, QR = RS and $P\hat{S}O = 42°$.

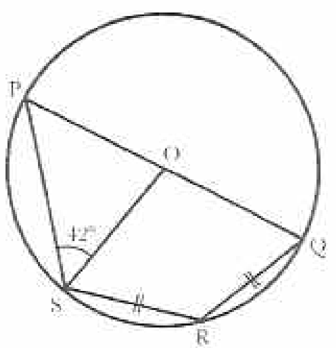


Fig. R31

Calculate the angles of quadrilateral PQRS. [4]

(b) The internal radius and height of a cylindrical oil drum are 30 cm and 90 cm respectively. The drum contains oil to a depth of 70 cm. Use the value 3.142 for π to calculate

(i) the area of the cylinder which is in contact with the oil, [4]

(ii) the extra volume of oil required to fill the drum to the top. [4]

3 (a) Simplify $\dfrac{2}{x+2} - \dfrac{5}{3x-1}$. [3]

(b) Solve $\dfrac{2}{x+2} = \dfrac{5}{3x-1}$. [3]

(c) Factorise $16a^2 - 9$. Hence or otherwise express 1591 as the product of two prime numbers. [4]

4 (a) In Fig. R32, O is the centre of rhombus PQRS, **PQ** = **a** and **PS** = **b**.

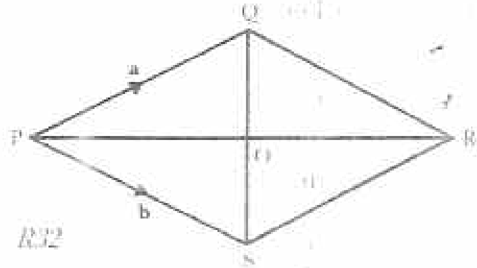


Fig. R32

Write down expressions for the following in terms of **a** and **b**.

(i) **PR** [2]

(ii) **OP** [2]

(iii) **OQ** [3]

(b) X sold an article to Y at a profit of 20%. Y then sold it to Z at a loss of 20% of what it cost her. Calculate the ratio *final price*:*original price* in its simplest form. [3]

5 (a) A triangle has sides of length x cm, $(2x-1)$ cm and $(2x+1)$ cm. If its perimeter is 40 cm,

(i) find the lengths of the sides of the triangle; [3]

(ii) state the size of the largest angle of the triangle, giving a reason; [2]

(iii) calculate the size of the smallest angle of the triangle. [3]

(b) Find the value of $\dfrac{2\sqrt{a}}{b}$ when

$a = 1.21 \times 10^{-4}$, $b = 4.4 \times 10^{-6}$, giving the answer in standard form. [4]

6 (a) In a lifting mechanism, the relation between the load W tonnes and the effort P kg is of the form $P = aW + b$, where a and b are constants. Table R6 gives the results of a lifting experiment.

Table R6

load, W tonnes	2	5
effort, P kg	2.32	3.19

(i) Find a and b and the equation connecting P and W. [3]

(ii) Sketch a graph of the relation showing where the line cuts the axes. [3]

(b) Find the values of d for which $(d-2)^2 : (d+3)^2 = 1:4$. [4]

Section B [36 marks]

Answer three questions in this section.
Each question carries 12 marks.

7 (a) The vertex V of a pyramid is vertically above the centre, O, of its rectangular base PQRS. If PQ = 9 cm, QR = 12 cm and VO = 6 cm, calculate

(i) $V\hat{P}O$, [3]

(ii) the angle between the faces PVS and QVR. [3]

(b) The matrix $\begin{pmatrix} -3 & 0 \\ 0 & 1 \end{pmatrix}$ represents a transformation C.

(i) Find the image of (2; 7) under C. [2]

(ii) If (6; −1) is the image of point (r; s) under C, find r and s. [2]

(iii) Describe C as completely as possible. [2]

8 The hourly earnings of 50 people are given in Table R7.

Table R7

earnings per hour (cents)	number of people (frequency)
151–175	1
176–200	4
201–225	17
226–250	15
251–275	11
276–300	2

(a) What is the modal class of hourly earnings? [1]

(b) Use mid-values to calculate the mean hourly earnings. [2]

(c) Construct a cumulative frequency table and draw the corresponding cumulative frequency curve. Use this to estimate the median hourly earnings. [5]

(d) One person is chosen at random. What is the probability that this person earns less than \$2,26 per hour? [1]

(e) Two people are chosen at random from the 50. Find the probability that

(i) both are in the 226–250 group. [1]

(ii) one is in that group and the other is n

9 Shapes P, Q, R and S are as given in Fig. R

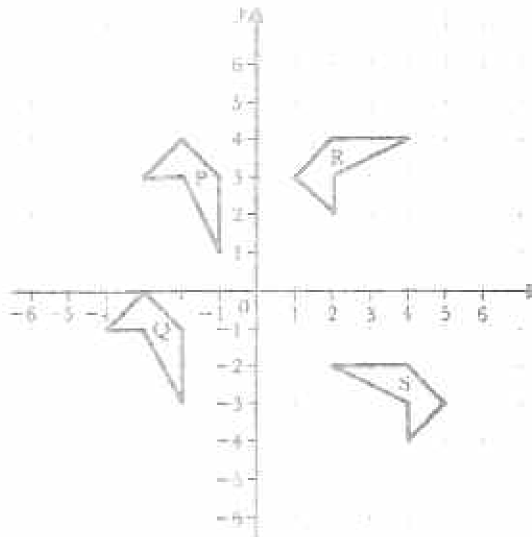


Fig. R33

(a) Write down the column vector rep senting the single translation which map onto Q.

(b) R is the image of P under a clockw rotation. Find

(i) the angle of rotation,

(ii) the coordinates of the centre of rotatio

(c) S is a reflection of P in a line *m*. Fi the equation of *m*.

(d) If P is transformed by a shear represen by the matrix $\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$,

(i) state the equation of the invariant li

(ii) find the coordinates of the vertices the image of P.

10 Fig. R34 represents a field ABCD.

(a) Use the given dimensions to calcula

(i) BD,

(ii) BC,

(iii) the area of the field in m².

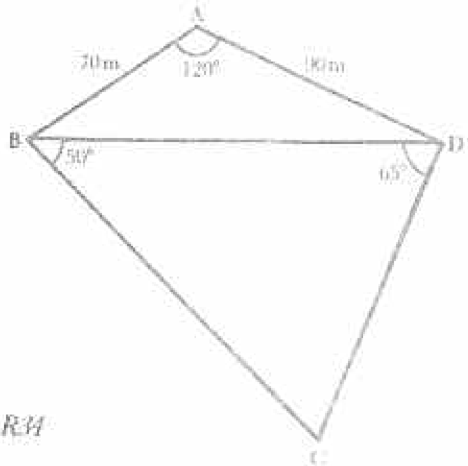


Fig. R34

(b) If D is due East of B, find the bearing of
(i) B from C, [2]
(ii) C from B, [1]
(iii) A from B. [2]

11 An object moves in a straight line so that its velocity v m/s after t seconds is given by

$$v = 2t^2 - 5t + 8.$$

Table R8 gives some corresponding values of v and t.

Table R8

t	0	1	2	3	4	5	6
v	8		6		20		50

(a) Copy and complete Table R8 by finding the values of v which correspond to $t = 1; 3; 5$. [3]

(b) Using 2 cm to 1 second horizontally and 5 m/s vertically, draw the graph of $v = 2t^2 - 5t + 8$ for the range $6 \geqslant t \geqslant 0$. [4]

(c) Use your graph to estimate
(i) the period of time during which the object is decelerating, [2]
(ii) the velocity when the acceleration is zero, [1]
(iii) the acceleration after 4 seconds. [2]

12 *Answer this question on a sheet of plain paper.*
Construct triangle ABC so that
BC = 10 cm, $A\hat{B}C = 60°$ and $B\hat{C}A = 45°$.
(a) On the diagram write the length of AC. [3]
(b) Draw the circumcircle of triangle ABC. [2]
(c) Construct the locus of a set of points which are equidistant from A and B. [2]
(d) Hence mark a point P such that $A\hat{P}B = 45°$ *and* AP = PB [2]
(e) Mark a point Q such that $A\hat{Q}B = 45°$ and AB = AQ. [2]
(f) Measure PQ. [1]

Mensuration tables and formulae, four-figure tables

SI units

Mass

The **gram** is the basic unit of mass.

unit	abbreviation	basic units
1 kilogram	1 kg	1 000 g
1 hectogram	1 hg	100 g
1 decagram	1 dag	10 g
1 gram	1 g	1 g
1 decigram	1 dg	0,1 g
1 centigram	1 cg	0,01 g
1 milligram	1 mg	0,001 g

The tonne (t) is used for large masses. The most common measures of mass are the milligram, the gram, the kilogram and the tonne.

1 g = 1 000 mg
1 kg = 1 000 g = 1 000 000 mg
1 t = 1 000 kg = 1 000 000 g

Time

The **second** is the basic unit of time.

unit	abbreviation	basic units
1 second	1 s	1 s
1 minute	1 min	60 s
1 hour	1 h	3 600 s

Length

The **metre** is the basic unit of length.

unit	abbreviation	basic unit
1 kilometre	1 km	1 000 m
1 hectometre	1 hm	100 m
1 decametre	1 dam	10 m
1 metre	1 m	1 m
1 decimetre	1 dm	0,1 m
1 centimetre	1 cm	0,01 m
1 millimetre	1 mm	0,001

The most common measures are the millimetr the metre and the kilometre.

1 m = 1 000 mm
1 km = 1 000 m = 1 000 000 mm

Area

The **square metre** is the basic unit of are Units of area are derived from units of lengt

unit	abbreviation	relation to othe units of area
square millimetre	mm^2	
square centimetre	cm^2	$1\ cm^2 = 100\ mm^2$
square metre	m^2	$1\ m^2 = 10\,000\ cm^2$
square kilometre	km^2	$1\ km^2 = 1\,000\,000$
hectare (for land measure)	ha	$1\ ha = 10\,000\ m^2$

Volume

The **cubic metre** is the basic unit of volume. Units of volume are derived from units of length.

unit	abbreviation	relation to other units of volume
cubic millimetre	mm^3	
cubic centimetre	cm^3	$1\ cm^3 = 1\,000\ mm^3$
cubic metre	m^3	$1\ m^3 = 1\,000\,000\ cm^3$

Capacity

The **litre** is the basic unit of capacity. 1 litre takes up the same space as $1\,000\ cm^3$.

unit	abbreviation	relation to other units of capacity	relation to units of volume
millilitre	$m\ell$		$1\ m\ell = 1\ cm^3$
litre	ℓ	$1\ \ell = 1\,000\ m\ell$	$1\ \ell = 1000\ cm^3$
kilolitre	$k\ell$	$1\ k\ell = 1\,000\ \ell$	$1\ k\ell = 1\ m^3$

Money

Some African currencies

Zimbabwe	100 cents (c)	= 1 Dollar ($)
Botswana	100 thebe (t)	= 1 Pula (P)
Kenya	100 cents (c)	= 1 Shilling (Sh)
Malawi	100 tambala (t)	= 1 Kwacha (K)
Mozambique	100 centavos (c)	= 1 Meticai (M)
Nigeria	100 kobo (k)	= 1 Naira (₦)
Zambia	100 ngwee (n)	= 1 Kwacha (K)

Other currencies

Britain	100 pence (p) = 1 Pound (£)
USA	100 cents (c) = 1 Dollar ($)

Exchange rates

At the time of going to press, $1 (Zimbabwe) was equivalent to the following.

US dollar	$0,50
UK sterling	£0,30
Botswana	P1,05
Kenya	Sh11,00
Mozambique	M318,00
Zambia	K5,20

Note: Exchange rates change from day to day. The above rates may be taken only as approximate.

The calendar

Remember this poem:

Thirty days has September,
April, June and November.
All the rest have thirty-one,
Excepting February alone;
This has twenty-eight days clear,
And twenty-nine in each Leap Year.

For a Leap Year, the date must be divisible by 4. Thus 1964 was a Leap Year.
Century year dates, such as 1900 and 2000, are Leap Years only if they are divisible by 400. Thus 1900 was not a Leap Year but 2000 will be a Leap Year.

Multiplication table

×	1	2	3	4	5	6	7	8	9	10
1	1	2	3	4	5	6	7	8	9	10
2	2	4	6	8	10	12	14	16	18	20
3	3	6	9	12	15	18	21	24	27	30
4	4	8	12	16	20	24	28	32	36	40
5	5	10	15	20	25	30	35	40	45	50
6	6	12	18	24	30	36	42	48	54	60
7	7	14	21	28	35	42	49	56	63	70
8	8	16	24	32	40	48	56	64	72	80
9	9	18	27	36	45	54	63	72	81	90
10	10	20	30	40	50	60	70	80	90	100

Divisibility tests

Any whole number is exactly divisible by

- 2 if its last digit is even
- 3 if the sum of its digits is divisible by 3
- 4 if its last two digits form a number divisible by 4
- 5 if its last digit is 5 or 0
- 6 if its last digit is even and the sum of its digits is divisible by 3
- 8 if its last three digits form a number divisible by 8
- 9 if the sum of its digits is divisible by 9
- 10 if its last digit is 0

Measuration formulae

Plane shapes

	perimeter	area
square side s	$4s$	s^2
rectangle length l, breadth b	$2(l+b)$	lb
triangle base b, height h		$\frac{1}{2}bh$
parallelogram base b, height h		bh
trapezium height h, parallels a and b		$\frac{1}{2}(a+b)h$
circle radius r	$2\pi r$	πr^2
sector of circle radius r, angle θ	$2r + \frac{\theta}{360}2\pi r$	$\frac{\theta}{360}\pi r^2$

Solid shapes

	surface area	volum
cube edge s	$6s^2$	s^3
cuboid length l, breadth b, height h	$2(lb + bh + lh)$	lbh
prism height h, base area A		Ah
cylinder radius r, height h	$2\pi rh + 2\pi r^2$	πr^2
cone radius r, slant height l, height h	$\pi rl + \pi r^2$	$\frac{1}{3}\pi r$
sphere radius r	$4\pi r^2$	$\frac{4}{3}\pi r$

Trigonometrical formulae

Right-angled triangles

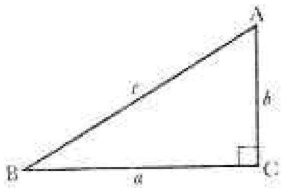


Fig. T1

In the right-angled triangle shown,
$c^2 = a^2 + b^2$ (*Pythagoras' theorem*)

$\tan B = \frac{b}{a}$ $\tan A = \frac{a}{b}$

$\sin B = \frac{b}{c}$ $\sin A = \frac{a}{c}$

$\cos B = \frac{a}{c}$ $\cos A = \frac{b}{c}$

Obtuse angles

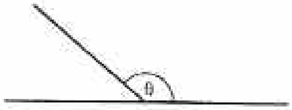


Fig. T2

$\sin\theta = \sin(180^\circ - \theta)$
$\cos\theta = -\cos(180^\circ - \theta)$
$\tan\theta = -\tan(180^\circ - \theta)$

Any triangle

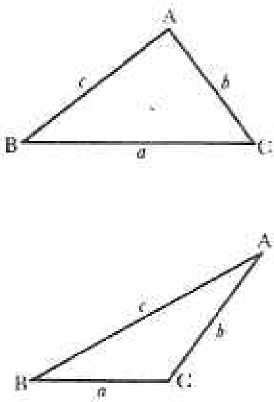


Fig. T3

In both triangles,

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} \qquad \textit{(sine rule)}$$

$$c^2 = a^2 + b^2 - 2ab\cos C \qquad \textit{(cosine rule)}$$
$$b^2 = a^2 + c^2 - 2ac\cos B$$
$$a^2 = b^2 + c^2 - 2bc\cos A$$

and

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

$$\cos B = \frac{c^2 + a^2 - b^2}{2ca}$$

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab}$$

Symbols

symbol	**meaning**
$=$	is equal to
$\neq$	is not equal to
$\simeq$	is approximately equal to
$\equiv$	is identical or congruent to
$\Rightarrow$	leads to
$\Leftrightarrow$	is equivalent to
$\propto$	is proportional to
$>$	is greater than
$<$	is less than
$\geqslant$	is greater than or equal to
$\leqslant$	is less than or equal to
$^\circ$	degree (angle)
$^\circ$C	degree Celsius (temperature)
A, B, C,	points (geometry)
AB	the line through points A and B, *or* the distance between points A and B
$\triangle$ABC	triangle ABC
$\parallel^{gm}$ABCD	parallelogram ABCD
$A\hat{B}C$	angle ABC
$\perp$	is perpendicular to
$\parallel$	is parallel to
π	pi
%	per cent
$A = \{p; q; r\}$	A is the set p, q, r
$B = \{1; 2; 3; \ldots\}$	B is the infinite set 1, 2, 3 and so on
$C = \{x : x$ is an integer$\}$	Set builder notation. C is the set of numbers x such that x is an integer
n(A)	number of elements in set A
$\in$	is an element of
$\notin$	is not an element of
A′	complement of A
$\{\ \}$ or $\varnothing$	the empty set
$\mathscr{E}$	the universal set
$A \subseteq B$	A is a subset of B
$A \supseteq B$	A contains B
$\nsubseteq$, $\nsupseteq$	negations of $\subseteq$ and $\supseteq$
$A \cup B$	union of A and B
$A \cap B$	intersection of A and B
a or $\tilde{a}$ or $\overline{a}$	vector **a**
AB or $\overrightarrow{AB}$	vector **AB**
|**AB**|	modulus of **AB**

Logarithms

$x \to \log x$

x	0	1	2	3	4	5	6	7	8	9	Diff. 1	2	3	4	5	6	7	8	9
10	0000	0043	0086	0128	0170	0212	0253	0294	0334	0374	4	8	12	17	21	25	29	33	37
11	0414	0453	0492	0531	0569	0607	0645	0682	0719	0755	4	8	11	15	19	23	26	30	34
12	0792	0828	0864	0899	0934	0969	1004	1038	1072	1106	3	7	10	14	17	21	24	28	31
13	1139	1173	1206	1239	1271	1303	1335	1367	1399	1430	3	6	10	13	16	19	23	26	29
14	1461	1492	1523	1553	1584	1614	1644	1673	1703	1732	3	6	9	12	15	18	21	24	27
15	1761	1790	1818	1847	1875	1903	1931	1959	1987	2014	3	6	8	11	14	17	20	22	25
16	2041	2068	2095	2122	2148	2175	2201	2227	2253	2279	3	5	8	11	13	16	18	21	24
17	2304	2330	2355	2380	2405	2430	2455	2480	2504	2529	2	5	7	10	12	15	17	20	22
18	2553	2577	2601	2625	2648	2672	2695	2718	2742	2765	2	5	7	9	12	14	16	19	21
19	2788	2810	2833	2856	2878	2900	2923	2945	2967	2989	2	4	7	9	11	13	16	18	20
20	3010	3032	3054	3075	3096	3118	3139	3160	3181	3201	2	4	6	8	11	13	15	17	19
21	3222	3243	3263	3284	3304	3324	3345	3365	3385	3404	2	4	6	8	10	12	14	16	18
22	3424	3444	3464	3483	3502	3522	3541	3560	3579	3598	2	4	6	8	10	12	14	15	17
23	3617	3636	3655	3674	3692	3711	3729	3747	3766	3784	2	4	6	7	9	11	13	15	17
24	3802	3820	3838	3856	3874	3892	3909	3927	3945	3962	2	4	5	7	9	11	12	14	16
25	3979	3997	4014	4031	4048	4065	4082	4099	4116	4133	2	3	5	7	9	10	12	14	15
26	4150	4166	4183	4200	4216	4232	4249	4265	4281	4298	2	3	5	7	8	10	11	13	15
27	4314	4330	4346	4362	4378	4393	4409	4425	4440	4456	2	3	5	6	8	9	11	13	14
28	4472	4487	4502	4518	4533	4548	4564	4579	4594	4609	2	3	5	6	8	9	11	12	14
29	4624	4639	4654	4669	4683	4698	4713	4728	4742	4757	1	3	4	6	7	9	10	12	13
30	4771	4786	4800	4814	4829	4843	4857	4871	4886	4900	1	3	4	6	7	9	10	11	13
31	4914	4928	4942	4955	4969	4983	4997	5011	5024	5038	1	3	4	6	7	8	10	11	12
32	5051	5065	5079	5092	5105	5119	5132	5145	5159	5172	1	3	4	5	7	8	9	11	12
33	5185	5198	5211	5224	5237	5250	5263	5276	5289	5302	1	3	4	5	6	8	9	10	12
34	5315	5328	5340	5353	5366	5378	5391	5403	5416	5428	1	3	4	5	6	8	9	10	11
35	5441	5453	5465	5478	5490	5502	5514	5527	5539	5551	1	2	4	5	6	7	9	10	11
36	5563	5575	5587	5599	5611	5623	5635	5647	5658	5670	1	2	4	5	6	7	8	10	11
37	5682	5694	5705	5717	5729	5740	5752	5763	5775	5786	1	2	3	5	6	7	8	9	10
38	5798	5809	5821	5832	5843	5855	5866	5877	5888	5899	1	2	3	5	6	7	8	9	10
39	5911	5922	5933	5944	5955	5966	5977	5988	5999	6010	1	2	3	4	5	7	8	9	10
40	6021	6031	6042	6053	6064	6075	6085	6096	6107	6117	1	2	3	4	5	6	8	9	10
41	6128	6138	6149	6160	6170	6180	6191	6201	6212	6222	1	2	3	4	5	6	7	8	9
42	6232	6243	6253	6263	6274	6284	6294	6304	6314	6325	1	2	3	4	5	6	7	8	9
43	6335	6345	6355	6365	6375	6385	6395	6405	6415	6425	1	2	3	4	5	6	7	8	9
44	6435	6444	6454	6464	6474	6484	6493	6503	6513	6522	1	2	3	4	5	6	7	8	9
45	6532	6542	6551	6561	6571	6580	6590	6599	6609	6618	1	2	3	4	5	6	7	8	9
46	6628	6637	6646	6656	6665	6675	6684	6693	6702	6712	1	2	3	4	5	6	7	7	8
47	6721	6730	6739	6749	6758	6767	6776	6785	6794	6803	1	2	3	4	5	5	6	7	8
48	6812	6821	6830	6839	6848	6857	6866	6875	6884	6893	1	2	3	4	4	5	6	7	8
49	6902	6911	6920	6928	6937	6946	6955	6964	6972	6981	1	2	3	4	4	5	6	7	8
50	6990	6998	7007	7016	7024	7033	7042	7050	7059	7067	1	2	3	3	4	5	6	7	8
51	7076	7084	7093	7101	7110	7118	7126	7135	7143	7152	1	2	3	3	4	5	6	7	8
52	7160	7168	7177	7185	7193	7202	7210	7218	7226	7235	1	2	2	3	4	5	6	7	7
53	7243	7251	7259	7267	7275	7284	7292	7300	7308	7316	1	2	2	3	4	5	6	6	7
54	7324	7332	7340	7348	7356	7364	7372	7380	7388	7396	1	2	2	3	4	5	6	6	7
55	7404	7412	7419	7427	7435	7443	7451	7459	7466	7474	1	2	2	3	4	5	5	6	7
56	7482	7490	7497	7505	7513	7520	7528	7536	7543	7551	1	2	2	3	4	5	5	6	7
57	7559	7566	7574	7582	7589	7597	7604	7612	7619	7627	1	2	2	3	4	5	5	6	7
58	7634	7642	7649	7657	7664	7672	7679	7686	7694	7701	1	1	2	3	4	4	5	6	7
59	7709	7716	7723	7731	7738	7745	7752	7760	7767	7774	1	1	2	3	4	4	5	6	7
60	7782	7789	7796	7803	7810	7818	7825	7832	7839	7846	1	1	2	3	4	4	5	6	6
61	7853	7860	7868	7875	7882	7889	7896	7903	7910	7917	1	1	2	3	4	4	5	6	6
62	7924	7931	7938	7945	7952	7959	7966	7973	7980	7987	1	1	2	3	3	4	5	6	6
63	7993	8000	8007	8014	8021	8028	8035	8041	8048	8055	1	1	2	3	3	4	5	5	6
64	8062	8069	8075	8082	8089	8096	8102	8109	8116	8122	1	1	2	3	3	4	5	5	6
65	8129	8136	8142	8149	8156	8162	8169	8176	8182	8189	1	1	2	3	3	4	5	5	6
66	8195	8202	8209	8215	8222	8228	8235	8241	8248	8254	1	1	2	3	3	4	5	5	6
67	8261	8267	8274	8280	8287	8293	8299	8306	8312	8319	1	1	2	3	3	4	5	5	6
68	8325	8331	8338	8344	8351	8357	8363	8370	8376	8382	1	1	2	3	3	4	4	5	6
69	8388	8395	8401	8407	8414	8420	8426	8432	8439	8445	1	1	2	2	3	4	4	5	6
70	8451	8457	8463	8470	8476	8482	8488	8494	8500	8506	1	1	2	2	3	4	4	5	6
71	8513	8519	8525	8531	8537	8543	8549	8555	8561	8567	1	1	2	2	3	4	4	5	5
72	8573	8579	8585	8591	8597	8603	8609	8615	8621	8627	1	1	2	2	3	4	4	5	5
73	8633	8639	8645	8651	8657	8663	8669	8675	8681	8686	1	1	2	2	3	4	4	5	5
74	8692	8698	8704	8710	8716	8722	8727	8733	8739	8745	1	1	2	2	3	4	4	5	5
75	8751	8756	8762	8768	8774	8779	8785	8791	8797	8802	1	1	2	2	3	3	4	5	5
76	8808	8814	8820	8825	8831	8837	8842	8848	8854	8859	1	1	2	2	3	3	4	5	5
77	8865	8871	8876	8882	8887	8893	8899	8904	8910	8915	1	1	2	2	3	3	4	4	5
78	8921	8927	8932	8938	8943	8949	8954	8960	8965	8971	1	1	2	2	3	3	4	4	5
79	8976	8982	8987	8993	8998	9004	9009	9015	9020	9025	1	1	2	2	3	3	4	4	5
80	9031	9036	9042	9047	9053	9058	9063	9069	9074	9079	1	1	2	2	3	3	4	4	5
81	9085	9090	9096	9101	9106	9112	9117	9122	9128	9133	1	1	2	2	3	3	4	4	5
82	9138	9143	9149	9154	9159	9165	9170	9175	9180	9186	1	1	2	2	3	3	4	4	5
83	9191	9196	9201	9206	9212	9217	9222	9227	9232	9238	1	1	2	2	3	3	4	4	5
84	9243	9248	9253	9258	9263	9269	9274	9279	9284	9289	1	1	2	2	3	3	4	4	5
85	9294	9299	9304	9309	9315	9320	9325	9330	9335	9340	1	1	2	2	3	3	4	4	5
86	9345	9350	9355	9360	9365	9370	9375	9380	9385	9390	1	1	2	2	3	3	4	4	5
87	9395	9400	9405	9410	9415	9420	9425	9430	9435	9440	0	1	1	2	2	3	3	4	4
88	9445	9450	9455	9460	9465	9469	9474	9479	9484	9489	0	1	1	2	2	3	3	4	4
89	9494	9499	9504	9509	9513	9518	9523	9528	9533	9538	0	1	1	2	2	3	3	4	4
90	9542	9547	9552	9557	9562	9566	9571	9576	9581	9586	0	1	1	2	2	3	3	4	4
91	9590	9595	9600	9605	9609	9614	9619	9624	9628	9633	0	1	1	2	2	3	3	4	4
92	9638	9643	9647	9652	9657	9661	9666	9671	9675	9680	0	1	1	2	2	3	3	4	4
93	9685	9689	9694	9699	9703	9708	9713	9717	9722	9727	0	1	1	2	2	3	3	4	4
94	9731	9736	9741	9745	9750	9754	9759	9763	9768	9773	0	1	1	2	2	3	3	4	4
95	9777	9782	9786	9791	9795	9800	9805	9809	9814	9818	0	1	1	2	2	3	3	4	4
96	9823	9827	9832	9836	9841	9845	9850	9854	9859	9863	0	1	1	2	2	3	3	4	4
97	9868	9872	9877	9881	9886	9890	9894	9899	9903	9908	0	1	1	2	2	3	3	4	4
98	9912	9917	9921	9926	9930	9934	9939	9943	9948	9952	0	1	1	2	2	3	3	4	4
99	9956	9961	9965	9969	9974	9978	9983	9987	9991	9996	0	1	1	2	2	3	3	3	4

Antilogarithms

$x \rightarrow 10^x$

x	0	1	2	3	4	5	6	7	8	9	Differences 1	2	3	4	5	6	7	8	9
.00	1000	1002	1005	1007	1009	1012	1014	1016	1019	1021	0	0	1	1	1	1	2	2	2
.01	1023	1026	1028	1030	1033	1035	1038	1040	1042	1045	0	0	1	1	1	1	2	2	2
.02	1047	1050	1052	1054	1057	1059	1062	1064	1067	1069	0	0	1	1	1	1	2	2	2
.03	1072	1074	1076	1079	1081	1084	1086	1089	1091	1094	0	0	1	1	1	1	2	2	2
.04	1096	1099	1102	1104	1107	1109	1112	1114	1117	1119	0	1	1	1	1	2	2	2	2
.05	1122	1125	1127	1130	1132	1135	1138	1140	1143	1146	0	1	1	1	1	2	2	2	2
.06	1148	1151	1153	1156	1159	1161	1164	1167	1169	1172	0	1	1	1	1	2	2	2	2
.07	1175	1178	1180	1183	1186	1189	1191	1194	1197	1199	0	1	1	1	1	2	2	2	2
.08	1202	1205	1208	1211	1213	1216	1219	1222	1225	1227	0	1	1	1	1	2	2	2	3
.09	1230	1233	1236	1239	1242	1245	1247	1250	1253	1256	0	1	1	1	1	2	2	2	3
.10	1259	1262	1265	1268	1271	1274	1276	1279	1282	1285	0	1	1	1	1	2	2	2	3
.11	1288	1291	1294	1297	1300	1303	1306	1309	1312	1315	0	1	1	1	2	2	2	2	3
.12	1318	1321	1324	1327	1330	1334	1337	1340	1343	1346	0	1	1	1	2	2	2	2	3
.13	1349	1352	1355	1358	1361	1365	1368	1371	1374	1377	0	1	1	1	2	2	2	3	3
.14	1380	1384	1387	1390	1393	1396	1400	1403	1406	1409	0	1	1	1	2	2	2	3	3
.15	1413	1416	1419	1422	1426	1429	1432	1435	1439	1442	0	1	1	1	2	2	2	3	3
.16	1445	1449	1452	1455	1459	1462	1466	1469	1472	1476	0	1	1	1	2	2	2	3	3
.17	1479	1483	1486	1489	1493	1496	1500	1503	1507	1510	0	1	1	1	2	2	2	3	3
.18	1514	1517	1521	1524	1528	1531	1535	1538	1542	1545	0	1	1	1	2	2	2	3	3
.19	1549	1552	1556	1560	1563	1567	1570	1574	1578	1581	0	1	1	1	2	2	3	3	3
.20	1585	1589	1592	1596	1600	1603	1607	1611	1614	1618	0	1	1	1	2	2	3	3	3
.21	1622	1626	1629	1633	1637	1641	1644	1648	1652	1656	0	1	1	2	2	2	3	3	3
.22	1660	1663	1667	1671	1675	1679	1683	1687	1690	1694	0	1	1	2	2	2	3	3	3
.23	1698	1702	1706	1710	1714	1718	1722	1726	1730	1734	0	1	1	2	2	2	3	3	4
.24	1738	1742	1746	1750	1754	1758	1762	1766	1770	1774	0	1	1	2	2	2	3	3	4
.25	1778	1782	1786	1791	1795	1799	1803	1807	1811	1816	0	1	1	2	2	2	3	3	4
.26	1820	1824	1828	1832	1837	1841	1845	1849	1854	1858	0	1	1	2	2	3	3	3	4
.27	1862	1866	1871	1875	1879	1884	1888	1892	1897	1901	0	1	1	2	2	3	3	3	4
.28	1905	1910	1914	1919	1923	1928	1932	1936	1941	1945	0	1	1	2	2	3	3	4	4
.29	1950	1954	1959	1963	1968	1972	1977	1982	1986	1991	0	1	1	2	2	3	3	4	4
.30	1995	2000	2004	2009	2014	2018	2023	2028	2032	2037	0	1	1	2	2	3	3	4	4
.31	2042	2046	2051	2056	2061	2065	2070	2075	2080	2084	0	1	1	2	2	3	3	4	4
.32	2089	2094	2099	2104	2109	2113	2118	2123	2128	2133	0	1	1	2	2	3	3	4	4
.33	2138	2143	2148	2153	2158	2163	2168	2173	2178	2183	0	1	1	2	2	3	3	4	4
.34	2188	2193	2198	2203	2208	2213	2218	2223	2228	2234	1	1	2	2	3	3	4	4	5
.35	2239	2244	2249	2254	2259	2265	2270	2275	2280	2286	1	1	2	2	3	3	4	4	5
.36	2291	2296	2301	2307	2312	2317	2323	2328	2333	2339	1	1	2	2	3	3	4	4	5
.37	2344	2350	2355	2360	2366	2371	2377	2382	2388	2393	1	1	2	2	3	3	4	4	5
.38	2399	2404	2410	2415	2421	2427	2432	2438	2443	2449	1	1	2	2	3	3	4	4	5
.39	2455	2460	2466	2472	2477	2483	2489	2495	2500	2506	1	1	2	2	3	3	4	5	5
.40	2512	2518	2523	2529	2535	2541	2547	2553	2559	2564	1	1	2	2	3	4	4	5	5
.41	2570	2576	2582	2588	2594	2600	2606	2612	2618	2624	1	1	2	2	3	4	4	5	5
.42	2630	2636	2642	2649	2655	2661	2667	2673	2679	2685	1	1	2	2	3	4	4	5	6
.43	2692	2698	2704	2710	2716	2723	2729	2735	2742	2748	1	1	2	3	3	4	4	5	6
.44	2754	2761	2767	2773	2780	2786	2793	2799	2805	2812	1	1	2	3	3	4	4	5	6
.45	2818	2825	2831	2838	2844	2851	2858	2864	2871	2877	1	1	2	3	3	4	5	5	6
.46	2884	2891	2897	2904	2911	2917	2924	2931	2938	2944	1	1	2	3	3	4	5	5	6
.47	2951	2958	2965	2972	2979	2985	2992	2999	3006	3013	1	1	2	3	3	4	5	5	6
.48	3020	3027	3034	3041	3048	3055	3062	3069	3076	3083	1	1	2	3	4	4	5	6	6
.49	3090	3097	3105	3112	3119	3126	3133	3141	3148	3155	1	1	2	3	4	4	5	6	6

x	0	1	2	3	4	5	6	7	8	9	Differences 1	2	3	4	5	6	7	8	9
.50	3162	3170	3177	3184	3192	3199	3206	3214	3221	3228	1	1	2	3	4	4	5	6	7
.51	3236	3243	3251	3258	3266	3273	3281	3289	3296	3304	1	2	2	3	4	5	5	6	7
.52	3311	3319	3327	3334	3342	3350	3357	3365	3373	3381	1	2	2	3	4	5	5	6	7
.53	3388	3396	3404	3412	3420	3428	3436	3443	3451	3459	1	2	2	3	4	5	6	6	7
.54	3467	3475	3483	3491	3499	3508	3516	3524	3532	3540	1	2	2	3	4	5	6	6	7
.55	3548	3556	3565	3573	3581	3589	3597	3606	3614	3622	1	2	2	3	4	5	6	7	7
.56	3631	3639	3648	3656	3664	3673	3681	3690	3698	3707	1	2	3	3	4	5	6	7	8
.57	3715	3724	3733	3741	3750	3758	3767	3776	3784	3793	1	2	3	3	4	5	6	7	8
.58	3802	3811	3819	3828	3837	3846	3855	3864	3873	3882	1	2	3	4	4	5	6	7	8
.59	3890	3899	3908	3917	3926	3936	3945	3954	3963	3972	1	2	3	4	5	5	6	7	8
.60	3981	3990	3999	4009	4018	4027	4036	4046	4055	4064	1	2	3	4	5	6	6	7	8
.61	4074	4083	4093	4102	4111	4121	4130	4140	4150	4159	1	2	3	4	5	6	7	8	9
.62	4169	4178	4188	4198	4207	4217	4227	4236	4246	4256	1	2	3	4	5	6	7	8	9
.63	4266	4276	4285	4295	4305	4315	4325	4335	4345	4355	1	2	3	4	5	6	7	8	9
.64	4365	4375	4385	4395	4406	4416	4426	4436	4446	4457	1	2	3	4	5	6	7	8	9
.65	4467	4477	4487	4498	4508	4519	4529	4539	4550	4560	1	2	3	4	5	6	7	8	9
.66	4571	4581	4592	4603	4613	4624	4634	4645	4656	4667	1	2	3	4	5	6	7	9	10
.67	4677	4688	4699	4710	4721	4732	4742	4753	4764	4775	1	2	3	4	5	7	8	9	10
.68	4786	4797	4808	4819	4831	4842	4853	4864	4875	4887	1	2	3	4	6	7	8	9	10
.69	4898	4909	4920	4932	4943	4955	4966	4977	4989	5000	1	2	3	5	6	7	8	9	10
.70	5012	5023	5035	5047	5058	5070	5082	5093	5105	5117	1	2	4	5	6	7	8	9	11
.71	5129	5140	5152	5164	5176	5188	5200	5212	5224	5236	1	2	4	5	6	7	8	10	11
.72	5248	5260	5272	5284	5297	5309	5321	5333	5346	5358	1	2	4	5	6	7	9	10	11
.73	5370	5383	5395	5408	5420	5433	5445	5458	5470	5483	1	3	4	5	6	8	9	10	11
.74	5495	5508	5521	5534	5546	5559	5572	5585	5598	5610	1	3	4	5	6	8	9	10	12
.75	5623	5636	5649	5662	5675	5689	5702	5715	5728	5741	1	3	4	5	7	8	9	10	12
.76	5754	5768	5781	5794	5808	5821	5834	5848	5861	5875	1	3	4	5	7	8	9	11	12
.77	5888	5902	5916	5929	5943	5957	5970	5984	5998	6012	1	3	4	5	7	8	10	11	12
.78	6026	6039	6053	6067	6081	6095	6109	6124	6138	6152	1	3	4	6	7	8	10	11	13
.79	6166	6180	6194	6209	6223	6237	6252	6266	6281	6295	1	3	4	6	7	9	10	11	13
.80	6310	6324	6339	6353	6368	6383	6397	6412	6427	6442	1	3	4	6	7	9	10	12	13
.81	6457	6471	6486	6501	6516	6531	6546	6561	6577	6592	2	3	5	6	8	9	11	12	14
.82	6607	6622	6637	6653	6668	6683	6699	6714	6730	6745	2	3	5	6	8	9	11	12	14
.83	6761	6776	6792	6808	6823	6839	6855	6871	6887	6902	2	3	5	6	8	9	11	13	14
.84	6918	6934	6950	6966	6982	6998	7015	7031	7047	7063	2	3	5	6	8	10	11	13	15
.85	7079	7096	7112	7129	7145	7161	7178	7194	7211	7228	2	3	5	7	8	10	12	13	15
.86	7244	7261	7278	7295	7311	7328	7345	7362	7379	7396	2	3	5	7	8	10	12	13	15
.87	7413	7430	7447	7464	7482	7499	7516	7534	7551	7568	2	3	5	7	9	10	12	14	16
.88	7586	7603	7621	7638	7656	7674	7691	7709	7727	7745	2	4	5	7	9	11	12	14	16
.89	7762	7780	7798	7816	7834	7852	7870	7889	7907	7925	2	4	5	7	9	11	13	14	16
.90	7943	7962	7980	7998	8017	8035	8054	8072	8091	8110	2	4	6	7	9	11	13	15	17
.91	8128	8147	8166	8185	8204	8222	8241	8260	8279	8299	2	4	6	8	9	11	13	15	17
.92	8318	8337	8356	8375	8395	8414	8433	8453	8472	8492	2	4	6	8	10	12	14	15	17
.93	8511	8531	8551	8570	8590	8610	8630	8650	8670	8690	2	4	6	8	10	12	14	16	18
.94	8710	8730	8750	8770	8790	8810	8831	8851	8872	8892	2	4	6	8	10	12	14	16	18
.95	8913	8933	8954	8974	8995	9016	9036	9057	9078	9099	2	4	6	8	10	12	15	17	19
.96	9120	9141	9162	9183	9204	9226	9247	9268	9290	9311	2	4	6	8	11	13	15	17	19
.97	9333	9354	9376	9397	9419	9441	9462	9484	9506	9528	2	4	7	9	11	13	15	17	20
.98	9550	9572	9594	9616	9638	9661	9683	9705	9727	9750	2	4	7	9	11	13	16	18	20
.99	9772	9795	9817	9840	9863	9886	9908	9931	9954	9977	2	5	7	9	11	14	16	18	20

Tangents of angles

θ → tan θ

θ	0′ 0,0°	6′ 0,1°	12′ 0,2°	18′ 0,3°	24′ 0,4°	30′ 0,5°	36′ 0,6°	42′ 0,7°	48′ 0,8°	54′ 0,9°	ADD Differences 1′	2′	3′	4′	5′
0°	0,0000	0017	0035	0052	0070	0087	0105	0122	0140	0157	3	6	9	12	15
1	0,0175	0192	0209	0227	0244	0262	0279	0297	0314	0332	3	6	9	12	15
2	0,0349	0367	0384	0402	0419	0437	0454	0472	0489	0507	3	6	9	12	15
3	0,0524	0542	0559	0577	0594	0612	0629	0647	0664	0682	3	6	9	12	15
4	0,0699	0717	0734	0752	0769	0787	0805	0822	0840	0857	3	6	9	12	15
5	0,0875	0892	0910	0928	0945	0963	0981	0998	1016	1033	3	6	9	12	15
6	0,1051	1069	1086	1104	1122	1139	1157	1175	1192	1210	3	6	9	12	15
7	0,1228	1246	1263	1281	1299	1317	1334	1352	1370	1388	3	6	9	12	15
8	0,1405	1423	1441	1459	1477	1495	1512	1530	1548	1566	3	6	9	12	15
9	0,1585	1602	1620	1638	1655	1673	1691	1709	1727	1745	3	6	9	12	15
10°	0,1763	1781	1799	1817	1835	1853	1871	1890	1908	1926	3	6	9	12	15
11	0,1944	1962	1980	1998	2016	2035	2053	2071	2089	2107	3	6	9	12	15
12	0,2126	2144	2162	2180	2199	2217	2235	2254	2272	2290	3	6	9	12	15
13	0,2309	2327	2345	2364	2382	2401	2419	2438	2456	2475	3	6	9	12	15
14	0,2493	2512	2530	2549	2568	2586	2605	2623	2642	2661	3	6	9	12	16
15	0,2679	2698	2717	2736	2754	2773	2792	2811	2830	2849	3	6	9	13	16
16	0,2867	2886	2905	2924	2943	2962	2981	3000	3019	3038	3	6	9	13	16
17	0,3057	3076	3096	3115	3134	3153	3172	3191	3211	3230	3	6	10	13	16
18	0,3249	3269	3288	3307	3327	3346	3365	3385	3404	3424	3	6	10	13	16
19	0,3443	3463	3482	3502	3522	3541	3561	3581	3600	3620	3	7	10	13	16
20°	0,3640	3659	3679	3699	3719	3739	3759	3779	3799	3819	3	7	10	13	17
21	0,3839	3859	3879	3899	3919	3939	3959	3979	4000	4020	3	7	10	13	17
22	0,4040	4061	4081	4101	4122	4142	4163	4183	4204	4224	3	7	10	14	17
23	0,4245	4265	4286	4307	4327	4348	4369	4390	4411	4431	3	7	10	14	17
24	0,4452	4473	4494	4515	4536	4557	4578	4599	4621	4642	4	7	11	14	18
25	0,4663	4684	4706	4727	4748	4770	4791	4813	4834	4856	4	7	11	14	18
26	0,4877	4899	4921	4942	4964	4986	5008	5029	5051	5073	4	7	11	15	18
27	0,5095	5117	5139	5161	5184	5206	5228	5250	5272	5295	4	7	11	15	18
28	0,5317	5340	5362	5384	5407	5430	5452	5475	5498	5520	4	8	11	15	19
29	0,5543	5566	5589	5612	5635	5658	5681	5704	5727	5750	4	8	12	15	19
30°	0,5774	5797	5820	5844	5867	5890	5914	5938	5961	5985	4	8	12	16	20
31	0,6009	6032	6056	6080	6104	6128	6152	6176	6200	6224	4	8	12	16	20
32	0,6249	6273	6297	6322	6346	6371	6395	6420	6445	6469	4	8	12	16	20
33	0,6494	6519	6544	6569	6594	6619	6644	6669	6694	6720	4	8	13	17	21
34	0,6745	6771	6796	6822	6847	6873	6899	6924	6950	6976	4	9	13	17	21
35	0,7002	7028	7054	7080	7107	7133	7159	7186	7212	7239	4	9	13	18	22
36	0,7265	7292	7319	7346	7373	7400	7427	7454	7481	7508	5	9	14	18	23
37	0,7536	7563	7590	7618	7646	7673	7701	7729	7757	7785	5	9	14	18	23
38	0,7813	7841	7869	7898	7926	7954	7983	8012	8040	8069	5	9	14	19	24
39	0,8098	8127	8156	8185	8214	8243	8273	8302	8332	8361	5	10	15	20	24
40°	0,8391	8421	8451	8481	8511	8541	8571	8601	8632	8662	5	10	15	20	25
41	0,8693	8724	8754	8785	8816	8847	8878	8910	8941	8972	5	10	16	21	26
42	0,9004	9036	9067	9099	9131	9163	9195	9228	9260	9293	5	11	16	21	27
43	0,9325	9358	9391	9424	9457	9490	9523	9556	9590	9623	6	11	17	22	28
44	0,9657	9691	9725	9759	9793	9827	9861	9896	9930	9965	6	11	17	23	29

θ	0′ 0,0°	6′ 0,1°	12′ 0,2°	18′ 0,3°	24′ 0,4°	30′ 0,5°	36′ 0,6°	42′ 0,7°	48′ 0,8°	54′ 0,9°	ADD Differences 1′	2′	3′	4′	5′
45°	1,0000	0035	0070	0105	0141	0176	0212	0247	0283	0319	6	12	18	24	30
46	1,0355	0392	0428	0464	0501	0538	0575	0612	0649	0686	6	12	18	25	31
47	1,0724	0761	0799	0837	0875	0913	0951	0990	1028	1067	6	13	19	25	32
48	1,1106	1145	1184	1224	1263	1303	1343	1383	1423	1463	7	13	20	27	33
49	1,1504	1544	1585	1626	1667	1708	1750	1792	1833	1875	7	14	21	28	34
50°	1,1918	1960	2002	2045	2088	2131	2174	2218	2261	2305	7	14	22	29	36
51	1,2349	2393	2437	2482	2527	2572	2617	2662	2708	2753	8	15	23	30	38
52	1,2799	2846	2892	2938	2985	3032	3079	3127	3175	3222	8	16	24	31	39
53	1,3270	3319	3367	3416	3465	3514	3564	3613	3663	3713	8	16	25	33	41
54	1,3764	3814	3865	3916	3968	4019	4071	4124	4176	4229	9	17	26	34	43
55	1,4281	4335	4388	4442	4496	4550	4605	4659	4715	4770	9	18	27	36	45
56	1,4826	4882	4938	4994	5051	5108	5166	5224	5282	5340	10	19	29	38	48
57	1,5399	5458	5517	5577	5637	5697	5757	5818	5880	5941	10	20	30	40	50
58	1,6003	6066	6128	6191	6255	6319	6383	6447	6512	6577	11	21	32	43	53
59	1,6643	6709	6775	6842	6909	6977	7045	7113	7182	7251	11	23	34	45	56
60°	1,7321	7391	7461	7532	7603	7675	7747	7820	7893	7966	12	24	36	48	60
61	1,8040	8115	8190	8265	8341	8418	8495	8572	8650	8728	13	26	38	51	64
62	1,8807	8887	8967	9047	9128	9210	9292	9375	9458	9542	14	27	41	55	68
63	1,9626	9711	9797	9883	9970	0057	0145	0233	0323	0413	15	29	44	58	73
64	2,0503	0594	0686	0778	0872	0965	1060	1155	1251	1348	16	31	47	63	78
65	2,1445	1543	1642	1742	1842	1943	2045	2148	2251	2355	17	34	51	68	85
66	2,2460	2566	2673	2781	2889	2998	3109	3220	3332	3445	18	37	55	73	92
67	2,3559	3673	3789	3906	4023	4142	4262	4383	4504	4627	20	40	60	79	99
68	2,4751	4876	5002	5129	5257	5386	5517	5649	5782	5916	22	43	65	87	108
69	2,6051	6187	6325	6464	6605	6746	6889	7034	7179	7326	24	47	71	95	119
70°	2,7475	7625	7776	7929	8083	8239	8397	8556	8716	8878	26	52	78	104	131
71	2,9042	9208	9375	9544	9714	9887	0061	0237	0415	0595	29	58	87	116	145
72	3,0777	0961	1146	1334	1524	1716	1910	2106	2305	2506	32	64	96	129	161
73	3,2709	2914	3122	3332	3544	3759	3977	4197	4420	4646	36	72	108	144	180
74	3,4874	5105	5339	5576	5816	6059	6305	6554	6806	7062	41	81	122	163	204
75	3,7321	7583	7848	8118	8391	8667	8947	9232	9520	9812	46	93	139	186	232
76	4,0108	0408	0713	1022	1335	1653	1976	2303	2635	2972					
77	4,3315	3662	4015	4374	4737	5107	5483	5864	6252	6645					
78	4,7046	7453	7867	8288	8716	9152	9594	0045	0504	0970					
79	5,1446	1929	2422	2924	3435	3955	4486	5026	5578	6140					
80°	5,6713	7297	7894	8502	9124	9758	0405	1066	1742	2432					
81	6,3138	3859	4596	5350	6122	6912	7720	8548	9395	0264	differences unreliable				
82	7,1154	2066	3002	3962	4947	5958	6996	8062	9158	0285					
83	8,1443	2636	3863	5126	6427	7769	9152	0579	2052	3572					
84	9,514	9,677	9,845	10,02	10,20	10,39	10,58	10,78	10,99	11,20					
85	11,43	11,66	11,91	12,16	12,43	12,71	13,00	13,30	13,62	13,95					
86	14,30	14,67	15,06	15,46	15,89	16,35	16,83	17,34	17,89	18,46					
87	19,08	19,74	20,45	21,20	22,02	22,90	23,86	24,90	26,03	27,27					
88	28,64	30,14	31,82	33,69	35,80	38,19	40,92	44,07	47,74	52,08					
89	57,29	63,66	71,62	81,85	95,49	114,6	143,2	191,0	286,5	573,0					

Sines of angles

$\theta \rightarrow \sin\theta$

θ	0′ 0.0°	6′ 0.1°	12′ 0.2°	18′ 0.3°	24′ 0.4°	30′ 0.5°	36′ 0.6°	42′ 0.7°	48′ 0.8°	54′ 0.9°	ADD Differences 1′	2′	3′	4′	5′
0°	0,0000	0017	0035	0052	0070	0087	0105	0122	0140	0157	3	6	9	12	15
1	0,0175	0192	0209	0227	0244	0262	0279	0297	0314	0332	3	6	9	12	15
2	0,0349	0366	0384	0401	0419	0436	0454	0471	0488	0506	3	6	9	12	15
3	0,0523	0541	0558	0576	0593	0610	0628	0645	0663	0680	3	6	9	12	15
4	0,0698	0715	0732	0750	0767	0785	0802	0819	0837	0854	3	6	9	12	15
5	0,0872	0889	0906	0924	0941	0958	0976	0993	1011	1028	3	6	9	12	14
6	0,1045	1063	1080	1097	1115	1132	1149	1167	1184	1201	3	6	9	12	14
7	0,1219	1236	1253	1271	1288	1305	1323	1340	1357	1374	3	6	9	12	14
8	0,1392	1409	1426	1444	1461	1478	1495	1513	1530	1547	3	6	9	12	14
9	0,1564	1582	1599	1616	1633	1650	1668	1685	1702	1719	3	6	9	12	14
10°	0,1736	1754	1771	1788	1805	1822	1840	1857	1874	1891	3	6	9	11	14
11	0,1908	1925	1942	1959	1977	1994	2011	2028	2045	2062	3	6	9	11	14
12	0,2079	2096	2113	2130	2147	2164	2181	2198	2215	2233	3	6	9	11	14
13	0,2250	2267	2284	2300	2317	2334	2351	2368	2385	2402	3	6	8	11	14
14	0,2419	2436	2453	2470	2487	2504	2521	2538	2554	2571	3	6	8	11	14
15	0,2588	2605	2622	2639	2656	2672	2689	2706	2723	2740	3	6	8	11	14
16	0,2756	2773	2790	2807	2823	2840	2857	2874	2890	2907	3	6	8	11	14
17	0,2924	2940	2957	2974	2990	3007	3024	3040	3057	3074	3	6	8	11	14
18	0,3090	3107	3123	3140	3156	3173	3190	3206	3223	3239	3	6	8	11	14
19	0,3256	3272	3289	3305	3322	3338	3355	3371	3387	3404	3	5	8	11	14
20°	0,3420	3437	3453	3469	3486	3502	3518	3535	3551	3567	3	5	8	11	14
21	0,3584	3600	3616	3633	3649	3665	3681	3697	3714	3730	3	5	8	11	14
22	0,3746	3762	3778	3795	3811	3827	3843	3859	3875	3891	3	5	8	11	14
23	0,3907	3923	3939	3955	3971	3987	4003	4019	4035	4051	3	5	8	11	14
24	0,4067	4083	4099	4115	4131	4147	4163	4179	4195	4210	3	5	8	11	13
25	0,4226	4242	4258	4274	4289	4305	4321	4337	4352	4368	3	5	8	11	13
26	0,4384	4399	4415	4431	4446	4462	4478	4493	4509	4524	3	5	8	10	13
27	0,4540	4555	4571	4586	4602	4617	4633	4648	4664	4679	3	5	8	10	13
28	0,4695	4710	4726	4741	4756	4772	4787	4802	4818	4833	3	5	8	10	13
29	0,4848	4863	4879	4894	4909	4924	4939	4955	4970	4985	3	5	8	10	13
30°	0,5000	5015	5030	5045	5060	5075	5090	5105	5120	5135	3	5	8	10	13
31	0,5150	5165	5180	5195	5210	5225	5240	5255	5270	5284	2	5	7	10	12
32	0,5299	5314	5329	5344	5358	5373	5388	5402	5417	5432	2	5	7	10	12
33	0,5446	5461	5476	5490	5505	5519	5534	5548	5563	5577	2	5	7	10	12
34	0,5592	5606	5621	5635	5650	5664	5678	5693	5707	5721	2	5	7	10	12
35	0,5736	5750	5764	5779	5793	5807	5821	5835	5850	5864	2	5	7	9	12
36	0,5878	5892	5906	5920	5934	5948	5962	5976	5990	6004	2	5	7	9	12
37	0,6018	6032	6046	6060	6074	6088	6101	6115	6129	6143	2	5	7	9	12
38	0,6157	6170	6184	6198	6211	6225	6239	6252	6266	6280	2	5	7	9	11
39	0,6293	6307	6320	6334	6347	6361	6374	6388	6401	6414	2	4	7	9	11
40°	0,6428	6441	6455	6468	6481	6494	6508	6521	6534	6547	2	4	7	9	11
41	0,6561	6574	6587	6600	6613	6626	6639	6652	6665	6678	2	4	7	9	11
42	0,6691	6704	6717	6730	6743	6756	6769	6782	6794	6807	2	4	6	9	11
43	0,6820	6833	6845	6858	6871	6884	6896	6909	6921	6934	2	4	6	8	11
44	0,6947	6959	6972	6984	6997	7009	7022	7034	7046	7059	2	4	6	8	10

θ	0′ 0.0°	6′ 0.1°	12′ 0.2°	18′ 0.3°	24′ 0.4°	30′ 0.5°	36′ 0.6°	42′ 0.7°	48′ 0.8°	54′ 0.9°	ADD Differences 1′	2′	3′	4′	5′
45°	0,7071	7083	7096	7108	7120	7133	7145	7157	7169	7181	2	4	6	8	10
46	0,7193	7206	7218	7230	7242	7254	7266	7278	7290	7302	2	4	6	8	10
47	0,7314	7325	7337	7349	7361	7373	7385	7396	7408	7420	2	4	6	8	10
48	0,7431	7443	7455	7466	7478	7490	7501	7513	7524	7536	2	4	6	8	10
49	0,7547	7559	7570	7581	7593	7604	7615	7627	7638	7649	2	4	6	8	9
50°	0,7660	7672	7683	7694	7705	7716	7727	7738	7749	7760	2	4	6	7	9
51	0,7771	7782	7793	7804	7815	7826	7837	7848	7859	7869	2	4	5	7	9
52	0,7880	7891	7902	7912	7923	7934	7944	7955	7965	7976	2	4	5	7	9
53	0,7986	7997	8007	8018	8028	8039	8049	8059	8070	8080	2	3	5	7	9
54	0,8090	8100	8111	8121	8131	8141	8151	8161	8171	8181	2	3	5	7	8
55	0,8192	8202	8211	8221	8231	8241	8251	8261	8271	8281	2	3	5	7	8
56	0,8290	8300	8310	8320	8329	8339	8348	8358	8368	8377	2	3	5	6	8
57	0,8387	8396	8406	8415	8425	8434	8443	8453	8462	8471	2	3	5	6	8
58	0,8480	8490	8499	8508	8517	8526	8536	8545	8554	8563	2	3	5	6	8
59	0,8572	8581	8590	8599	8607	8616	8625	8634	8643	8652	1	3	4	6	7
60°	0,8660	8669	8678	8686	8695	8704	8712	8721	8729	8738	1	3	4	6	7
61	0,8746	8755	8763	8771	8780	8788	8796	8805	8813	8821	1	3	4	6	7
62	0,8829	8838	8846	8854	8862	8870	8878	8886	8894	8902	1	3	4	5	7
63	0,8910	8918	8926	8934	8942	8949	8957	8965	8973	8980	1	3	4	5	6
64	0,8988	8996	9003	9011	9018	9026	9033	9041	9048	9056	1	3	4	5	6
65	0,9063	9070	9078	9085	9092	9100	9107	9114	9121	9128	1	2	4	5	6
66	0,9135	9143	9150	9157	9164	9171	9178	9184	9191	9198	1	2	3	5	6
67	0,9205	9212	9219	9225	9232	9239	9245	9252	9259	9265	1	2	3	4	6
68	0,9272	9278	9285	9291	9298	9304	9311	9317	9323	9330	1	2	3	4	5
69	0,9336	9342	9348	9354	9361	9367	9373	9379	9385	9391	1	2	3	4	5
70°	0,9397	9403	9409	9415	9421	9426	9432	9438	9444	9449	1	2	3	4	5
71	0,9455	9461	9466	9472	9478	9483	9489	9494	9500	9505	1	2	3	4	5
72	0,9511	9516	9521	9527	9532	9537	9542	9548	9553	9558	1	2	3	3	4
73	0,9563	9568	9573	9578	9583	9588	9593	9598	9603	9608	1	2	2	3	4
74	0,9613	9617	9622	9627	9632	9636	9641	9646	9650	9655	1	2	2	3	4
75	0,9659	9664	9668	9673	9677	9681	9686	9690	9694	9699	1	1	2	3	4
76	0,9703	9707	9711	9715	9720	9724	9728	9732	9736	9740	1	1	2	3	3
77	0,9744	9748	9751	9755	9759	9763	9767	9770	9774	9778	1	1	2	3	3
78	0,9781	9785	9789	9792	9796	9799	9803	9806	9810	9813	1	1	2	2	3
79	0,9816	9820	9823	9826	9829	9833	9836	9839	9842	9845	1	1	2	2	3
80°	0,9848	9851	9854	9857	9860	9863	9866	9869	9871	9874	0	1	1	2	2
81	0,9877	9880	9882	9885	9888	9890	9893	9895	9898	9900	0	1	1	2	2
82	0,9903	9905	9907	9910	9912	9914	9917	9919	9921	9923	0	1	1	2	2
83	0,9925	9928	9930	9932	9934	9936	9938	9940	9942	9943	0	1	1	1	2
84	0,9945	9947	9949	9951	9952	9954	9956	9957	9959	9960	0	1	1	1	2
85	0,9962	9963	9965	9966	9968	9969	9971	9972	9973	9974	0	0	1	1	1
86	0,9976	9977	9978	9979	9980	9981	9982	9983	9984	9985	0	0	1	1	1
87	0,9986	9987	9988	9989	9990	9990	9991	9992	9993	9993	0	0	0	1	1
88	0,9994	9995	9995	9996	9996	9997	9997	9997	9998	9998	0	0	0	0	0
89	0,9998	9999	9999	9999	9999	1,000	1,000	1,000	1,000	1,000	0	0	0	0	0

Cosines of angles

$\theta \rightarrow \cos\theta$

θ	0' 0,0°	6' 0,1°	12' 0,2°	18' 0,3°	24' 0,4°	30' 0,5°	36' 0,6°	42' 0,7°	48' 0,8°	54' 0,9°	SUBTRACT Differences 1'	2'	3'	4'	5'
0°	1,000	1,000	1,000	1,000	1,000	1,000	,9999	9999	9999	9999	0	0	0	0	0
1	0,9998	9998	9998	9997	9997	9997	9996	9996	9995	9995	0	0	0	0	0
2	0,9994	9993	9993	9992	9991	9990	9990	9989	9988	9987	0	0	0	1	1
3	0,9986	9985	9984	9983	9982	9981	9980	9979	9978	9977	0	0	1	1	1
4	0,9976	9974	9973	9972	9971	9969	9968	9966	9965	9963	0	0	1	1	1
5	0,9962	9960	9959	9957	9956	9954	9952	9951	9949	9947	0	1	1	1	2
6	0,9945	9943	9942	9940	9938	9936	9934	9932	9930	9928	0	1	1	1	2
7	0,9925	9923	9921	9919	9917	9914	9912	9910	9907	9905	0	1	1	2	2
8	0,9903	9900	9898	9895	9893	9890	9888	9885	9882	9880	0	1	1	2	2
9	0,9877	9874	9871	9869	9866	9863	9860	9857	9854	9851	0	1	1	2	2
10°	0,9848	9845	9842	9839	9836	9833	9829	9826	9823	9820	1	1	2	2	3
11	0,9816	9813	9810	9806	9803	9799	9796	9792	9789	9785	1	1	2	2	3
12	0,9781	9778	9774	9770	9767	9763	9759	9755	9751	9748	1	1	2	3	3
13	0,9744	9740	9736	9732	9728	9724	9720	9715	9711	9707	1	1	2	3	3
14	0,9703	9699	9694	9690	9686	9681	9677	9673	9668	9664	1	1	2	3	4
15	0,9659	9655	9650	9646	9641	9636	9632	9627	9622	9617	1	2	2	3	4
16	0,9613	9608	9603	9598	9593	9588	9583	9578	9573	9568	1	2	2	3	4
17	0,9563	9558	9553	9548	9542	9537	9532	9527	9521	9516	1	2	3	3	4
18	0,9511	9505	9500	9494	9489	9483	9478	9472	9466	9461	1	2	3	4	5
19	0,9455	9449	9444	9438	9432	9426	9421	9415	9409	9403	1	2	3	4	5
20°	0,9397	9391	9385	9379	9373	9367	9361	9354	9348	9342	1	2	3	4	5
21	0,9336	9330	9323	9317	9311	9304	9298	9291	9285	9278	1	2	3	4	5
22	0,9272	9265	9259	9252	9245	9239	9232	9225	9219	9212	1	2	3	4	6
23	0,9205	9198	9191	9184	9178	9171	9164	9157	9150	9143	1	2	3	5	6
24	0,9135	9128	9121	9114	9107	9100	9092	9085	9078	9070	1	2	4	5	6
25	0,9063	9056	9048	9041	9033	9026	9018	9011	9003	8996	1	3	4	5	6
26	0,8988	8980	8973	8965	8957	8949	8942	8934	8926	8918	1	3	4	5	6
27	0,8910	8902	8894	8886	8878	8870	8862	8854	8846	8838	1	3	4	5	7
28	0,8829	8821	8813	8805	8796	8788	8780	8771	8763	8755	1	3	4	6	7
29	0,8746	8738	8729	8721	8712	8704	8695	8686	8678	8669	1	3	4	6	7
30	0,8660	8652	8643	8634	8625	8616	8607	8599	8590	8581	1	3	4	6	7
31	0,8572	8563	8554	8545	8536	8526	8517	8508	8499	8490	2	3	5	6	8
32	0,8480	8471	8462	8453	8443	8434	8425	8415	8406	8396	2	3	5	6	8
33	0,8387	8377	8368	8358	8348	8339	8329	8320	8310	8300	2	3	5	6	8
34	0,8290	8281	8271	8261	8251	8241	8231	8221	8211	8202	2	3	5	7	8
35	0,8192	8181	8171	8161	8151	8141	8131	8121	8111	8100	2	3	5	7	8
36	0,8090	8080	8070	8059	8049	8039	8028	8018	8007	7997	2	3	5	7	9
37	0,7986	7976	7965	7955	7944	7934	7923	7912	7902	7891	2	4	5	7	9
38	0,7880	7869	7859	7848	7837	7826	7815	7804	7793	7782	2	4	5	7	9
39	0,7771	7760	7749	7738	7727	7716	7705	7694	7683	7672	2	4	6	7	9
40°	0,7660	7649	7638	7627	7615	7604	7593	7581	7570	7559	2	4	6	8	9
41	0,7547	7536	7524	7513	7501	7490	7478	7466	7455	7443	2	4	6	8	10
42	0,7431	7420	7408	7396	7385	7373	7361	7349	7337	7325	2	4	6	8	10
43	0,7314	7302	7290	7278	7266	7254	7242	7230	7218	7206	2	4	6	8	10
44	0,7193	7181	7169	7157	7145	7133	7120	7108	7096	7083	2	4	6	8	10

θ	0' 0,0°	6' 0,1°	12' 0,2°	18' 0,3°	24' 0,4°	30' 0,5°	36' 0,6°	42' 0,7°	48' 0,8°	54' 0,9°	SUBTRACT Differences 1'	2'	3'	4'	5'
45°	0,7071	7059	7046	7034	7022	7009	6997	6984	6972	6959	2	4	6	8	10
46	0,6947	6934	6921	6909	6896	6884	6871	6858	6845	6833	2	4	6	8	11
47	0,6820	6807	6794	6782	6769	6756	6743	6730	6717	6704	2	4	6	9	11
48	0,6691	6678	6665	6652	6639	6626	6613	6600	6587	6574	2	4	7	9	11
49	0,6561	6547	6534	6521	6508	6494	6481	6468	6455	6441	2	4	7	9	11
50°	0,6428	6414	6401	6388	6374	6361	6347	6334	6320	6307	2	4	7	9	11
51	0,6293	6280	6266	6252	6239	6225	6211	6198	6184	6170	2	5	7	9	11
52	0,6157	6143	6129	6115	6101	6088	6074	6060	6046	6032	2	5	7	9	12
53	0,6018	6004	5990	5976	5962	5948	5934	5920	5906	5892	2	5	7	9	12
54	0,5878	5864	5850	5835	5821	5807	5793	5779	5764	5750	2	5	7	9	12
55	0,5736	5721	5707	5693	5678	5664	5650	5635	5621	5606	2	5	7	10	12
56	0,5592	5577	5563	5548	5534	5519	5505	5490	5476	5461	2	5	7	10	12
57	0,5446	5432	5417	5402	5388	5373	5358	5344	5329	5314	2	5	7	10	12
58	0,5299	5284	5270	5255	5240	5225	5210	5195	5180	5165	2	5	7	10	12
59	0,5150	5135	5120	5105	5090	5075	5060	5045	5030	5015	3	5	8	10	13
60°	0,5000	4985	4970	4955	4939	4924	4909	4894	4879	4863	3	5	8	10	13
61	0,4848	4833	4818	4802	4787	4772	4756	4741	4726	4710	3	5	8	10	13
62	0,4695	4679	4664	4648	4633	4617	4602	4586	4571	4555	3	5	8	10	13
63	0,4540	4524	4509	4493	4478	4462	4446	4431	4415	4399	3	5	8	10	13
64	0,4384	4368	4352	4337	4321	4305	4289	4274	4258	4242	3	5	8	11	13
65	0,4226	4210	4195	4179	4163	4147	4131	4115	4099	4083	3	5	8	11	13
66	0,4067	4051	4035	4019	4003	3987	3971	3955	3939	3923	3	5	8	11	14
67	0,3907	3891	3875	3859	3843	3827	3811	3795	3778	3762	3	5	8	11	14
68	0,3746	3730	3714	3697	3681	3665	3649	3633	3616	3600	3	5	8	11	14
69	0,3584	3567	3551	3535	3518	3502	3486	3469	3453	3437	3	5	8	11	14
70°	0,3420	3404	3387	3371	3355	3338	3322	3305	3289	3272	3	5	8	11	14
71	0,3256	3239	3223	3206	3190	3173	3156	3140	3123	3107	3	6	8	11	14
72	0,3090	3074	3057	3040	3024	3007	2990	2974	2957	2940	3	6	8	11	14
73	0,2924	2907	2890	2874	2857	2840	2823	2807	2790	2773	3	6	8	11	14
74	0,2756	2740	2723	2706	2689	2672	2656	2639	2622	2605	3	6	8	11	14
75	0,2588	2571	2554	2538	2521	2504	2487	2470	2453	2436	3	6	8	11	14
76	0,2419	2402	2385	2368	2351	2334	2317	2300	2284	2267	3	6	8	11	14
77	0,2250	2233	2215	2198	2181	2164	2147	2130	2113	2096	3	6	9	11	14
78	0,2079	2062	2045	2028	2011	1994	1977	1959	1942	1925	3	6	9	11	14
79	0,1908	1891	1874	1857	1840	1822	1805	1788	1771	1754	3	6	9	11	14
80°	0,1736	1719	1702	1685	1668	1650	1633	1616	1599	1582	3	6	9	12	14
81	0,1564	1547	1530	1513	1495	1478	1461	1444	1426	1409	3	6	9	12	14
82	0,1392	1374	1357	1340	1323	1305	1288	1271	1253	1236	3	6	9	12	14
83	0,1219	1201	1184	1167	1149	1132	1115	1097	1080	1063	3	6	9	12	14
84	0,1045	1028	1011	0993	0976	0958	0941	0924	0906	0889	3	6	9	12	14
85	0,0872	0854	0837	0819	0802	0785	0767	0750	0732	0715	3	6	9	12	14
86	0,0698	0680	0663	0645	0628	0610	0593	0576	0558	0541	3	6	9	12	15
87	0,0523	0506	0488	0471	0454	0436	0419	0401	0384	0366	3	6	9	12	15
88	0,0349	0332	0314	0297	0279	0262	0244	0227	0209	0192	3	6	9	12	15
89	0,0175	0157	0140	0122	0105	0087	0070	0052	0035	0017	3	6	9	12	15

Logarithms of tangents

$\theta \rightarrow \log \tan \theta$

θ	0′ 0.0°	6′ 0.1°	12′ 0.2°	18′ 0.3°	24′ 0.4°	30′ 0.5°	36′ 0.6°	42′ 0.7°	48′ 0.8°	54′ 0.9°	Add Differences 1′	2′	3′	4′	5′
0°	−∞	$\bar{3}$,2419	5429	7190	8439	9408	0200	0870	1450	1962					
1	$\bar{2}$,2419	2833	3211	3559	3881	4181	4461	4725	4973	5208					
2	$\bar{2}$,5431	5643	5845	6038	6223	6401	6571	6736	6894	7046					
3	$\bar{2}$,7194	7337	7475	7609	7739	7865	7988	8107	8223	8336					
4	$\bar{2}$,8446	8554	8659	8762	8862	8960	9056	9150	9241	9331	16	32	48	64	81
5	$\bar{2}$,9420	9506	9591	9674	9756	9836	9915	9992	0068	0143	13	26	40	53	66
6	$\bar{1}$,0216	0289	0360	0430	0499	0567	0633	0699	0764	0828	11	22	34	45	56
7	$\bar{1}$,0891	0954	1015	1076	1135	1194	1252	1310	1367	1423	10	20	29	39	49
8	$\bar{1}$,1478	1533	1587	1640	1693	1745	1797	1848	1898	1948	9	17	26	35	43
9	$\bar{1}$,1997	2046	2094	2142	2189	2236	2282	2328	2374	2419	8	16	23	31	39
10°	$\bar{1}$,2463	2507	2551	2594	2637	2680	2722	2764	2805	2846	7	14	21	28	35
11	$\bar{1}$,2887	2927	2967	3006	3046	3085	3123	3162	3200	3237	6	13	19	26	32
12	$\bar{1}$,3275	3312	3349	3385	3422	3458	3493	3529	3564	3599	6	12	18	24	30
13	$\bar{1}$,3634	3668	3702	3736	3770	3804	3837	3870	3903	3935	6	11	17	22	28
14	$\bar{1}$,3968	4000	4032	4064	4095	4127	4158	4189	4220	4250	5	10	16	21	26
15	$\bar{1}$,4281	4311	4341	4371	4400	4430	4459	4488	4517	4546	5	10	15	20	25
16	$\bar{1}$,4575	4603	4632	4660	4688	4716	4744	4771	4799	4826	5	9	14	19	23
17	$\bar{1}$,4853	4880	4907	4934	4961	4987	5014	5040	5066	5092	4	9	13	18	22
18	$\bar{1}$,5118	5143	5169	5195	5220	5245	5270	5295	5320	5345	4	8	13	17	21
19	$\bar{1}$,5370	5394	5419	5443	5467	5491	5516	5539	5563	5587	4	8	12	16	20
20°	$\bar{1}$,5611	5634	5658	5681	5704	5727	5750	5773	5796	5819	4	8	12	15	19
21	$\bar{1}$,5842	5864	5887	5909	5932	5954	5976	5998	6020	6042	4	7	11	15	18
22	$\bar{1}$,6064	6086	6108	6129	6151	6172	6194	6215	6236	6257	4	7	11	14	18
23	$\bar{1}$,6279	6300	6321	6341	6362	6383	6404	6424	6445	6465	3	7	10	14	17
24	$\bar{1}$,6486	6506	6527	6547	6567	6587	6607	6627	6647	6667	3	7	10	13	17
25	$\bar{1}$,6687	6706	6726	6746	6765	6785	6804	6824	6843	6863	3	7	10	13	16
26	$\bar{1}$,6882	6901	6920	6939	6958	6977	6996	7015	7034	7053	3	6	9	13	16
27	$\bar{1}$,7072	7090	7109	7128	7146	7165	7183	7202	7220	7238	3	6	9	12	15
28	$\bar{1}$,7257	7275	7293	7311	7330	7348	7366	7384	7402	7420	3	6	9	12	15
29	$\bar{1}$,7438	7455	7473	7491	7509	7526	7544	7562	7579	7597	3	6	9	12	15
30°	$\bar{1}$,7614	7632	7649	7667	7684	7701	7719	7736	7753	7771	3	6	9	12	14
31	$\bar{1}$,7788	7805	7822	7839	7856	7873	7890	7907	7924	7941	3	6	8	11	14
32	$\bar{1}$,7958	7975	7992	8008	8025	8042	8059	8075	8092	8109	3	6	8	11	14
33	$\bar{1}$,8125	8142	8158	8175	8191	8208	8224	8241	8257	8274	3	5	8	11	14
34	$\bar{1}$,8290	8306	8323	8339	8355	8371	8388	8404	8420	8436	3	5	8	11	14
35	$\bar{1}$,8452	8468	8484	8501	8517	8533	8549	8565	8581	8597	3	5	8	11	13
36	$\bar{1}$,8613	8629	8644	8660	8676	8692	8708	8724	8740	8755	3	5	8	11	13
37	$\bar{1}$,8771	8787	8803	8818	8834	8850	8865	8881	8897	8912	3	5	8	10	13
38	$\bar{1}$,8928	8944	8959	8975	8990	9006	9022	9037	9053	9068	3	5	8	10	13
39	$\bar{1}$,9084	9099	9115	9130	9146	9161	9176	9192	9207	9223	3	5	8	10	13
40°	$\bar{1}$,9238	9254	9269	9284	9300	9315	9330	9346	9361	9376	3	5	8	10	13
41	$\bar{1}$,9392	9407	9422	9438	9453	9468	9483	9499	9514	9529	3	5	8	10	13
42	$\bar{1}$,9544	9560	9575	9590	9605	9621	9636	9651	9666	9681	3	5	8	10	13
43	$\bar{1}$,9697	9712	9727	9742	9757	9772	9788	9803	9818	9833	3	5	8	10	13
44	$\bar{1}$,9848	9864	9879	9894	9909	9924	9939	9955	9970	9985	3	5	8	10	13

θ	0′ 0.0°	6′ 0.1°	12′ 0.2°	18′ 0.3°	24′ 0.4°	30′ 0.5°	36′ 0.6°	42′ 0.7°	48′ 0.8°	54′ 0.9°	Add Differences 1′	2′	3′	4′	5′
45°	0,0000	0015	0030	0045	0061	0076	0091	0106	0121	0136	3	5	8	10	13
46	0,0152	0167	0182	0197	0212	0228	0243	0258	0273	0288	3	5	8	10	13
47	0,0303	0319	0334	0349	0364	0379	0395	0410	0425	0440	3	5	8	10	13
48	0,0456	0471	0486	0501	0517	0532	0547	0562	0578	0593	3	5	8	10	13
49	0,0608	0624	0639	0654	0670	0685	0700	0716	0731	0746	3	5	8	10	13
50°	0,0762	0777	0793	0808	0824	0839	0854	0870	0885	0901	3	5	8	10	13
51	0,0916	0932	0947	0963	0978	0994	1010	1025	1041	1056	3	5	8	10	13
52	0,1072	1088	1103	1119	1135	1150	1166	1182	1197	1213	3	5	8	10	13
53	0,1229	1245	1260	1276	1292	1308	1324	1340	1356	1371	3	5	8	11	13
54	0,1387	1403	1419	1435	1451	1467	1483	1499	1516	1532	3	5	8	11	13
55	0,1548	1564	1580	1596	1612	1629	1645	1661	1677	1694	3	5	8	11	14
56	0,1710	1726	1743	1759	1776	1792	1809	1825	1842	1858	3	5	8	11	14
57	0,1875	1891	1908	1925	1941	1958	1975	1992	2008	2025	3	6	8	11	14
58	0,2042	2059	2076	2093	2110	2127	2144	2161	2178	2195	3	6	9	11	14
59	0,2212	2229	2247	2264	2281	2299	2316	2333	2351	2368	3	6	9	12	14
60°	0,2386	2403	2421	2438	2456	2474	2491	2509	2527	2545	3	6	9	12	15
61	0,2562	2580	2598	2616	2634	2652	2670	2689	2707	2725	3	6	9	12	15
62	0,2743	2762	2780	2798	2817	2835	2854	2872	2891	2910	3	6	9	12	15
63	0,2928	2947	2966	2985	3004	3023	3042	3061	3080	3099	3	6	9	13	16
64	0,3118	3137	3157	3176	3196	3215	3235	3254	3274	3294	3	6	10	13	16
65	0,3313	3333	3353	3373	3393	3413	3433	3453	3473	3494	3	7	10	13	17
66	0,3514	3535	3555	3576	3596	3617	3638	3659	3679	3700	3	7	10	14	17
67	0,3721	3743	3764	3785	3806	3828	3849	3871	3892	3914	4	7	11	14	18
68	0,3936	3958	3980	4002	4024	4046	4068	4091	4113	4136	4	7	11	15	18
69	0,4158	4181	4204	4227	4250	4273	4296	4319	4342	4366	4	8	12	15	19
70°	0,4389	4413	4437	4461	4484	4509	4533	4557	4581	4606	4	8	12	16	20
71	0,4630	4655	4680	4705	4730	4755	4780	4805	4831	4857	4	8	13	17	21
72	0,4882	4908	4934	4960	4986	5013	5039	5066	5093	5120	4	9	13	18	22
73	0,5147	5174	5201	5229	5256	5284	5312	5340	5368	5397	5	9	14	19	23
74	0,5425	5454	5483	5512	5541	5570	5600	5629	5659	5689	5	10	15	20	25
75	0,5719	5750	5780	5811	5842	5873	5905	5936	5968	6000	5	10	16	21	26
76	0,6032	6065	6097	6130	6163	6196	6230	6264	6298	6332	6	11	17	22	28
77	0,6366	6401	6436	6471	6507	6542	6578	6615	6651	6688	6	12	18	24	30
78	0,6725	6763	6800	6838	6877	6915	6954	6994	7033	7073	6	13	19	26	32
79	0,7113	7154	7195	7236	7278	7320	7363	7406	7449	7493	7	14	21	28	35
80°	0,7537	7581	7626	7672	7718	7764	7811	7858	7906	7954	8	16	23	31	39
81	0,8003	8052	8102	8152	8203	8255	8307	8360	8413	8467	9	17	26	35	43
82	0,8522	8577	8633	8690	8748	8806	8865	8924	8985	9046	10	20	29	39	49
83	0,9109	9172	9236	9301	9367	9433	9501	9570	9640	9711	11	22	34	45	56
84	0,9784	9857	9932	0008	0085	0164	0244	0326	0409	0494	13	26	40	53	66
85	1,0580	0669	0759	0850	0944	1040	1138	1238	1341	1446	16	32	48	64	81
86	1,1554	1664	1777	1893	2012	2135	2261	2391	2525	2663					
87	1,2806	2954	3106	3264	3429	3599	3777	3962	4155	4357					
88	1,4569	4792	5027	5275	5539	5819	6119	6441	6789	7167					
89	1,7581	8038	8550	9130	9800	0591	1561	2810	4571	7581					

Logarithms of sines

$\theta \to \log \sin \theta$

θ	0′ 0.0°	6′ 0.1°	12′ 0.2°	18′ 0.3°	24′ 0.4°	30′ 0.5°	36′ 0.6°	42′ 0.7°	48′ 0.8°	54′ 0.9°	ADD Differences 1′	2′	3′	4′	5′
0°	−∞	$\bar{3}$.2419	5429	7190	8439	9408	$\bar{2}$.0200	0870	1450	1961					
1	$\bar{2}$.2419	2832	3210	3558	3880	4179	4459	4723	4971	5206					
2	$\bar{2}$.5428	5640	5842	6035	6220	6397	6567	6731	6889	7041					
3	$\bar{2}$.7188	7330	7468	7602	7731	7857	7979	8098	8213	8326					
4	$\bar{2}$.8436	8543	8647	8749	8849	8946	9042	9135	9226	9315	16	32	48	64	80
5	$\bar{2}$.9403	9489	9573	9655	9736	9816	9894	9970	0046	0120	13	26	39	52	65
6	$\bar{1}$.0192	0264	0334	0403	0472	0539	0605	0670	0734	0797	11	22	33	44	55
7	$\bar{1}$.0859	0920	0981	1040	1099	1157	1214	1271	1326	1381	10	19	29	38	48
8	$\bar{1}$.1436	1489	1542	1594	1646	1697	1747	1797	1847	1895	8	17	25	34	42
9	$\bar{1}$.1943	1991	2038	2085	2131	2176	2221	2266	2310	2353	8	15	23	30	38
10°	$\bar{1}$.2397	2439	2482	2524	2565	2606	2647	2687	2727	2767	7	14	20	27	34
11	$\bar{1}$.2806	2845	2883	2921	2959	2997	3034	3070	3107	3143	6	12	19	25	31
12	$\bar{1}$.3179	3214	3250	3284	3319	3353	3387	3421	3455	3488	6	11	17	23	28
13	$\bar{1}$.3521	3554	3586	3618	3650	3682	3713	3745	3775	3806	5	11	16	21	26
14	$\bar{1}$.3837	3867	3897	3927	3957	3986	4015	4044	4073	4102	5	10	15	20	24
15	$\bar{1}$.4130	4158	4186	4214	4242	4269	4296	4323	4350	4377	5	9	14	18	23
16	$\bar{1}$.4403	4430	4456	4482	4508	4533	4559	4584	4609	4634	4	9	13	17	21
17	$\bar{1}$.4659	4684	4709	4733	4757	4781	4805	4829	4853	4876	4	8	12	16	20
18	$\bar{1}$.4900	4923	4946	4969	4992	5015	5037	5060	5082	5104	4	8	11	15	19
19	$\bar{1}$.5126	5148	5170	5192	5213	5235	5256	5278	5299	5320	4	7	11	14	18
20°	$\bar{1}$.5341	5361	5382	5402	5423	5443	5463	5484	5504	5523	3	7	10	14	17
21	$\bar{1}$.5543	5563	5583	5602	5621	5641	5660	5679	5698	5717	3	6	10	13	16
22	$\bar{1}$.5736	5754	5773	5792	5810	5828	5847	5865	5883	5901	3	6	9	12	15
23	$\bar{1}$.5919	5937	5954	5972	5990	6007	6024	6042	6059	6076	3	6	9	12	15
24	$\bar{1}$.6093	6110	6127	6144	6161	6177	6194	6210	6227	6243	3	6	8	11	14
25	$\bar{1}$.6259	6276	6292	6308	6324	6340	6356	6371	6387	6403	3	5	8	11	13
26	$\bar{1}$.6418	6434	6449	6465	6480	6495	6510	6526	6541	6556	3	5	8	10	13
27	$\bar{1}$.6570	6585	6600	6615	6629	6644	6659	6673	6687	6702	2	5	7	10	12
28	$\bar{1}$.6716	6730	6744	6759	6773	6787	6801	6814	6828	6842	2	5	7	9	12
29	$\bar{1}$.6856	6869	6883	6896	6910	6923	6937	6950	6963	6977	2	4	7	9	11
30°	$\bar{1}$.6990	7003	7016	7029	7042	7055	7068	7080	7093	7106	2	4	6	9	11
31	$\bar{1}$.7118	7131	7144	7156	7168	7181	7193	7205	7218	7230	2	4	6	8	10
32	$\bar{1}$.7242	7254	7266	7278	7290	7302	7314	7326	7338	7349	2	4	6	8	10
33	$\bar{1}$.7361	7373	7384	7396	7407	7419	7430	7442	7453	7464	2	4	6	8	10
34	$\bar{1}$.7476	7487	7498	7509	7520	7531	7542	7553	7564	7575	2	4	6	7	9
35	$\bar{1}$.7586	7597	7607	7618	7629	7640	7650	7661	7671	7682	2	4	5	7	9
36	$\bar{1}$.7692	7703	7713	7723	7734	7744	7754	7764	7774	7785	2	3	5	7	9
37	$\bar{1}$.7795	7805	7815	7825	7835	7844	7854	7864	7874	7884	2	3	5	7	8
38	$\bar{1}$.7893	7903	7913	7922	7932	7941	7951	7960	7970	7979	2	3	5	6	8
39	$\bar{1}$.7989	7998	8007	8017	8026	8035	8044	8053	8063	8072	2	3	5	6	8
40°	$\bar{1}$.8081	8090	8099	8108	8117	8125	8134	8143	8152	8161	1	3	4	6	7
41	$\bar{1}$.8169	8178	8187	8195	8204	8213	8221	8230	8238	8247	1	3	4	6	7
42	$\bar{1}$.8255	8264	8272	8280	8289	8297	8305	8313	8322	8330	1	3	4	6	7
43	$\bar{1}$.8338	8346	8354	8362	8370	8378	8386	8394	8402	8410	1	3	4	5	7
44	$\bar{1}$.8418	8426	8433	8441	8449	8457	8464	8472	8480	8487	1	3	4	5	6

θ	0′ 0.0°	6′ 0.1°	12′ 0.2°	18′ 0.3°	24′ 0.4°	30′ 0.5°	36′ 0.6°	42′ 0.7°	48′ 0.8°	54′ 0.9°	ADD Difference 1′	2′	3′	4′	5′
45	$\bar{1}$.8495	8502	8510	8517	8525	8532	8540	8547	8555	8562	1	2	4	5	6
46	$\bar{1}$.8569	8577	8584	8591	8598	8606	8613	8620	8627	8634	1	2	4	5	6
47	$\bar{1}$.8641	8648	8655	8662	8669	8676	8683	8690	8697	8704	1	2	3	5	6
48	$\bar{1}$.8711	8718	8724	8731	8738	8745	8751	8758	8765	8771	1	2	3	4	6
49	$\bar{1}$.8778	8784	8791	8797	8804	8810	8817	8823	8830	8836	1	2	3	4	5
50°	$\bar{1}$.8843	8849	8855	8862	8868	8874	8880	8887	8893	8899	1	2	3	4	5
51	$\bar{1}$.8905	8911	8917	8923	8929	8935	8941	8947	8953	8959	1	2	3	4	5
52	$\bar{1}$.8965	8971	8977	8983	8989	8995	9000	9006	9012	9018	1	2	3	4	5
53	$\bar{1}$.9023	9029	9035	9041	9046	9052	9057	9063	9069	9074	1	2	3	4	5
54	$\bar{1}$.9080	9085	9091	9096	9101	9107	9112	9118	9123	9128	1	2	3	4	5
55	$\bar{1}$.9134	9139	9144	9149	9155	9160	9165	9170	9175	9181	1	2	3	3	4
56	$\bar{1}$.9186	9191	9196	9201	9206	9211	9216	9221	9226	9231	1	2	3	3	4
57	$\bar{1}$.9236	9241	9246	9251	9255	9260	9265	9270	9275	9279	1	2	2	3	4
58	$\bar{1}$.9284	9289	9294	9298	9303	9308	9312	9317	9322	9326	1	2	2	3	4
59	$\bar{1}$.9331	9335	9340	9344	9349	9353	9358	9362	9367	9371	1	1	2	3	4
60°	$\bar{1}$.9375	9380	9384	9388	9393	9397	9401	9406	9410	9414	1	1	2	3	4
61	$\bar{1}$.9418	9422	9427	9431	9435	9439	9443	9447	9451	9455	1	1	2	3	3
62	$\bar{1}$.9459	9463	9467	9471	9475	9479	9483	9487	9491	9495	1	1	2	3	3
63	$\bar{1}$.9499	9503	9507	9510	9514	9518	9522	9525	9529	9533	1	1	2	3	3
64	$\bar{1}$.9537	9540	9544	9548	9551	9555	9558	9562	9566	9569	1	1	2	2	3
65	$\bar{1}$.9573	9576	9580	9583	9587	9590	9594	9597	9601	9604	1	1	2	2	3
66	$\bar{1}$.9607	9611	9614	9617	9621	9624	9627	9631	9634	9637	1	1	2	2	3
67	$\bar{1}$.9640	9643	9647	9650	9653	9656	9659	9662	9666	9669	1	1	2	2	3
68	$\bar{1}$.9672	9675	9678	9681	9684	9687	9690	9693	9696	9699	0	1	1	2	2
69	$\bar{1}$.9702	9704	9707	9710	9713	9716	9719	9722	9724	9727	0	1	1	2	2
70°	$\bar{1}$.9730	9733	9735	9738	9741	9743	9746	9749	9751	9754	0	1	1	2	2
71	$\bar{1}$.9757	9759	9762	9764	9767	9770	9772	9775	9777	9780	0	1	1	2	2
72	$\bar{1}$.9782	9785	9787	9789	9792	9794	9797	9799	9801	9804	0	1	1	2	2
73	$\bar{1}$.9806	9808	9811	9813	9815	9817	9820	9822	9824	9826	0	1	1	2	2
74	$\bar{1}$.9828	9831	9833	9835	9837	9839	9841	9843	9845	9847	0	1	1	1	2
75	$\bar{1}$.9849	9851	9853	9855	9857	9859	9861	9863	9865	9867	0	1	1	1	2
76	$\bar{1}$.9869	9871	9873	9875	9876	9878	9880	9882	9884	9885	0	1	1	1	2
77	$\bar{1}$.9887	9889	9891	9892	9894	9896	9897	9899	9901	9902	0	1	1	1	1
78	$\bar{1}$.9904	9906	9907	9909	9910	9912	9913	9915	9916	9918	0	1	1	1	1
79	$\bar{1}$.9919	9921	9922	9924	9925	9927	9928	9929	9931	9932	0	0	1	1	1
80°	$\bar{1}$.9934	9935	9936	9937	9939	9940	9941	9943	9944	9945	0	0	1	1	1
81	$\bar{1}$.9946	9947	9949	9950	9951	9952	9953	9954	9955	9956	0	0	1	1	1
82	$\bar{1}$.9958	9959	9960	9961	9962	9963	9964	9965	9966	9967	0	0	1	1	1
83	$\bar{1}$.9968	9968	9969	9970	9971	9972	9973	9974	9975	9975	0	0	1	1	1
84	$\bar{1}$.9976	9977	9978	9978	9979	9980	9981	9981	9982	9983	0	0	0	0	1
85	$\bar{1}$.9983	9984	9985	9985	9986	9987	9987	9988	9988	9989	0	0	0	0	0
86	$\bar{1}$.9989	9990	9990	9991	9991	9992	9992	9993	9993	9994	0	0	0	0	0
87	$\bar{1}$.9994	9994	9995	9995	9996	9996	9996	9996	9997	9997	0	0	0	0	0
88	$\bar{1}$.9997	9998	9998	9998	9998	9999	9999	9999	9999	9999	0	0	0	0	0
89	$\bar{1}$.9999	9999	0000	0000	0000	0000	0000	0000	0000	0000	0	0	0	0	0

Logarithms of cosines

θ → log cos θ

θ	0' 0.0°	6' 0.1°	12' 0.2°	18' 0.3°	24' 0.4°	30' 0.5°	36' 0.6°	42' 0.7°	48' 0.8°	54' 0.9°	SUBTRACT Differences 1'	2'	3'	4'	5'
0°	0,0000	0000	0000	0000	0000	0000	0000	0000	0000	$\bar{1}$,9999	0	0	0	0	0
1	$\bar{1}$,9999	9999	9999	9999	9999	9999	9998	9998	9998	9998	0	0	0	0	0
2	$\bar{1}$,9997	9997	9997	9996	9996	9996	9996	9995	9995	9994	0	0	0	0	0
3	$\bar{1}$,9994	9994	9993	9993	9992	9992	9991	9991	9990	9990	0	0	0	0	0
4	$\bar{1}$,9989	9989	9988	9988	9987	9987	9986	9985	9985	9984	0	0	0	0	0
5	$\bar{1}$,9983	9983	9982	9981	9981	9980	9979	9978	9978	9977	0	0	0	0	1
6	$\bar{1}$,9976	9975	9975	9974	9973	9972	9971	9970	9969	9968	0	0	0	1	1
7	$\bar{1}$,9968	9967	9966	9965	9964	9963	9962	9961	9960	9959	0	0	1	1	1
8	$\bar{1}$,9958	9956	9955	9954	9953	9952	9951	9950	9949	9947	0	0	1	1	1
9	$\bar{1}$,9946	9945	9944	9943	9941	9940	9939	9937	9936	9935	0	0	1	1	1
10°	$\bar{1}$,9934	9932	9931	9929	9928	9927	9925	9924	9922	9921	0	0	1	1	1
11	$\bar{1}$,9919	9918	9916	9915	9913	9912	9910	9909	9907	9906	0	1	1	1	1
12	$\bar{1}$,9904	9902	9901	9899	9897	9896	9894	9892	9891	9889	0	1	1	1	1
13	$\bar{1}$,9887	9885	9884	9882	9880	9878	9876	9875	9873	9871	0	1	1	1	2
14	$\bar{1}$,9869	9867	9865	9863	9861	9859	9857	9855	9853	9851	0	1	1	1	2
15	$\bar{1}$,9849	9847	9845	9843	9841	9839	9837	9835	9833	9831	0	1	1	1	2
16	$\bar{1}$,9828	9826	9824	9822	9820	9817	9815	9813	9811	9808	0	1	1	2	2
17	$\bar{1}$,9806	9804	9801	9799	9797	9794	9792	9789	9787	9785	0	1	1	2	2
18	$\bar{1}$,9782	9780	9777	9775	9772	9770	9767	9764	9762	9759	0	1	1	2	2
19	$\bar{1}$,9757	9754	9751	9749	9746	9743	9741	9738	9735	9733	0	1	1	2	2
20°	$\bar{1}$,9730	9727	9724	9722	9719	9716	9713	9710	9707	9704	0	1	1	2	2
21	$\bar{1}$,9702	9699	9696	9693	9690	9687	9684	9681	9678	9675	0	1	1	2	2
22	$\bar{1}$,9672	9669	9665	9662	9659	9656	9653	9650	9647	9643	1	1	2	2	3
23	$\bar{1}$,9640	9637	9634	9631	9627	9624	9621	9617	9614	9611	1	1	2	2	3
24	$\bar{1}$,9607	9604	9601	9597	9594	9590	9587	9583	9580	9576	1	1	2	2	3
25	$\bar{1}$,9573	9569	9566	9562	9558	9555	9551	9548	9544	9540	1	1	2	2	3
26	$\bar{1}$,9537	9533	9529	9525	9522	9518	9514	9510	9507	9503	1	1	2	3	3
27	$\bar{1}$,9499	9495	9491	9487	9483	9479	9475	9471	9467	9463	1	1	2	3	3
28	$\bar{1}$,9459	9455	9451	9447	9443	9439	9435	9431	9427	9422	1	1	2	3	3
29	$\bar{1}$,9418	9414	3410	9406	9401	9397	9393	9388	9384	9380	1	1	2	3	4
30°	$\bar{1}$,9375	9371	9367	9362	9358	9353	9349	9344	9340	9335	1	1	2	3	4
31	$\bar{1}$,9431	9326	9322	9317	9312	9308	9303	9298	9294	9289	1	2	2	3	4
32	$\bar{1}$,9284	9279	9275	9270	9265	9260	9255	9251	9246	9241	1	2	2	3	4
33	$\bar{1}$,9236	9231	9226	9221	9216	9211	9206	9201	9196	9191	1	2	3	3	4
34	$\bar{1}$,9186	9181	9175	9170	9165	9160	9155	9149	9144	9139	1	2	3	3	4
35	$\bar{1}$,9134	9128	9123	9118	9112	9107	9101	9096	9091	9085	1	2	3	4	5
36	$\bar{1}$,9080	9074	9069	9063	9057	9052	9046	9041	9035	9029	1	2	3	4	5
37	$\bar{1}$,9023	9018	9012	9006	9000	8995	8989	8983	8977	8971	1	2	3	4	5
38	$\bar{1}$,8965	8959	8953	8947	8941	8935	8929	8923	8917	8911	1	2	3	4	5
39	$\bar{1}$,8905	8899	8893	8887	8880	8874	8868	8862	8855	8849	1	2	3	4	5
40°	$\bar{1}$,8843	8836	8830	8823	8817	8810	8804	8797	8791	8784	1	2	3	4	5
41	$\bar{1}$,8778	8771	8765	8758	8751	8745	8738	8731	8724	8718	1	2	3	5	6
42	$\bar{1}$,8711	8704	8697	8690	8683	8676	8669	8662	8655	8648	1	2	3	5	6
43	$\bar{1}$,8641	8634	8627	8620	8613	8606	8598	8591	8584	8577	1	2	4	5	6
44	$\bar{1}$,8569	8562	8555	8547	8540	8532	8525	8517	8510	8502	1	2	4	5	6

θ	0' 0.0°	6' 0.1°	12' 0.2°	18' 0.3°	24' 0.4°	30' 0.5°	36' 0.6°	42' 0.7°	48' 0.8°	54' 0.9°	SUBTRACT Differences 1'	2'	3'	4'	5'
45°	$\bar{1}$,8495	8487	8480	8472	8464	8457	8449	8441	8433	8426	1	3	4	5	6
46	$\bar{1}$,8418	8410	8402	8394	8386	8378	8370	8362	8354	8346	1	3	4	5	7
47	$\bar{1}$,8338	8330	8322	8313	8305	8297	8289	8280	8272	8264	1	3	4	6	7
48	$\bar{1}$,8255	8247	8238	8230	8221	8213	8204	8195	8187	8178	1	3	4	6	7
49	$\bar{1}$,8169	8161	8152	8143	8134	8125	8117	8108	8099	8090	1	3	4	6	7
50°	$\bar{1}$,8081	8072	8063	8053	8044	8035	8026	8017	8007	7998	2	3	5	6	8
51	$\bar{1}$,7989	7979	7970	7960	7951	7941	7932	7922	7913	7903	2	3	5	6	8
52	$\bar{1}$,7893	7884	7874	7864	7854	7844	7835	7825	7815	7805	2	3	5	7	8
53	$\bar{1}$,7795	7785	7774	7764	7754	7744	7734	7723	7713	7703	2	3	5	7	9
54	$\bar{1}$,7692	7682	7671	7661	7650	7640	7629	7618	7607	7597	2	4	5	7	9
55	$\bar{1}$,7586	7575	7564	7553	7542	7531	7520	7509	7498	7487	2	4	6	7	9
56	$\bar{1}$,7476	7464	7453	7442	7430	7419	7407	7396	7384	7373	2	4	6	8	10
57	$\bar{1}$,7361	7349	7338	7326	7314	7302	7290	7278	7266	7254	2	4	6	8	10
58	$\bar{1}$,7242	7230	7218	7205	7193	7181	7168	7156	7144	7131	2	4	6	8	10
59	$\bar{1}$,7118	7106	7093	7080	7068	7055	7042	7029	7016	7003	2	4	6	9	11
60°	$\bar{1}$,6990	6977	6963	6950	6937	6923	6910	6896	6883	6869	2	4	7	9	11
61	$\bar{1}$,6856	6842	6828	6814	6801	6787	6773	6759	6744	6730	2	5	7	9	12
62	$\bar{1}$,6716	6702	6687	6673	6659	6644	6629	6615	6600	6585	2	5	7	10	12
63	$\bar{1}$,6570	6556	6541	6526	6510	6495	6480	6465	6449	6434	3	5	8	10	13
64	$\bar{1}$,6418	6403	6387	6371	6356	6340	6324	6308	6292	6276	3	5	8	11	13
65	$\bar{1}$,6259	6243	6227	6210	6194	6177	6161	6144	6127	6110	3	6	8	11	14
66	$\bar{1}$,6093	6076	6059	6042	6024	6007	5990	5972	5954	5937	3	6	9	12	15
67	$\bar{1}$,5919	5901	5883	5865	5847	5828	5810	5792	5773	5754	3	6	9	12	15
68	$\bar{1}$,5736	5717	5698	5679	5660	5641	5621	5602	5583	5563	3	6	10	13	16
69	$\bar{1}$,5543	5523	5504	5484	5463	5443	5423	5402	5382	5361	3	7	10	14	17
70°	$\bar{1}$,5341	5320	5299	5278	5256	5235	5213	5192	5170	5148	4	7	11	14	18
71	$\bar{1}$,5126	5104	5082	5060	5037	5015	4992	4969	4946	4923	4	8	11	15	19
72	$\bar{1}$,4900	4876	4853	4829	4805	4781	4757	4733	4709	4684	4	8	12	16	20
73	$\bar{1}$,4659	4634	4609	4584	4559	4533	4508	4482	4456	4430	4	9	13	17	21
74	$\bar{1}$,4403	4377	4350	4323	4296	4269	4242	4214	4186	4158	5	9	14	18	23
75	$\bar{1}$,4130	4102	4073	4044	4015	3986	3957	3927	3897	3867	5	10	15	20	24
76	$\bar{1}$,3837	3806	3775	3745	3713	3682	3650	3618	3586	3554	5	11	16	21	26
77	$\bar{1}$,3521	3488	3455	3421	3387	3353	3319	3284	3250	3214	6	11	17	23	28
78	$\bar{1}$,3179	3143	3107	3070	3034	2997	2959	2921	2883	2845	6	12	19	25	31
79	$\bar{1}$,2806	2767	2727	2687	2647	2606	2565	2524	2482	2439	7	14	20	27	34
80°	$\bar{1}$,2397	2353	2310	2266	2221	2176	2131	2085	2038	1991	8	15	23	30	38
81	$\bar{1}$,1943	1895	1847	1797	1747	1697	1646	1594	1542	1489	8	17	25	34	42
82	$\bar{1}$,1436	1381	1326	1271	1214	1157	1099	1040	0981	0920	10	19	29	38	48
83	$\bar{1}$,0859	0797	0734	0670	0605	0539	0472	0403	0334	0264	11	22	33	44	55
84	$\bar{1}$,0192	0120	0046	$\bar{9}$970	$\bar{9}$894	$\bar{9}$816	$\bar{9}$736	$\bar{9}$655	$\bar{9}$573	$\bar{9}$489	13	26	39	52	65
85	$\bar{2}$,9403	9315	9226	9135	9042	8946	8849	8749	8647	8543	16	32	48	64	80
86	$\bar{2}$,8436	8326	8213	8098	7979	7857	7731	7602	7468	7330					
87	$\bar{2}$,7188	7041	6889	6731	6567	6397	6220	6035	5842	5640					
88	$\bar{2}$,5428	5206	4971	4723	4459	4179	3880	3558	3210	2832					
89	$\bar{2}$,2419	1961	1450	0870	0200	$\bar{9}$408	$\bar{8}$439	$\bar{7}$190	$\bar{5}$429	$\bar{2}$419					

Reciprocals

$x \rightarrow \frac{1}{x}$

x	0	1	2	3	4	5	6	7	8	9	SUBTRACT Differences 1	2	3	4	5	6	7	8	9
1,0	1,000	9901	9804	9709	9615	9524	9434	9346	9259	9174	9	18	28	37	46	55	64	74	83
1,1	0,9091	9009	8929	8850	8772	8696	8621	8547	8475	8403	8	15	23	31	38	46	53	61	69
1,2	0,8333	8264	8197	8130	8065	8000	7937	7874	7813	7752	7	13	20	26	33	39	46	52	59
1,3	0,7692	7634	7576	7519	7463	7407	7353	7299	7246	7194	6	11	17	22	28	33	39	44	50
1,4	0,7143	7092	7042	6993	6944	6897	6849	6803	6757	6711	5	10	14	19	24	29	33	38	43
1,5	0,6667	6623	6579	6536	6494	6452	6410	6369	6329	6289	4	8	13	17	21	25	29	33	38
1,6	0,6250	6211	6173	6135	6098	6061	6024	5988	5952	5917	4	7	11	15	18	22	26	29	33
1,7	0,5882	5848	5814	5780	5747	5714	5682	5650	5618	5587	3	7	10	13	16	20	23	26	29
1,8	0,5556	5525	5495	5464	5435	5405	5376	5348	5319	5291	3	6	9	12	15	17	20	23	26
1,9	0,5263	5236	5208	5181	5155	5128	5102	5076	5051	5025	3	5	8	11	13	16	18	21	24
2,0	0,5000	4975	4950	4926	4902	4878	4854	4831	4808	4785	2	5	7	10	12	14	17	19	21
2,1	0,4762	4739	4717	4695	4673	4651	4630	4608	4587	4566	2	4	7	9	11	13	15	17	19
2,2	0,4545	4525	4505	4484	4464	4444	4425	4405	4386	4367	2	4	6	8	10	12	14	16	18
2,3	0,4348	4329	4310	4292	4274	4255	4237	4219	4202	4184	2	4	5	7	9	11	13	14	16
2,4	0,4167	4149	4132	4115	4098	4082	4065	4049	4032	4016	2	3	5	7	8	10	12	13	15
2,5	0,4000	3984	3968	3953	3937	3922	3906	3891	3876	3861	2	3	5	6	8	9	11	12	14
2,6	0,3846	3831	3817	3802	3788	3774	3759	3745	3731	3717	1	3	4	6	7	9	10	11	13
2,7	0,3704	3690	3676	3663	3650	3636	3623	3610	3597	3584	1	3	4	5	7	8	9	11	12
2,8	0,3571	3559	3546	3534	3521	3509	3497	3484	3472	3460	1	2	4	5	6	7	9	10	11
2,9	0,3448	3436	3425	3413	3401	3390	3378	3367	3356	3344	1	2	3	5	6	7	8	9	10
3,0	0,3333	3322	3311	3300	3289	3279	3268	3257	3247	3236	1	2	3	4	5	6	8	9	10
3,1	0,3226	3215	3205	3195	3185	3175	3165	3155	3145	3135	1	2	3	4	5	6	7	8	9
3,2	0,3125	3115	3106	3096	3086	3077	3067	3058	3049	3040	1	2	3	4	5	6	7	8	9
3,3	0,3030	3021	3012	3003	2994	2985	2976	2967	2959	2950	1	2	3	4	4	5	6	7	8
3,4	0,2941	2933	2924	2915	2907	2899	2890	2882	2874	2865	1	2	3	3	4	5	6	7	8
3,5	0,2857	2849	2841	2833	2825	2817	2809	2801	2793	2786	1	2	2	3	4	5	6	6	7
3,6	0,2778	2770	2762	2755	2747	2740	2732	2725	2717	2710	1	2	2	3	4	5	5	6	7
3,7	0,2703	2695	2688	2681	2674	2667	2660	2653	2646	2639	1	1	2	3	4	4	5	6	6
3,8	0,2632	2625	2618	2611	2604	2597	2591	2584	2577	2571	1	1	2	3	3	4	5	5	6
3,9	0,2564	2558	2551	2545	2538	2532	2525	2519	2513	2506	1	1	2	3	3	4	4	5	6
4,0	0,2500	2494	2488	2481	2475	2469	2463	2457	2451	2445	1	1	2	2	3	4	4	5	5
4,1	0,2439	2433	2427	2421	2415	2410	2404	2398	2392	2387	1	1	2	2	3	3	4	5	5
4,2	0,2381	2375	2370	2364	2358	2353	2347	2342	2336	2331	1	1	2	2	3	3	4	4	5
4,3	0,2326	2320	2315	2309	2304	2299	2294	2288	2283	2278	1	1	2	2	3	3	4	4	5
4,4	0,2273	2268	2262	2257	2252	2247	2242	2237	2232	2227	1	1	2	2	3	3	4	4	5
4,5	0,2222	2217	2212	2208	2203	2198	2193	2188	2183	2179	0	1	1	2	2	3	3	4	4
4,6	0,2174	2169	2165	2160	2155	2151	2146	2141	2137	2132	0	1	1	2	2	3	3	4	4
4,7	0,2128	2123	2119	2114	2110	2105	2101	2096	2092	2088	0	1	1	2	2	3	3	4	4
4,8	0,2083	2079	2075	2070	2066	2062	2058	2053	2049	2045	0	1	1	2	2	3	3	3	4
4,9	0,2041	2037	2033	2028	2024	2020	2016	2012	2008	2004	0	1	1	2	2	2	3	3	4
5,0	0,2000	1996	1992	1988	1984	1980	1976	1972	1969	1965	0	1	1	2	2	2	3	3	4
5,1	0,1961	1957	1953	1949	1946	1942	1938	1934	1931	1927	0	1	1	2	2	2	3	3	3
5,2	0,1923	1919	1916	1912	1908	1905	1901	1898	1894	1890	0	1	1	1	2	2	3	3	3
5,3	0,1887	1883	1880	1876	1873	1869	1866	1862	1859	1855	0	1	1	1	2	2	2	3	3
5,4	0,1852	1848	1845	1842	1838	1835	1832	1828	1825	1821	0	1	1	1	2	2	2	3	3

x	0	1	2	3	4	5	6	7	8	9	SUBTRACT Differences 1	2	3	4	5	6	7	8	9
5,5	0,1818	1815	1812	1808	1805	1802	1799	1795	1792	1789	0	1	1	1	2	2	2	3	3
5,6	0,1786	1783	1779	1776	1773	1770	1767	1764	1761	1757	0	1	1	1	2	2	2	3	3
5,7	0,1754	1751	1748	1745	1742	1739	1736	1733	1730	1727	0	1	1	1	1	2	2	2	3
5,8	0,1724	1721	1718	1715	1712	1709	1706	1704	1701	1698	0	1	1	1	1	2	2	2	3
5,9	0,1695	1692	1689	1686	1684	1681	1678	1675	1672	1669	0	1	1	1	1	2	2	2	3
6,0	0,1667	1664	1661	1658	1656	1653	1650	1647	1645	1642	0	1	1	1	1	2	2	2	3
6,1	0,1639	1637	1634	1631	1629	1626	1623	1621	1618	1616	0	1	1	1	1	2	2	2	2
6,2	0,1613	1610	1608	1605	1603	1600	1597	1595	1592	1590	0	1	1	1	1	2	2	2	2
6,3	0,1587	1585	1582	1580	1577	1575	1572	1570	1567	1565	0	0	1	1	1	1	2	2	2
6,4	0,1562	1560	1558	1555	1553	1550	1548	1546	1543	1541	0	0	1	1	1	1	2	2	2
6,5	0,1538	1536	1534	1531	1529	1527	1524	1522	1520	1517	0	0	1	1	1	1	2	2	2
6,6	0,1515	1513	1511	1508	1506	1504	1502	1499	1497	1495	0	0	1	1	1	1	2	2	2
6,7	0,1493	1490	1488	1486	1484	1481	1479	1477	1475	1473	0	0	1	1	1	1	2	2	2
6,8	0,1471	1468	1466	1464	1462	1460	1458	1456	1453	1451	0	0	1	1	1	1	2	2	2
6,9	0,1449	1447	1445	1443	1441	1439	1437	1435	1433	1431	0	0	1	1	1	1	2	2	2
7,0	0,1429	1427	1425	1422	1420	1418	1416	1414	1412	1410	0	0	1	1	1	1	1	2	2
7,1	0,1408	1406	1404	1403	1401	1399	1397	1395	1393	1391	0	0	1	1	1	1	1	2	2
7,2	0,1389	1387	1385	1383	1381	1379	1377	1376	1374	1372	0	0	1	1	1	1	1	2	2
7,3	0,1370	1368	1366	1364	1362	1361	1359	1357	1355	1353	0	0	1	1	1	1	1	2	2
7,4	0,1351	1350	1348	1346	1344	1342	1340	1339	1337	1335	0	0	1	1	1	1	1	1	2
7,5	0,1333	1332	1330	1328	1326	1325	1323	1321	1319	1318	0	0	1	1	1	1	1	1	2
7,6	0,1316	1314	1312	1311	1309	1307	1305	1304	1302	1300	0	0	1	1	1	1	1	1	2
7,7	0,1299	1297	1295	1294	1292	1290	1289	1287	1285	1284	0	0	0	1	1	1	1	1	1
7,8	0,1282	1280	1279	1277	1276	1274	1272	1271	1269	1267	0	0	0	1	1	1	1	1	1
7,9	0,1266	1264	1263	1261	1259	1258	1256	1255	1253	1252	0	0	0	1	1	1	1	1	1
8,0	0,1250	1248	1247	1245	1244	1242	1241	1239	1238	1236	0	0	0	1	1	1	1	1	1
8,1	0,1235	1233	1232	1230	1229	1227	1225	1224	1222	1221	0	0	0	1	1	1	1	1	1
8,2	0,1220	1218	1217	1215	1214	1212	1211	1209	1208	1206	0	0	0	1	1	1	1	1	1
8,3	0,1205	1203	1202	1200	1199	1198	1196	1195	1193	1192	0	0	0	1	1	1	1	1	1
8,4	0,1190	1189	1188	1186	1185	1183	1182	1181	1179	1178	0	0	0	1	1	1	1	1	1
8,5	0,1176	1175	1174	1172	1171	1170	1168	1167	1166	1164	0	0	0	1	1	1	1	1	1
8,6	0,1163	1161	1160	1159	1157	1156	1155	1153	1152	1151	0	0	0	1	1	1	1	1	1
8,7	0,1149	1148	1147	1145	1144	1143	1142	1140	1139	1138	0	0	0	1	1	1	1	1	1
8,8	0,1136	1135	1134	1133	1131	1130	1129	1127	1126	1125	0	0	0	1	1	1	1	1	1
8,9	0,1124	1122	1121	1120	1119	1117	1116	1115	1114	1112	0	0	0	1	1	1	1	1	1
9,0	0,1111	1110	1109	1107	1106	1105	1104	1103	1101	1100	0	0	0	1	1	1	1	1	1
9,1	0,1099	1098	1096	1095	1094	1093	1092	1090	1089	1088	0	0	0	1	1	1	1	1	1
9,2	0,1087	1086	1085	1083	1082	1081	1080	1079	1078	1076	0	0	0	0	1	1	1	1	1
9,3	0,1075	1074	1073	1072	1071	1070	1068	1067	1066	1065	0	0	0	0	1	1	1	1	1
9,4	0,1064	1063	1062	1060	1059	1058	1057	1056	1055	1054	0	0	0	0	1	1	1	1	1
9,5	0,1053	1052	1050	1049	1048	1047	1046	1045	1044	1043	0	0	0	0	1	1	1	1	1
9,6	0,1042	1041	1039	1038	1037	1036	1035	1034	1033	1032	0	0	0	0	1	1	1	1	1
9,7	0,1031	1030	1029	1028	1027	1026	1025	1024	1022	1021	0	0	0	0	1	1	1	1	1
9,8	0,1020	1019	1018	1017	1016	1015	1014	1013	1012	1011	0	0	0	0	1	1	1	1	1
9,9	0,1010	1009	1008	1007	1006	1005	1004	1003	1002	1001	0	0	0	0	1	1	1	1	1

Squares

$x \rightarrow x^2$

x	0	1	2	3	4	5	6	7	8	9	Differences 1	2	3	4	5	6	7	8	9
10	1000	1020	1040	1061	1082	1103	1124	1145	1166	1188	2	4	6	8	10	13	15	17	19
11	1210	1232	1254	1277	1300	1323	1346	1369	1392	1416	2	5	7	9	11	14	16	18	21
12	1440	1464	1488	1513	1538	1563	1588	1613	1638	1664	2	5	7	10	12	15	17	20	22
13	1690	1716	1742	1769	1796	1823	1850	1877	1904	1932	3	5	8	11	13	16	19	22	24
14	1960	1988	2016	2045	2074	2103	2132	2161	2190	2220	3	6	9	12	14	17	20	23	26
15	2250	2280	2310	2341	2372	2403	2434	2465	2496	2528	3	6	9	12	15	19	22	25	28
16	2560	2592	2624	2657	2690	2723	2756	2789	2822	2856	3	7	10	13	16	20	23	26	30
17	2890	2924	2958	2993	3028	3063	3098	3133	3168	3204	3	7	10	14	17	21	24	28	31
18	3240	3276	3312	3349	3386	3423	3460	3497	3534	3572	4	7	11	15	18	22	26	30	33
19	3610	3648	3686	3725	3764	3803	3842	3881	3920	3960	4	8	12	16	19	23	27	31	35
20	4000	4040	4080	4121	4162	4203	4244	4285	4326	4368	4	8	12	16	20	25	29	33	37
21	4410	4452	4494	4537	4580	4623	4666	4709	4752	4796	4	9	13	17	21	26	30	34	39
22	4840	4884	4928	4973	5018	5063	5108	5153	5198	5244	4	9	13	18	22	27	31	36	40
23	5290	5336	5382	5429	5476	5523	5570	5617	5664	5712	5	9	14	19	23	28	33	38	42
24	5760	5808	5856	5905	5954	6003	6052	6101	6150	6200	5	10	15	20	24	29	34	39	44
25	6250	6300	6350	6401	6452	6503	6554	6605	6656	6708	5	10	15	20	25	31	36	41	46
26	6760	6812	6864	6917	6970	7023	7076	7129	7182	7236	5	11	16	21	26	32	37	42	48
27	7290	7344	7398	7453	7508	7563	7618	7673	7728	7784	5	11	16	22	27	33	38	44	49
28	7840	7896	7952	8009	8066	8123	8180	8237	8294	8352	6	11	17	23	28	34	40	46	51
29	8410	8468	8526	8585	8644	8703	8762	8821	8880	8940	6	12	18	24	29	35	41	47	53
30	9000	9060	9120	9181	9242	9303	9364	9425	9486	9548	6	12	18	24	30	37	43	49	55
31	9610	9672	9734	9797	9860	9923	9986	—	—	—	6	13	19	25	31	38	44	50	56
31	—	—	—	—	—	—	—	1005	1011	1018	1	1	2	3	3	4	5	5	6
32	1024	1030	1037	1043	1050	1056	1063	1069	1076	1082	1	1	2	3	3	4	5	5	6
33	1089	1096	1102	1109	1116	1122	1129	1136	1142	1149	1	1	2	3	3	4	5	5	6
34	1156	1163	1170	1176	1183	1190	1197	1204	1211	1218	1	1	2	3	3	4	5	6	6
35	1225	1232	1239	1246	1253	1260	1267	1274	1282	1289	1	1	2	3	4	4	5	6	6
36	1296	1303	1310	1318	1325	1332	1340	1347	1354	1362	1	1	2	3	4	4	5	6	7
37	1369	1376	1384	1391	1399	1406	1414	1421	1429	1436	1	2	2	3	4	5	5	6	7
38	1444	1452	1459	1467	1475	1482	1490	1498	1505	1513	1	2	2	3	4	5	5	6	7
39	1521	1529	1537	1544	1552	1560	1568	1576	1584	1592	1	2	2	3	4	5	6	6	7
40	1600	1608	1616	1624	1632	1640	1648	1656	1665	1673	1	2	2	3	4	5	6	6	7
41	1681	1689	1697	1706	1714	1722	1731	1739	1747	1756	1	2	2	3	4	5	6	7	7
42	1764	1772	1781	1789	1798	1806	1815	1823	1832	1840	1	2	3	3	4	5	6	7	8
43	1849	1858	1866	1875	1884	1892	1901	1910	1918	1927	1	2	3	3	4	5	6	7	8
44	1936	1945	1954	1962	1971	1980	1989	1998	2007	2016	1	2	3	4	5	5	6	7	8
45	2025	2034	2043	2052	2061	2070	2079	2088	2098	2107	1	2	3	4	5	5	6	7	8
46	2116	2125	2134	2144	2153	2162	2172	2181	2190	2200	1	2	3	4	5	6	7	7	8
47	2209	2218	2228	2237	2247	2256	2266	2275	2285	2294	1	2	3	4	5	6	7	8	9
48	2304	2314	2323	2333	2343	2352	2362	2372	2381	2391	1	2	3	4	5	6	7	8	9
49	2401	2411	2421	2430	2440	2450	2460	2470	2480	2490	1	2	3	4	5	6	7	8	9
50	2500	2510	2520	2530	2540	2550	2560	2570	2581	2591	1	2	3	4	5	6	7	8	9
51	2601	2611	2621	2632	2642	2652	2663	2673	2683	2694	1	2	3	4	5	6	7	8	9
52	2704	2714	2725	2735	2746	2756	2767	2777	2788	2798	1	2	3	4	5	6	7	8	9
53	2809	2820	2830	2841	2852	2862	2873	2884	2894	2905	1	2	3	4	5	6	7	9	10
54	2916	2927	2938	2948	2959	2970	2981	2992	3003	3014	1	2	3	4	5	7	8	9	10
55	3025	3036	3047	3058	3069	3080	3091	3102	3114	3125	1	2	3	4	6	7	8	9	10
56	3136	3147	3158	3170	3181	3192	3204	3215	3226	3238	1	2	3	5	6	7	8	9	10
57	3249	3260	3272	3283	3295	3306	3318	3329	3341	3352	1	2	3	5	6	7	8	9	10
58	3364	3376	3387	3399	3411	3422	3434	3446	3457	3469	1	2	4	5	6	7	8	9	10
59	3481	3493	3505	3516	3528	3540	3552	3564	3576	3588	1	2	4	5	6	7	8	10	11
60	3600	3612	3624	3636	3648	3660	3672	3684	3697	3709	1	2	4	5	6	7	8	10	11
61	3721	3733	3745	3758	3770	3782	3795	3807	3819	3832	1	2	4	5	6	7	9	10	11
62	3844	3856	3869	3881	3894	3906	3919	3931	3944	3956	1	2	4	5	6	7	9	10	11
63	3969	3982	3994	4007	4020	4032	4045	4058	4070	4083	1	3	4	5	6	8	9	10	11
64	4096	4109	4122	4134	4147	4160	4173	4186	4199	4212	1	3	4	5	6	8	9	10	12
65	4225	4238	4251	4264	4277	4290	4303	4316	4330	4343	1	3	4	5	7	8	9	10	12
66	4356	4369	4382	4396	4409	4422	4436	4449	4462	4476	1	3	4	5	7	8	9	11	12
67	4489	4502	4516	4529	4543	4556	4570	4583	4597	4610	1	3	4	5	7	8	9	11	12
68	4624	4638	4651	4665	4679	4692	4706	4720	4733	4747	1	3	4	5	7	8	10	11	12
69	4761	4775	4789	4802	4816	4830	4844	4858	4872	4886	1	3	4	6	7	8	10	11	13
70	4900	4914	4928	4942	4956	4970	4984	4998	5013	5027	1	3	4	6	7	8	10	11	13
71	5041	5055	5069	5084	5098	5112	5127	5141	5155	5170	1	3	4	6	7	9	10	11	13
72	5184	5198	5213	5227	5242	5256	5271	5285	5300	5314	1	3	4	6	7	9	10	11	13
73	5329	5344	5358	5373	5388	5402	5417	5432	5446	5461	1	3	4	6	7	9	10	12	13
74	5476	5491	5506	5520	5535	5550	5565	5580	5595	5610	1	3	4	6	7	9	10	12	13
75	5625	5640	5655	5670	5685	5700	5715	5730	5746	5761	2	3	5	6	8	9	11	12	14
76	5776	5791	5806	5822	5837	5852	5868	5883	5898	5914	2	3	5	6	8	9	11	12	14
77	5929	5944	5960	5975	5991	6006	6022	6037	6053	6068	2	3	5	6	8	9	11	12	14
78	6084	6100	6115	6131	6147	6162	6178	6194	6209	6225	2	3	5	6	8	9	11	13	14
79	6241	6257	6273	6288	6304	6320	6336	6352	6368	6384	2	3	5	6	8	10	11	13	14
80	6400	6416	6432	6448	6464	6480	6496	6512	6529	6545	2	3	5	6	8	10	11	13	14
81	6561	6577	6593	6610	6626	6642	6659	6675	6691	6708	2	3	5	7	8	10	11	13	15
82	6724	6740	6757	6773	6790	6806	6823	6839	6856	6872	2	3	5	7	8	10	12	13	15
83	6889	6906	6922	6939	6956	6972	6989	7006	7022	7039	2	3	5	7	8	10	12	13	15
84	7056	7073	7090	7106	7123	7140	7157	7174	7191	7208	2	3	5	7	8	10	12	14	15
85	7225	7242	7259	7276	7293	7310	7327	7344	7362	7379	2	3	5	7	9	10	12	14	15
86	7396	7413	7430	7448	7465	7482	7500	7517	7534	7552	2	3	5	7	9	10	12	14	16
87	7569	7586	7604	7621	7639	7656	7674	7691	7709	7726	2	3	5	7	9	10	12	14	16
88	7744	7762	7779	7797	7815	7832	7850	7868	7885	7903	2	4	5	7	9	11	12	14	16
89	7921	7939	7957	7974	7992	8010	8028	8046	8064	8082	2	4	5	7	9	11	13	14	16
90	8100	8118	8136	8154	8172	8190	8208	8226	8245	8263	2	4	5	7	9	11	13	14	16
91	8281	8299	8317	8336	8354	8372	8391	8409	8427	8446	2	4	5	7	9	11	13	15	16
92	8464	8482	8501	8519	8538	8556	8575	8593	8612	8630	2	4	6	7	9	11	13	15	17
93	8649	8668	8686	8705	8724	8742	8761	8780	8798	8817	2	4	6	7	9	11	13	15	17
94	8836	8855	8874	8892	8911	8930	8949	8968	8987	9006	2	4	6	8	9	11	13	15	17
95	9025	9044	9063	9082	9101	9120	9139	9158	9178	9197	2	4	6	8	10	11	13	15	17
96	9216	9235	9254	9274	9293	9312	9332	9351	9370	9390	2	4	6	8	10	12	14	15	17
97	9409	9428	9448	9467	9487	9506	9526	9545	9565	9584	2	4	6	8	10	12	14	16	18
98	9604	9624	9643	9663	9683	9702	9722	9742	9761	9781	2	4	6	8	10	12	14	16	18
99	9801	9821	9841	9860	9880	9900	9920	9940	9960	9980	2	4	6	8	10	12	14	16	18

Square roots from 1 to 10

$x \rightarrow \sqrt{x}$

x	0	1	2	3	4	5	6	7	8	9	Differences 1 2 3 4 5 6 7 8 9 0
1,0	1,000	1,005	1,010	1,015	1,020	1,025	1,030	1,034	1,039	1,044	0 1 1 2 2 3 3 4 4
1,1	1,049	1,054	1,058	1,063	1,068	1,072	1,077	1,082	1,086	1,091	0 1 1 2 2 3 3 4 4
1,2	1,095	1,100	1,105	1,109	1,114	1,118	1,122	1,127	1,131	1,136	0 1 1 2 2 3 3 4 4
1,3	1,140	1,145	1,149	1,153	1,158	1,162	1,166	1,170	1,175	1,179	0 1 1 2 2 3 3 3 4
1,4	1,183	1,187	1,192	1,196	1,200	1,204	1,208	1,212	1,217	1,221	0 1 1 2 2 2 3 3 4
1,5	1,225	1,229	1,233	1,237	1,241	1,245	1,249	1,253	1,257	1,261	0 1 1 2 2 2 3 3 4
1,6	1,265	1,269	1,273	1,277	1,281	1,285	1,288	1,292	1,296	1,300	0 1 1 2 2 2 3 3 3
1,7	1,304	1,308	1,311	1,315	1,319	1,323	1,327	1,330	1,334	1,338	0 1 1 2 2 2 3 3 3
1,8	1,342	1,345	1,349	1,353	1,356	1,360	1,364	1,367	1,371	1,375	0 1 1 1 2 2 3 3 3
1,9	1,378	1,382	1,386	1,389	1,393	1,396	1,400	1,404	1,407	1,411	0 1 1 1 2 2 3 3 3
2,0	1,414	1,418	1,421	1,425	1,428	1,432	1,435	1,439	1,442	1,446	0 1 1 1 2 2 2 3 3
2,1	1,449	1,453	1,456	1,460	1,463	1,466	1,470	1,473	1,477	1,480	0 1 1 1 2 2 2 3 3
2,2	1,483	1,487	1,490	1,493	1,497	1,500	1,503	1,507	1,510	1,513	0 1 1 1 2 2 2 3 3
2,3	1,517	1,520	1,523	1,526	1,530	1,533	1,536	1,539	1,543	1,546	0 1 1 1 2 2 2 3 3
2,4	1,549	1,552	1,556	1,559	1,562	1,565	1,568	1,572	1,575	1,578	0 1 1 1 2 2 2 3 3
2,5	1,581	1,584	1,587	1,591	1,594	1,597	1,600	1,603	1,606	1,609	0 1 1 1 2 2 2 3 3
2,6	1,612	1,616	1,619	1,622	1,625	1,628	1,631	1,634	1,637	1,640	0 1 1 1 2 2 2 2 3
2,7	1,643	1,646	1,649	1,652	1,655	1,658	1,661	1,664	1,667	1,670	0 1 1 1 2 2 2 2 3
2,8	1,673	1,676	1,679	1,682	1,685	1,688	1,691	1,694	1,697	1,700	0 1 1 1 1 2 2 2 3
2,9	1,703	1,706	1,709	1,712	1,715	1,718	1,720	1,723	1,726	1,729	0 1 1 1 1 2 2 2 3
3,0	1,732	1,735	1,738	1,741	1,744	1,746	1,749	1,752	1,755	1,758	0 1 1 1 1 2 2 2 3
3,1	1,761	1,764	1,766	1,769	1,772	1,775	1,778	1,780	1,783	1,786	0 1 1 1 1 2 2 2 2
3,2	1,789	1,792	1,794	1,797	1,800	1,803	1,806	1,808	1,811	1,814	0 1 1 1 1 2 2 2 2
3,3	1,817	1,819	1,822	1,825	1,828	1,830	1,833	1,836	1,839	1,841	0 1 1 1 1 2 2 2 2
3,4	1,844	1,847	1,849	1,852	1,855	1,857	1,860	1,863	1,865	1,868	0 1 1 1 1 2 2 2 2
3,5	1,871	1,874	1,876	1,879	1,882	1,884	1,887	1,889	1,892	1,895	0 1 1 1 1 2 2 2 2
3,6	1,897	1,900	1,903	1,905	1,908	1,910	1,913	1,916	1,918	1,921	0 1 1 1 1 2 2 2 2
3,7	1,924	1,926	1,929	1,931	1,934	1,936	1,939	1,942	1,944	1,947	0 1 1 1 1 2 2 2 2
3,8	1,949	1,952	1,955	1,957	1,960	1,962	1,965	1,967	1,970	1,972	0 1 1 1 1 2 2 2 2
3,9	1,975	1,977	1,980	1,982	1,985	1,988	1,990	1,992	1,995	1,997	0 1 1 1 1 2 2 2 2
4,0	2,000	2,003	2,005	2,008	2,010	2,013	2,015	2,017	2,020	2,022	0 0 1 1 1 1 2 2 2
4,1	2,025	2,027	2,030	2,032	2,035	2,037	2,040	2,042	2,045	2,047	0 0 1 1 1 1 2 2 2
4,2	2,049	2,052	2,054	2,057	2,059	2,062	2,064	2,066	2,069	2,071	0 0 1 1 1 1 2 2 2
4,3	2,074	2,076	2,078	2,081	2,083	2,086	2,088	2,090	2,093	2,095	0 0 1 1 1 1 2 2 2
4,4	2,098	2,100	2,102	2,105	2,107	2,110	2,112	2,114	2,117	2,119	0 0 1 1 1 1 2 2 2
4,5	2,121	2,124	2,126	2,128	2,131	2,133	2,135	2,138	2,140	2,142	0 0 1 1 1 1 2 2 2
4,6	2,145	2,147	2,149	2,152	2,154	2,156	2,159	2,161	2,163	2,166	0 0 1 1 1 1 2 2 2
4,7	2,168	2,170	2,173	2,175	2,177	2,179	2,182	2,184	2,186	2,188	0 0 1 1 1 1 2 2 2
4,8	2,191	2,193	2,195	2,198	2,200	2,202	2,205	2,207	2,209	2,211	0 0 1 1 1 1 2 2 2
4,9	2,214	2,216	2,218	2,220	2,223	2,225	2,227	2,229	2,232	2,234	0 0 1 1 1 1 2 2 2
5,0	2,236	2,238	2,241	2,243	2,245	2,247	2,249	2,252	2,254	2,256	0 0 1 1 1 1 2 2 2
5,1	2,258	2,261	2,263	2,265	2,267	2,269	2,272	2,274	2,276	2,278	0 0 1 1 1 1 2 2 2
5,2	2,280	2,283	2,285	2,287	2,289	2,291	2,293	2,296	2,298	2,300	0 0 1 1 1 1 2 2 2
5,3	2,302	2,304	2,307	2,309	2,311	2,313	2,315	2,317	2,319	2,322	0 0 1 1 1 1 2 2 2
5,4	2,324	2,326	2,328	2,330	2,332	2,335	2,337	2,339	2,341	2,343	0 0 1 1 1 1 1 2 2

x	0	1	2	3	4	5	6	7	8	9	Differences 1 2 3 4 5 6 7 8 9
5,5	2,345	2,347	2,349	2,352	2,354	2,356	2,358	2,360	2,362	2,364	0 0 1 1 1 1 1 2 2
5,6	2,366	2,369	2,371	2,373	2,375	2,377	2,379	2,381	2,383	2,385	0 0 1 1 1 1 1 2 2
5,7	2,387	2,390	2,392	2,394	2,396	2,398	2,400	2,402	2,404	2,406	0 0 1 1 1 1 1 2 2
5,8	2,408	2,410	2,412	2,415	2,417	2,419	2,421	2,423	2,425	2,427	0 0 1 1 1 1 1 2 2
5,9	2,429	2,431	2,433	2,435	2,437	2,439	2,441	2,443	2,445	2,447	0 0 1 1 1 1 1 2 2
6,0	2,449	2,452	2,454	2,456	2,458	2,460	2,462	2,464	2,466	2,468	0 0 1 1 1 1 1 2 2
6,1	2,470	2,472	2,474	2,476	2,478	2,480	2,482	2,484	2,486	2,488	0 0 1 1 1 1 1 2 2
6,2	2,490	2,492	2,494	2,496	2,498	2,500	2,502	2,504	2,506	2,508	0 0 1 1 1 1 1 2 2
6,3	2,510	2,512	2,514	2,516	2,518	2,520	2,522	2,524	2,526	2,528	0 0 1 1 1 1 1 2 2
6,4	2,530	2,532	2,534	2,536	2,538	2,540	2,542	2,544	2,546	2,548	0 0 1 1 1 1 1 2 2
6,5	2,550	2,551	2,553	2,555	2,557	2,559	2,561	2,563	2,565	2,567	0 0 1 1 1 1 1 2 2
6,6	2,569	2,571	2,573	2,575	2,577	2,579	2,581	2,583	2,585	2,587	0 0 1 1 1 1 1 2 2
6,7	2,588	2,590	2,592	2,594	2,596	2,598	2,600	2,602	2,604	2,606	0 0 1 1 1 1 1 2 2
6,8	2,608	2,610	2,612	2,613	2,615	2,617	2,619	2,621	2,623	2,625	0 0 1 1 1 1 1 2 2
6,9	2,627	2,629	2,631	2,632	2,634	2,636	2,638	2,640	2,642	2,644	0 0 1 1 1 1 1 2 2
7,0	2,646	2,648	2,650	2,651	2,653	2,655	2,657	2,659	2,661	2,663	0 0 1 1 1 1 1 2 2
7,1	2,665	2,666	2,668	2,670	2,672	2,674	2,676	2,678	2,680	2,681	0 0 1 1 1 1 1 1 2
7,2	2,683	2,685	2,687	2,689	2,691	2,693	2,694	2,696	2,698	2,700	0 0 1 1 1 1 1 1 2
7,3	2,702	2,704	2,706	2,707	2,709	2,711	2,713	2,715	2,717	2,718	0 0 1 1 1 1 1 1 2
7,4	2,720	2,722	2,724	2,726	2,728	2,729	2,731	2,733	2,735	2,737	0 0 1 1 1 1 1 1 2
7,5	2,739	2,740	2,742	2,744	2,746	2,748	2,750	2,751	2,753	2,755	0 0 1 1 1 1 1 1 2
7,6	2,757	2,759	2,760	2,762	2,764	2,766	2,768	2,769	2,771	2,773	0 0 1 1 1 1 1 1 2
7,7	2,775	2,777	2,778	2,780	2,782	2,784	2,786	2,787	2,789	2,791	0 0 1 1 1 1 1 1 2
7,8	2,793	2,795	2,796	2,798	2,800	2,802	2,804	2,805	2,807	2,809	0 0 1 1 1 1 1 1 2
7,9	2,811	2,812	2,814	2,816	2,818	2,820	2,821	2,823	2,825	2,827	0 0 1 1 1 1 1 1 2
8,0	2,828	2,830	2,832	2,834	2,835	2,837	2,839	2,841	2,843	2,844	0 0 1 1 1 1 1 1 2
8,1	2,846	2,848	2,850	2,851	2,853	2,855	2,857	2,858	2,860	2,862	0 0 1 1 1 1 1 1 2
8,2	2,864	2,865	2,867	2,869	2,871	2,872	2,874	2,876	2,877	2,879	0 0 1 1 1 1 1 1 2
8,3	2,881	2,883	2,884	2,886	2,888	2,890	2,891	2,893	2,895	2,897	0 0 1 1 1 1 1 1 2
8,4	2,898	2,900	2,902	2,903	2,905	2,907	2,909	2,910	2,912	2,914	0 0 1 1 1 1 1 1 2
8,5	2,915	2,917	2,919	2,921	2,922	2,924	2,926	2,927	2,929	2,931	0 0 1 1 1 1 1 1 2
8,6	2,933	2,934	2,936	2,938	2,939	2,941	2,943	2,944	2,946	2,948	0 0 1 1 1 1 1 1 2
8,7	2,950	2,951	2,953	2,955	2,956	2,958	2,960	2,961	2,963	2,965	0 0 1 1 1 1 1 1 2
8,8	2,966	2,968	2,970	2,972	2,973	2,975	2,977	2,978	2,980	2,982	0 0 1 1 1 1 1 1 2
8,9	2,983	2,985	2,987	2,988	2,990	2,992	2,993	2,995	2,997	2,998	0 0 1 1 1 1 1 1 2
9,0	3,000	3,002	3,003	3,005	3,007	3,008	3,010	3,012	3,013	3,015	0 0 0 1 1 1 1 1 1
9,1	3,017	3,018	3,020	3,022	3,023	3,025	3,027	3,028	3,030	3,032	0 0 0 1 1 1 1 1 1
9,2	3,033	3,035	3,036	3,038	3,040	3,041	3,043	3,045	3,046	3,048	0 0 0 1 1 1 1 1 1
9,3	3,050	3,051	3,053	3,055	3,056	3,058	3,059	3,061	3,063	3,064	0 0 0 1 1 1 1 1 1
9,4	3,066	3,068	3,069	3,071	3,072	3,074	3,076	3,077	3,079	3,081	0 0 0 1 1 1 1 1 1
9,5	3,082	3,084	3,085	3,087	3,089	3,090	3,092	3,094	3,095	3,097	0 0 0 1 1 1 1 1 1
9,6	3,098	3,100	3,102	3,103	3,105	3,106	3,108	3,110	3,111	3,113	0 0 0 1 1 1 1 1 1
9,7	3,114	3,116	3,118	3,119	3,121	3,122	3,124	3,126	3,127	3,129	0 0 0 1 1 1 1 1 1
9,8	3,130	3,132	3,134	3,135	3,137	3,138	3,140	3,142	3,143	3,145	0 0 0 1 1 1 1 1 1
9,9	3,146	3,148	3,150	3,151	3,153	3,154	3,156	3,158	3,159	3,161	0 0 0 1 1 1 1 1 1

Square roots from 10 to 100

$x \to \sqrt{x}$

x	0	1	2	3	4	5	6	7	8	9	Differences 1	2	3	4	5	6	7	8	9
10	3.162	3.178	3.194	3.209	3.225	3.240	3.256	3.271	3.286	3.302	2	3	5	6	8	9	11	12	14
11	3.317	3.332	3.347	3.362	3.376	3.391	3.406	3.421	3.435	3.450	1	3	4	6	7	9	10	12	13
12	3.464	3.479	3.493	3.507	3.521	3.536	3.550	3.564	3.578	3.592	1	3	4	6	7	8	10	11	13
13	3.606	3.619	3.633	3.647	3.661	3.674	3.688	3.701	3.715	3.728	1	3	4	5	7	8	10	11	12
14	3.742	3.755	3.768	3.782	3.795	3.808	3.821	3.834	3.847	3.860	1	3	4	5	7	8	9	11	12
15	3.873	3.886	3.899	3.912	3.924	3.937	3.950	3.962	3.975	3.988	1	3	4	5	6	8	9	10	11
16	4.000	4.012	4.025	4.037	4.050	4.062	4.074	4.087	4.099	4.111	1	2	4	5	6	7	9	10	11
17	4.123	4.135	4.147	4.159	4.171	4.183	4.195	4.207	4.219	4.231	1	2	4	5	6	7	8	10	11
18	4.243	4.254	4.266	4.278	4.290	4.301	4.313	4.324	4.336	4.347	1	2	3	5	6	7	8	9	10
19	4.359	4.370	4.382	4.393	4.405	4.416	4.427	4.438	4.450	4.461	1	2	3	5	6	7	8	9	10
20	4.472	4.483	4.494	4.506	4.517	4.528	4.539	4.550	4.561	4.572	1	2	3	4	6	7	8	9	10
21	4.583	4.594	4.604	4.615	4.626	4.637	4.648	4.658	4.669	4.680	1	2	3	4	5	6	8	9	10
22	4.690	4.701	4.712	4.722	4.733	4.743	4.754	4.765	4.775	4.785	1	2	3	4	5	6	7	8	9
23	4.796	4.806	4.817	4.827	4.837	4.848	4.858	4.868	4.879	4.889	1	2	3	4	5	6	7	8	9
24	4.899	4.909	4.919	4.930	4.940	4.950	4.960	4.970	4.980	4.990	1	2	3	4	5	6	7	8	9
25	5.000	5.010	5.020	5.030	5.040	5.050	5.060	5.070	5.079	5.089	1	2	3	4	5	6	7	8	9
26	5.099	5.109	5.119	5.128	5.138	5.148	5.158	5.167	5.177	5.187	1	2	3	4	5	6	7	8	9
27	5.196	5.206	5.215	5.225	5.235	5.244	5.254	5.263	5.273	5.282	1	2	3	4	5	6	7	8	9
28	5.292	5.301	5.310	5.320	5.329	5.339	5.348	5.357	5.367	5.376	1	2	3	4	5	6	7	7	8
29	5.385	5.394	5.404	5.413	5.422	5.431	5.441	5.450	5.459	5.468	1	2	3	4	5	5	6	7	8
30	5.477	5.486	5.495	5.505	5.514	5.523	5.532	5.541	5.550	5.559	1	2	3	4	4	5	6	7	8
31	5.568	5.577	5.586	5.595	5.604	5.612	5.621	5.630	5.639	5.648	1	2	3	3	4	5	6	7	8
32	5.657	5.666	5.675	5.683	5.692	5.701	5.710	5.718	5.727	5.736	1	2	3	3	4	5	6	7	8
33	5.745	5.753	5.762	5.771	5.779	5.788	5.797	5.805	5.814	5.822	1	2	3	3	4	5	6	7	8
34	5.831	5.840	5.848	5.857	5.865	5.874	5.882	5.891	5.899	5.908	1	2	3	3	4	5	6	7	8
35	5.916	5.925	5.933	5.941	5.950	5.958	5.967	5.975	5.983	5.992	1	2	2	3	4	5	6	7	8
36	6.000	6.008	6.017	6.025	6.033	6.042	6.050	6.058	6.066	6.075	1	2	2	3	4	5	6	7	7
37	6.083	6.091	6.099	6.107	6.116	6.124	6.132	6.140	6.148	6.156	1	2	2	3	4	5	6	7	7
38	6.164	6.173	6.181	6.189	6.197	6.205	6.213	6.221	6.229	6.237	1	2	2	3	4	5	6	6	7
39	6.245	6.253	6.261	6.269	6.277	6.285	6.293	6.301	6.309	6.317	1	2	2	3	4	5	6	6	7
40	6.325	6.332	6.340	6.348	6.356	6.364	6.372	6.380	6.387	6.395	1	2	2	3	4	5	6	6	7
41	6.403	6.411	6.419	6.427	6.434	6.442	6.450	6.458	6.465	6.473	1	2	2	3	4	5	5	6	7
42	6.481	6.488	6.496	6.504	6.512	6.519	6.527	6.535	6.542	6.550	1	2	2	3	4	5	5	6	7
43	6.557	6.565	6.573	6.580	6.588	6.595	6.603	6.611	6.618	6.626	1	2	2	3	4	5	5	6	7
44	6.633	6.641	6.648	6.656	6.663	6.671	6.678	6.686	6.693	6.701	1	2	2	3	4	5	5	6	7
45	6.708	6.716	6.723	6.731	6.738	6.745	6.753	6.760	6.768	6.775	1	1	2	3	4	4	5	6	7
46	6.782	6.790	6.797	6.804	6.812	6.819	6.826	6.834	6.841	6.848	1	1	2	3	4	4	5	6	7
47	6.856	6.863	6.870	6.878	6.885	6.892	6.899	6.907	6.914	6.921	1	1	2	3	4	4	5	6	7
48	6.928	6.935	6.943	6.950	6.957	6.964	6.971	6.979	6.986	6.993	1	1	2	3	4	4	5	6	6
49	7.000	7.007	7.014	7.021	7.029	7.036	7.043	7.050	7.057	7.064	1	1	2	3	4	4	5	6	6
50	7.071	7.078	7.085	7.092	7.099	7.106	7.113	7.120	7.127	7.134	1	1	2	3	4	4	5	6	6
51	7.141	7.148	7.155	7.162	7.169	7.176	7.183	7.190	7.197	7.204	1	1	2	3	4	4	5	6	6
52	7.211	7.218	7.225	7.232	7.239	7.246	7.253	7.259	7.266	7.273	1	1	2	3	3	4	5	6	6
53	7.280	7.287	7.294	7.301	7.308	7.314	7.321	7.328	7.335	7.342	1	1	2	3	3	4	5	5	6
54	7.349	7.355	7.362	7.369	7.376	7.382	7.389	7.396	7.403	7.410	1	1	2	3	3	4	5	5	6

x	0	1	2	3	4	5	6	7	8	9	Differences 1	2	3	4	5	6	7	8	9
55	7.416	7.423	7.430	7.436	7.443	7.450	7.457	7.463	7.470	7.477	1	1	2	3	3	4	5	5	6
56	7.483	7.490	7.497	7.503	7.510	7.517	7.523	7.530	7.537	7.543	1	1	2	3	3	4	5	5	6
57	7.550	7.556	7.563	7.570	7.576	7.583	7.589	7.596	7.603	7.609	1	1	2	3	3	4	5	5	6
58	7.616	7.622	7.629	7.635	7.642	7.649	7.655	7.662	7.668	7.675	1	1	2	3	3	4	5	5	6
59	7.681	7.688	7.694	7.701	7.707	7.714	7.720	7.727	7.733	7.740	1	1	2	3	3	4	4	5	6
60	7.746	7.752	7.759	7.765	7.772	7.778	7.785	7.791	7.797	7.804	1	1	2	3	3	4	4	5	6
61	7.810	7.817	7.823	7.829	7.836	7.842	7.849	7.855	7.861	7.868	1	1	2	3	3	4	4	5	6
62	7.874	7.880	7.887	7.893	7.899	7.906	7.912	7.918	7.925	7.931	1	1	2	3	3	4	4	5	6
63	7.937	7.944	7.950	7.956	7.962	7.969	7.975	7.981	7.987	7.994	1	1	2	3	3	4	4	5	6
64	8.000	8.006	8.012	8.019	8.025	8.031	8.037	8.044	8.050	8.056	1	1	2	3	3	4	4	5	6
65	8.062	8.068	8.075	8.081	8.087	8.093	8.099	8.106	8.112	8.118	1	1	2	2	3	4	4	5	6
66	8.124	8.130	8.136	8.142	8.149	8.155	8.161	8.167	8.173	8.179	1	1	2	2	3	4	4	5	5
67	8.185	8.191	8.198	8.204	8.210	8.216	8.222	8.228	8.234	8.240	1	1	2	2	3	4	4	5	5
68	8.246	8.252	8.258	8.264	8.270	8.276	8.283	8.289	8.295	8.301	1	1	2	2	3	4	4	5	5
69	8.307	8.313	8.319	8.325	8.331	8.337	8.343	8.349	8.355	8.361	1	1	2	2	3	4	4	5	5
70	8.367	8.373	8.379	8.385	8.390	8.396	8.402	8.408	8.414	8.420	1	1	2	2	3	4	4	5	5
71	8.426	8.432	8.438	8.444	8.450	8.456	8.462	8.468	8.473	8.479	1	1	2	2	3	4	4	5	5
72	8.485	8.491	8.497	8.503	8.509	8.515	8.521	8.526	8.532	8.538	1	1	2	2	3	3	4	5	5
73	8.544	8.550	8.556	8.562	8.567	8.573	8.579	8.585	8.591	8.597	1	1	2	2	3	3	4	5	5
74	8.602	8.608	8.614	8.620	8.626	8.631	8.637	8.643	8.649	8.654	1	1	2	2	3	3	4	5	5
75	8.660	8.666	8.672	8.678	8.683	8.689	8.695	8.701	8.706	8.712	1	1	2	2	3	3	4	5	5
76	8.718	8.724	8.729	8.735	8.741	8.746	8.752	8.758	8.764	8.769	1	1	2	2	3	3	4	5	5
77	8.775	8.781	8.786	8.792	8.798	8.803	8.809	8.815	8.820	8.826	1	1	2	2	3	3	4	5	5
78	8.832	8.837	8.843	8.849	8.854	8.860	8.866	8.871	8.877	8.883	1	1	2	2	3	3	4	4	5
79	8.888	8.894	8.899	8.905	8.911	8.916	8.922	8.927	8.933	8.939	1	1	2	2	3	3	4	4	5
80	8.944	8.950	8.955	8.961	8.967	8.972	8.978	8.983	8.989	8.994	1	1	2	2	3	3	4	4	5
81	9.000	9.006	9.011	9.017	9.022	9.028	9.033	9.039	9.044	9.050	1	1	2	2	3	3	4	4	5
82	9.055	9.061	9.066	9.072	9.077	9.083	9.088	9.094	9.099	9.105	1	1	2	2	3	3	4	4	5
83	9.110	9.116	9.121	9.127	9.132	9.138	9.143	9.149	9.154	9.160	1	1	2	2	3	3	4	4	5
84	9.165	9.171	9.176	9.182	9.187	9.192	9.198	9.203	9.209	9.214	1	1	2	2	3	3	4	4	5
85	9.220	9.225	9.230	9.236	9.241	9.247	9.252	9.257	9.263	9.268	1	1	2	2	3	3	4	4	5
86	9.274	9.279	9.284	9.290	9.295	9.301	9.306	9.311	9.317	9.322	1	1	2	2	3	3	4	4	5
87	9.327	9.333	9.338	9.343	9.349	9.354	9.359	9.365	9.370	9.375	1	1	2	2	3	3	4	4	5
88	9.381	9.386	9.391	9.397	9.402	9.407	9.413	9.418	9.423	9.429	1	1	2	2	3	3	4	4	5
89	9.434	9.439	9.445	9.450	9.455	9.460	9.466	9.471	9.476	9.482	1	1	2	2	3	3	4	4	5
90	9.487	9.492	9.497	9.503	9.508	9.513	9.518	9.524	9.529	9.534	1	1	2	2	3	3	4	4	5
91	9.539	9.545	9.550	9.555	9.560	9.566	9.571	9.576	9.581	9.586	1	1	2	2	3	3	4	4	5
92	9.592	9.597	9.602	9.607	9.613	9.618	9.623	9.628	9.633	9.638	1	1	2	2	3	3	4	4	5
93	9.644	9.649	9.654	9.659	9.664	9.670	9.675	9.680	9.685	9.690	1	1	2	2	3	3	4	4	5
94	9.695	9.701	9.706	9.711	9.716	9.721	9.726	9.731	9.737	9.742	1	1	2	2	3	3	4	4	5
95	9.747	9.752	9.757	9.762	9.767	9.772	9.778	9.783	9.788	9.793	1	1	2	2	3	3	4	4	5
96	9.798	9.803	9.808	9.813	9.818	9.823	9.829	9.834	9.839	9.844	1	1	2	2	3	3	4	4	5
97	9.849	9.854	9.859	9.864	9.869	9.874	9.879	9.884	9.889	9.894	1	1	1	2	3	3	4	4	5
98	9.900	9.905	9.910	9.915	9.920	9.925	9.930	9.935	9.940	9.945	0	1	1	2	2	3	3	4	4
99	9.950	9.955	9.960	9.965	9.970	9.975	9.980	9.985	9.990	9.995	0	1	1	2	2	3	3	4	4

Index

nswers

ercise 1a (p. 2)

(a) 3 m (b) 1 m (c) 86 ha
(d) 342 cm (e) 496 km (f) 165 mm
(g) 53 (h) \$18 (i) 130 litres
(j) \$100

(a) 0,1 (b) 18,0 (c) 19,0
(d) 0,8 (e) 7,9 (f) 20,1

(a) 37 (b) 110 (c) 0,009 3
(d) 5,0 (e) 0,024 (f) 86 000
(g) 140 (h) 9,0 (i) 3,0
(j) 0,030

(a) 7 580 (b) 52 100 (c) 352 000
(d) 1 790 (e) 0,089 2 (f) 170
(g) 83,4 (h) 0,906 (i) 828 000
(j) 8,01

\$389 million, \$390 million,
Almost \$400 million, \$0,39 billion

(a)

town	population
Bulawayo	500 000
Chitungwiza	200 000
Harare	700 000

(b)

town	population
Bulawayo	500 000
Chitungwiza	170 000
Harare	660 000

\$1,45 billion and \$1,55 billion

(a)

1984	1985	1986	1987	1988
00 000	400 000	400 000	500 000	500 000

(b) 2 100 000 (true value is 2 141 872)

(a) 28 (b) 300 (c) 900 (d) 0,002

(a) 34,62 (b) 327,9 (c) 742,8 (d) 0,002 135

Exercise 1b (p. 4)

1 (a) 8,5 cm to 9,5 cm (b) 1,745 m to 1,755 m
(c) 6 350 km to 6 450 km (d) 9,75 m to 9,85 m
(e) 8,5 million to 9,5 million
(f) 7,45 litres to 7,55 litres
(g) \$19 500 to \$20 500 (h) 75,55 kg to 75,65 kg
(i) 30,15 °C to 30,25 °C
(j) 859,5 cm^3 to 860,5 cm^3

2 (a) 0,667% (b) 0,83% (c) 0,4%
(d) 2% (e) 2,5% (f) 0,806%
(g) 6,25% (h) 2,38% (i) 1,25%
(j) 5,6%

3 (a) 4 000 g (4 kg) (b) 4 140 g (4,14 kg)
(c) −3,4%

4 (a) \$350 (b) \$340,06
(c) +2,9%

5 −8,0% **6** 102 g, 100 g

7 244,8 kg, 235,2 kg

8 6 175 m^2, 7 875 m^2

9 (a) 11π cm, 13π cm (b) $30\frac{1}{4}\pi$ cm^2, $42\frac{1}{4}\pi$ cm^2

10 (a) 337,5 cm^2, 433,5 cm^2
(b) 421,875 cm^3, 614,125 cm^3

Exercise 1c (p. 6)

1 (a) $1\frac{3}{4}$ h (b) 60 km/h

2 39 000 cm^3

3 132 cm

4 (a) 5,442 (b) 33,236 (c) 14 (d) 0,11

5 (a) 0,095 or 0,10 (b) 0,83 or 0,8 (c) 23

6 250 mm to 280 mm, 590 to 670

7 (a) 5,6 m (b) 5,9 m

8 800 km/h

9 1 000 cm^3

10 2 800 cm^2

Exercise 2a (p. 7)

1 (c) XY = 4,1 cm, CY = 6,25 cm

2 (d) a rhombus (e) 11,9 cm, 3,3 cm

3 (c) 3,5 cm **4** (b) 8,7 cm

5 (b) 8,25 cm **6** (b) 9,9 cm

Exercise 2b (p. 8)

1 (c) The altitudes meet at one point inside the triangle.

2 (c) Yes (outside the triangle)

3 (c) 43 mm **4** (c) 8,9 cm
5 (d) The circle touches all three sides of $\triangle ABC$
(e) 2,05 cm
6 (c) 1,7 cm
7 (d) (i) 3,4 cm, (ii) the circle touches all four sides of ABCD
8 (c) X, Y, Z lie in a straight line (d) yes

Exercise 2c (p. 10)
1 a spiral
2 an oval shape or **ellipse**
3 an arc of a circle
4 (a), (b): major arc of a circle; (c) a semicircle
5 a circle of diameter 2 cm
6 (a) a semicircle (b) a circle
(c) an arc of a circle subtending 30° at its centre
(d) a straight line (e) a circular arc
(f)

Fig. A1

7 a straight line at an angle to the direction in which the car is moving
8 a hemispherical surface (smaller than that of the bowl)
9 a straight line 1,5 cm from AB
10 a circle with the foot of the pole at its centre
11 a semicircle
12 the perpendicular bisector of the line joining the two points
13 two straight lines, parallel to and on either side of AB
14 42 m
15 88 cm

Exercise 2d (p. 13)
1 two; 6,7 cm **2** four
3 2,7 cm, 8,7 cm **4** 1,8 cm, 6,2 cm
5 23°, $56\frac{1}{2}$° **6** 5,7 cm
7 6 cm **8** 42 m
10 8 cm **11** (b) 11,3 cm

Exercise 2e (p. 16)
1 P is the point of intersection of side YZ and the bisector of $\hat{X}$
2 two
4 3,2 cm or 9,2 cm
8 AB = 4,1 cm; AC = 6,4 cm
10 (c) 0,8 cm or 1,1 cm

12 the circumcentres lie (a) inside (b) outside (c) at the mid-point of the hypotenuse of the triangles respectively
13 PZ = 4,6 cm
14 (b) AC = 9,6 cm

Exercise 3a (p. 18)
1 (a) 18° (b) 45° (c) 63° (d) 70°
2 (a) 36° (b) 42° (c) 124° (d) 70°
3 (a) 15 cm (b) 5 cm
4 (a) 43° (b) 63° (c) 48° (d) 32°
5 4,5 cm **6** 3,3 cm
7 141° **8** 8 cm
9 (a) (i) $(90 - x)°$, (ii) $(90 - x)°$
(b) $O\hat{B}A = A\hat{B}C$
10 (a) 90° (b) $(90 - x)°$
(c) $A\hat{D}B = x°$ (*sum of angles of* $\triangle A\hat{D}B$)
$\therefore B\hat{A}T = A\hat{D}B = x°$

Exercise 3a (p. 18)
1 (a) 36° (b) 57° (c) 46° (d) 12 cm
2 $T\hat{A}O = T\hat{B}O = 90°$ (*radius* ⊥ *tangent*)
$\therefore$ TAOB is cyclic (*opposite angles are supplementary*)
3 43° **4** 106°
5 59° or 121° **10** 134°

Exercise 3c (p. 21)
1 5 cm; 4 cm; 2 cm
2 2 cm; 3 cm; 4 cm
3 5 cm; 6 cm; 14 cm
4 (a) 14 cm (b) 4 cm
8 86 cm

Exercise 3d (p. 23)
1 (a) $A\hat{B}Y = A\hat{C}Y$ (b) $C\hat{B}Y$, $C\hat{A}Y$
(c) $B\hat{C}Y$ (d) $B\hat{A}Y$
(e) 58° (f) 112°
(g) 55° (h) 80°
2 (a) 56° (b) 52° (c) 43°
(d) 102° (e) 78°
3 ZYQ **4** 84°
5 72° **6** 82°; 49°; 49°
7 54° **8** 40°
9 101°; 74° **10** 34°; 77°; 69°
11 50°; 60°; 70° **12** 44°; 108°

Exercise 4a (p. 26)
1 0,939 7 **2** −0,342 0 **3** −2,747
4 0,454 0 **5** 0,990 3 **6** −0,275 6
7 −0,788 0 **8** −0,230 9 **9** −19,08

10 −0,615 7   11 0,398 7   12 −0,113 9
13 −0,947 8   14 −1,122   15 0,998 4
16 −0,714 5   17 −0,226 7   18 0,877 1
19 0,028 8   20 −21,45   21 0,271 9
22 −0,482 2   23 −0,522 8   24 0,996 0

Exercise 4b (p. 27)

1 36°   **2** 144°   **3** 75°
4 105°   **5** 67°, 113°   **6** 117°
7 160°   **8** 25°, 155°   **9** 98°
10 148°   **11** 136,7°
12 37,5°, 142,5°   **13** 74,7°; 105,3°
14 115,4°   **15** 162,43°   **16** 115,17°
17 56,4°, 123,6°   **18** 144,13°
19 73,62°, 106,38°   **20** 141,68°
21 120,8°   **22** 15,23°; 164,77°
23 48,15°; 131,85°   **24** 94,6°

Exercise 4c (p. 29)

1 $\dfrac{x}{\sin 80^\circ} = \dfrac{4}{\sin 30^\circ}$ or $x = \dfrac{4 \sin 80^\circ}{\sin 30^\circ}$

The answer to questions 2–12 have been rounded to 3 s.f.

2 13,0 cm   **3** 14,5 cm   **4** 21,5 cm
5 48,8°   **6** 41,6°
7 B = 26,5°, C = 38,5°, c = 44,6 m
8 A = 30°, a = 23,8 m, b = 20,5 m
9 X = 98,8°, y = 9,9 cm, z = 15,5 cm
10 B = 29°9′, C = 112°33′, c = 376 m
11 A = 67,4°, B = 16,4°, a = 36,6 cm
12 A = 29°26′, B = 34°58′

Exercise 4d (p. 30)

1 38,1 m   **2** 8,15 km
3 294,8°, 188 m   **4** 190 m
5 4,06 m   **6** 240 km
7 23,6 m, 21,7 m   **8** 165 m, 116 m
9 2,54 km
10 (a) 386 km   (b) 141 km
11 (a) 63°3′   (b) 7,51 cm
12 (a) 060° (b) 340° (c) 9,54 n.mi. (d) 5 n.mi.

Exercise 5a (p. 33)

1 $\frac{2}{7}$ **2** $\frac{3}{2}$ **3** $\frac{3}{5}$ **4** $-\frac{4}{5}$ **5** $-\frac{4}{3}$
6 $\frac{2}{7}$ **7** -2 **8** $-\frac{3}{4}$ **9** 3   **10** $-\frac{1}{2}$

Exercise 5b (p. 34)

1 3   **2** 3   **3** -2 **4** 2   **5** $-\frac{2}{3}$
6 $-\frac{2}{3}$ **7** $\frac{4}{3}$ **8** $\frac{2}{5}$ **9** $\frac{5}{2}$ **10** $-\frac{7}{4}$

Exercise 5c (p. 35)

1

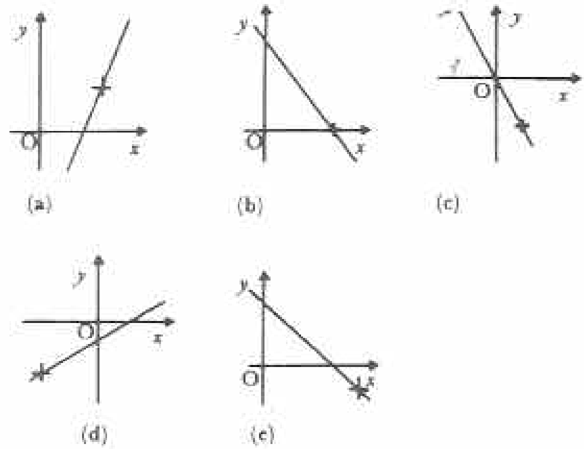


Fig. A2

2 (a) 2 (b) $\frac{1}{3}$ (c) $\frac{3}{4}$ (d) $-\frac{3}{7}$ (e) $\frac{4}{7}$
3 (a) (0; 2), (1; 0)   (b) (0; 1), (−3; 0)
(c) (0; −6), (10; 0)   (d) $(0; \frac{2}{3})$, $(\frac{1}{2}; 0)$
(e) $(0; \frac{4}{5})$, $(\frac{1}{2}; 0)$
4 (a) 1   (b) −2   (c) $1\frac{1}{3}$ (d) $-\frac{5}{4}$
5 (a) −1   (b) undefined
(c) 1   (d) 0   (e) $\frac{1}{2}$

Exercise 5d (p. 37)

1 (a) $y = 3x - 3$ (b) $y = 3x$
(c) $x + y = 6$ (d) $3x + 4y = 18$
(e) $4x + y + 5 = 0$ (f) $2y = 5x + 9$
2 (a) $7x - 3y = 0$ (b) $7x + 3y = 0$
(c) $x + y = 3$ (d) $3x - 10y = 42$
(e) $5x + 16y = 67$ (f) $5x + 2y + 12 = 0$
3 (a) $\frac{1}{3}$ (b) $y = \frac{1}{3}x - \frac{7}{3}$
4 (a) $2y = 9x - 42$ (b) $2y = 9x + 1$
5 l: $3y = x - 13$ m: $5y = 13x - 3$
6 (a) 2   (b) $4x + y = 17$
7 (a) −1
8 (a) $\dfrac{k-1}{2}$; $k = 2\frac{1}{5}$ (b) $2x + y + 3 = 0$
9 AB: $y = 0$, BC: $3x + y = 21$, AC: $y = 1\frac{1}{5}x$
10 (a) $k = -4$ (b) $2y = x + 14$

Exercise 5e (p. 39)

1 (a) $x = \frac{1}{2}$ (b) $x = 0$
2 (a) (i) 6 (ii) −2   (b) −5   (c) $x = 3$
3 (a) (i) 2 (ii) −4   (b) 4   (c) $x = -1$
4 (c) (i) 3 (ii) −1   (d) $x = 1\frac{1}{2}$ (e) $2\frac{1}{4}$
5 (a) 1,2   (b) 0,6   (c) −0,8
6 (a) $x = 1$ (b) 8   (c) 1
7 (a) 4   (b) 0   (c) −4

8 (a) (i) 5 (ii) 1 (iii) −7 (b) $x = 1\frac{1}{4}$
9 (a) 2 (b) 0 (c) −3 (d) −4
10 (a) −4 (b) −2 (c) 0 (d) 3

Exercise 6a (p. 42)

1 $A\hat{O}K$ ($X\hat{O}K$), $A\hat{O}L$ ($X\hat{O}L$), $A\hat{O}M$ ($X\hat{O}M$)
2 (a) $V\hat{O}A$, $V\hat{O}B$, $V\hat{O}C$ (b) $V\hat{B}O$
3 (a) OA, OB, OC, OD, OK, OL, OM, ON
(b) VBD (c) VKM
(d) $V\hat{B}O$ (or $V\hat{B}D$) (e) $V\hat{A}O$, $V\hat{C}O$, $V\hat{D}O$
(f) $V\hat{K}O$ (or $V\hat{K}M$) (g) $V\hat{L}O$, $V\hat{M}O$, $V\hat{N}O$
(h) $M\hat{V}O$
4 (a) PQRS, TUVW (b) SP, RQ, WV, TU
(c) all except $Q\hat{V}S$
(d) (i) VT (ii) QW (iii) RT
(e) (i) $P\hat{R}U$ (ii) $R\hat{U}S$ (iii) $R\hat{U}Q$
5 (a) BCDF (b) AFD
(c) (i) AD (ii) CF
(d) (i) $B\hat{E}C$ (ii) $E\hat{B}D$
6 (a) true (b) true (c) false
(d) true (e) true

Exercise 6b (p. 44)

1 $D\hat{E}H$ (or $A\hat{F}G$), 90°
2 $E\hat{C}H$ 3 $A\hat{O}B$
4 (a) $V\hat{O}D$ (or $V\hat{O}B$) (b) $V\hat{O}M$ (or $V\hat{O}K$)
(c) $V\hat{M}O$ (d) $V\hat{N}O$
(e) $A\hat{O}N$ (or $C\hat{O}L$) (f) $N\hat{V}L$
5 (a) $D\hat{C}B$ (b) $B\hat{A}C$ (or $F\hat{E}D$)
(c) $A\hat{C}B$ (or $E\hat{D}F$) (d) $A\hat{B}C$ (or $E\hat{F}D$)
6 (c) $Q\hat{W}X$ (d) $Q\hat{W}P$ (e) 90°
7 (a) $B\hat{P}Q$ (or $C\hat{S}R$) (b) $\frac{2}{3}$
8 (a) $P\hat{N}O$ (b) 0.3 (0.3333)
9 (a) $Q\hat{M}O$ (or $Q\hat{M}N$) (b) $\frac{1}{2}$ (i.e. $\frac{3}{6}$)
(c) $P\hat{N}O$ (or $P\hat{X}Y$) (d) $\frac{3}{4}$
10 (b) $V\hat{M}O$ (c) $Q\hat{N}S$

Exercise 6c (p. 49)

1 (a) 19 cm (b) 6 cm
(c) 15 cm (d) 21 cm
2 (a) 10 cm (b) 15.6 cm ($\sqrt{244}$)
(c) 33.7°
3 18.4° 4 41.8°
5 (a) 38.9° (b) 28.3 cm
6 25°
7 (a) 50 cm (b) 67.4° (c) 130 cm
8 (a) 44.7 cm (b) 63.4°
9 (a) 7 cm (b) (i) 51.1° (ii) 60.3°
10 (a) 5 cm (b) 53.1° (c) 57°
11 (a) 49.2° (b) 60.3°
12 (a) 15 cm (b) 17 cm
(c) (i) 28.1° (ii) 44.9° (iii) 32°
(d) 41.6° (e) 56.3°
13 (a) 1.5 m (b) 2.5 m (c) 37°
14 (a) 10.3 cm (b) 64.1° (c) 343 cm^2
15 (a) 144 cm^2 (b) 8.5 cm
(c) 407 cm^3 (d) 45°
16 (b) 12.7 cm (c) 48.3° (d) 57.8°
17 (a) 26.6° (b) 36.9°
18 (a) 3 m (b) 36.9°
19 (a) 6 cm (b) 3.07 cm ($10 - 4\sqrt{3}$)
(c) 48.6°
20 (a) 100 cm (b) 88° (c) 56°

Exercise 6d (p. 52)

1 (a) AB or EF (b) PB or SC
(c) (i) NP (ii) MQ
(d) (i) MV (ii) NV
(e) ST (f) CB or DA
2 (a) AV or DV (b) AP or BP
(c) PA (d) SA or RB
3 45°; 31° 4 (a) 23.6° (b) 22.2°
5 8° 6 16.3°; 11.2°

Exercise 7a (p. 54)

1 $\frac{m}{r}$ 2 $\frac{4x}{5y}$ 3 No simpler form
4 $\frac{b+c}{d+e}$ 5 $\frac{a+b}{a+c}$ 6 No simpler form
7 $\frac{u}{v}$ 8 $\frac{h-k}{k}$ 9 $\frac{1}{3dw}$
10 $-\frac{c}{d}$ 11 $-\frac{a+b}{b}$ 12 $\frac{x}{x-y}$
13 $\frac{4cd^2}{3e^2}$ 14 No simpler form 15 $\frac{c-d}{c}$
16 $\frac{m+n}{m-n}$ 17 $\frac{c+3}{c+2}$ 18 $\frac{d+3}{d-4}$
19 $-\frac{m}{n}$ 20 $\frac{y}{x-y}$ 21 No simpler form
22 No simpler form 23 $\frac{h+k}{h-k}$ 24 $\frac{u-2v}{u+4v}$

25 $\frac{x+3y}{x-y}$ 26 $-\frac{x+3}{x+5}$ 27 $-\frac{3a+m}{a+m}$

28 $-\frac{a+4}{2a+1}$ 29 $\frac{a-n}{a+n}$ 30 $\frac{a-n}{a+n}$

31 $\frac{w+u-v}{w-u+v}$ 32 $\frac{a+b+c}{b-a-c}$ 33 $\frac{m(m-a)}{a(m+a)}$

34 $\frac{3}{5}$ 35 $\frac{b+c}{b-c}$ 36 $\frac{a-b}{c-a}$

Exercise 7b (p. 55)

1 $\frac{a}{c}$ 2 $\frac{8n^2}{9c^2}$ 3 $\frac{n}{3}$ 4 $\frac{4}{3u}$

5 $\frac{b}{2}$ 6 $\frac{a+2}{a}$ 7 $\frac{m+3}{m}$ 8 $\frac{3u^3v}{16m^2}$

9 $-2a$ 10 $\frac{d(d-2)}{2}$ 11 $\frac{a}{2}$ 12 $\frac{n-3}{n}$

13 $-\frac{n}{m}$ 14 $\frac{a+2b}{a+3b}$ 15 $2c$ 16 $-\frac{c^2}{d^2}$

17 $\frac{2e(e-2)}{3(e-1)}$ 18 $\frac{u-2}{3u}$

19 $\frac{(x+1)(x+2)}{x(x-2)}$ 20 $\frac{a}{b}$

Exercise 7c (p. 56)

1 $\frac{8a+9c}{6abc}$ 2 $\frac{5c-a+b}{c}$ 3 $\frac{3c-14b}{30bc}$

4 $\frac{7}{6(x+y)}$ 5 $\frac{2}{a-2b}$ 6 $\frac{3a-b}{a-b}$

7 $\frac{x+4y}{x+2y}$ 8 $\frac{9}{20(u-v)}$ 9 $\frac{u-9v}{2u+3v}$

10 $\frac{6a-b}{2(2a+b)}$ 11 $\frac{19mn}{6(m^2+n^2)}$ 12 $\frac{3}{2(2x-y)}$

13 $\frac{c-a}{ac}$ 14 $\frac{u(u-2v)}{v^2}$ 15 $\frac{5d+7}{6(d-4)}$

16 $\frac{5a+7}{(a+1)(a+2)}$ 17 $\frac{3x^2+2x+4}{(x-1)(x+2)}$

18 $-\frac{4}{(e+2)(e+3)}$ 19 $\frac{4m}{(m-n)^2}$

20 $\frac{c-d}{(2c+d)^2}$ 21 $\frac{2a-b}{(a-2b)^2}$ 22 $\frac{3}{c-d}$

23 $\frac{2}{4m+3n}$ 24 $-\frac{m+n}{n}$ 25 $\frac{2}{(c+5)(c-2)}$

26 $\frac{7d-6}{(d+2)(d-2)(d-4)}$ 27 $-\frac{a+11b}{6(a+b)(a-b)}$

28 $\frac{7d+12}{(d-4)(d+4)^2}$ 29 $\frac{2}{x-3}$

30 $\frac{a^2+7a+5}{(a-2)(a-3)(a+4)}$

Exercise 7d (p. 58)

1 $\frac{1}{5}$ 2 1 3 $8\frac{1}{4}$ 4 21

5 d 6 $\frac{4a+5}{5a+8}$ 7 $-\frac{5}{3m}$

8 $\frac{3w-8}{8w-5}$ 9 $\frac{5-a}{7a+4}$ 10 $\frac{m+5}{3m+1}$

Exercise 7e (p. 59)

1 3; -1 2 $\frac{1}{2}$; 2 3 3; $-\frac{2}{3}$ 4 1; -3

5 -2; $1\frac{1}{3}$ 6 -1; -6 7 $-\frac{1}{3}$; 2 8 5; 1

9 -1; -2 10 0; 2 11 5; $-\frac{2}{3}$ 12 1

13 $2\frac{1}{5}$ 14 1; 3 15 6; $-1\frac{1}{3}$ 16 1

17 $-\frac{1}{4}$ 18 5; $8\frac{1}{2}$ 19 3; $1\frac{3}{4}$ 20 -2; $1\frac{3}{5}$

21 3; $-7\frac{3}{8}$ 22 3; $1\frac{1}{19}$ 23 $2\frac{2}{3}$ 24 2; -3

25 $-\frac{3}{4}$ 26 $-1\frac{1}{6}$ 27 0 28 -9

29 $-\frac{1}{3}$ 30 -2

Exercise 7f (p. 61)

1 3 2 4 3 -7 4 0

5 $2\frac{1}{2}$ 6 -5 7 $6\frac{2}{3}$ 8 0; -2

9 0; $\frac{1}{2}$ 10 -4; -3 11 ± 1 12 6

13 0; -4; 9 14 1; 2 15 10; -2

16 5; -2 17 -6 18 0; ± 3

19 0; ± 2 20 1; -3; -5

Exercise 7g (p. 62)

1 $\frac{y+z}{y-z}$ 2 $\frac{2m-5}{m-3}$; 3 3 $\frac{3(2a-1)}{2(a-1)}$; 1

4 $\frac{6-x^2+10x}{2x}$ 5 $\frac{5}{6(x-2)}$

6 (a) $-\frac{a+2b}{b(a-b)}$ (b) $\frac{1}{a+b}$ 7 $\frac{3x+2}{4}$

8 $\frac{2}{x}$ 9 $\frac{2y-5x}{4}$ 10 $\frac{6m+1}{2m+3}$

11 $m = 1$ (*Note:* $m = 2$ is *not* a solution since the terms of the equation are not defined when $m = 2$)
12 $x = \frac{1}{2}$ 13 $a = 4$ or $4\frac{1}{2}$
14 10
15 (a) -3 (b) $x < 4$ (but not -3)
16 (a) $-5; 4$ (b) $-5\frac{1}{2}$
17 (a) $a = 6$ (b) $\dfrac{3(a-6)}{(a+4)(a-2)}$
18 (a) $\dfrac{3(b-5)}{2(b+1)(b-3)}$ (b) $b = 5$
19 $\dfrac{1}{a+2}$; -2
20 (a) -2 (b) (i) 0; 3 (ii) $k = -2$

Exercise 8a (p. 64)
1 4,5 unit^2 2 2,25 unit^2 3 10,7 unit^2
4 14,9 unit^2 5 36 unit^2 6 9 unit^2

Exercise 8b (p. 66)
1 (b) 0,9 m/s^2; 0,15 m/s^2 (c) 1 600 m
2 (b) 1 100 m^2 (c) $1,2 \times 10^6$ litres/second
3 (b) 3; -1
(c) velocity in m/s; the object is not moving
4 (b) 0,18 cm/s; 0,10 cm/s
(c) round-bottomed flask with thin neck
5 when $t = 1$, $v = -2$; when $t = 7$, $v = 40$
(i) 0,65; 3,85 (ii) 2,25 s (iii) 15 m/s^2
6

t	0	1	2	3	4	5	6
v	6	10	12	12	10	6	0

(c) $v = 12,25$ m/s when $t = 2,5$ s
(d) -5 m/s^2 (a deceleration) (e) 31,5 m
7 (b) 1,2 m/s^2; 3,68 m/s^2 (c) 6,4 m
8 (a) 29,8 m/s when $t = 27$
(b) 0,38 m/s^2 (c) 1 360 m
9 0,067 m/s^2; $-0,047$ m/s^2; 5 600 m
10 600 m; 0,60 m/s^2; $-0,15$ m/s^2; 45,3 km/h after 43 s

Exercise 9a (p. 69)
1 $25x$ g 2 \$$4y$ 3 $15t$ km 4 $16n$ cents
5 $8d$ mℓ 6 $C = 7n$ 7 $D = 16t$ 8 $x = \frac{1}{4}y$
9 $d = 4s$ 10 $a = 0,8b$
11 (a) $D = 4s$ (b) 44
12 (a) $x = 2\frac{1}{2}y$ (b) 25 (c) 5,6
13 (a) $P = \frac{3}{8}Q$ (b) 6 (c) 6,4
14 (a) 0,9 (b) $3\frac{1}{3}$
15 (a) 0,36 (b) 55
16 (a) $D = 4S$ (b) $S = 48$
17 (a) $y = \frac{1}{5}x$ (b) $y = 8$ (c) $x = 50$
18 $p = 25q$
19 (a) $x = \frac{3}{5}y$ (b) $x = 20$
20 225 cm^2 21 $y \propto x$ (or $x \propto y$)
22

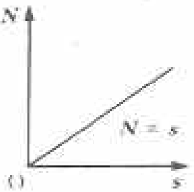


Fig. A3

23 (b) $B = 56\frac{2}{3}$
24 (b) £1 = \$1,60 (c) \$11,20 (d) £5
25 (b) $H = \frac{13}{15}V$ (or, by graph, $H = 0,9V$)
(c) $V = 7,5$ (d) $H = 3,9$

Exercise 9b (p. 71)
1 $a = 2$ 2 $y = 50$
3 (a) $x = 5y^2$ (b) $x = 80$ (c) $y = 5$
4 (a) $A = \frac{1}{2}B^3$ (b) $A = 108$ (c) $B = 3$
5 (a) $P = \frac{5}{2}\sqrt{Q}$ (b) $P = 7\frac{1}{2}$ (c) $Q = \frac{9}{16}$
6 (a) $Z^2 = 3T$ (b) $Z = 12$ (c) $T = 12$
7 (a) $V = \frac{1}{2}D^3$ (b) $V = 13\frac{1}{2}$ (c) $D = 1,6$
8 (a) $D = \sqrt{\dfrac{3H}{2}}$ (b) $D = 15$ (c) $H = 73\frac{1}{2}$
9 $y \propto x^2$
10

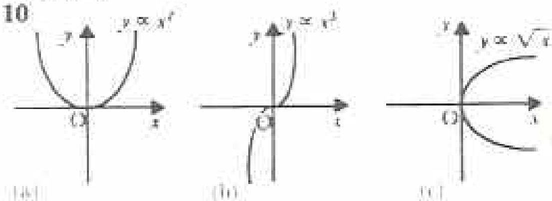


Fig. A4

11

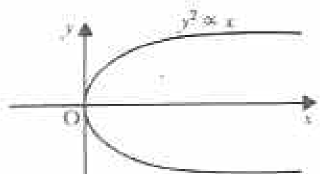


Fig. A5

12 $y = 4x^2$, $y = 1$ when $x = \frac{1}{2}$
13 8,8 amps 14 35 km

15 (a) x increases by 21%
(b) x decreases by 19%
16 x increases by 20%
17 72,8%
18 $27\frac{3}{4}$%

Exercise 9c (page 73)

1 $d \propto \frac{1}{t}$

2 (a) inversely (b) $n \propto \frac{1}{l}$

3 (a) $l = \frac{A}{b}$ (b) $b = \frac{A}{l}$ (c) inversely

4 $x = \frac{66}{y}$
5 $R = \frac{32}{T}$

6 $y = 1$
7 $Q = \frac{4}{5}$

8 $y = \frac{1}{\sqrt{5}}$
9 $x \propto \frac{1}{y^2}$

10 $y = k + \frac{h}{x^2}$
11 $R = \frac{k}{r^2}$

12 $v = 2,25$

13

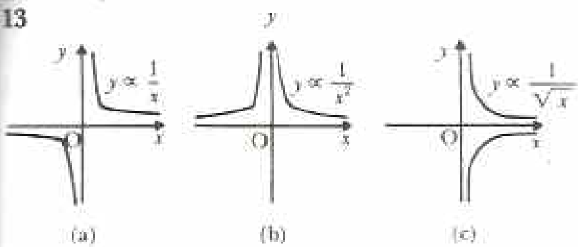


Fig. A6

14 $Y = 4$
15 11,25 km

Exercise 9d (p. 75)

1 (a) $x = 5yz$ (b) 120

2 (a) $x = \frac{6y}{z}$ (b) 7

3 (a) $p = \frac{6q}{r^2}$ (b) $p = 37\frac{1}{2}$

4 $k = \frac{kV}{r^2}$

5 (a) $A = 5$ (b) $C = 8$ (c) A decreases by 1%

6 (a) $x = 37\frac{1}{2}$ (b) $y = \pm 4$ (c) x is doubled

7 $x = 900$
8 $P \propto \frac{1}{R^2}$

9 $x \propto z^3$
10 $x \propto z^4$

11 $A \propto \frac{1}{C}$

13 (a) $y = \frac{1}{2x}$ (b) $z = \frac{x^3}{24}$ (c) 33,1%

14 26,1 kg

15 From the given values, $E = \frac{4R}{d^2}$

(a) $R = 36$ (b) $d = \sqrt{24}$ (c) 9,2%

Exercise 9e (p. 76)

1 (a) $x = 20 + 5y$ (b) 35
2 (a) $x = 2 + 3y$ (b) 32
3 (a) $x = 5 + \frac{2}{3}y$ (b) $7\frac{2}{3}$
4 (a) $D = 30 + 3V$ (b) 249
5 \$35,50
6 $A = 100$
7 (a) $k = 2$ (b) $y = 10$
8 (a) $P = \frac{1}{2}Q - 20$

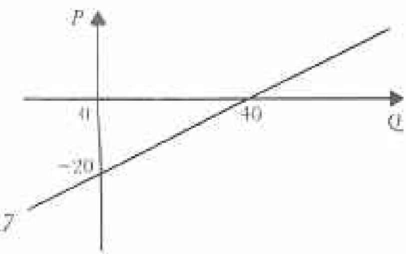


Fig. A7

9 costs: \$75 000, profit: \$15 000
10 \$1 600
11 \$63 500
12 45 km/h
13 90 m
14 \$795
15 $15\frac{4}{9}$

Exercise 10a (p. 79)

1 (a) 120 cm³ (b) 616 cm³ (c) 480 cm³
(d) 48 cm³ (e) 77 cm³ (f) 420 cm³
2 (a) 158 cm² (b) 484 cm² (c) 528 cm²
3 152 litres
4 108 times
5 16 cm
6 (a) 5 000 cm³ (b) 5 500 cm³ (c) 7 500 cm³
7 1,44 tonnes
8 (a) 27 cm³ (b) $1\frac{2}{3}$ g/cm³ (c) 19 g
9 42π cm²
10 396 cm²
11 9 cm
12 (a) 275 cm³ (b) 1,09 g/cm³
13 6,2 g/cm³
14 9 cm
15 28,8 cm
16 64
17 (a) 15 400 cm³/s (b) 924 ℓ/min
18 5
19 1,4 cm
20 64 cm³
21 (a) 16π cm³ (b) 5 cm
(c) 20π cm² (d) 288°
22 1 232 cm³, 550 cm²
23 100,8°
24 6 750 m³
25 38,9°

Exercise 10b (p. 82)

1 (a) 4 190 cm^3, 1 260 cm^2 (b) 2 140 cm^3, 804 cm^2 (c) 16,8 cm^3; 37,7 cm^2 (d) 191 cm^3, 191 cm^2
2 7,06 kg 3 27 cm 4 1 728
5 (a) 6,20 cm (b) 484 cm^2

Exercise 10c (p. 84)

1 38 125 cm^3
2 (a) 3 000 litres (b) 1,224 cm^3
3 $121\frac{1}{2}$ tonnes 4 57,6 kg
5 (a) 7,08 litres (b) 3,24 kg
6 67,0 cm^3 7 352 cm^3 8 (b) $V = 880$
9 52,4% $\left(\frac{\pi}{6} \times 100\%\right)$
10 9 610 cm^3 11 16,5 kg 12 11,4 cm
13 9 cm 14 $31\frac{1}{4}$ cm 15 $2\frac{1}{4}$ cm
16 $3\frac{25}{77}$ cm 17 1 cm 18 2 cm
19 420 g 20 0,343 kg

Exercise 10d (p. 87)

1 218π cm^3
2 (a) 500 m^3 (b) 468 m^3
3 1 552 cm^3
4 (a) 195π cm^2 (b) 700π cm^3
5 cone: 185 mℓ, frustum: 4 815 mℓ
6 16 m^3 7 12,7 cm 8 $5\frac{1}{3}$ cm

Exercise 11a (p. 89)

1 $a = 3{,}79$ cm 2 $b = 1{,}49$ cm
3 $c = 1{,}58$ m 4 $a = 2{,}31$ m
5 $a = 16{,}6$ cm 6 $b = 4{,}18$ m
7 $c = 5{,}72$ m 8 $b = 12{,}1$ cm
9 $a = 6{,}68$ cm 10 $c = 18{,}2$ m

Exercise 11b (p. 90)

1 $a = 4{,}97$ cm	$B = 47{,}3°$	$C = 66{,}7°$
2 $c = 9{,}87$ cm	$B = 29{,}2°$	$A = 76{,}8°$
3 $c = 12{,}2$ m	$A = 25{,}4°$	$B = 15{,}6°$
4 $b = 375$ m	$C = 32{,}1°$	$A = 52{,}9°$
5 $a = 9{,}08$ m	$B = 69{,}3°$	$C = 52{,}6°$
6 $b = 9{,}45$ cm	$C = 28{,}7°$	$A = 25{,}3°$
7 $c = 3{,}23$ cm	$B = 28{,}1°$	$A = 126{,}2°$
8 $a = 65{,}3$ m	$B = 26{,}2°$	$C = 13{,}6°$
9 $c = 5{,}81$ cm	$A = 13{,}7°$	$B = 23°$
10 $b = 3{,}21$ cm	$C = 29{,}6°$	$A = 115{,}9°$

Exercise 11c (p. 91)

1 29,9°; 63,9°; 86,2° 2 81,8°; 38,2°; 60°
3 62,7°; 81°; 36,3° 4 108,2°; 22,3°; 49,5°
5 33,6°; 50,7°; 95,7° 6 110,9°; 43,2°; 25,9°
7 41,4°; 55,8°; 82,8° 8 46,5°; 39,4°; 94,1°
9 98,2°; 50,3°; 31,5° 10 45,7°; 79°; 55,3°

Exercise 11d (p. 92)

1 (a) $\frac{2}{3}$ (b) 7 cm
2 (a) 117° (b) 9,40 cm
3 $x = 6{,}36$; $\theta = 61{,}4°$
4 (a) 120,4° (b) 29,8°
5 $-\frac{23}{40}$; 4,24 cm 6 $\frac{1}{16}$; 7,42 m
7 7,63 cm, 3,12 cm 8 $y = 5$
9 4:1:1
10 (a) 82,8° (b) 8,89 cm

Exercise 11e (p. 93)

1 5,99 km 2 226 m
3 11° 4 171 km
5 82,0 m 6 067,2°
7 36,1 km, 073,2° 8 19,2 km
9 (a) 8,72 km (b) 054,1°
10 154 m 11 0,706 m, 56,4°
12 (a) 700 km (b) 021,8°
13 (a) 44,5 km (b) 134,2°
14 193 km, 339,9°
15 5,65 km, N 9,5°W (or 350,5°)

Exercise 12a (p. 96)

1 three piece suite: $960, table and chairs: $42 black pots: $16,80, bicycle: $418,80
2 $2; $1; $4; $1,50; $3; $4,40; $5,11; $0,37 respectively
3 $15,22 4 $45
5 (a) (i) $900 (ii) $1 140
(b) (i) $1 620 (ii) $1 980
(c) (i) $3 220 (ii) $3 780
(d) (i) $1 240 (ii) $1 540
(e) (i) $1 516 (ii) $1 860
(f) (i) $2 134 (ii) $2 565,20
6 (a) $2 200 (b) $4 800 (c) $4 000 (d) $6 800
7 $1 265
8 (a) $1 800 (b) $3 508 (c) $268 (d) $3 240 (e) $486 (f) $3 726 (g) $9 594
9 (a) $5 800 (b) $9 688,75 (c) $1 509,27
10 $46 912,82

Exercise 12b (p. 98)

1 (a) $53,10 (b) $367,50 (c) $23,80 (d) $36
2 $165
3 (a) 35% (b) $25,35
4 $1,52
5 $1,96 and $1,30, a saving of 66 cents per kg
6 $28,25 7 $38 700
8 (a) $825 (b) $126

9 \$858 and \$988 (the higher price reflects the longer hire period)
10 \$833,65

Exercise 12c (p. 101)

2 \$29,24 **3** \$87,11
5 \$14,93 **6** \$28,15
7 (a) \$35,95 (b) \$33,95 (c) \$40,25
8 \$142,03 **9** \$156,80
10 \$356,90 (i.e. \$250,32 rates, \$32,35 water and \$74,23 electricity)

Exercise 12d (p. 103)

1 \$37,80 each
2 benefit C, \$3 422 (= \$2 215 + \$1 207)
3 \$858
4 (a) \$910 (b) \$364
5 (a) \$67 (b) \$99 (c) \$164
6 (a) \$173,10 (b) \$2 077,20 (c) \$51 930
7 (a) \$132,50 (b) \$11,04
8 (a) \$45 400 (b) \$90 975,60
9 \$66 375
10 (a) \$45 000 (b) \$622,35 (c) \$149 364

Exercise 12e (p. 105)

1 (a) mortgage, insurance, school costs
(b) \$320,78 (c) \$150,84
2 (a) \$19 593,33 (b) \$980
(c) \$15 401,20 (d) \$3 608,28
(e) \$124,90
3 (b) \$6 380,70
(c) operating loss of \$3 283,37
4 (a) \$1 386,7 million (b) 1988/89
(c) income tax (d) 1987/88
5 (a) yes (b) yes (c) no (d) no
6 it is probably a rounding off error, for example,

possible real value	*rounded value*
210,77	210,8
+786,06	786,1
996,83 (sum)	996,9

7 \$4 831,3 million **8** \$13 078 million
9 (a) 21,1% (b) 11,6%
(c) the proportion of deficit financed by foreign loan was virtually halved during the two-year period
10 (a) 142,5% (b) 134,9%
(c) both are rising; however, revenue (income) is rising at a greater rate than expenditure; this means that Government had a firm control of Zimbabwe's financial affairs during the given two-year period

Exercise 13a (p. 108)

1 (a) $\begin{pmatrix} 7 & 9 \\ -4 & 3 \end{pmatrix}$ (b) $\begin{pmatrix} 12 \\ 9 \end{pmatrix}$

(c) $\begin{pmatrix} 3 & -7 \\ 0 & 6 \\ -5 & 7 \end{pmatrix}$ (d) $\begin{pmatrix} -10 & -7 \\ 3 & 1 \end{pmatrix}$

(e) $\begin{pmatrix} 5 \\ 10 \end{pmatrix}$ (f) $\begin{pmatrix} 0 & -3 \\ -8 & 6 \end{pmatrix}$

2 (a) $\begin{pmatrix} -3 & 15 \\ 6 & 9 \end{pmatrix}$ (b) $\begin{pmatrix} 12 & 0 \\ 8 & -16 \end{pmatrix}$

(c) $\begin{pmatrix} 2 & -10 \\ -4 & -6 \end{pmatrix}$ (d) $\begin{pmatrix} 3 & 0 \\ 2 & -4 \end{pmatrix}$

(e) $\begin{pmatrix} 4 & 10 \\ 8 & -12 \end{pmatrix}$ (f) $\begin{pmatrix} -19 & 5 \\ -10 & 27 \end{pmatrix}$

3 (a) (29) (b) (-8)
(c) $(3x - y)$ (d) $(4x + 5y + 6z)$
4 $x = 8$, $y = -5$

5 (a) $\begin{pmatrix} -7 \\ -10 \end{pmatrix}$ (b) $(7; 4; 11)$

(c) $\begin{pmatrix} 4 & 3 \\ -2 & -11 \end{pmatrix}$ (d) $\begin{pmatrix} -2 & 24 \\ 2 & -5 \end{pmatrix}$

(e) $\begin{pmatrix} 33 & 24 \\ -8 & 1 \end{pmatrix}$ (f) $\begin{pmatrix} 2 & 3 & 0 \\ 5 & -1 & 6 \end{pmatrix}$

Exercise 13b (p. 110)

1 $\frac{1}{9}\begin{pmatrix} 2 & -3 \\ -1 & 6 \end{pmatrix}$ **2** $\frac{1}{9}\begin{pmatrix} 3 & -3 \\ -2 & 5 \end{pmatrix}$

3 $-\frac{1}{2}\begin{pmatrix} 1 & -3 \\ -2 & 4 \end{pmatrix}$ **4** $-\frac{1}{6}\begin{pmatrix} 3 & -3 \\ -5 & 4\frac{1}{2} \end{pmatrix}$

5 $\begin{pmatrix} 2 & -3 \\ -1 & 2 \end{pmatrix}$ **6** $-\begin{pmatrix} 1 & -2 \\ -2 & 3 \end{pmatrix}$

7 singular **8** $-\frac{1}{17}\begin{pmatrix} 4 & -3 \\ 1 & -5 \end{pmatrix}$

9 $\frac{1}{60}\begin{pmatrix} -6 & 9 \\ -8 & 2 \end{pmatrix}$ **10** $-\frac{1}{8}\begin{pmatrix} 0 & 4 \\ 2 & -5 \end{pmatrix}$

11 $-\frac{1}{2}\begin{pmatrix} 3 & -1 \\ -16 & 4 \end{pmatrix}$ **12** $\begin{pmatrix} 0 & 3 \\ -2 & 0 \end{pmatrix}$

Exercise 13c (p. 111)

1 $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$

2 $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$

3 $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -1 \\ 5 \end{pmatrix}$

4 $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$

5 $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2\frac{1}{2} \\ \frac{1}{2} \end{pmatrix}$

6 $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ -\frac{1}{2} \end{pmatrix}$

7 $\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$

8 $\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 12 \\ 6 \end{pmatrix}$

9 $\begin{pmatrix} s \\ t \end{pmatrix} = \begin{pmatrix} \frac{1}{3} \\ \frac{1}{6} \end{pmatrix}$

10 $\begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ 5 \end{pmatrix}$

Exercise 13d (p. 111)

1 $\begin{pmatrix} -7 & -6 \\ 0 & 12 \end{pmatrix}$

2 (a) 2 (b) $\frac{1}{2}\begin{pmatrix} 4 & 6 \\ 1 & 2 \end{pmatrix}$

3 14, $\frac{1}{14}\begin{pmatrix} 3 & 4 \\ -5 & -2 \end{pmatrix}$

4 (a) $x = 3$ (b) $\begin{pmatrix} 3 & 2 \\ 4 & 5 \end{pmatrix}$ 5 $k = 5$

6 (a) $\begin{pmatrix} -6 & 15 & 0 \\ 2 & -5 & 0 \\ -4 & 10 & 0 \end{pmatrix}$ (b) (11)

7 (a) $\begin{pmatrix} 14 \\ -4 \\ 7 \end{pmatrix}$ (b) $\begin{pmatrix} 0 & 12 & 15 & 3 \\ 0 & 3 & 3 & 0 \end{pmatrix}$

8 $m = 5, n = 3$ 9 $P = \begin{pmatrix} -1 & 0 \\ 1 & 2 \end{pmatrix}$

10 (a) $\begin{pmatrix} 5 & -2 \\ 0 & -4 \end{pmatrix}$ (b) $p = 9, q = -6$

11 $p = 1, q = -3, r = 5$ 12 $a = 5, b = 2$

13 (a) $\begin{pmatrix} 16 & 14 \\ 0 & 9 \end{pmatrix}$ (b) $k = -\frac{1}{6}$
(c) $m = -2$

14 (a) $\frac{1}{41}\begin{pmatrix} 7 & 4 \\ -5 & 3 \end{pmatrix}$ (b) $x = 2, y = -1$

15 $\begin{pmatrix} 5 & 3 \\ 3 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 \\ -1 \end{pmatrix}$, $x = 9$ and $y = -14$

Exercise 14a (p. 116)

1

transformation	matrix
identity	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$
reflection in x-axis	$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$
reflection in y-axis	$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$
rotation of 180° about origin	$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$

2 (a) $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ (b) $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

3 (1; −1), (4; −2), (7; −3)
4 P(−1; 5)
5 (0; 0), (−1; 1), (−1; −1)
6 (0; 0), (−1; −1), (−1; 1)
7 (a) (2; −2), (1; 1), (−2; 2), (−1; −1)
(b) (−2; 2), (−1; −1), (2; −2), (−1; 1)

8 (a) $\begin{pmatrix} 7 \\ -5 \end{pmatrix}$ (b) (i) 270°,
(ii) $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ (c) $y = -1$

10 (a) (−2; 1) (b) (−3; −4) (c) C(−1; 7)

Exercise 14b (p. 119)

1 A′(−3; −3), B′(−6; −9), C′(−12; −6)

2 A′(4; 1), B′(11; 3), C′(10; 2), $\begin{pmatrix} 1 & -3 \\ 0 & 1 \end{pmatrix}$

3 A′(2; 5), B′(4; 15), C′(8; 10), a two-way stretch of factor 2 in the x-direction and factor 5 in the y-direction.

4 $\begin{pmatrix} 1\frac{1}{2} & 0 \\ 0 & 1\frac{1}{2} \end{pmatrix}$

5 (0; 0), (4½; 0), (0; 3), (4½; 3)
6 (a) (2; 15) (b) (−2; −5)
(c) a shear of factor 5 in the y-direction with the y-axis invariant

7 (a) −3 (b) 2 (c) $\begin{pmatrix} -3 & 0 \\ 0 & 2 \end{pmatrix}$

(a) an enlargement of scale factor 4 with the origin as centre; $\begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix}$

(b) $P_2(16; 4)$, $Q_2(23; 5)$

(a) $(-4; -2)$ (b) $(-5; 4)$ (c) $y = 0$

$P_1(0; 0)$, $Q_1(4; 2)$, $R_1(-2; 6)$

$P_2(0; 0)$, $Q_2(2; -4)$, $R_2(6; 2)$

Exercise 14c (p. 122)

$X_1(-1; 2)$, $Y_1(4; 5)$, $Z_1(7; 4)$

$X_2(-2; 15)$, $Y_2(-5; 31)$, $Z_2(-4; 21)$

(a) enlargement, scale factor 4 with origin as centre followed by a translation of vector $\begin{pmatrix} -3 \\ 1 \end{pmatrix}$; shear of factor 3 with $y = 0$ invariant followed by a two-way stretch

(b) $(-7; 17)$, $(22; 12)$

(a) $(-10; 30)$ (b) $(-4; 14)$ (c) $(-1\frac{1}{2}; \frac{2}{3})$

(a) $(-5; 13)$ (b) $(-11; -3)$ (c) $(-22; 5)$

(a) 90° (b) $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ (c) $\begin{pmatrix} -1 \\ -7 \end{pmatrix}$

(d) $x + y = -1$

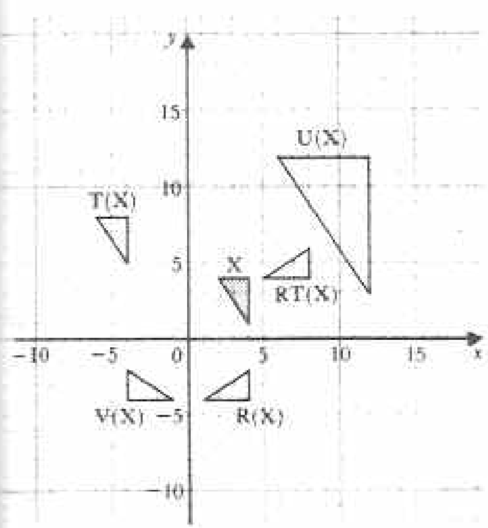
(a) 2 (b) $\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$ (c) $\begin{pmatrix} -3 \\ -7 \end{pmatrix}$

(d) $\begin{pmatrix} -1\frac{1}{2} \\ -3\frac{1}{2} \end{pmatrix}$ (e) $h = 7$, $k = 3$

$(-1; 0)$

(a) $(-1; 5)$ (b) $(10; 7)$ (c) $m = -4$, $n = 3$

(a)



A8

(b) U is an enlargement of scale factor 3 with the origin as centre.

(c) V is a reflection in the line $y = -x$.

Exercise 15a (p. 126)

1 (c) $x \simeq 3{,}7$ **2** $x = -2$; $-0{,}4$ or $2{,}4$

3 (c) $x \simeq 0{,}5$ **4** $x \simeq -2{,}5$; $0{,}7$; $1{,}8$

5 $x \simeq 1{,}4$ **6** (b) $x \simeq -3{,}1$; $-0{,}4$; $3{,}5$

7 $x \simeq 3{,}8$

8 (a) $x = 0$; 1 or 3 (b) $1 < x < 3$ (c) (i) 3 (ii) 13

9 (a) 32 cm (b) 3,9 s

10 (c) (i) 27 cm^3 (ii) 3 cm × 3 cm × 3 cm

Exercise 15b (p. 128)

2 (a) $x \approx 2{,}1$ (b) $x^3 - x^2 - 5 = 0$

3 (c) $y \simeq 3{,}9$ **4** (b) $x \simeq -1{,}45$; $3{,}45$

5 $x \simeq 0{,}2$; $-2{,}7$ **6** $x \simeq -0{,}6$; $-6{,}4$

7 (b) $x \simeq -3{,}1$; $1{,}4$ or $1{,}7$

8 $x \simeq 2$, 6 **9** (b) $x \simeq 0{,}2$; $4{,}3$

10 (a) $x = 5$; 10 (b) $-1{,}8$

11 (a) $y \simeq 3{,}2$ (b) $x \simeq 4{,}5$

12 (a) $x \simeq 2{,}2$; $0{,}1$; $1{,}55$ (b) 1,5 (c) 1,4

Exercise 15c (p. 131)

2 (a) $5x - 6y = 30$ (b) $2x + 3y = 18$
(c) $x + 2y = -8$ (d) $4x - y = -12$

3 -11 **5** (a) $x = 3$ (b) $\tan O\hat{M}N = 2$

6 $y = x^2 - x - 2$

7 (a) $y = 4 - 3x - x^2$ (b) $y = 2x^2 + 2x - 12$

9 (a) $y = \dfrac{1}{2 - x}$ (b) $y = \dfrac{1}{x + 7}$

10 l: $y = x + 1$; p: $y = x^2 - 2x - 3$; h: $y = \dfrac{1}{x + 1}$

Exercise 16a (p. 133)

1 (a) music (b) 3 hours
(c) 16 hours (d) $\frac{1}{4}$

2 (a) 6 km (b) 28 students
(c) 305 km (d) 4 km

3 (b) mode = median = 2 (c) 2,1

4 (b) 31 people (c) 42 marks

5 (b) 52,85%
(c) mode = 60%, median = 50%

Exercise 16b (p. 135)

1

1–100	101–200	201–300	301–400	401–500
5	17	19	5	4

There is very little change in the pattern of the distribution.

2

0–49	50–99	100–149	150–199	200–249
0	5	7	9	11

250–299	300–349	350–399	400–449	450–499
8	4	2	2	2

3

55–59	60–64	65–69	70–74	75–79	80–84
3	7	16	14	6	4

Exercise 16c (p. 138)

1 11–15
2 12,7
3 25,5
4 mode: 32, mean: 31,4
5 6 plants
6 $224,50
7 69,5°
8 22,2 years
9 $27\frac{3}{4}$ absentees (28, to nearest whole person)
10 (a) 25 students (b) 50,1%
11 yes (mean = 199,75 ≃ 200 to the nearest whole nail)

12 (a)

40–44	45–49	50–54	55–59	60–65
5	9	10	5	1

(c) 50 kg

13 (a)

135–144	144–154	155–164	165–174
4	12	8	6

(b) 154,8 cm

14 (a)

51–55	50–60	61–65	66–70
4	6	10	13

71–75	76–80	81–85
9	4	4

(c) 67,9 kg

15 (a) 70 mm to 79 mm

(b)

70–79	80–89	90–94	95–9
4	9	12	16

100–104	105–109	110–119	120–1
22	15	8	4

(d) modal group 100 mm–104 mm, mode ≃ 102 mm

Exercise 16d (p. 141)

1 (a) 28% (b) 20 students (c) 5%
2

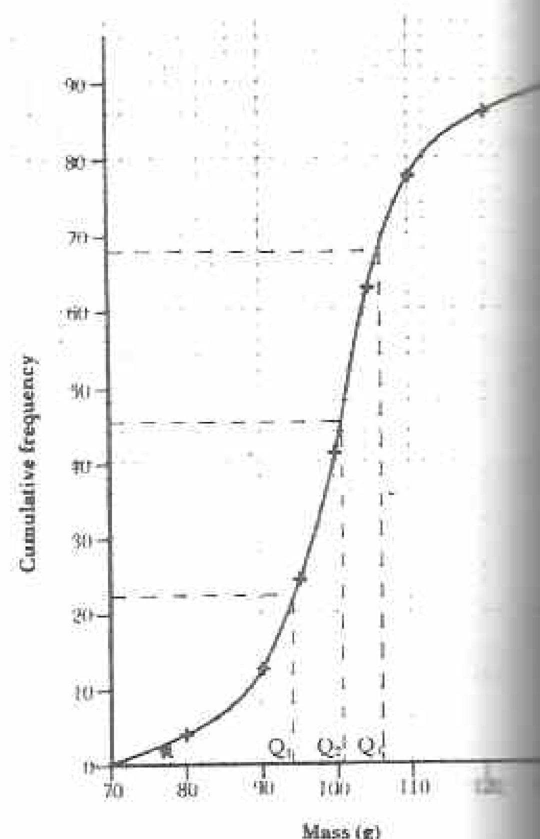
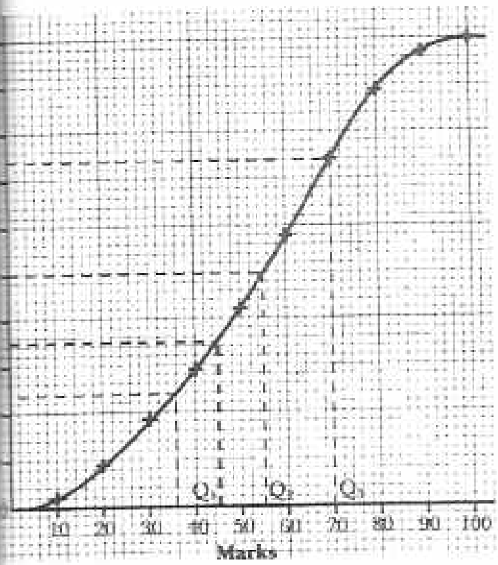


Fig. A9

(a) 101 mm (b) about $6\frac{1}{4}$ mm

ss interval	frequency	cumulative frequency
1–10	2	2
11–20	7	9
21–30	9	18
31–40	11	29
41–50	13	42
51–60	16	58
61–70	16	74
71–80	15	89
81–90	8	97
91–100	3	100



.10

Q_2(median) $\simeq 55$, $Q_1 \simeq 36\frac{1}{2}$, $Q_3 \simeq 70\frac{1}{2}$
17 marks
65%

4 (a)

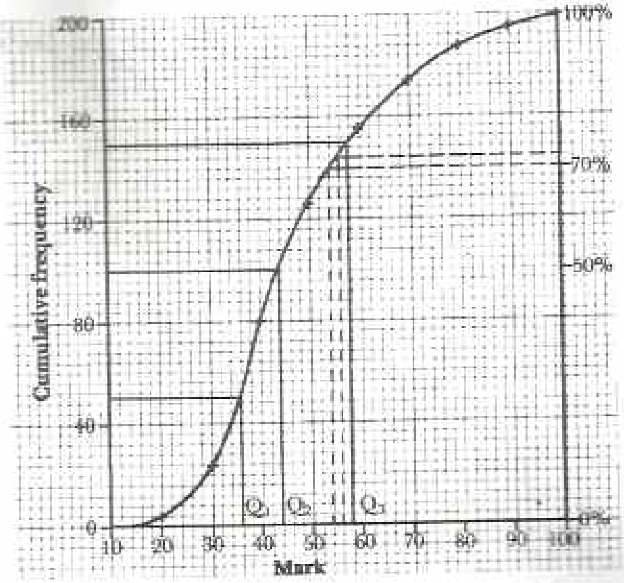


Fig. A.11

(b) 44; 11 (c) 28% (d) 54

5 (a)

class interval	tally	frequency	cumulative frequency
551–560	//	2	2
561–570	//	2	4
571–580	////	4	8
581–590	//// //	7	15
591–600	~~////~~ ~~////~~ ~~////~~	15	30
601–610	~~////~~ ///	8	38
611–620	~~////~~	5	43
621–630	~~////~~	5	48
631–640	//	2	50

modal group: 591–600, mode: 596 h, to nearest hour

(b)

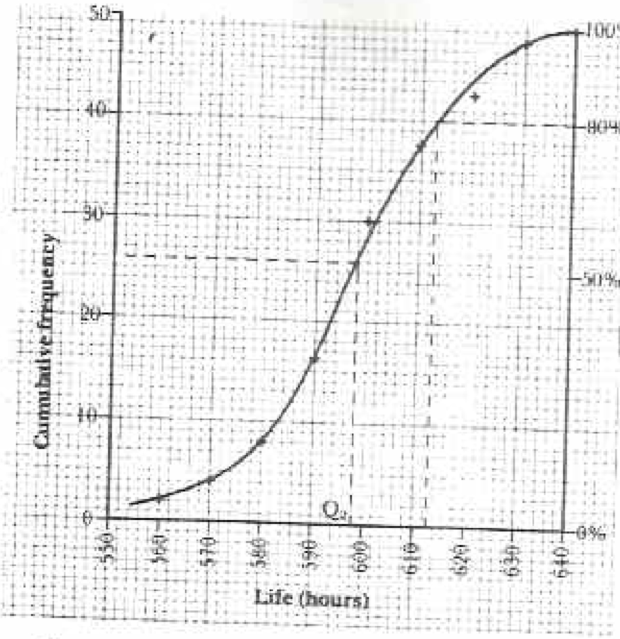


Fig. A12

median: 597 h, to nearest hour
80th percentile: 613 h, to nearest hour

6 (a) 3

(b)

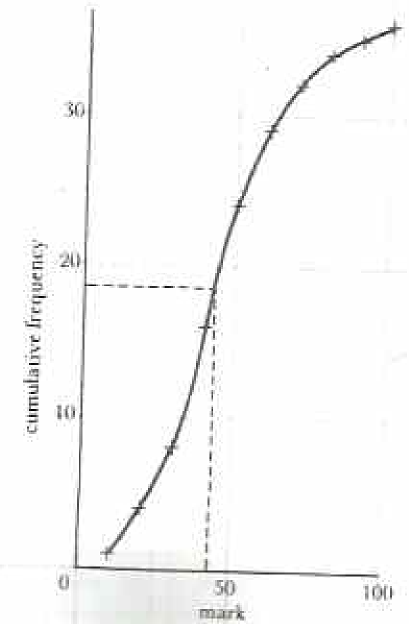


Fig. A13

(c) median mark = 43 (approx)

Exercise 17a (p. 144)

2 (a) $\{(1;2), (1;3), (2;3)\}$ (b) $\{(0;2$ ...

3 $x > 1$, $x + y < 5$, $3y > x$

4 $y > 2$, $y > 3x$, $x + y > -2$

5 (a) P(0; 4) (b) $y = -2x$ (or $2x +$ ...
(c) $x \leqslant 0$; $2x + y \geqslant 0$; $2y < x + 8$

Exercise 17b (p. 146)

1 14 buses, 11 minibuses

2 (a) two ways: 3 notebooks, 5 pencils or 4 notebooks, 3 pencils
(b) yes, the 2nd way: 12 c change

3 (a) 93; either (72; 21) or (73; 20)
(b) 60 cheap; 30 dear

4 (a) (i) (96; 34) (ii) (140; 20) (b) ...

5 (a) $4x + y > 20$, $4x + 3y > 30$
(c) 4 Feelgood pills and 5 Getbetta pi...

6 15 lorries (5 Landmasters; 10 Sandr...

7 (a) 37 (10; 27) (b) either (10; 27) or ...
(c) $7,05

8 (a) 10 (b) 4 of A and 6 of ...

9 (a) 20
(b) (i) (16; 38) (ii) either (20; 30) or ...

10 $x + y \leqslant 1000$; $y \geqslant 2x$; $x \geqslant 100$; $y <$ ...
333 cans of Kula, 667 cans of Sundow...

Exercise 18a (p. 150)

1 $\begin{pmatrix} 0 \\ -2 \end{pmatrix}$

2 (a) $\begin{pmatrix} 3 \\ -1 \end{pmatrix}$ (b) $\begin{pmatrix} -4 \\ 5 \end{pmatrix}$

(c) $\begin{pmatrix} 8 \\ 6 \end{pmatrix}$ (d) $\begin{pmatrix} -2 \\ -7 \end{pmatrix}$

3 (a) $\sqrt{89}$ (b) $\begin{pmatrix} 8 \\ -5 \end{pmatrix}$

4 (a) $\begin{pmatrix} 7 \\ 5 \end{pmatrix}$ (b) $\sqrt{74}$

5 **PR** $= \begin{pmatrix} -3 \\ 7 \end{pmatrix}$, **RP** $= \begin{pmatrix} 3 \\ -7 \end{pmatrix}$

6 (a) $\begin{pmatrix} 15 \\ 20 \end{pmatrix}$ (b) $\begin{pmatrix} -9 \\ 3 \end{pmatrix}$

(c) $\begin{pmatrix} -2 \\ 0 \end{pmatrix}$ (d) $\begin{pmatrix} 6 \\ 3 \end{pmatrix}$

(e) $\begin{pmatrix} -7 \\ -4 \end{pmatrix}$ (f) $\begin{pmatrix} 7 \\ 4 \end{pmatrix}$

(g) $\begin{pmatrix}5\\12\end{pmatrix}$ (h) $\begin{pmatrix}-3\\6\end{pmatrix}$

(i) $\begin{pmatrix}-1\\24\end{pmatrix}$ (j) $\begin{pmatrix}36\\11\end{pmatrix}$

7 (a) 5 (b) $\sqrt{10}$ (c) 4
(d) $\sqrt{17}$ (e) $\sqrt{2}$ (f) 5

8 (a) A′(4; 4), B′(3; 6), C′(4; 7)
(b) 5

9 (a) $\begin{pmatrix}-8\\6\end{pmatrix}$ (b) 10

10 P(6; 6), Q(1; 3), $|\mathbf{OA}| = 5$

Exercise 18b (p. 153)

1 (a) $\begin{pmatrix}-5\\-9\end{pmatrix}$ (b) $\begin{pmatrix}5\\9\end{pmatrix}$

2 (a) $\begin{pmatrix}3\\8\end{pmatrix}$ (b) $\begin{pmatrix}10\\3\end{pmatrix}$ (c) $\begin{pmatrix}2\\3\end{pmatrix}$

(d) $\begin{pmatrix}4\\2\end{pmatrix}$ (e) $\begin{pmatrix}7\\-5\end{pmatrix}$ (f) $\begin{pmatrix}-6\\-5\end{pmatrix}$

3 (a) $\begin{pmatrix}-1\\3\end{pmatrix}$ (b) $\begin{pmatrix}8\\-1\end{pmatrix}$

(c) $\begin{pmatrix}6\\1\end{pmatrix}$ (d) $\begin{pmatrix}3\\-5\end{pmatrix}$

(e) $\begin{pmatrix}5\\4\end{pmatrix}$ (f) $\begin{pmatrix}9\\-4\end{pmatrix}$

(g) $\begin{pmatrix}5\\4\end{pmatrix}$ (h) $\begin{pmatrix}8\\-1\end{pmatrix}$

(i) $\begin{pmatrix}-3\\5\end{pmatrix}$ (j) $\begin{pmatrix}-8\\1\end{pmatrix}$

4 (a) A(1; 4), B(6; 6), C(5; 2)
(b) (i) $\begin{pmatrix}6\\6\end{pmatrix}$ (ii) $\begin{pmatrix}-4\\2\end{pmatrix}$

5 $\mathbf{QR} = \begin{pmatrix}4\\-1\end{pmatrix}$ and $\mathbf{PS} = \begin{pmatrix}8\\-2\end{pmatrix} = 2\begin{pmatrix}4\\-1\end{pmatrix}$

$\Rightarrow$ QR ∥ PS

6 AB = BC = CD = DA $(= \sqrt{125}) \Rightarrow$ ABCD is a rhombus

7 $\mathbf{OA} = \mathbf{CB}\left[= \begin{pmatrix}4\\0\end{pmatrix}\right]$

$\Rightarrow$ OA ∥ CB and OA = CB
$\Leftrightarrow$ OABC is a parallelogram

8 $\mathbf{PQ} = \mathbf{SR}\left[= \begin{pmatrix}6\\3\end{pmatrix}\right]$

$\Leftrightarrow$ PQ ∥ SR and PQ = SR
$\Leftrightarrow$ PQRS is a parallelogram
diagonals intersect at (7; 5)

9 Q(12; 8), diagonals intersect at (6; 4)

10 (a) $\begin{pmatrix}8\\6\end{pmatrix}$ (b) 10 (c) P(−7; −10)

Exercise 18c (p. 155)

1 (a) **PR** (b) **PS** (c) **PT** (d) **PT**
(e) **PS** (f) **PS** (g) **PT** (h) **PS**

2 $\frac{1}{2}(\mathbf{a} + \mathbf{b})$

3 $\mathbf{XY} = \mathbf{b} - \mathbf{a}$, $\mathbf{YZ} = \mathbf{c} - \mathbf{b}$, $\mathbf{ZX} = \mathbf{a} - \mathbf{c}$

4 (a) $2\mathbf{b} - \mathbf{a}$ (b) $\mathbf{a} + \mathbf{b}$

5 (a) $\mathbf{t}$ (b) $-\mathbf{r}$ (c) $-\frac{1}{4}\mathbf{r}$
(d) $\mathbf{t} - \frac{1}{4}\mathbf{r}$ (e) $\frac{3}{4}\mathbf{r} + \mathbf{t}$

6 (a) $\mathbf{b} - \mathbf{a}$ (b) $\frac{1}{2}\mathbf{b}$ (c) $\frac{1}{2}\mathbf{b} - \mathbf{a}$

7 (a) $2\mathbf{a}$ (b) $\mathbf{b} - \mathbf{a}$
(c) $1\frac{1}{2}\mathbf{a} - \frac{1}{2}\mathbf{b}$ (d) $1\frac{1}{2}\mathbf{a} + \frac{1}{2}\mathbf{b}$

8 (a) $2\mathbf{x}$ (b) $\mathbf{x} + \mathbf{y}$ (c) $\mathbf{x} + \mathbf{y}$
(d) $\mathbf{x} + 2\mathbf{y}$ (e) $2\mathbf{x} + 2\mathbf{y}$ (f) $2\mathbf{x} + \mathbf{y}$

9 (a) $\frac{4}{5}\mathbf{x}$ (b) $\frac{1}{2}(\mathbf{x} + \mathbf{y})$ (c) $1\frac{1}{2}\mathbf{y}$
(d) $\frac{1}{2}\mathbf{y} - \frac{3}{10}\mathbf{x}$ (e) $\mathbf{y} - \frac{1}{2}\mathbf{x}$

10 (a) $2\mathbf{a} + \mathbf{b}$ (b) $-\mathbf{b}$ (c) $\mathbf{a}$

13 (a) $\mathbf{b} - \mathbf{a}$, $\frac{1}{2}\mathbf{a}$, $\frac{1}{2}\mathbf{b}$, $\frac{1}{2}(\mathbf{b} - \mathbf{a})$
(b) MN ∥ AB and MN = $\frac{1}{2}$AB

16 (a) $\mathbf{RP} = \mathbf{a} + \mathbf{b}$, $\mathbf{QS} = 3\mathbf{a} - \mathbf{b}$

17 (a) (i) $\mathbf{b} - \mathbf{a}$, (ii) $\frac{1}{4}\mathbf{a} + \frac{3}{4}\mathbf{b}$, (iii) $\frac{1}{2}\mathbf{a} - \mathbf{b}$
(b) $\frac{1}{2}h\mathbf{a} + (1 - h)\mathbf{b}$
(c) $h = \frac{2}{5}$, $k = \frac{4}{5}$ (d) $\frac{1}{5}\mathbf{a} + \frac{3}{5}\mathbf{b}$

18 (a) (i) $\frac{1}{2}\mathbf{a}$, (ii) $\mathbf{c}$, (iii) $\mathbf{c} - \mathbf{a}$, (iv) $\frac{2}{7}(\mathbf{c} - \mathbf{a})$
(b) $\frac{3}{14}\mathbf{a} + \frac{2}{7}\mathbf{c}$ (c) $\frac{1}{2}\mathbf{a} + p\mathbf{c}$
(d) $\frac{3}{14}q\mathbf{a} + \frac{2}{7}q\mathbf{c}$
(e) $p = \frac{2}{3}$, $q = 2\frac{1}{3}$, AY : YB = 2:1

19 (a) $\mathbf{a} - \mathbf{b}$ (c) $3\mathbf{a} + 2\mathbf{b}$
(d) $h = \frac{3}{5}$, $k = \frac{1}{5}$ (e) $\frac{2}{3}$

20 (a) $\frac{1}{2}(\mathbf{x} + \mathbf{y})$ (b) $\frac{1}{2}h\mathbf{x} + \frac{1}{2}h\mathbf{y}$
(c) $\frac{1}{2}\mathbf{y} - \mathbf{x}$ (d) $(1 - k)\mathbf{x} + \frac{1}{2}k\mathbf{y}$
(e) $h = k = \frac{2}{3}$
(f) the lines meet each other at a single point and divide each other in the ratio 2:1

(i) $\frac{2\sqrt{2}}{3}$ (j) $\frac{\sqrt{6}}{3}$ (k) $\frac{\sqrt{3}}{3}$ (l) $\frac{6}{7}$
(m) $\sqrt{5}$

4 (a) $3\sqrt{5}$ (b) $2\sqrt{3}$ (c) $3\sqrt{7}$ (d) 0
(e) $4\sqrt{2}$ (f) $\sqrt{3}$ (g) 9 (h) 12
(i) $9\sqrt{5}$ (j) 60 (k) $3\sqrt{3}$ (l) 8
(m) $54\sqrt{2}$

Exercise 21g (p. 183)

1 (a) 300 (3×10^2) (b) 75 (3×5^2)
2 41_{five}, 22_{ten}, $10\,111_{two}$
3 8
4 (a) 16 (b) 5 (c) 625
5 (a) 27 (b) 65 (c) 108 (d) 11
6 (a) 100 001 (b) 11 111 (c) 1 100 001
(d) 1 100 1111
7 (a) 222 (b) 2 222 (c) 10 001
(d) 4 210
8 (a) 10 000 (b) 11 101 (c) 110 11
(d) 11 111
9 (a) 131 (b) 433 (c) 340
(d) 441
10 (a) 14 125 (b) 1 000 100
11 (a) 2 324 (b) 111 101
12 1 321
13 (a) 1 021 (b) 10 111 101 (c) 4
(d) 110
14 (a) 100 111 (b) 11 111 101
(c) 10 100 010 110
(d) 11 111 100
15 (a) 101 (b) 101 (c) 110 (d) 111
16 (a) CALCULATE X
(b)

(i)	(ii)
00001	00011
10000	01111
10000	01100
10010	01100
01111	00101
11000	00011
01001	10100
01101	00000
00001	00100
10100	00001
00101	10100
	00001

Exercise 21h (p. 184)

(Answers given to ready-reckoner accuracy only.)

1 (a) \$5,52 (b) \$15,18 (c) \$33,81
(d) \$60,03 (e) \$91,77 (f) \$73,83
2 (a) \$966 (b) \$200,10 (c) \$34,500
(d) \$17,25 (e) \$220,80 (f) \$54,51
3 \$20,70 **4** \$43,10 **5** \$24,84
6 (a) \$79,17 (b) \$258,33 (c) \$666,67
(d) \$700 (e) \$1 500 (f) \$1 446,67
7 (a) \$86,54 (b) \$144,23 (c) \$173,08
(d) \$230,77 (e) \$336,54 (f) \$164,90
8 \$945
9 \$1 403,89
10 \$17 648,08
11 (a) 18,2 (b) 31,8 (c) 40,9
(d) 45,5 (e) 272,8 (f) 3,6
12 (a) 0,7 (b) 1,1 (c) 1,8
(d) 2,2 (e) 19,8 (f) 1,4
13 9,24 gallons
14 (a) 1,39 (b) 2,22 (c) 16,7
(d) 19,4 (e) 24,7 (f) 37,4
15 (a) 2,49 (b) 4,35 (c) 6,2
(d) 49,7 (e) 5,28 (f) 27,4
16 30 mph (since 50 km/h = 31,1 mph)
17 (a) 0 (b) 10 (c) 30
(d) 20 (e) −5 (f) 37
18 (a) 14 (b) 68 (c) 86
(d) 41 (e) 93 (f) 57
19 (a) 18 °F (b) 122 °F
20 (a) US\$7,01 (b) US\$18,70
(c) US\$4,20 (d) US\$10,28
(e) US\$63,09 (f) US\$12,16
(g) US\$17,76 (h) US\$35,99

Exercise 22a (p. 188)

1 (a) {l; i; o; n; t; g; e; r} (b) {i}
(c) {r; e; v; t; g} (d) {v; o; l; n}
(e) {r; e; v; o; l; t; n; g} (f) {v}
(g) {v} (h) {r; e; v; o; l; t; n; g}
(i) {t; g; e; r}
2

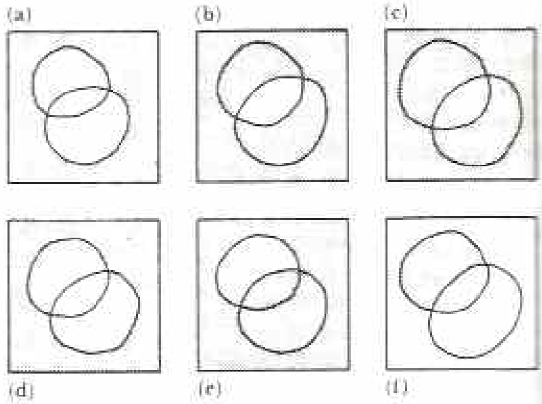


Fig. A14

3

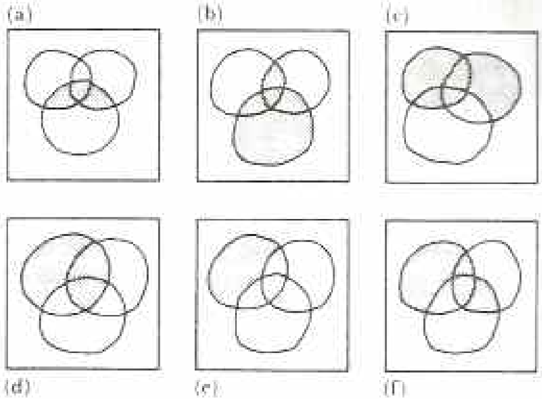


Fig. A15

4 (a) $\{1;2\}$ (b) $\{5;10\}$
(c) $\{4;5;6;7\}$ (d) $\{2\}$
(e) $\{1;2;3;4;5;6;8;9;10\}$ (f) $\{2;4\}$

5 (a) 6 (b) $\{4;8;10\}$

6

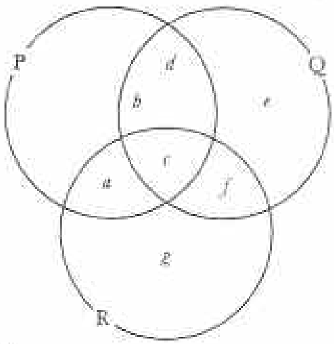


Fig. A16

$\{a;c;f\}$

7

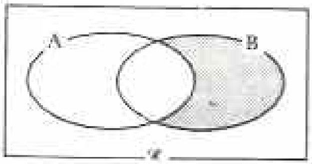


Fig. A.17

8 11 men

9 (a) $k=3$ (b) 23 (c) 0 (d) 13

10 (a) 7 (b) $\{y: 0 \geqslant y \geqslant 5$, y is an integer$\}$

11 (a) $p = 75 - x$ (b) 90 (c) 35 (d) 5

12 (a) 2 (b) 16

13 (a) 8 (b) 11

14 37

15 some boys are unhappy; no girls are unhappy

Exercise 22b (p. 190)

1 $16x - 9y$ **2** $2p^2 - 5q$
3 $3a - 10b$ **4** $23 + 15x^2$
5 $x^2 - 10xy + 9y^2$ **6** $2u^2v - 4uv^2$
7 $2d^2 - 10ad$ **8** $5a^2 + 3a$
9 $7x^2 + 4x^3 + 2x^4$ **10** $m^2 - mn + 4n^2$

Exercise 22c (p. 191)

1 $36a + 4$ **2** $-30x + 10y - 5$
3 $42x - 3xy$ **4** $-10a^2 + 2ab - 6ac$
5 $5 - x$ **6** $8x^2 - 4x$
7 $6x^2 - 17x + 10$ **8** $32 - 7b^2$
9 $3ac + 4ad + 6bc + 8bd$
10 $x^2 + 9x + 8$ **11** $10a^2 + 21ab + 8b^2$
12 $10a^2 - 11ab - 8b^2$ **13** $10a^2 - 11ab - 8b^2$
14 $10a^2 - 21ab + 8b^2$ **15** $2x^2 - 11x + 15$
16 $12 + 11x - 5x^2$ **17** $9a^2 + 6a + 1$
18 $b^2 - 8b + 16$ **19** $4c^2 - 20cd + 25d^2$
20 $16m^2 - 24mn + 9n^2$

Exercise 22d (p. 192)

1 $(3c - d)(3m - 4n)$ **2** $(a + b)(c + 2d)$
3 $(x + 4a)(x - 4a)$ **4** $(2c + 5d)(2c - 5d)$
5 $(a + 2)(a - 5)$ **6** $(a + 2b)(a - 5b)$
7 $(ab + 2)(ab - 5)$ **8** $5(m - 3n)(m + 3n)$
9 $(h - 2k)(m - 2n)$ **10** $(2a + b)(a + 4b)$
11 $(3m - 1)(m - 3)$ **12** $(cd - 9)(cd + 9)$
13 $(4x + 3am)(4x - 3am)$ **14** $(2n + 3)(3n + 2)$
15 $(2a + 5)^2$ **16** $9(h - 2k)(h + 2k)$
17 $(a - 2b)(a - 4b)$
18 $(5abc + 3d)(5abc - 3d)$
19 $\left(\frac{m}{3} - \frac{n}{2}\right)\left(\frac{m}{3} + \frac{n}{2}\right)$
20 $(3x + t)(u - 2v)$
21 $(3x - 2y)^2$ **22** $(4d - 1)(3d + 2)$
23 $(x^2 + y)(x^2 - y)$ **24** $(2mn - 3)(5mn + 4)$
25 $(4 + n^2)(2 + n)(2 - n)$
26 $(a + b + c)(a + b - c)$
27 $(x + m - n)(x - m + n)$
28 $2(m + 2n)(a + b)$
29 $(2 + 5h)(1 - 3h)$
30 $(2a - b)(m - 3n)$
31 $(a - 6b)(a - 9b)$
32 $(m - 18n)(m + 3n)$
33 $(c - 2d - 3e)(c - 2d + 3e)$
34 $(3x + 2y)(4x + 9y)$
35 $(h - k)(3h - 4k)$
36 $2a(3x + 2y)(c - 2d)$
37 $(2a - 9x)(3a + 4x)$
38 $(5a + 2m + 4n)(5a - 2m - 4n)$
39 $(7a - 4b)(3a + 4b)$
40 $4b(3a - b)$

Exercise 22e (p. 192)

1 1 800 **2** 2 300 **3** 2 430 **4** 28 000 **5** 940
6 3 050 **7** 2 400 **8** 75.6 **9** 308 **10** 10 560

Exercise 22f (p. 193)

1 $\frac{5x+1}{6}$ 2 $\frac{12b+11}{35}$ 3 $\frac{19d+48}{30}$

4 $\frac{13-22x}{21}$ 5 $\frac{8c-9a}{12abc}$ 6 $\frac{6q-p}{q}$

7 $\frac{b+c}{b-c}$ 8 $\frac{a+b}{a}$ 9 $\frac{a}{b}$

10 $\frac{m+n}{m-n}$ 11 $2x+1$ 12 $\frac{x+2}{x+3}$

13 $-\frac{x+5}{x+3}$ 14 $\frac{3-a}{3+a}$ 15 $\frac{y-8x}{x-2y}$

16 $-\frac{a+b}{ab}$ 17 $\frac{1}{x-2}$ 18 $\frac{a}{b(a-b)}$

19 $\frac{(x-11)(x+1)}{6(x-5)}$ 20 $\frac{5m(6-m)}{(m-1)(2m+3)(m+4)}$

Exercise 22g (p. 194)

1 24 2 11 3 ± 15 4 -9
5 (a) 8 (b) 8 (c) 12 (d) 36
6 (a) -2 (b) -4 (c) 0 (d) 10
7 18 8 -30
9 (a) 1 (b) 1 260 (c) $1\frac{1}{3}$
10 (a) 4 900 (b) $7\frac{1}{7}$

Exercise 22h (p. 196)

1 (a) $l \propto \frac{1}{b}$ or $l = \frac{k}{b}$

(b) $r \propto \frac{1}{\sqrt{c}}$ or $r = \frac{k}{\sqrt{c}}$

(c) $G \propto \frac{m_1 m_2}{d^2}$ or $G = \frac{k m_1 m_2}{d^2}$

(d) $E = ah + bv^2$

2 $y = 20x$, $x = 1\frac{3}{4}$ 3 $A = 6$, $M = 17\frac{1}{2}$
4 $P = 10\frac{1}{2}$, $Q = 16$ 5 $D = 135$, $V = 5\frac{1}{4}$
6 $P = \frac{3}{4}Q^2$; $P = 75$, $Q = 5$
7 $xy = 30$; $x = 2\frac{1}{2}$, $y = 1\frac{1}{2}$
8 $M = \frac{5}{8}R^3$; $M = 625$, $R = 1{,}6$
9 $\sqrt{Y} = \frac{2}{3}Z$; $Y = 100$, $Z = 6$
10 $A = 30$, $B = 7\frac{1}{2}$ 11 $P = 13\frac{1}{2}$, $R = 5$
12 $x = 4\frac{1}{2} + \frac{1}{2}y$; $x = 10$ 13 $x = 115{,}2$

14 $x \propto z^2$ 15 $x \propto \frac{1}{z^2}$

16 (a) 12 h (b) 2 h (c) 1 h

17

x	10	15	20	25	30	35
y	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{5}$	$\frac{1}{6}$	$\frac{1}{7}$

18 31,4 cm 19 766 N 20 84 min

Exercise 22i (p. 198)

1

x	-2	-1	0	1	2	3	4
y	25	11	3	1	5	15	31

(b) 0,92 (c) $x = 0{,}83$
(d) $x = 2$, $x = -0{,}3$
2 (a) (i) 2,75 (ii) $-5{,}25$
(b) (i) 4,7 or $-1{,}7$ (ii) -1 or 4
(c) $x = 1{,}5$
3 (a) $-2{,}25$ (b) 4,65 or $-0{,}65$ (c) $x = 2; 4$
4 (a) $x > -2$ (b) $(-2, -3)$
5 (a) $x < \frac{1}{2}$ (b) $-1 < x < 2$
7 (a) $(4;0)$ and $(-2;0)$ (b) $(0;8)$
(c) maximum (d) $x = 1$
(e) $y = 9$
8 (a) $(0;0)$ and $(5;0)$
(b) $y = 0$ (the x-axis), $x = 2\frac{1}{2}$
9 $(2{,}64; 0{,}76)$, $(-1{,}14; -1{,}75)$
10 $(-1; -1)$, $(-3; 1)$

Exercise 23a (p. 199)

1 -2 2 $3\frac{1}{5}$ 3 5 4 $-4\frac{1}{2}$ 5 -2
6 2 7 -2 8 4 9 $3\frac{1}{3}$ 10 $-1\frac{1}{2}$

Exercise 23b (p. 200)

1 $T = \frac{100I}{PR}$ 2 $h = \sqrt{\frac{4V}{\pi d^2}}$, $d = 2\sqrt{\frac{V}{\pi h}}$

3 $t = \frac{mv - mu}{F}$ 4 $M = N - RD$

5 $N = \sqrt{\frac{T-a}{b}}$ 6 $r = \sqrt[3]{\frac{3V}{4\pi}}$

7 $h = \frac{S - 2\pi r^2}{2\pi r}$

8 $a = \frac{x}{\sqrt{1-w^2}}$, $w = \sqrt{1 - \frac{x^2}{a^2}} = \frac{\sqrt{a^2 - x^2}}{a}$

9 $f = \frac{uv}{u+v}$, $u = \frac{vf}{v-f}$

10 $a = \frac{2s}{n} - (n-1)d$

11 $Q = \frac{P(y + 3x)}{x - 3y}$

12 $s = \frac{u^2 - v^2}{2a}$, $u = \pm\sqrt{v^2 + 2as}$

13 $u = \pm\sqrt{v^2 - \frac{2Eg}{m}}$

14 $p = \frac{P(1 - eT)}{1 - et}$

15 $W = \frac{Tgx}{gx + v^2}$

16 $g = \frac{4\pi^2 l}{t^2}$

17 $h = \frac{V}{\pi r^2} - \frac{2}{3}r$

18 $h = \frac{v^2 - gd}{3g}$, $d = \frac{v^2 - 3gh}{g}$

19 $h = \pm\sqrt{\frac{A^2}{\pi^2 r^2} + r^2}$

20 $Q = \frac{P}{d^3} + P$, $P = \frac{d^3 Q}{1 + d^3}$

21 (a) $T = \frac{100}{R}\left(\frac{A}{P} - 1\right)$ (b) 8 years

22 (b) $39\frac{1}{3}$ cm 23 $a = \frac{x}{\sqrt{1 - w^2}}$, $\pm\frac{20\sqrt{3}}{3}$

24 (a) $a = \frac{1}{t}\left(\frac{l}{L} - 1\right)$

(b) $\frac{1}{24\,000}$

25 (a) $r = \frac{C}{2\pi}$ (b) $V = \frac{C^2 h}{4\pi}$

Exercise 23c (p. 201)

1 (a) $x \leqslant 1$ (b) $x < 4$ (c) $x \geqslant 2$
(d) $x \geqslant 4$ (e) $x > -6$
(f) $x > -5$

2 (a) $x \leqslant 2$ (b) $x \geqslant 6$ (c) $p < 47$
(d) $x \leqslant 3$ (e) $x > 1$ (f) $x < 11$
(g) $x \leqslant -10$ (h) $y < -47$ (i) $x \leqslant 4$
(j) $z < 1\frac{1}{2}$

4

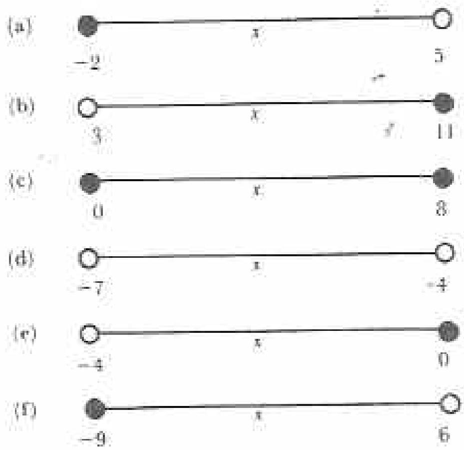


Fig. A18

5 C 6 (a) $-3 \leqslant x < 1$

7 (a) $\{-6; -5; -4\}$ (b) $\{-2; -1; 0; 1; 2\}$

8 $\{-1; 0; 1; 2; 3; 4; 5\}$

9 (a) (i) 6 (ii) −14 (iii) −9 (iv) −45
(b) (i) 14 (ii) −6 (iii) 16 (iv) 27

10 17, 19, 23

Exercise 23d (p. 202)

1 4; 2 2 3; −5 3 −3; −4 4 1; 2
5 −3; 4 6 −2; −5 7 5; 2 8 −7; −2
9 $2\frac{1}{2}$; 3 10 −2; $1\frac{1}{4}$ 11 3; 0 12 5; 1
13 −2; $1\frac{1}{2}$ 14 0; −2 15 $2\frac{1}{2}$; $-3\frac{1}{2}$ 16 4; 5
17 2; −5 18 8; −12 19 −2; 4 20 1; 2
21 −2; −2 22 3; 1 23 −3; −2 24 −3; −2
25 $1\frac{1}{2}$; $-\frac{2}{3}$

Exercise 23e (p. 203)

1 (a) $y = 2x$ (b) $x + y = 5$
(c) $y < 2x$, $x + y \geqslant 5$, $y \leqslant 0$

2 $x > -4$, $x + y < -5$, $y \leqslant x - 2$

3 $y \leqslant 1$, $x + y < 5$, $y \geqslant 2x$

4 $x \leqslant 0$, $y > 1$, $x + y \geqslant 4$, $x + 2y \geqslant 6$

5 greatest value is $4\frac{1}{2}$ at $(2\frac{1}{2}; 2)$

6 (a) (i)

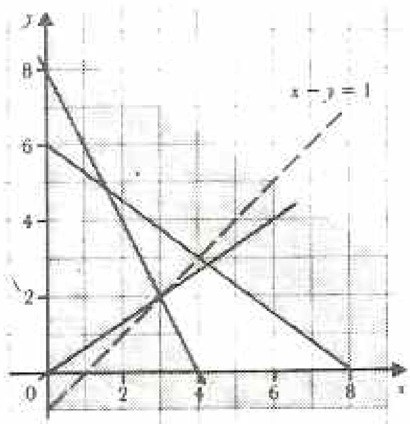


Fig. A19

(ii) The least value of $x - y$ is 1
(b) $h + b \geqslant 36$, $h \leqslant 20$, $h \leqslant 2b$

7 9 shirts, 6 dresses

8 Maximum 37 (10 large; 27 small)
Greatest profit from either (10 large; 27 small) or (11 large; 25 small) \$4,70

9 (a) (14 of A; 9 of B) (b) (12 of A; 11 of B)

10 (i) $h \leqslant 15$ (ii) $p > 25$ (iii) $45 \geqslant h + p < 60$

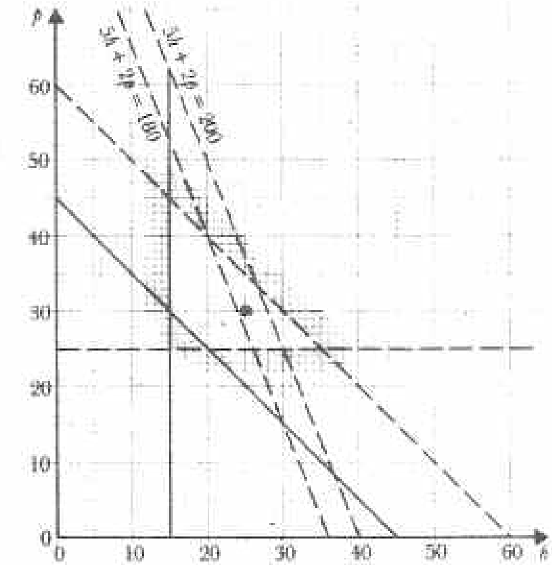


Fig. A20

25 hardback, 30 paperback

Exercise 23f (p. 205)

1 3; 7 **2** -1; -2 **3** 2; -3
4 5; -2 **5** 1; -2 **6** 0; -3
7 2; $1\frac{1}{2}$ **8** -2; $-\frac{3}{4}$ **9** 2; -2
10 0; 4 **11** -1; $-2\frac{1}{2}$ **12** 4; $-\frac{2}{3}$
13 $1\frac{1}{2}$; $\frac{3}{4}$ **14** $2\frac{1}{4}$; $-\frac{2}{3}$ **15** $1\frac{1}{2}$; $-1\frac{3}{4}$

Exercise 23g (p. 206)

1

x	-2	-1	0	1	2	3	4	5
y	14	7	2	-1	-2	-1	2	7

(a) $x = 3{,}41$ or $0{,}59$ (b) -2

2

x	-2	-1	0	1	2	3	4
y	25	10	1	-2	1	10	25

$x = 0{,}18$; $1{,}82$

3 (a) 3,56; 0,56 (b) 2,62; 0,38
(c) 4; -1

4 (a) 0; $-0{,}67$ (b) 1,23; $-1{,}9$
(c) 0,72; $-1{,}39$ (d) 1,69; $-2{,}36$

5 (a) imaginary roots (b) 0; 1,67
(c) 1,85; $-0{,}18$

6 $x = 1{,}77$ or $0{,}57$ **7** $\pm 1{,}73$, $x^2 - 3 = 0$

8 $\pm 1{,}87$

9

x	-2	$-1{,}5$	-1	$-0{,}5$	0	0,5
y	-4	$-1{,}75$	0	1,25	2	2,25

x	1	1,5	2	2,5	3
y	2	1,25	0	$-1{,}75$	-4

(b) $y_{max} = 2{,}25$ when $x = 0{,}5$ (d) 2,41; $-0{,}41$

10 2,27; $-1{,}77$

Exercise 23h (p. 207)

1 (0;5), (4; -3) **2** (3;5), ($-1\frac{1}{4}$; -12)
3 (1;2) **4** (3; -5), ($-2\frac{1}{2}$;6)
5 (2; -1) **6** (1; -2), ($-\frac{1}{7}$; $1\frac{5}{7}$)
7 (-4;3) **8** (4; -1)
9 (-1;1), ($1\frac{8}{11}$; $-\frac{1}{11}$) **10** (5;6), (2;15)
11 (2;4) **12** (4;1), (-8; -3)
13 (1;2), ($-2\frac{3}{7}$; $-\frac{2}{7}$) **14** ($1\frac{1}{3}$; -1), ($-1\frac{2}{15}$; $\frac{2}{3}$)
15 (2;1), ($-1\frac{1}{3}$; $-1\frac{1}{2}$)

Exercise 23i (p. 208)

1 \$1,03; 29c **2** 12; 15 **3** 18 cm
4 40 yr, 12 yr **5** 17 cm, 4 cm **6** 10 yr
7 $x = 12$, $y = 10$ **8** 2 m
9 5 **10** $\frac{5}{8}$ **11** 3 cm
12 6 years ago **13** 4:1
14 12 cm, 2 cm **15** 15

Exercise 24a (p. 210)

1 $a = 68°$, $b = 68°$, $c = 112°$
2 $d = 70°$, $e = 95°$, $f = 15°$
3 $g = 135°$, $h = 45°$, $i = 45°$
4 $j = 48°$, $k = 48°$, $l = 132°$
5 $n = 38°$, $m = 142°$
6 $p = 55°$ **7** $q = 130°$
8 $r = 130°$, $s = 80°$

Exercise 24b (p. 211)

1 $P\hat{A}C = 105°$, $Q\hat{B}A = 135°$, $R\hat{C}A = 120°$
2 (a) 65°, scalene (b) 71°, isosceles
(c) 43°, isosceles (d) 60°, equilateral
(e) 90°, right-angled (f) 97°, obtuse-angled
3 (a) $h = 65°$, $k = 84°$ (b) $m = n = 71°$
(c) $p = 35°$, $q = 60°$, $r = 25°$
(d) $s = 68°$, $t = 44°$, $u = 24°$
(e) $v = 47°$, $w = 29°$, $x = 151°$
(f) $y = 70°$, $z = 35°$
4 $x = 12°$
5 (a) $x = 25°$, isosceles (b) $x = 21°$, scalene
6 (a) △XAZ (RHS) (b) △BYZ (SAS)
(c) △YXC (SSS) (d) △ZDX (AAS)
(e) △XEZ (SAS) (f) △GHF (SAS)
7 84°
8 (a) SAS (b) RHS (c) ASA or AAS (d) ASA
9 79°
10 △ABC ≡ △RPQ (SSS),
△KLM ≡ △XZY (AAS)

Exercise 24c (p. 215)

1

polygon		sum of interior angles
triangle	(3)	180°
quadrilateral	(4)	360°
pentagon	(5)	540°
hexagon	(6)	720°
heptagon	(7)	900°
octagon	(8)	1 080°
decagon	(10)	1 440°
dodecagon	(12)	1 800°

2 (a) 100° (b) 170° (c) 80° (d) 50°, 150°
3 156° each 4 18 sides
5 $B\hat{C}D = 112°$, $E\hat{A}B = 62°$, $A\hat{E}D = 118°$
6 11 7 23 cm 8 55° 9 11 sides
10 $x = 18°$; 72°, 90°, 108°, 126°, 144°

Exercise 24d (p. 218)

1 (a) $g = 95°$, $h = 33°$
(b) $i = j = 63°$, $k = 108°$, $l = 9°$, $m = 45°$
(c) $n = 226°$, $p = 113°$, $q = 67°$
(d) $r = 60°$, $s = 80°$, $t = 28°$, $u = 72°$
(e) $w = 90°$, $x = 42°$, $y = 12°$
(f) $z = 62°$
2 7 cm, 6 cm, 5 cm 3 3 cm, 4 cm, 6 cm
4 AY = 4 cm, BX = 5 cm
5 (a) $h = 90°$, $i = 90°$, $j = 40°$, $k = 50°$
(b) $m = n = 61°$
(c) $p = 23°$, $q = 67°$, $r = 67°$, $s = 23°$
(d) $t = u = 90°$, $v = 113°$
(e) $w = 128°$, $x = 64°$
(f) $y = 57°$, $z = 66°$
6 62°, 56°, 62°
7 (a) 62° (b) 120° (c) 58° (d) 106°
8 (a) $f = 56°$, $g = 68°$, $h = 56°$
(b) $x = 76°$, $y = 70°$, $z = 34°$
(c) $p = 38°$ (d) $r = 86°$ (e) $q = 25°$
9 $y = 2x - 180°$
10 (a) 58° (b) 72° (c) 26° (d) 19°

Exercise 24e (p. 220)

2 6,2 cm 3 5,9 cm 4 6,9 cm
6 185 mm 7 5,8 cm 8 10 cm
9 7,4 cm 10 8,5 cm

Exercise 24f (p. 223)

1 hemispherical surface 2 circle
3 construct the bisectors of the angles between the lines
5 perpendicular bisector of line joining the fixed points
6 4 7 4 8 4 cm or 6,2 cm
10 YZ = 8,7 cm; maximum area of △XMY is 23,3 cm^2

Exercise 25a (p. 224)

1 62,8 cm 2 15,84 m (1 584 cm)
3 150°, 16,5 cm 4 50
5 22 cm 6 1 cm
7 $14\frac{1}{3}$ cm 8 6,3 cm
9 120 cm 10 553,5 cm

Exercise 25b (p. 226)

1 (a) 88 cm^2 (b) 84 m^2
(c) 105 cm^2 (d) 14 400 m^2
2 19,7 cm^2 3 1,113 ha
4 3 850 cm^2, 294 cm^2
5 5,28 kg 6 $31\frac{1}{4}$%
7 113 m 8 14,2 cm^2
9 27,5 m 10 15 510 cm^2
11 90,1 cm^2 12 6,19%

Exercise 25c (p. 229)

1 18,48 litres 2 1,98 kg
3 1,6 m (160 cm) 4 1,188 kg
5 221,57 ($70,5\pi$) cm^2 6 1,12 m
7 (a) 20 250 cm^3 (b) 16,2 kg
8 0,79 litres 9 $4\frac{3}{4}$ cm
10 (a) 10 cm (b) 60π cm^2 (c) 96π cm^3
11 216° 12 195π cm^2
13 1 571 (500π)l 14 1 728

15 $3\frac{3}{8}$ cm **16** 336 cm^2
17 (a) (i) $\frac{2}{3}$ (ii) $\frac{1}{3}$ (b) 1:1
18 20,0 cm^2 **19** $31\frac{1}{4}$ cm
20 1,49 kg

Exercise 25d (p. 230)
1 (a) 3 cm (b) 25:1 **2** 32 kg
3 16 kg **4** 9 times **5** 135 cm^3
6 (a) 5:3 (b) 25:9 (c) 125:27
7 (a) 6 cm (b) 27:8
8 (a) 9 km^2 (b) 3 km (c) 300 000:1
9 38,5 cm **10** (a) 5:3 (b) 25:9

Exercise 26a (p. 231)
1 (a) 3 (b) 9 (c) 7 (d) 9 (e) 14 (f) 44
2 37 m **3** 4 m
4 (a) 7 cm (b) 84 cm^2 **5** 20 cm
6 (a) 5,39 (b) 3,74 (c) 6,48
(d) 7,28 (e) 7,55 (f) 9,54
7 58,7 cm **8** (a) 2,74 cm (b) 11,8 cm^2
9 4,36 cm **10** (a) 5,6 cm (b) 6,5 cm

Exercise 26b (p. 234)
1 181 m **2** 147 m
3 (a) 3,64 km (b) 10,6 km
4 $a = 11{,}0$ cm, $b = 11{,}4$ cm, $c = 15{,}9$ cm
5 (a) 5,54 (b) 6,08 cm
6 (a) 5; $5\sqrt{2}$ (b) $2\sqrt{2}$; $2\sqrt{2}$
(c) $\frac{7\sqrt{2}}{2}$; $\frac{7\sqrt{2}}{2}$ (d) 6; $3\sqrt{3}$
(e) $2\sqrt{3}$; $4\sqrt{3}$ (f) $2\frac{1}{2}$; $2\frac{1}{2}\sqrt{3}$
(g) 10; $5\sqrt{3}$ (h) 4; $4\sqrt{3}$
7 (a) 12 (b) $6\sqrt{2}$
(c) $6\sqrt{3}$ (d) 8
(e) $2\sqrt{6}$ (f) $\sqrt{6}$
(g) $3\sqrt{2} - \sqrt{6}$ (h) $4\sqrt{6}$
8 $10\sqrt{3}$ m **9** 59°
10 (a) 030°35′ (b) 12,8 km **11** 3,35 m
12 (a) 7,81 km (b) 085,2°
13 $47\frac{1}{2}$ m (45,9 m + 1,6 m)
14 3 835 m, 61 m
15 (a) 800 m (b) $400\sqrt{3}$ m

Exercise 26c (p. 237)
1 (a) 6,93 cm ($\sqrt{48}$ cm) (b) 35°21′
2 (a) 11 cm (b) 39°31′ (c) 49°24′
3 (a) 14,5° (b) 9°
4 4,48 m **5** 54°44′
6 6°54′, 17°38′ **7** 2,87 m, 60,5°
8 46°41′
9 (a) 53°8′ (b) 57°
10 (a) 33°4′ (b) $6\frac{6}{11}$ cm

Exercise 26d (p. 240)
1 13,1 cm **2** 20,4 cm **3** 10°4′
4 16°25′ **5** 4,36 cm **6** 1,98 cm
7 4,42 m **8** 61,8 cm **9** 57,1°
10 60° **11** 106,6° **12** 112,4°
13 240,5 m **14** 135 m
15 $d = 12{,}3$ cm, $\alpha = 50°3'$
16 $x = 8{,}90$ m, $\theta = 107{,}7°$
17 (a) 20,2 cm (b) 68,1° **18** 110,8°
19 1 041 km, 017°52′ **20** 23,1 km, 246°51′

Exercise 27a (p. 242)
1 7, $\frac{1}{7}\begin{pmatrix} -3 & -2 \\ -4 & -5 \end{pmatrix}$ **2** $x = -8\frac{1}{2}$

3 (a) $\begin{pmatrix} 4 & 4 \\ 1 & 7 \end{pmatrix}$ (b) $x = 4$, $y = 7$

4 (a) $\begin{pmatrix} 7 & -3 \\ 6 & -2 \end{pmatrix}$ (b) $m = 1\frac{1}{2}$ (c) $n = -2$

5 $x = 4$, $\frac{1}{5}\begin{pmatrix} 2 & 3 \\ 1 & 4 \end{pmatrix}$

6 $a = 4$, $b = -3$, $c = 1$

7 (a) $\begin{pmatrix} 1 & -8 \\ 3 & 0 \end{pmatrix}$ (b) $\frac{1}{2}\begin{pmatrix} 4 & -5 \\ -2 & 3 \end{pmatrix}$

(c) $\frac{1}{3}\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$

8 $k = -\frac{1}{2}$ or 6

9 (a) $-\frac{1}{7}\begin{pmatrix} 3 & -4 \\ -1 & -1 \end{pmatrix}$ (b) $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -5 \\ 2 \end{pmatrix}$

10 $\begin{pmatrix} 2 & 5 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 7 \\ 7 \end{pmatrix}$, $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 6 \\ -1 \end{pmatrix}$

Exercise 27b (p. 245)
1 Reflection in the line $x = y$
2 (a) 7 cm (b) 4,2 cm (c) 3,5 cm
3 (a) P (3; 5), Q (3; 3), R (9; 1) (b) $\frac{1}{3}$
4 (a) (9; −2) (b) (1; 2)
(c) A shear of factor −2 with the x-axis invariant
5 (a) (5; −3) (b) (−4; −1) (c) (1; 2)

6 (a) B′ (−7; 4), A (−2; −3)
(b) any point of the form $(k; 0)$, i.e. any point on the x-axis

7 (b) enlargement, centre at origin, factor k, $k = 2\frac{1}{2}$
(c) A″ (3; 1), B″ (1; −3), C″ (−3; 2)

8 (a) $\begin{pmatrix} 1 & 0 \\ 4 & 1 \end{pmatrix}$ (b) $\begin{pmatrix} 1 & 0 \\ -4 & 1 \end{pmatrix}$

9 (a) (i) (7; 5), (ii) (−3; −1)
(b) Rotation of 90° anticlockwise about (2; 2)

10 (a) 90° (b) $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ (c) $\begin{pmatrix} -8 \\ -4 \end{pmatrix}$
(d) (6; 2) (f) $x + y = -5$

Exercise 27c (p. 246)

1

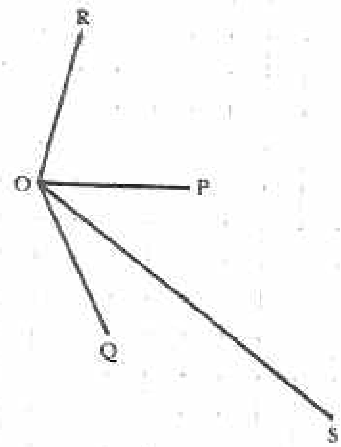


Fig. A21

2 (b) (i) $\begin{pmatrix} -1 \\ -4 \end{pmatrix}$, (ii) $\begin{pmatrix} 3 \\ -3 \end{pmatrix}$, (iii) $\begin{pmatrix} 5 \\ 5 \end{pmatrix}$

3 (b) (i) $\begin{pmatrix} 5 \\ 6 \end{pmatrix}$, (ii) $\begin{pmatrix} 1 \\ -4 \end{pmatrix}$, (iii) $\begin{pmatrix} -4 \\ 3 \end{pmatrix}$

4 (a) $\begin{pmatrix} 10 \\ -12 \end{pmatrix}$ (b) $\begin{pmatrix} 5 \\ -6 \end{pmatrix}$ (c) $\begin{pmatrix} 1 \\ 3 \end{pmatrix}$

5 (a) $\mathbf{b}$ (b) $-\frac{1}{2}\mathbf{a}$ (c) $\frac{1}{2}\mathbf{a} + \frac{1}{2}\mathbf{b}$
(d) $\frac{1}{2}\mathbf{b}$ (e) $\frac{1}{2}\mathbf{a} + \mathbf{b}$ (f) $\mathbf{b} - \frac{1}{2}\mathbf{a}$

6 (a) 5 (b) $q = 7$

7 $h = 2, k = 9$

8 $m = -1, n = -10$

9 (a) $\mathbf{XY} = \frac{1}{4}\mathbf{a} + \frac{1}{2}\mathbf{c}$, $\mathbf{CZ} = \mathbf{a} - \frac{1}{2}\mathbf{c}$
(b) $\mathbf{XP} = \frac{1}{2}h\mathbf{a} + \frac{1}{2}h\mathbf{c}$
(c) $\mathbf{CP} = k\mathbf{a} - \frac{1}{2}k\mathbf{c}$
(d) $h = \frac{2}{3}, k = \frac{1}{3}$

10 (a) (i) $\begin{pmatrix} -6 \\ -2\frac{1}{2} \end{pmatrix}$, (ii) 13
(b) (i) $3\mathbf{a} - 3\mathbf{b}$, (ii) $-\mathbf{a} + 3\mathbf{b}$
$4\mathbf{b} + m(3\mathbf{a} - \mathbf{b}) = n(4\mathbf{a})$; $m = 4$, $n = 3$

Exercise 28a (p. 250)

1 (a) 5 (b) 145 km (c) 58 km/h (d) 36 km/h
(e) 100 km/h (f) 70 km/h (g) 10 km

2 (a) 280 m (b) 4 min (c) 1 217
(d) 400 m (e) 80 m/min (f) 120 m

3 (a) 1 250 (b) 1 300 (c) 1 310

4 (a) 4 600 km (b) 400 km

5 (a) 17,9 km/h (b) 2 min

6 (a) 9 km/h (b) 3,4 km

7 (a) 5 m/s² (b) $\frac{1}{9}$ m/s² (c) 172 m

8 (a) (i) constant speed of 120 km/h for 3 min
(ii) uniform acceleration from 120 km/h to 150 km/h in 2 min
(iii) uniform deceleration from 150 km/h to rest in 1 min
(b) (ii) 15 km/h per min
(iii) −150 km/h per min
(c) $11\frac{3}{4}$ km

9 (a) 1,5 m/s² (b) (i) 225 (ii) 6 675 km

10 (a) 19 m/s (b) 24,95 m/s

11 (b) 227 min (c) 2 639 km

12 (a) 0 m/s² (b) −6 m/s² (c) 116 m

13 (a) −6 m/s² (b) 212 m

14 (a) $m = 2, n = 18$ (c) 0,3 s and 2,7 s
(d) 12 m/s² (e) 33 m

15 (a) $a = 16, b = 0$ (c) 14,3 m/s
(d) (i) 4 m/s², (ii) −6 m/s² (e) 82 m

Exercise 28b (p. 255)

3 (b) 1 434

5 (a) 4,6; 4; 4 (b) 6,5; 7; 7 (c) $6\frac{1}{3}$; 6; 6

6 117,1

7 18 yr 3 mo

8 158 cm, 159,5 cm

9 $47\frac{7}{15}$ kg, 46 kg

10 (a) 20 students (b) 15 yr (d) 15 yr

Exercise 28c (p. 257)

1 (a) 5 yr (b) 40 children (c) 4,7 yr

2 (b) 547,5 h

3 (b) 0–4 days (c) 6,51 days

4 (c) 60 kg (d) $59\frac{1}{4}$ kg

5 (c) (i) 51 (ii) 9,5 (d) 68%

6 (b) 151–160 cm (c) 155,75 cm (e) 155 cm

Exercise 28d (p. 259)

1 (a) $\frac{7}{10}$ (b) $\frac{3}{10}$

2 (a) $\frac{1}{2}$ (b) $\frac{1}{5}$ (c) $\frac{3}{10}$

3 (a) $\frac{1}{52}$ (b) $\frac{1}{26}$ (c) $\frac{1}{13}$ (d) 0

4 $\frac{5}{12}$

5 (a) $\frac{1}{6}$ (b) $\frac{1}{2}$ (c) $\frac{2}{3}$

6 (a) HHH, HHT, HTH, HTT, THH, THT, TTH, TTT (b) $\frac{3}{8}$

7 15

8 $\frac{102}{125}$

9 (a) 1 (b) $\frac{1}{3}$ (c) $\frac{1}{3}$ (d) $\frac{2}{3}$
(e) $\frac{1}{3}$ (f) $\frac{1}{6}$ (g) 0 (h) $\frac{1}{6}$

10 $\frac{3}{16}$

11 (a) (i) $\frac{1}{4}$, (ii) $\frac{5}{9}$, (iii) $\frac{31}{36}$
(b) 17 yr 4 mo (c) $\frac{1}{63}$

12 (a) $\frac{3}{5}$ (b) $\frac{1}{15}$

13 (a) $\frac{2}{11}$ (b) $\frac{1}{22}$ (c) $\frac{15}{44}$

14 $\frac{3}{5}, \frac{1}{2}$ (a) $\frac{3}{10}$ (b) $\frac{1}{10}$ (c) $\frac{3}{5}$

15 (a) $\frac{1}{10}$ (b) $\frac{4}{15}$ (c) $\frac{12}{145}$ (d) $\frac{63}{145}$

Chapter 29

Note: answers to investigations are generally not given.

Number, algebra and pattern (p. 261)

1 (a) 15 (b) 8 (c) 24
(d) $n^2 + 2n$ or $(n + 1)^2 - 1$
(e) 21 for 3 boys and 3 girls

4 2 178

5 431 × 52

6

4	2	3
2	3	4
3	4	2

7 here are two solutions; there are others

B	R	B
W	W	W
R	B	R

R	W	B
W	B	R
R	B	W

8

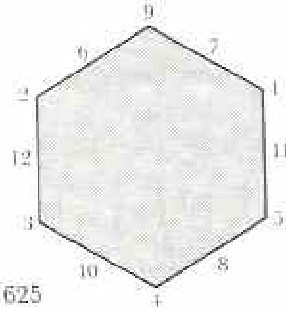


Fig. A22

9 64 × 15 625

10 11, 16, 27; $\frac{(m+n)}{3}$

11 (a) 2 (b) 16 (c) 1, 9, 16
(d) 1, 2, 9 (e) 1, 2, 9 (f) 5, 9, 16
(g) 2, 9, 1

12 (b) $S_1 = \{1; 2; 4; 8; 16; 32\}$
$S_2 = \{3; 5; 6; 9; 10; 12; 17; 18; 20; 24; 33\}$
$S_3 = \{7; 11; 13; 14; 19; 21; 22; 25; 26; 28\}$
$S_4 = \{15; 23; 27; 29; 30\}$
$S_5 = \{31\}$

13 50 cents

14 plan (b)

15 0

16 18 hens

17 7 cattle or 22 cattle or 37 cattle or ... 7 + 15*n* cattle where *n* = 0, 1, 2, 3, ...

18 (a)

```
 x x            x x x x
m m m    or    m m m m m    etc.
 x x            x x x x
```

(b) 28 (c) 46 (d) $X = 2M - 2$ (e) 20
(f) $7\frac{1}{3}$, it would be impossible to make this pattern

19 (a) the first two numbers are added and only the units digit of their sum is written down to become the 3rd number; the 2nd and 3rd numbers are added, writing down the units digit of their sum as the 4th number; and so on
(b) the chain repeats itself after a while

20 (a) 11 (b) 44

Spatial awareness and pattern (p. 263)

1

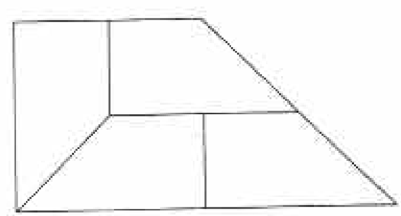


Fig. A23

2 (a) 1:4 (b) 1:4

3 $1^2 + 2^2 + 3^2 + \ldots + 8^2 = 204$

4

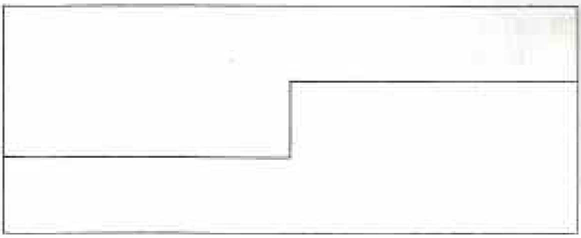


Fig. A24

5 (a) here is one of many solutions:

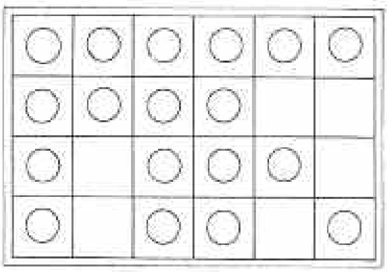


Fig. A25

6 10

8

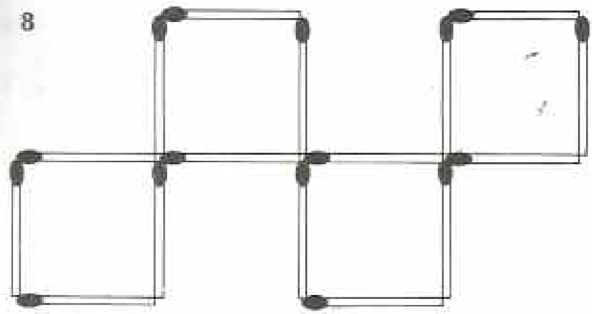


Fig. A26

9 $\alpha + \omega = \phi$

10 $m + n -$ (highest common factor of m and n)

Miscellaneous (p. 265)

1 1 in 5

3 36, 10, 124 750 $\left(= \dfrac{499 \times 500}{2}\right)$

4 Bernadette took the exam paper (Cynthia is truthful)

5 (a) 1,5 cm (b) 11,8 cm (c) 3,2 km, 23,6 km (d) 20,4 km (e) $10\frac{1}{2}$ min

6 24 000 km

7 Tembo, the history student, did not eat lunch

8 only 10 times

9 X overtakes Y inside the last 5 m and wins

10 741

Examination 1

Paper 1 (p. 267)

1 (a) $1\frac{5}{12}$ (b) 4 (c) $2\frac{1}{4}$
2 (a) 9,17 (b) 16 (c) 0,000 168
3 (a) {a; e; i; u} (b) 4 (c) {a; e; i; u; r; v}
4 (a) 25 (b) 26%
5 (a) $12\,144_{\text{five}}$ (b) $1\,000\,00_{\text{two}}$ (c) 154_{ten}
6 (a) $4{,}9 \times 10^7$ (b) 3×10^3 (c) $2{,}5 \times 10^{-8}$
7 $x = 39$, $y = 19$, $z = 58$
8 (a) $\frac{1}{450}$ (b) 0,004 (c) $\frac{5}{16}$
9 (a) 133,5 kg (b) 11; 12 or 13
10 (a) 5 units (b) $1\frac{1}{3}$ (c) $k = 4$
11 (a) $\begin{pmatrix} 5 \\ -4 \end{pmatrix}$ (b) $\begin{pmatrix} 8 \\ 2 \end{pmatrix}$ (c) $\begin{pmatrix} -5 \\ 4 \end{pmatrix}$
12 $x = 108°$, $y = 36°$, $z = 108°$
13 (a) 20 °C (b) 7 °C
14 (a) $(x - 7)(x - 7)$ or $(x - 7)^2$
(b) $(3a - b)(2b + 4) = 2(3c - b)(b + z)$
15 $x = 2{,}5$; $y = -3$
16 (a) 2.30 pm (b) 3 km (c) 2,5 km/h
17 (a) \$240 (b) \$192
18 308 mℓ
19 (a) (ii) (b) $m = \dfrac{5\,400}{n}$ (c) 400
20 (a) $n \geqslant 540$ (b) 416 (c) \$316,20
21 (a) 323° or N37°W (b) 143° or S37°E
(c) 176 km
22 (a) 120° (b) $k = 25$
23 (a) 2,8 cm (b) 8,29 cm²
24 (a) $x = 6$ (b) $x = 1$ or 7
25 (a) 0 (b) 2 (c) 11
26 (a)

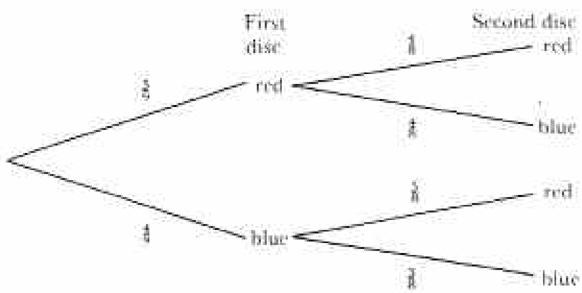


Fig. A27

(b) $\frac{5}{18}$
27 (a) $21 + 23 + 25 + 27 + 29$
(b) 100 (c) 441

28 (a) 3 (b) B′ (9, 5), C′ (6, 7)
(c) $\begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}$

29 (a) (i) 30 Francs (ii) Z\$18,30 approx.
(b) (i) line should join origin to (20,65)
(ii) Z\$16,90 approx.

Paper 2 (p. 271)

1 (a) (i) \$3 900 (ii) \$78 000
(b) (i) \$6,00 (ii) \$369,60 (iii) 45 years
(iv) 31–35 years, Plan C
2 (a) (i) $(x + 3)(x + 4) = 2(x + 7)(x - 2)$
(ii) $x = 5$ or -8, only $x = 5$ is realistic
(iii) 72 units²
(b) (i) $c = 1{,}5a - 18$ (ii) $-9{,}8$ °C
3 (a) 85° (b) 8,97 cm
4 (a) (i) 5,41 cm² (ii) 59,5 cm³
(iii) 55,5 cm³ (iv) 52,4 cm³
(b) (i) 18 m (ii) 4 000 cm³
5 (a) (i) 12° (base angles, isos $\triangle$)
(ii) 204° ($= 360° -$ obtuse $B\hat{O}A$ (angles at point)
$= 360° - 156°$ (angles of $\triangle$O…
(iii) 102° (angle at circumference = half angle at centre)
(iv) 12° ($= B\hat{A}T$ (alt. segment) $= O\hat{B}A$ (alt. angles))
(v) 66° (angles on a straight line)
(b) (i) $2\mathbf{a}$ (ii) $2\mathbf{a} - 3\mathbf{b}$ (iii) $\frac{1}{2}\mathbf{a} + \frac{1}{2}\mathbf{b}$
6 (a) $A_1(0; 0)$, $B_1(2; 0)$, $C_1(3; 2)$
(b) $A_2(0; 0)$, $B_2(6; 0)$, $C_2(9; 6)$
(c) $\begin{pmatrix} 3 & 3 \\ 0 & 3 \end{pmatrix}$
7 (a) $x \leqslant 6$, $y \leqslant 6$, $2x + y \geqslant 34$, $x + 3y < 42$
(b)

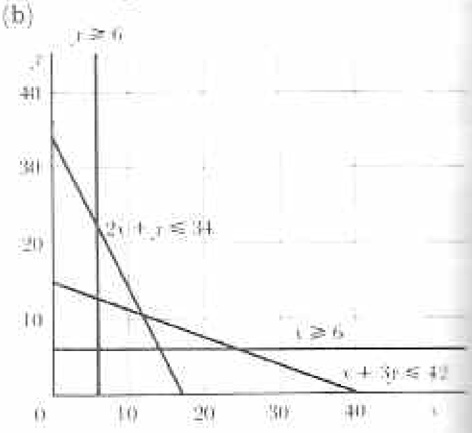


Fig. A28

(c) (i) 12 Standard and 10 Special (ii) $688

8 (a) $x = 3$ (b) 8 (c) (i) 17, (ii) 7

(d)

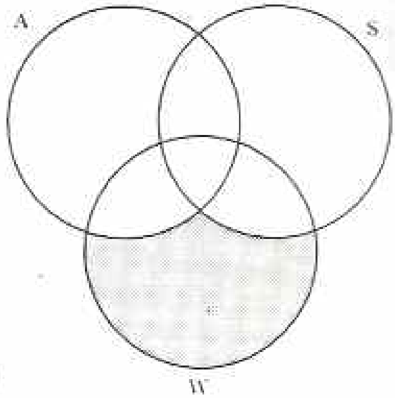


Fig. A29

those who can do only word-processing

9 (a) 9,33 cm (b) 9,98 cm

(c) 69,25° or 69°15′ (d) 121,6 cm²

10 (a) 120° (b) 2,36 m² (c) 1,38 m²

(d) 6 760 kg (or 6,76 tonnes)

11 (a)

chest (cm)	⩾80	⩾85	⩾90	⩾95	⩾100	⩾105	⩾110	⩾115
cum. freq.	3	13	27	47	70	85	94	100

(c) 96 cm (d) 77 people (e) $\frac{29}{330}$

12 (a)

x	1	2	3	4	5	6	7	8	9
y	9	4,5	3	2,25	1,8	1,5	1,29	1,13	1

(d) $x = 1,35$ and $6,65$ (e) $x^2 - 8x + 9 = 0$

Examination 2

Paper 1 (p. 275)

1 (a) 60 000
(b) $6{,}03 \times 10^4$
(c) $2^2 \times 3^2 \times 5^2 \times 67$
2 (a) $\frac{1}{10}$
(b) 3
3 (a) 0
(b) 15
(c) 25
4 $x < -1$
5 (a) $(a + 5x)(a - 3x)$
(b) $100m + n$
6 (a) 19°
(b) 38°
7 (a) $r + r\sqrt{2}$
(b) $3 + 2\sqrt{2}$
8 (a) 0,05
(b) $2\frac{2}{3}$
(c) $5d^{18}$
9 6,75 kg
10 (a) 8
(b) 125
(c) $1\frac{1}{4}$
11 43 kg
12 0,7 kg (700 g)
13 (a) 5 cm
(b) 0,6
14 2 or 4
15 98 women
16 (a) 5
(b) $b = \dfrac{ac}{a^2 - 1}$
17 $a = 108°$, $b = 36°$
18 (a) 11,9 cm
(b) 15,6 cm
19 420 g
20 (b) (i) $\{2; 5; 7; 8\}$, (ii) 7
21 (b) $p = -1$, $q = -3$
(c) $\begin{pmatrix} 3 \\ -1 \end{pmatrix}$ (d) $\begin{pmatrix} 4 \\ 2 \end{pmatrix}$
22 $x = 2$ or $\frac{2}{7}$
23 (a) $-\frac{1}{3}$
(b) $y = -\frac{1}{3}x + 1\frac{2}{3}$ (i.e. $x + 3y = 5$)
(c) $(7; -\frac{2}{3})$
24 (a) 9 cm (b) $\frac{4}{9}$
25 (a) 112° (b) 124°

26

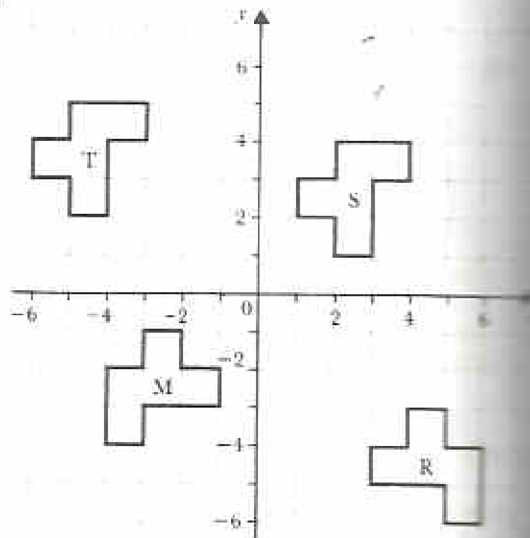


Fig. A30

27 (a) $y = \dfrac{8}{\sqrt{x}}$ (b) $y = 2\frac{2}{7}$ (c) $n = 4$
28 (a) 7,25 (b) $-3{,}2; 2{,}2$
(c) area in the range 25–26 unit²
29 (a) 1 272 m (b) 3 m/s² (c) 16 m/s

Paper 2 (p. 278)

1 (a) 6%
(b) (i) $\begin{pmatrix} 4 & -20 \\ 0 & 6 \end{pmatrix}$ (ii) $\begin{pmatrix} -6 & -10 \\ 8 & -4 \end{pmatrix}$
(iii) $x = -\frac{1}{2}, y = 3$
2 (a) $\hat{P} = 42°$, $\hat{Q} = 69°$, $\hat{R} = 138°$, $P\hat{S}R = 111°$
(b) (i) 16 000 (16 024,2) cm²
(ii) 56 600 (56 556) cm³
3 (a) $\dfrac{x - 12}{(x + 2)(3x - 1)}$ (b) $x = 12$
(c) $(4a - 3)(4a + 3)$
$1\,591 = 1\,600 - 9 = (40 - 3)(40 + 3)$
$= 37 \times 43$
4 (a) (i) $\mathbf{a} + \mathbf{b}$ (ii) $-\frac{1}{2}\mathbf{a} - \frac{1}{2}\mathbf{b}$ (iii) $\frac{1}{2}\mathbf{a} - \ldots$
(b) 24:25
5 (a) (i) 8 cm, 15 cm, 17 cm
(ii) 90° ($17^2 = 15^2 + 8^2 \Rightarrow$ rt-angled △)
(iii) 28°4′ or 28,07°
(b) 5×10^3

6 (a) (i) $a = 0{,}29$; $b = 1{,}74$; $P = 0{,}29W + 1{,}74$
(ii) straight line cutting W-axis at $(-6;0)$ and P-axis at $(0;1{,}74)$
(b) $d = \frac{1}{3}$ or 7

7 (a) (i) 38°40′ or 38,66° (ii) 73°44′ or 73,74°
(b) (i) $(-6;7)$ (ii) $r = -2, s = -1$
(iii) one-way stretch of factor -3 parallel to the x-axis with the y-axis invariant

8 (a) 101c − 125c (b) 131,5c (c) 131c
(d) $\frac{11}{25}$ (e) (i) $\frac{3}{35}$, (ii) $\frac{3}{7}$

9 (a) $\begin{pmatrix}-1\\-4\end{pmatrix}$ (b) (i) 270°, (ii) $(0;5)$
(c) $y = x - 1$
(d) (i) $x = 0$, (ii) $(-1;-2)$, $(-1;0)$ $(-2;-3)$, $(-2;-2)$, $(-3;-6)$

10 (a) (i) 139 m,
(ii) 139 m,
(iii) 10 000 m^2
(b) (i) 320° (or N40°W),
(ii) 140° (or S40°E),
(iii) 055,87° (or N55°52′E)

11 (a) (1;5), (3;11), (5;33)
(c) (i) during first 1,25 seconds,
(ii) 4.9 m/s,
(iii) 11 m/s^2

12 (a) 9,0 cm
(f) 2,7 cm (answers to nearest 0,1 cm)