New General Mathematics 3

An 'O' Level Course

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Preface to 1993 edition

The *New General Mathematics* series has been revised to reflect the content and philosophy of mathematical education in Zimbabwe's secondary schools. Books 1 and 2 provide a full course at Junior Certificate Level. Books 3 and 4 contain a substantial course leading to the General Certificate in 'O' Level Mathematics.

In Books 3 and 4, special emphasis is given to the development of mathematical skills which school leavers need to acquire for their everyday lives, and for further study and training. To achieve this there are extensive new sections on consumer arithmetic and on the use of calculators.

In order to develop problem solving skills and provide opportunities for independent thought, users of Books 3 and 4 will find sections which include 'non-routine problems'. Students and their teachers are urged to approach these with an open mind, and with the aim of finding unexpected joy and self-fulfilment in mathematics.

The need to consolidate and practice problem solving skills is maintained through periodic revision tests. These can be used to diagnose performance in mathematics and provide a basis for further improvement.

Whilst revising Books 3 and 4, opportunities were used to make corrections and to update statistical information. The authors and publishers are grateful to the Central Statistical Office, Harare, for providing valuable data. Above all, we are grateful to the many readers who have made helpful suggestions and who have provided so much encouragement.

M.F. Macrae
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FOREWORD

This textbook is a gift from Zimbabwe's friends in the International Community which include (in alphabetical order): Australia, Denmark, the European Commission, Finland, Germany, Japan, Netherlands, Norway, Sweden, the United Kingdom and the United States of America. Through the wonderful generosity of these countries, and our ability to work together in the Education Transition Fund (established by the Ministry of Education, Sports, Arts and Culture in September 2009), we have already distributed more than seventeen million textbooks to all primary schools. It is now the turn of secondary schools and several million textbooks like this one are being distributed countrywide to schools.

Please would you treat this book with care so that students following on from you can also benefit from it. Accordingly, please could you ensure that it is covered and kept out of rain and dirt. Under no circumstances should this book ever be sold because it is a gift.

The future of Zimbabwe depends on your generation. My hope is that the provisions of this textbook will inspire you to study hard and, through that, you will play a significant role in transforming Zimbabwe into the jewel of Africa.

Senator David Coltart

MINISTER OF EDUCATION, SPORT, ARTS AND CULTURE

July 2011

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Preliminary chapter

Review of Books 1 and 2

Before beginning Book 3, readers should be familiar with the contents of Books 1 and 2. The following summary contains those parts of Books 1 and 2 which appear in the GCE 'O' Level Mathematics syllabus.

Number and numeration

(a) Numbers are normally written in the decimal **place value** system (Fig. P1):

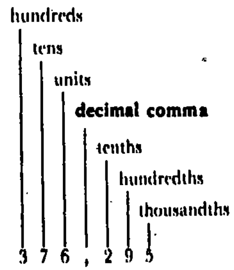


Fig. P1

The symbols 0; 1; 2; 3; 4; 5; 6; 7; 8; 9 are called **digits.**

(b) 28 ÷ 7 = 4. 7 is a whole number which divides exactly into another whole number, 28. 7 is a **factor** of 28. 28 is a **multiple** of 7.

(c) A **prime number** has only two factors, itself and 1. 1 *is not* a prime number. 2; 3; 5; 7; 11; 13; 17; ..., are prime numbers. They continue without end. The **prime factors** of a number are those factors which are prime. For example, 2 and 5 are the prime factors of 40. 40 can be written as a **product of prime factors**; either 2 × 2 × 2 × 5 = 40, or, in **index form**, $2^3 \times 5 = 40$.

(d) The numbers 18, 24 and 30 all have 3 as a factor. 3 is a **common factor** of all the numbers. The **highest common factor** (HCF) is the largest of the common factors of a given set of numbers. For example, 2, 3 and 6 are the common factors of 18, 24 and 30; 6 is the HCF.

The number 48 is a multiple of 4 and a multiple of 6. 48 is a **common multiple** of 4 and 6. The **lowest common multiple** (LCM) is the smallest of the common multiples of a given set of numbers. For example, 12 is the LCM of 4 and 6.

(e) A **fraction** is the number obtained when one number (the **numerator**) is divided by another number (the **denominator**). The fraction $\frac{5}{8}$ means 5 ÷ 8 (Fig. P2).

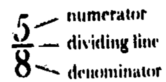


Fig. P2

Fractions are used to describe parts of quantities (Fig. P3).

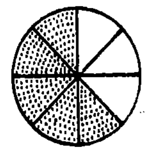


Fig. P3 $\frac{5}{8}$ *of the circle is shaded*

The fractions $\frac{5}{8}$, $\frac{10}{16}$, $\frac{15}{24}$ all represent the same amount; they are **equivalent fractions**. $\frac{5}{8}$ is the **simplest form** of $\frac{15}{24}$, i.e. $\frac{15}{24}$ in its **lowest terms** is $\frac{5}{8}$.

To add or subtract fractions, change them to equivalent fractions with a **common denominator**. For example:

$$\frac{5}{8} + \frac{2}{3} = \frac{15}{24} + \frac{16}{24} = \frac{15 + 16}{24} = \frac{31}{24} \ (= 1\tfrac{7}{24})$$

$$\frac{13}{16} - \frac{5}{8} = \frac{13}{16} - \frac{10}{16} = \frac{13 - 10}{16} = \frac{3}{16}$$

To multiply fractions, multiply numerator by numerator and denominator by denominator. For example:

$$\frac{5}{8} \times \frac{2}{3} = \frac{5 \times 2}{8 \times 3} = \frac{10}{24} \ (= \tfrac{5}{12} \text{ in simplest form})$$

$$12 \times \frac{5}{8} = \frac{12}{1} \times \frac{5}{8} = \frac{12 \times 5}{1 \times 8} = \frac{60}{8} \ (= \tfrac{15}{2} = 7\tfrac{1}{2})$$

To divide by a fraction, multiply by the **reciprocal** of the fraction. For example:

$$35 \div \tfrac{5}{8} = \frac{35}{1} \times \frac{8}{5} = \frac{35 \times 8}{1 \times 5} = \frac{7 \times 8}{1} = 56$$

$$\tfrac{5}{8} \div 3\tfrac{3}{4} = \frac{5}{8} \div \frac{15}{4} = \frac{5}{8} \times \frac{4}{15} = \frac{5 \times 4}{8 \times 15}$$

$$= \frac{20}{120} \ (= \tfrac{1}{6})$$

(f) $x\%$ is short for $\frac{x}{100}$. 64% means $\frac{64}{100}$. To change a fraction to an equivalent percentage, multiply the fraction by 100. For example, $\frac{5}{8}$ as a percentage $= \frac{5}{8} \times 100\% = \frac{125}{2}\% = 62\frac{1}{2}\%$.

(g) To change a fraction to a **decimal fraction**, divide the numerator by the denominator. For example:

$\frac{5}{8} = 0{,}625$

```
   0,625
8)5,000
  4 8
    20
    16
     40
     40
```


When adding or subtracting decimals, write the numbers in a column with the decimal commas exactly under each other. For example: *Add* 2,29, 0,084 *and* 4,3, *then subtract the result from* 11,06

```
   2,29
   0,084        11,06
 + 4,3        − 6,674
   6,674        4,386
```

To multiply decimals, first ignore the decimal commas and multiply the given numbers as if they are whole numbers. Then place the decimal comma so that the answer has as many digits after the comma as there are in the given numbers together. For example: 0,08 × 0,3.

8 × 3 = 24

There are 3 digits after the decimal comma in the given numbers, so 0,08 × 0,3 = 0,024.

To divide by decimals, make an equivalent division such that the divisor is a whole number. For example 5,6 ÷ 0,07:

$$5{,}6 \div 0{,}07 = \frac{5{,}6}{0{,}07} = \frac{5{,}6 \times 100}{0{,}07 \times 100} = \frac{560}{7} = 80$$

(h) Numbers may be positive or negative. Positive and negative numbers are called **directed numbers**. Directed numbers can be shown on a **number line** (Fig. P4).

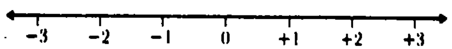


Fig. P4

The following examples show how directed numbers are added, subtracted, multiplied and divided.

addition	*subtraction*
(+ 8) + (+ 3) = + 11	(+ 9) − (+ 4) = + 5
(+ 8) + (− 3) = + 5	(+ 9) − (− 4) = + 13
(− 8) + (+ 3) = − 5	(− 9) − (+ 4) = − 13
(− 8) + (− 3) = − 11	(− 9) − (− 4) = − 5

multiplication	*division*
(+ 2) × (+ 7) = + 14	(+ 6) ÷ (+ 3) = + 2
(+ 2) × (− 7) = − 14	(+ 6) ÷ (− 3) = − 2
(− 2) × (+ 7) = − 14	(− 6) ÷ (+ 3) = − 2
(− 2) × (− 7) = + 14	(− 6) ÷ (− 3) = + 2

An **integer** is any positive or negative *whole* number.

Algebraic processes

(a) A **set** is a collection of objects. The **members** or **elements** of a set may be defined in a number of ways:

by description:
A = {first five counting numbers}
by listing elements:
A = {1; 2; 3; 4; 5}
in set-builder notation:
A = $\{x: 1 \leqslant x \leqslant 5, x \in Z\}$

The last statement may be read as: A is the set of values x such that x lies between 1 and 5 inclusively, where x is an integer. Note the use of curly brackets to contain sets, the use of semi-colons to separate the elements of a set and the use of Z as an abbreviation for the set of integers.

A set can be represented on a **Venn diagram** (Fig. P5).

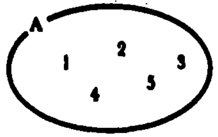


Fig. P5

Given a **universal set**, $\mathscr{E} = \{a; b; c; d; e\}$ and sets $X = \{a; b; e\}$ and $Y = \{d; e\}$:

X is a **subset** of $\mathscr{E}$ written $X \subseteq \mathscr{E}$. Also $Y \subseteq \mathscr{E}$.

The **union** of X and Y, written $X \cup Y$, is the set whose elements are members of X or Y or both X and Y.

$X \cup Y = \{a; b; d; e\}$

The **intersection** of X and Y, written $X \cap Y$, is the set whose elements are members of both X and Y.

$X \cap Y = \{e\}$

The **numbers** of elements in X, written n(X), is 3.

n(X) = 3

(b) $3y^2 + 2x - 7x$ is an example of an **algebraic expression**. The letters y and x stand for numbers. $3y^2$, $2x$ and $7x$ are the **terms** of the expression. $3y^2$ is short for $3 \times y \times y$. 3 is the **coefficient** of y^2. Algebraic terms may be **simplified** by combining **like terms**. Thus $3y^2 + 2x - 7x = 3y^2 - 5x$ since $2x$ and $7x$ are like terms (i.e. both terms in x).

(c) $3(5x - 2) = 11x$ is an **algebraic sentence** containing an equals sign; it is an **equation** in x. x is the **unknown** of the equation. To **solve an equation** means to find the value of the unknown which makes the equation true. We can use the **balance method** to solve simple equations.

$3(5x - 2) = 11x$
clear brackets
$15x - 6 = 11x$
subtract $11x$ from both sides
$15x - 11x - 6 = 11x - 11x$
$4x - 6 = 0$
add 6 to both sides
$4x - 6 + 6 = 0 + 6$
$4x = 6$
divide both sides by 4
$\frac{4x}{4} = \frac{6}{4}$
$x = 1\frac{1}{2}$

In general, when solving equations, (i) first clear brackets and fractions, (ii) using equal additions and/or subtractions, collect unknown terms on one side of the equals sign and known terms on the other, (iii) where necessary, divide or multiply both sides of the equation by the same number to find the unknown.

(d) An **inequality** is an algebraic sentence which contains an inequality sign:

$<$ is less than
$\leqslant$ is less than or equal to
$>$ is greater than
$\geqslant$ is greater than or equal to

Inequalities are solved in much the same way as equations. However, when both sides of an inequality are multiplied or divided by a negative number, the inequality sign is *reversed*. For example,

If $-3a \leqslant 12$

divide both sides by -3 and reverse the inequality

Then $a \geqslant -4$

(e) $2x - 5y = 16$ is a **linear equation** with two **variables**, x and y. There are many pairs of values of x and y which **satisfy** this equation (i.e. make the equation true). For example, if $x = 13$ and $y = 2$ or if $x = 8$ and $y = 0$ the equation will be true. Given two equations, such as $2x - 5y = 16$ and $x + 4y = -5$, it is usually possible to find values of x and y which satisfy *both* equations simultaneously (i.e. at the same time). To solve a pair of **simultaneous linear equations**, either use a graphical method or the method of elimination and substitution. For example,

$2x - 5y = 16$ (1)

$x + 4y = -5$ (2)

multiply (2) by 2

$2x + 8y = -10$ (3)

subtract (3) from (1) to eliminate terms in x

$-13y = 26$

$y = -2$

substitute -2 for y in (2) to find x

$x + 4(-2) = -5$

$x - 8 = -5$

$x = 3$

The solution is $x = 3$ and $y = -2$.

(f) A **graph** of an algebraic sentence is a picture representing the meaning of the sentence. Graphs of equations and inequalities in one variable can be shown on the number line (Fig. P6).

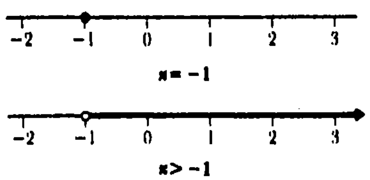


Fig. P6

For graphs connecting two variables, two **axes** are drawn at right angles to each other to give a **cartesian plane**. The horizontal **x-axis** and the vertical **y-axis** cross at their zero-point, the **origin** of the plane. Fig. P7 is the graph of the equation $y = 2x - 3$.

To draw a straight-line graph, plot at least three points which satisfy the given equation. See the table of values in Fig. P7. At point A in Fig. P7, $x = 2$ and $y = 1$. The **coordinates** of A are A(2; 1). The *order* of the coordinates is important: the **x-coordinate** is given first, the **y-coordinate** second.

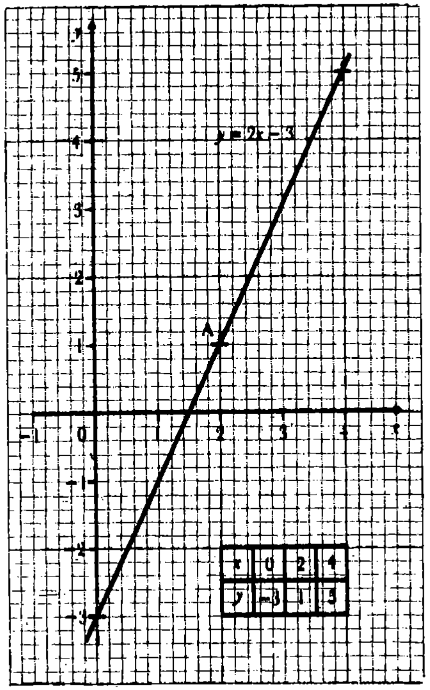


Fig. P7

Straight-line graphs can be drawn to represent any two connected variables, for example, cost and quantity, distance and time, temperature and time. Straight-line graphs can also be drawn to show conversions between currencies or between marks and percentages.

(g) Algebraic expressions may be factorised or expanded in accordance with the basic rules of arithmetic. Some examples follow.

expansion

$3(a - 2b) = 3a - 6b$

$(5 + 8x)x = 5x + 8x^2$

$(a + b)(c + d) = c(a + b) + d(a + b)$
$= ac + bc + ad + bd$

$(3x + 2)(x - 4) = 3x^2 + 2x - 12x - 8$
$= 3x^2 - 10x - 8$

$(a - 5b)^2 = a^2 - 10ab + 25b^2$

factorisation

common factor

$5y - 10y^2 = 5y(1 - 2y)$

$4x - 8 + 3bx - 6b = 4(x - 2) + 3b(x - 2)$
$= (x - 2)(4 + 3b)$

difference of two squares

$a^2 - b^2 = (a + b)(a - b)$

perfect square

$x^2 + 2xy + y^2 = (x + y)^2$

$a^2 - 16a + 64 = (a - 8)^2$

quadratic

$x^2 + 13x + 12 = (x + 12)(x + 1)$

$n^2 - 7n - 18 = (n - 9)(n + 2)$

(h) The following laws of indices are true for all values of a, b and x.

$x^a \times x^b = x^{a+b}$ $\quad x^a \div x^b = x^{a-b}$

$x^0 = 1$ $\quad x^{-a} = \frac{1}{x^a}$

Geometry and mensuration

(a) Fig. P8 gives sketches and names of some **common solids**.

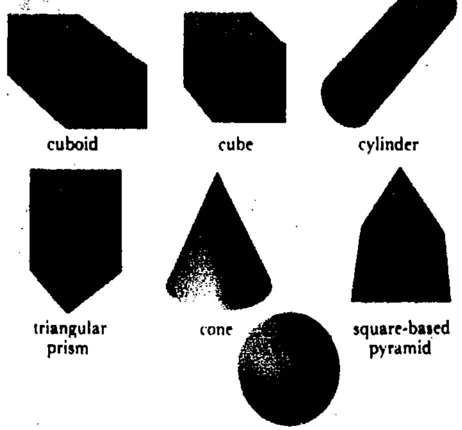


Fig. P8

All solids have **faces**; most solids have **edges** and **vertices** (Fig. P9).

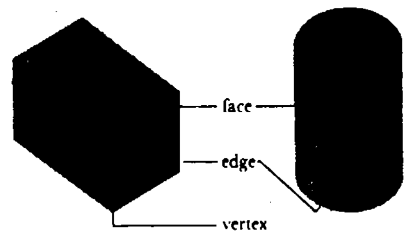


Fig. P9

Formulae for the **surface area** and **volume** of common solids are given in the table on page 278.

Fig. P10 shows two common methods of **drawing** a cuboid. These methods may be used with any solid.

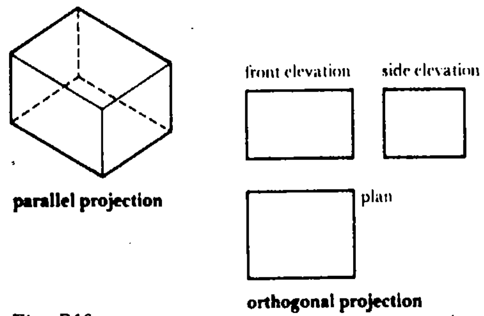


Fig. P10

In **parallel projections**, lines which are parallel on the solid appear parallel on the drawing. In **orthogonal projections**, the solid is represented by separate scale drawings of its **plan** and **elevations**.

The **net** of a solid is the plane shape which can be folded to make the solid.

(b) **Angle** is a measure of rotation or turning.

1 **revolution** = 360 **degrees** (1 rev = 360°)

1 **degree** = 60 **minutes** (1° = 60′)

The names of angles change with their size. See Fig. P11 overleaf.

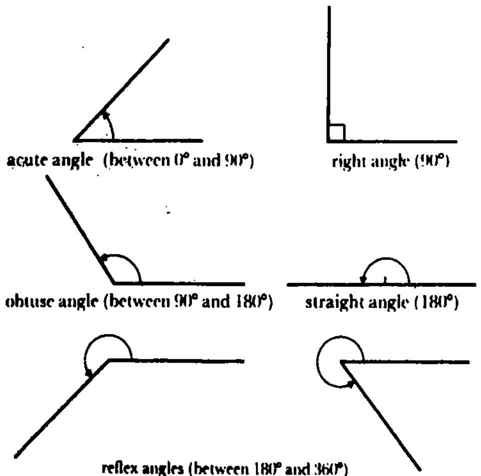


Fig. P11

Angles are measured and constructed using **a protractor**.

Fig. P12 shows some properties of angles formed when straight lines meet.

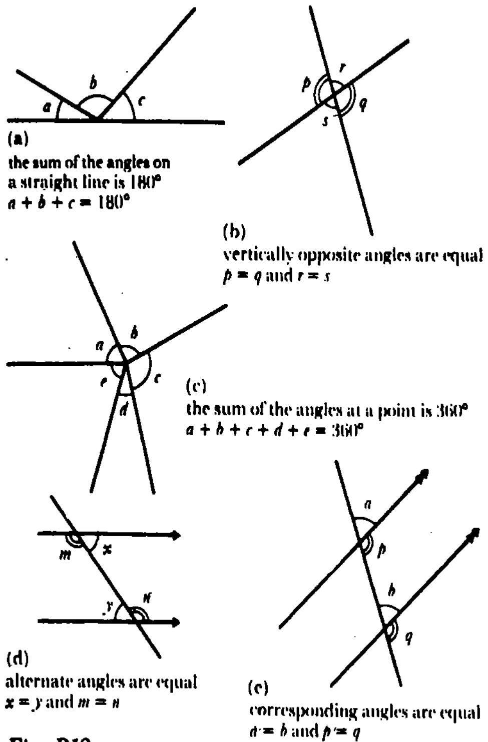


Fig. P12

In Fig. P13, α is the **angle of elevation** of the top of the flag-pole from the girl and β is the **angle of depression** of the girl from the boy.

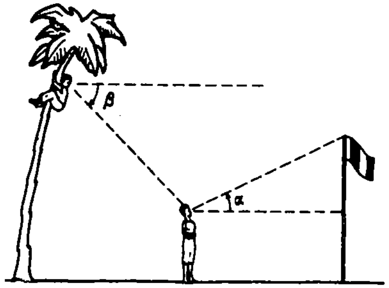


Fig. P13

Directions are taken from the **points of the compass** (Fig. P14).

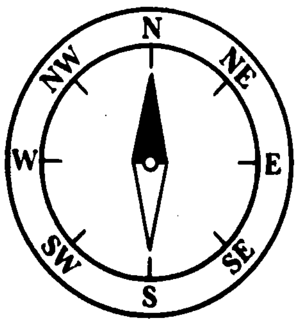


Fig. P14

A **3-figure bearing** is a direction given as the number of degrees from north measured in a clockwise direction. See Fig. P15.

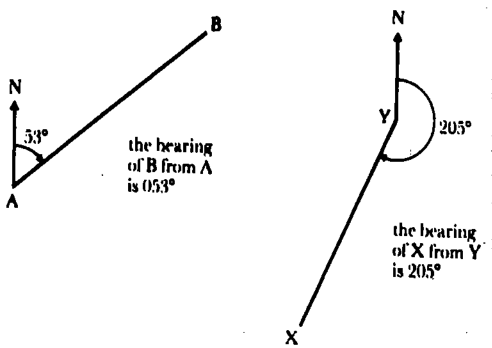


Fig. P15

(c) Fig. P16 shows the names and properties of some **common triangles**.

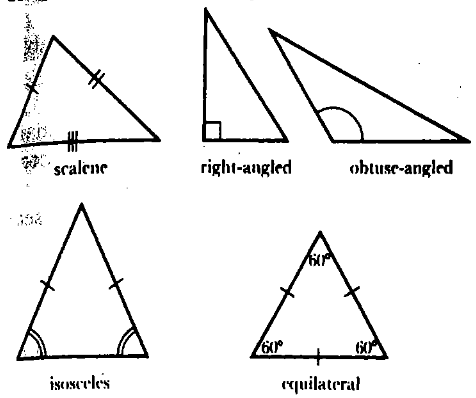


Fig. P16

Fig. P17 shows the names and properties of some **common quadrilaterals**.

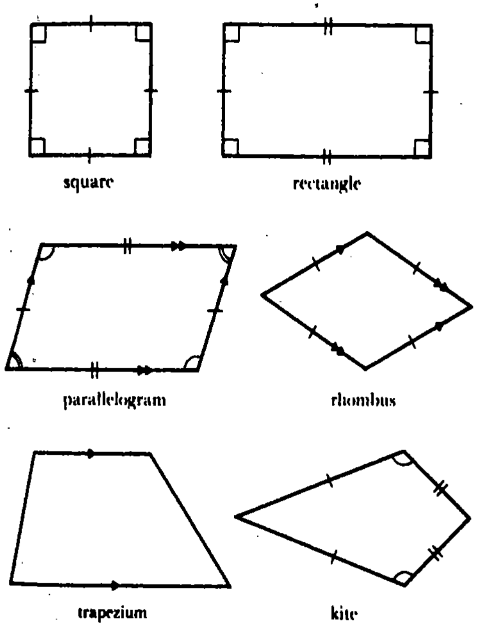


Fig. P17

Fig. P18 gives the names of lines and regions in a **circle**.

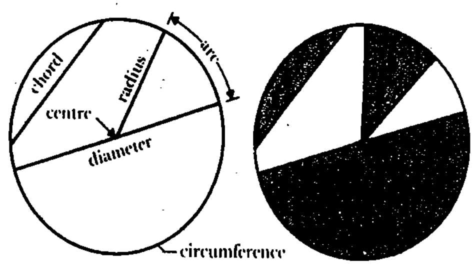


Fig. P18

The **sum of the angles of an** n**-sided polygon** is $(n - 2) \times 180°$. In particular, the sum of the angles of a triangle is 180° and the sum of the angles of a quadrilateral is 360°.

A **polygon** is a plane shape with three or more straight sides. A **regular polygon** has all its sides of equal length and all its angles of equal size. Fig. P19 gives the names of some common regular polygons.

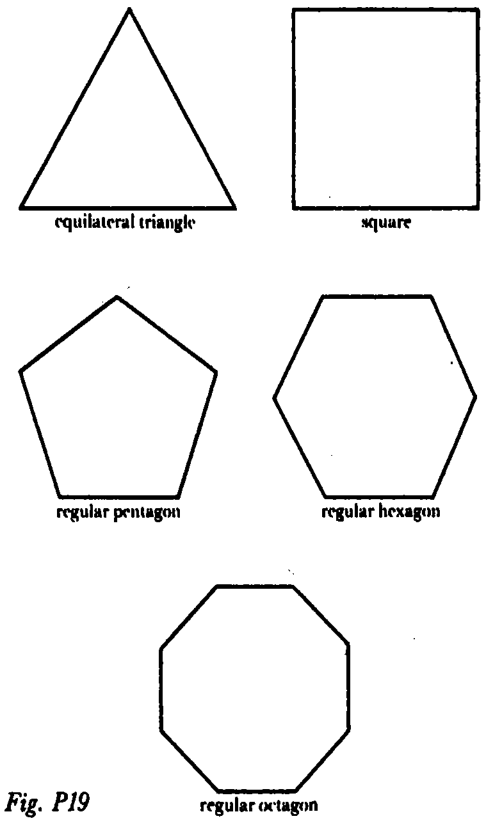


Fig. P19

(d) A shape is **transformed** when its position or dimensions (or both) change. The **image** of a shape is the figure that results after a transformation. If the image has the same dimensions as the original shape, the transformation is called a **congruency.** Fig. P20 shows the three basic congruencies, (a) **translation**, (b) **reflection**, (c) **rotation**.

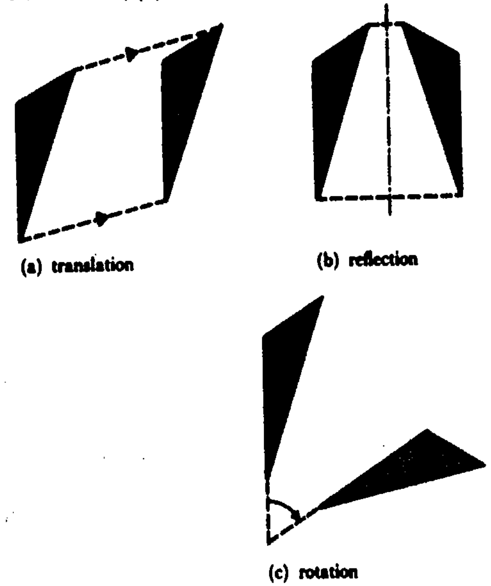


Fig. P20

An **enlargement** is a transformation in which the image and original shape are **equiangular** and have corresponding sides in the same ratio. Such shapes are geometrically **similar**. Any two triangles are similar if they are equiangular (Fig. P21).

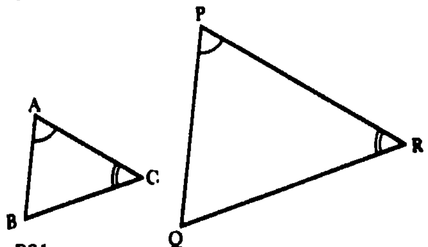


Fig. P21

In Fig. P21, $\frac{AB}{PQ} = \frac{BC}{QR} = \frac{AC}{PR}$.

(f) Formulae for the **perimeter and area of plane shapes** are given in the table on page 278.

The **SI system of units** is given in the tables on pages 276 and 277.

(g) The sketches in Fig. P22 show the main features of the common **geometrical constructions.**

To construct angles of 45° and 30°, bisect angles of 90° and 60° respectively.

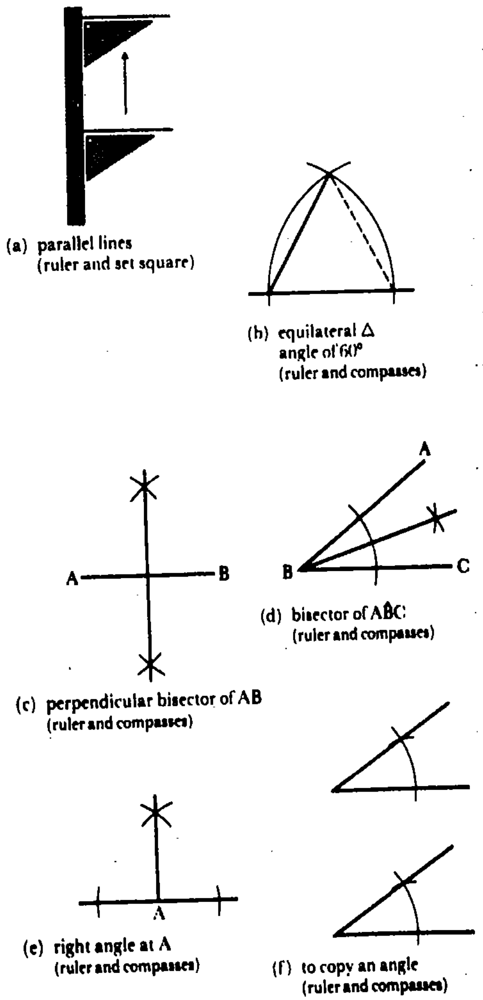


Fig. P22

Statistics

(a) Information in numerical form is called **statistics.** Statistical **data** may be given in **rank order** (i.e. in order of increasing size) such as in the following marks obtained in a test out of 5:

0; 1; 1; 2; 2; 2; 3; 3; 5

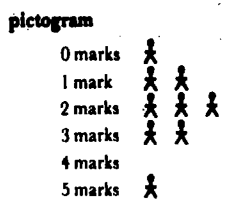
Data may also be given in a **frequency table** (Table P1).

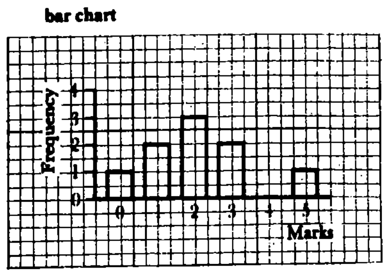
Table P1

mark	0	1	2	3	4	5
frequency	1	2	3	2	0	1

The **frequency** is the number of times each piece of data occurs.

Statistics can also be presented in graphical form. Fig. P23 shows the above data in a **pictogram**, a **bar chart** and a **pie chart.**





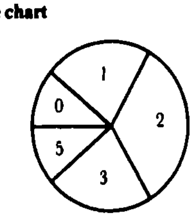


Fig. P23

(b) The **average** of a set of statistics is a number which is representative of the whole set. The three most common averages are the **mean,** the **median** and the **mode.** For the 9 numbers given in rank order in paragraph (a) above,

$$mean = \frac{0 + 1 + 1 + 2 + 2 + 2 + 3 + 3 + 5}{9}$$

$$= 2\tfrac{1}{9};$$

the *median* is the middle number when the data is arranged in order of size (2);
the *mode* is the number with the greatest frequency (also 2 in this case).

Chapter 1

General arithmetic (1)

Standard form

Decimal fractions in standard form

Decimal fractions such as 0,001 and 0,000 001 can be expressed as powers of 10. For example,

$$0{,}000\,001 = \frac{1}{1\,000\,000} = \frac{1}{10^6} = 10^{-6}$$

Any decimal fraction can be expressed in standard form. For example,

$$0{,}008 = \frac{8}{1\,000} = \frac{8}{10^3} = 8 \times 10^{-3}$$

$$0{,}000\,03 = \frac{3}{100\,000} = \frac{3}{10^5} = 3 \times 10^{-5}$$

$$0{,}000\,25 = \frac{2{,}5}{10\,000} = \frac{2{,}5}{10^4} = 2{,}5 \times 10^{-4}$$

The numbers 8×10^{-3}, 3×10^{-5} and $2{,}5 \times 10^{-4}$ are all in **standard form** $A \times 10^n$, where A is a number between 1 and 10 and n is a whole number. Notice that for decimal fractions, n is a negative whole number.

Example 1

Express the following fractions in standard form.
(*a*) 0,000 07 (*b*) 0,075
(*c*) 0,000 000 022 (*d*) 0,000 006 3

(a) $0{,}000\,07 = \frac{7}{100\,000} = \frac{7}{10^5} = 7 \times 10^{-5}$

(b) $0{,}075 = \frac{7{,}5}{100} = \frac{7{,}5}{10^2} = 7{,}5 \times 10^{-2}$

(c) $0{,}000\,000\,022 = \frac{2{,}2}{100\,000\,000} = \frac{2{,}2}{10^8}$
$= 2{,}2 \times 10^{-8}$

(d) $0{,}000\,006\,3 = \frac{6{,}3}{1\,000\,000} = \frac{6{,}3}{10^6}$
$= 6{,}3 \times 10^{-6}$

Exercise 1a (Oral)

Express the following in standard form.

1 0,005 **2** 0,08
3 0,000 6 **4** 0,000 004
5 0,000 02 **6** 0,000 000 9
7 0,3 **8** 0,003
9 0,000 03 **10** 0,038
11 0,006 2 **12** 0,71
13 0,000 88 **14** 0,000 45
15 0,000 026 **16** 0,000 072
17 0,000 005 5 **18** 0,011
19 0,000 000 91 **20** 0,000 000 067
21 0,005 7 **22** 0,000 15
23 0,001 5 **24** 0,015

Example 2

Express the following numbers as decimal fractions.
(*a*) 9×10^{-4} (*b*) $9{,}4 \times 10^{-4}$ (*c*) $5{,}3 \times 10^{-7}$

(a) $9 \times 10^{-4} = \frac{9}{10\,000} = 0{,}000\,9$

(b) $9{,}4 \times 10^{-4} = \frac{9{,}4}{10\,000} = 0{,}000\,94$

(c) $5{,}3 \times 10^{-7} = \frac{5{,}3}{10\,000\,000} = 0{,}000\,000\,53$

Exercise 1b

Express the following as decimal fractions.

1 2×10^{-4} **2** 8×10^{-6}
3 5×10^{-3} **4** 4×10^{-2}
5 7×10^{-1} **6** 3×10^{-5}
7 6×10^{-3} **8** 9×10^{-5}
9 2×10^{-7} **10** $2{,}8 \times 10^{-4}$
11 $8{,}3 \times 10^{-6}$ **12** $5{,}1 \times 10^{-3}$
13 $4{,}5 \times 10^{-2}$ **14** $7{,}9 \times 10^{-1}$
15 $3{,}3 \times 10^{-5}$ **16** $6{,}2 \times 10^{-3}$
17 $9{,}4 \times 10^{-5}$ **18** $2{,}6 \times 10^{-4}$
19 $1{,}8 \times 10^{-1}$ **20** $8{,}8 \times 10^{-3}$
21 $4{,}1 \times 10^{-2}$ **22** $2{,}4 \times 10^{-3}$
23 $2{,}4 \times 10^{-2}$ **24** $2{,}4 \times 10^{-1}$

Adding and subtracting numbers in standard form

Example 3

Find the sum of $6{,}28 \times 10^3$ *and* $9{,}5 \times 10^4$. *Give the sum in standard form.*

Either by changing to ordinary form:
$6{,}28 \times 10^3 + 9{,}5 \times 10^4 = 6\,280 + 95\,000$
$= 101\,280$
$= 1{,}012\,8 \times 100\,000$
$= 1{,}012\,8 \times 10^5$

or by factorising:
$6{,}28 \times 10^3 + 9{,}5 \times 10^4$
$= 10^3(6{,}28 + 9{,}5 \times 10)$
$= 10^3(6{,}28 + 95)$
$= 10^3(101{,}28)$
$= 10^3 \times 1{,}012\,8 \times 10^2$
$= 1{,}012\,8 \times 10^5$

Example 4

Find the value of $2{,}9 \times 10^6 - 3{,}8 \times 10^5$. *Give the answer in standard form.*

Either by changing to ordinary form:
$2{,}9 \times 10^6 - 3{,}8 \times 10^5$
$= 2\,900\,000 - 380\,000$
$= 2\,520\,000$
$= 2{,}52 \times 10^6$

or by factorising:
$2{,}9 \times 10^6 - 3{,}8 \times 10^5$
$= 10^5(2{,}9 \times 10 - 3{,}8)$
$= 10^5(29 - 3{,}8)$
$= 10^5 \times 25{,}2$
$= 10^5 \times 2{,}52 \times 10$
$= 2{,}52 \times 10^6$

Example 5

Express $1{,}6 \times 10^{-2} - 8{,}4 \times 10^{-3}$ *as a single number in standard form.*

$1{,}6 \times 10^{-2} - 8{,}4 \times 10^{-3}$
$= 0{,}016 - 0{,}008\,4$
$= 0{,}007\,6$
$= 7{,}6 \times 10^{-3}$

or by factorising:
$1{,}6 \times 10^{-2} - 8{,}4 \times 10^{-3}$
$= 10^{-2}(1{,}6 - 8{,}4 \times 10^{-1})$
$= 10^{-2}(1{,}6 - 0{,}84)$
$= 10^{-2} \times 0{,}76$
$= 10^{-2} \times 7{,}6 \times 10^{-1}$
$= 7{,}6 \times 10^{-3}$

Numbers in standard form can be added or subtracted by taking out the power of 10 which is a common factor. If necessary, the working can be checked by changing the given numbers to ordinary form.

Exercise 1c

Simplify the following. Give all answers in standard form.

1 $3{,}4 \times 10^3 + 6{,}2 \times 10^3$
2 $5{,}7 \times 10^8 + 1{,}8 \times 10^8$
3 $4{,}62 \times 10^9 + 3{,}75 \times 10^9$
4 $8{,}7 \times 10^4 - 3{,}5 \times 10^4$
5 $4{,}3 \times 10^2 - 2{,}8 \times 10^2$
6 $9{,}37 \times 10^4 - 6{,}51 \times 10^4$
7 $9{,}9 \times 10^5 + 6{,}8 \times 10^5$
8 $4{,}1 \times 10^6 + 5{,}9 \times 10^6$
9 $7{,}95 \times 10^3 + 3{,}06 \times 10^3$
10 $5{,}8 \times 10^4 - 5{,}2 \times 10^4$
11 $1{,}75 \times 10^9 - 1{,}25 \times 10^9$
12 $8{,}49 \times 10^6 - 8{,}44 \times 10^6$
13 $3{,}6 \times 10^{-2} + 4 \times 10^{-2}$
14 $2{,}9 \times 10^{-4} + 3{,}5 \times 10^{-4}$
15 $7{,}8 \times 10^{-3} - 3{,}4 \times 10^{-3}$
16 $8{,}65 \times 10^{-5} - 5{,}76 \times 10^{-5}$
17 $1{,}7 \times 10^4 + 6{,}5 \times 10^3$
18 $9{,}17 \times 10^5 + 7{,}45 \times 10^6$
19 $6{,}9 \times 10^{-2} + 5 \times 10^{-3}$
20 $8{,}31 \times 10^3 - 9{,}73 \times 10^2$
21 $6{,}4 \times 10^5 - 1{,}5 \times 10^4$
22 $5{,}9 \times 10^{-4} - 4{,}1 \times 10^{-5}$
23 $3{,}18 \times 10^{-2} + 9{,}73 \times 10^{-1}$
24 $1{,}1 \times 10^{-3} - 8{,}7 \times 10^{-4}$

Multiplying and dividing numbers in standard form

Use the laws of indices when simplifying powers of 10 that are multiplied or divided:
$10^a \times 10^b = 10^{a+b}$
$10^a \div 10^b = 10^{a-b}$

Example 6

Simplify $(6 \times 10^9) \times (8 \times 10^2)$.

$(6 \times 10^9) \times (8 \times 10^2) = 6 \times 8 \times 10^9 \times 10^2$
$= 48 \times 10^{9+2}$
$= 48 \times 10^{11}$
$= 4{,}8 \times 10 \times 10^{11}$
$= 4{,}8 \times 10^{12}$

Example 7

Divide 6×10^3 *by* 8×10^{-2}.

$$(6 \times 10^3) \div (8 \times 10^{-2}) = \frac{6 \times 10^3}{8 \times 10^{-2}}$$
$$= \frac{6}{8} \times 10^{3-(-2)}$$
$$= 0,75 \times 10^5$$
$$= 7,5 \times 10^{-1} \times 10^5$$
$$= 7,5 \times 10^4$$

Example 8

Simplify $(1,4 \times 10^{-5}) \times (2,4 \times 10^6)$.

$$(1,4 \times 10^{-5}) \times (2,4 \times 10^6)$$
$$= 1,4 \times 2,4 \times 10^{-5} \times 10^6$$
$$= 3,36 \times 10^{-5+6}$$
$$= 3,36 \times 10^1$$
$$= 33,6$$

working:

$$\begin{array}{r} 14 \\ \times\ 24 \\ \hline 56 \\ 28\ \\ \hline 336 \end{array}$$

Exercise 1d

Simplify the following. Give all answers in standard form.

1 $(3 \times 10^8) \times (2 \times 10^3)$
2 $(2,8 \times 10^6) \div (1,4 \times 10^2)$
3 $(2 \times 10^{-5}) \times (4 \times 10^{-2})$
4 $(6,3 \times 10^{-2}) \div (2,1 \times 10^4)$
5 $(5 \times 10^2) \times (8 \times 10^5)$
6 $(4,8 \times 10^7) \div (8 \times 10^3)$
7 $(7 \times 10^6) \times (4 \times 10^{-4})$
8 $(3,6 \times 10^2) \div (9 \times 10^{-5})$
9 $(9 \times 10^{-7}) \times (5 \times 10^4)$
10 $(4,2 \times 10^{-9}) \div (7 \times 10^5)$
11 $(6 \times 10^{-3}) \times (6 \times 10^{-3})$
12 $(5,4 \times 10^{-3}) \div (2,7 \times 10^{-7})$
13 $(8,7 \times 10^2) \times (5 \times 10^2)$
14 $(8 \times 10^3) \times (1,5 \times 10^{-3})$
15 $(1,6 \times 10^8) \div (6,4 \times 10^7)$
16 $(1,3 \times 10^{-5}) \times (1,9 \times 10^4)$
17 $(9,1 \times 10^{-2}) \div (1,3 \times 10^{-2})$
18 $(5,5 \times 10^{-6}) \times (4,2 \times 10^{-4})$
19 $(1,92 \times 10^{-6}) \div (1,6 \times 10^{-3})$
20 $(1,05 \times 10^{-7}) \div (1,68 \times 10^{-9})$
21 $(6,2 \times 10^{-5}) \times (8,1 \times 10^6)$
22 $(1,404 \times 10^3) \div (2,6 \times 10^{-2})$
23 $(1,12 \times 10^{-1}) \times (2,43 \times 10^5)$
24 $(8,51 \times 10^{-3}) \div (3,7 \times 10^{-1})$

Problems with large and small numbers

In science and astronomy, many measurements are given in very small or very large numbers. For this reason, most scientists prefer to do calculations in standard form. Scientists use standard form so often, it is sometimes called **scientific notation**.

Example 9

A light year is a distance of $9,456 \times 10^{12}$ km. *Express this number to 2 significant figures, then write it out in full.*

$9,456 \times 10^{12}$ km $= 9,5 \times 10^{12}$ km to 2 s.f.
$9,5 \times 10^{12}$ km $= 9\,500\,000\,000\,000$ km

Example 10

The density of hydrogen is $8,89 \times 10^{-5}$ g/cm³. *(a) Find the mass of* 1 m³ *of hydrogen. (b) Argon is approximately* 20 *times as dense as hydrogen. Find the density of argon, giving the answer in standard form correct to 3 s.f.*

(a) $1 \text{ m}^3 = 10^6 \text{ cm}^3$

mass of 1 m³ of hydrogen
$$= 8,89 \times 10^{-5} \times 10^6 \text{ g}$$
$$= 8,89 \times 10^1 \text{ g}$$
$$= 88,9 \text{ g}$$

(b) density of argon
$$= 20 \times 8,89 \times 10^{-5} \text{ g/cm}^3$$
$$= 2 \times 10^1 \times 8,89 \times 10^{-5} \text{ g/cm}^3$$
$$= 2 \times 8,89 \times 10^{-4} \text{ g/cm}^3$$
$$= 17,78 \times 10^{-4} \text{ g/cm}^3$$
$$= 1,778 \times 10^{-3} \text{ g/cm}^3$$
$$= 1,78 \times 10^{-3} \text{ g/cm}^3 \text{ to 3 s.f.}$$

Example 11

The diameters of the earth and moon are $1,28 \times 10^4$ km *and* $3,5 \times 10^3$ km *respectively. Find the ratio, diameter of earth: diameter of moon in the form* $n : 1$ *where n is correct to 2 s.f.*

$$\frac{\text{diameter of earth}}{\text{diameter of moon}} = \frac{1,28 \times 10^4}{3,5 \times 10^3}$$
$$= \frac{1,28}{3,5} \times 10$$
$$= \frac{128}{35}$$
$$= \frac{3,65}{1}$$

$$\begin{array}{r} 3,65\ldots \\ 35\overline{)128,00} \\ 105 \\ \hline 23\,0 \\ 21\,0 \\ \hline 2\,00 \\ 1\,75 \\ \hline 25 \end{array}$$

ratio = 3,7:1

Example 12

The pages of a book are numbered from 1 *to* 400. *The thickness of the book is* 24 mm. *Calculate the thickness of* 1 *leaf (i.e.* 2 *pages) of the book. Give the answer in metres in standard form.*

A book with 400 pages contain 200 leaves of paper.

Thickness of 200 leaves = 24 mm

Thickness of 1 leaf = $\frac{24}{200}$ mm

$$\text{Thickness of 1 leaf in metres} = \frac{24}{200 \times 1\,000} \text{ m}$$
$$= \frac{12}{100\,000} \text{ m}$$
$$= \frac{1,2}{10\,000} \text{ m}$$
$$= 1,2 \times 10^{-4} \text{ m}$$

Exercise 1e

Give all the answers in standard form unless told otherwise.

1 An atom of caesium 133 vibrates 9 192 631 770 times per second. Give this number in standard form correct to 2 s.f.

2 The area of Zimbabwe is 390 750 km². Express this area in standard form correct to 3 s.f.

3 1 hectare (ha) $= 10^4$ m².
(a) Find the number of ha in 1 km².
(b) Use the data of question 2 to find the area of Zimbabwe in hectares in standard form correct to 3 s.f.

4 The distance between two points is $2,54 \times 10^{-2}$ m. Express this distance in km in standard form.

5 A room measures 4 m by 3 m by $2\frac{1}{2}$ m. Calculate its volume in cm³ in standard form.

6 The density of air is $1,3 \times 10^{-3}$ g/cm³. Calculate the mass of air in the room in question 5. Give your answer in kg in ordinary form.

7 Express 1 hour in seconds in standard form.

8 The velocity of light is approximately 3×10^5 km/s. Use your answer to question 7 to find the distance travelled by light in 1 hour.

9 The height of Mount Everest is $8,85 \times 10^3$ m. The height of Mount Kilimanjaro is $5,89 \times 10^3$ m. Write these heights in ordinary form and find the difference in height between the two mountains.

10 In 1920 the population of the world was 1,81 billion. By 1990 it was 5,32 billion. Find the increase in world population during those 70 years.

11 The distance of the moon from the earth varies between $3,843 \times 10^5$ km and $3,563 \times 10^5$ km. Find the difference between these two distances.

12 Mount Everest is the highest point on the earth's surface: $8,848 \times 10^3$ m above sea level. The lowest point on the earth's surface is the Mariana Trench: $1,103 \times 10^4$ m *below* sea level. Find the vertical distance between the lowest and highest points on the earth's surface.

13 The wavelength of sodium light is 5 893 Å, where $1 \text{ Å} = 10^{-10}$ m. Give the wavelength of sodium light in metres in standard form.

14 The diameters of the sun and earth are approximately $1,4 \times 10^6$ km and $1,3 \times 10^4$ km respectively. Express both numbers correct to 1 s.f. and hence find the approximate ratio, diameter of the sun : diameter of the earth.

15 Atmospheric pressure at the earth's surface is approximately $1,013 \times 10^5$ newtons/m². At a height of 6 km, the atmospheric pressure is about $\frac{1}{2}$ of that at the surface of the earth. Find the atmospheric pressure at this height, giving your answer correct to 2 s.f.

16 The masses of the earth and the sun are approximately 6×10^{24} kg and 2×10^{30} kg respectively. Express the ratio, mass of earth : mass of sun in the form $x : 1$, where x is a number in standard form.

17 1 barrel of oil has a capacity of $1,65 \times 10^{-1}$ m³. Find the total volume, in m³, of $6,7 \times 10^7$ barrels of oil. Give the volume in standard form correct to 2 s.f.

18 A packet of paper contains 500 sheets. The thickness of the packet is 56 mm. Calculate the thickness of 1 sheet of paper. Give your answer in metres in standard form.

19 The pages of a book are numbered 1 to 300.
   (a) How many leaves of paper make 300 pages?
   (b) If the thickness of the book is 15 mm, calculate the thickness of one leaf. Give your answer in metres in standard form.

20 The pages of a dictionary are numbered from 1 to 1 322. The dictionary is 7 cm thick (neglecting the covers).
   (a) How many leaves of paper make 1 322 numbered pages?
   (b) Find the thickness of 1 sheet of paper. Give your answer in metres in standard form correct to 1 s.f.

Reciprocals of numbers

1 The **reciprocal** of 4 is $\frac{1}{4} = 0{,}25$. Similarly, the reciprocal of 0,25 is $\frac{1}{0{,}25} = \frac{1}{\frac{1}{4}} = 4$.
Thus, if the reciprocal of x is y, then the reciprocal of y is x.

2 If a number is multiplied by a power of 10, its reciprocal is divided by that power of 10. Table 1.1 shows the products of 4 and some powers of 10 and their corresponding reciprocals.

Table 1.1

number	reciprocal
0,04	25
0,4	2,5
4	0,25
40	0,025
400	0,002 5

Example 13

Find the reciprocals of the following.
(*a*) 6 (*b*) 25 (*c*) $3\frac{1}{3}$ (*d*) 0,000 2

(a) Reciprocal of 6 is $\frac{1}{6} = 0{,}16$
$= 0{,}167$ to 3 s.f.

(b) Reciprocal of 25 is $\frac{1}{25} = \frac{4}{100} = 0{,}04$.

(c) Reciprocal of $3\frac{1}{3}$ is $\frac{1}{3\frac{1}{3}} = \frac{1}{\frac{10}{3}} = \frac{3}{10} = 0{,}3$.

(d) Reciprocal of 0,002 is $\frac{1}{0{,}002} = \frac{1\,000}{2}$
$= 500$.

Reciprocals can also be calculated by long division.

Example 14

Calculate the reciprocal of 0,58.

Reciprocal of 0,58 is $\frac{1}{0{,}58} = \frac{100}{58}$

```
             1,724 ...
working: 58)100,000 ...
             58
             42 0
             40 6
              1 40
              1 16
                240
                232
                ...
```

Reciprocal of 0,58 is 1,72 to 3 s.f.

Exercise 1f

1 Copy and complete Table 1.2. Give each reciprocal as a decimal number correct to 3 s.f.

Table 1.2

n	$\frac{1}{n}$	n	$\frac{1}{n}$	n	$\frac{1}{n}$
1		10		0,1	
2		20		0,2	
3		30		0,3	
4		40		0,4	
5		50		0,5	
6		60		0,6	
7		70		0,7	
8		80		0,8	
9		90		0,9	

Table 1.3

x	0	1	2	3	4	5	6	7	8	9	SUBTRACT Differences 1	2	3	4	5	6	7	8	9
1,7	0,5882	5848	5814	5780	5747	5714	5682	5650	5618	5587	3	7	10	13	16	20	23	26	29
4,0	0,2500	2494	2488	2481	2475	2469	2463	2457	2451	2445	1	1	2	2	3	4	4	5	5

2 Find the reciprocals of the following. Give each answer as a decimal, rounding off to 3 s.f. where necessary.
(a) 2 (b) 0,2 (c) 0,02
(d) 8 (e) 80 (f) 800
(g) $\frac{2}{3}$ (h) $1\frac{2}{3}$ (i) $6\frac{2}{3}$
(j) $\frac{1}{4}$ (k) $1\frac{1}{4}$ (l) $6\frac{1}{4}$
(m) $1\frac{1}{2}$ (n) $\frac{3}{4}$ (o) $4\frac{1}{2}$
(p) 0,05 (q) 0,625 (r) 15

3 Use long division to find the reciprocals of the following correct to 3 s.f.
(a) 0,32 (b) 0,47 (c) 8,9
(d) 65 (e) 0,018 (f) 290

Use of reciprocal tables

Four-figure reciprocal tables are given on page 290. Use these to find reciprocals quickly and easily, Table 1.3 shows two lines from the reciprocal table.

Notice the following:
1 The table gives the reciprocals of numbers from 1 to 9,999 in intervals of 0,001.
2 Each reciprocal is given correct to 4 s.f.
3 As the numbers increase, their reciprocals decrease.
4 The differences in the right-hand columns must be *subtracted.*
5 The table can be used to find the reciprocal of *any* 4-digit number. However, the decimal comma must be placed by inspection. (See Example 15 below.)

Example 15

Use Table 1.3 to find the reciprocal of (*a*) 1,76, (*b*) 176, (*c*) 0,403, (*d*) 0,4037

(a) The reciprocal of 1,76 is found across from **1,7** and under **6**.
$\frac{1}{1{,}76} = 0{,}5682$ (to 4 s.f.)

(b) $176 = 1{,}76 \times 100$.
$\frac{1}{176} = \frac{1}{1{,}76} \div 100 = 0{,}5682 \div 100$
$= 0{,}005\,682$ (to 4 s.f.)

(c) $0{,}403 = 4{,}03 \div 10$.
$\frac{1}{0{,}403} = \frac{1}{4{,}03} \times 10 = 0{,}2481 \times 10$
$= 2{,}481$ (to 4 s.f.)

(d) $40{,}37 = 4{,}037 \times 10$
$\frac{1}{40{,}37} = \frac{1}{4{,}037} \div 10 = 0{,}247\,7^* \div 10$
$= 0{,}024\,77$

*Note how the differences were *subtracted:* $2481 - 4 = 2477$

Exercise 1g (Oral)

Use the tables on page 290 to find the reciprocals of the following.

1 1,7	**2** 4,0	**3** 5,2
4 9,0	**5** 6,9	**6** 3,8
7 1,75	**8** 4,06	**9** 8,13
10 7,52	**11** 2,49	**12** 6,06
13 15	**14** 22	**15** 93
16 0,35	**17** 0,58	**18** 0,71
19 2,89	**20** 289	**21** 0,002 89
22 48,3	**23** 4,830	**24** 0,048 3
25 7,75	**26** 0,129	**27** 3,16
28 0,897	**29** 1,11	**30** 0,303
31 1,738	**32** 4,055	**33** 13,62
34 5 669	**35** 0,630 4	**36** 0,056 27

Calculations using reciprocals

Example 16

Find the value of $\frac{4}{7{,}41}$.

Treat $\frac{4}{7{,}41}$ as $4 \times \frac{1}{7{,}41}$

$\frac{4}{7{,}41} = 4 \times 0{,}1350$ (from tables)
$= 0{,}5400 = 0{,}540$ (to 3 s.f.)

It is possible to find the value of $\frac{4}{7,41}$ by long division. However, this would mean a long and tiring calculation. By using reciprocal tables, the work can be done mentally. Notice that the final answer is correct to only 3 s.f. This is accurate enough for most purposes.

Example 17

Find the value of f if $\frac{1}{f} = \frac{1}{9,5} + \frac{1}{4,4}$.

$\frac{1}{f} = \frac{1}{9,5} + \frac{1}{4,4}$

Use tables to find the reciprocals of the RHS:

$\frac{1}{f} = 0,1053 + 0,2273$

$\frac{1}{f} = 0,3326$

Take the reciprocal of both sides,

$f = 3,007$

$= 3,0$ to 2 s.f.

The answer in Example 17 is given to the same degree of accuracy as the number given in the question.

Formulae such as that given in Example 17 are often found in science. Use of reciprocal tables can simplify calculations.

Exercise 1h

Use reciprocal tables to simplify any calculation. Give all answers correct to 2 s.f.

1 Use tables to find the value of the following.

(a) $\frac{5}{6,6}$ (b) $\frac{2}{0,34}$ (c) $\frac{4}{170}$

(d) $\frac{2}{2,38}$ (e) $\frac{3}{0,477}$ (f) $\frac{6}{\pi}$

2 A car travels 80 km in 1,17 hours. Calculate its average speed.

3 A trader sells 13 oranges for $1. Find the average cost of an orange to the nearest tenth of a cent.

4 A car uses 12,8 litres of petrol to travel 100 km. Find the number of km the car travels on 1 litre.

5 A circle has a circumference of 40 cm. Use the value 3,14 for π to calculate its radius.

6 A rectangle has an area of 50 cm². Calculate its length if its breadth is 8,33 cm.

7 A machine makes 5 000 pencils in 42 hours. Calculate its production rate in pencils/hour.

8 A boarding school uses an average of 30 000 litres of water per week. Find the average number of litres/hour used.

9 Find the value of f if $\frac{1}{f} = \frac{1}{19} + \frac{1}{26}$.

10 Find the value of R if $\frac{1}{R} = \frac{1}{2,7} + \frac{1}{6,4}$.

Number bases

For most purposes, numbers are written in the **base ten** or **denary** system. The placing of the digits shows their values. For example 2 053 means **2** thousands, **0** hundreds, **5** tens, **3** units:

$2\,053 = 2 \times 1\,000 + 0 \times 100 + 5 \times 10 + 3 \times 1$
$= 2 \times 10^3 + 0 \times 10^2 + 5 \times 10^1 + 3 \times 1$

Remember that 10^3 is a short way of writing $10 \times 10 \times 10$ or 1 000. The digit 3 in 10^3 is called the **index** or **power** (Fig. 1.1).

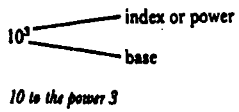


Fig. 1.1

It is possible to expand any denary number in powers of ten:

$25 = 2 \times 10 + 5 \times 1$
$= 2 \times 10^1 + 5 \times 1$
$147 = 1 \times 100 + 4 \times 10 + 7 \times 1$
$= 1 \times 10^2 + 4 \times 10^1 + 7 \times 1$
$3\,706 = 3 \times 1000 + 7 \times 100 + 0 \times 10 + 6 \times 1$
$= 3 \times 10^3 + 7 \times 10^2 + 0 \times 10^1 + 6 \times 1$

Other number systems are sometimes used. For example, the **base five** system is based on powers of five. In base five, **23** means **2** fives and **3** units.

$23_{five} = 2 \times 5 + 3 \times 1$
$142_{five} = 1$ twenty-five, 4 fives, 2 units
$= 1 \times 25 + 4 \times 5 + 2 \times 1$
$= 1 \times 5^2 + 4 \times 5^1 + 2 \times 1$
$3\,204_{five} = 3 \times 5^3 + 2 \times 5^2 + 0 \times 5^1 + 4 \times 1$

Fig. 1.2 shows the place values of the digits in the number 142_{five}.

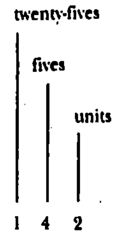


Fig. 1.2

Notice that 142_{five} is short for 142 in base five. Also, just as there are ten digits in the base ten system, there are just five digits (0, 1, 2, 3, 4) in the base five system.

Example 18

Expand (a) $25\,024_{eight}$ (b) $1\,001_{two}$ in powers of their bases.

(a) $25\,024_{eight}$
$= 2 \times 8^4 + 5 \times 8^3 + 0 \times 8^2 + 2 \times 8^1 + 4 \times 1$

(b) $1\,001_{two}$
$= 1 \times 2^3 + 0 \times 2^2 + 0 \times 2^1 + 1 \times 1$

Exercise 1i (Oral)

Expand the following in the powers of their bases.

1 $2\,389_{ten}$	2 647_{eight}
3 $35\,154_{eight}$	4 $4\,102_{five}$
5 $1\,011_{two}$	6 $22\,010_{three}$
7 $26\,523_{eight}$	8 $1\,100_{two}$
9 $2\,102_{three}$	10 $71\,062_{eight}$
11 $30\,312_{five}$	12 $107\,824_{ten}$

In theory there is no limit to the choice of number bases. (There can be as many number bases as there are numbers!) However, for the purposes of this course, examples will be chosen only from bases ten, five and two. There are some reasons for this:

1 Counting in base ten is common throughout the world.

2 Base five counting systems are fairly common in many countries (e.g. in Chapter 1 of Book 1 it was noted that Tonga-speaking people use a base five system).

3 The base two, or **binary**, system has only two digits: 0 and 1. These may be taken to represent the off (0) and on (1) states of an electrical circuit. As a result, binary numbers have become part of the language of computers.

Converting base ten numbers to other bases

To convert from base ten to another base, express the given number in powers of the new base.

Example 19

Convert 37_{ten} to base five.

Since 37 is greater than 25, there must be a 25 in 37.

$37 \div 25 = 1$, remainder 12
$37 = 1$ twenty-five $+ 12$ units

Consider the 12 units. Since $12 > 5$, there must be some 5's in 12.

$12 \div 5 = 2$, remainder 2
$12 = 2$ fives $+ 2$ units

Hence $37 = 1$ twenty-five $+ 2$ fives $+ 2$ units
$= 1 \times 5^2 + 2 \times 5^1 + 2 \times 1$
$37_{ten} = 122_{five}$

Check: $122_{five} = 1 \times 25 + 2 \times 5 + 2 \times 1$
$= 25 + 10 + 2 = 37_{ten}$

The method in Example 19 can be shortened as follows:

```
5 | 37   rem
5 |  7 + 2 (i.e. 7 × 5  + 2 × 1)
5 |  1 + 2 (i.e. 1 × 5² + 2 × 5¹)
     0 + 1 (i.e. 0 × 5³ + 1 × 5²)
         ↑                   ↑
```

Repeated division by 5 gives remainders. Reading the remainders *upwards* gives $37_{ten} = 122_{five}$ (see the arrows above).

To change from base ten to another base:

1 Divide the base ten number by the new base number.

2 Continue dividing until 0 (zero) is reached, writing down the remainder each time.

3 Start at the last remainder and read upwards to get the answer.

Example 20

Convert 75_{ten} *(a) to base five, (b) to base two.*

(a)

5	75	*rem*
5	15 + 0	↑
5	3 + 0	
	0 + 3	

$75_{ten} = 300_{five}$

(b)

2	75	*rem*
2	37 + 1	↑
2	18 + 1	
2	9 + 0	
2	4 + 1	
2	2 + 0	
2	1 + 0	
	0 + 1	

$75_{ten} = 1\ 001\ 011_{two}$

Example 20 shows that if a remainder is 0 it must be written down.

Exercise 1j

Each of the given numbers is in base ten. Convert the numbers to the bases shown.

1 15 to base *five* **2** 20 to base *five*
3 11 to base *two* **4** 12 to base *two*
5 64 to base *five* **6** 27 to base *two*
7 76 to base *five* **8** 18 to base *five*
9 35 to base *two* **10** 31 to base *two*
11 49 to base *two* **12** 49 to base *five*
13 99 to base *five* **14** 56 to base *five*
15 88 to base *two* **16** 98 to base *two*
17 128 to base *five* **18** 115 to base *five*
19 725 to base *two* **20** 129 to base *two*
21 100 to base *five* **22** 733 to base *five*
23 256 to base *two* **24** 543 to base *two*

Converting from other bases to base ten

Example 21

Convert $1\ 234_{five}$ *to base ten.*

1st method: By expanding the given number:

$$\mathbf{1\ 234}_{five} = \mathbf{1} \times 5^3 + \mathbf{2} \times 5^2 + \mathbf{3} \times 5^1 + \mathbf{4} \times 1$$
$$= 1 \times 125 + 2 \times 25 + 3 \times 5 + 4$$
$$= 125 + 50 + 15 + 4$$
$$= 194_{ten}$$

2nd method: By repeated multiplication:

1 2 3 4

$1 \times 5 = 5$; $5 + 2 = 7$
$7 \times 5 = 35$; $35 + 3 = 38$
$38 \times 5 = 190$; $190 + 4 = 194$

$1\ 234_{five} = 194_{ten}$

Notice that the method of repeated multiplication is the reverse of the repeated division method for conversion from base ten.

Example 22

Convert the following to base ten. (a) $11\ 101_{two}$, *(b)* 432_{five}

(a) By expanding in powers:

$$11\ 101_{two}$$
$$= \mathbf{1} \times 2^4 + \mathbf{1} \times 2^3 + \mathbf{1} \times 2^2 + \mathbf{0} \times 2^1 + \mathbf{1} \times 1$$
$$= 1 \times 16 + 1 \times 8 + 1 \times 4 + 0 \times 2 + 1 \times 1$$
$$= 16 + 8 + 4 + 0 + 1 = 29_{ten}$$

(b) By repeated multiplication:

4 3 2

$4 \times 5 = 20$; $20 + 3 = 23$
$23 \times 5 = 115$; $115 + 2 = 117_{ten}$

When converting to base ten, either expand the given number in powers of its base and evaluate, or use repeated multiplication. The first of these methods is recommended.

Exercise 1k

Convert the following to base ten.

1 10_{five}	**2** 11_{five}	**3** 20_{five}
4 22_{five}	**5** 31_{five}	**6** 43_{five}
7 204_{five}	**8** 102_{five}	**9** 324_{five}
10 $1\ 212_{five}$	**11** $2\ 403_{five}$	**12** $4\ 104_{five}$
13 111_{two}	**14** $1\ 101_{two}$	**15** $1\ 001_{two}$
16 $1\ 010_{two}$	**17** $11\ 110_{two}$	**18** $10\ 100_{two}$
19 $110\ 100_{two}$	**20** $10\ 011_{two}$	**21** $10\ 101_{two}$
22 $101\ 011_{two}$	**23** $111\ 000_{two}$	**24** $100\ 101_{two}$

Further examples of arithmetic in various number bases are given in Book 4, Chapter 21.

Chapter 2

Solving triangles (1) Pythagoras' theorem

To **solve** a triangle means to find the sizes of its sides and angles by calculation.

In any triangle, the sum of the angles is 180°. Thus if two angles are known, it is easy to calculate the third angle.

In any right-angled triangle, if two of the sides are known, it is possible to calculate the length of the third side. Work carefully through Exercise 2a.

Exercise 2a

1 The longest side of a right-angled triangle is called the **hypotenuse**. The hypotenuse is opposite the right angle.

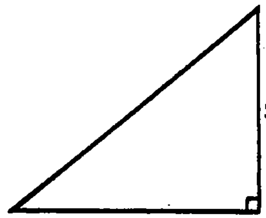


Fig. 2.1

Measure the length of the hypotenuse of the triangle in Fig. 2.1.

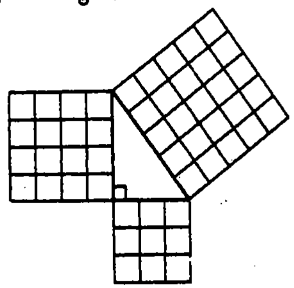


Fig. 2.2

2 Fig. 2.2 shows a right-angled triangle with sides of 3, 4 and 5 units. A square has been drawn on each side of the triangle. Each square is divided into small squares of area 1 unit².

(a) Count the number of unit² in the square on the hypotenuse.
(b) Count the number of unit² in the squares on the other two sides. Add these together.
(c) What do you notice?

3 Repeat question 2 with the triangle in Fig. 2.3.

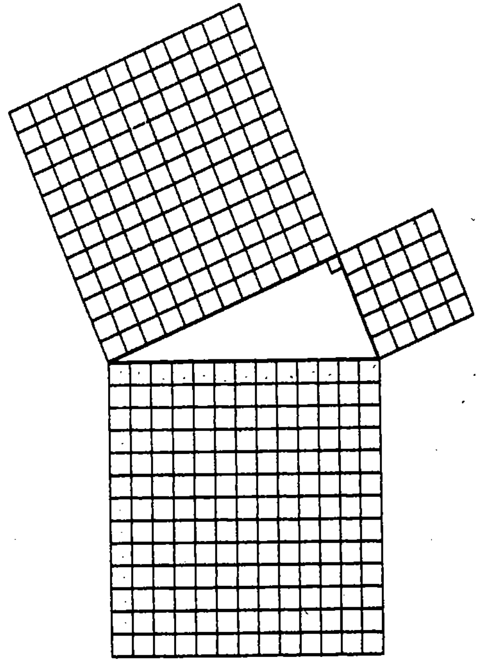


Fig. 2.3

4 (a) On a large sheet of paper, draw a right-angled triangle such that the sides containing the right angle are 8 cm and 6 cm.
(b) Measure the hypotenuse.

(c) Draw squares on the three sides of the triangle as in Figs 2.2 and 2.3.
(d) Divide the squares into 1 cm^2 small squares. Count the small squares as in question 2.

5 Repeat question 4 for a right-angled triangle such that the sides containing the right angle are 8 cm and 15 cm.

Pythagoras' theorem

From the work of Exercise 2a, it can be seen that the square on the hypotenuse is equal in area to the sum of the squares on the other two sides. In Fig. 2.4, $AB^2 = BC^2 + AC^2$.

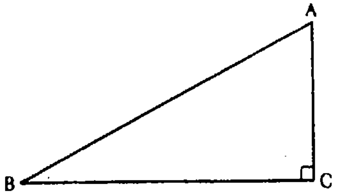


Fig. 2.4

This rule is very famous. It is called **Pythagoras' theorem.** Pythagoras was a Greek philosopher who lived about 2 500 years ago. The theorem was proved by Pythagoras and his friends at that time. However, the rule was used in northern Africa long before then and has been proved independently in many other parts of the world.

There are many ways of proving Pythagoras' theorem. The proof which follows is sometimes known as the Chinese proof.

In Fig. 2.5, PQRS is a square of side $a + b$ units. W is a point on PQ such that PW = a units and WQ = b units. Similarly for X, Y and Z. Lines joining these points give a square of side c and 4 right-angled triangles (shaded) within the large square.

The area of square PQRS can be found in two ways:

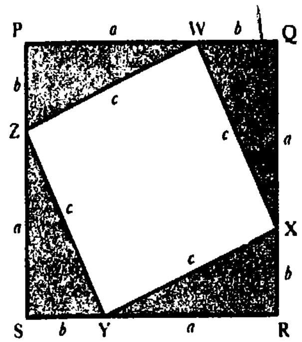


Fig. 2.5

1 area of PQRS = length × breadth
$= (a + b)(a + b)$
expanding brackets, this gives
area of PQRS $= a^2 + 2ab + b^2$*

2 area of PQRS = area of square WXYZ + area of 4 triangles
$= c^2 + 4 \times \frac{1}{2}ab$
$= c^2 + 2ab$
thus, $c^2 + 2ab = a^2 + 2ab + b^2$
subtract $2ab$ from both sides
$c^2 = a^2 + b^2$

* *Note:* $(a + b)(a + b) = a(a + b) + b(a + b)$
$= a^2 + ab + ab + b^2$
$= a^2 + 2ab + b^2$
Expansion of brackets was covered in Book 1 of the course.

Look at triangle PWZ in Fig. 2.5. c is the hypotenuse of $\triangle$PWZ and a and b are its other two sides. See Fig. 2.6.

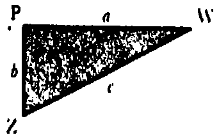


Fig. 2.6

For any right-angled triangle with hypotenuse c and other sides a and b, $\mathbf{c^2 = a^2 + b^2}$

Example 1
Given the data of Fig. 2.7, calculate the value of c.

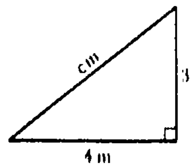


Fig. 2.7

Using Pythagoras' theorem,
$c^2 = 3^2 + 4^2$
$= 9 + 16$
$= 25$
$c = \sqrt{25} = 5$
The hypotenuse is 5 m long.

Example 2
Calculate the length of the third side of the triangle in Fig. 2.8.

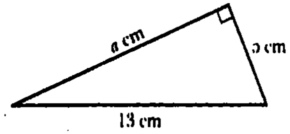


Fig. 2.8

Using Pythagoras' theorem,
$13^2 = a^2 + 5^2$
$169 = a^2 + 25$
Subtract 25 from both sides.
$169 - 25 = a^2$
$a^2 = 144$
$a = \sqrt{144} = 12$
The length of the third side of the triangle is 12 cm.

Example 3
In Fig. 2.9, calculate the length of PS.

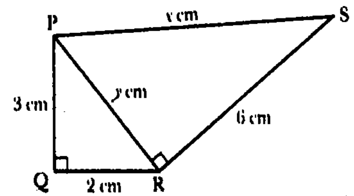


Fig. 2.9

PS is in right-angled triangle PRS.
Let PR be y cm.
In $\triangle$PQR, $y^2 = 3^2 + 2^2$
$= 9 + 4$
$= 13$

Let PS be x cm.
In $\triangle$PRS, $x^2 = y^2 + 6^2$
$= 13 + 36$
$= 49$
$x = \sqrt{49} = 7$
PS = 7 cm

Example 4
In Fig. 2.10, calculate the length of AD.

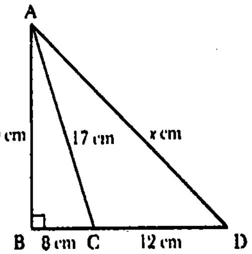


Fig. 2.10

AD is in right-angled triangle ABD.
Let AB be y cm.
In $\triangle$ABC, $y^2 = 17^2 - 8^2$
$= 289 - 64$
$= 225$
Let AD be x cm.
In $\triangle$ABD, $x^2 = y^2 + (8 + 12)^2$
$= 225 + 20^2$
$= 225 + 400$
$= 625$
$x = \sqrt{625} = 25$
AD = 25 cm

In Examples 3 and 4, notice that the sides labelled y are *intermediate* sides. We do not have to find their values. When y^2 has been found, there is no need to find y. y^2 is used in the second part of the working.

Exercise 2b

1 ABC is a triangle in which $\hat{B} = 90°$. In each of the following, draw and label a sketch then calculate the length of the third side of the triangle.
(a) AB = 6 m, BC = 8 m
(b) AB = 9 cm, BC = 12 cm
(c) AB = 5 m, BC = 12 m
(d) AB = 15 cm, BC = 8 cm
(e) AC = 25 m, BC = 24 m
(f) AC = 25 cm, BC = 20 cm
(g) AC = 100 m, AB = 80 m
(h) AC = 26 cm, AB = 24 cm
(i) AC = 41 mm, AB = 40 mm
(j) AC = 29 m, BC = 21 m

2 Find the value of x in each part of Fig. 2.11 overleaf. It will be necessary to find a value for y^2 before finding x.

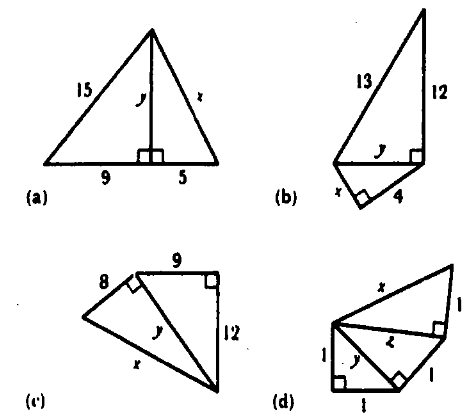


Fig. 2.11

Pythagorean triples

The sides of the triangle in Example 1 are 3 m, 4 m and 5 m. We call this a 3; 4; 5 triangle. The numbers (3; 4; 5) are called a **Pythagorean triple.** A Pythagorean triple is a set of 3 whole numbers which give the lengths of the sides of right-angled triangles. (5; 12; 13), (7; 24; 25), (8; 15; 17) are some other common Pythagorean triples. You discovered these and others in Exercise 2b.

(6; 8; 10) and (30; 40; 50) are multiples of (3; 4; 5). They are also Pythagorean triples.

Example 5

Which of the following is a Pythagorean triple?
(*a*) (33; 56; 65) (*b*) (15; 30; 35)

(a) $33^2 + 56^2 = 1\,089 + 3\,136$
$= 4\,225$
$65^2 = 4\,225$
Hence, $33^2 + 56^2 = 65^2$
(33; 56; 65) is a Pythagorean triple.

(b) $15^2 + 30^2 = 225 + 900$
$= 1\,125$
$35^2 = 1\,225$
Hence $15^2 + 30^2 \neq 35^2$
(15; 30; 35) is *not* a Pythagorean triple.

The results of Example 5 show that a triangle with sides of length 33, 56 and 65 units will be right-angled, whereas one with sides of length 15; 30 and 35 will *not* be right-angled. This method, therefore, can be used as a test for right-angled triangles.

Exercise 2c

1 Write down four multiples of each of the following Pythagorean triples.
(a) (3; 4; 5) (b) (5; 12; 13)
(c) (7; 24; 25) (d) (8; 15; 17)

2 Find out which of the following are Pythagorean triples.
(a) (20; 21; 29) (b) (15; 22; 27)
(c) (28; 45; 53) (d) (11; 60; 61)

3 Try to complete the following pattern of Pythagorean triples.
(3; 4; 5) → $3^2 = 4 + 5$
(5; 12; 13) → $5^2 = 12 + 13$
(7; 24; 25) → $7^2 = 24 + 25$
(9; ...; ...) → $9^2 =$
(11; ...; ...) → $11^2 =$
Hint: notice that the difference between the last two numbers of each triple is 1.

4 Try to complete the following pattern of Pythagorean triples.
(6; 8; 10) → $\frac{1}{2}$ of $6^2 = 8 + 10$
(8; 15; 17) → $\frac{1}{2}$ of $8^2 = 15 + 17$
(10; 24; 26) → $\frac{1}{2}$ of $10^2 = 24 + 26$
(12; ...; ...) → $\frac{1}{2}$ of $12^2 =$
(14; ...; ...) → $\frac{1}{2}$ of $14^2 =$
Hint: notice that the difference between the last two terms of each triple is 2.

5 Try to extend the patterns of questions 3 and 4 for five more terms.

Everyday use of Pythagoras' theorem

So far the exercises in this chapter have been arranged so that when a square root of a number was needed, it could be found exactly. However, this does not often happen with numbers in everyday situations. More often we have to find squares and square roots, either from tables or by using a calculator.

Using tables

Squares

The table of squares on page 287 can be used to convert 4-digit numbers to squares of those numbers.

Example 6

Use the table of squares to find $4{,}16^2$.

The digits 4,1 appear in the left-hand column of the table of squares. 6 is the third digit. Look for the column headed 6. Find the number which is across from 4,1 and under 6. See Fig. 2.12:

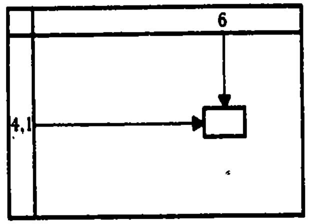


Fig. 2.12

The number is 1731.
The decimal comma is placed by inspection. In this case since 4,16 is between 4 and 5 then $4{,}16^2$ must lie between 16 (4^2) and 25 (5^2).
Hence $4{,}16^2 = 17{,}31$.
This result is correct to 4 s.f.

Example 7

Use the table of squares to find (*a*) 19^2, (*b*) 190^2

(a) $19 = 1{,}9 \times 10$
$19^2 = (1{,}9 \times 10)^2$
$= 1{,}9^2 \times 10^2$
From the table of squares,
$1{,}9^2 = 3{,}610$
Hence $19^2 = 3{,}610 \times 100$
$= 361{,}0$

(b) $190 = 1{,}9 \times 100$
$190^2 = (1{,}9 \times 100)^2$
$= 1{,}9^2 \times 100^2$
$= 3{,}610 \times 10\,000$
$= 36\,100$

In Example 7, notice that
$1{,}9^2 = 3{,}610$
$19^2 = 361{,}0$
$190^2 = 36\,100$

When a number is multiplied by increasing powers of 10, its square is multiplied by increasing powers of 100.

Exercise 2d

Use the table of squares on page 287 in this exercise.

1 Find the values of the following.
(a) $1{,}4^2$ (b) $2{,}3^2$ (c) $6{,}8^2$
(d) $7{,}2^2$ (e) $4{,}9^2$ (f) $8{,}6^2$
(g) $5{,}63^2$ (h) $9{,}08^2$ (i) $3{,}15^2$
(j) $1{,}88^2$ (k) $5{,}71^2$ (l) $4{,}54^2$

2 Find the values of the following.
(a) 18^2 (b) 31^2 (c) 32^2
(d) 15^2 (e) 29^2 (f) 44^2
(g) $70{,}5^2$ (h) $20{,}6^2$ (i) $62{,}7^2$
(j) $59{,}8^2$ (k) $81{,}3^2$ (l) $90{,}9^2$

3 Round off the following to 3 s.f. and then find the approximate square of each number.
(a) 1,733 (b) 2,808 (c) 78,65
(d) 52,14 (e) 96,47 (f) 49,57
(g) 632,6 (h) 805,3 (i) 303,6

4 Find the values of the following.
(a) 130^2 (b) 410^2 (c) 870^2
(d) 504^2 (e) $2\,700^2$ (f) $8\,350^2$

5 Look at the following pattern:
$1{,}5^2 = 2{,}25 = 1 \times 2 + 0{,}25$
$2{,}5^2 = 6{,}25 = 2 \times 3 + 0{,}25$
$3{,}5^2 = 12{,}25 = 3 \times 4 + 0{,}25$
Find out if the pattern continues in the same way.

Square roots

Tables of square roots are given on pages 288 and 289. Notice that there are *two* tables.

Example 8

Use square root tables to find (*a*) $\sqrt{5{,}7}$, (*b*) $\sqrt{57}$.

(a) 5,7 lies between 1 and 9,999. Use the first table (page 288).
$\sqrt{5{,}7} = 2{,}388$

(b) 57 lies between 10 and 99,99. Use the second table (page 289).
$\sqrt{57} = 7{,}550$

The square root tables give results rounded to 4 s.f. For example $\sqrt{5{,}7} = 2{,}388$ to 4 s.f.

However, if $2{,}388^2$ is worked out, the result is 5,702 544, *not* 5,7. Nevertheless, 4 significant figures are accurate enough for most purposes.

Example 9

Use square root tables to find (a) $\sqrt{875}$, (b) $\sqrt{3\,827}$

(a) $875 = 8{,}75 \times 100$

$\sqrt{875} = \sqrt{8{,}75 \times 100} = \sqrt{8{,}75} \times \sqrt{100}$

$= \sqrt{8{,}75} \times 10$

From the first table $\sqrt{8{,}75} = 2{,}958$

Hence $\sqrt{875} = 2{,}958 \times 10 = 29{,}58$ to 4 s.f.

(b) $3\,827 = 38{,}27 \times 100$

$\sqrt{3\,827} = \sqrt{38{,}27 \times 100}$

$= \sqrt{38{,}27} \times \sqrt{100}$

$= \sqrt{38{,}27} \times 10$

From the second table $\sqrt{38{,}27} = 6{,}187$*

Hence $\sqrt{3\,827} = 6{,}187 \times 10 = 61{,}87$ to 4 s.f.

* Use the differences column to obtain the correction for the fourth figure. The four figures under 38,2 are 6,181. The fourth figure is 7 and a correction of 6 appears under the 7 in the difference column. The 6 is added to give 6,187.

Notice again that the final results are not exact. For example, $61{,}87^2 = 3\,827{,}896\,9$, *not* 3 827.

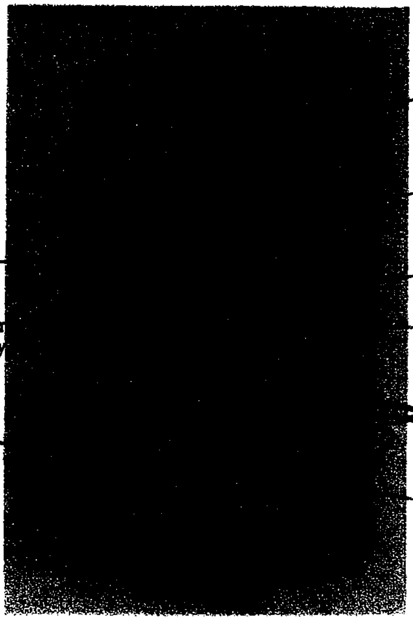


Fig. 2.13.

Exercise 2e

Use the square root tables on pages 288 and 289 in this exercise.

1 Find the square roots of the following.
(a) 9 (b) 90 (c) 2,8 (d) 28
(e) 4,7 (f) 47 (g) 5,04 (h) 50,4
(i) 36,2 (j) 3,62 (k) 25,7 (l) 2,57

2 Find the square roots of the following.
(a) 7 (b) 70 (c) 700 (d) 7 000
(e) 2,9 (f) 29 (g) 290 (h) 2 900
(i) 38,2 (j) 382 (k) 3 820 (l) 38 200
(m) 10 (n) 100 (o) 1 000 (p) 10 000
(q) 2 (r) 439 (s) 8 450 (t) 72 100

3 Find the square roots of the following.
(a) 9,286 (b) 78,23 (c) 463,8
(d) 8,455 (e) 61,27 (f) 612,7
(g) 59,03 (h) 5,806 (i) 5 003
(j) 500,3 (k) 63 945 (l) 1 982

4 Find out if $\sqrt{10}$ is a good approximation for π.

5 (a) Use square root tables to find m if $m = \sqrt{50}$.
(b) Using the value of m found in part (a), find the value of m^2 from the table of squares.
(c) What do you notice? Explain.

Using a calculator

To solve problems involving Pythagoras' theorem, you will need a calculator which has memory keys and a square root key. A scientific calculator or one like that in Fig. 2.13 will be necessary.

Example 10

In a right-angled triangle, the two shorter sides are 61,8 cm and 77,9 cm respectively. Use a calculator to find the length of its hypotenuse.

The problem is to find the value of $\sqrt{61{,}8^2 + 77{,}9^2}$. The following gives a key-stroke sequence that works on most calculators which do not have a x^2 key.

Key	*Display*
CA CM	0. (Clears memory and display)
6 1 [illegible] 8 [illegible] [illegible] = M	3819.24 ($61{,}8^2$ into memory)
7 7 [illegible] 9 [illegible] [illegible] = M	6068 41 ($77{,}9^2$ added to memory)
RM	9887.65 ($61{,}8^2 + 77{,}9^2$)
√	99.436663 ($\sqrt{(61{,}8^2 + 77{,}9^2)}$)

The hypotenuse is 99,4 cm to 1 decimal place.

Example 11

In $\triangle PQR$, $P = 90°$, $QR = 8{,}3$ m *and* $PR = 3{,}8$ m. *Calculate* PQ.

Since $\triangle PQR$ is right-angled at P,

$QR^2 = PQ^2 + PR^2$

i.e. $PQ^2 = QR^2 - PR^2$

$= 8{,}3^2 - 3{,}8^2$

$PQ = \sqrt{8{,}3^2 - 3{,}8^2}$

Using a calculator with x^2 and +/− keys:

Key	*Display*
CA CM	0.
8 [illegible] 3 x^2 M	68.89
3 [illegible] 8 x^2 +/− M	-14.44
RM	54.45
√	7.3790243

PQ = 7,4 m

Note:

1 In Examples 10 and 11, the final answers are given to the same degree of accuracy as the given data.

2 Calculators vary. If you have a calculator, get to know how it works.

3 In this book, the symbol indicates that an answer has been obtained using a calculator.

Exercise 2f (Calculator exercise)

Use a calculator to evaluate the following. Give each answer correct to 3 significant figures.

1 In each of the following, the two shorter sides of a right-angled triangle are given; calculate the hypotenuse.
(a) 5,6 m, 9,2 m (b) 739 mm, 613 mm
(c) 9,34 cm, 1,72 cm (d) 45 km, 29 km

2 Calculate the third side in the following right-angled triangles.
(a) $\triangle ABC$, $B = 90°$, $AC = 8{,}1$ m, $BC = 7{,}4$ m
(b) $\triangle XYZ$, $Z = 90°$, $YZ = 17{,}8$ cm, $XY = 23{,}6$ cm
(c) $\triangle KLM$, $K = 90°$, $KM = 55$ mm, $LM = 66$ mm
(d) $\triangle PQR$, $R = 90°$, $PR = 4{,}3$ km, $PQ = 5{,}2$ km

Applications of Pythagoras

Example 12

Fig. 2.14 shows a ladder leaning against a wall. The ladder is 7,3 m long and the foot of the ladder is 1,8 m from the wall. Find how far up the wall the ladder reaches.

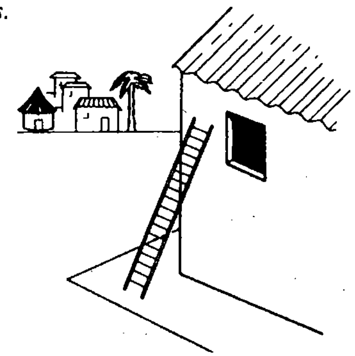


Fig. 2.14

Draw a sketch of the right-angled triangle which contains the ladder (Fig. 2.15).

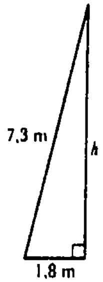


Fig. 2.15

By Pythagoras' theorem,

$h^2 = 7{,}3^2 - 1{,}8^2$
$= 53{,}29 - 3{,}240$ (from squares table)
$= 50{,}05$
$h = \sqrt{50{,}05}$
$= 7{,}075$ (from square root table)

The ladder reaches about 7,1 m up the wall.

The problem can also be answered by scale drawing. Fig. 2.16 is a scale drawing of the data on a scale 1 cm to 1 m.

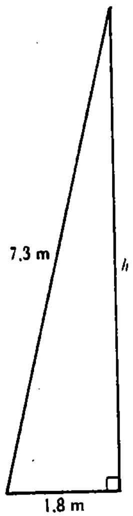


Fig. 2.16 Ladder and wall
Scale: 1 cm to 1 m

From the drawing, $h \simeq 7{,}1$ cm
Height of ladder up the wall $\simeq 7{,}1 \times 1$ m
$\simeq 7{,}1$ m

If done on a calculator, the outcome is 7.074 602 5 ( ), which gives $h = 7{,}1$ m to an appropriate degree of accuracy.

Exercise 2g

In each question, sketch the right-angled triangle which contains the unknown. Either use Pythagoras' theorem or make a scale drawing to solve the triangle. You may use a calculator.

1 A pencil which has been sharpened at each end just fits along the diagonal of the base of a box. See Fig. 2.17.

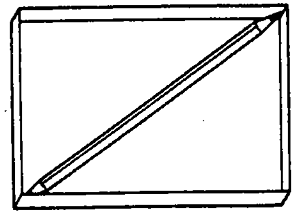


Fig. 2.17

If the box measures 14 cm by 8 cm, find the length of the pencil.

2 A telegraph pole is supported by a wire as shown in Fig. 2.18.

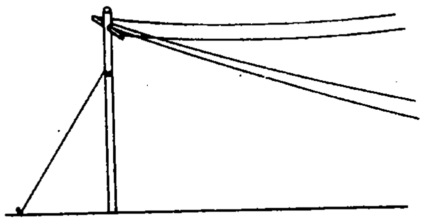


Fig. 2.18

The wire is attached to the pole 6 m above the ground and to a point on the ground 2,5 m from the foot of the pole. Calculate the total length of wire needed if an extra 0,8 m of wire is needed for the attachments.

3 Fig. 2.19 shows a straight pipe which carries water from a reservoir at R to a tap at T.

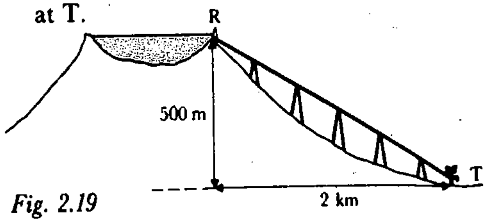


Fig. 2.19

R and T are 2 km apart horizontally and R is 500 m above the level of T. Find the length of the pipe.

4 Find the length of the longest straight line which can be drawn on a rectangular chalkboard which measures 2,2 m by 1,2 m.

5 A plane flies northwards for 430 km. It then flies eastwards for 380 km. How far is it from its starting point? (Neglect its height above the ground.)

6 Fig. 2.20 shows a simple bridge over a ditch.

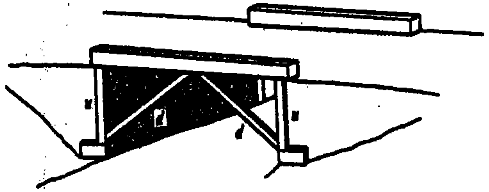


Fig. 2.20

The bridge is supported by uprights, *u*, and diagonals, *d*. Find the length of a diagonal support, if the upright supports are 4 m long and the bridge is 20 m long.

7 A ladder 7 m long leans against a wall as in Fig. 2.21.

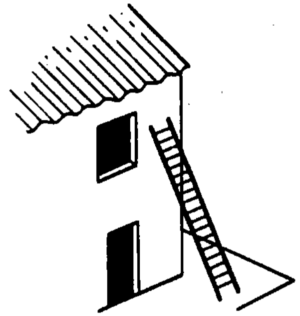


Fig. 2.21

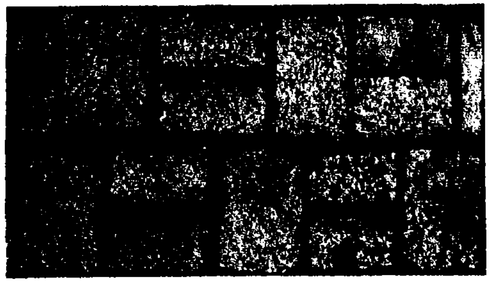
Its foot is 2 m from the wall. Calculate how far up the wall the ladder reaches.

Chapter 3

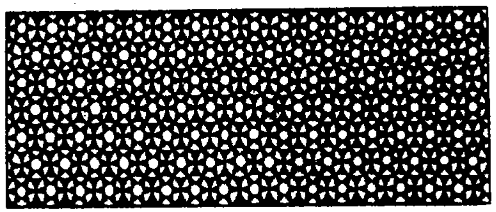
Geometrical transformations (1) Congruencies

Congruency

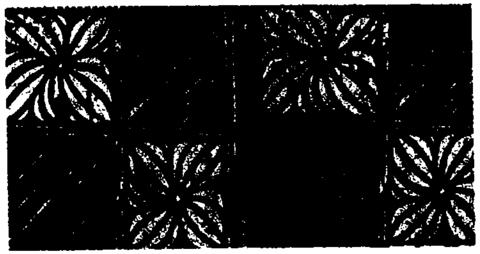
Look at the patterns in Fig. 3.1.



(*a*) *brick wall*



(*b*) *wall pattern*



(*c*) *cloth pattern*

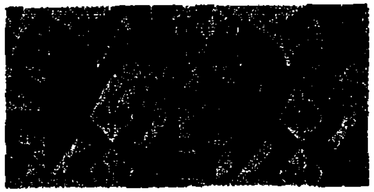


Fig. 3.1 (*d*) *print*

The patterns in Fig. 3.1 all have something in common. They are made by taking a **basic shape** and repeating it to build up the pattern.

Look at the patterns in Fig. 3.2.

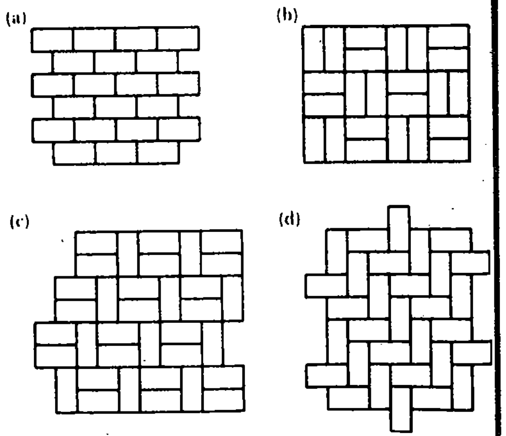


Fig. 3.2

The basic shape which makes each pattern is a 2 × 1 rectangle. The patterns appear different because the rectangles have been arranged in different ways.

Exercise 3a

You will need graph paper and a ruler for this exercise.

1 Copy patterns (a), (b) and (c) of Fig. 3.2 on to graph paper. Make each rectangle 2 cm by 1 cm. Extend each pattern by drawing more rectangles until the patterns are about 10 cm wide by 8 cm long.

2 (a) Name the basic shapes which make the patterns in Fig. 3.3.

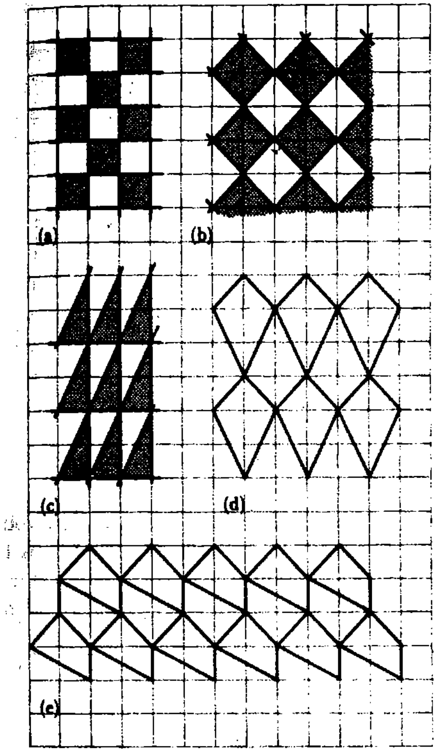


Fig. 3.3

(b) Copy each pattern on to graph paper. Draw more shapes until each pattern is about 8 cm wide and 6 cm long.

When the position or dimensions (or both) of a shape change, we say that it is **transformed**. The **image** of a shape is the figure which results after transformation (Fig. 3.4).

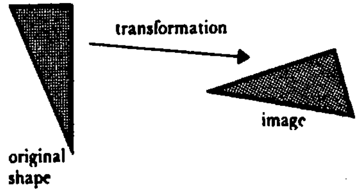


Fig. 3.4

The patterns in Exercise 3a were made by building up images of basic shapes on a plane surface. In every case, each image has the same dimensions as the original given shape. Transformations of this kind are called **congruencies**. Two shapes are congruent if their corresponding dimensions are identical. There are three basic congruencies: translations, reflections and rotations.

Translation

A **translation** is a movement in a straight line. Fig. 3.5 shows the letter p being given translations of 1 cm steps across the page.

Fig. 3.5

Translations may be in any direction (Fig. 3.6).

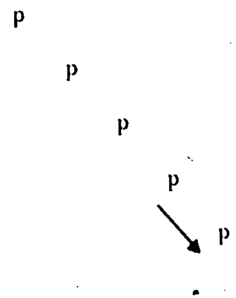


Fig. 3.6

Two or more translations of a basic shape may give a pattern which fills the plane (Fig. 3.7).

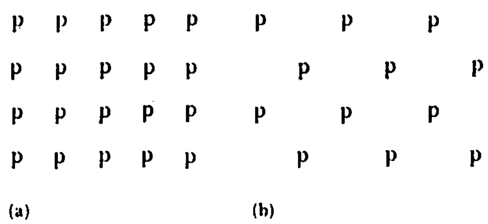


Fig. 3.7

Exercise 3b

1 Which of the patterns in Figs 3.1, 3.2, 3.3 are translation patterns?

2 Copy each pattern in Fig. 3.8. Use translation to draw at least 6 more basic shapes on each pattern.

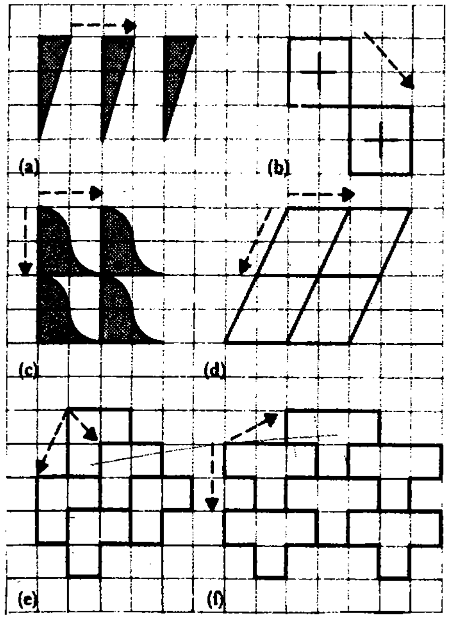


Fig. 3.8

3 Fig. 3.9 shows △ABC drawn on the cartesian plane.
  (a) State the coordinates of A, B and C.
  (b) Find the coordinates of the images of A, B and C if △ABC is translated 4 units to the left.
  (c) △ABC is translated so that the image of C is the point (− 3; − 2). Find the images of A and B.

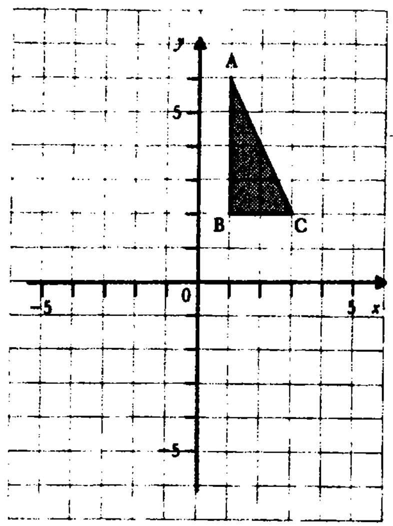


Fig. 3.9

Reflection

A **reflection** is the image you see when you look in a mirror. We have already seen in Book 1 that a line of symmetry acts like a mirror line. Fig. 3.10 shows the letter p and its reflection in a line of symmetry.

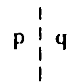


Fig. 3.10

Two or more mirror lines will reflect a shape many times to give a pattern which may fill the plane (Fig. 3.11).

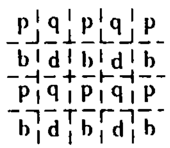


Fig. 3.11

Every line of symmetry in Fig. 3.11 is a mirror line for the whole pattern. Place a mirror along any of the dotted lines and look into it. You will see that it continues the pattern.

Exercise 3c

1 Which of the patterns in Figs 3.1, 3.2, 3.3 are reflection patterns? If possible, use a mirror to help you to decide.

2 In each of the following, copy the given figure on to graph paper. Take dotted lines to be mirror lines.
  (a) Reflect the word HAND (i) across the page, (ii) down the page (Fig. 3.12).

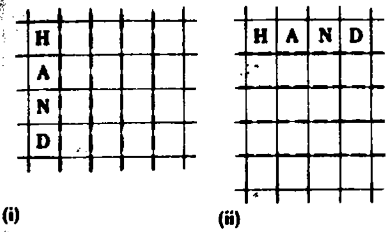


Fig. 3.12

  (b) In Fig. 3.13 draw the missing lines of symmetry. Continue the pattern for at least 3 more rows and columns.

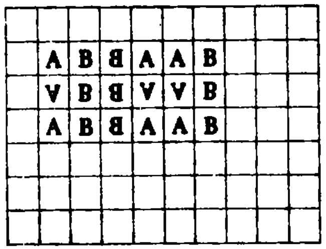


Fig. 3.13

  (c) In Fig. 3.14, use the given shape and mirror lines to make a reflection pattern.

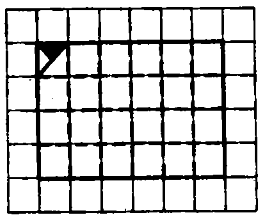


Fig. 3.14

3 In Fig. 3.9 on page 30, state the coordinates of the vertices A, B and C after reflection in (a) the x-axis, (b) the y-axis.

Rotation

When a shape turns about a point, we say it **rotates**. Fig. 3.15 shows four positions of the letter p when it is rotated about the small cross.

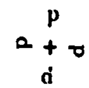


Fig. 3.15

In Fig. 3.15, p has been rotated through 90° each time. Of course there are many other angles through which p could be rotated.

The pattern in Fig. 3.15 can be continued by rotating the p about other centres of rotation. Fig. 3.16 gives one example.

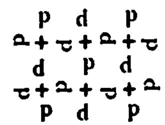


Fig. 3.16

Rotation is often combined with other movements. Fig. 3.17 is a pattern where the basic shape has been rotated through 90° about its centre then translated horizontally and vertically.

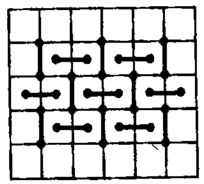


Fig. 3.17

Exercise 3d

1 Which of the patterns in Figs 3.1, 3.2, 3.3 are rotation patterns?

2 Use rotation and translation to continue the patterns in Fig. 3.18 overleaf.

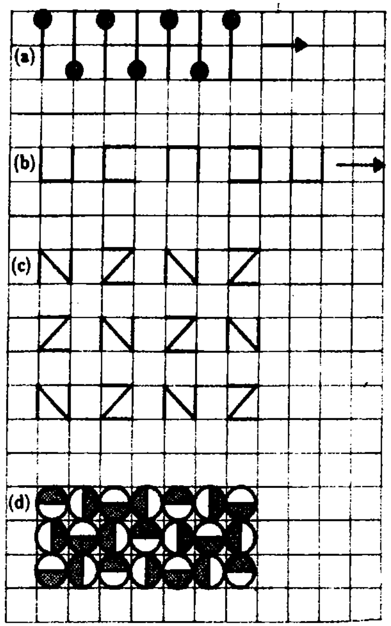


Fig. 3.18

3 In Fig. 3.9 on page 30 state the coordinates of the vertices A, B and C after a clockwise rotation about the origin of (a) 90°, (b) 180°, (c) 270°.

4 As question 3, with the point (– 2; 1) as the centre of rotation.

Combining transformations

Example 1

In Fig. 3.19 describe how $\triangle$ABC *is transformed to* $\triangle$DEF.

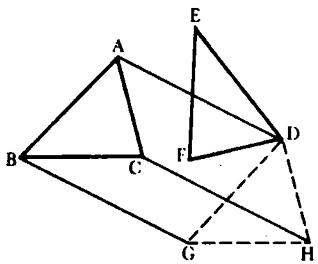


Fig. 3.19

The transformation may be described as follows:

(a) Translate $\triangle$ABC along the line AD to position DGH (dotted).

(b) With centre D, rotate $\triangle$DGH clockwise through an angle equal to H$\hat{D}$F to the position of $\triangle$DEF.

Exercise 3e

1 Copy the patterns of Fig. 3.20 on to graph paper. Extend each pattern by repeating the sequence of basic shapes.

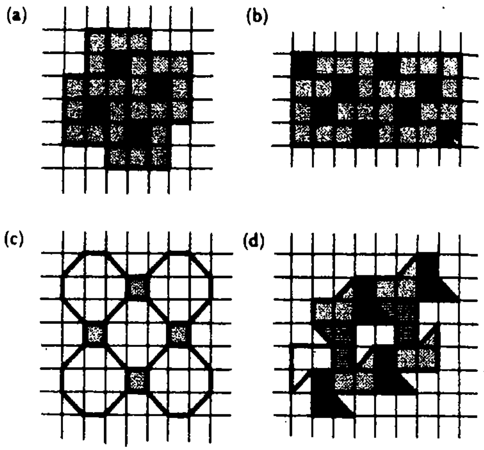


Fig. 3.20

2 Fig. 3.21 shows a grid with a shape in the top left-hand corner.

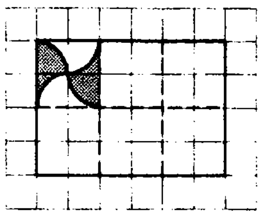


Fig. 3.21

Make three copies of Fig. 3.21. Make a pattern on each copy by

(a) translation of the basic shape to adjacent squares;

(b) reflection of the basic shape across the dotted lines;

(c) rotation of 90° of the basic shape about its centre, followed by translation to adjacent squares.

3 Each part of Fig. 3.22 contains a pair of congruent triangles. Sketch each figure, showing the line of reflection or the centre of rotation.

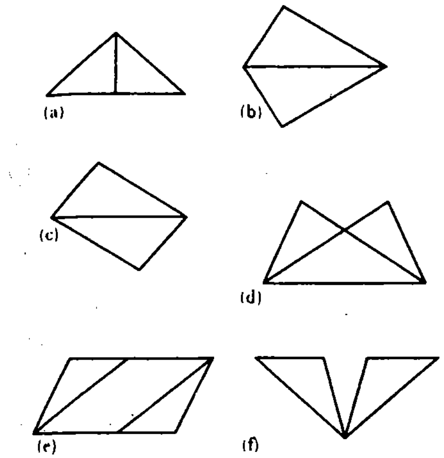


Fig. 3.22

4 In each part of Fig. 3.23 describe how $\triangle$ABC is transformed to $\triangle$DEF.

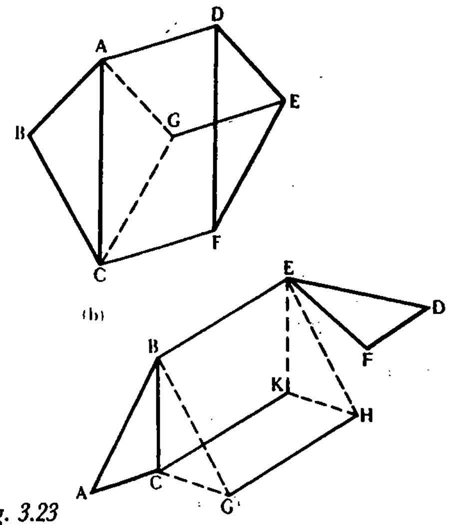


Fig. 3.23

5 In Fig. 3.9 on page 30, find the coordinates of the images of A, B and C if $\triangle$ABC is

(a) first reflected in the *x*-axis then rotated anticlockwise through 90° about the origin;

(b) first rotated through 180° about (1; 1) then reflected in the *y*-axis.

Chapter 4

Basic processes of algebra

Use of letters

Algebra uses letters of the alphabet to represent general numbers. The easiest way of dealing with letters which represent numbers is to imagine first that a letter stands for a particular number.

Example 1

Express p metres in centimetres.

If $1 \text{ m} = 100 \text{ cm}$,
$2 \text{ m} = 100 \times 2 \text{ cm}$,
and $3 \text{ m} = 100 \times 3 \text{ cm}$,
then $p \text{ m} = 100 \times p \text{ cm}$
$= 100p \text{ cm}$

In Example 1 notice that $100p$ is short for $100 \times p$. In the expression $100p$, the 100 is called the **coefficient** of p.

Example 2

In a class of p students the average mark is x and in another class of n students the average mark is y. What is the average mark for both classes?

$$\text{average mark} = \frac{\text{total marks scored}}{\text{number of students}}$$

In 1st class, total marks scored $= p \times x = px$
In 2nd class, total marks scored $= n \times y = ny$
For both classes, total marks scored $= px + ny$
Number of students $= p + n$

$$\text{Average mark} = \frac{px + ny}{p + n}$$

Exercise 4a

1 A girl has 20 cents. If someone gives her 16 cents, how much will she have? If someone gives her x cents how much will she have?
2 A boy is 15 years old. How old will he be in 4 years' time? How old will he be in y years' time?
3 A woman has \$20. If she spends \$8, how much will she have left? If she spends \$$a$ instead, how much will she have left?
4 6 years ago a man was 20 years old. How old is he now? p years ago another man was 23 years old. How old is he now?
5 A lorry carries 8 t of blocks.
(a) If it delivers 3 t, what mass of blocks is left on the lorry?
(b) If it delivers d t, what mass remains on the lorry?
6 What number is (a) 5 times as big as 20, (b) 5 times as big as x?
7 How many cents in (a) \$3, (b) \$$e$?
8 How many hours are there in (a) 120 min, (b) m min?
9 A sheet of paper has an area of 300 cm^2.
(a) How long is the paper if its breadth is 15 cm?
(b) How long is the paper if its breadth is b cm?
10 Express x Dollars and y cents in cents.
11 Express a Dollars and b cents in Dollars.
12 The perimeter of a rectangle is 14 m and the length is x m. Express the breadth of the rectangle in terms of x.
13 The perimeter of a rectangle is 20 m and the length is x m. Find the area of the rectangle in terms of x.
14 A rectangle has one side equal to $(2x + 4)$ cm and a perimeter of $(6x + 4)$ cm. What is the area of the rectangle in terms of x?
15 A car travels d km at an average speed of u km/h. How long does it take?
16 A train moves at an average speed of x km/h. How many hours will it take to cover 340 km?

17 A train travels for t hours at a speed of v km/h. How far does it go?
18 Two motor cars start together from the same place and travel in the same direction along a road. If the faster one has an average speed of u km/h and the other v km/h, how far ahead is the faster one after t hours?
19 A girl's age is x years and her father is 4 times as old. Find the father's age in y years' time.
20 p years ago a woman's age was $(q + r)$ times that of her son who was then r years old. How old is the woman now?

Simplification

Grouping terms

Expressions such as $2x$, x, $\frac{1}{3}x$ and $-5x$ are called **terms in x**. Expressions containing a number of terms can be simplified by grouping terms together.

Example 3

Simplify $9a - 5a - 8a + 10a$.

Either, by grouping positive and negative terms together.

$9a - 5a - 8a + 10a$
$= 9a + 10a - 5a - 8a$
$= 19a - 13a$
$= 6a$

or, by treating the terms as directed numbers,

$9a - 5a - 8a + 10a$
$= 4a - 8a + 10a$
$= -4a + 10a$
$= 6a$

Example 4

Simplify $5x - 8x + x + 3y$.

$5x - 8x + x + 3y$
$= -3x + x + 3y$
$= -2x + 3y = 3y - 2x$

In Example 4, notice that $2x$ and $3y$ are **unlike terms**. $3y - 2x$ cannot be simplified any further.

Exercise 4b

Simplify the following expressions as far as possible.

1 $9x - 2x$
2 $2a - 9a$
3 $-3y - 30y$
4 $2y + 5y - 3y$
5 $4x - 8x + 9x$
6 $9z - 3z - z$
7 $3a - 7a - 2a$
8 $b + 4b - 12b$
9 $6c - 17c + 3c$
10 $2a + 5x - 3a$
11 $-3h - 6g + 10g$
12 $8d - 3 - 7d$
13 $3a - 5a + 11a - 4a$
14 $7x + 3x - x - 5x$
15 $8k - 4k + 3k - 7k$
16 $6x - 9x + 2x + 4y$
17 $2a - 3b + 5b - 8a$
18 $3x + 8y - 5x - y$
19 $2m - 9n - 5m + 4n$
20 $r - 3s - 3t - 4r + 10s + 8t$

Brackets

If any quantity multiplies the terms inside a bracket, *every* term inside the bracket must be multiplied by that quantity when the bracket is removed.

In general, $a(x + y) = ax + ay$
and $a(x - y) = ax - ay$

Example 5

Remove the brackets from $4(3x - 5y + z)$.

$4(3x - 5y + z)$
$= 4 \times 3x - 4 \times 5y + 4 \times z$
$= 12x - 20y + 4z$

Example 6

Remove brackets and simplify $3 - (a - \overline{5 - 6a})$.

Note: the line over the terms $5 - 6a$ acts like a bracket.

$3 - (a - \overline{5 - 6a})$
$= 3 - (a - 5 + 6a)$
$= 3 - (7a - 5)$
$= 3 - 7a + 5$
$= 8 - 7a$

In Example 6, notice that the inner bracket is removed *first*. Notice also that a negative sign outside a bracket is equivalent to -1 outside the bracket, i.e. $-(5-6a) = -1(5-6a)$.

Example 7

Remove brackets and simplify
$3x(2-x) - 5(6+x-2x^2)$.

$$\begin{aligned} &3x(2-x) - 5(6+x-2x^2)\\ &= 6x - 3x^2 - 30 - 5x + 10x^2\\ &= 10x^2 - 3x^2 + 6x - 5x - 30\\ &= 7x^2 + x - 30 \end{aligned}$$

Example 8

Simplify (a) $3 \times 2a - 5$, *(b)* $3(2a-5)$.

(a) $3 \times 2a - 5 = 6a - 5$
(b) $3(2a-5) = 3 \times 2a - 3 \times 5$
$= 6a - 15$

Exercise 4c

Remove brackets from the following and simplify where possible.

1 $3(a+b-2)$ **2** $-2(3m-n+4)$
3 $2a + 3(a+3b)$ **4** $5(2y-x)+6x$
5 $\frac{1}{2}(2u-8)+5$ **6** $3c - \frac{2}{3}(12c-3d)$
7 $4 \times 3m - 2$ **8** $4 \times (3m-2)$
9 $4u - \overline{7u-3v}$ **10** $5d - \overline{e+2d}$
11 $2(a-3b)+3(a+b)$
12 $5m + 5(2n-m) - 8n$
13 $2(h+5k)+5(h+2k)$
14 $5(2u-3v+4w) - 7(u-2v+3w)$
15 $4(r-3s-3t) - 2(2r-5s-4t)$
16 $6x - (3 - \overline{x-8})$
17 $2a - (4a - \overline{5a-7})$
18 $2x(3x-1) + 7(x^2+3)$
19 $6(a^2-2a-1) - 3a(2a-5)$
20 $4x(3x-2y) + 5y(x+3y) - 7xy$

Expansion of brackets

The expression $(a+b)(c+d)$ means $(a+b) \times (c+d)$. The terms in the first bracket multiply each term in the second bracket.

$$\begin{aligned} &(a+b)(c+d)\\ &= (a+b)c + (a+b)d\\ &= ac + bc + ad + bd \end{aligned}$$

Alternatively, each term in the first bracket multiplies the terms in the second bracket.

$$\begin{aligned} &(a+b)(c+d)\\ &= a(c+d) + b(c+d)\\ &= ac + ad + bc + bd\\ &= ac + bc + ad + bd \end{aligned}$$

In either case, $ac + bc + ad + bd$ is the **expansion** of $(a+b)(c+d)$.

Example 9

Expand the following, simplifying where possible.
(a) $(r-x)(r+y)$ *(b)* $(x-5)(x+8)$
(c) $(2a+3b)(5a-2b)$ *(d)* $(m+4n)^2$

(a) $(r-x)(r+y) = r(r-x) + y(r-x)$
$= r^2 - rx + ry - xy$
(b) $(x-5)(x+8) = x(x-5) + 8(x-5)$
$= x^2 - 5x + 8x - 40$
$= x^2 + 3x - 40$
(c) $(2a+3b)(5a-2b)$
$= 5a(2a+3b) - 2b(2a+3b)$
$= 10a^2 + 15ab - 4ab - 6b^2$
$= 10a^2 + 11ab - 6b^2$
(d) $(m+4n)^2 = (m+4n)(m+4n)$
$= m(m+4n) + 4n(m+4n)$
$= m^2 + 4mn + 4mn + 16n^2$
$= m^2 + 8mn + 16n^2$

In Example 9(*d*), the expression $(m+4n)^2$ is called a **perfect square**. The expansions in parts (*b*), (*c*) and (*d*) of Example 9 were simplified by collecting the like terms together.

Exercise 4d

Expand the following, collecting like terms where possible.

1 $(p+q)(r+s)$ **2** $(6+y)(5+x)$
3 $(a-b)(c-d)$ **4** $(2x+1)(3x-1)$
5 $(2n+3)(2n+5)$ **6** $(4m-9)(2m-3)$
7 $(5x-2)(x+4)$ **8** $(a+5)(3a-2)$
9 $(x-8)(x+9)$ **10** $(3h+4)(h-2)$
11 $(b-5)(b+5)$ **12** $(c+7)(c-7)$
13 $(3t-2)^2$ **14** $(2x-5y)^2$
15 $(5x-y)(2x-3y)$
16 $(4a-1)(5a-2)$
17 $(5a+2d)(3a-8d)$
18 $(2d+e)(3d+e)$
19 $(6m+n)(5m-n)$
20 $(2k-7)(4k+5)$

Example 10

Find the coefficient of n in
$(4n+3)(n-5) - (2n-3)^2$.

Either
$(4n+3)(n-5) - (2n-3)^2$
$= n(4n+3) - 5(4n+3) - [2n(2n-3) - 3(2n-3)]$
$= 4n^2 + 3n - 20n - 15 - (4n^2 - 6n - 6n + 9)$
$= 4n^2 - 17n - 15 - (4n^2 - 12n + 9)$
$= 4n^2 - 17n - 15 - 4n^2 + 12n - 9$
$= -5n - 24$

The coefficient of n is -5.
Or, by inspection,
for $(4n+3)(n-5)$, the term in n is found by adding the products of the inner and outer pairs of terms:

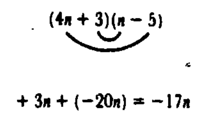


Fig. 4.1

$(2n-3)^2 = (2n-3)(2n-3)$. Again, the term in n is the sum of the products of the inner and outer terms of the brackets.

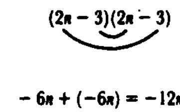


Fig. 4.2

For the whole expression,
term in $n = -17n - (-12n)$
$= -17n + 12n$
$= -5n$
Coefficient of n is -5.

Fig. 4.3 shows how to expand brackets directly by multiplying the terms inside the brackets in pairs.

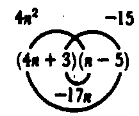


Fig. 4.3

Exercise 4e

1 Either by expanding the brackets or otherwise, write down the coefficient of x in the following.
(a) $(x+7)(x+9)$
(b) $(x-3)(x+8)$
(c) $(x+4)(x-17)$
(d) $(2x+5)(4x+7)$
(e) $(3x-1)(5x+2)$
(f) $(2x-8)(2x+8)$
2 Multiply $(3+a)$ by $(5-2a)$.
3 Simplify the following.
(a) $(a-3)(a-4) + (a-1)(a+5)$
(b) $(m+6)(m-2) - (m-3)(m+1)$
(c) $(2x+3)(x-2) + 2x(x-1)$
(d) $3d(2d+3) - (3d+1)(2d+1)$.
4 Simplify $(f+g)^2 + (f-2g)^2$.
5 Simplify $(3b-1)^2 - (2b+1)^2$.
6 Simplify $(2a-3b)^2 - (a-b)^2$.
7 Simplify $(y+z)^2 - 2yz$.
8 Find the coefficient of (a) x^2, (b) x in $(x+1)(2x-3) + (x-4)^2$.
9 Simplify $(a+b)^2 - (a-b)^2$.
10 Simplify
$(3n+t)(n+3t) - 3(n-2t)(n-4t)$.

Adding and subtracting algebraic fractions

Just as in arithmetic, fractions in algebraic form must have a common denominator before they can be added or subtracted.

Example 11

Simplify the following.
(a) $\frac{3x}{7} + \frac{2x}{7}$ *(b)* $\frac{5}{2a} - \frac{3}{2a}$
(c) $\frac{1}{4x} + \frac{2}{3x}$ *(d)* $\frac{7}{3a} - \frac{6}{5c}$

(a) $\frac{3x}{7} + \frac{2x}{7} = \frac{3x+2x}{7} = \frac{5x}{7}$
(b) $\frac{5}{2a} - \frac{3}{2a} = \frac{5-3}{2a} = \frac{2}{2a} = \frac{2 \div 2}{2a \div 2} = \frac{1}{a}$

(c) The LCM of $4x$ and $3x$ is $12x$. $12x$ is the common denominator.

$$\frac{1}{4x}+\frac{2}{3x}=\frac{1\times 3}{4x\times 3}+\frac{2\times 4}{3x\times 4}$$
$$=\frac{3}{12x}+\frac{8}{12x}$$
$$=\frac{3+8}{12x}$$
$$=\frac{11}{12x}$$

(d) The LCM of $3a$ and $5c$ is $15ac$. $15ac$ is the common denominator.

$$\frac{7}{3a}-\frac{6}{5c}=\frac{7\times 5c}{3a\times 5c}-\frac{6\times 3a}{5c\times 3a}$$
$$=\frac{35c}{15ac}-\frac{18a}{15ac}$$
$$=\frac{35c-18a}{15ac}$$

This does not simplify further.

$\frac{x-2}{4}$ is a short way of writing $\frac{(x-2)}{4}$ or $\frac{1}{4}(x-2)$. Similarly $\frac{3x}{x-y}$ is a short way of writing $\frac{3x}{(x-y)}$. In each case, consider the terms inside the brackets to be a single term until the brackets can be properly removed.

Example 12

Simplify the following.

(a) $\frac{4a+13}{5}-\frac{2a+3}{3}$

(b) $2+\frac{5(2a+1)}{6a}-\frac{4b-3}{2b}$

(a) The LCM of 5 and 3 is 15. 15 is the common denominator.

$$\frac{4a+13}{5}-\frac{2a+3}{3}$$
$$=\frac{3(4a+13)}{5\times 3}-\frac{5(2a+3)}{3\times 5}$$
$$=\frac{3(4a+13)-5(2a+3)}{15}$$
$$=\frac{12a+39-10a-15}{15}$$
$$=\frac{2a+24}{15}$$

(b) The LCM of $6a$ and $2b$ is $6ab$. Make equivalent fractions with denominators of $6ab$.

$$2+\frac{5(2a+1)}{6a}-\frac{4b-3}{2b}$$
$$=\frac{6ab\times 2}{6ab}+\frac{b\times 5(2a+1)}{6ab}-\frac{3a\times(4b-3)}{6ab}$$
$$=\frac{12ab+5b(2a+1)-3a(4b-3)}{6ab}$$
$$=\frac{12ab+10ab+5b-12ab+9a}{6ab}$$
$$=\frac{10ab+9a+5b}{6ab}$$

This does not simplify further.

Example 13

Express each of the following as a single fraction in its simplest form.

(a) $\frac{3}{m+2n}-\frac{2}{m-3n}$ *(b)* $\frac{x}{x+2}-\frac{x-2}{x-3}$

(a) The denominators are $(m+2n)$ and $(m-3n)$. Their LCM is $(m+2n)(m-3n)$. Make equivalent fractions with a common denominator of $(m+2n)(m-3n)$.

$$\frac{3}{m+2n}-\frac{2}{m-3n}$$
$$=\frac{3(m-3n)}{(m+2n)(m-3n)}-\frac{2(m+2n)}{(m+2n)(m-3n)}$$
$$=\frac{3(m-3n)-2(m+2n)}{(m+2n)(m-3n)}$$
$$=\frac{3m-9n-2m-4n}{(m+2n)(m-3n)}$$
$$=\frac{m-13n}{(m+2n)(m-3n)}$$

This does not simplify further.

(b) $\frac{x}{x+2}-\frac{x-2}{x-3}$
$$=\frac{x(x-3)-(x+2)(x-2)}{(x+2)(x-3)}$$
$$=\frac{x^2-3x-(x^2-4)}{(x+2)(x-3)}$$
$$=\frac{x^2-3x-x^2+4}{(x+2)(x-3)}$$
$$=\frac{4-3x}{(x+2)(x-3)}$$

Exercise 4f

1 Express each of the following as a single fraction.

(a) $\frac{4x}{9}+\frac{x}{9}$ (b) $\frac{5y}{2}-\frac{2y}{3}$

(c) $\frac{3}{a}-\frac{2}{b}$ (d) $\frac{1}{2x}+\frac{1}{7x}$

(e) $\frac{3}{4a}-\frac{4}{3b}$ (f) $\frac{3}{2ab}+\frac{4}{3bc}$

2 Express $\frac{2}{a}+\frac{7}{b}-\frac{3}{c}$ as a single fraction.

3 Reduce $\frac{3}{x}-\frac{x}{2}+5$ to a single fraction.

4 Simplify $\frac{5}{2cd}+\frac{4}{3de}$.

5 Simplify each of the following.

(a) $\frac{2x-3}{3}+\frac{x-1}{3}$

(b) $\frac{7x+2}{5}-\frac{5x+3}{5}$

(c) $\frac{a-2}{3}-\frac{a-4}{5}$

(d) $\frac{1}{2}(3x+1)+\frac{1}{3}(x+2)$

(e) $\frac{2(3x-2)}{5}+\frac{x-5}{6}$

(f) $\frac{3a-4}{5}-\frac{2a+19}{15}$

6 Simplify the following expression.
$\frac{3x+2}{3}-\frac{x-1}{4}-\frac{5}{12}$

7 Simplify the following.

(a) $\frac{6x+1}{2a}-\frac{x-2}{2a}$

(b) $\frac{x-2}{6x}+\frac{2x+1}{3x}$

(c) $\frac{4a+1}{a}+\frac{3b-2}{b}$

(d) $3-\frac{6a-5}{2a}$

(e) $\frac{2a-1}{5a}-\frac{4b-3}{10b}$

(f) $\frac{2x+1}{x}+\frac{3y-2}{y}-5$

8 Express $\frac{a-4}{2b}-\frac{b-3}{6b}+4$ as a single fraction in its lowest terms.

9 Simplify the following.

(a) $3+\frac{2b}{a-b}$ (b) $2-\frac{x}{x+2y}$

(c) $\frac{5}{a+4}-\frac{2}{a-2}$ (d) $\frac{2}{t+1}+\frac{3}{t+2}$

(e) $\frac{3x}{x-1}-\frac{x}{x-2}$ (f) $\frac{x}{x-3}-\frac{8}{x}-2$

(g) $\frac{1}{n-6}+\frac{1}{n-4}-\frac{2}{n-5}$

(h) $\frac{x-2}{x+2}-\frac{x-1}{x+3}$

10 If $X=\frac{2a+4}{3a-2}$, express $\frac{X-1}{2X+1}$ in terms of a.

Common factors

Example 14

Complete the bracket in the statement
$8p-20q=4(\quad)$.

4 is the HCF of $8p$ and $20q$. Divide $8p$ and $20q$ to find the terms inside the bracket.
$8p\div 4=2p$ and $20q\div 4=5q$
$8p-20q=4(2p-5q)$
In Example 14, 4 and $(2p-5q)$ are the **factors** of $8p-20q$. 4 is the **common factor** of the given terms.

Example 15

Factorise the following.

(a) $2a^3-5a^2-a$

(b) $15a^3b^4c-6a^2b^5c^2$

(a) The common factor is a.

$2a^3 - 5a^2 - a = a(2a^2 - 5a - 1)$

(b) The HCF of the two terms is $3a^2b^4c$.

$15a^3b^4c - 6a^2b^5c^2 = 3a^2b^4c(5a - 2bc)$

Check the results of Example 15 by multiplication.

Exercise 4g (Oral)

Factorise the following either by completing the brackets or by finding the highest common factors of the given terms.

1 $2m + 8n = 2(\quad)$
2 $3a - 15b = 3(\quad)$
3 $10x - 5 = 5(\quad)$
4 $-3h - 12k = -3(\quad)$
5 $-2x + 18 = -2(\quad)$
6 $5a - 8ab = a(\quad)$
7 $9x + 3xz = 3x(\quad)$
8 $8cm + 12dm - 16em = 4m(\quad)$
9 $3x^3 - 12x^2 - 9x = 3x(\quad)$
10 $10a^2b^2 - 15a^2b + 20ab^2 = 5ab(\quad)$
11 $4a - 8b$ **12** $9x + 12y$
13 $3ab - 6ac + 3ad$ **14** $8px - 4qx + 8rx$
15 $3m^3 - 2m^2 + m$ **16** $6n^4 - 2n^3 + 4n^2$
17 $5ab + 4a^2b - 6ab^2$
18 $2abx + 7acx - 3a^2x$
19 $4a^4 + 2a^3b - 10a^2b^2$
20 $24a^2bc^2 + 30a^2c^2x - 18a^3cx^2$

Example 16

Factorise $12x^2 + 3x - 4x - 1$.

The terms $12x^2$ and $3x$ have $3x$ in common.
The terms $-4x$ and -1 have -1 in common.

$12x^2 + 3x - 4x - 1$
$= 3x(4x + 1) - 1(4x + 1)$
$= (4x + 1)(3x - 1)$

(since the expressions $3x(4x + 1)$ and $-1(4x + 1)$ both have $(4x + 1)$ in common).

Example 17

Factorise $3x - 2dy + 3y - 2dx$.

The terms $3x$ and $3y$ both have 3 in common.
The terms $2dx$ and $2dy$ both have $2d$ in common.

Grouping the given terms in this order,

$3x - 2dy + 3y - 2dx$
$= 3x + 3y - 2dx - 2dy$
$= 3(x + y) - 2d(x + y)$
$= (x + y)(3 - 2d)$

Exercise 4h

Factorise the following by grouping terms in pairs.

1 $3x + 9b + 5ax + 15ab$
2 $2ce - 2cf + de - df$
3 $4u + 4v + vx + ux$
4 $mn - 3my - 3nx + 9xy$
5 $6a^2 - 3a + 4a - 2$
6 $cd - ce - d^2 + de$
7 $12eg - 4eh - 6fg + 2fh$
8 $4ab + 6bn - 2a - 3n$
9 $3ax - 2a - 6bx + 4b$
10 $ax - a + x - 1$
11 $ac + ad - bc - bd$
12 $4a - 7b + 28bx - 16ax$
13 $p + q + 5ap + 5aq$
14 $2c^2 - 8cm - 3cm + 12m^2$
15 $x^2 + 2x - 5x - 10$
16 $2pr - sq + 2qr - ps$
17 $y^2 - 5y + 4y - 20$
18 $6ac + bd - 3bc - 2ad$
19 $2ab - 10cd - 5bc + 4ad$
20 $6xy - 2z - 4y + 3xz$

Substitution

Example 18

Find the value of $4(3d - e) - 2f$ *when* $d = 2$, $e = 4$ *and* $f = 3$.

Either

$4(3d - e) - 2f = 4(3 \times 2 - 4) - 2 \times 3$
$= 4(6 - 4) - 6$
$= 4 \times 2 - 6$
$= 8 - 6$
$= 2$

or

$4(3d - e) - 2f = 12d - 4e - 2f$
$= 12 \times 2 - 4 \times 4 - 2 \times 3$
$= 24 - 16 - 6$
$= 2$

Example 19

Evaluate $(m - n)(u + v)$ *if* $m = 5$, $n = 3$, $u = 1$ *and* $v = 2$.

$(m - n)(u + v) = (5 - 3)(1 + 2)$
$= 2 \times 3 = 6$

Notice the order of operations in Examples 18 and 19: first evaluate the contents of brackets, then do multiplication (or division) before addition or subtraction.

Exercise 4i

If $a = 1$, $b = 2$, $c = 3$, $m = 4$ and $n = 5$, find the value of the following.

1 bc **2** $bc + n$
3 $4n - 2b$ **4** $4(n - 2b)$
5 $3(a + 2m)$ **6** $3a + 2m$
7 $2(a + 3c) - 4n$ **8** $3(a - b + c)$
9 $(a + c)(m - b)$ **10** $(2m - n)(c + 2a)$
11 $b(3m - n)$ **12** $(m - 1)(m + 1)$
13 $\dfrac{bcn}{b + c}$ **14** $\dfrac{mn + b}{cm - a}$
15 $a + \dfrac{m}{b}$ **16** $\dfrac{a + m}{b}$
17 $\dfrac{c^2 + m}{n^2 + a}$ **18** $cm^2 - c^2m$
19 $\sqrt{c^2 + m^2}$ **20** $\sqrt{n(n + b) + a}$

Example 20

Evaluate $\dfrac{2a^2bc}{2b - c}$ *when* $a = 3$, $b = -4$, $c = -5$.

$$\frac{2a^2bc}{2b - c} = \frac{2 \times (3)^2 \times (-4) \times (-5)}{2 \times (-4) - (-5)}$$
$$= \frac{2 \times 9 \times 20}{-8 + 5}$$
$$= \frac{2 \times 9 \times 20}{-3} = -120$$

Notice in Example 20 that the denominator, $(2b - c)$, must be reduced to a single number before division is possible.

Exercise 4j

1 Find the value of the following when $a = 1$, $b = 0$ and $c = -3$.
(a) ac (b) abc
(c) $a - c$ (d) $ab - ac$
(e) $a + b + c$ (f) $b - (a + c)$
(g) $\dfrac{c - a}{2a}$ (h) $(a + 2c)^b$
(i) $c(a + c)$ (j) $\sqrt{a - 5c}$

2 Evaluate the following when $x = 4$, $y = -5$ and $z = -2$.
(a) $x + y + z$ (b) $x - y + z$
(c) $x - (y + z)$ (d) $x + y - z$
(e) xyz (f) $z(x + y)$
(g) $y^2 - 2x$ (h) $3xy - y^3$
(i) $\dfrac{x + yz}{y + z}$ (j) $\dfrac{y^2 - 1}{x - z}$

3 What is the value of $p^2q - q^2p$ if $p = 3$ and $q = -1$?

4 Evaluate $x^2 + 3x + 2$ when $x =$ (a) 1, (b) 0, (c) -1, (d) -2.

5 Evaluate $p^2 - 2p - 3$ when $p =$ (a) 4, (b) 3, (c) 1, (d) 0, (e) -1, (f) -2.

6 Evaluate $3a^2 - 2a + 5$ if $a =$ (a) 2, (b) 0, (c) -2, (d) -4.

7 Evaluate $z^3 - z$ if $z =$ (a) 2, (b) 1, (c) 0, (d) -1, (e) -2.

8 Evaluate $ab\sqrt{c^2 + b^2}$, given that $a = 2$, $b = -3$, $c = 4$.

9 If $a = -3$, $b = 2$, $c = 1$ and $d = -4$, find the values of
(a) $5b - ad$, (b) $(ba)^2 - bd^2$,
(c) $\left(\dfrac{c - a}{b - d}\right)^2$.

10 If $x = -7$ and $y = 3$, calculate the values of
(a) $\left(\dfrac{x + y}{x - y}\right)^2$,
(b) $2x^2y + y^2x$,
(c) $27x^2 \div 12y^2$.

Chapter 5

Plane shapes

Polygons

A **polygon** is any plane figure with straight sides. Thus a triangle is a 3-sided polygon and a quadrilateral is a 4-sided polygon. Polygons are named after the number of sides they have. Table 5.1 gives the names of the first 8 polygons.

Table 5.1

triangle	3 sides
quadrilateral	4 sides
pentagon	5 sides
hexagon	6 sides
heptagon	7 sides
octagon	8 sides
nonagon	9 sides
decagon	10 sides

A regular polygon has all sides and all angles equal. Fig. 5.1 shows the first 6 regular polygons.

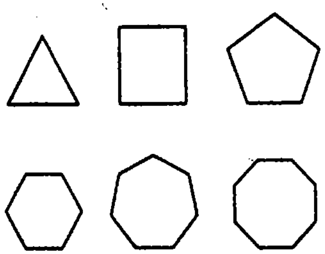


Fig. 5.1

Exercise 5a

1 How many lines of symmetry has each polygon in Fig. 5.1?

2 What is the order of rotational symmetry of each polygon in Fig. 5.1?

3 Fig. 5.2 is a regular pentagon ABCDE. O is the centre of point symmetry.

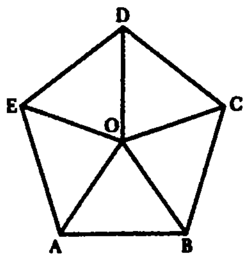


Fig. 5.2

(a) What is the sum of the angles at O?
(b) How many angles are at O?
(c) Calculate the size of each angle at O.
(d) What kind of Δ is Δ AOB?
(e) Calculate OÂB and OB̂A.
(f) What is the size of AB̂C?
(g) Calculate the sum of the angles of pentagon ABCDE.

4 Fig. 5.3 shows a pattern made from regular hexagons.

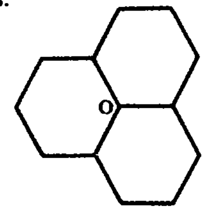


Fig. 5.3

(a) What is the order of symmetry of Fig. 5.3 about O?
(b) What is the size of each angle at O?
(c) Calculate the sum of the angles of a regular hexagon.

5 In Fig. 5.4, ABCDEFGH is regular octagon.

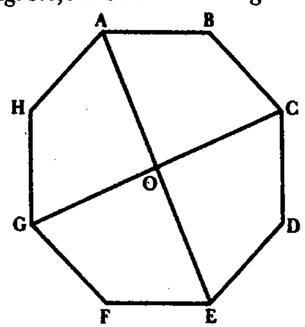


Fig. 5.4

(a) What is the order of symmetry of Fig. 5.4 about O?
(b) What is the size of each angle at O?
(c) Show that OÂB + AB̂C + OĈB = 270°.
(d) Hence find the sum of the angles of ABCDEFGH.
(e) Hence find the size of each angle of a regular octagon.

6 Fig. 5.5 shows polygons labelled (a), (b), (c), (d), (e).

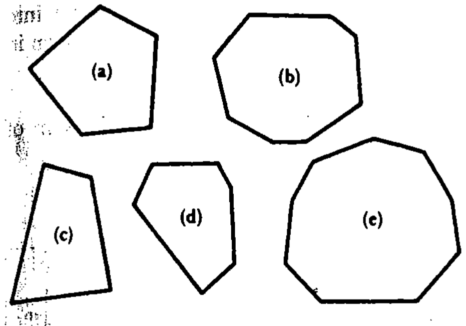


Fig. 5.5

Name each polygon correctly.

7 (a) Draw any large non-regular pentagon. Each side should be greater than 5 cm long.
(b) Use a protractor to measure the angles of your pentagon.
(c) Find the sum of the five angles.
(d) Compare your result in (c) with your classmates. What do you notice?

8 Repeat question 7 with any large non-regular hexagon.

9 Using your results in questions 7 and 8, copy and complete Table 5.2.

Table 5.2

polygon	sum of angles
triangle	a
quadrilateral	b
pentagon	c
hexagon	d

10 (a) Write your results in question 9 as a ratio $a: b: c: d$.
(b) Simplify the ratio as far as possible.
(c) Hence guess the sum of the angles of any heptagon.

Interior angles of a polygon

Look at the quadrilateral (a), the pentagon (b) and the hexagon (c) in Fig. 5.6.

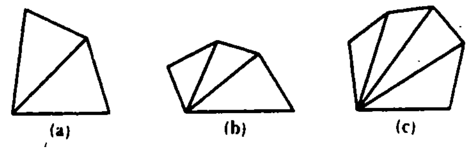


Fig. 5.6

In each polygon, one vertex is joined to all the other vertices. This divides the polygons into triangles. The number of triangles depends on the number of sides of the polygon. Table 5.3 shows the numbers of triangles for the polygons in Fig. 5.6.

Table 5.3

polygon	number of sides	number of triangles
quadrilateral	4	2
pentagon	5	3
hexagon	6	4

In each case, the number of triangles is 2 less than the number of sides. For a polygon with n sides there will be $n - 2$ triangles. The sum of the angles of a triangle is 180°. Thus,

the sum of the angles of an n-sided polygon = $(n - 2) \times 180°$

Notice that this formula is true for the polygons we know. In a triangle, $n = 3$.

Sum of angles $= (3 - 2) \times 180°$
$= 1 \times 180° = 180°$

In a quadrilateral, $n = 4$.

Sum of angles $= (4 - 2) \times 180°$
$= 2 \times 180° = 360°$

Example 1

Calculate the size of each angle of a regular heptagon.

A regular heptagon has 7 equal sides and 7 equal angles.
Use the formula,
sum of angles of polygon = $(n - 2) \times 180°$.
In a heptagon, $n = 7$.
Sum of angles = $(7 - 2) \times 180°$
$= 5 \times 180° = 900°$
There are 7 angles.
Each angle = $\frac{900°}{7} = 128\frac{4}{7}°$

Example 2

The sum of 7 of the angles of a nonagon is 1 000°. The other two angles are equal to each other. Calculate the sizes of the other two angles.

A nonagon has 9 sides and 9 angles.
Use the formula,
sum of angles of polygon = $(n - 2) \times 180°$.
In a nonagon, $n = 9$.
Sum of angles = $(9 - 2) \times 180°$
$= 7 \times 180° = 1\,260°$
Sum of 7 of the angles = 1 000°
Sum of 2 other angles = 1 260° − 1 000°
= 260°
Size of each angle = $\frac{260°}{2} = 130°$

Notice that $(n - 2) \times 180° = (n - 2) \times 2 \times 90°$
$= (2n - 4) \times 90°$
Hence **the sum of the angles of a polygon with n sides is $(2n - 4)$ right-angles.**

Example 3

The sum of the angles of a polygon is 1 980°. How many sides has the polygon?

Let the polygon have n sides.
Use the formula, sum of angles of polygon
$= (2n - 4)$ right angles
then $(2n - 4) \times 90° = 1\,980°$
$\Leftrightarrow \quad 2n - 4 = 22$
$\Leftrightarrow \quad 2n = 26$
$\Leftrightarrow \quad n = 13$
The polygon has 13 sides.

Exercise 5b

1 (a) In Fig. 5.7, O is any point inside each polygon. Straight lines join O to the vertices of each polygon. This divides the polygons into triangles.

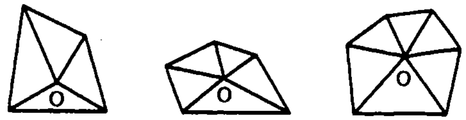


Fig. 5.7

Draw a heptagon (7 sides) and an octagon (8 sides). Divide these into triangles in the same way as shown in Fig. 5.7.

(b) Copy and complete Table 5.4.

(c) Hence find a formula for the sum of the angles of an n-sided polygon.

Table 5.4

polygon	number of sides	number of triangles	sum of angles at O	sum of angles of polygon
quadrilateral	4	4	360°	4 × 180° − 360°
pentagon	5	5	360°	5 × 180° − 360°
hexagon				
heptagon				
octagon				
n-gon	n			

2 Calculate the size of each angle of (a) a regular hexagon, (b) a regular decagon.

3 Fig. 5.8 is a *regular pentagram*. A regular pentagram has a regular pentagon at its centre. Calculate each angle in the pentagram.

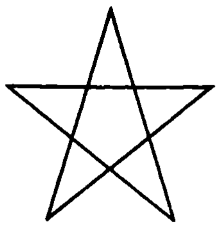


Fig. 5.8

4 Fig. 5.9 shows a pattern made with squares and regular octagons.

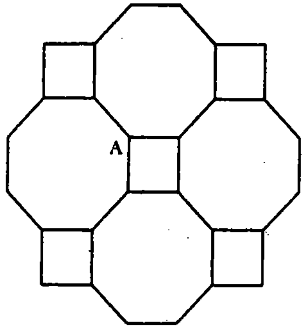


Fig. 5.9

Find the sizes of the three angles at point A, (a) from knowledge of the sum of the angles at a point, (b) by calculating the sizes of the angles of a regular octagon using the formula: $(n - 2) \times 180°$.

5 The pattern in Fig. 5.10 is made from four regular pentagons.

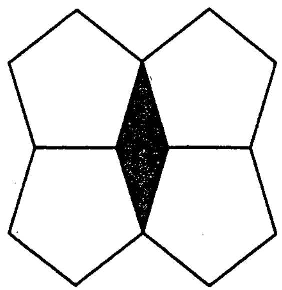


Fig. 5.10

(a) What kind of quadrilateral is shown shaded?

(b) Calculate the sizes of the four angles of the shaded quadrilateral.

6 In Fig. 5.11, first find the value of x, then find the unknown angles in each polygon.

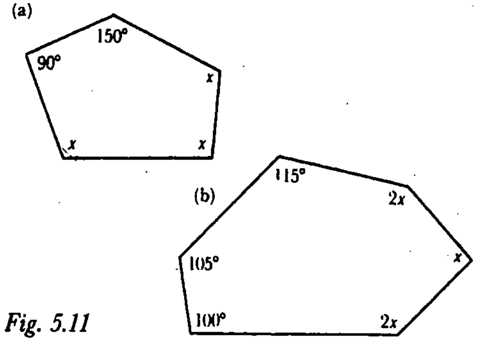


Fig. 5.11

7 The sum of seven of the angles of a decagon is 1 170°. The other three angles are all equal to each other. Calculate the sizes of the other three angles.

8 If the angles of a pentagon could be x, $2x$, $3x$, $4x$ and $5x$, what would be the value of x? Calculate the size of the largest angle. What do you notice? Sketch the pentagon. What type of shape is it?

9 How many sides has a polygon if the sum of its angles is (a) 3 240°, (b) 2 340°?

10 (a) How many sides has a polygon if the sum of its angles is 3 960°?
(b) If the polygon is regular, what is the size of each angle?

Exterior angles of a polygon

Fig. 5.12 represents an n-sided polygon with each side produced.

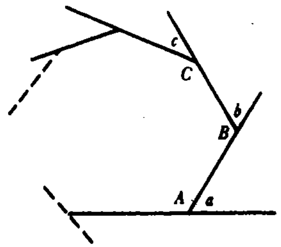


Fig. 5.12

The interior angles of the polygon are A, B, C, ... and the exterior angles are a, b, c, ... It follows that
$(A + a) = 180°$ (*straight angle*)
and $(B + b) = 180°$
etc.

Since there are n sides and n vertices

$$(A + a) + (B + b) + \ldots = n \times 180°$$
$$\Leftrightarrow (A + B + \ldots) + (a + b + \ldots) = n \times 180°$$

But $(A + B + \ldots) = (n - 2) \times 180°$

Hence, by subtraction,

$$(a + b + c + \ldots) = n \times 180° - (n - 2) \times 180°$$
$$= 180n° - 180n° + 360°$$
$$= 360°$$

The sum of the exterior angles of any polygon is 360°.

Example 4

Calculate the interior angles of a regular decagon (10 sides).

A decagon has 10 sides, 10 interior angles and 10 exterior angles.
The 10 exterior angles add up to 360°.
Since the polygon is regular the exterior angles are equal.

$$\text{Each exterior angle} = \frac{360°}{10} = 36°$$
$$\Leftrightarrow \text{Each interior angle} = 180° - 36° = 144°$$

Compare the working of Example 4 with your calculation for question 2(b) in Exercise 5b.

Example 5

How many sides has a regular polygon if each interior angle is 135°?

$$\text{Each interior angle} = 135°$$
$$\Leftrightarrow \text{Each exterior angle} = 180° - 135° = 45°$$

The sum of the exterior angles = 360°

$$\text{Hence the number of exterior angles} = \frac{360°}{45°} = 8$$

The polygon has 8 sides (since it has 8 exterior angles).

Exercise 5c

1 Use the method of Example 4 to calculate the interior angles of regular polygons with (a) 9, (b) 12, (c) 20 sides.

2 Use the method of Example 5 to find the number of sides that a regular polygon has if its interior angles are (a) 144°, (b) 168°, (c) 156°.

3 The angles of a pentagon are $2x°$, $3x°$, $4x°$, $5x°$, $6x°$.
(a) Calculate x.
(b) Hence calculate the angles of the pentagon.

4 Calculate x in Fig. 5.13 (a) and (b).

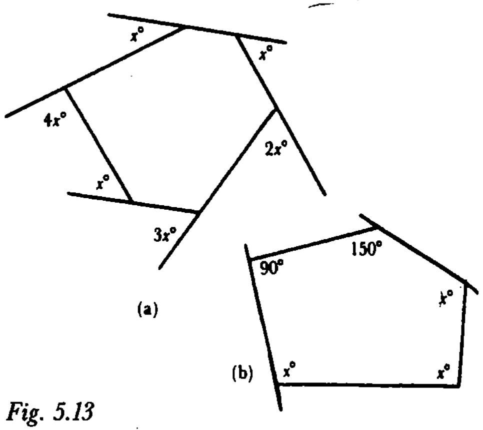


Fig. 5.13

5 In Fig. 5.14, AB, BC, CD are three sides of a regular pentagon. Calculate $B\hat{X}C$.

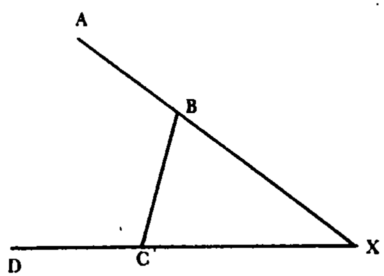


Fig. 5.14

Formal geometry (Optional section)

Some geometrical facts are more important than others. These basic facts are called **theorems**. Theorems form the foundations upon which **formal geometry** is built. The following section shows how the proof of the theorem that the sum of the angles of a triangle is 180° leads to the proof of other angle properties of triangles and polygons.

Interior and exterior angles of triangles and polygons

Theorem

The sum of the angles of a triangle is 180°.

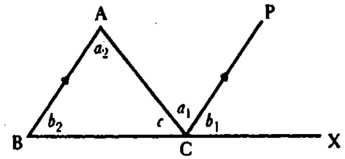


Fig. 5.15

Given: any △ABC.
To prove: $\hat{A} + \hat{B} + \hat{C} = 180°$.
Construction: Produce BC to a point X. Draw CP parallel to BA.
Proof:
With the lettering of Fig. 5.15,

$a_1 = a_2$ (*alternate angles*)
$b_1 = b_2$ (*corresponding angles*)
$c + a_1 + b_1 = 180°$ ($B\hat{C}X$ *is a straight angle*)
$\therefore c + a_2 + b_2 = 180°$
$\therefore A\hat{C}B + \hat{A} + \hat{B} = 180°$
$\therefore \hat{A} + \hat{B} + \hat{C} = 180°$

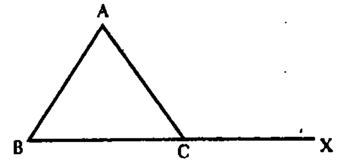
Notice how the proof of the theorem is set out:
1st: A **general statement**, in words, of the theorem.
2nd: The **diagram** or figure with suitable lettering.
3rd: A statement of what is **given**, using the letters of the diagram.
4th: A statement of what is **to be proved** in terms of the letters of the diagram.
5th: Any **constructions**, where necessary. Sometimes constructions are added to the diagram to make the proof possible. Often there is no need for construction.
6th: The **proof** in the form of a reasoned argument which shows that the theorem is true for any general case. Where necessary, reasons for statements in the proof should be given in brackets. The last line of the proof should be the same as the 'to prove' line. The symbol ∴ is short for 'therefore'.

Theorem

The exterior angle of a triangle is equal to the sum of the opposite interior angles.

Given: any △ABC with BC produced to X.
To prove: $A\hat{C}X = \hat{A} + \hat{B}$.

Fig. 5.16



Proof:
With the lettering of Fig. 5.16,

$A\hat{C}X + A\hat{C}B = 180°$ (*$B\hat{C}X$ is a straight angle*)
$\hat{A} + \hat{B} + A\hat{C}B = 180°$ (*angle sum of △*)
$\therefore A\hat{C}X = \hat{A} + \hat{B}$ (= *180° − $A\hat{C}B$*)

Notice that since the theorem that the sum of the angles of a triangle is 180° has already been proved, it can be used in the proof of the above theorem.

Theorem

The sum of the interior angles of any n-sided convex polygon is $(2n - 4)$ right angles.

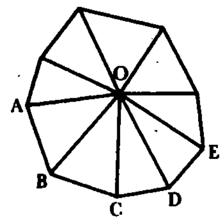


Fig. 5.17

Given: any convex polygon* ABCDE ... with n sides.
To prove: $\hat{A} + \hat{B} + \hat{C} + \ldots = (2n - 4)$ right angles.
Construction: Join the vertices A, B, C, ... to any point O inside the polygon.
Proof:
By construction there are n triangles. (*Polygon ABCDE ... has n sides*)

Sum of angles of 1 △ = 180° or 2 right angles
∴ Sum of angles of n △s = $2n$ right angles
Sum of angles at O = 360° or 4 right angles
(*sum of angles at a point = 360°*)

Sum of angles of polygon ABCDE ...
= Sum of angles of n $\triangle$s − sum of angles at O

$\therefore \hat{A} + \hat{B} + \hat{C} + \ldots = (2n - 4)$ right angles

* *Note:* A **convex polygon** does not contain any reflex angles. A polygon which contains reflex angles is called a **re-entrant polygon** (Fig. 5.18).

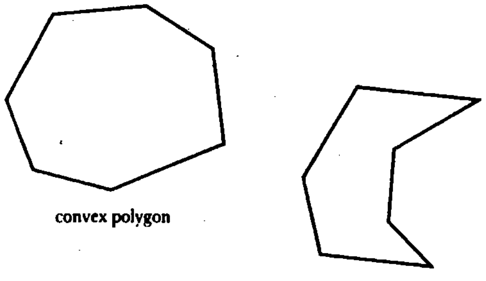


Fig. 5.18

Theorem

The sum of the exterior angles of any convex polygon is 4 right angles.

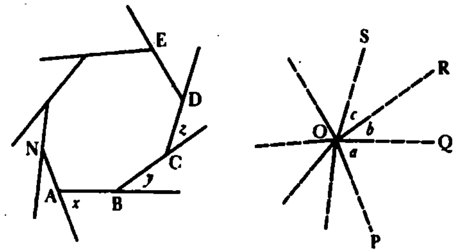


Fig. 5.19

Given: any convex polygon ABCDE...N with n sides, each side produced to give exterior angles $x, y, z, \ldots$.

To prove: $x + y + z + \ldots = 4$ right angles.

Construction: From any point O, draw lines OP, OQ, OR, OS, ... parallel to the sides of ABCDE...N in turn.

Proof: With the lettering of Fig. 5.19,

$a = x$ (OP∥NA *and* OQ∥AB)

Similarly, $b = y$, $c = z, \ldots$

But $a + b + c + \ldots = 4$ right angles (*angles at a point*)

$\therefore x + y + z + \ldots = 4$ right angles

Example 6

ABCDE is a regular pentagon. The sides AB *and* DC *are produced to meet at* X. *Show that* $\triangle$BDX *has two equal angles.*

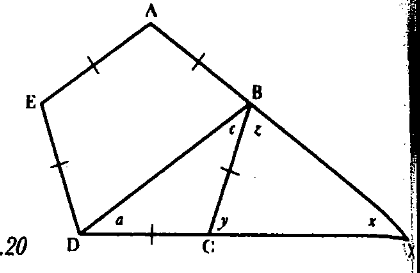


Fig. 5.20

With the lettering of Fig. 5.20:

In $\triangle$BXC,

$y = z$ (*regular polygon, ∴ equal exterior angles*)

$y = z = \dfrac{360°}{5}$ (*5 ext. angles add to 360°*)

$\therefore y = z = 72°$

$x + y + z = 180°$ (*angle sum of* $\triangle$)

$\therefore x = 180° - y - z$

$= 180° - 72° - 72° = 36°$

In $\triangle$BCD,

$y = a + c$ (*ext. angle of* $\triangle$)

But $a = c$ (*isosceles* $\triangle$)

$\therefore y = 2a$

$\therefore a = \frac{1}{2}y$

$= \frac{1}{2} \times 72° = 36°$

In $\triangle$BDX, $a = x = 36°$.

$\therefore$ $\triangle$BDX has two equal angles.

Example 7

In Fig. 5.21, side BC *of* $\triangle$ABC *is produced to* D. *The bisector of* $\mathrm{A\hat{C}D}$ *meets* BA *produced at* A. *Prove that* $\mathrm{A\hat{X}C} = \frac{1}{2}(\hat{A} - \hat{B})$.

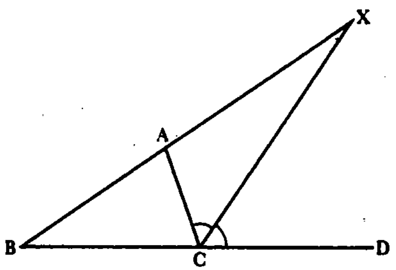


Fig. 5.21

Letter the angles of the given figure for convenience (Fig. 5.22).

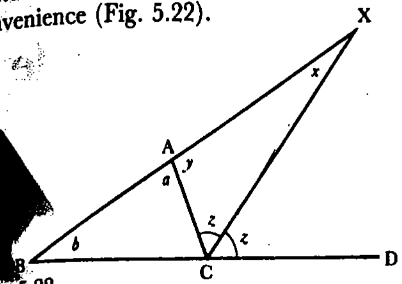


Fig. 5.22

Notice in Fig. 5.22 that since CX is the bisector of $\mathrm{A\hat{C}D}$, $\mathrm{A\hat{C}X} = \mathrm{D\hat{C}X} = z$.

In $\triangle$XAC,

$x + y + z = 180°$ (*angle sum of* $\triangle$)

$\therefore x = 180° - y - z$ (1)

At A, $a + y = 180°$ (*angles on str. line*)

$\therefore y = 180° - a$

In $\triangle$ABC,

$2z = a + b$ (*ext. angle of* $\triangle$)

$\therefore z = \frac{1}{2}a + \frac{1}{2}b$

Substitute for y and z in (1)

$x = 180° - (180° - a) - (\frac{1}{2}a + \frac{1}{2}b)$

$= 180° - 180° + a - \frac{1}{2}a - \frac{1}{2}b$

$= \frac{1}{2}a - \frac{1}{2}b = \frac{1}{2}(a - b)$

$\therefore \mathrm{A\hat{X}C} = \frac{1}{2}(\hat{A} - \hat{B})$.

When solving geometrical problems, always draw a large clear diagram. It is common to let small letters of the alphabet stand for the angles.

Exercise 5d

1 The angles of a triangle are x, $2x$ and $3x$. Find the value of x in degrees.

2 An isosceles triangle is such that each of the base angles is twice the vertical angle. Find the angles of the triangle.

3 In a right-angled triangle one of the acute angles is 20° greater than the other. Find the angles of the triangle.

4 The angles of a quadrilateral, taken in order, are x, $5x$, $4x$ and $2x$. Find these angles. Draw a rough sketch of the angles. What kind of quadrilateral is it?

5 Find the interior angles of a regular polygon which has (a) 6, (b) 10, (c) 20 sides.

6 Find to the nearest degree, the size of the angles of a regular heptagon (7 sides).

7 A regular polygon has angles of size 150°. How many sides has the polygon?

8 ABCDE is a regular pentagon. Find the angles of $\triangle$ADC.

9 Four angles of a pentagon are equal and the fifth is 60°. Find the equal angles and show that two sides of the pentagon are parallel.

10 In $\triangle$ABC, the side BC is produced to D. If the bisector of $\mathrm{A\hat{C}D}$ is parallel to AB, prove that two angles of $\triangle$ABC are equal.

11 In $\triangle$ABC, the bisectors of $\hat{B}$ and $\hat{C}$ meet at I. Prove that $\mathrm{B\hat{I}C} = 90° + \frac{1}{2}\hat{A}$.

12 In Fig. 5.23, what is the value of the angle marked m?

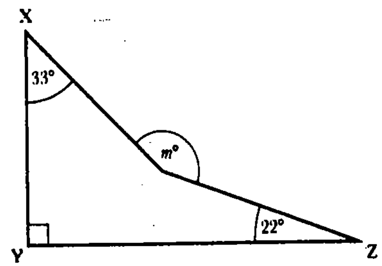


Fig. 5.23

13 In Fig. 5.24, BX is the bisector of $\mathrm{A\hat{B}C}$ and CX is the bisector of $\mathrm{A\hat{C}B}$. If $\hat{A} = 68°$, find the size of $\mathrm{B\hat{X}C}$.

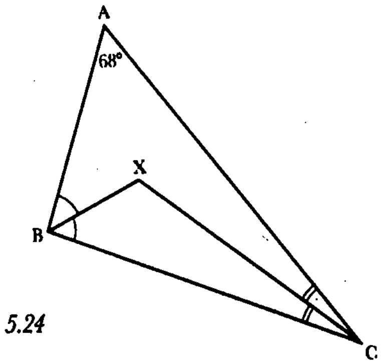


Fig. 5.24

14 (a) Prove that the sum of the angles of any triangle PQR is two right angles.
(b) In $\triangle$PQR, X is a point on QR such that $\mathrm{R\hat{P}X} = \hat{Q}$. Prove that $\mathrm{P\hat{X}R} = \mathrm{Q\hat{P}R}$.

Chapter 6

Solving triangles (2) Tangent of an angle

Tangent of an angle

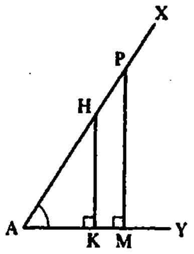


Fig. 6.1

Fig. 6.1 shows an angle A with arms AX and AY. H and P are any two points on AX. HK and PM meet AY perpendicularly at K and M. Thus Δs AHK and APM are equiangular and therefore similar.

Thus $\frac{HK}{KA} = \frac{PM}{MA}$

If any number of points are taken on AX and the perpendiculars drawn, the ratio $\frac{\text{length of perpendicular}}{\text{length of base-line}}$ is the same for each.

Hence the value of the ratio $\frac{HK}{KA}$ depends only on the size of $\hat{A}$.

The ratio $\frac{HK}{KA}$ is called the **tangent** of the angle A. This is usually shortened to **tan A.** Fig. 6.2 shows ΔAHK in various positions.

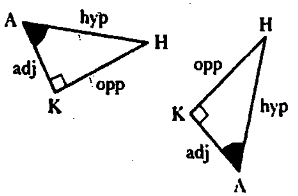


Fig. 6.2

The sides of the triangle are as follows:

AH the hypotenuse,
HK the side opposite to $\hat{A}$,
KA the side adjacent to $\hat{A}$.

These are abbreviated to **hyp, opp, adj** respectively, so that

$$\tan A = \frac{\text{opp}}{\text{adj}}$$

Finding the tangent of an angle by measurement

In Fig. 6.3, $X\hat{A}Y = 41°$. Perpendiculars BP, CQ, DR, ES have been drawn so that AP = 3 cm, AQ = 4 cm, AR = 5 cm and AS = 6 cm.

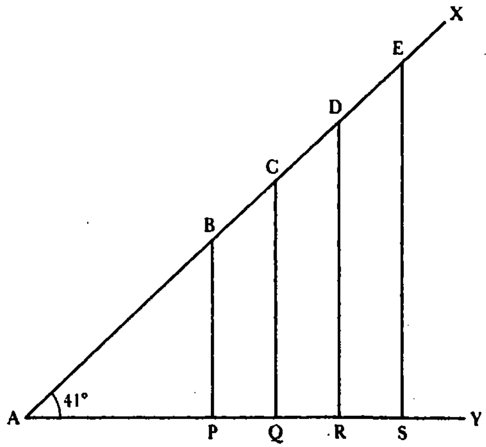


Fig. 6.3

By measurement, BP = 2,6 cm, CQ = 3,5 cm, DR = 4,3 cm and ES = 5,2 cm. Use a ruler to check these measurements.

Hence $\frac{BP}{PA} = \frac{2,6}{3} = 0,87$

$\frac{CQ}{QA} = \frac{3,5}{4} = 0,87$

$\frac{DR}{RA} = \frac{4,3}{5} = 0,86$

$\frac{ES}{SA} = \frac{5,2}{6} = 0,87$

The value of the ratio is roughly the same each time, i.e. tan 41° ≃ 0,87.

The working is made easier if the base-line (adj) is a convenient length such as 10 cm.

Exercise 6a (Class assignment)

1 Copy Fig. 6.3, making $X\hat{A}Y = 30°$, AP = 7 cm, PQ = QR = RS = 1 cm. Measure BP, CQ, DR, ES. Hence calculate the values of tan 30° by the above method.

2 Use the method of question 1 with $X\hat{A}Y = 51°$. Hence find four values of tan 51°.

Example 1

Find the value of tan 57° by drawing and measurement.

Fig. 6.4 is a scale drawing of the method.

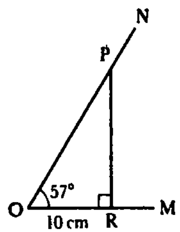


Fig. 6.4

Draw an angle MON of 57°. On OM, mark off OR equal to 10 cm. From R, draw a line perpendicular to OM to meet ON at P. Measure RP. It is found that RP = 15,4 cm (approx.).

$\tan 57° = \frac{PR}{RO} = \frac{15,4}{10} = 1,54$ (approx.)

Exercise 6b

Find the tangents of the following angles by drawing and measurement.

1 42° **2** 62° **3** 38° **4** 71° **5** 45°
6 27° **7** 77° **8** 14° **9** 33°

Example 2

Find by drawing and measurement the angle whose tangent is $\frac{3}{7}$.

The lengths of the opp and adj sides are to be in the ratio 3:7. Thus the lengths could be 3 cm and 7 cm, or 6 cm and 14 cm, and so on. The bigger the drawing, the better the chance of accurate measurement. Fig. 6.5 is a scale drawing of the required triangle.

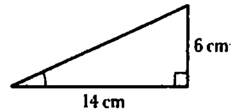


Fig. 6.5

By measurement, the angle whose tangent is $\frac{3}{7}$ is 23° (approx.).

Exercise 6c

Find by drawing and measurement the angles whose tangents are as follows.

1 $\frac{5}{9}$ **2** $\frac{2}{7}$ **3** $\frac{4}{3}$
4 $\frac{8}{5}$ **5** $\frac{11}{4}$ **6** $\frac{5}{8}$
7 $\frac{9}{10}$ **8** $\frac{10}{9}$ **9** $\frac{5}{7}$

Use of tangent of an angle

The use of tables for finding the tangent of an angle will be explained later in this chapter. Meanwhile, Table 6.1 gives the tangents of some chosen angles.

Table 6.1

angle A	tan A
25°	0,4663
30°	0,5774
35°	0,7002
40°	0,8391
45°	1,0000
50°	1,1918
55°	1,4281
60°	1,7321
65°	2,1445
70°	2,7475

The values in Table 6.1 are given correct to 4 decimal places.

Example 3

The angle of elevation of the top of a building is 25° from a point 70 m away on level ground. Calculate the height of the building.

In Fig. 6.6, HK represents the height of the building, AK is on level ground.

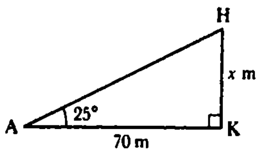


Fig. 6.6

$\frac{HK}{KA} = \tan 25°.$

Let HK be x m. KA = 70 m and, from Table 6.1, tan 25° = 0,4663.

Hence, $\frac{x}{70} = 0{,}4663$

$x = 0{,}4663 \times 70$
$= 4{,}663 \times 7$
$= 32{,}641$

However, the answer cannot be given to this **degree of accuracy.** The working depends on a value taken from four-figure tables. The fourth significant figure is only *approximate*, so that when 4,663 is multiplied by 7, the best that can be obtained is accuracy to 3 significant figures.

The height of the building is 32,6 m to 3 s.f.

Exercise 6d

Use the values in Table 6.1 in this exercise. Give all answers correct to 3 s.f.

Note: Angles of elevation are fully explained in Chapter 16 of Book 2.

1 Find the value of x in each of the triangles in Fig. 6.7.

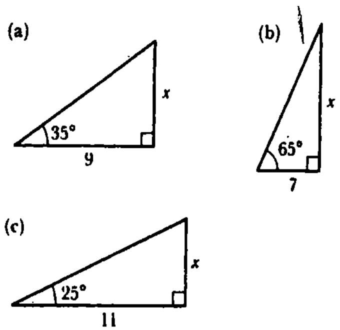


Fig. 6.7

2 Find the value of y in each of the triangles in Fig. 6.8.

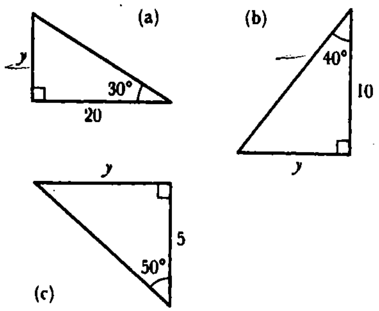


Fig. 6.8

3 Find the value of z in each of the triangles in Fig. 6.9.

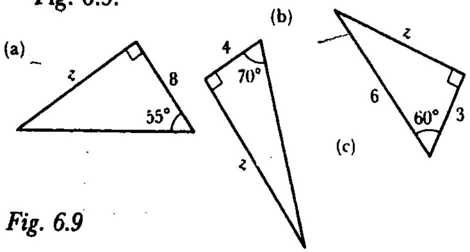


Fig. 6.9

4 When the angle of elevation of the sun is 45°, a student's shadow on level ground is 1,6 m long. Find the height of the student.

5 An aerial mast has a shadow 40 m long on level ground when the elevation of the sun is 20°. Calculate the height of the mast.

6 The angle of elevation of the top of a building from a point 80 m away on level ground is 25°. Calculate the height of the building.

Complement of an angle

In Exercise 6d, the given length was always *adjacent* to the given angle. The side to be found was always *opposite* the given angle.

Suppose, as in Fig. 6.10, the opposite side is given and the adjacent side is to be found.

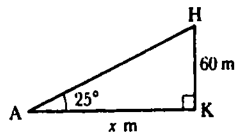


Fig. 6.10

In Fig. 6.10, Â = 25°, HK = 60 m and AK = x m. Then, as before

$\tan 25° = \frac{60}{x}$

Multiply both sides by x

$x \times \tan 25° = 60$

Divide both sides by tan 25°

$x = \frac{60}{\tan 25°}$
$= \frac{60}{0{,}4663}$

This involves tedious calculation. A different method, involving less demanding calculation, would be better.

Since ΔAHK is right-angled at K,
Ĥ = 90° − 25° = 65°

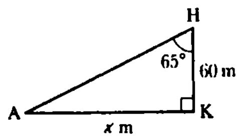


Fig. 6.11

With the data of Fig. 6.11,

$\frac{x}{60} = \tan 65°$

$x = 60 \tan 65°$
$= 60 \times 2{,}1445$
$= 128{,}67$
$= 129$ to 3 s.f.

Thus AK = 129 m.

65° is the **complement** of 25°, i.e. 65° + 25° = 90°. To avoid dividing by the tangent of an angle, find the complement of the angle and use its tangent.

Example 4

A post is standing vertically in horizontal ground such that 80 cm of the post is above ground. Find the length of its shadow when the elevation of the sun is 40°.

In Fig. 6.12, AB is the post and BC, d cm long, is its shadow.

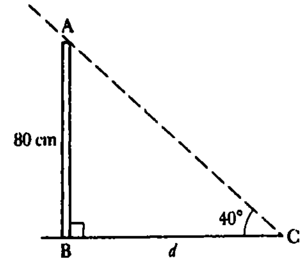


Fig. 6.12

If Ĉ = 40°, then Â = 90° − 40° = 50°

$\frac{BC}{AB} = \tan 50°$

$\frac{d}{80} = 1{,}1918$

$d = 1{,}1918 \times 80$
$= 95{,}344$
$= 95{,}3$ to 3 s.f.

The shadow is 95,3 cm long.

Exercise 6e

Use the values in Table 6.1 on page 51 in this exercise. Give all answers correct to 3 s.f.

Note: Angle of depression is fully defined in Chapter 16 of Book 2.

1 Find the value of a in each of the triangles in Fig. 6.13.

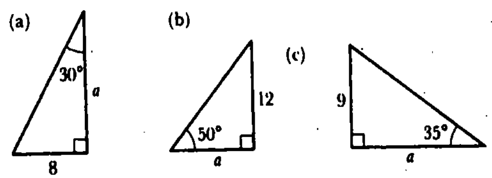


Fig. 6.13

2 Find the value of b in each of the triangles in Fig. 6.14.

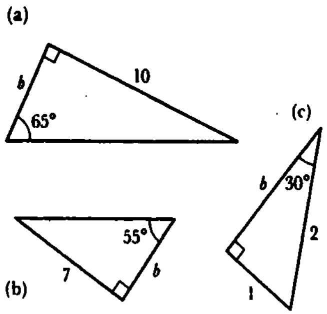


Fig. 6.14

3 Find the value of c in each of the triangles in Fig. 6.15.

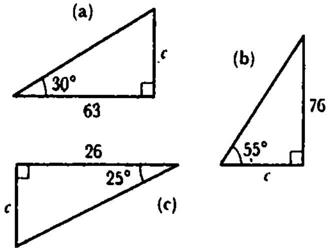


Fig. 6.15

4 Calculate the length of the shadow cast on level ground by a radio mast 90 m high when the elevation of the sun is 40°.

5 From a window 15 m up, the angle of depression of an object on the ground is 20°. Find the distance of the object from the base of the building.

6 A point A is 5 km due east of a point B. B is due north of a point C and A is on a bearing 025° from C. Calculate the distance between B and C.

Degrees and minutes

Angles are usually measured to the nearest degree. However, it is possible to calculate with angles which contain fractions of a degree. There are 60 **minutes** in 1 degree. (These are different minutes from those used for measuring time.)
In symbols 1° = 60′

Example 5

Find the value of (a) 15° 56′ + 8° 23′, (b) 26° 48′ − 15° 59′.

(a) 15° 56′ + 8° 23′ = 24° 19′

method: 56′ + 23′ = 79′ = 1° 19′
Write down 19′ and carry 1°

(b) 26° 48′ − 15° 59′ → 25° 108′ − 15° 59′ = 10° 49′

method: 59′ cannot be subtracted from 48′; write 26° as 25° and add 60′ to 48′ giving 25° 108′ in the top line.

With practice it is not necessary to write down every step when adding or subtracting degrees and minutes.

Example 6

(a) Change 26,8° to degrees and minutes.
(b) Express 53° 27′ as a decimal number of degrees.

(a) 26,8° = 26° + 0,8°
= 26° + 0,8 × 60′
= 26° + 48′
= 26° 48′

(b) 53° 27′ = 53° + $\frac{27}{60}$°
= 53° + 0,45°
= 53,45°

Thus
(a) to change degrees to minutes, multiply by 60,
(b) to change minutes to degrees, divide by 60.

Exercise 6f

1 Change the following to minutes.
(a) 2° (b) 3½° (c) 5°
(d) 8¼° (e) 22½° (f) 90°

2 Change the following to degrees.
(a) 180′ (b) 90′ (c) 240′
(d) 20′ (e) 600′ (f) 450′

3 Find the value of the following.
(a) 28° 22′ + 42° 31′
(b) 36° 42′ + 18° 53′
(c) 44° 43′ − 21° 18′
(d) 65° 11′ − 58° 32′
(e) 18° 44′ × 3
(f) 25° 52′ ÷ 4

4 Convert the following into minutes.
0,1°; 0,2°; 0,3°; 0,4°; 0,5°; 0,6°; 0,7°; 0,8°; 0,9°

5 Convert the following into decimal parts of a degree.
(a) 6′ (b) 48′ (c) 15′
(d) 21′ (e) 33′ (f) 57′

6 Change the following into degrees and minutes.
(a) 18,2° (b) 77,7° (c) 45,75°
(d) 67½° (e) 28⅔° (f) 32⅙°

7 Express the following as a decimal number of degrees.
(a) 51° 36′ (b) 13° 54′
(c) 32° 15′ (d) 80° 39′

8 Express the following as a decimal number of degrees, giving each answer to the nearest 0,1 of a degree.
(a) 47° 47′ (b) 8° 52′
(c) 38° 8′ (d) 63° 26′

Tangent tables

The tangent tables on page 283 can be used to find the tangents of angles from 0° to 90°. Table 6.2 gives three lines taken from the tangent table.

Notice the following:
1 The table gives the tangent of any angle from 0° to 90° in intervals of 0,1° or 6′.
2 Each tangent is given correct to 4 decimal places.
3 As angles increase towards 90°, the sizes of their tangents increase rapidly.
4 The *Differences* column gives increments in intervals of 1′. These are added as required. For example tan 32°32′ = 0,6379 (i.e. 6371 + 8 = 6379).

Table 6.2

θ	0′ 0,0°	6′ 0,1°	12′ 0,2°	18′ 0,3°	24′ 0,4°	30′ 0,5°	36′ 0,6°	42′ 0,7°	48′ 0,8°	54′ 0,9°	1′	2′	3′	4′	5′
											ADD Differences				
32	0,6249	6273	6297	6322	6346	6371	6395	6420	6445	6469	4	8	12	16	20
59	1,6643	6709	6775	6842	6909	6977	7045	7113	7182	7251	11	23	34	45	56
71	2,9042	9208	9375	9544	9714	9887	0061	0237	0415	0595	29	58	87	116	145

Using a calculator

If you have a scientific calculator you can use it to find tangents of angles. Note that you must express angles in degrees. To convert minutes to degrees, divide by 60,
e.g:
tan32° 32′ = tan32$\frac{32}{60}$° = tan32,533 333 3°

Key	*Display*
32.5333333 tan	0.6378884

tan32° 32′ = 0,6379 to 4 d.p.

Exercise 6g (Oral or written)

Use the table on page 283 to find the tangents of the following. If you have a calculator, use it to check your answers.

1 13°	**2** 64°	**3** 35°
4 56°	**5** 74°	**6** 88°
7 23,1°	**8** 36° 6′	**9** 45,1°
10 32½°	**11** 42,5°	**12** 19° 30′
13 56° 12′	**14** 63,8°	**15** 18,3°
16 27,7°	**17** 48,6°	**18** 67° 24′
19 78,6°	**20** 78,8°	**21** 25,9°
22 87° 6′	**23** 87° 12′	**24** 87° 18′
25 68° 12′	**26** 39,4°	**27** 11,9°
28 71,9°	**29** 80° 24′	**30** 55,7°
31 85° 36′	**32** 81,8°	**33** 3,2°
34 12,5°	**35** 42° 42′	**36** 45° 54′

Example 7

Use tables to find the angles whose tangents are (*a*) 0,9556; (*b*) 0,6395; (*c*) $\frac{5}{3}$.

(a) Let the angle be A, then $\tan A = 0{,}9556$. Looking within the table entries, 0,9556 is opposite 43° and under 0,7°.
Thus $A = 43{,}7°$

(b) Let $\tan B = 0{,}6395$
The value 6395 appears in the 32° row under the 0,6° column.
Hence $B = 32{,}6°$ (or 32° 36′)

(c) Let $\tan C = \frac{5}{3} = 1{,}\mathit{6667}$ to 4 d.p.
In the table, $\tan 59° = 1{,}\mathit{6643}$
Compare the decimal fractions in italics:
$$6667 - 6643 = 24$$
In the differences column the value 23 (in the 2′ column) is closest to 24. Add 2′ to the 59°: $C = 59°\ 2'$

Part (c) can be done on a scientific calculator as follows.

Key	*Display*	*Comment*
5 ÷ 3 = SHIFT tan	59.036243	degrees
− 59 = × 60 =	2.1746081	minutes

In the first line above, the SHIFT tan sequence gives the angle which corresponds to the given tangent. This is shown as $\tan^{-1}$ on most calculators.
The second line of working converts the decimal part of the answer to minutes.
The final outcome is 59° 2′ to the nearest minute.

Exercise 6h

Use the table on page 283 to find the angles whose tangents are as follows. If you have a calculator, use it to check your answers.

1 0,9325	**2** 0,4452	**3** 0,5543
4 1,8807	**5** 2,3559	**6** 19,08
7 1,5697	**8** 0,8816	**9** 0,8847
10 2,1943	**11** 0,0524	**12** 1,7113
13 0,3581	**14** 3,5816	**15** 35,80
16 0,1022	**17** 10,20	**18** 13,95
19 $\frac{5}{8}$	**20** $\frac{3}{8}$	**21** $\frac{7}{8}$
22 $\frac{5}{2}$	**23** $\frac{2}{5}$	**24** $\frac{17}{9}$
25 0,4982	**26** 0,6835	**27** 2,7651
28 0,9408	**29** 0,9413	**30** 150

In geometry, capital letters are mostly used for naming points and angles; small letters often stand for the lengths of lines. Single Greek letters are often used for sizes of angles. Some of the most common Greek letters are α (alpha), β (beta), γ (gamma), δ (delta), θ (theta) and φ (phi).

Example 8

A cone is 6 cm *high and its vertical angle is* 54°. *Calculate the radius of its base.*

In Fig. 6.16, the **vertical angle** is the angle between opposite slant heights VA and VB.

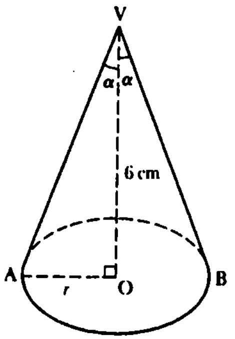


Fig. 6.16

Thus, with the lettering of the diagram, the vertical angle is 2α.
$2\alpha = 54°$
thus $\alpha = 27°$

In $\triangle$AVO, $\tan \alpha = \frac{r}{6}$
$$\Leftrightarrow \quad r = 6 \tan 27° = 6 \times 0{,}5095 = 3{,}057 = 3{,}06 \text{ to 3 s.f.}$$
The radius of the base of the cone is 3,06 cm.

Example 9

An aerial is $83\frac{1}{2}$ m *high. Calculate the angle of elevation of its top from a point* 120 m *away on level ground.*

Let the angle of elevation be θ.

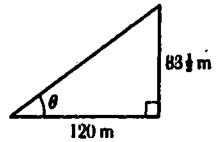


Fig. 6.17

From Fig. 6.17,
$$\tan\theta = \frac{83{,}5}{120} = \frac{8{,}35}{12} = \frac{4{,}175}{6} = 0{,}6958 \text{ to 4 s.f.}$$
From tables, $\theta = 34°\ 50'$ to nearest minute.
The angle of elevation is approximately 34° 50′.

Exercise 6i

Give all calculated lengths correct to 2 significant figures. Give all calculated angles correct to the nearest minute.
Note: Angle of elevation is fully defined in Chapter 16 of Book 2.

1 Calculate the lengths marked x in the triangles shown in Fig. 6.18, all lengths being in metres.

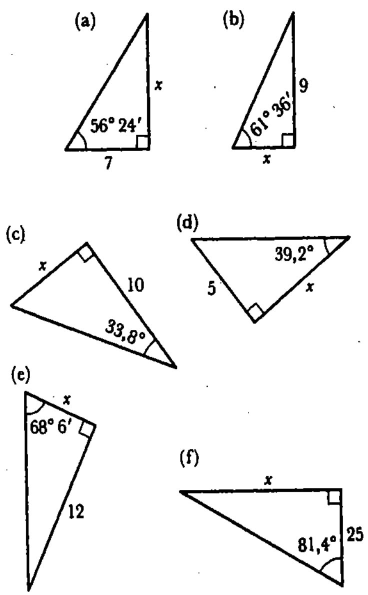


Fig. 6.18

2 Calculate the angles marked θ in the triangles in Fig. 6.19.

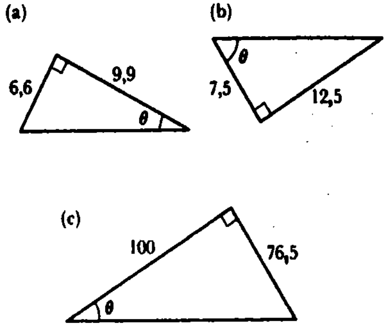


Fig. 6.19

3 Calculate the angles marked α and β in the triangles in Fig. 6.20.

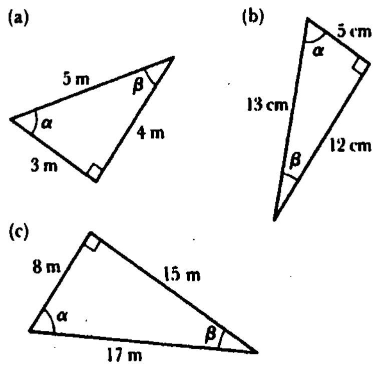


Fig. 6.20

4 A cone is 8 cm high and its vertical angle is 62°. Find the diameter of its base.

5 An isosceles triangle has a vertical angle of 116°, and its base is 8 cm long. Calculate its height.

6 Find the angle of elevation of the top of a flag-pole 31,9 m high, from a point 55 m away on level ground.

7 The gradient of a road is 1 (vertically) in $4\frac{1}{2}$ (horizontally). See Fig. 6.21.

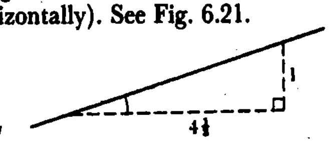


Fig. 6.21

Find the angle that the road makes with the horizontal.

8 From a point on level ground 40 m away, the angle of elevation of the top of a tree is 32½°. Calculate the height of the tree.

9 In question 8, if the tree had been 21,6 m high, what would have been the angle of elevation?

10 A rectangle has sides of length 2,2 m and 8 m. Calculate the angle between a diagonal and a longer side.

11 A student travels 8 km north and then 5 km east. What is then her bearing from her starting point?

12 Find the angle of elevation of the sun when a tower 93 m high has a shadow 62 m long.

13 The roof of a round hut 3,6 m in diameter rises symmetrically to a vertex. If the roof slopes at 48° to the horizontal, calculate the height of the vertex above the top of the hut's wall.

14 In Fig. 6.22 O is the centre of the circle.

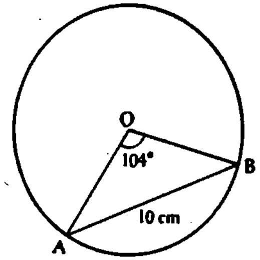


Fig. 6.22

Calculate the perpendicular distance of O from AB.

15 An aeroplane, coming in to land, passes over a point 1 km away from its landing place on level ground. If its angle of elevation is 15°, calculate the height of the plane in metres.

16 From a point 100 m from the foot of a building, the angle of elevation of the top of the building is 18° 42′. Find the height of the building.

17 Fig. 6.23 shows how a surveyor finds the width of a river.

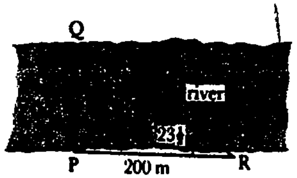


Fig. 6.23

He places a stone at P on one bank directly opposite a post Q on the other bank. From P he walks 200 m along the bank to R. He finds that $P\hat{R}Q = 23\frac{1}{2}°$. Calculate the width of the river.

18

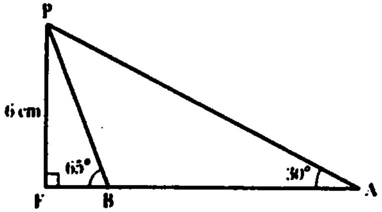


Fig. 6.24

Given the data of Fig. 6.24, calculate the length of AB. (*Hint*: first find FA, then FB, and subtract.)

19 A flag-pole is 20 m high. The angle of elevation of its top from a point A on level ground is 37°. From another point B, in line with A and the foot of the pole, the angle of elevation is 52°. Calculate the distance AB. (*Hint*: make a diagram like that of Fig 6.24.)

20 Fig. 6.25 represents a football player, P, kicking at goal AB.

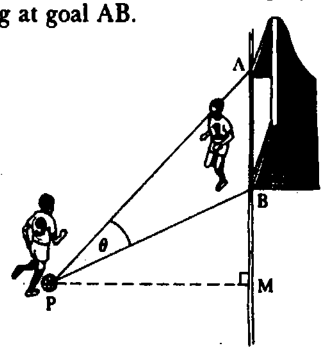


Fig. 6.25

If AB = 7,2 m, BM = 4,8 m, PM = 12 m, find the angle θ which the goal subtends at P. (*Hint*: find the difference between $A\hat{P}M$ and $B\hat{P}M$.)

Chapter 7

Matrices (1)

Matrices

It is quite common to store information in lists and tables. For example, Table 7.1 shows the amounts of bread, sugar and milk used by the Moyo and Phiri families in one week.

Table 7.1

	Moyo family	Phiri family
bread (loaves)	16	15
sugar (kg)	4	5½
milk (bottles)	22	20

The numbers in Table 7.1 can also be written as a **matrix:**

$$\begin{pmatrix} 16 & 15 \\ 4 & 5\frac{1}{2} \\ 22 & 20 \end{pmatrix}$$

A matrix (plural *matrices*) is simply a set of numbers or elements arranged in a rectangular pattern. In the matrix above there are 3 **rows** and 2 **columns**; we say that the **order** of the matrix is **3 by 2**. The order of the matrix gives its size in terms of rows and columns. The number of rows is always written first.

The following are other examples of matrices:

$$\begin{pmatrix} 1 & 12 & 1 \\ 3 & -1 & 2 \end{pmatrix} \quad \begin{pmatrix} 4 \\ 7 \end{pmatrix} \quad (5;\ 7;\ 6) \quad \begin{pmatrix} 2 & 4 \\ 5 & 0 \end{pmatrix}$$

(*a*) (*b*) (*c*) (*d*)

(*a*) is a 2 × 3 matrix
(*b*) is a 2 × 1 **column matrix**
(*c*) is a 1 × 3 **row matrix**
(*d*) is a 2 × 2 **square matrix** of order 2

Notice that the elements of a row matrix are often separated by semi-colons.

Exercise 7a

1 Table 7.2 shows the amounts of bread, sugar and milk used by the Moyo and Phiri families in the next week.

Table 7.2

	Moyo family	Phiri family
bread (loaves)	15	10
sugar (kg)	3½	4½
milk (bottles)	18	20

(a) Write the numbers in Table 7.2 as a matrix.
(b) Write Mrs Moyo's shopping list as a 3 × 1 column matrix.
(c) Write Mrs Phiri's shopping list as a 1 × 3 row matrix.
(d) Write the numbers of bottles of milk as a row matrix.

2 State the orders of the following matrices.

(a) $\begin{pmatrix} 1 & 0 & -1 & 2 & 4 \\ 2 & 2 & 1 & -3 & 1 \end{pmatrix}$ (b) $\begin{pmatrix} x \\ y \end{pmatrix}$

(c) $\begin{pmatrix} x & 1 & 0 \\ y & 2 & 1 \\ z & 3 & 0 \end{pmatrix}$ (d) $\begin{pmatrix} 8 & 2 \\ 9 & -3 \\ 3 & 1 \end{pmatrix}$

(e) (4; 8; 10; −5) (f) $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

3 Write down examples of
(a) any 3 × 5 matrix,
(b) any 2 × 4 matrix,
(c) any 4 × 2 matrix,
(d) any 4 × 4 matrix.

4 Write down any square matrix of order
(a) 1, (b) 2, (c) 3, (d) 4.

5 How many elements are in a matrix of order
(a) 2×3, (b) 3×2, (c) 4 by 3,
(d) m by n, (e) x by x, (f) 1×4?

6 Given the matrix $\begin{pmatrix} a & b & c & d \\ e & f & g & h \end{pmatrix}$ name the element in the
(a) first row, first column,
(b) second row, third column,
(c) second row, second column,
(d) first row, last column.

7 $\begin{pmatrix} 9 & s & t \\ x & 0 & 5 \\ 1 & y & -2 \\ -1 & 8 & k \end{pmatrix}$

Given the above matrix, name the row and column in which the following elements appear.
(a) 5 (b) -1 (c) s (d) k
(e) 1 (f) 0 (g) t (h) y

8 Two used-car dealers have the following cars for sale: *Mike's Motors* has 11 Peugeots, 3 Fords and 5 VWs; *Pete's Cars* has 8 Peugeots and 2 VWs only. Show this information in a 2×3 matrix.

Addition and subtraction

The numbers in Tables 7.1 and 7.2 give the following matrices:

$$\begin{pmatrix} 16 & 15 \\ 4 & 5\frac{1}{2} \\ 22 & 20 \end{pmatrix} \text{ and } \begin{pmatrix} 15 & 10 \\ 3\frac{1}{2} & 4\frac{1}{2} \\ 18 & 20 \end{pmatrix}$$

To find the total amounts of food used by the families in the two weeks, add each number in one matrix to the corresponding number in the other matrix:

$$\begin{pmatrix} 16 & 15 \\ 4 & 5\frac{1}{2} \\ 22 & 20 \end{pmatrix} + \begin{pmatrix} 15 & 10 \\ 3\frac{1}{2} & 4\frac{1}{2} \\ 18 & 20 \end{pmatrix}$$

$$= \begin{pmatrix} 16+15 & 15+10 \\ 4+3\frac{1}{2} & 5\frac{1}{2}+4\frac{1}{2} \\ 22+18 & 20+20 \end{pmatrix} = \begin{pmatrix} 31 & 25 \\ 7\frac{1}{2} & 10 \\ 40 & 40 \end{pmatrix}$$

Matrices can be added only if they are of the same order. The resulting matrix is also of that order.

Example 1

If $A = \begin{pmatrix} 2 & 3 & 0 \\ 4 & 2 & 1 \\ 3 & 1 & -2 \end{pmatrix}$ *and* $B = \begin{pmatrix} 3 & -1 & 2 \\ 1 & 3 & 1 \\ 4 & 2 & -1 \end{pmatrix}$

show that $A + B = B + A$.

$$A + B = \begin{pmatrix} 2 & 3 & 0 \\ 4 & 2 & 1 \\ 3 & 1 & -2 \end{pmatrix} + \begin{pmatrix} 3 & -1 & 2 \\ 1 & 3 & 1 \\ 4 & 2 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} 5 & 2 & 2 \\ 5 & 5 & 2 \\ 7 & 3 & -3 \end{pmatrix}$$

$$B + A = \begin{pmatrix} 3 & -1 & 2 \\ 1 & 3 & 1 \\ 4 & 2 & -1 \end{pmatrix} + \begin{pmatrix} 2 & 3 & 0 \\ 4 & 2 & 1 \\ 3 & 1 & -2 \end{pmatrix}$$

$$= \begin{pmatrix} 5 & 2 & 2 \\ 5 & 5 & 2 \\ 7 & 3 & -3 \end{pmatrix}$$

Hence $A + B = B + A$.

The method of subtraction follows the same pattern as that for addition. For example, if it is required to find how much more food the Moyo and Phiri families used during the first week, each number in the second matrix is taken from the corresponding number in the first matrix:

$$\begin{pmatrix} 16 & 15 \\ 4 & 5\frac{1}{2} \\ 22 & 20 \end{pmatrix} - \begin{pmatrix} 15 & 10 \\ 3\frac{1}{2} & 4\frac{1}{2} \\ 18 & 20 \end{pmatrix}$$

$$= \begin{pmatrix} 16-15 & 15-10 \\ 4-3\frac{1}{2} & 5\frac{1}{2}-4\frac{1}{2} \\ 22-18 & 20-20 \end{pmatrix} = \begin{pmatrix} 1 & 5 \\ \frac{1}{2} & 1 \\ 4 & 0 \end{pmatrix}$$

Example 2

Using the matrices A and B of Example 1, show that $A - B \neq B - A$.

$$A - B = \begin{pmatrix} 2 & 3 & 0 \\ 4 & 2 & 1 \\ 3 & 1 & -2 \end{pmatrix} - \begin{pmatrix} 3 & -1 & 2 \\ 1 & 3 & 1 \\ 4 & 2 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} -1 & 4 & -2 \\ 3 & -1 & 0 \\ -1 & -1 & -1 \end{pmatrix}$$

$$B - A = \begin{pmatrix} 3 & -1 & 2 \\ 1 & 3 & 1 \\ 4 & 2 & -1 \end{pmatrix} - \begin{pmatrix} 2 & 3 & 0 \\ 4 & 2 & 1 \\ 3 & 1 & -2 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & -4 & 2 \\ -3 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}$$

Hence $A - B \neq B - A$. [However, it can be seen that $A - B = -(B - A)$].

Exercise 7b

Combine the following matrices where possible.

1 $\begin{pmatrix} 1 & 2 \\ 3 & 0 \end{pmatrix} + \begin{pmatrix} 2 & -1 \\ 4 & 5 \end{pmatrix}$

2 $\begin{pmatrix} 3 & 1 \\ 2 & 0 \\ 4 & 7 \end{pmatrix} + \begin{pmatrix} 1 & 2 \\ 3 & 1 \\ 2 & -3 \end{pmatrix}$

3 $\begin{pmatrix} 3 & -1 & -2 \\ 2 & 3 & 1 \end{pmatrix} - \begin{pmatrix} 2 & -2 & 3 \\ 1 & 3 & 1 \end{pmatrix}$

4 $(2;\ 1;\ 3) - (3;\ 4)$

5 $\begin{pmatrix} 4 \\ 3 \\ 1 \end{pmatrix} - \begin{pmatrix} 6 \\ 1 \\ 3 \end{pmatrix}$

6 $\begin{pmatrix} 1{,}3 & 4{,}2 \\ 3{,}1 & 6{,}1 \end{pmatrix} + \begin{pmatrix} 7{,}1 & -3{,}2 \\ -2{,}9 & 4{,}3 \end{pmatrix}$

7 $\begin{pmatrix} 3 \\ 1 \end{pmatrix} + \begin{pmatrix} -1 \\ 3 \\ 2 \end{pmatrix}$

8 $\begin{pmatrix} 1 & 3 & 1 \\ 2 & 4 & 1 \end{pmatrix} + \begin{pmatrix} 3 & -2 & 1 & 5 \\ 4 & 2 & 3 & 2 \end{pmatrix}$

9 $\begin{pmatrix} 2 & 1 & 3 \\ 1 & 3 & 2 \\ 4 & 1 & 1 \end{pmatrix} + \begin{pmatrix} 7 & -1 & 0 \\ 2 & 3 & -2 \\ 1 & 4 & 6 \end{pmatrix} + \begin{pmatrix} 3 & -2 & -1 \\ 4 & -1 & 3 \\ 2 & 3 & 1 \end{pmatrix}$

10 $\begin{pmatrix} 3 & 2 \\ 1 & 0 \\ -1 & 1 \end{pmatrix} - \begin{pmatrix} 2 & 3 \\ -1 & -2 \\ -1 & 3 \end{pmatrix}$

11 $\begin{pmatrix} 2 & 1{,}3 \\ 3 & 2{,}1 \end{pmatrix} - \begin{pmatrix} 4{,}2 & 3 \\ 2{,}1 & 4 \end{pmatrix}$

12 $\begin{pmatrix} 2 \\ 7 \end{pmatrix} + \begin{pmatrix} 0 \\ -9 \end{pmatrix} - \begin{pmatrix} 5 \\ -4 \end{pmatrix}$

13 $(1;\ 5;\ 8) + (6;\ -4;\ 0) - \begin{pmatrix} 2 \\ 4 \\ 3 \end{pmatrix}$

14 $\begin{pmatrix} 2 & 8 \\ 9 & 3 \end{pmatrix} + \begin{pmatrix} 1 & 5 \\ 2 & 9 \end{pmatrix} - \begin{pmatrix} 3 & 7 \\ 6 & 5 \end{pmatrix}$

15 $\begin{pmatrix} 4 & -1 \\ 2 & 2 \end{pmatrix} - \begin{pmatrix} -2 & 9 \\ 6 & -8 \end{pmatrix} + \begin{pmatrix} -5 & 0 \\ 4 & 1 \end{pmatrix}$

Scalar multiplication

Example 3

If $A = \begin{pmatrix} 2 & x \\ y & 0 \end{pmatrix}$, *find* 3A.

$$3A = A + A + A$$

$$= \begin{pmatrix} 2 & x \\ y & 0 \end{pmatrix} + \begin{pmatrix} 2 & x \\ y & 0 \end{pmatrix} + \begin{pmatrix} 2 & x \\ y & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 4 & 2x \\ 2y & 0 \end{pmatrix} + \begin{pmatrix} 2 & x \\ y & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 6 & 3x \\ 3y & 0 \end{pmatrix}$$

In Example 3 the working can be shortened by multiplying each element of A by 3:

$$3A = 3\begin{pmatrix} 2 & x \\ y & 0 \end{pmatrix} = \begin{pmatrix} 3 \times 2 & 3 \times x \\ 3 \times y & 3 \times 0 \end{pmatrix}$$

$$= \begin{pmatrix} 6 & 3x \\ 3y & 0 \end{pmatrix}$$

The number multiplying the matrix is called a **scalar**. To multiply a matrix by a scalar, multiply each element of the matrix by the scalar.

Example 4

Find x if

$$5\begin{pmatrix} 4 & 3 \\ 7 & 3 \end{pmatrix} - 2\begin{pmatrix} 1 & 4 \\ 3 & 2 \end{pmatrix} = \begin{pmatrix} w & x \\ y & z \end{pmatrix}$$

$$5\begin{pmatrix} 4 & 3 \\ 7 & 3 \end{pmatrix} - 2\begin{pmatrix} 1 & 4 \\ 3 & 2 \end{pmatrix} = \begin{pmatrix} w & x \\ y & z \end{pmatrix}$$

$\Leftrightarrow \begin{pmatrix}20 & 15\\ 35 & 15\end{pmatrix} - \begin{pmatrix}2 & 8\\ 6 & 4\end{pmatrix} = \begin{pmatrix}w & x\\ y & z\end{pmatrix}$

$\Leftrightarrow \qquad \begin{pmatrix}18 & 7\\ 29 & 11\end{pmatrix} = \begin{pmatrix}w & x\\ y & z\end{pmatrix}$

Since the final matrices are equal, corresponding elements must be equal: $x = 7$.

Notice that two matrices are equal if they are of the same order and their corresponding elements are equal.

Exercise 7c

1 If $A = \begin{pmatrix}3\\ 2\end{pmatrix}$, $\quad B = \begin{pmatrix}2 & 1\\ 0 & 1\end{pmatrix}$,

$C = (1; 2; 8)$, $\quad D = \begin{pmatrix}6 & 8\\ 2 & 14\end{pmatrix}$

find
(a) 3A (b) 4B (c) $-2C$
(d) $\frac{1}{2}D$ (e) $D + 3B$ (f) $D - 3B$

2 Find n if

$n\begin{pmatrix}1 & 2\\ 1 & 0\end{pmatrix} + 2\begin{pmatrix}0 & 1\\ 3 & 5\end{pmatrix} = \begin{pmatrix}3 & 8\\ 9 & 10\end{pmatrix}$

3 Find the matrix M which satisfies

(a) $7M = 3\begin{pmatrix}-4 & 5\\ 0 & 1\end{pmatrix} + \begin{pmatrix}-2 & 6\\ 0 & 4\end{pmatrix}$

(b) $\begin{pmatrix}9 & 1\\ 2 & 6\end{pmatrix} + M = \begin{pmatrix}3 & 7\\ 6 & 8\end{pmatrix} - M$

4 Find x and y if

$5\begin{pmatrix}1 & 2\\ 0 & -1\end{pmatrix} - \begin{pmatrix}w & x\\ y & z\end{pmatrix} = 3\begin{pmatrix}3 & 4\\ -1 & 7\end{pmatrix}$

5 Find p and q if

$3\begin{pmatrix}p & 5\\ 4 & 8\end{pmatrix} - \begin{pmatrix}-2 & 7\\ 0 & 2\end{pmatrix} = 2\begin{pmatrix}-5 & 4\\ 6 & q\end{pmatrix}$

Multiplication

Row and column matrices

Look at the foods given in Table 7.1 on page 59. If bread costs 50c per loaf, sugar 60c per kg and milk 30c per bottle, then the row matrix (50; 60; 30) represents their respective costs. The total cost of the Moyos' food is the following matrix product:

$(50; 60; 30)\begin{pmatrix}16\\ 4\\ 22\end{pmatrix}$
$= (50 \times 16 + 60 \times 4 + 30 \times 22)$
$= (800 + 240 + 660)$
$= (1\ 700)$

The Moyos' food costs 1 700 cents, or $17,00.

Similarly, the total cost of the Phiris' food is the following product:

$(50; 60; 30)\begin{pmatrix}15\\ 5\frac{1}{2}\\ 20\end{pmatrix}$
$= (50 \times 15 + 60 \times 5\frac{1}{2} + 30 \times 20)$
$= (750 + 330 + 600)$
$= (1\ 680)$

The Phiris' food costs 1 680 cents, or $16,80.

The two calculations above can be combined as a single matrix product:

$(50; 60; 30)\begin{pmatrix}16 & 15\\ 4 & 5\frac{1}{2}\\ 22 & 20\end{pmatrix} = (1\ 700; 1\ 680)$

Notice that each element in the resulting matrix is made up by multiplying the elements of the row matrix by the elements of the columns in turn. It follows that for multiplication to be possible there must be as many columns in the first matrix as there are rows in the second matrix.

Example 5

Find the product $(2; 2; -4; 5)\begin{pmatrix}-3\\ 7\\ -9\\ -8\end{pmatrix}$

$(2; 2; -4; 5)\begin{pmatrix}-3\\ 7\\ -9\\ -8\end{pmatrix}$
$= 2 \times (-3) + 2 \times 7 + (-4) \times (-9) + 5 \times (-8)$
$= -6 + 14 + 36 - 40$
$= 4$

Example 6

A hotel has 8 *single rooms and* 14 *double rooms. The costs per night of single and double rooms are* $30 *and* $50 *respectively. Use a matrix method to show how much money the hotel makes per night when full.*

The row matrix $(30; 50) represents the costs. The column matrix $\begin{pmatrix}8\\ 14\end{pmatrix}$ represents the numbers of rooms.

Total income per night when full

$= \$(30; 50)\begin{pmatrix}8\\ 14\end{pmatrix}$
$= \$(30 \times 8 + 50 \times 14)$
$= \$(240 + 700)$
$= \$940$

Exercise 7d

1 Calculate the following products.

(a) $(2; 3)\begin{pmatrix}5\\ 6\end{pmatrix}$

(b) $(4; -3)\begin{pmatrix}-3\\ 9\end{pmatrix}$

(c) $(1; 2; 1)\begin{pmatrix}0\\ 0\\ 1\end{pmatrix}$

(d) $(2; 3;1)\begin{pmatrix}5\\ 6\\ 7\end{pmatrix}$

(e) $(1; 2; -1)\begin{pmatrix}\frac{1}{2}\\ 2\\ 2\frac{1}{2}\end{pmatrix}$

(f) $(6; 8; -3, \frac{1}{2})\begin{pmatrix}2\\ -3\\ -5\\ -6\end{pmatrix}$

2 Do the following multiplications, *where possible.*

(a) $(3; 6; 4)\begin{pmatrix}12\\ 10\\ 15\end{pmatrix}$

(b) $(3; 6; 4)\begin{pmatrix}11\\ 9\end{pmatrix}$

(c) $(1; 1; 0)\begin{pmatrix}1\\ 0\end{pmatrix}$

(d) $(1; 0)\begin{pmatrix}0\\ 1\end{pmatrix}$

3 One night the hotel in Example 6 had guests in only 5 single rooms and 3 double rooms. Use a matrix method to find the income of the hotel for that night.

4 Given that bread costs 60c per loaf, sugar 70c per kg and milk 35c per bottle, find the costs for the Moyo and Phiri families of the food stuffs given in Table 7.1 on page 59.

5 A cinema has 400 seats upstairs and 600 seats downstairs, each seat costing $3,00 and $1,80 respectively. Use a matrix method to compare the income of the cinema on a night when it is full with that when only 305 upstairs and 420 downstairs seats were sold.

General matrix multiplication

We have already seen that

$(50; 60; 30)\begin{pmatrix}16 & 15\\ 4 & 5\frac{1}{2}\\ 22 & 20\end{pmatrix} = (1\ 700; 1\ 680)$

If the row matrix is changed to (52; 70; 35), check that

$(52; 70; 35)\begin{pmatrix}16 & 15\\ 4 & 5\frac{1}{2}\\ 22 & 20\end{pmatrix} = (1\ 882; 1\ 865)$

These two results can be combined as a single matrix product:

$\begin{pmatrix}50 & 60 & 30\\ 52 & 70 & 35\end{pmatrix}\begin{pmatrix}16 & 15\\ 4 & 5\frac{1}{2}\\ 22 & 20\end{pmatrix} = \begin{pmatrix}1\ 700 & 1\ 680\\ 1\ 882 & 1\ 865\end{pmatrix}$

Notice, in this case that a 2×3 matrix multiplies a 3×2 matrix to give a 2×2 matrix product:

$(2 \times 3) \times (3 \times 2) \longrightarrow (2 \times 2)$

Example 7

Multiply $\begin{pmatrix}3 & 2\\ 4 & 3\\ 6 & 0\\ 3 & 1\end{pmatrix}$ *by* $\begin{pmatrix}6 & 0\\ 1 & -1\end{pmatrix}$.

$$\begin{pmatrix}3 & 2\\4 & 3\\6 & 0\\3 & 1\end{pmatrix}\begin{pmatrix}6 & 0\\1 & -1\end{pmatrix}=\begin{pmatrix}20 & -2\\27 & -3\\36 & 0\\19 & -1\end{pmatrix}$$

method:
The first element of the first row of the product is given by:

$$(3\quad 2)\begin{pmatrix}6\\1\end{pmatrix} = 3\times 6 + 2\times 1$$
$$= 18 + 2 = 20$$

The second element of the first row is given by:

$$(3\quad 2)\begin{pmatrix}0\\-1\end{pmatrix} = 3\times 0 + 2\times(-1)$$
$$= 0 + (-2) = -2$$

The first element of the second row is given by:

$$(4\quad 3)\begin{pmatrix}6\\1\end{pmatrix} = 4\times 6 + 3\times 1$$
$$= 24 + 3 = 27$$

... and so on.

Notice in Example 7 that a 4×2 matrix multiplies a 2×2 matrix to give a 4×2 product:

$$(4\times \boxed{2)\times(2} \times 2) \longrightarrow (4\times 2)$$

In order for it to be possible to multiply two matrices the first matrix must have the same number of columns that the second matrix has of rows. The product will have the same number of rows as the first matrix and the same number of columns as the second matrix.

Hence a $p \times q$ matrix will multiply a $q \times r$ matrix to give a $p \times r$ product:

$$(p\times \boxed{q)\times(q} \times r) \longrightarrow (p\times r)$$

Example 8

If $A = \begin{pmatrix}3 & 2\\1 & 4\end{pmatrix}$ *and* $B = \begin{pmatrix}2 & -1\\0 & 3\end{pmatrix}$

show that AB ≠ BA.

$$AB = \begin{pmatrix}3 & 2\\1 & 4\end{pmatrix}\begin{pmatrix}2 & -1\\0 & 3\end{pmatrix}$$
$$= \begin{pmatrix}3\times 2 + 2\times 0 & 3\times(-1) + 2\times 3\\1\times 2 + 4\times 0 & 1\times(-1) + 4\times 3\end{pmatrix}$$
$$= \begin{pmatrix}6 & 3\\2 & 11\end{pmatrix}$$
$$BA = \begin{pmatrix}2 & -1\\0 & 3\end{pmatrix}\begin{pmatrix}3 & 2\\1 & 4\end{pmatrix}$$
$$= \begin{pmatrix}2\times 3 + (-1)\times 1 & 2\times 2 + (-1)\times 4\\0\times 3 + \quad 3\times 1 & 0\times 2 + \quad 3\times 4\end{pmatrix}$$
$$= \begin{pmatrix}5 & 0\\3 & 12\end{pmatrix}$$

Hence AB ≠ BA.

Exercise 7e
Find the following matrix products.

1 $\begin{pmatrix}3 & 2\\1 & 4\end{pmatrix}\begin{pmatrix}2\\3\end{pmatrix}$

2 $\begin{pmatrix}4 & 1\\3 & 2\end{pmatrix}\begin{pmatrix}3 & 1\\2 & 1\end{pmatrix}$

3 $\begin{pmatrix}3 & 1\\2 & 1\end{pmatrix}\begin{pmatrix}4 & 1\\3 & 2\end{pmatrix}$

4 $\begin{pmatrix}11 & 2\\5 & 1\end{pmatrix}\begin{pmatrix}1 & -2\\-5 & 11\end{pmatrix}$

5 $\begin{pmatrix}2 & -1\\-3 & 6\end{pmatrix}\begin{pmatrix}6 & 1\\3 & 2\end{pmatrix}$

6 $\begin{pmatrix}2 & 3 & 1\\4 & 2 & 1\\3 & 1 & 1\end{pmatrix}\begin{pmatrix}2\\1\\3\end{pmatrix}$

7 $\begin{pmatrix}2 & 2\\0 & -4\end{pmatrix}\begin{pmatrix}2 & 2\\0 & -4\end{pmatrix}$

8 $\begin{pmatrix}1 & 2 & -1\\3 & 1 & -2\end{pmatrix}\begin{pmatrix}1 & 2\\-2 & 1\\-3 & 0\end{pmatrix}$

9 $\begin{pmatrix}1 & 2\\-2 & 1\end{pmatrix}\begin{pmatrix}1 & 2 & -1\\3 & 1 & -2\end{pmatrix}$

10 $\begin{pmatrix}1 & 0\\3 & 1\end{pmatrix}\begin{pmatrix}1 & 4 & -5\\1 & 7 & -2\end{pmatrix}$

Chapter 8

Indices and logarithms (1)

Laws of indices

The following laws of indices are true for all non-zero values of a, b and x.

1 $x^a \times x^b = x^{a+b}$
2 $x^a \div x^b = x^{a-b}$
3 $x^0 = 1$
4 $x^{-a} = \frac{1}{x^a}$

Example 1
Simplify (*a*) $10^2 \times 10^3$ (*b*) $22n^7 \div 2n^3$ (*c*) 19^0 (*d*) 5^{-2} (*e*) $2^3 \times (\frac{1}{6})^{-1}$ (*f*) $r \times r^0 \times r^{-5}$

(a) $10^2 \times 10^3 = 10^{2+3} = 10^5$ (Law 1)

Check:
$$10^2 \times 10^3 = (10\times 10)\times(10\times 10\times 10)$$
$$= 10\times 10\times 10\times 10\times 10$$
$$= 10^5$$

(b) $22n^7 \div 2n^3 = \frac{22}{2} \times n^{7-3}$ (Law 2)
$$= 11n^4$$

Check:
$$22n^7 \div 2n^3 = \frac{22\times n\times n\times n\times n\times n\times n\times n}{2\times n\times n\times n}$$
$$= 11\times n\times n\times n\times n$$
$$= 11\times n^4$$
$$= 11n^4$$

(c) $19^0 = 1$ (Law 3)

Check:
$19^0 = 19^{a-a}$ (since $a - a = 0$)
$= \frac{19^a}{19^a}$ (Law 2)
$= 1$ (num. and denom. are equal)

(d) $5^{-2} = \frac{1}{5^2} = \frac{1}{25}$ (Law 4)

Check:
$5^{-2} = 5^{0-2} = \frac{5^0}{5^2}$ (Law 2)
$= \frac{1}{5^2}$ (Law 3)
$= \frac{1}{25}$

(e) $2^3 \times (\frac{1}{6})^{-1} = 8 \times \frac{1}{\frac{1}{6}}$ (Law 4)
$= 8 \times 6 = 48$

(f) $r \times r^0 \times r^{-5} = r^{1+0+(-5)}$ (Law 1)
$= r^{-4} = \frac{1}{r^4}$

or

$r \times r^0 \times r^{-5} = r^1 \times 1 \times \frac{1}{r^5}$ (Laws 3 and 4)
$= \frac{1}{r^4}$

Exercise 8a (Revision)
Simplify the following.

1 $10^5 \times 10^4$	**2** $a^3 \times a^9$
3 $5y \times 4y^4$	**4** $2^2 \times 2^4$
5 $m^8 \div m^5$	**6** $c^7 \div c$
7 $\frac{24x^6}{8x^4}$	**8** $\frac{9\times 10^9}{3\times 10^3}$
9 2^0	**10** $6 \times z^0$
11 4^{-3}	**12** $3x^{-3}$
13 $(\frac{1}{4})^{-2}$	**14** $(\frac{2}{3})^{-1}$
15 $x^3 \div x^{-5}$	**16** $a^{-9} \div b^0$
17 $(3x)^{-3}$	**18** $9a^{-5} \times 4a^6$
19 $5x^2 \times 4x^0 \times 2x^{-6}$	
20 $15 \times 10^4 \div (3 \times 10^{-2})$	

Product of indices, $(x^a)^b$

Example 2

Simplify (*a*) $(x^2)^3$ (*b*) $(y^4)^2$ (*c*) $(z^3)^5$

(a) $(x^2)^3 = x^2 \times x^2 \times x^2$
$= x^{2+2+2} = x^{2\times3}$
$= x^6$

(b) $(y^4)^2 = y^4 \times y^4$
$= y^{4\times2}$
$= y^8$

(c) $(z^3)^5 = z^3 \times z^3 \times z^3 \times z^3 \times z^3$
$= z^{3\times5}$
$= z^{15}$

Notice that in each part of Example 2, the final index in the result is the **product** of the given indices, e.g. $(x^2)^3 = x^{2\times3} = x^6$.

In general, $(x^a)^b = x^{a\times b} = x^{ab}$.

Exercise 8b (Oral)

Simplify the following.

1 $(a^3)^2$	**2** $(b^2)^4$	**3** $(c^5)^3$
4 $(d^4)^3$	**5** $(e^3)^3$	**6** $(f^0)^8$
7 $(g^{-2})^5$	**8** $(h^4)^{-5}$	**9** $(5^2)^{-1}$
10 $(3^{-2})^{-2}$	**11** $(10^2)^7$	**12** $(2^{-3})^2$

Example 3

Simplify the following.

(*a*) $(-3d^3)^2$ (*b*) $-3(d^3)^2$
(*c*) $(-4g^5)^3$ (*d*) $(a^3b)^4$

(a) $(-3d^3)^2 = (-3d^3) \times (-3d^3)$
$= -3 \times -3 \times d^3 \times d^3$
$= +9d^6$

(b) $-3(d^3)^2 = -3 \times d^3 \times d^3$
$= -3d^6$

(c) $(-4g^5)^3 = (-4g^5) \times (-4g^5) \times (-4g^5)$
$= -64 \times g^{5\times3}$
$= -64g^{15}$

Or, more quickly:
$(-4g^5)^3 = (-4)^3 \times g^{5\times3} = -64g^{15}$

(d) $(a^3b)^4 = (a^3b^1)^4 = a^{3\times4} \times b^{1\times4}$
$= a^{12}b^4$

Notice the following in Example 3:

1 The power outside the bracket raises everything inside the bracket to that power.
2 A negative number raised to an odd power is negative. A negative number raised to an even power is positive.

Exercise 8c

Simplify the following.

1 $(3m^4)^2$	**2** $(2n^5)^3$	**3** $(4v^3)^2$
4 $4(v^3)^2$	**5** $-2(a^2)^3$	**6** $(-2b^2)^3$
7 $(-c^3)^2$	**8** $(-e^4)^3$	**9** $(-u^2)^5$
10 $-(c^5)^4$	**11** $(-d^5)^4$	**12** $(-f^4)^5$
13 $(mn^2)^4$	**14** $(a^2b)^3$	**15** $(x^2y^3)^4$
16 $(-u^3v^2)^4$	**17** $(5mn^3)^3$	**18** $(-4u^2v)^3$
19 $(-a^2m)^4$	**20** $-3(de^3)^4$	**21** $\dfrac{(-x^3)^2}{-x^4}$
22 $\dfrac{a^6}{(-a)^4}$	**23** $\dfrac{(-c)^2 \times c^4}{(-c)^5}$	
24 $\dfrac{-(d^2)^3}{d^4 \times (-d)}$		

Fractional indices, $x^{\frac{1}{a}}$ and $x^{\frac{a}{b}}$

$\sqrt{x}$ is short for the square root of x.

$\sqrt{x} \times \sqrt{x} = x$
Let $\sqrt{x} = x^p$
then, $x^p \times x^p = \sqrt{x} \times \sqrt{x} = x^1$
$x^{2p} = x^1$
Therefore $2p = 1$
and $p = \frac{1}{2}$
Thus $\sqrt{x} = x^{\frac{1}{2}}$

Similarly, $\sqrt[3]{x}$ is short for the **cube root** of x.
For example, $\sqrt[3]{8} = 2$ since $2 \times 2 \times 2 = 8$ and $\sqrt[3]{-27} = -3$ since $-3 \times -3 \times -3 = -27$.

$\sqrt[3]{x} \times \sqrt[3]{x} \times \sqrt[3]{x} = x$
Let $\sqrt[3]{x} = x^q$
then, $x^q \times x^q \times x^q = x^1$
$x^{3q} = x^1$
Therefore $3q = 1$
and $q = \frac{1}{3}$
Thus $\sqrt[3]{x} = x^{\frac{1}{3}}$

$x^{\frac{1}{2}} = \sqrt{x}$ and $x^{\frac{1}{3}} = \sqrt[3]{x}$.

In general, $x^{\frac{1}{a}} = \sqrt[a]{x}$.

Also $x^{\frac{2}{3}} = x^{2\times\frac{1}{3}} = (x^2)^{\frac{1}{3}} = \sqrt[3]{x^2}$
or $x^{\frac{2}{3}} = x^{\frac{1}{3}\times2} = (x^{\frac{1}{3}})^2 = (\sqrt[3]{x})^2$

In general, $x^{\frac{a}{b}} = \sqrt[b]{x^a}$ or $(\sqrt[b]{x})^a$.

The laws of indices are summarised in Table 8.1.

Table 8.1

1	$x^a \times x^b = x^{a+b}$
2	$x^a \div x^b = x^{a-b}$
3	$x^0 = 1$
4	$x^{-a} = \dfrac{1}{x^a}$
5	$(x^a)^b = x^{ab}$
6	$x^{\frac{1}{a}} = \sqrt[a]{x}$
7	$x^{\frac{a}{b}} = \sqrt[b]{x^a}$ or $(\sqrt[b]{x})^a$

Example 4

Simplify (*a*) $9^{\frac{1}{2}}$ (*b*) $8^{\frac{1}{3}}$ (*c*) $8^{-\frac{2}{3}}$ (*d*) $4 \times 4^{-\frac{1}{2}}$ (*e*) $(\frac{16}{81})^{-\frac{3}{4}}$

(a) $9^{\frac{1}{2}} = \sqrt{9} = \pm 3$

(b) $8^{\frac{1}{3}} = \sqrt[3]{8} = 2$

(c) $8^{-\frac{2}{3}} = \dfrac{1}{8^{\frac{2}{3}}} = \dfrac{1}{(\sqrt[3]{8})^2} = \dfrac{1}{2^2} = \frac{1}{4}$

(d) $4 \times 4^{-\frac{1}{2}} = 4^{1+(-\frac{1}{2})} = 4^{\frac{1}{2}} = \sqrt{4} = \pm 2$

(e) $(\frac{16}{81})^{-\frac{3}{4}} = \dfrac{1}{(\frac{16}{81})^{\frac{3}{4}}} = (\frac{81}{16})^{\frac{3}{4}} = (\sqrt[4]{\frac{81}{16}})^3 = (\pm\frac{3}{2})^3$
$= \pm\frac{27}{8}$

Notice the following in Example 4:

1 The square root of a positive number may be positive or negative.
$\sqrt{9} = \pm3$. ±3 means $+3$ or -3.
2 $\sqrt[4]{n}$ means the 4th root of n, i.e. the number which multiplied by itself 4 times gives n. For example $\sqrt[4]{16} = \pm2$ since $2 \times 2 \times 2 \times 2 = 16$ and $-2 \times -2 \times -2 \times -2 = 16$.
3 A number raised to a negative power is equivalent to the reciprocal of the number raised to a positive power of the same numerical value.
For example, $(\frac{3}{4})^{-2} = (\frac{4}{3})^2 = \frac{16}{9}$.

Example 5

Simplify (*a*) $(2\frac{1}{4})^{1\frac{1}{2}}$, (*b*) $(\frac{8}{50})^{-\frac{1}{2}}$, (*c*) $\sqrt{\dfrac{72a^3b^{-2}}{2a^5b^{-6}}}$.

(a) Change mixed numbers to fractions.
$(2\frac{1}{4})^{1\frac{1}{2}} = (\frac{9}{4})^{\frac{3}{2}} = (\sqrt{\frac{9}{4}})^3 = (\pm\frac{3}{2})^3 = \pm\frac{27}{8}$
$= \pm3\frac{3}{8}$

(b) Reduce the given fraction to its lowest terms.
$(\frac{8}{50})^{-\frac{1}{2}} = (\frac{4}{25})^{-\frac{1}{2}} = (\frac{25}{4})^{\frac{1}{2}} = \sqrt{\frac{25}{4}} = \pm\frac{5}{2} = \pm2\frac{1}{2}$

(c) $\sqrt{\dfrac{72a^3b^{-2}}{2a^5b^{-6}}} = \sqrt{36 \times a^{3-5} \times b^{-2-(-6)}}$
$= \sqrt{36 \times a^{-2} \times b^4}$
$= \sqrt{36} \times (a^{-2})^{\frac{1}{2}} \times (b^4)^{\frac{1}{2}}$
$= \pm6 \times a^{-1} \times b^2$
$= \pm\dfrac{6b^2}{a}$

Exercise 8d

Simplify the following.

1 $2a \times 3a^2$	**2** $2a \times (3a)^2$	**3** $(2a)^2 \times 3a$
4 $4^{\frac{1}{2}}$	**5** $27^{\frac{1}{3}}$	**6** $125^{\frac{1}{3}}$
7 $\sqrt[3]{2^6}$	**8** $8^{\frac{2}{3}}$	**9** 2^{-2}
10 3^{-3}	**11** $9^{\frac{1}{2}}$	**12** $9^{-\frac{1}{2}}$
13 $(25a^2)^{\frac{1}{2}}$	**14** $2a^{-1}$	**15** $(2a)^{-1}$
16 $4^{\frac{3}{2}}$	**17** $2^{-2} \times 2^3$	**18** $(2^2)^2$
19 10^{-2}	**20** $\sqrt{1\frac{9}{16}}$	**21** $3a^{-2}$
22 $(3a)^{-2}$	**23** $\sqrt{3^4}$	**24** $(a^2)^{-\frac{1}{2}}$
25 $(\frac{1}{5})^{-1}$	**26** $(\frac{1}{4})^{-\frac{1}{2}}$	**27** $3^{\frac{1}{2}} \times 3^{\frac{3}{2}}$
28 $(\frac{1}{27})^{-\frac{2}{3}}$	**29** $3^{\frac{1}{2}} \times 3^{-\frac{5}{2}}$	**30** $0{,}04^{\frac{1}{2}}$
31 $2a^{-1} \times 3a^2$	**32** $(2a)^{-1} \times 3a^2$	
33 $2a^{-1} \times (3a)^2$	**34** $16^{-\frac{3}{4}}$	
35 $2^{\frac{1}{2}} \times 2^{\frac{3}{2}}$	**36** $(2^6)^{-\frac{2}{3}}$	**37** $125^{-\frac{2}{3}}$
38 $3^x \times 3^{-x}$	**39** $16^{-\frac{1}{4}}$	**40** $0{,}027^{\frac{2}{3}}$
41 $2a \times 3a^{-2}$	**42** $2a \times (3a)^{-2}$	
43 $4^{-\frac{3}{2}}$	**44** $(\frac{8}{27})^{-\frac{1}{3}}$	**45** $\dfrac{1}{3^{-2}}$

46 2×2^{-3} **47** $\sqrt[3]{4^{1,5}}$
48 $3^{n-1} \times 3^{1-n}$ **49** $64^{-\frac{1}{3}}$
50 $\sqrt[3]{8a^{-6}}$ **51** $2x^{\frac{1}{2}} \times 3x^{-\frac{1}{2}}$
52 $0,125^{-\frac{1}{3}}$ **53** $(\frac{16}{9})^{-\frac{1}{2}}$ **54** $\sqrt[4]{16a^{-12}}$
55 $4a^3b \times 3ab^{-2}$ **56** $4a^3b \times (3ab)^{-2}$
57 $\sqrt{(125^2)^{-\frac{1}{3}}}$ **58** $\dfrac{75a^2b^{-2}}{5a^3b^{-3}}$
59 $(2x)^{\frac{1}{2}} \times (2x^3)^{\frac{1}{2}}$ **60** $(\frac{18}{32})^{-\frac{1}{2}}$

Example 6
Rewrite the following expressions with positive indices only.

(*a*) pq^{-2} (*b*) $\left(\frac{2a}{b}\right)^{-1}$

(a) $pq^{-2} = \dfrac{p}{q^2}$

Note that the index -2 refers to q only.

(b) $\left(\dfrac{2a}{b}\right)^{-1} = \dfrac{b}{2a}$

Exercise 8e
Rewrite the following expressions using positive indices only.

1 a^{-2} **2** b^{-1} **3** $c^{-\frac{1}{2}}$
4 xy^{-1} **5** $(xy)^{-1}$ **6** $a^{-2}b^3$
7 ab^{-3} **8** $(ab)^{-3}$ **9** $2x^{-\frac{1}{2}}$
10 $3y^{-\frac{1}{2}}$ **11** $\left(\dfrac{a}{3b}\right)^{-2}$ **12** $\left(\dfrac{1}{n}\right)^{-\frac{1}{2}}$

Example 7
Solve the following equations.
(*a*) $x^{\frac{1}{3}} = 4$ (*b*) $2a^{-\frac{1}{2}} = -14$ (*c*) $8^x = 32$

(a) *Either:* $x^{\frac{1}{3}} = 4$
$\sqrt[3]{x} = 4$
Take the cube of both sides
$(\sqrt[3]{x})^3 = 4^3 = 4 \times 4 \times 4$
$x = 64$
Or: $x^{\frac{1}{3}} = 4$
$(x^{\frac{1}{3}})^3 = 4^3$
$x^{\frac{1}{3} \times 3} = 64$
$x = 64$

(b) $2a^{-\frac{1}{2}} = -14$
Divide both sides by 2
$a^{-\frac{1}{2}} = -7$
$\dfrac{1}{a^{\frac{1}{2}}} = -7$
$a^{\frac{1}{2}} = -\frac{1}{7}$
$\sqrt{a} = -\frac{1}{7}$
Square both sides
$a = \frac{1}{49}$

(c) $8^x = 32$
Express 8 and 32 as powers of 2
$(2^3)^x = 2^5$
$2^{3x} = 2^5$
Equate the powers of 2
$3x = 5$
$x = \frac{5}{3} = 1\frac{2}{3}$

Exercise 8f
Solve the following equations.

1 $x^{\frac{1}{2}} = 2$ **2** $x^{\frac{1}{3}} = 3$ **3** $a^{-1} = 2$
4 $a^{-2} = 9$ **5** $2x^3 = 54$ **6** $x^{-\frac{1}{2}} = 5$
7 $n^{-\frac{1}{2}} = 9$ **8** $2r^{-3} = -16$ **9** $5x = 40x^{-\frac{1}{2}}$
10 $5^x = 25$ **11** $9^x = 27$ **12** $4^{c-1} = 64$

Logarithms

Table 8.2

Number	Power of 10
1 000	10^3
100	10^2
10	10^1
1	10^0

Table 8.2 shows that 1 000 is 10 to the power 3. This can also be expressed as: 'the **logarithm**, to the base 10, of 1 000 is 3', or, in abbreviated form, '$\log_{10}$ 1 000 = 3'. In this case, the logarithm of a number is the power to which 10 is raised to give that number. Thus 'logarithm' is another word for 'power'.

Logarithms can be in bases other than 10. For example, since $32 = 2^5$, then $\log_2 32 = 5$, i.e. in base 2, the logarithm of 32 is 5. Chapter ... shows how to simplify expressions containing logarithms in bases other than 10.

Logarithms used in calculations are normally expressed in base 10. Log 1 000 = 3 is taken to mean $\log_{10}$ 1 000 = 3. There is no need to write down the base (Table 8.3).

Table 8.3

Powers	Logarithms
$1000 = 10^3$	log 1 000 = 3
$100 = 10^2$	log 100 = 2
$10 = 10^1$	log 10 = 1
$1 = 10^0$	log 1 = 0

Since log 1 = 0 and log 10 = 1, the logarithm of any number between 1 and 10 must lie between 0 and 1. Logarithms of numbers are found with the help of tables. See page 279 near the back of this book. Logarithm tables were first published about 350 years ago by Henry Briggs following original work by John Napier. Table 8.4 shows a typical line from a set of 4-figure logarithm tables.

Example 8
Use Table 8.4 to find the logarithms of (*a*) 3,7 (*b*) 37 (*c*) 3 700.

(a) 3,7 lies between 1 and 10, therefore log 3,7 lies between 0 and 1.
log 3,7 = 0, 'something'
In Table 8.4, the figures 5682 appear immediately after 37.

log 3,7 = 0,5682

(b) 37 $= 3,7 \times 10$
$= 10^{0,5682} \times 10^1$
$= 10^{0,5682+1}$
$= 10^{1,5682}$
Hence, log 37 = 1,5682

(c) 3 700 $= 3,7 \times 1\,000$
$= 10^{0,5682} \times 10^3$
$= 10^{0,5682+3}$
$= 10^{3,5682}$
Hence, log 3 700 = 3,5682

Table 8.4

x	0	1	2	3	4	5	6	7	8	9	differences 1	2	3	4	5	6	7	8	9
37	5682	5694	5705	5717	5729	5740	5752	5763	5775	5786	1	2	3	5	6	7	8	9	10

In the logarithm of any number there are two parts, an integer (whole number) before the decimal comma and a fraction after the comma (Fig. 8.1).

3·5682
integer *decimal fraction*

Fig. 8.1

Look at the integers and the fractions of the results in Example 8. The logarithms of 3,7; 37 and 3 700 all have the same decimal fraction. Notice that in each case the integer is 1 less than the number of digits to the left of the decimal comma in the given number. For example, there are 8 digits before the decimal comma in 37 000 000, therefore the integer part of log 37 000 000 is 7.
Log 37 000 000 = 7,5682.

Example 9
Use Table 8.4 to find the logarithms of (*a*) 37,4, (*b*) 3,746.

(a) First, write down the integer.
Log 37,4 = 1, 'something'
To find the fractional part, go along the row beginning with 37 and stop at the column with **4** at the top. This gives the figures 5729.
Log 37,4 = 1,5729

(b) First, write down the integer.
Log 3,746 = 0, 'something'
To find the fractional part, find the figures along from 37 and under **4** as before (5729). Now find the number in the 'differences' column headed **6**. This number is 7. Add 7 to 5729. 5729 + 7 = 5736
Log 3,746 = 0,5736
(Note: when using tables, always make sure that the number taken from the differences column is in the same row as the rest of the figures.)

Exercise 8g

1 Use the logarithm tables on page 279 to check whether the following are correct or incorrect.

(a) log 7,7 = 0,8865
(b) log 2,2 = 0,3424
(c) log 2,0 = 0,3010
(d) log 9 = 0,9542
(e) log 5,6 = 0,7404
(f) log 1,4 = 0,1461
(g) log 6,45 = 0,8096
(h) log 4,44 = 0,6439
(i) log 8,03 = 0,9047
(j) log 9,57 = 0,9809
(k) log 5,02 = 0,7160
(l) log 3,09 = 0,4900
(m) log 3,142 = 0,4972
(n) log 7,153 = 0,8545
(o) log 3,704 = 0,5687
(p) log 2,603 = 0,4200
(q) log 9,989 = 0,9995
(r) log 8,008 = 0,9035

2 Write down the integer parts of the logarithms of the following numbers.

(a) 6,5 (b) 24,12 (c) 10,65
(d) 560 (e) 5 600 (f) 56
(g) 15 000 (h) 418 000 (i) 6,008
(j) 6 000 000 (k) 374 (l) 94,5

3 Use Table 8.4 to express the following as powers of 10.

(a) 3,75
(b) 37,5
(c) 375
(d) 3 750
(e) 37 500

4 Use logarithm tables to express the following as powers of 10.

(a) 9,13
(b) 9 130
(c) 913
(d) 9 130 000

5 Write down the logarithms of the following.

(a) 4,58; 458; 458 000; 45,8
(b) 7,06; 70 600 000; 70,6; 7 060

6 Express the following as powers of 10.

(a) 7,241 (b) 7,248 (c) 7,246 (d) 7,243

7 Write down the logarithms of the following.

(a) 5,136; 5 136; 51,36; 513 600
(b) 8,403; 840,3; 8 403 000; 84 030

8 Write down the logarithms of the following.

(a) 75,12 (b) 4 137 (c) 209,5
(d) 294 100 (e) 82 460 000 (f) 65 160
(g) 12,05 (h) 397,3 (i) 40,02
(j) 100,6 (k) 2 709 (l) 1,903

9 Each of the following is the logarithm of a number. Use tables to find each number.

(a) 0,9345 (b) 0,4624 (c) 0,7709
(d) 0,2480 (e) 0,9009 (f) 0,6010

10 (a) Copy and complete Table 8.5 for values of x in 10s from 1 to 100. Round off the logarithms to 2 decimal places.

Table 8.5

x	1	10	20	30	...	100
Log x	0,00	1,00	1,30	1,48	...	2,0

(b) Draw the graph of log x against x using the values in your table. Use scales of 1 cm to 10 units on the x-axis and 1 cm to 0,1 unit on the log x axis.
(c) Read off the logarithms of 15, 47, 73, 9, 91.
(d) Use your graph to find the numbers whose logarithms are 0,5; 1,1; 1,75; 1,33; 0,86.
(e) Use log tables to check your answers to (c) and (d).

Antilogarithms

To find a number whose logarithm is given, it is possible to use log tables in reverse. However, it is more convenient to use tables of **antilogarithms**. See page 280.

When finding an antilog, look up the fractional part *only* in antilog tables. Then use the integer to place the decimal comma correctly in the final number.

Example 10

Use antilog tables to find (a) $10^{2,7547}$, *(b) the number whose logarithms is* 5,3914.

(a) $10^{2,7547}$ is the number whose logarithm is 2,7547.
The fractional part of 2,7547 is ,7547.
,7547 in the antilog tables gives 5684.
The integer part of 2,7547 is 2. This shows that there are three digits before the decimal comma.
Hence, $10^{2,7547}$ = 568,4.

(b) The fractional part of 5,3914 is ,3914.
,3914 in the antilog tables gives 2462.
The integer 5 shows that there are 6 digits before the decimal comma.
Hence the number whose logarithm is 5,3914 is 246 200.

Exercise 8h

1 Write down the values of the following.

(a) $10^{0,6382}$, $10^{3,6382}$, $10^{2,6382}$, $10^{5,6382}$
(b) $10^{0,9517}$, $10^{2,9517}$, $10^{4,9517}$, $10^{7,9517}$

2 Write down the numbers whose logarithms are 0,7142; 1,7142; 6,7142; 3,7142.

3 Use antilog tables to find the numbers whose logarithms are:

(a) 2,1814 (b) 4,2105 (c) 1,5638
(d) 6,2983 (e) 3,4485 (f) 5,0813
(g) 1,1091 (h) 2,0088

4 (a) Copy and complete Table 8.6 for values of x in intervals of 0,1 from 0 to 1. Round off the values of 10^x to 1 d.p.

Table 8.6

x	0	0,1	0,2	0,3	...	1
10^x	1	1,3	1,6	2,0	...	10

(b) Use the values in your table to draw the graph of 10^x against x. Use scales of 1 cm to 0,1 unit on the x axis and 1 cm to 1 unit on the 10^x axis.
(c) Use your graph to read off the antilogarithms of 0,15; 0,36 and 0,87.
(d) Also use the graph to read off the logarithms of 5, 8 and 9.
(e) Compare the graph with that of question 10 in Exercise 8g (Table 8.5).

Multiplication and division

The basic principles of calculation using logarithms depend on the laws of indices. Read Examples 11 and 12 carefully.

If you have a calculator, you may disregard the methods used in the remainder of this chapter. However, you should work through the examples and exercises using your calculator. In some cases, a *scientific calculator* may be necessary. The use of the scientific calculator is explained in Book 4. In the examples, 8-figure calculator outcomes are given as a check.

Example 11

Evaluate 34,83 × 5,427.

34,83 × 5,427
= $10^{1,5420} \times 10^{0,7346}$ (from log tables)
= $10^{1,5420 + 0,7346}$ ($x^a \times x^b = x^{a+b}$)
= $10^{2,2766}$
= 189,1 (from antilog tables)
(= 189.02241)
Rough check: 34,83 × 5,427 ≃ 35 × 5
= 175

Note: Always make a rough check. The result of a rough check is not usually very close to the correct answer. However, it is close enough to show any serious errors and it will always point out whether or not the decimal point is in the correct place. A rough check will also help to reduce calculator keystroke errors.

Example 12

Work out 4 562 ÷ 98,76.

4 562 ÷ 98,76
= $10^{3,6592} \div 10^{1,9946}$ (from log tables)
= $10^{3,6592 - 1,9946}$ ($x^a \div x^b = x^{a-b}$)
= $10^{1,6646}$
= 46,19 (from antilog tables)
(= 46.192791)
Rough check: 4 562 ÷ 98,76 ≃ 4 600 ÷ 100
= 46

Exercise 8i

Use tables or a calculator to work out the following. Make a rough check in every case.

1 2,413 × 3,092	**2** 9,475 ÷ 6,13
3 3,802 × 2,09	**4** 8,735 ÷ 3,909
5 3,338 × 2,074	**6** 98,15 × 7,264
7 46,31 ÷ 8,742	**8** 45,34 × 16,21
9 176,3 ÷ 92,48	**10** 26,52 × 9,184
11 16,83 ÷ 8,992	**12** 912,4 ÷ 53,55
13 18,1 × 60,27	**14** 527,2 ÷ 94,35
15 43,14 × 8,932	**16** 43,14 ÷ 8,932
17 286,3 ÷ 17,08	**18** 34,07 × 1,007
19 705,6 × 85,04	**20** 45,80 ÷ 6,392

Calculation of powers and roots

Example 13

Evaluate $53{,}75^3$.

$53{,}75^3 = (10^{1,7304})^3$ (from log tables)
$= 10^{1,7304 \times 3}$ $((x^a)^b = x^{a \times b})$
$= 10^{5,1912}$
$= 155\ 300$ (from antilog tables)
(= 155 287.11)

Rough check:
$53{,}75^3 \simeq 50^3 = 125\ 000$

Example 14

Evaluate $\sqrt[3]{350}$.

$\sqrt[3]{350} = (350)^{\frac{1}{3}}$ $(\sqrt[a]{x} = x^{\frac{1}{a}})$
$= (10^{2,5441})^{\frac{1}{3}}$ (from log tables)
$= 10^{2,5441 \div 3}$
$= 10^{0,8480}$ (0,8480 = 2,5441 ÷ 3, correct to the 4th d.p.)
$= 7{,}047$ (from antilog tables)
(= 7.0472987)

Rough check using the answer:
$7{,}047 \simeq 7$
$7^3 = 343$
and $343 \simeq 350$, the given number.

Exercise 8j

Use tables or a calculator to calculate the following.

1 $5{,}037^2$	**2** $61{,}03^2$	**3** $2{,}938^3$
4 $12{,}94^3$	**5** $3{,}572^5$	**6** $252{,}8^2$
7 $7{,}214^3$	**8** $2{,}539^5$	**9** $5{,}632^4$
10 $19{,}18^3$	**11** $\sqrt{26{,}21}$	**12** $\sqrt[3]{26{,}21}$
13 $\sqrt[5]{31{,}60}$	**14** $\sqrt{2\ 621}$	**15** $\sqrt[3]{6{,}392}$
16 $\sqrt[5]{3\ 160}$	**17** $\sqrt[4]{35{,}81}$	**18** $\sqrt[3]{927{,}8}$
19 $\sqrt[10]{2{,}882}$	**20** $\sqrt[6]{35{,}81}$	

Setting out logarithm work

Calculations using logarithms depend on the laws of indices. Since all logarithms are based on powers of 10, there is no need to write out the base 10 every time. It is preferable to write the given numbers and their logarithms in tables. This method is shown below, where Examples 11, 12, 13 and 14 are re-worked in tabular form.

Example 11
Evaluate 34,83 × 5,427.

No	Log
34,83	1,5420
5,427	0,7346
189,1	2,2766

34,83 × 5,427 = 189,1

Example 12
Evaluate 4 562 ÷ 98,76.

No	Log
4 562	3,6592
98,76	1,9946
46,19	1,6646

4 562 ÷ 98,76 = 46,19

Example 13
Evaluate $53{,}75^3$.

No	Log
53,75	1,7304
$53{,}75^3$	1,7304 × 3
155 300	5,1912

or

No	Log
$53{,}75^3$	1,7304 × 3
155 300	5,1912

$53{,}75^3 = 155\ 300$

Example 14
Evaluate $\sqrt[3]{350}$.

No	Log
350	2,5441
$\sqrt[3]{350}$	2,5441 ÷ 3
7,047	0,8480

or

No	Log
$\sqrt[3]{350}$	2,5441 ÷ 3
7,047	0,8480

$\sqrt[3]{350} = 7{,}047$

Example 15

Calculate 42,87 × 23,82 × 1,127.

Method: First make a table of sufficient size to include the given numbers. Enter the numbers and the integer parts of their logarithms in the table.

No	Log
42,87	1,
23,82	1,
1,127	0,

Use tables to find the fractional part of each logarithm. Since the given numbers are multiplied, add the three logarithms. Use antilog tables to find the answer.

No	Log
42,87	1,6321
23,82	1,3770
1,127	0,0518
1 150	3,0609

42,87 × 23,82 × 1,127 = 1 150
(= 1150.8512)

Rough check: 40 × 25 × 1 = 1 000

Example 16

Evaluate $\sqrt[3]{\dfrac{218}{3{,}12}}$.

Work out 218 ÷ 3,12 before taking the cube root.

No	Log
218	2,3385
3,12	0,4942
$(\frac{218}{3,12})^{\frac{1}{3}}$	1,8443 ÷ 3
4,119	0,6148

$\sqrt[3]{\dfrac{218}{3{,}12}} = 4{,}119$ (= 4.1187677)

Rough check: $\sqrt[3]{\dfrac{218}{3{,}12}} \simeq \sqrt[3]{\dfrac{200}{3}}$

$\simeq \sqrt[3]{67} \simeq 4 \ (4^3 = 64)$

Example 17

Evaluate $\dfrac{17{,}83 \times 246{,}9}{256{,}2 \times 3{,}28}$.

Find a single logarithm representing the numerator and a single logarithm representing the denominator. Subtract log-denominator from log-numerator.

No	Log	
17,83	1,2511	
246,9	2,3925	
Numerator	3,6436	3,6436
256,2	2,4085	
3,28	0,5159	
Denominator	2,9244	2,9244
5,238		0,7192

$\dfrac{17{,}83 \times 246{,}9}{256{,}2 \times 3{,}28} = 5{,}238$ (= 5.238651)

Rough check: $\dfrac{18 \times 250}{250 \times 3} = 6$

Example 18

Evaluate $\dfrac{(36{,}12)^3 \times 750{,}9}{(113{,}2)^2 \times \sqrt{92{,}5}}$.

No	Log	
$36{,}12^3$	1,5577 × 3 = 4,6731	
750,9	2,8756	
Numerator	7,5487	7,5487
$113{,}2^2$	2,0539 × 2 = 4,1078	
$\sqrt{92{,}5}$	1,9661 ÷ 2 = 0,9831	
Denominator	5,0909	5,0909
286,9		2,4578

$\dfrac{(36{,}12)^3 \times 750{,}9}{(113{,}2)^2 \times \sqrt{92{,}5}} = 286{,}9$ (= 287.11879)

Rough check: $\dfrac{40 \times 40 \times 40 \times 750}{120 \times 120 \times 10} \simeq 300$

Exercise 8k

Calculate the following. Check the answers wherever possible.

1 6,26 × 23,83	**2** 14,28 × 843,7
3 675,2 ÷ 35,81	**4** 1 200 ÷ 85,25
5 409 × 6,932	**6** 63,75 ÷ 8,946

7 $5{,}932 \times 8{,}164 \times 18{,}51$

8 $8{,}4 \times 19{,}7 \times 51{,}5$

9 $\dfrac{86{,}23 \times 4\,058}{913{,}6}$ 10 $\dfrac{29{,}86 \times 105{,}2}{685{,}3}$

11 $3{,}925^2$ 12 $5{,}103^3$ 13 $2{,}895^4$

14 $\sqrt[3]{210{,}4}$ 15 $\sqrt[4]{83{,}64}$ 16 $\sqrt[5]{31{,}64}$

17 $2{,}96^2 \times 8{,}542$ 18 $\left(\dfrac{95{,}32}{8{,}971}\right)^2$

19 $\sqrt[3]{3{,}172 \times 19{,}86}$

20 $\sqrt{56{,}3 \times 39{,}5 \times 8{,}64}$

21 $2{,}973^3$ 22 $\sqrt[3]{128{,}7}$

23 $85{,}73 \div 39{,}63$ 24 $\sqrt[5]{3{,}865 \times 8{,}835}$

25 $3{,}86^3 \times 8{,}63$ 26 $(11{,}62 \div 3{,}95)^2$

27 $\dfrac{36{,}84 \times 2{,}95}{18{,}52}$ 28 $8{,}3 \times 22{,}4 \times 19{,}6$

29 $\sqrt[3]{19{,}63 \times 12{,}28 \times 74}$

30 $\sqrt[5]{6{,}838^3}$

31 $\sqrt[4]{83{,}67}$

32 $1{,}084^{10}$

33 $\sqrt{31{,}87 \times 1{,}863}$

34 $3{,}95^3 \times 62{,}5$

35 $\sqrt[3]{\dfrac{1\,067}{29{,}4}}$

36 $5{,}836^2 \times 1{,}283^3$

37 $\sqrt{3{,}892^3}$

38 $\left(\dfrac{403{,}9}{79{,}62}\right)^3$

39 $\sqrt[3]{\dfrac{218 \times 37{,}2}{95{,}43}}$

40 $\sqrt{254{,}6 \times 3{,}876}$

41 $(36{,}92 \div 8{,}15)^3$

42 $29{,}3 \times \sqrt[5]{3{,}87}$

43 $\sqrt[3]{65\,210 \div 8{,}673}$

44 $\sqrt{28{,}4 \times 3{,}82 \times 15{,}64 \times 9{,}47}$

45 $\dfrac{28{,}61 \times 74{,}23}{355{,}9 \times 2{,}547}$

46 $\dfrac{315{,}6 \times 95{,}47}{456{,}2 \times 31{,}88}$

47 $\dfrac{943}{11{,}64 \times 7{,}189}$

48 $\dfrac{35{,}2}{7{,}165 \times 3{,}92}$

49 $\dfrac{(17{,}2)^2 \times 4{,}93}{\sqrt[3]{6\,750\,000}}$

50 $\sqrt[3]{\left(\dfrac{38{,}32 \times 2{,}964}{8{,}637 \times 6{,}285}\right)^2}$

Accuracy of logarithm tables

The results obtained when calculating with 4-figure tables are accurate only to the first three figures. The 4th digit is not likely to be accurate and is used for rounding off a result to three significant figures. Three-figure accuracy is sufficient for most practical purposes.

Example 19

Calculate, to three significant figures, the area of a flat circular washer 6,84 cm *in diameter with a hole* 2,96 cm *in diameter. Take* log π *to be* 0,4971.

Outer radius $= \dfrac{6{,}84}{2}$ cm $= 3{,}42$ cm

Inner radius $= \dfrac{2{,}96}{2}$ cm $= 1{,}48$ cm

$$\begin{aligned} \text{Area} &= \pi 3{,}42^2 - \pi 1{,}48^2 \text{ cm}^2 \\ &= \pi(3{,}42^2 - 1{,}48^2) \text{ cm}^2 \\ &= \pi(3{,}42 + 1{,}48)\,(3{,}42 - 1{,}48) \text{ cm}^2 \\ &= \pi \times 4{,}9 \times 1{,}94 \text{ cm}^{2*} \\ &= 29{,}86 \text{ cm}^2 \\ &= 29{,}9 \text{ cm}^2 \text{ to 3 s.f.} \end{aligned}$$

(= 29.86398)

Check from line marked:*

Area $\approx 3 \times 5 \times 2$ cm²

$= 30$ cm²

working:

No	Log
π	0,4971
4,9	0,6902
1,94	0,2878
29,86	1,4751

Notice in Example 19 that the logarithm working is set at the side of the main answer to the question. In this way, it does not get mixed up with the method used to solve the problem. Notice also how factorisation (common factor and difference of two squares) simplifies the working.

Exercise 8l

Use tables or a calculator to give the answers to the following correct to 3 significant figures. Use the value 0,4971 for log π where necessary.

1 Calculate the area of a rectangle 3,85 m long and 2,37 m wide.

2 Calculate the area of a square of side 2,83 cm.

3 Calculate the volume of a cube of edge 8,24 cm.

4 Calculate the area of a circular disc 5,86 cm in diameter.

5 Calculate the mass, in kg, of a rectangular sheet of metal 15,7 cm long and 12,9 cm wide, if its mass is 38,1 g per cm².

6 A rectangle of area 209,8 cm² is 10,4 cm long. Calculate its width.

7 Calculate the length of the side of a square of area 508,5 cm².

8 Calculate the length of the edge of a cube of volume 129,7 cm³.

9 Calculate the volume of a cylinder of diameter 5,93 cm and length 10 cm.

10 A room contains 156,1 m³ of air. If the room is of length 8,31 m and breadth 5,72 m, calculate its height.

11 Calculate the area in hectares of a rectangular field 126 m long and 97 m wide. (1 ha = 10 000 m²).

12 If the area of a square field is 3,95 hectares, calculate the length of a side of the field in metres.

13 Calculate the area of a circular washer of diameter 3,42 cm if the hole in the centre is 1 cm in diameter.

14 Calculate the length of a solid cylinder of diameter 7,3 cm and volume 435 cm³.

15 Calculate the capacity in litres of a cylinder 23,6 cm in diameter and 37,8 cm deep.

Chapter 9

Vectors (1)

Translation vectors

In Chapter 3 a translation was described as a movement in a certain direction, without turning. For example, in Fig. 9.1 $\triangle ABC$ is translated to $\triangle PQR$.

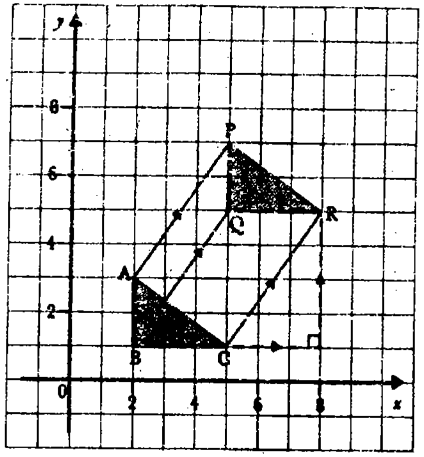


Fig. 9.1

In Fig. 9.1 point A moves to point P. This movement can be written as $\overrightarrow{AP}$. $\overrightarrow{AP}$ is a **translation vector**. A **vector** is any quantity which has direction as well as size. In Fig. 9.1 the translation vectors $\overrightarrow{BQ}$ and $\overrightarrow{CR}$ have the same size and direction as $\overrightarrow{AP}$.

Hence $\overrightarrow{AP} = \overrightarrow{BQ} = \overrightarrow{CR}$

Any one of these vectors describes the translation that takes $\triangle ABC$ to the position shown by $\triangle PQR$.

In Fig. 9.1 the dotted lines show that the vector $\overrightarrow{CR}$ is equivalent to a movement of 3 units to the right followed by a movement of 4 units upwards. These movements can be combined as a single column matrix or **column vector**:

$\overrightarrow{CR} = \begin{pmatrix}3\\4\end{pmatrix}$

In general any translation of the cartesian plane can be written as a column vector $\begin{pmatrix}x\\y\end{pmatrix}$ where x represents a movement parallel to the x-axis and y represents a movement parallel to the y-axis. Movements to the right and movements upwards are positive. Movements to the left and movements downwards are negative.

Example 1

In Fig. 9.2 the line segments represent vectors $\overrightarrow{AB}$, $\overrightarrow{CD}$, …, $\overrightarrow{IJ}$.

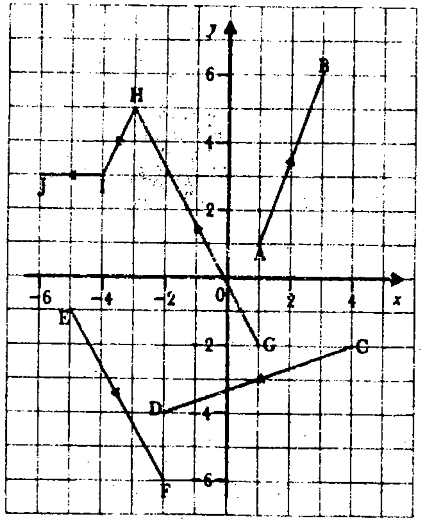


Fig. 9.2

Write these vectors in the form $\begin{pmatrix}x\\y\end{pmatrix}$.

$\overrightarrow{AB} = \begin{pmatrix}2\\5\end{pmatrix}$ $\quad$ $\overrightarrow{CD} = \begin{pmatrix}-6\\-2\end{pmatrix}$

$\overrightarrow{EF} = \begin{pmatrix}3\\-5\end{pmatrix}$ $\quad$ $\overrightarrow{GH} = \begin{pmatrix}-4\\7\end{pmatrix}$

$\overrightarrow{HI} = \begin{pmatrix}-1\\-2\end{pmatrix}$ $\quad$ $\overrightarrow{IJ} = \begin{pmatrix}-2\\0\end{pmatrix}$

Example 2

A square OABC *has coordinates* O(0; 0), A(3; 0), B(3; 3), C(0; 3). *It is translated by vector* $\overrightarrow{OP}$ *to square* PQRS. *Find the coordinates of* P, Q, R *and* S *if* $\overrightarrow{OP} = \begin{pmatrix}2\\-4\end{pmatrix}$.

Fig. 9.3 shows the translation from square OABC to square PQRS.

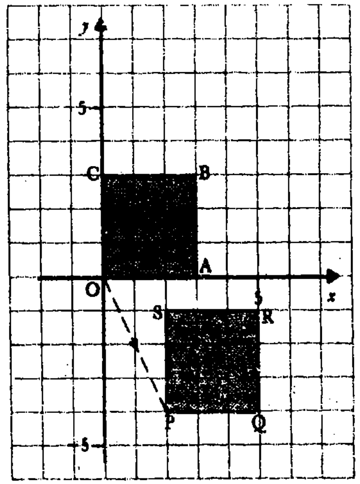


Fig. 9.3

The coordinates of PQRS are P(2; − 4), Q(5; − 4), R(5; − 1) and S(2; − 1).

Exercise 9a

Use graph paper when answering questions 3–10. Use a scale of 1 cm to 1 unit throughout.

1 In Fig. 9.4 the line segments represent vectors $\overrightarrow{AB}$, $\overrightarrow{CD}$, …, $\overrightarrow{KL}$. Write these vectors in the form $\begin{pmatrix}x\\y\end{pmatrix}$.

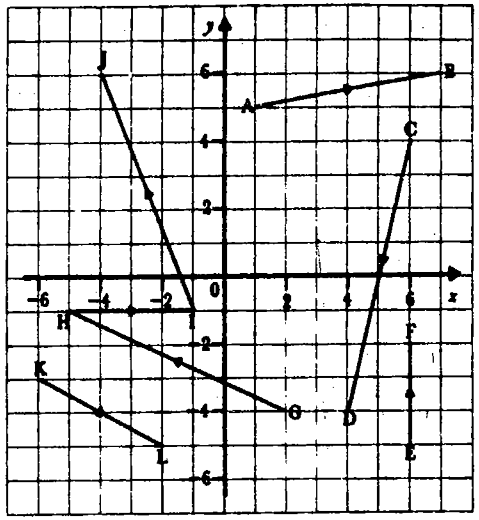


Fig. 9.4

2 If $\overrightarrow{AB} = \begin{pmatrix}9\\-2\end{pmatrix}$, what is $\overrightarrow{BA}$?

3 Draw line segments to represent the following vectors.

(a) $\overrightarrow{AB} = \begin{pmatrix}4\\1\end{pmatrix}$ $\quad$ (b) $\overrightarrow{CD} = \begin{pmatrix}-6\\2\end{pmatrix}$

(c) $\overrightarrow{EF} = \begin{pmatrix}2\\-5\end{pmatrix}$ $\quad$ (d) $\overrightarrow{GH} = \begin{pmatrix}-3\\-2\end{pmatrix}$

4 Draw line segments to represent the following vectors.

(a) $\overrightarrow{KL} = \begin{pmatrix}4\\0\end{pmatrix}$ $\quad$ (b) $\overrightarrow{LM} = \begin{pmatrix}0\\4\end{pmatrix}$

(c) $\overrightarrow{MN} = \begin{pmatrix}-2\\4\end{pmatrix}$ $\quad$ (d) $\overrightarrow{NK} = \begin{pmatrix}-2\\-8\end{pmatrix}$

5 (a) Draw line segments to represent the following vectors.

$\overrightarrow{AB} = \begin{pmatrix}3\\-1\end{pmatrix}$, $\overrightarrow{BC} = \begin{pmatrix}0\\-2\end{pmatrix}$, $\overrightarrow{CD} = \begin{pmatrix}-5\\-3\end{pmatrix}$

(b) Hence give the vector AD in the form $\begin{pmatrix} a \\ b \end{pmatrix}$.

(c) What is $\overrightarrow{DA}$?

6 (a) Draw the following vectors.

$\overrightarrow{PQ} = \begin{pmatrix} 1 \\ 4 \end{pmatrix}$ $\quad \overrightarrow{QR} = \begin{pmatrix} -7 \\ 5 \end{pmatrix}$

(b) What is the vector $\overrightarrow{PR}$?

(c) What is the vector $\overrightarrow{RP}$?

7 $\triangle$OAB has coordinates O(0; 0), A(4; 2), B(2; 5).
Find the coordinates of the images of O, A and B when the triangle is translated by vector

(a) $\begin{pmatrix} 3 \\ 6 \end{pmatrix}$, (b) $\overrightarrow{OA}$, (c) $\begin{pmatrix} -1 \\ 2 \end{pmatrix}$,

(d) $\overrightarrow{AB}$, (e) $\overrightarrow{BA}$.

8 The coordinates of the vertices of rectangle ABCD are A(1; 2), B(1; 5), C(2; 5) and D(2; 2).
Find the coordinates of the images of A, B, C, D when the rectangle is translated by vector

(a) $\begin{pmatrix} -1 \\ 3 \end{pmatrix}$, (b) $\begin{pmatrix} 3 \\ 1 \end{pmatrix}$, (c) $\overrightarrow{AC}$.

9 A cartesian plane is translated so that (4; 5) is the image of the point (3; 1).

(a) What is the vector of translation?

(b) What are the coordinates of the image of the origin?

(c) What are the coordinates of the point whose image is (− 1; 2)?

10 A quadrilateral has vertices P(1; 4), Q(5; 7), R(9; 4), S(5; 1).

(a) What kind of quadrilateral is PQRS?

(b) Find the coordinates of the images of P, Q, R, S after translation of PQRS through

(i) $\begin{pmatrix} 3 \\ -1 \end{pmatrix}$, (ii) $\begin{pmatrix} -5 \\ -4 \end{pmatrix}$, (iii) $\overrightarrow{PR}$.

Sum of vectors

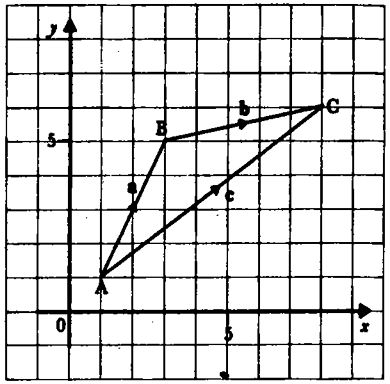


Fig. 9.5

In Fig. 9.5 it is clear that a translation $\overrightarrow{AB}$ followed by a translation $\overrightarrow{BC}$ is equivalent to the single translation $\overrightarrow{AC}$. We write this as the **vector sum**:

$\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$

or, writing the vector sum with the small letters given in Fig. 9.5,

$\mathbf{a} + \mathbf{b} = \mathbf{c}$

By counting units in Fig. 9.5,

$\mathbf{a} = \begin{pmatrix} 2 \\ 4 \end{pmatrix}, \mathbf{b} = \begin{pmatrix} 5 \\ 1 \end{pmatrix}, \mathbf{c} = \begin{pmatrix} 7 \\ 5 \end{pmatrix}$

By matrix addition,

$$\mathbf{a} + \mathbf{b} = \begin{pmatrix} 2 \\ 4 \end{pmatrix} + \begin{pmatrix} 5 \\ 1 \end{pmatrix} = \begin{pmatrix} 2+5 \\ 4+1 \end{pmatrix} = \begin{pmatrix} 7 \\ 5 \end{pmatrix} = \mathbf{c}$$

In general if $\mathbf{a} = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$

then $\mathbf{a} + \mathbf{b} = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} x_1 + x_2 \\ y_1 + y_2 \end{pmatrix}$

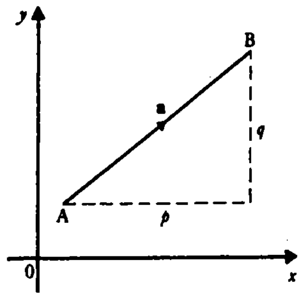


Fig. 9.6

Note: There are many ways of writing vectors. The vector in Fig. 9.6 can be written,

(a) using capital letters:
$\overrightarrow{AB}$ or **AB** or $\overline{AB}$ or $\underset{\sim}{AB}$

(b) using small letters:
$\vec{a}$ or **a** or $\bar{a}$ or $\underset{\sim}{a}$

(c) using a column matrix:
$\overrightarrow{AB} = \mathbf{a} = \begin{pmatrix} p \\ q \end{pmatrix}$

Example 3

If $\mathbf{p} = \begin{pmatrix} 3 \\ 5 \end{pmatrix}, \mathbf{q} = \begin{pmatrix} -4 \\ 3 \end{pmatrix}, \mathbf{r} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$, *find*

(*a*) **p** + **q**, (*b*) **p** + **r**, (*c*) **q** + **r**, (*d*) **p** + **q** + **r**, *showing the vector sum in part (d) on a diagram.*

(a) $\mathbf{p} + \mathbf{q} = \begin{pmatrix} 3 \\ 5 \end{pmatrix} + \begin{pmatrix} -4 \\ 3 \end{pmatrix} = \begin{pmatrix} -1 \\ 8 \end{pmatrix}$

(b) $\mathbf{p} + \mathbf{r} = \begin{pmatrix} 3 \\ 5 \end{pmatrix} + \begin{pmatrix} 1 \\ -2 \end{pmatrix} = \begin{pmatrix} 4 \\ 3 \end{pmatrix}$

(c) $\mathbf{q} + \mathbf{r} = \begin{pmatrix} -4 \\ 3 \end{pmatrix} + \begin{pmatrix} 1 \\ -2 \end{pmatrix} = \begin{pmatrix} -3 \\ 1 \end{pmatrix}$

(d) $\mathbf{p} + \mathbf{q} + \mathbf{r} = \begin{pmatrix} 3 \\ 5 \end{pmatrix} + \begin{pmatrix} -4 \\ 3 \end{pmatrix} + \begin{pmatrix} 1 \\ -2 \end{pmatrix} = \begin{pmatrix} 0 \\ 6 \end{pmatrix}$

In Fig. 9.7, the broken line represents the vector **p** + **q** + **r**.

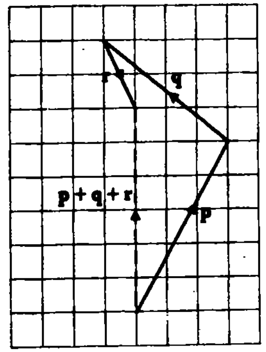


Fig. 9.7

This shows that $\mathbf{p} + \mathbf{q} + \mathbf{r} = \begin{pmatrix} 0 \\ 6 \end{pmatrix}$.

Exercise 9b

When using graph paper, take a scale of 1 cm to 1 unit.

1 In each part of Fig. 9.8, find **p** + **q**, giving the results in the form $\begin{pmatrix} a \\ b \end{pmatrix}$.

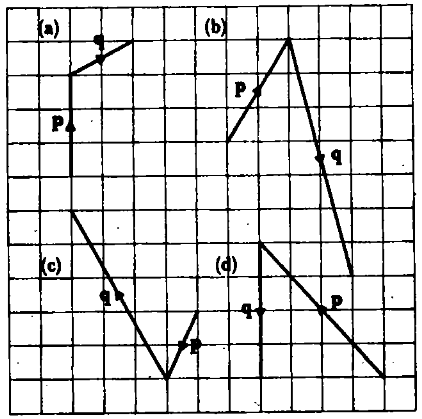


Fig. 9.8

2 If vectors **p**, **q**, **r**, **s** are as given in Fig. 9.9, express each of the following as a single vector in the form $\begin{pmatrix} a \\ b \end{pmatrix}$.

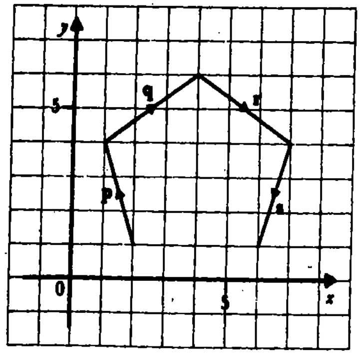


Fig. 9.9

(a) $\mathbf{p} + \mathbf{q}$ (b) $\mathbf{p} + \mathbf{q} + \mathbf{r}$
(c) $\mathbf{q} + \mathbf{r}$ (d) $\mathbf{q} + \mathbf{r} + \mathbf{s}$
(e) $\mathbf{r} + \mathbf{s}$ (f) $\mathbf{p} + \mathbf{q} + \mathbf{r} + \mathbf{s}$

3 (a) Draw vectors **PQ** and **QR** such that

$$\mathbf{PQ} = \begin{pmatrix} 2 \\ -1 \end{pmatrix} \text{ and } \mathbf{QR} = \begin{pmatrix} -5 \\ 3 \end{pmatrix}.$$

(b) Find **PR** (i) by drawing, (ii) by matrix addition.

4 If $\mathbf{p} = \begin{pmatrix} 0 \\ 4 \end{pmatrix}$, $\mathbf{q} = \begin{pmatrix} -2 \\ -3 \end{pmatrix}$, $\mathbf{r} = \begin{pmatrix} 4 \\ -2 \end{pmatrix}$, $\mathbf{s} = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$, find

(a) $\mathbf{p} + \mathbf{q}$ (b) $\mathbf{p} + \mathbf{r}$
(c) $\mathbf{p} + \mathbf{s}$ (d) $\mathbf{q} + \mathbf{r}$
(e) $\mathbf{q} + \mathbf{s}$ (f) $\mathbf{r} + \mathbf{s}$
(g) $\mathbf{p} + \mathbf{q} + \mathbf{r}$ (h) $\mathbf{p} + \mathbf{q} + \mathbf{s}$
(i) $\mathbf{q} + \mathbf{r} + \mathbf{s}$ (j) $\mathbf{p} + \mathbf{q} + \mathbf{r} + \mathbf{s}$

5 A vector **b** is such that

$$\begin{pmatrix} 7 \\ 2 \end{pmatrix} + \mathbf{b} = \begin{pmatrix} 4 \\ -5 \end{pmatrix}. \text{ Find } \mathbf{b}.$$

6 The cartesian plane is translated through $\begin{pmatrix} 6 \\ -1 \end{pmatrix}$. It is then translated by $\begin{pmatrix} -2 \\ -2 \end{pmatrix}$.

(a) What single translation is this equivalent to?
(b) What are the coordinates of the final image of the point (− 2; 7)?
(c) The final image of a point P is at (− 2; 0). What are the coordinates of P?

Difference of vectors

Magnitude of a vector

In Fig. 9.10, $\mathbf{AB} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$ and $\mathbf{CD} = \begin{pmatrix} -4 \\ 3 \end{pmatrix}$.

The **magnitude**, or size, of **AB** is the length of the line segment AB. This is often written as $|\mathbf{AB}|$.

$$|\mathbf{AB}| = \sqrt{3^2 + 4^2} \text{ (Pythagoras)}$$
$$= 5 \text{ units}$$

Similarly,

$$|\mathbf{CD}| = \sqrt{(-4)^2 + 3^2}$$
$$= 5 \text{ units}$$

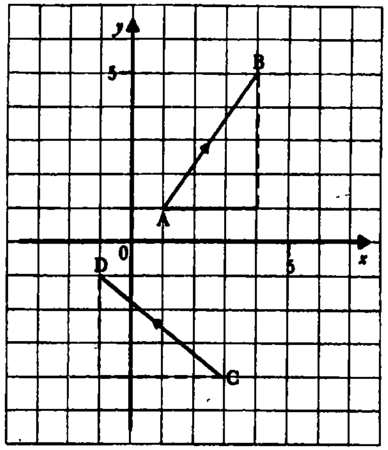


Fig. 9.10

Hence different vectors may have the same magnitude.

In general, if $\mathbf{a} = \begin{pmatrix} x \\ y \end{pmatrix}$

then $|\mathbf{a}| = \sqrt{x^2 + y^2}$

where $|\mathbf{a}|$ is the magnitude of **a**. Notice that the magnitude of a vector is always given as a positive number of units.

Subtraction

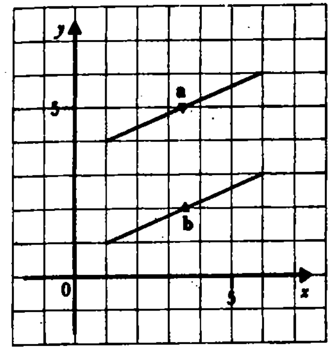


Fig. 9.11

In Fig. 9.11, $\mathbf{a} = \begin{pmatrix} 5 \\ 2 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} -5 \\ -2 \end{pmatrix}$.

Notice that **b** is a vector which has the same magnitude as **a** but which is in the opposite direction. We say that $\mathbf{b} = -\mathbf{a}$.

In general, if $\mathbf{a} = \begin{pmatrix} x \\ y \end{pmatrix}$, then $-\mathbf{a} = \begin{pmatrix} -x \\ -y \end{pmatrix}$.

Example 4

If $\mathbf{p} = \begin{pmatrix} 4 \\ 1 \end{pmatrix} - \begin{pmatrix} -2 \\ 7 \end{pmatrix}$ *find* **p** *(a) by calculation, (b) by drawing. (c) Hence find the magnitude of* **p**.

(a) Using matrix arithmetic:

$$\mathbf{p} = \begin{pmatrix} 4 \\ 1 \end{pmatrix} - \begin{pmatrix} -2 \\ 7 \end{pmatrix} = \begin{pmatrix} 4 - (-2) \\ 1 - 7 \end{pmatrix}$$
$$= \begin{pmatrix} 6 \\ -6 \end{pmatrix}$$

(b) If $-\begin{pmatrix} -2 \\ 7 \end{pmatrix} = +\begin{pmatrix} 2 \\ -7 \end{pmatrix}$

then $\begin{pmatrix} 4 \\ 1 \end{pmatrix} - \begin{pmatrix} -2 \\ 7 \end{pmatrix} = \begin{pmatrix} 4 \\ 1 \end{pmatrix} + \begin{pmatrix} 2 \\ -7 \end{pmatrix}$

i.e. to subtract a vector is equivalent to adding a vector of the same size in the opposite direction.

Fig. 9.12 shows the vector sum

$$\begin{pmatrix} 4 \\ 1 \end{pmatrix} + \begin{pmatrix} 2 \\ -7 \end{pmatrix}.$$

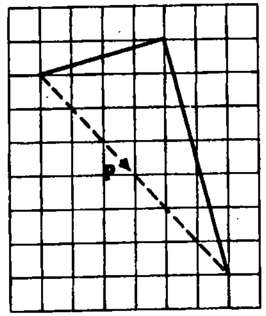


Fig. 9.12

In Fig. 9.12 the broken line represents the vector **p**.

$$\mathbf{p} = \begin{pmatrix} 6 \\ -6 \end{pmatrix}$$

(c) $|\mathbf{p}| = \sqrt{6^2 + (-6)^2}$
$= \sqrt{36 + 36}$
$= \sqrt{72}$ units

Exercise 9c

1 Find the magnitudes of the following vectors.

(a) $\begin{pmatrix} 4 \\ -3 \end{pmatrix}$ (b) $\begin{pmatrix} -5 \\ -12 \end{pmatrix}$
(c) $\begin{pmatrix} 8 \\ -6 \end{pmatrix}$ (d) $\begin{pmatrix} 0 \\ 7 \end{pmatrix}$
(e) $\begin{pmatrix} -2 \\ 0 \end{pmatrix}$ (f) $\begin{pmatrix} -15 \\ 8 \end{pmatrix}$

2 Express the following as positive vectors.

(a) $-\begin{pmatrix} 8 \\ -1 \end{pmatrix}$ (b) $-\begin{pmatrix} -2 \\ 3 \end{pmatrix}$
(c) $-\begin{pmatrix} 0 \\ 2 \end{pmatrix}$ (d) $-\begin{pmatrix} -6 \\ -5 \end{pmatrix}$
(e) $-\begin{pmatrix} -4 \\ 0 \end{pmatrix}$ (f) $-\begin{pmatrix} 7 \\ 3 \end{pmatrix}$

3 If $\vec{AB} = \begin{pmatrix} 11 \\ 5 \end{pmatrix} + \begin{pmatrix} -2 \\ 7 \end{pmatrix}$, find $|\vec{AB}|$.

4 Find vector **p** such that

$\begin{pmatrix}5\\-6\end{pmatrix} + \mathbf{p} = \begin{pmatrix}1\\-2\end{pmatrix}$

5 Find vector $\vec{q}$ such that

$\begin{pmatrix}-3\\1\end{pmatrix} - \vec{q} = \begin{pmatrix}-8\\4\end{pmatrix}$. Hence find $|\vec{q}|$.

6 Express each of the following as a single vector.

(a) $\begin{pmatrix}5\\1\end{pmatrix} - \begin{pmatrix}4\\2\end{pmatrix}$

(b) $\begin{pmatrix}8\\-3\end{pmatrix} - \begin{pmatrix}-1\\0\end{pmatrix}$

(c) $\begin{pmatrix}-1\\4\end{pmatrix} - \begin{pmatrix}-9\\-1\end{pmatrix}$

(d) $\begin{pmatrix}-3\\-5\end{pmatrix} - \begin{pmatrix}-4\\-9\end{pmatrix}$

7 $\vec{AB} = \begin{pmatrix}4\\-3\end{pmatrix}$ and $\vec{BC} = \begin{pmatrix}-6\\4\end{pmatrix}$. Find $\vec{AB} - \vec{BC}$ (a) using matrix arithmetic, (b) by drawing.

8 If $\mathbf{p} = \begin{pmatrix}7\\-3\end{pmatrix}$ and $\mathbf{q} = \begin{pmatrix}-6\\2\end{pmatrix}$ find

(a) $\mathbf{p} - \mathbf{q}$, (b) $\mathbf{p} + \mathbf{q}$, (c) $\mathbf{q} - \mathbf{p}$, (d) $|\mathbf{p} + \mathbf{q}|$, (e) $|\mathbf{p} - \mathbf{q}|$.

9 If $\mathbf{p} = \begin{pmatrix}6\\3\end{pmatrix}$, $\mathbf{q} = \begin{pmatrix}-1\\5\end{pmatrix}$, $\mathbf{r} = \begin{pmatrix}-4\\2\end{pmatrix}$,

$\mathbf{s} = \begin{pmatrix}0\\-3\end{pmatrix}$, find

(a) $\mathbf{p} - \mathbf{q}$ (b) $\mathbf{p} + \mathbf{q} - \mathbf{r}$
(c) $\mathbf{p} - \mathbf{r}$ (d) $\mathbf{p} + \mathbf{r} - \mathbf{s}$
(e) $\mathbf{s} - \mathbf{q}$ (f) $\mathbf{q} + \mathbf{r} - \mathbf{s}$
(g) $(\mathbf{p} - \mathbf{q}) - \mathbf{r}$ (h) $\mathbf{p} - (\mathbf{q} - \mathbf{r})$

10 Draw $\mathbf{OP} = \begin{pmatrix}3\\-5\end{pmatrix}$ and $\mathbf{OQ} = \begin{pmatrix}6\\7\end{pmatrix}$

(a) Use your drawing to find **PQ**.
(b) Use any method to find **OQ** − **OP**.
(c) What do you notice?
(d) Does this happen for any two vectors **OP** and **OQ**?

Multiplication by a scalar

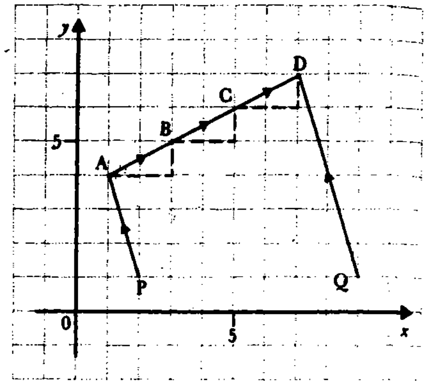


Fig. 9.13

In Fig. 9.13,

$$\vec{AB} = \vec{BC} = \vec{CD} = \begin{pmatrix}2\\1\end{pmatrix}$$

Also $\vec{AD} = \vec{AB} + \vec{BC} + \vec{CD}$

$$= \begin{pmatrix}2\\1\end{pmatrix} + \begin{pmatrix}2\\1\end{pmatrix} + \begin{pmatrix}2\\1\end{pmatrix}$$

$$= \begin{pmatrix}6\\3\end{pmatrix} = 3\begin{pmatrix}2\\1\end{pmatrix}$$

Hence $\vec{AD} = 3\vec{AB}$ or $3\vec{BC}$ or $3\vec{CD}$.

If any of the vectors $\vec{AB}$, $\vec{BC}$ or $\vec{CD}$ are multiplied by the **scalar** 3, the result is a vector 3 times as big. *Note:* A scalar is simply a numerical multiplier.

Also in Fig. 9.13,

$$\vec{QD} = \begin{pmatrix}-2\\6\end{pmatrix}$$

and $\vec{PA} = \begin{pmatrix}-1\\3\end{pmatrix} = \frac{1}{2}\begin{pmatrix}-2\\6\end{pmatrix}$

Hence $\vec{PA} = \frac{1}{2}\vec{QD}$.

In the first case the vectors $\vec{AB}$, $\vec{BC}$, $\vec{CD}$ lie along the line of vector $\vec{AD}$. In the second case vectors $\vec{PA}$ and $\vec{QD}$ are *not* in the same straight line. In both cases the resultant vector is parallel to the original vector.

In general, if a vector **a** is multiplied by a scalar k, the result is a new vector $k\mathbf{a}$ which is in the same direction as **a** but which is k times as big.

Example 5

Vectors $\vec{OA}$, $\vec{OB}$, $\vec{OC}$, $\vec{OP}$, $\vec{OQ}$, $\vec{OR}$ *are such that*

$\vec{OA} = \begin{pmatrix}2\\1\end{pmatrix}$, $\vec{OB} = \begin{pmatrix}2\\4\end{pmatrix}$, $\vec{OC} = \begin{pmatrix}3\\1\end{pmatrix}$ *and*

$\vec{OP} = -2\vec{OA}$, $\vec{OQ} = -2\vec{OB}$, $\vec{OR} = -2\vec{OC}$

Take O *as origin and draw the vectors on a cartesian plane. Compare* △PQR *with* △ABC.

The coordinates of A, B, C are (2; 1), (2; 4), (3; 1) respectively.

$$\vec{OP} = -2\vec{OA} = -2\begin{pmatrix}2\\1\end{pmatrix} = \begin{pmatrix}-4\\-2\end{pmatrix}$$

Similarly $\vec{OQ} = \begin{pmatrix}-4\\-8\end{pmatrix}$ and $\vec{OR} = \begin{pmatrix}-6\\-2\end{pmatrix}$

The coordinates of P, Q, R are (−4; −2), (−4; −8), (−6; −2) respectively.
Fig. 9.14 shows the vectors and △s ABC and PQR.

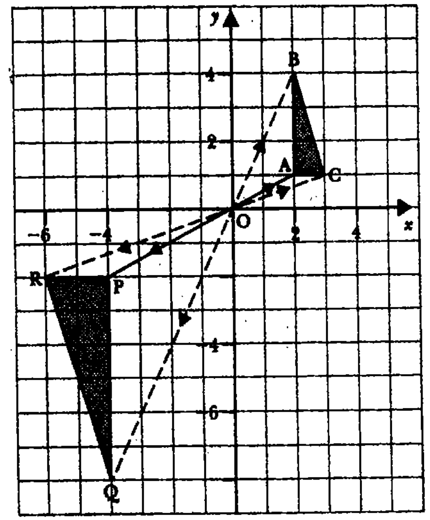


Fig. 9.14

Comparing the triangles,
1 △PQR is an enlargement of △ABC.*
2 Each side of △PQR is twice as long as the corresponding side of △ABC.
3 △PQR is four times the area of △ABC.
4 △PQR appears to be rotated with respect to △ABC.

* See Chapter 19 for a full explanation of enlargement.

Exercise 9d

In questions 8 and 9 use a scale of 1 cm to 1 unit.
Use Fig. 9.15 when answering questions 1 to 7.

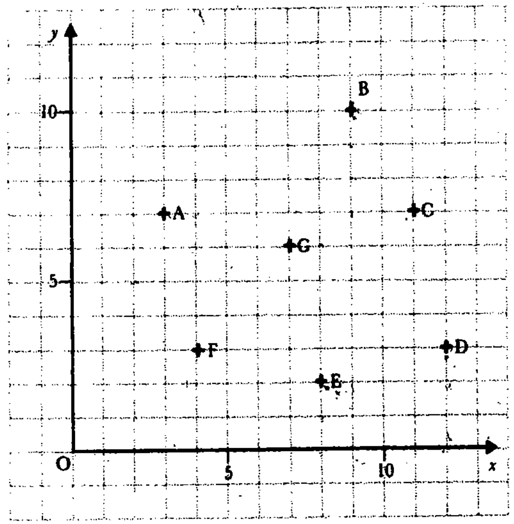


Fig. 9.15

1 Express each of the following as a column vector.

(a) $3\vec{OA}$ (b) $5\vec{ED}$ (c) $6\vec{CA}$
(d) $2\vec{CE}$ (e) $4\vec{AF}$ (f) $4\vec{FA}$
(g) $3\vec{GC}$ (h) $3\vec{ED}$

2 Express each of the following as a column vector.

(a) $\frac{1}{2}\vec{EO}$ (b) $\frac{1}{4}\vec{AC}$ (c) $\frac{1}{3}\vec{BA}$ (d) $\frac{1}{3}\vec{OD}$

3 Express each of the following as a column vector.

(a) $-2\overrightarrow{AG}$ (b) $-5\overrightarrow{BC}$

(c) $-\frac{3}{4}\overrightarrow{AC}$ (d) $-\frac{1}{2}\overrightarrow{EG}$

4 Calculate the following.

(a) $2\overrightarrow{OA} + \overrightarrow{BA}$ (b) $5\overrightarrow{ED} + 2\overrightarrow{DO}$

(c) $3\overrightarrow{OF} - 2\overrightarrow{AB}$ (d) $4\overrightarrow{EG} - \overrightarrow{ED}$

5 Calculate the following.

(a) $\frac{1}{4}$**OF** + $\frac{3}{4}$**AG** (b) $\frac{2}{3}$**AB** + $\frac{1}{2}$**EO**

(c) $\frac{1}{2}$**BC** − $1\frac{1}{2}$**ED** (d) $\frac{1}{3}$**OD** − $\frac{3}{4}$**CA**

6 Calculate the following.

(a) $\frac{1}{8}$(**OA** + **EC**) (b) $\frac{1}{2}$(**OA** + **AG** + **GE**)

(c) 2(**EC** − **EG**) (d) $\frac{1}{2}$(**AG** − **GC**)

7 Calculate the following.

(a) 3(2**OA** + **GE**) (b) 6**OA** + 3**GE**

(c) 5(**OE** − 3**EC**) (d) 5**OE** − 15**EC**

(e) $\frac{1}{2}$(2**BC** − 3**OE**) (f) **BC** + $1\frac{1}{2}$**OE**

(g) −2(3**ED** + **BC**) (h) −6**ED** − 2**BC**

8 Vectors $\overrightarrow{OA}$, $\overrightarrow{OB}$, $\overrightarrow{OC}$, $\overrightarrow{OP}$, $\overrightarrow{OQ}$, $\overrightarrow{OR}$ are such that

$$\overrightarrow{OA} = \begin{pmatrix}3\\4\end{pmatrix}, \overrightarrow{OB} = \begin{pmatrix}1\\-2\end{pmatrix}, \overrightarrow{OC} = \begin{pmatrix}4\\-3\end{pmatrix}$$

and $\overrightarrow{OP} = 3\overrightarrow{OA}$, $\overrightarrow{OQ} = 3\overrightarrow{OB}$, $\overrightarrow{OR} = 3\overrightarrow{OC}$. Take O as origin and draw the vectors on a cartesian plane.

Compare △PQR with △ABC.

9 Given the same vectors $\overrightarrow{OA}$, $\overrightarrow{OB}$, $\overrightarrow{OC}$ as in question 8, find, by drawing, the positions of K, L, M such that $\overrightarrow{OK} = -2\overrightarrow{OA}$, $\overrightarrow{OL} = -2\overrightarrow{OB}$, $\overrightarrow{OM} = -2\overrightarrow{OC}$. Compare △KLM with △ABC.

Chapter 10

Equations and formulae

Equations

Solving linear equations (revision)

An **equation** is a statement that two algebraic expressions are equal in value. For example, $4 - 4x = 9 - 12x$ is a linear equation with an **unknown** x. This equation is only true when x has a particular numerical value. To **solve** an equation means to find the real number value of the unknown which makes the equation true.

Example 1

Solve $4 - 4x = 9 - 12x$.

$4 - 4x = 9 - 12x$

Add $12x$ to both sides of the equation.

$4 - 4x + 12x = 9 - 12x + 12x$

$4 + 8x = 9$

Subtract 4 from both sides of the equation.

$4 + 8x - 4 = 9 - 4$

$8x = 5$

Divide both sides of the equation by 8.

$\frac{8x}{8} = \frac{5}{8}$

$x = \frac{5}{8}$

$\frac{5}{8}$ is the solution or **root** of the equation.

Check: When $x = \frac{5}{8}$,

LHS $= 4 - 4 \times \frac{5}{8} = 4 - 2\frac{1}{2} = 1\frac{1}{2}$

RHS $= 9 - 12 \times \frac{5}{8} = 9 - 7\frac{1}{2} = 1\frac{1}{2} =$ LHS

The equation in Example 1 was solved by the **balance method**. Compare the equation with a pair of scales. If the expressions on opposite sides of the equals sign 'balance', they will continue to do so if the same amounts are added to or subtracted from both sides. They will also balance if both sides are multiplied or divided by the same amounts.

Exercise 10a (Revision)

Solve the following equations.

1 $4x = 3x + 5$
2 $7m = 8 + 5m$
3 $2a = 9 - a$
4 $\frac{1}{4}d = 3$
5 $5y + 6 = 21$
6 $2n - 3 = 6$
7 $4 + 3x = 17$
8 $15 = 4a + 3$
9 $3 = 3m - 4$
10 $5 = \frac{1}{3}b$
11 $\frac{2}{3}t = 8$
12 $6x + 5 = 13 + 4x$
13 $2a + 4 = 16 - 3a$
14 $\frac{3}{4}x = 18$
15 $1\frac{1}{2}y = 9$
16 $7b - 9 = 5 + 3b$
17 $2 - 5t = 20 - 8t$
18 $2\frac{1}{3}d = 28$
19 $22 = 7 + 2\frac{1}{2}x$
20 $8a - 19 = 5 + 3a$
21 $3 + 2y - 24 = 14 - 3y$
22 $2k = 3k + 4 - 5k$
23 $8 - h = 5 - 4h + 3$
24 $2a + 20 = 5a + 6$
25 $8 + 4d - 7 = 4 - d$
26 $2e = 20 - 3e - 9$
27 $2x + 19 - 5x = x - 5$
28 $12 - 3a - 3 = 9 - 5a$
29 $1\frac{1}{2}b - 4\frac{1}{2} = 1\frac{1}{2} + \frac{3}{4}b$
30 $a - 10\frac{1}{2} = 10\frac{1}{2} - \frac{5}{8}a$

Equations with brackets (Revision)

If an equation contains brackets, remove the brackets before collecting terms.

Example 2

Solve $3(4c - 7) - 4(4c - 1) = 0$.

$3(4o - 7) - 4(4c - 1) = 0$

Remove brackets.

$12c - 21 - 16c + 4 = 0$

Collect like terms.

$-4c - 17 = 0$

Add 17 to both sides.

$-4c = 17$

Divide both sides by -4.

$c = -\frac{17}{4}$

$c = -4\frac{1}{4}$

Check: When $c = -4\frac{1}{4}$,

$$\begin{aligned} \text{LHS} &= 3(-17 - 7) - 4(-17 - 1) \\ &= 3(-24) - 4(-18) \\ &= 72 + 72 = 0 = \text{RHS} \end{aligned}$$

Exercise 10b (Revision)

Solve the following equations.

1 $4a - (3 - a) = 17$
2 $8b - (3b + 4) = 11$
3 $3 - (3m - 7) = 43$
4 $8n - (5n + 13) = 7$
5 $12t + (1 - 7t) = 31$
6 $0 = 5 - (2x - 17)$
7 $9 - (5 - 7y) = 13 + 4y$
8 $d = 12 - (11 + 4d)$
9 $2 - 3(a + 5) = -10$
10 $3x - 2(x + 3) = 0$
11 $3(5c - 1) = 4(3c + 2)$
12 $5(3m + 4) = 3(4m + 7)$
13 $2(2x - 5) = 3(x - 6)$
14 $5(a + 2) - 3(3a - 5) = 1$
15 $7(5n - 4) - 10(3n - 2) = 0$
16 $4(3x - 1) = 11x - 3(x - 4)$
17 $5(v + 2) + 3(v - 5) = 19$
18 $3(6 + 7y) + 2(1 - 5y) = 42$
19 $2 = 5(5z - 2) - 9(3z - 2)$
20 $3x - [3(1 + x) - 2x] = 3$

Equations with fractions (Revision)

Always clear fractions before beginning to solve an equation. To clear fractions, multiply each term in the equation by the LCM of the denominators of the fraction.

Example 3

Solve the equation $\frac{3}{4}x - 1\frac{2}{3} = \frac{2}{3}x$.

Express the given equation as follows.

$$\frac{3x}{4} - \frac{5}{3} = \frac{2x}{3}$$

The denominators are 4, 3 and 3. Their LCM is 12. Multiply every term by 12.

$$\frac{12 \times 3x}{4} - \frac{12 \times 5}{3} = \frac{12 \times 2x}{3}$$

$$3 \times 3x - 4 \times 5 = 4 \times 2x$$

$$9x - 20 = 8x$$

Subtract $8x$ from both sides.

$$x - 20 = 0$$

Add 20 to both sides.

$$x = 20$$

Check: When $x = 20$,

LHS $= \frac{3}{4} \times 20 - 1\frac{2}{3} = 15 - 1\frac{2}{3} = 13\frac{1}{3}$

RHS $= \frac{2}{3} \times 20 = 13\frac{1}{3} =$ LHS

Example 4

Solve the equation $\frac{3x + 2}{6} - \frac{2x - 7}{9} = 0$.

The LCM of 6 and 9 is 18. Multiply every term by 18, remembering that the dividing line of the given fraction acts like a bracket on the numerators.

$$\frac{18(3x + 2)}{6} - \frac{18(2x - 7)}{9} = 18 \times 0$$

$$3(3x + 2) - 2(2x - 7) = 0$$

Clear brackets.

$$\begin{aligned} 9x + 6 - 4x + 14 &= 0 \\ 5x + 20 &= 0 \\ 5x &= -20 \\ x &= -4 \end{aligned}$$

Check: When $x = -4$,

$$\begin{aligned} \text{LHS} &= \frac{-12 + 2}{6} - \frac{-8 - 7}{9} \\ &= \frac{-10}{6} - \frac{-15}{9} \\ &= -\frac{5}{3} + \frac{5}{3} = 0 = \text{RHS} \end{aligned}$$

Example 5

Solve the equation $\frac{2}{5}(6x - 1) = \frac{3}{4}(3x + 2) - 2$.

Express the given equation as follows.

$$\frac{2(6x - 1)}{5} = \frac{3(3x + 2)}{4} - 2$$

The LCM of 5 and 4 is 20. Multiply every term by 20.

$$\frac{20 \times 2(6x - 1)}{5} = \frac{20 \times 3(3x + 2)}{4} - 20 \times 2$$

$$\Leftrightarrow 4 \times 2(6x - 1) = 5 \times 3(3x + 2) - 40$$

$$\Leftrightarrow \quad 8(6x - 1) = 15(3x + 2) - 40$$

Clear brackets.

$$48x - 8 = 45x + 30 - 40$$

Collect terms.

$$\begin{aligned} 48x - 45x &= 30 - 40 + 8 \\ 3x &= -2 \\ x &= -\frac{2}{3} \end{aligned}$$

Check: When $x = -\frac{2}{3}$,

LHS $= \frac{2}{5}(-4 - 1) = -2$

RHS $= \frac{3}{4}(-2 + 2) - 2$
$= 0 - 2 = -2 =$ LHS

Notice in each example that *every* term is multiplied by the LCM of the denominators, whether the term is a fraction or not. Also notice that the solution can be checked by substituting the value of the unknown into the original equation.

Exercise 10c

Solve the following equations.

1 $\frac{x}{2} = \frac{x}{3} + \frac{1}{2}$

2 $\frac{1}{2}a + 1\frac{1}{4} = \frac{1}{4}a$

3 $\frac{5d}{3} + 1 = \frac{d}{6} + 3$

4 $3y - \frac{2}{9} = 4\frac{7}{9}$

5 $\frac{1}{5}x - \frac{1}{7}x = \frac{7}{15}$

6 $\frac{9y}{10} + \frac{3}{5} = \frac{2y}{5} + \frac{7}{10}$

7 $\frac{2d + 7}{6} + \frac{d - 5}{3} = 0$

8 $\frac{6a + 3}{7} = \frac{2a - 1}{3}$

9 $\frac{7x + 2}{3} - \frac{9x - 2}{5} = 2$

10 $\frac{3x + 2}{5} - \frac{2x + 3}{3} = 3$

11 $\frac{2x - 1}{3} - \frac{3 - x}{2} = \frac{x}{4}$

12 $\frac{4n + 1}{3} - 1\frac{1}{2} = \frac{2n + 5}{6}$

13 $\frac{3}{8}(4a - 5) - \frac{5}{12}(3a - 4) = \frac{1}{6}$

14 $\frac{4}{5}(2y - 5) = \frac{2}{3}(2y - 7) + \frac{2}{15}$

15 $\frac{7}{6}(3x - 1) - 8\frac{1}{3} = \frac{3}{2}(2x - 5)$

16 $\frac{5x + 16}{4} + \frac{x}{2} = \frac{4x + 2}{3}$

17 $\frac{2(5z - 3)}{3} - \frac{3(5z - 2)}{5} = \frac{8}{15}$

18 $\frac{4m}{3} - \frac{17}{21} = \frac{6m - 1}{7}$

19 $\frac{x}{5} + 1\frac{1}{2}x + \frac{11}{20} = \frac{5x + 1}{4}$

20 $7(x + 4) - 2[x - 3(5 + x)] = \frac{2}{3}(x - 6)$

Equations from word problems

Example 6

A man walks for 2 hours at a certain speed. He then cycles at 3 times that speed for 4 hours. He goes 77 km altogether. Find the speed at which he walks.

Since we are to find walking speed, let this be v km/h.

From the 1st sentence,
distance walked $= v \times 2$ km $= 2v$ km
From the second sentence,
cycling speed $= 3v$ km/h
distance cycled $= 3v \times 4$ km $= 12v$ km
From the 3rd sentence,
total distance $= 77$ km
Hence $2v + 12v = 77$

$$\Leftrightarrow \quad 14v = 77$$

$$\Leftrightarrow \quad v = \frac{77}{14} = \frac{11}{2} = 5\frac{1}{2}$$

The man walks at $5\frac{1}{2}$ km/h.

Check:

Distance in 2 h at $5\frac{1}{2}$ km/h = 11 km
Distance in 4 h at $16\frac{1}{2}$ km/h = 66 km
Total distance = 77 km

Example 7

Aquilina is 11 years older than Zodwa. In 5 years' time, Aquilina will be twice as old as Zodwa. Find their present ages.

Let Zodwa's age be x years.
Aquilina's age is $(x + 11)$ years (from the 1st sentence)
In 5 years' time, Aquilina's age
$= (x + 11) + 5$ years
Zodwa's age $= (x + 5)$ years

Hence, from the second sentence,
$(x + 11) + 5 = 2(x + 5)$
$\Leftrightarrow \quad x + 16 = 2x + 10$
$\Leftrightarrow \quad 6 = x$
$\Leftrightarrow \quad x = 6$
Zodwa is 6 years old and Aquilina is 17 years old.

Check: In 5 years' time,
Zodwa's age = 6 + 5 = 11 yr
Aquilina's age = 17 + 5 = 22 yr = 2 × Zodwa's age.

Example 8

A trader buys n oranges at the rate of 5 *oranges for* 50 *cents.* 8 *of the oranges are bad so she sells the rest at the rate of* 4 *oranges for* 50 *cents and makes a profit of* \$2,30. *Find n.*

If 5 oranges cost 50 cents
1 orange costs $\frac{50}{5} = 10$ cents
$\therefore$ n oranges cost $10n$ cents
8 oranges are bad,
$\therefore$ number of oranges sold $= n - 8$
If 4 oranges sell for 50 cents
1 orange sells for $\frac{50}{4}$ cents
$\therefore$ $(n - 8)$ oranges sell for $(n - 8) \times \frac{50}{4}$ cents
$= \frac{25(n-8)}{2}$ cents

Profit = Selling Price − Cost Price
In cents,
$$230 = \frac{25(n-8)}{2} - 10n$$
Multiply each term by 2.
$460 = 25(n - 8) - 20n$
$460 = 25n - 200 - 20n$
$660 = 5n$
$n = \frac{660}{5} = 132$

Check:
C.P. of 132 oranges $= \frac{132}{5} \times 50\text{c} = 1320\text{c}$
S.P. of 124 oranges $= \frac{124}{4} \times 50\text{c} = 1550\text{c}$
Profit = 1550c − 1320c = 230c = \$2,30

Where necessary, choose a letter to represent the unknown quantity. Express the data of the question in terms of the letter. Form an equation and solve the equation. Check that the solution agrees with the data of the question.

Exercise 10d

1 x represents a certain number. When that number is multiplied by 3 the result is the same as that of adding 34 to the number. Find x.
2 A rectangle is 3 times as long as it is wide. If its perimeter is 56 cm, find the width of the rectangle.
3 Thabo and Nomsa share 147 cents between them so that Thabo gets 19 cents more than Nomsa. Calculate the amount of money each gets.
4 A woman is 3 times as old as her daughter. 6 years ago the sum of their ages was 36. Find the age of the daughter.
5 A man walked for 2 hours at 6 km/h. He then cycled for a certain time at 16 km/h. If he travelled 36 km altogether, for how many hours did he cycle?
6 A sum of \$1,24 is made up of 5-cent coins and 1-cent coins. There are 6 times as many 5-cent coins as there are 1-cent coins. Find the number of 5-cent coins.
7 Rufaro has 30 cents and Tom has \$1,86. If Rufaro saves 5c a day and Tom spends 7c a day, after how many days will they have equal amounts?
8 Divide 59 mℓ into two parts so that one part is 7 mℓ less than 5 times the other part.
9 The result of taking 3 from x and multiplying the answer by 4 is the same as taking 3 from 5 times x.
(a) Express this statement as an algebraic equation.
(b) Hence find the value of x.
10 The sum of 6 and one third of n is one more than twice n.
(a) Express this statement in algebraic terms.
(b) Hence find the value of n.
11 A boy is 10 years old and his father is 37 years old. In how many years' time will the father be twice as old as his son?
12 One farmer has 119 goats and another has 73. After they each sell the same number of goats, one is left with 3 times as many goats as the other. How many goats did each sell?
A motorist travels regularly between two towns. She usually takes 5 hours when travelling at a certain speed. She finds that if she increases her average speed by 15 km/h the journey takes 1 hour less. Find her usual speed.
14 A water tank contains 5 times as much as another water tank. When 20 litres of water are poured from the first tank into the second, the first contains 3 times as much as the second. How much water did each tank contain originally?
15 A trader buys some eggs at 22c each. She finds that 6 of them are broken. She sells the rest at 34c each and makes a profit of \$6,24. How many eggs did she buy?
16 The result of adding 15 to x and dividing the answer by 4 is the same as taking x from 80. (a) Express this statement as an algebraic equation. (b) Hence find the value of x.
17 One stick is 9 cm longer than another. $\frac{2}{5}$ of the longer stick is equal to $\frac{1}{2}$ of the shorter stick. Find the length of the longer stick.
18 A man cycles to a village at 18 km/h and returns at 12 km/h. If he takes $6\frac{1}{4}$ hours for the double journey, how far does he ride altogether?
19 A total of m matches are needed to fill 30 matchboxes with the same number of matches in each box.
(a) How many matches are in each box?
(b) If 3 fewer matches are put into each box, there are enough for 32 boxes. What is the value of m?
20 A train travels a certain journey and is supposed to arrive at midday. When its average speed is 40 km/h, it arrives at 1 p.m. When its average speed is 48 km/h it arrives at 11 a.m. What is the length of the journey?

Formulae

Substitution in formulae

A **formula** is an equation in which letters represent quantities. For example, the area, A, of a triangle of base length b and height h is given by the formula $A = \frac{1}{2}bh$ where b and h are in the same units. In this example, A is the **subject** of the formula. By substituting various values of b and h into the formula, corresponding values of A can be found.

Example 9

Use the formula $A = \frac{1}{2}bh$ *to calculate the area of a triangle (a) of base* 3,2 cm *and height* 5 cm, *(b) of base* 4 km *and height* 600 m.

(a) $A = \frac{1}{2}bh$
When $b = 3{,}2$ and $h = 5$
$A = \frac{1}{2} \times 3{,}2 \times 5$
$= 1{,}6 \times 5$
$= 8$
The area is 8 cm^2.

(b) Working in km, 600 m = 0,6 km.
When $b = 4$ and $h = 0{,}6$,
$A = \frac{1}{2} \times 4 \times 0{,}6$
$= 2 \times 0{,}6$
$= 1{,}2$
The area is 1,2 km^2.

Example 10

The sum of the squares of the first n integers is given by
$$S_n = \frac{n(n+1)(2n+1)}{6}$$
Calculate (a) S_{20}, *(b) the sum of the squares from* 21 *to* 40 *inclusive.*

(a) S_{20} means the value of S_n when $n = 20$.
$$S_{20} = \frac{20(20+1)(2 \times 20 + 1)}{6}$$
$$= \frac{20 \times 21 \times 41}{6}$$
$$= 10 \times 7 \times 41$$
$$= 2\,870$$
(b) the sum of squares from 21 to 40
= sum of squares from 1 to 40
− sum of squares from 1 to 20
$= S_{40} - S_{20}$
$$S_{40} = \frac{40(40+1)(2 \times 40 + 1)}{6}$$

$= \dfrac{40 \times 41 \times 81}{6}$

$= 20 \times 41 \times 27 = 540 \times 41$

$= 22\,140$

	540
×	41
	540
	21 60
	22 140

$S_{40} - S_{20} = 22\,140 - 2\,870 = 19\,270$

The next example shows how a formula can be used to find the value of a quantity which is not the subject of the formula.

Example 11

The formula $F = \frac{9C}{5} + 32$ *shows the relationship between temperature in degrees Fahrenheit* (F) *and degrees Celsius* (C). *Find* (*a*) F *when* $C = 40$, (*b*) C *when* $F = 100$, (*c*) *the temperature when* $F = C$.

$F = \dfrac{9C}{5} + 32$

(a) When $C = 40$,

$F = \dfrac{9 \times 40}{5} + 32$

$= 9 \times 8 + 32 = 72 + 32 = 104$

The temperature is 104 °F.

(b) When $F = 100$,

$100 = \dfrac{9C}{5} + 32$

Solve this equation for C.

Multiply each term by 5.

$500 = 9C + 160$

$340 = 9C$

$C = \frac{340}{9} = 37\frac{7}{9}$

The temperature is $37\frac{7}{9}$ °C.

(c) When $F = C$,

$C = \dfrac{9C}{5} + 32$

Multiply each term by 5.

$5C = 9C + 160$

$-4C = 160$

$C = \dfrac{160}{-4} = -40$

$F = C = -40$

The temperature is -40 °F (or -40 °C).

Exercise 10e

1 If $y = 2x^2 - 5x - 3$, find the value of y when $x =$ (a) -1, (b) 0, (c) 1, (d) 2, (e) 3.

2 The formula $A = P\left(1 + \dfrac{RT}{100}\right)$ gives the total money, A, that a principal, P, amounts to in T years at R% simple interest per annum. Find the amount that a principal of \$750 becomes if invested for 5 years at $6\frac{1}{2}$% simple interest per annum.

3 The formula $d = \sqrt{(l^2 + b^2 + h^2)}$ gives the length d of the longest diagonal in a cuboid of length l, breadth b and height h. (d, l, b and h are in the same units.) Find the length of the diagonal of a cuboid which is 6 cm long, 2 cm wide and 3 cm high.

4 Find the value of $2\pi\sqrt{\dfrac{l}{g}}$ when $\pi = 3\frac{1}{7}$, $l = 98$ and $g = 32$.

5 The formula $A = \pi r(r + s)$ gives the surface area, A, of a cone of base radius r cm and slant height s cm. Find the surface area of a cone of base radius 6 cm and slant height 22 cm, using the value $\frac{22}{7}$ for π.

6 Given that $S_n = \dfrac{4n^3 - 3n^2 + 6}{n}$, evaluate $S_{20} - S_{10}$.

7 Given that $c = 2\pi r$, (a) find c when $\pi = 3{,}142$ and $r = 50$, (b) find r when $c = 286$ and $\pi = 3\frac{1}{7}$.

8 In a certain country, the cost, c cents, of sending a telegram of 12 words or over is given by the formula $c = 3(w - 2)$ where w is the number of words in the telegram.
(a) Find the cost of sending a telegram of 35 words.
(b) If it costs \$1,41 to send a telegram, how many words does it contain?

9 The formula $b = 4 + \dfrac{3W}{50}$ is used to work out the electricity bill, b Dollars, for a month in which W kilowatt-hours of electricity are used. (a) Find the bill for a month in which 705 kilowatt-hours are used. (b) Find the number of kilowatt-hours used by a consumer who receives a bill for \$13,06.

10 The formula $d \approx 5\sqrt{\dfrac{h}{2}}$ gives the approximate distance, d km, of the horizon which can be seen from a point h m above ground level.
(a) Find the approximate distance of the horizon from the top of a building 72 m high.
(b) From the top of a tower, a person can see for about 35 km. How high is the person above ground level?

Change of subject of formulae

It is often necessary to change the subject of a formula. To do this, think of the formula as an equation. Solve the equation for the letter which is to become the subject. The following examples show how different formulae can be rearranged in order to change the subject.

Example 12

Make x the subject of the formula $a = b(1 - x)$.

$a = b(1 - x)$

Clear brackets.

$a = b - bx$

Rearrange to give terms in x on one side of the equation.

$bx = b - a$

Divide both sides by b.

$x = \dfrac{b - a}{b}$

Example 13

Make x the subject of the formula $a = \dfrac{b + x}{b - x}$.

$a = \dfrac{b + x}{b - x}$

Clear fractions. Multiply both sides by $(b - x)$.

$a(b - x) = b + x$

Clear brackets.

$ab - ax = b + x$

Collect terms in x on one side of the equation.

$ab - b = x + ax$

Take x outside a bracket.

$ab - b = x(1 + a)$

Divide both sides by $(1 + a)$.

$\dfrac{ab - b}{1 + a} = x$

$\therefore x = \dfrac{b(a - 1)}{1 + a}$

Example 14

Make x the subject of the formula $b = \frac{1}{2}\sqrt{a^2 - x^2}$.

$b = \frac{1}{2}\sqrt{a^2 - x^2}$

Clear fractions.

$2b = \sqrt{a^2 - x^2}$

Square both sides.

$(2b)^2 = a^2 - x^2$

$4b^2 = a^2 - x^2$

Rearrange to give the term in x on one side of the equation.

$x^2 = a^2 - 4b^2$

Take the square root of both sides.

$x = \sqrt{a^2 - 4b^2}$

The general method of Examples 12, 13 and 14 is to treat the formula as an equation and the new subject as the unknown of the equation. There are many different formulae and it is not possible to give general rules for changing their subject. However, the following points should be remembered:

1 Begin by clearing fractions, brackets and root signs.
2 Rearrange the formula so that all the terms which contain the new subject are on one side of the equals sign and the rest on the other. Do not try to place the subject on the left-hand side if it comes more naturally on the right (see Example 13).
3 If more than one term contains the subject, take it outside a bracket.
4 Divide both sides by the bracket, then simplify as far as possible.

Exercise 10f

Make x the subject of the following equations.

1 $x + a = b$ **2** $a - x = b$ **3** $ax = b$

4 $ax + bx = c$ **5** $ax + b = x$ **6** $\dfrac{a}{x} = b$

7 $\frac{a}{x} + b = c$ 8 $\frac{x}{a} + b = c$ 9 $\frac{x}{a} + \frac{x}{b} = 1$

10 $\frac{a}{x} + \frac{b}{x} = 1$ 11 $a(x + b) = c$

12 $ax = b(c + x)$ 13 $a(b - x) = cx$

14 $\frac{x}{2a} + \frac{x}{3a} = b$ 15 $x(a - b) = b(c - x)$

16 $\frac{a}{b - x} = c$ 17 $a = \frac{2b + 3x}{3b - 2x}$

18 $\sqrt{x} = a$ 19 $\sqrt{2x} = a$ 20 $2\sqrt{x} = a$

21 $\sqrt{\frac{x}{2}} = a$ 22 $\frac{\sqrt{x}}{2} = a$ 23 $a\sqrt{x} = b$

24 $\sqrt{ax} = b$ 25 $\sqrt[3]{\frac{x}{a}} = b$ 26 $x^2 = a^4$

27 $x^2 = a$ 28 $\sqrt{x + a} = b$

29 $\sqrt{x} + a = b$ 30 $\sqrt{x^2 + a^2} = b$

31 $\sqrt{x^2 + a^2} = 3a$ 32 $\frac{a}{x} - 1 = \frac{b}{2x}$

33 $a\sqrt{x - 1} = b$ 34 $a\sqrt{x} - 1 = b$

35 $(ax - b)(bx + a) = (bx^2 + a)a$

36 $\frac{a}{a - x} = \frac{b}{b + x}$ 37 $\frac{a}{b - x} = \frac{b}{a + x}$

38 $a(a^2 - x) = b(b^2 - x)$

39 $\frac{x}{x + a} - \frac{a}{x - b} = 1$

40 $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

Example 15

If $u = 1 - \frac{3v}{vt - w}$, express t in terms of the other letters.

$$u = 1 - \frac{3v}{vt - w}$$

Clear fractions.

$$v(vt - w) = 1(vt - w) - 3v$$

Clear brackets.

$$uvt - uw = vt - w - 3v$$

Collect terms in t.

$$uvt - vt = uw - w - 3v$$

Take t outside a bracket.

$$t(uv - v) = uw - w - 3v$$

$$\Leftrightarrow \quad t = \frac{uw - w - 3v}{uv - v}$$

Example 16

The period of a compound pendulum is given by $T = 2\pi\sqrt{\frac{h^2 + k^2}{gh}}$. Express k in terms of T, h and g, taking π^2 as 10.

$$T = 2\pi\sqrt{\frac{h^2 + k^2}{gh}}$$

$$\Rightarrow \quad T^2 = 4\pi^2\left(\frac{h^2 + k^2}{gh}\right)$$

$$\Leftrightarrow \quad \frac{T^2}{4\pi^2} = \frac{h^2 + k^2}{gh}$$

$$\Leftrightarrow \frac{T^2gh}{4\pi^2} = h^2 + k^2$$

$$\Leftrightarrow \quad k^2 = \frac{T^2gh}{4\pi^2} - h^2$$

$$\Leftarrow \quad k = \sqrt{\frac{T^2gh}{4\pi^2} - h^2}$$

Using the value 10 for π^2,

$$k = \sqrt{\left(\frac{T^2gh}{40} - h^2\right)}$$

Exercise 10g

In each question a formula is given. A letter is printed in heavy type after it. Make the letter the subject of the formula. If more than one letter is given, make each letter the subject in turn.

1 $c = 2\pi r$ **r**
2 $P = aW + b$ **W**
3 $P = \frac{N + 2}{D}$ **N, D**
4 $A = P + \frac{PRT}{100}$ **T, P**
5 $v^2 = u^2 + 2as$ **s, u**
6 $S = \frac{n}{2}(a + l)$ **n, l**
7 $S = 2\pi r(r + h)$ **h**
8 $k = \frac{brt}{v - b}$ **b**
9 $S = 4\pi r^2$ **r**
10 $V = \pi h^2\left(r - \frac{h}{3}\right)$ **r**
11 $L = \frac{Wh}{a(W + P)}$ **W**
12 $\frac{L}{E} = \frac{2a}{R - r}$ **R**
13 $T = \frac{mu^2}{K} - 5mg$ **K**
14 $D = \sqrt{\frac{3h}{2}}$ **h**
15 $t = \frac{3p}{r} + s$ **r**
16 $\frac{a}{p} - \frac{b}{q} = c$ **q**
17 $H = \frac{m(v^2 - u^2)}{2gx}$ **v**
18 $R = \sqrt{\frac{ax - P}{Q + bx}}$ **x**
19 $T = 2\pi\sqrt{\frac{I}{MH}}$ **M**
20 $A = \frac{1}{2}m(v^2 - u^2)$ **u**
21 $I = \frac{nE}{R + nr}$ **n**
22 $S = \frac{wd}{h}\left(h - \frac{d}{2}\right)$ **h**
23 $T = \sqrt{\frac{Pbh}{4 + a^2}}$ **b, a**
24 $H = \frac{w^2}{2g}(R^2 - r^2)$ **r**
25 $I = \frac{E}{\sqrt{R^2 + W^2L^2}}$ **R**

Exercise 10h

1 The simple interest, \$$I$, on a sum of money, \$$P$ after T years at R% is given by the formula $I = \frac{PRT}{100}$
(a) Make T the subject of the formula.
(b) Find T if $I = 51$, $P = 340$ and $R = 2\frac{1}{2}$.

2 (a) Make x the subject of the formula
$$\frac{x}{a} + \frac{y}{b} = 1.$$
(b) Hence, if $a = 4$, $b = 1$ and $y = -2$, evaluate x.

3 The volume of a square-based pyramid is given by $V = \frac{a^2h}{3}$, where h is the height of the pyramid and a is the length of one of the base edges.
(a) Make a the subject of the formula.
(b) Find a when $V = 162$ and $h = 24$.

4 The volume V of a cone of height h and base radius r is $\frac{1}{3}\pi r^2h$.
(a) Obtain a formula for r in terms of V, π and h.
(b) Calculate the base radius of a cone of height 14 cm and volume $91\frac{2}{3}$ cm^3 using the value $\frac{22}{7}$ for π.

5 The length of the hypotenuse, h, in a right-angled triangle is given by the formula $h = \sqrt{a^2 + b^2}$ where a and b are the lengths of the other two sides of the triangle.
(a) Make a the subject of this formula.
(b) Hence find a if $h = 34$ and $b = 16$.

6 (a) Make d the subject of the formula
$$S_n = \frac{n}{2}[2a + (n - 1)d].$$
(b) Hence find d if $S_{32} = 56$ and $a = 25$.
(*Note:* S_{32} is the value of S_n when $n = 32$.)

7 The energy E possessed by an object of mass m kg travelling at a height h m with a velocity v m/s is given by $E = \frac{mv^2}{2} + mgh$ joules.
(a) Express v in terms of the other letters.
(b) If the energy of a 20 kg mass at a height of 15 m is 4 900 joules and $g = 9,8$, how fast is the mass moving?

8 The formula $V = \frac{1}{3}\pi r^2(2r + h)$ gives the volume V of a solid consisting of a cone of height h and base radius r attached to a hemisphere of the same radius.
(a) Change the subject of the formula to h.
(b) Hence calculate the height of the cone, if the solid has a volume of 55 cm^3 and the common radius is $2\frac{1}{2}$ cm. (Use the value $\frac{22}{7}$ for π.)

9 If a wire L metres long is stretched tightly between two points at the same level d metres apart, the sag in the middle of the wire is s metres,
where $s = \sqrt{\frac{3d(L - d)}{8}}$.
(a) Change the subject of the formula to L.
(b) Find the length of the wire if $d = 16$ and $s = 0{,}6$.

10 The formula $T = \frac{4b^2}{21}(d - \frac{3}{8}b)$ gives the approximate mass, T tonnes, of a ship d metres long and b metres wide.
(a) Make d the subject of the formula.
(b) Calculate the length of a 4 500 tonne ship if it is 20 m wide, giving your answer to the nearest 5 m.

Revision exercises and tests

Chapters 1–10

Revision exercise 1

1 Express the following numbers in standard form.
(a) 3 500 000 (b) 5 700 (c) 28
(d) 0,47 (e) 0,085 (f) 0,000 003

2 A rectangle measures 8 cm by 15 cm. Make a sketch and use Pythagoras' theorem to calculate the length of one of its diagonals.

3 Copy patterns (a) and (b) of Fig. R1 on to graph paper.

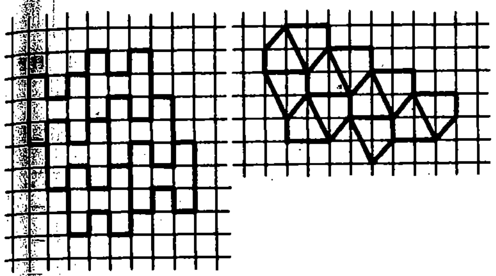


Fig. R1

Extend each pattern by repeating the sequence of basic shapes.

4 A lorry travels at a speed of 50 km/h. How many hours will it take to cover
(a) 200 km, (b) n km?

5 How many sides has a regular pentagon if each interior angle is 171°?

6 Find the tangent of 60° by construction and measurement. Check your result from tables.

7 Simplify the following matrices.

(a) $\begin{pmatrix} 5 & -1 & 3 \\ -9 & 3 & 0 \\ 8 & 0 & -7 \end{pmatrix} + \begin{pmatrix} 2 & 6 & 1 \\ -6 & -2 & 8 \\ -5 & 5 & 0 \end{pmatrix}$

(b) $\begin{pmatrix} 0 & -1 \\ 9 & 2 \end{pmatrix} - \begin{pmatrix} 3 & 3 \\ 5 & 4 \end{pmatrix} + \begin{pmatrix} 4 & 11 \\ -2 & -5 \end{pmatrix}$

8 Simplify the following.
(a) $(2x)^3 \times 2x^4$ (b) $(5x)^2 \div x^4$
(c) $(256)^{0,5}$ (d) $\left(\frac{1}{243}\right)^{\frac{1}{5}}$

9 (a) On a piece of graph paper, draw line segments to represent the following vectors:
(i) $\mathbf{AB} = \begin{pmatrix} 4 \\ 7 \end{pmatrix}$, (ii) $\mathbf{BC} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$,
(iii) $\mathbf{CD} = \begin{pmatrix} -8 \\ -3 \end{pmatrix}$.
(b) Hence give vector **AD** as a column matrix.

10 Evaluate the following when $x = -3$, $y = 4$ and $z = -2$.
(a) $2y + x - 5z$ (b) $2x^2 - 4y$
(c) $x^3 - z^2y$ (d) $\frac{x + y}{y + z}$

Revision test 1

1 Express 0,000 263 in standard form.
A $2{,}63 \times 10^{-6}$ B $2{,}63 \times 10^{-5}$
C $2{,}63 \times 10^{-4}$ D $2{,}63 \times 10^{4}$
E $2{,}63 \times 10^{5}$

2 Which of the following are Pythagorean triples?
I (3; 4; 5), II (5; 12; 13), III (8; 13; 17)
A I only B I and II only
C II only D II and III only
E III only

3 15° 44′ to the nearest 0,1° is
A 15,3° B 15,4°
C 15,6° D 15,7°
E 15,8°

4 One factor of $6ax - 10ay + 3x - 5y$ is $3x - 5y$. The other factor is
A $2a + 1$ B $2a - 1$
C $2a$ D $6ax - 10ay$
E $1 - 2a$

Fig. R2 overleaf shows trapezium PQRS drawn within cartesian axes. Use it to answer questions 5, 6, 7 and 8.

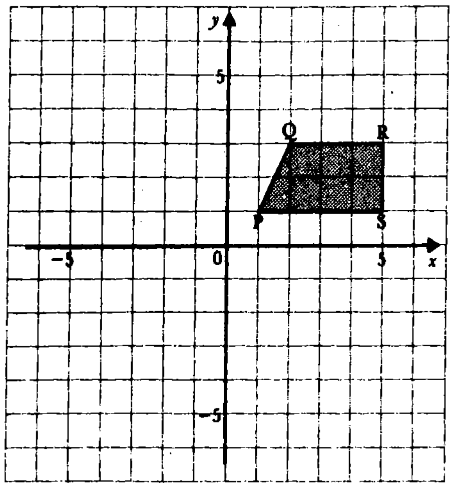


Fig. R2

5 In Fig. R2, PQRS is reflected first in the x-axis. Its image is then reflected in the y-axis. What are the coordinates of the final image of Q?
A (2; − 3) B (3; − 2) C (− 2; 3)
D (− 3; − 2) E (− 2; − 3)

6 PQRS is translated so that the image of S is the point (− 3; − 1). (a) Write down the column matrix which represents the translation. (b) Find the images of P, Q and R.

7 Find the coordinates of the image of point R if PQRS is given an anti-clockwise rotation of 90° about (a) the origin, (b) point P, (c) point (4; − 5).

8 Find the coordinates of the final image of PQRS if the trapezium is given a translation represented by vector $\begin{pmatrix}-3\\0\end{pmatrix}$ then rotated through 180° about the point (1; − 2).

9 (a) Find the sum of the angles of a pentagon.

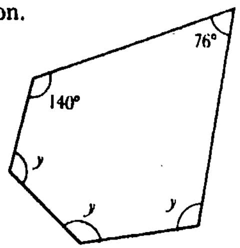


Fig. R3

(b) Hence calculate y in Fig. R3.

10 Solve the following equations.

(a) $\frac{x}{4} + \frac{x}{10} = \frac{7}{8}$

(b) $\frac{2x-5}{2} - \frac{3x+7}{5} + \frac{3x-1}{10} + 4\frac{3}{8} = 0$

Revision exercise 2

1 $\frac{0{,}0001 \times 1{,}11}{0{,}1 \times 10^4} = A \times 10^n$ where A is a number between 1 and 10 and n is an integer. Find the values of A and n.

2 In Fig. R4 calculate XY.

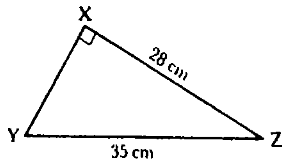


Fig. R4

3 Fig. R5 shows △PQR drawn on the cartesian plane.

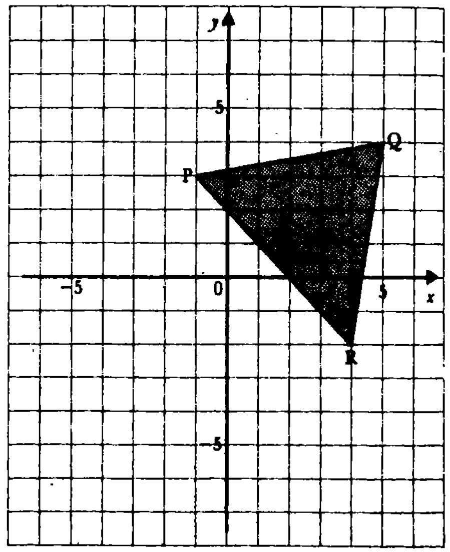


Fig. R5

Find the coordinates of the images of P, Q and R

(a) if the triangle is translated 5 units downwards,

(b) after reflection of △PQR in the x-axis,

(c) after reflection of △PQR in the y-axis,

(d) after a rotation of 270° anti-clockwise about the origin,

(e) after a rotation of 180° about (0; 3).

4 Simplify the following.

(a) $(x+3)(x-2) - x(x-1)$

(b) $(a-b)^2 - (a^2 - b^2)$

5 Calculate the size of each angle of a regular 15-sided polygon.

6 In △ABC, B = 90°, A = 33° and BC = [illegible]6 cm. (a) Calculate C. (b) Hence or otherwise calculate AB correct to 2 d.p.

7 If A = $\begin{pmatrix}2 & -1\\0 & 1\end{pmatrix}$ and B = $\begin{pmatrix}-1 & 0\\3 & 1\end{pmatrix}$, find 3A − 2B.

8 Use either logarithms or a calculator to compute each of the following. Give your answers correct to 3 s.f.

(a) $608 \times 4{,}85$ (b) $\frac{107}{39}$

(c) $(6{,}297)^3$ (d) $\sqrt[3]{6{,}297}$

9 Given $\mathbf{p} = \begin{pmatrix}3\\5\end{pmatrix}$ and $\mathbf{q} = \begin{pmatrix}-2\\4\end{pmatrix}$:

(a) Find $\mathbf{p} - \mathbf{q}$ and $\mathbf{q} - \mathbf{p}$.

(b) Calculate $|\mathbf{p} - \mathbf{q}|$ and $|\mathbf{q} - \mathbf{p}|$.

(c) What do you notice?

10 Make d the subject of the following.

(a) $V = \frac{1}{3}\pi d^2 h$ (b) $y = \sqrt{\frac{5dx}{7}}$

Revision test 2

1 Without using tables or a calculator, calculate 82,5 ÷ 0,025, expressing the answer in standard form.
A $3{,}3 \times 10^{-3}$ B $3{,}3 \times 10^{-2}$
C $3{,}3 \times 10^{1}$ D $3{,}3 \times 10^{2}$
E $3{,}3 \times 10^{3}$

2 PQRS is a rectangle with sides 3 cm and 4 cm. If its diagonals cross at O, calculate the length of PO.
A 2,5 cm B 3,0 cm C 3,5 cm
D 4,0 cm E 5,0 cm

3 The coefficient of x in the expansion of $(x-2)(x+9)$ is
A − 18 B − 2 C + 1
D + 7 E + 9

4 How many sides has a polygon if the sum of its angles is 1620°?
A 7 B 9 C 11
D 16 E 20

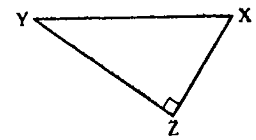
5


Fig. R6

In Fig. R6 the tangent of X is given by the ratio

A $\frac{XY}{YZ}$ B $\frac{XZ}{XY}$ C $\frac{XZ}{YZ}$

D $\frac{YZ}{XZ}$ E $\frac{YZ}{XY}$

6 The vertex, V, of a triangle is at position (5; 3). The triangle is translated 3 units to the left and then rotated through 180° about the point (1; − 2). Find the co-ordinates of the final image of V.

7 Find x, y and z if

$$2\begin{pmatrix}2 & x\\-1 & 9\end{pmatrix} + \begin{pmatrix}5 & 0\\z & 3\end{pmatrix} = 3\begin{pmatrix}y & 4\\z & 7\end{pmatrix}.$$

8 Simplify the following.

(a) $\left(\frac{y^{-3} \times y^2}{y^{-7}}\right)^{\frac{1}{2}}$ (b) $(x^{1\frac{1}{3}})^3 \times \frac{1}{x^4}$

9 Given vectors $\mathbf{OA} = \begin{pmatrix}4\\2\end{pmatrix}$, $\mathbf{OB} = \begin{pmatrix}2\\4\end{pmatrix}$ and $\mathbf{OE} = \begin{pmatrix}3\\3\end{pmatrix}$ on the cartesian plane:

(a) Find the vector **OC** such that **OC** = 2**OE**.

(b) What kind of quadrilateral is OACB?

10 The formula for converting a temperature in degrees Fahrenheit (F) to degrees Celsius (C) is $C = \frac{5}{9}(F - 32)$.

(a) Find C when $F = 149$.

(b) Make F the subject of the formula.

(c) Hence find F when C is (i) 35, (ii) − 5.

Revision exercise 3

1 (a) Convert 213_{ten} to (i) base five, (ii) base two.

(b) Convert $101\,011_{two}$ to (i) base ten, (ii) base five.

2 In Fig. R7, AP is perpendicular to BC. AB = 13 cm, BP = 5 cm, AC = 15 cm. Calculate the lengths of (a) AP, (b) BC.

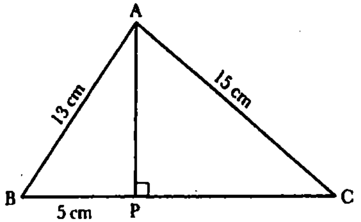


Fig. R7

3 Factorise
(a) $x^3 + x^2 + x + 1$
(b) $2 - b^3 - 2b^2 + b$
(c) $x^2 + px - qx - pq$

4 Find the value of R if $\frac{1}{R} = \frac{1}{15} + \frac{1}{23}$. Give your answer correct to 2 s.f.

5 The sum of 3 of the angles of a nonagon (9 sides) is 462°. The other 6 angles are all equal to each other. Calculate the size of each of the other angles.

6 When the angle of elevation of the sun is 64°, a boy's shadow on level ground is 80 cm long. Calculate the height of the boy to the nearest 5 cm.

7 Use tables to calculate the following correct to 3 s.f.
(a) $\frac{863}{7{,}15}$ (b) $\frac{36{,}7}{\sqrt{2{,}8}}$
(c) $\sqrt{(2{,}65)^2 + (5{,}92)^2}$

8 Given that $v = u + ft$,
(a) Find v if $u = 10$, $f = 6$ and $t = 2{,}5$;
(b) Find u if $v = 20$, $f = 10$ and $t = 1{,}5$;
(c) find t if $v = 0$, $u = 10$ and $f = -2$.

9 A quadrilateral has vertices A(−1; −1), B(1; 3), C(3; 4), D(1; 0).
(a) Express **AB** and **DC** as column vectors.
(b) Hence find the lengths of AB and DC.
(c) What kind of quadrilateral is ABCD?

10 If $M = \begin{pmatrix} 1 & 3 \\ 2 & 0 \end{pmatrix}$ and $N = \begin{pmatrix} 5 & -2 \\ -1 & 4 \end{pmatrix}$, find the value of the following.
(a) M + N (b) M − N
(c) MN (d) NM

Revision test 3

1 If $412_{five} = 1{*}7_{ten}$, what digit does the * stand for?
A 0 B 1 C 2 D 3 E 4

2 The angles of a pentagon are $x°$, $2x°$, $(x + 30)°$, $(x - 10)°$ and $(x + 40)°$. $x =$
A 30 B 50 C 60 D 80 E 108

3 Simplify $3^3 \times 6^{-3} \times 2^5$.
A 0 B 1 C 2 D 4 E 12

4 If $p = \sqrt{x - y}$, then, in terms of p and y, $x =$
A $\sqrt{p - y}$ B $p^2 + y$ C $\sqrt{p^2 + y}$
D $p^2 - y^2$ E $(p - y)^2$

5 Express $\begin{pmatrix} 4 \\ -2 \end{pmatrix} - \begin{pmatrix} -3 \\ -1 \end{pmatrix}$ as a single vector.
A $\begin{pmatrix} 1 \\ -3 \end{pmatrix}$ B $\begin{pmatrix} 7 \\ 3 \end{pmatrix}$ C $\begin{pmatrix} 7 \\ -3 \end{pmatrix}$
D $\begin{pmatrix} -1 \\ 3 \end{pmatrix}$ E $\begin{pmatrix} 7 \\ -1 \end{pmatrix}$

6 If $3M = \begin{pmatrix} 0 & -2 \\ 1 & 1 \end{pmatrix} - \begin{pmatrix} -3 & 1 \\ 7 & -2 \end{pmatrix}$ find M. Hence find M^2.

7 In Fig. R8, △PQR is right-angled at Q and QR = 5 cm.

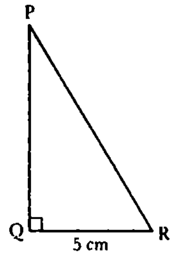


Fig. R8

(a) If the area of △PQR is 30 cm², find the length of PQ.
(b) Hence find the length of PR.

8 Simplify the following. Express the answers in standard form.
(a) $(6{,}3 \times 10^{-3}) + (3 \times 10^{-4})$
(b) $(7{,}42 \times 10^4) - (6{,}8 \times 10^3)$
(c) $(4 \times 10^5) \times (3{,}9 \times 10^{-1})$
(d) $(9{,}1 \times 10^{-5}) \div (7 \times 10^{-3})$

9 Simplify the following.
(a) $\frac{x - 4}{5} - \frac{x - 5}{3}$
(b) $\frac{2a + 3}{4} + \frac{6 - 5a}{6}$

10 In △ABC, $\hat{B} = 90°$, $\hat{A} = 23°$ and BC = 6 cm. (a) Calculate $\hat{C}$. (b) Hence or otherwise calculate AB, correct to 2 decimal places.

Revision exercise 4

1 Simplify the following, giving the answers in standard form.
(a) $(5{,}2 \times 10^2) + (6{,}24 \times 10^3)$
(b) $(7 \times 10^{-3}) - (8 \times 10^{-4})$
(c) $(4 \times 10^{-4}) \times (8 \times 10^5)$
(d) $(5{,}4 \times 10^2) \div (9 \times 10^{-3})$

2 Which of the following are Pythagorean triples?
(a) (10; 24; 26) (b) (12; 29; 31)
(c) (14; 49; 50) (d) (16; 30; 34)

3 Solve the following equations.
(a) $5x - 6(2 - x) = 32$
(b) $1\frac{1}{3} - 4\frac{1}{3} = \frac{5}{6}y + \frac{2}{3}$
(c) $\frac{x}{3} - \frac{2x - 1}{5} = \frac{1}{3}$
(d) $\frac{3x - 7}{5} - \frac{5x - 1}{3} = 4$

4 Expand the following.
(a) $(3 - x)(3 + 3x)$
(b) $(1 - y)(2y + 1)$
(c) $(7z + 9)(8z - 9)$
(d) $(2n - 5)(6 - 5n)$

5 (a) Find the sum of the angles of a hexagon.

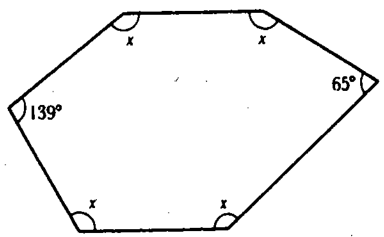


Fig. R9

(b) Hence find the value of x in Fig. R9.

6

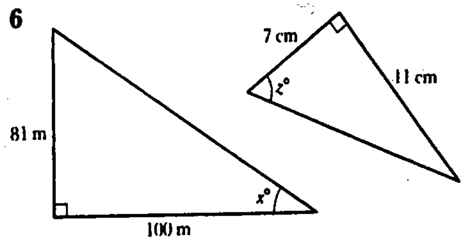


Fig. R10

Use calculation and tangent tables to find x and z in Fig. R10.

7 A woman had a son when she was x years old. When the son was y years old, the woman was p times as old as her son. Express x in terms of p and y.

8 Use logarithms to find the value of πrl, where $r = 3{,}27$, $l = 7{,}35$ and $\log \pi = 0{,}4971$.

9 Vectors **OA**, **OB**, **OC** are such that
$\mathbf{OA} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$, $\mathbf{OB} = \begin{pmatrix} 1 \\ -3 \end{pmatrix}$, $\mathbf{OC} = \begin{pmatrix} -3 \\ -4 \end{pmatrix}$.
(a) Take O as the origin and draw the vectors on a cartesian plane.
(b) Express **AB**, **BC** and **CA** as column vectors.
(c) Draw the positions of A′, B′, C′ such that **OA′** = −**OA**, **OB′** = −**OB**, **OC′** = −**OC**.
(d) Describe the transformation that would change △ABC into △A′B′C′

10 Express each of the following as a single matrix.
(a) $\begin{pmatrix} 5 & 3 \\ -1 & 7 \end{pmatrix}\begin{pmatrix} 0 \\ 3 \end{pmatrix}$
(b) $\begin{pmatrix} 1 & 2 \\ 3 & 0 \end{pmatrix}\begin{pmatrix} 1 & 4 & -2 \\ 6 & -1 & 3 \end{pmatrix}$
(c) $\begin{pmatrix} 0 & 2 & -2 \\ 7 & 5 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 3 & -1 \\ 2 & 4 \end{pmatrix}$

Revision test 4

1 Express 22_{ten} in base two.
A 1 010 B 1 110 C 10 110
D 11 010 E 101 011

2 How many minutes in x hours and y minutes?
A $x + y$ B $60x + y$ C $x + 60y$
D $\frac{x}{60} + y$ E $x + \frac{y}{60}$
3 The exterior angle of a regular decagon (ten sides) is
A 10° B 18° C 20° D 30° E 36°
4 Calculate the value of $(\frac{27}{125})^{-\frac{1}{3}} \times (\frac{4}{9})^{\frac{1}{2}}$.
A $\frac{12}{125}$ B $\frac{2}{5}$ C $\frac{3}{5}$ D $\frac{9}{10}$ E $\frac{10}{9}$
5 Evaluate $3y^2 - 5y - 6$ when $y = -2$.
A -8 B -4 C 4 D 8 E 16
6 Use tables to find the value of the following.
(a) $4{,}8^2$ (b) 48^2 (c) 480^2
(d) $\sqrt{8}$ (e) $\sqrt{80}$ (f) $\sqrt{8\,520}$
7 Find by drawing and measurement, the angle whose tangent is $\frac{5}{8}$.
8 The following numbers are in standard form. Change them to ordinary form.
(a) 9×10^2 (b) $3{,}6 \times 10^5$
(c) $6{,}1 \times 10^7$ (d) 8×10^{-4}
(e) 6×10^{-1} (f) $3{,}4 \times 10^{-3}$
9 If $\mathbf{a} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} -3 \\ -2 \end{pmatrix}$ and $\mathbf{c} = \begin{pmatrix} 0 \\ 3 \end{pmatrix}$ evaluate each of the following as a single vector.
(a) $\mathbf{a} + \mathbf{b}$ (b) $\mathbf{a} - \mathbf{b}$
(c) $\mathbf{b} - \mathbf{c}$ (d) $\mathbf{c} - \mathbf{a}$
(e) $\mathbf{a} - \mathbf{c} - \mathbf{b}$ (f) $\mathbf{a} + 3\mathbf{b} - 2\mathbf{c}$
10 $P = \begin{pmatrix} 3 & a \\ b & -1 \end{pmatrix}$ and $Q = \begin{pmatrix} -2 & 2 \\ 1 & 3 \end{pmatrix}$.
If $P + Q = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$, find the values of a and b.

on-routine problems (1)

e are three sets of non-routine problems is book: here, on page 194 and on page The problems are included to encourage to think independently, to develop skills in blem solving and to demonstrate some of enjoyable and recreational features of hematics. Some of the problems are in the of puzzles; others ask you to investigate a en situation.

here is no 'right way' to approach these blems. You may have to consider unusual n-routine) approaches. For example, the thod of trial and error is often a good first ategy. Many of the investigations are open ded. Therefore, not only is it sometimes ficult to know where to start, it is also difficult know when to stop!

Investigations can often be approached stematically. If a problem seems too complex, y a simpler example of the same kind. Find a elpful way of recording your results; this often means completing lists or tables. You may then discover a pattern which might suggest a rule. A few of the following problems give tables to get you started.

You will not normally need any materials other than paper and pencil. However, for many of the problems, a supply of cm squared paper will be very useful. Also, some problems ask you to cut shapes from cardboard or paper.

The problems are not presented in order of difficulty. Look at a few, then choose one which looks interesting to you. Then choose another, and another, ...

1 From some cardboard, cut out three pieces as shown in Fig. Q1.

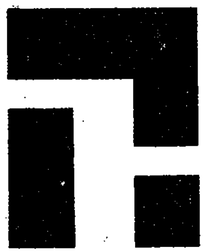


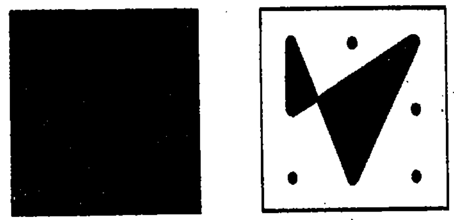
Fig. Q1

Fit the three pieces together to make a shape which has bilateral symmetry. Two examples are shown in Fig. Q2.

Fig. Q2

(a) How many symmetrical shapes can you make from the three pieces?
(b) Think of some other activities with these pieces.

2 Nine nails are arranged 5 cm apart to make a pin board as shown in Fig. Q3 (a). An elastic band is put round some of the pins as shown in Fig. Q3 (b).



(a) (b)

Fig. Q3

What is the area enclosed by the band?

3 A number is 'interesting' if the sum of its digits divides the number without remainder, e.g. 247 is 'interesting' because 2 + 4 + 7 = 13 and 247 is divisible by 13. Investigate 'interesting' numbers.

4 The rectangle in Fig. Q4 is made up of squares.

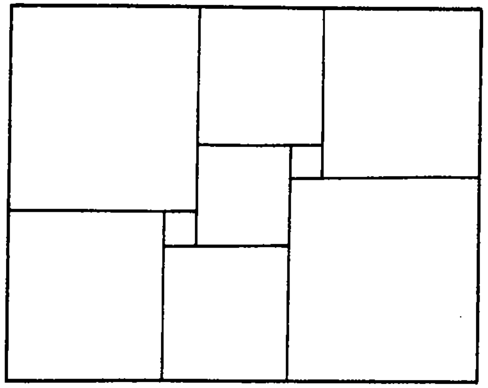


Fig. Q4

The length of the sides of each square is a whole number of centimetres. The two smallest squares have sides of length 3 cm.
(a) Find the lengths of the sides of the other squares and the area of the rectangle.
(b) Is it possible to find another way of arranging the squares to form a different rectangle?

5 Find the smallest number which leaves a remainder of 1 when divided by 5, a remainder of 5 when divided by 6 and a remainder of 3 when divided by 7. Find a rule which will give all the higher numbers with this property.

6 A cube of edge 2 cm is placed inside a sphere, so that its eight corners lie on the surface of the sphere. Find the radius of the sphere.

7 Here is a number pattern:

$$9 - 1 = 8$$
$$98 - 12 = 86$$
$$987 - 123 = 864$$

(a) Continue this pattern until you reach

$$987654321 - 123456789 =$$

(b) Can you see anything special about the final number?

8 Fig. Q5 shows a 6×6 square and a 1×4 rectangle.

(a) Your task is to make a 5×8 rectangle from these pieces. You can cut the square

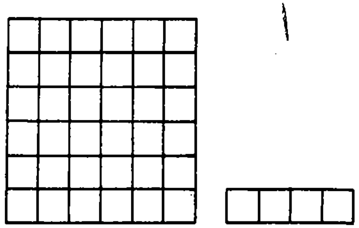


Fig. Q5

into *two pieces only*, but you can *not* cut the 1×4 strip.
(b) Investigate for other cases where you are given an $n \times n$ square and a $1 \times (n-2)$ rectangle to make a $(n-1) \times (n+2)$ rectangle.

9 Place the numbers 1 to 9 in the circles in Fig. Q6 so that each side adds up to the same total.

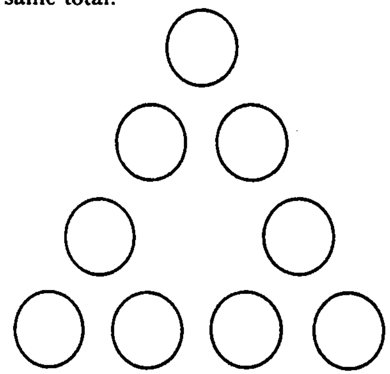


Fig. Q6

10 Fig. Q7 shows some 2×2 squares which have been shaded in different ways using up to three colours (black, white and grey).

Fig. Q7

In how many different ways can a 2×2 square be shaded in this way by three colours?
Investigate for other squares and other numbers of colours.

Fig. Q8 shows how matchsticks can be arranged to make chains of squares.

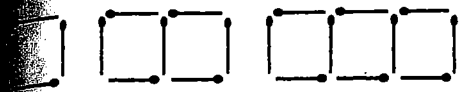


Fig. Q8

Copy and complete Table Q1.

Table Q1

number of squares	number of matches
1	4
2	7
3	10
4	
n	

12 (a) Find a 2-digit number in base five such that when the order of its digits are reversed, the number is doubled.
(b) Does such a 2-digit number exist in base ten?

13 Fig. Q9 shows an unusual snooker table which measures 4 units by 3 units. It has

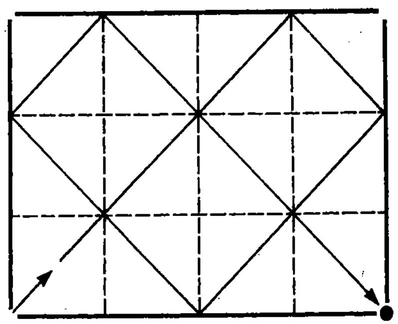


Fig. Q9

four pockets, one at each corner. A ball is projected across the table from the bottom left-hand corner at 45° to the side. It bounces off the sides, always at 45°, until it finishes in the bottom right-hand corner (BRH) as shown.

Investigate for tables of other sizes. Given a table of a certain size, is it possible to predict (a) the number of bounces the ball will make, (b) the distance it travels on the table, (c) the pocket which it finishes in?
It may help if you record your results in a table such as Table Q2.

Table Q2

table size	number of bounces	distance	final pocket
3×4	5	$12\sqrt{2}$	BRH
2×3			
3×3			
4×3			
1×1			
etc.			

14 The divisors of 12 (not counting 12 itself) are 1, 2, 3, 4, 6. The sum of these divisors is $1 + 2 + 3 + 4 + 6 = 16$. Similarly, the sum of the divisors of 16 is $1 + 2 + 4 + 8 = 15$. These numbers can be shown as a chain:

12 ⟶ 16 ⟶ 15 ⟶

(a) Continue the chain as far as you can.
(b) Investigate chains starting with other numbers, e.g. start with 10, 18, 20, 26, 37, 28 or 24.

15 Ms M asked four students A, B, C, D to tell her their marks in a test. The students told Ms M that they wanted to give *her* a test. They would each tell her two marks, one true and one false in either possible order:
A said, 'B scored 16. C scored 11.'
B said, 'D got 12. C got 16.'
C said, 'A's mark was 16. D's mark was 21.'
Before D could speak, Ms M said, 'I've just remembered that you all got different marks and one of you got 21. Now I know your marks!'
What were the marks?

Chapter 11

General arithmetic (2) Fractions, decimals, percentages

Fractions

Exercise 11a (Oral revision)

1 Find the number of cents in the following.
(a) $\frac{1}{2}$ of \$1 (b) $\frac{1}{4}$ of \$1
(c) $\frac{1}{5}$ of \$1 (d) $\frac{7}{10}$ of \$1
(e) $\frac{1}{4}$ of \$2 (f) $\frac{3}{5}$ of \$2

2 Find the number of minutes in the following.
(a) $\frac{1}{2}$ hour (b) $\frac{1}{4}$ hour
(c) $\frac{1}{10}$ hour (d) $\frac{3}{4}$ hour
(e) $\frac{1}{20}$ of 2 hours (f) $\frac{7}{15}$ of 2 hours

3 Find the number of seconds in the following.
(a) $\frac{1}{4}$ min (b) $\frac{3}{4}$ min
(c) $\frac{5}{6}$ min (d) $\frac{9}{10}$ min
(e) $\frac{1}{15}$ of 2 min (f) $\frac{5}{6}$ of 2 min

4 Find the number of grammes in the following.
(a) $\frac{3}{10}$ kg (b) $\frac{3}{8}$ kg
(c) $\frac{3}{4}$ kg (d) $\frac{71}{100}$ kg
(e) $\frac{889}{1000}$ of 2 kg (f) $\frac{5}{8}$ of 2 kg

5 Find $\frac{1}{4}$ of \$3 in cents.
6 Find $\frac{1}{8}$ of 3 km in metres.
7 Find $\frac{1}{3}$ of $2\frac{1}{4}$ hours in minutes.
8 Find $\frac{1}{4}$ of 6 cm in millimetres.
9 Find $\frac{4}{7}$ of 2,1 kg in grammes.
10 Find $\frac{1}{6}$ of 4,5 m in millimetres.
11 In each of the following, express the first quantity as a fraction of the second. Give the fraction in its lowest terms.
(a) 15 s, 1 min (b) 50 s, 1 min
(c) 10 min, $1\frac{1}{2}$ h (d) 25 cm, 3 m
(e) 40c, \$1 (f) 500 g, 2 kg
(g) 5 days, 5 weeks (h) 800 m, 3 km
12 Calculate the value of the following.
(a) $\frac{3}{8}$ of \$4 (b) $\frac{2}{3}$ of 3 h
(c) $\frac{7}{10}$ of 5 kg (d) $\frac{2}{5}$ of 3 m
(e) $\frac{5}{6}$ of 4 min (f) $\frac{5}{8}$ of 4,8 cm
(g) $\frac{5}{9}$ of \$2,70 (h) $\frac{3}{4}$ of 2 h 24 min
(i) $\frac{4}{7}$ of 6,3 km (j) $\frac{5}{6}$ of 1,8 m
(k) $\frac{5}{9}$ of 5,4 cm (l) $\frac{3}{10}$ of 5,6 g
(m) $\frac{5}{7}$ of 3 weeks (n) $\frac{2}{11}$ of \$3,41

Example 1
Which fraction is the greater, $\frac{2}{5}$ or $\frac{5}{13}$?

The common denominator of $\frac{2}{5}$ and $\frac{5}{13}$ is 65 (i.e. 5 × 13).

$$\frac{2}{5} = \frac{2 \times 13}{5 \times 13} = \frac{26}{65}$$

$$\frac{5}{13} = \frac{5 \times 5}{13 \times 5} = \frac{25}{65}$$

Since $\frac{26}{65} > \frac{25}{65}$, $\frac{2}{5}$ is the greater fraction.

Example 2
Simplify $(1\frac{5}{8} + 1\frac{3}{5}) \div 5\frac{3}{8}$.

$$(1\tfrac{5}{8} + 1\tfrac{3}{5}) \div 5\tfrac{3}{8} = \left(\frac{13}{8} + \frac{8}{5}\right) \div \frac{43}{8}$$

$$= \left(\frac{65}{40} + \frac{64}{40}\right) \times \frac{8}{43}$$

$$= \frac{129}{40} \times \frac{8}{43} = \frac{3 \times 43}{5 \times 8} \times \frac{8}{43}$$

$$= \tfrac{3}{5}$$

Example 3
$\frac{3}{5}$ of a class of 40 students study history and $\frac{3}{4}$ study geography. Every student studies at least one of these subjects. (a) How many students study both subjects? (b) What fraction of the class studies history but not geography?

(a) Number of students
studying history $= \frac{3}{5} \times 40 = 24$
Number of students
studying geography $= \frac{3}{4} \times 40 = 30$
Total number of students studying history and geography
$= 24 + 30 = 54$
Since there are only 40 students in the class, number studying both subjects
$= 54 - 40 = 14$

(b) Of the 24 students studying history, 14 also study geography.
Number studying history only $= 24 - 14$
$= 10$
Fraction studying history only $= \frac{10}{40} = \frac{1}{4}$
or:
Since $\frac{3}{4}$ of the class study geography, $\frac{1}{4}$ of the class study history but not geography.

Exercise 11b

1 Simplify the following.
(a) $\frac{11}{12} + \frac{5}{6}$ (b) $\frac{3}{4} - \frac{3}{8}$
(c) $4\frac{2}{3} - 1\frac{5}{8}$ (d) $3\frac{4}{7} + 1\frac{5}{8}$
(e) $\frac{1}{2} - \frac{3}{8} + \frac{3}{4}$ (f) $9\frac{5}{6} + 5\frac{3}{4} - 6\frac{2}{3}$
(g) $\frac{4}{5} \times \frac{3}{8}$ (h) $\frac{14}{15} \div \frac{28}{35}$
(i) $1\frac{3}{5} \div 2\frac{2}{15}$ (j) $\frac{16}{21} \times 1\frac{3}{32}$
(k) $2\frac{1}{4} \times 3\frac{1}{3} \div 4\frac{2}{5}$ (l) $1\frac{1}{6} \div 1\frac{1}{6} \times 3\frac{3}{5}$
2 Find the fraction which is the greater, $\frac{7}{15}$ or $\frac{8}{17}$.
3 Determine which is the greatest of $\frac{7}{8}$, $\frac{3}{4}$, $\frac{10}{13}$.
4 Simplify the following.
(a) $(2\frac{1}{3} + 3\frac{11}{18}) \div 6\frac{1}{4}$
(b) $(7\frac{3}{8} - 1\frac{1}{2}) \div (4\frac{3}{8} + 4\frac{1}{8})$
(c) $1\frac{1}{2} + 2\frac{1}{4} \times \frac{3}{4} - \frac{1}{2}$
(d) $\dfrac{2\frac{1}{4} + 3\frac{3}{4}}{2\frac{1}{4} - 1\frac{1}{2}}$
(e) $6\frac{1}{6} - 2\frac{3}{4} + 1\frac{1}{3}$
(f) $\dfrac{\frac{1}{3} - \frac{1}{6}}{\frac{1}{3} + \frac{1}{6}}$
5 Express 0,575 as a fraction in its lowest terms.
6 A student has 80c. She spends 16c. What fraction of her original money does she have left?
7 A flag-pole 6,3 m long is driven 1,4 m into the ground. What fraction of the pole is above the ground?
8 There are 572 students in a mixed school. $\frac{6}{11}$ of them are boys. How many girls are there?
9 How much less than 6 is the sum of $2\frac{3}{4}$ and $2\frac{1}{8}$?
10 A notebook has 128 pages and 88 of them have been used. What fraction of the notebook remains?
11 After spending $\frac{3}{8}$ of her money on sweets and $\frac{1}{6}$ on cosmetics, a woman was left with \$6,50. How much money did she have originally?
12 A woman spent $\frac{1}{3}$ of her money at the market, $\frac{1}{4}$ at the chemist's, $\frac{1}{6}$ at the electrical shop and had \$1,85 left. How much money had she first?
13 Kudzai can hoe a garden in 5 hours and Farai can hoe it in 4 hours. What fraction of the garden can each of them hoe in 1 hour? If they work together, what fraction of the garden will they hoe in 1 hour? How long will it take them to hoe the whole garden working together?
14 How many pieces of string each $8\frac{1}{2}$ cm long can be cut from a string $42\frac{1}{2}$ cm long?
15 (a) What is the product of $1\frac{7}{9}$ and $3\frac{3}{4}$?
(b) What must be added to the product of $1\frac{7}{9}$ and $3\frac{3}{4}$ to make a total of $8\frac{1}{3}$?
16 Divide $20\frac{15}{16}$ by $6\frac{1}{4}$ and add the result to the product of $1\frac{3}{4}$ and $3\frac{1}{6}$.
17 $\frac{3}{4}$ of the girls in a 3rd form play netball and $\frac{4}{7}$ play volleyball. Every girl plays at least one of these games. If 27 girls play both games, how many girls are there in the 3rd form?
18 In an election there were 3 candidates. $\frac{3}{7}$ of the electors voted for the winner. The runner-up received $\frac{5}{8}$ of the remaining votes.
(a) What fraction of the electors voted for the third candidate? (b) If the winner received 3 021 votes more than the runner-up, how many electors voted?
19 A man made a will in which he left $\frac{4}{9}$ of his money to his wife and $\frac{2}{5}$ of the remainder to his eldest child. The rest was to be shared equally among his four younger children. If each of the younger children received \$540, what was the wife's share?
20 During one year in a school, $\frac{5}{8}$ of the students had measles, $\frac{1}{3}$ had chicken-pox and $\frac{1}{6}$ had neither. What fraction of the school had both measles and chicken-pox?

Decimals

Exercise 11c (Oral revision)

1 Evaluate the following.
(a) 3,42 × 100 (b) 0,725 × 1 000
(c) 22,55 × 1 000 (d) 0,041 × 100
(e) 9,84 ÷ 10 (f) 9,84 ÷ 100

(g) $38{,}6 \div 1\,000$ (h) $7\,905 \div 100$
(i) $2{,}7 \times 10^4$ (j) $35{,}8 \div 10^3$
(k) $650 \div 10^2$ (l) $0{,}0005 \times 10^3$

2 Express the following as decimal fractions.
(a) $\frac{1}{2}$ (b) $\frac{1}{4}$ (c) $\frac{3}{4}$ (d) $\frac{1}{5}$
(e) $\frac{3}{5}$ (f) $\frac{9}{10}$ (g) $\frac{7}{20}$ (h) $\frac{21}{50}$

3 Evaluate the following.
(a) $0{,}9 \times 3$ (b) $6 \times 0{,}4$
(c) $8 \times 0{,}2$ (d) $0{,}5 \times 5$
(e) $0{,}7 \times 0{,}4$ (f) $0{,}3 \times 0{,}8$
(g) $0{,}1 \times 0{,}5$ (h) $0{,}2 \times 0{,}3$
(i) $0{,}6 \times 0{,}08$ (j) $0{,}07 \times 0{,}09$

4 Evaluate the following.
(a) $3{,}5 \div 7$ (b) $3{,}5 \div 0{,}7$
(c) $2{,}4 \div 0{,}8$ (d) $2{,}4 \div 8$
(e) $0{,}036 \div 9$ (f) $3{,}6 \div 0{,}09$
(g) $\dfrac{100}{2{,}5}$ (h) $\dfrac{9{,}6}{0{,}32}$
(i) $\dfrac{0{,}99}{11}$ (j) $\dfrac{0{,}056}{0{,}8}$

When adding or subtracting decimals, always place the numbers so that decimal commas are under each other, units are under units, and so on.

Example 4

Simplify the following correct to 3 decimal places.
$11{,}416 + 0{,}463\,4 + 25{,}400\,33 - 32{,}378\,91$

addition:
$$\begin{array}{r} 11{,}416 \\ +\ 0{,}463\,4 \\ +\ 25{,}400\,33 \\ \hline 37{,}279\,73 \end{array}$$

subtraction:
$$\begin{array}{r} 37{,}279\,73 \\ -\ 32{,}378\,91 \\ \hline 4{,}900\,82 \end{array}$$

$11{,}416 + 0{,}463\,4 + 25{,}400\,33 - 32{,}378\,91$
$= 4{,}900\,82$
$= 4{,}901$ to 3 d.p.

Notice, in Example 4, that rounding off is done *after* addition and subtraction.

When dividing by decimals, simplify where possible by making equal divisions and multiplications of the numerator and denominator.

Example 5

Simplify $\dfrac{20{,}3 \times 0{,}88}{2{,}24}$

$\dfrac{20{,}3 \times 0{,}88}{2{,}24} = \dfrac{2{,}9 \times 0{,}11}{0{,}04}$ (after equal division by 7 and 8)

$= \dfrac{2{,}9 \times 11}{4}$ (multiplying num. and denom. by 100)

$= \dfrac{31{,}9}{4}$

$= 7{,}975$

Example 6

Convert \$352 *into Francs if* £1 = \$1,65 *and* £1 = 11,10 *Francs.*

From the given exchange rates,
\$1,65 = 11,10 Francs (= £1)
By unitary method,

$\$1 = \dfrac{11{,}10}{1{,}65}$ Francs

$\$352 = \dfrac{11{,}10 \times 352}{1{,}65}$ Francs

$= \dfrac{1110 \times 352}{165}$ Francs

$= 2\,368$ Francs

Exercise 11d

1 Add the following and give your answers correct to 3 significant figures. (Do not round off until addition has been done.)
(a) 18,058; 0,302; 7,82; 0,004
(b) 13,66; 4,318; 0,7; 26,85; 5,2
(c) 0,213; 0,008 7; 7,32; 0,614 39
(d) 0,065; 0,304 4; 0,735 6; 0,997
(e) 30,043; 0,74; 12,375; 0,823; 6,005

2 Simplify the following correct to 2 decimal places.
(a) $0{,}678 + 0{,}742\,9 - 0{,}820\,5$
(b) $7{,}382 - 0{,}795\,3 + 0{,}058\,9$
(c) $2{,}683 - 6{,}808 + 5{,}316$
(d) $0{,}71 + 98{,}438 - (8{,}263 + 1{,}737)$
(e) $9{,}205 - 3{,}142 - 3{,}25 - 1{,}094$

3 Simplify the following.
(a) $\dfrac{0{,}1}{0{,}001}$ (b) $\dfrac{6{,}75 \times 7{,}5}{0{,}375}$
(c) $\dfrac{(0{,}2)^3 \times 30}{(0{,}4)^2}$ (d) $\dfrac{2{,}25 \times 7{,}5}{3{,}75}$
(e) $\dfrac{10{,}2 \times 1{,}4}{4{,}8}$
(f) $\dfrac{13{,}2 \times 0{,}051}{0{,}198}$
(g) $\dfrac{0{,}97 + 0{,}56}{0{,}97 - 0{,}12}$
(h) $\dfrac{12 - 2{,}76}{0{,}87 + 2{,}49}$
(i) $0{,}64 \times (1{,}184\,2 - 0{,}809\,2)$
(j) $1{,}103 + 0{,}42 \times 2{,}85$

4 Without using tables, divide 0,689 85 by 3,15.

5 Evaluate $0{,}047 \times 70$ correct to 1 decimal place.

6 Write $10 + \dfrac{1}{10} + \dfrac{8}{100} + \dfrac{3}{10\,000}$ as a decimal number.

7 Given that $4{,}928 \times 37{,}5 = 184{,}8$, evaluate $49{,}28 \times 0{,}375$.

8 How many notebooks at \$0,95 each can a trader buy for \$17,10? If she sells them at \$1,25 each, find her total profit.

9 A man walks at the rate of 88 paces to the minute. If the average length of his pace is 0,875 m, find the time he takes to walk 2,31 km.

10 Posts at the side of a road are 3,75 m apart and extend for three-quarters of a kilometre. How many posts are there?

11 A quantity of identical plastic blocks weighs 10,98 kg. If each block weighs 54,9 g, how many blocks are there?

12 Divide the sum of 3,19 and 2,39 by 3,6.

13 By how much is the product of 0,2 and 15,4 less than 10?

14 In a school, 0,6 of the total number of children have had chicken-pox, while 162 children have not had chicken-pox. What is the enrolment of the school?

15 Which is greater, $(0{,}2)^3$ or $(0{,}3)^2$? How much greater?

Percentages

Exercise 11e (Oral revision)

1 Express the following percentages (i) as fractions in their lowest form, (ii) as decimals.
(a) 25% (b) 50% (c) 75%
(d) 20% (e) 10% (f) 30%
(g) 90% (h) 5% (i) 3%
(j) 55% (k) $33\frac{1}{3}$% (l) $66\frac{2}{3}$%

2 Express the following fractions and decimals as percentages.
(a) $\frac{3}{5}$ (b) 0,25 (c) $\frac{7}{10}$
(d) 0,4 (e) $\frac{9}{20}$ (f) 0,01
(g) $\frac{1}{25}$ (h) 0,95 (i) $\frac{17}{50}$
(j) 0,73 (k) $\frac{5}{8}$ (l) 0,325

3 Express the first quantity as a percentage of the second.
(a) \$1, \$2 (b) 20c, \$1
(c) 3 km, 5 km (d) 270°, 360°
(e) 30°, 90° (f) 12,5 cm, 50 cm
(g) 300 m, 1 km (h) 1,2 litres, 3 litres
(i) 60c, \$2 (j) \$1, \$1,50

4 Calculate the following.
(a) 10% of \$1 (b) 25% of \$1
(c) 40% of \$2 (d) 60% of \$1,50
(e) 80% of 1 kg (f) $12\frac{1}{2}$% of 60 mℓ
(g) 5% of \$8 (h) 70% of 2 m
(i) $33\frac{1}{3}$% of 21c (j) $66\frac{2}{3}$% of \$24

5 In a box of 200 mangoes 44 are bad. What percentage is bad?

6 In a test a student obtained 30 marks out of a possible 40. What percentage was this?

7 A woman gets a 10% pay rise. If her present wage is \$45 per week, calculate her new weekly wage.

8 A man pays 15% of his taxable income as tax. If his taxable income is \$3 000, how much tax does he pay?

9 A man buys a car costing \$8 500. He pays a 20% deposit. How much is the deposit?

10 A trader reduces all her prices by 10%. What will be the price of a pair of shoes originally marked at \$31?

When finding percentage increase or decrease, the percentage is calculated on the **original amount**. First express the increase (or decrease) as a fraction of the original amount, then convert the fraction to a percentage by multiplying it by 100.

Example 7

The price of a carpet is given as $450 or $210 down and the balance in twelve monthly payments of $24,75. What percentage more will be paid if the carpet is bought on hire purchase?

Hire purchase price = $210 + 12 × $24,75
= $210 + $297
= $507

Extra cost of hire purchase
= $507 − $450
= $57

Extra cost as fraction of original cost
$= \frac{\$57}{\$450} = \frac{57}{450}$

Extra cost as a percentage of original cost
$= \frac{57}{450} \times 100\%$
$= \frac{19 \times 100}{150}\%$
$= 12\frac{2}{3}\%$

There are a number of ways of finding the result of increasing or decreasing a given quantity by a given percentage. Example 8 shows some of these methods.

Example 8

Increase 180 mℓ *by* 40%.

1st method (unitary method):
100% of the quantity = 180 mℓ
1% of the quantity = $\frac{180}{100}$ mℓ
140% of the quantity = $\frac{180}{100} \times 140$ mℓ
= 252 mℓ

2nd method:
40% of 180 mℓ = $\frac{40}{100}$ of 180 mℓ = 72 mℓ
Required quantity = 180 mℓ + 72 mℓ
= 252 mℓ

3rd method:
Required quantity = $(1 + \frac{40}{100})$ of 180 mℓ
= $\frac{140}{100}$ of 180 mℓ
= 252 mℓ

Example 9

Decrease $360 *by* 15%.

Either:
15% of $360 = $\frac{15}{100}$ of $360
$= \$\frac{15 \times 360}{100}$
$= \$\frac{3 \times 36}{2} = \$54$

Required amount = $360 − $54
= $306

or:
Required amount = 85% of $360
$= \frac{85}{100} \times \$360$
$= \$\frac{17 \times 36}{2}$
= $306

Example 10

186 is the result of increasing a number by 20%. *Find the number.*

By unitary method
120% of the number = 186
1% of the number $= \frac{186}{120}$
100% of the number $= \frac{186}{120} \times 100$
$= \frac{186 \times 5}{6}$
= 31 × 5
= 155

The number is 155.

Unless instructed otherwise, profit and loss are always calculated as a percentage of the *cost price.*

Example 11

A mat is bought for $14,00 *and sold at a profit of* 35%. *Calculate the selling price.*

Selling price = 135% of cost price
$= \frac{135}{100}$ of $14,00
$= \$\frac{27 \times 14{,}00}{20}$
= $27 × 0,7
= $18,90

Example 12

By selling an item for $6 900 *a dealer makes a profit of 15%. How much did the item cost?*

By unitary method,
115% of cost price = $6 900
1% of cost price $= \frac{\$6\,900}{115}$
100% of cost price $= \$\frac{6\,900 \times 100}{115}$
$= \$\frac{6\,900 \times 20}{23}$
= $300 × 20
= $6 000

The item cost $6 000.

Example 13

A bicycle is sold for $567 *at a loss of* 12%. *Calculate the cost price.*

Since 100 − 12½ = 87½,
selling price = 87½% of cost price.
87½% of cost price = $567
100% of cost price $= \$567 \times \frac{100}{87\frac{1}{2}}$
$= \$567 \times \frac{200}{175}$
$= \$567 \times \frac{8}{7}$
= $81 × 8
= $648

or:
12½% = ⅛
⅞ of cost price = $567
$\frac{8}{8}$ of cost price = $567 × $\frac{8}{7}$
= $648

Notice that in all of the above examples the working out and simplification is left as late as possible.

Exercise 11f

1 By what fraction should a quantity be multiplied to (a) increase it, (b) decrease it, by: 20%, 50%, 37%, 6%, 130%? (Do not simplify the fractions: i.e. leave each fraction with a denominator of 100.)
2 Find the selling price of an article which is bought for
(a) $1,25 and sold at a profit of 24%,
(b) 75c and sold at a profit of 12%,
(c) $8,50 and sold at a profit of 46%,
(d) $340 and sold at a loss of 6%,
(e) $9,20 and sold at a loss of 7½%.
3 Find the gain or loss per cent when an article is bought for
(a) 30c and sold for 42c,
(b) $7,60 and sold for $6,65,
(c) $3,50 and sold for $4,34,
(d) $16,40 and sold for $19,27,
(e) $184 and sold for $101,20.
4 Find the cost price of an article which is sold for
(a) $77 at a profit of 10%,
(b) $5,85 at a profit of 30%,
(c) $9,40 at a profit of 17½%,
(d) $15,75 at a loss of 16%,
(e) $319,60 at a loss of 6%.
5 What percentage of 4 is 5?
6 When a farm of 325 ha is increased by 16%, what is its new area?
7 A woman's income increases from $6 020 to $6 923. Calculate the increase per cent.
8 A boy spends 32% of his money and has 34c left. How much had he at first?
9 By selling a radio for $84,00, a dealer gained 12%. How much money did she gain?
10 In an examination a girl gets 425 marks out of 625. Calculate her percentage.
11 In an examination a student scores 78 out of a maximum 90 marks. Express this result as a percentage, correct to three significant figures.
12 A spring which was 35 cm long is stretched so that its length is increased by 16%. Calculate its new length.
13 A storekeeper decided to give a 10% discount on all purchases during the Christmas season. How much would a

customer pay for an item that originally cost $240?

14 A tailor gives a discount of $2\frac{1}{2}$% for cash payments. Calculate the reduced price of a suit marked $81,60.

15 A car dealer gained $600 on a sale. If this was equivalent to an 8% profit, what was the cost price of the car?

16 By selling an article for $35,00, a dealer lost 30%. For how much should she have sold it to gain 30%?

17 A trader made a loss of 10% on a bicycle he sold for $810. If he had sold it for $864 what would have been his percentage loss or gain?

18 A factory increases its annual production of radios from 4 325 to 4 671.
(a) Calculate the increase per cent.
(b) Calculate the number of radios it would have had to produce for an increase of 12%.

19 By selling an article for $21,75 a woman makes a profit of 16%.
(a) Calculate how much the article cost.
(b) For how much should she have sold it to make a profit of 28%?

20 In April, Shupikai bought a bag of corn for $24,00. When she wanted to buy more in June she found that she could only buy $\frac{3}{4}$ of a bag for $24,00. What was the percentage increase in price from April to June?

21 A man earning $5 625 per annum is awarded a pay rise of 8%. Calculate his new annual salary. [Camb]

22 Express 1,26 metres as a percentage of 4,5 metres. [Camb]

23 The total cost of a car service consists of a basic price plus a tax of 15%. Given that the total cost is $690, calculate the basic price of the service. [Camb]

24 In a batch of 150 articles 6% were defective. Calculate the number which were not defective. [Camb]

25 A man is deciding whether to buy or rent a new radio set. The model he wants costs $400 and the dealer charges an additional $3\frac{1}{2}$% of this cost to install it. During the first year no charge will be made for repairs. After this the man estimates that repairs will cost $20 for each of the next four years, and then $35 for each of the following three years. At the end of these eight years he expects to receive a trade-in value of $20 for the set when he buys a new one. Calculate (i) the installation charge, (ii) the total estimated repair cost, (iii) the estimated net cost of the set over the eight years (that is, the total he expects to pay less the trade-in value).

The cost to rent the same set is $8,40 per month during the first year but $7\frac{1}{2}$% discount is allowed if the year's rental is paid in advance. Calculate the rental for this year if it is paid in advance. [Camb]

Simple interest

When money is borrowed, the price paid for using the money is called **interest**. Similarly, a person who saves money receives payments of interest proportional to the money he saves.

The money borrowed (or saved) is called the **principal**. The interest is usually a percentage of the principal for each year the money is borrowed. For example, if someone borrows $100 at 12% **p.a.**, the interest is $12 each year. p.a. is short for *per annum* and means 'yearly'. In 1 year the interest is $12, in 2 years the interest is $24, and so on. When interest is paid in regular intervals like this, it is called **simple interest**.

Any one of simple interest, I, principal, P, percentage rate p.a., R, or time in years, T, may be calculated by substitution in the formula:

$$I = \frac{PRT}{100}$$

Example 14

Calculate the simple interest on $450 which is borrowed for $1\frac{1}{2}$ years at 8% p.a.

$$I = \frac{PRT}{100} \quad \text{where } P = \$450,\ R = 8,\ T = 1\tfrac{1}{2}$$

$$I = \$\frac{450 \times 8 \times 1\frac{1}{2}}{100} = \$\frac{9 \times 12}{2} = \$54$$

The simple interest is $54.

Example 15

At what rate per cent per annum is simple interest paid when $5 680,00 yields an interest of $170,40 in 6 months?

From the data of the question,
$P = \$5\,680$, $I = \$170{,}40$, $T = \frac{1}{2}$
Note that T must be expressed in years.

Using the formula, $I = \dfrac{PRT}{100}$,

$$\$170{,}40 = \$\frac{5\,680 \times R \times \frac{1}{2}}{100}$$

Make R the subject of the formula.

$$R = \frac{100 \times 170{,}4}{5\,680 \times \frac{1}{2}} = \frac{17\,040}{2\,840} = \frac{1\,704}{284} = \frac{426}{71}$$

$$= 6$$

The rate is 6% p.a.

Exercise 11g

1 Calculate the simple interest on the following.
(a) $250 for 8 years at 6% p.a.,
(b) $28,20 for 7 years at 5% p.a.,
(c) $74,40 for 2 yr 7 mo at 10% p.a.,
(d) $108 for 8 months at 9% p.a.,
(e) $221,40 for 1 yr 4 mo at $8\frac{1}{3}$% p.a.

2 Calculate the principal that will earn interest of
(a) $81 in 12 years at $7\frac{1}{2}$% p.a.,
(b) $297,50 in 7 years at 5% p.a.,
(c) $63,12 in 10 years at $5\frac{1}{3}$% p.a.,
(d) $162,89 in 4 yr 4 mo at 4% p.a.,
(e) $130,32 in 9 years at $12\frac{1}{2}$% p.a.

3 Find the time in which
(a) $250 will earn $52,50 at 6% p.a.,
(b) $216,25 will earn $121,10 at 7% p.a.,
(c) $294,25 will earn $117,70 at 8% p.a.,
(d) $159,60 will amount* to $191,52 at 5% p.a.,
(e) $150 will amount* to $277,50 at $8\frac{1}{2}$% p.a.
(*The **amount** is the sum of the principal and the interest.)

4 Calculate the rate per cent p.a. at which
(a) $245 will earn $117,80 in 8 years,
(b) $360 will earn $81 in 3 years,
(c) $162 will earn $43,20 in 4 years,
(d) $485,40 will amount to $631,02 in 6 years,
(e) $372,68 will amount to $652,19 in 10 years.

5 For how long must I leave $400 in a bank to earn an interest of $48, the rate being 3% p.a. simple interest?

6 Calculate the principal which earns $750,75 simple interest in 11 years at 7% p.a.

7 Calculate the simple interest on $146 from 6 March 1992 to 19 May 1992 (inclusive) at 7% p.a. (Use the calendar given in the Tables on page 277 to find the number of days in March and April.)

8 A woman invests $50,00 at 5% simple interest for 4 years. How much was her investment worth by the end of this period?

9 If simple interest on loans is increased from 5% to $5\frac{1}{2}$% per annum, what would be the increase in interest charged on a loan of $1 200 for a period of 4 months?

10 Calculate which gives the greater interest and by how much: $400 invested for 10 years at 8% p.a. simple interest, or $400 invested for 9 years at 9% p.a. simple interest.

11 A person invests $840 for 3 years and receives $63 simple interest. At what rate per cent per annum was the money invested?

12 A bank charges $28 simple interest on a sum of money which is borrowed for four months. Given that the rate of interest is 15% per annum, calculate the sum of money. [Camb]

13 If $3 900 is borrowed at 11% p.a. simple interest, what total payment will settle the debt after two weeks?

14 A woman wants to buy a refrigerator costing $642. If she has it now and pays for it by instalments for a year, the price is increased by 15%. However, if she saves a certain sum now at 7% per annum simple interest, it will amount to $642 in 1 year. Calculate how much money she saves by waiting a year for the refrigerator.

Chapter 12

Solving triangles (3) Sine and cosine

Sine and cosine

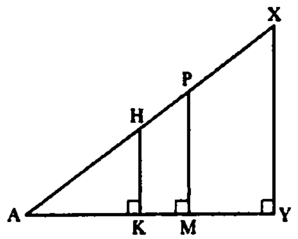


Fig. 12.1

In Fig. 12.1, △s HAK, PAM, XAY are similar.

Thus, $\frac{KH}{AH} = \frac{MP}{AP} = \frac{YX}{AX}$

The value of this ratio depends only on the size of Â. The ratio is called the **sine of Â.** This is usually shortened to **sin A.**

Similarly, $\frac{AK}{AH} = \frac{AM}{AP} = \frac{AY}{AX}$ is a ratio whose size depends only on the size of Â. This ratio is called the **cosine of Â**, usually shortened to **cos A**.

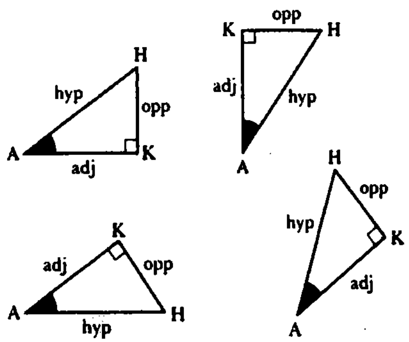


Fig. 12.2

Fig. 12.2 shows △AHK in various positions. The sides of the triangle are as follows:
AH, the hypotenuse,
KH, the side opposite to Â,
AK, the side adjacent to Â.
These are abbreviated to **hyp**, **opp**, **adj** respectively, so that

$$\sin A = \frac{\text{opp}}{\text{hyp}} \qquad \cos A = \frac{\text{adj}}{\text{hyp}}$$

Example 1

Find, by drawing and measurement, approximate values for sin 25°, *cos* 25°, *sin* 48°, *cos* 48°, *sin* 70°, *cos* 70°.

On graph paper, draw a quadrant of a circle of radius 10 units.

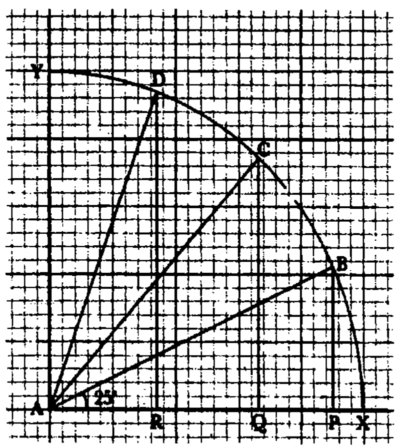


Fig. 12.3

Note: In Fig. 12.3, to save space, a scale of 1 cm to 2 units has been used. In practice, it is better to use a scale of 1 cm to 1 unit.

Draw BÂX = 25°. Construct P on AX so that BP̂A = 90°. Then, in △APB,

$$\sin 25° = \frac{PB}{AB} = \frac{4{,}2}{10} = 0{,}42$$

$$\cos 25° = \frac{AP}{AB} = \frac{9}{10} = 0{,}9$$

Similarly, in △ACQ, CÂQ = 48° and

$$\sin 48° = \frac{QC}{AC} = \frac{7{,}4}{10} = 0{,}74$$

$$\cos 48° = \frac{AQ}{AC} = \frac{6{,}7}{10} = 0{,}67$$

In △ADR, DÂR = 70° and

$$\sin 70° = \frac{RD}{AD} = \frac{9{,}4}{10} = 0{,}94$$

$$\cos 70° = \frac{AR}{AD} = \frac{3{,}4}{10} = 0{,}34$$

Example 2

Find by drawing (a) the angle whose sine is 0,56, *(b) the angle whose cosine is* 0,60.

(a) $0{,}56 = \frac{5{,}6}{10}$

It is necessary to construct a right-angled triangle with a hypotenuse of 10 units and a side of 5,6 units. Draw an arc of a circle centre O and radius 10 cm. Draw perpendicular radii OX and OY. Draw a line parallel to OX and 5,6 cm from OX to cut the arc at A. Draw AB perpendicular to OX.

Fig. 12.4 is a scale drawing of the construction.

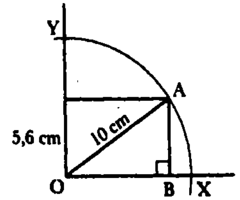


Fig. 12.4

In Fig. 12.4,

$$\sin A\hat{O}B = \frac{BA}{OA} = \frac{5{,}6}{10} = 0{,}56$$

By measurement, AÔB ≃ 34°.

(b) $0.60 = \frac{6}{10}$

Draw an arc of radius 10 cm and perpendicular radii OX and OY as before. Mark off OB = 6 cm along OX. Construct BA perpendicular to OX to cut the arc at A. Fig. 12.5 is a scale drawing of the construction.

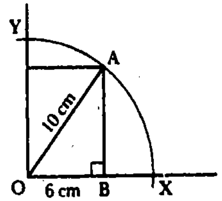


Fig. 12.5

In Fig. 12.5, $\cos A\hat{O}B = \frac{6}{10} = 0{,}60$

By measurement, AÔB ≃ 53°.

Exercise 12a

1 Find by drawing and measurement, as in Example 1, approximate values for (a) sin 20°, cos 20°, (b) sin 40°, cos 40°, (c) sin 65°, cos 65°.

2 Find by drawing and measurement, as in Example 2, approximate sizes of angles A, B, C, D, E, F where
(a) sin A = $\frac{1}{2}$, (b) cos B = $\frac{3}{8}$,
(c) sin C = $\frac{3}{5}$, (d) cos D = 0,95,
(e) sin E = 0,26, (f) cos F = 0,34.

Use of sine and cosine

Sines and cosines of angles are used to find the lengths of unknown sides in triangles. Table 12.1 gives the sines and cosines of some chosen angles.

Table 12.1

angle A	sin A	cos A
30°	0,5000	0,8660
35°	0,5736	0,8192
40°	0,6428	0,7660
45°	0,7081	0,7071
50°	0,7660	0,6428
55°	0,8192	0,5736
60°	0,8660	0,5000

The values in Table 12.1 are given to 4 significant figures.

Example 3

Calculate the value of x in Fig. 12.6.

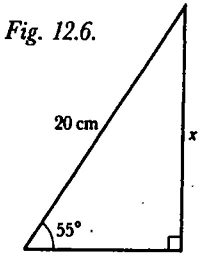


Fig. 12.6

In Fig. 12.6, the hypotenuse is given and x is *opposite* the given angle. Use the *sine* of the given angle.

$$\sin 55° = \frac{x}{20}$$

$$\Leftrightarrow \quad x = 20 \times \sin 55° \text{ cm}$$
$$= 20 \times 0{,}8192 \text{ cm}$$
$$= 16{,}384 \text{ cm}$$
$$= 16{,}4 \text{ cm to 3 s.f.}$$

Example 4

A village is 8 km on a bearing of 040° from a point O. Calculate how far the village is north of O.

Fig. 12.7 shows the position of the village, V, in relation to O.

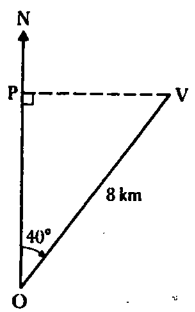


Fig. 12.7

It is required to find the length of OP. OP is *adjacent* to the known angle. Use the *cosine* of 40°.

$$\cos 40° = \frac{OP}{8}$$

$$\Leftrightarrow \quad OP = 8 \times \cos 40° \text{ km}$$
$$= 8 \times 0{,}7660 \text{ km}$$
$$= 6{,}128 \text{ km}$$
$$= 6{,}13 \text{ km to 3 s.f.}$$

The village is 6,13 km north of O.

Notice that if the unknown side is *opposite* the given angle, use the *sine* of the angle; if the unknown side is *adjacent* to the given angle, use the *cosine* of the angle.

Exercise 12b

Use the values in Table 12.1 in this exercise. Give all answers correct to 3 s.f.

1 Find the value of x in each of the triangles in Fig. 12.8.

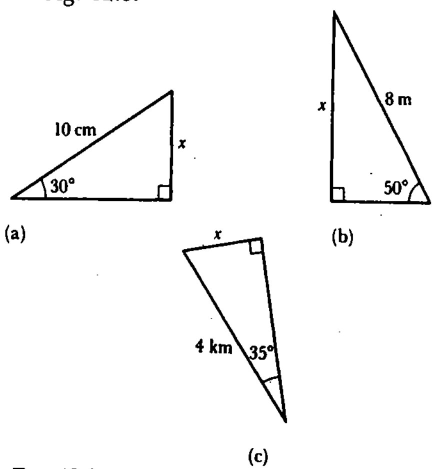


Fig. 12.8

2 Find the value of y in each of the triangles in Fig. 12.9.

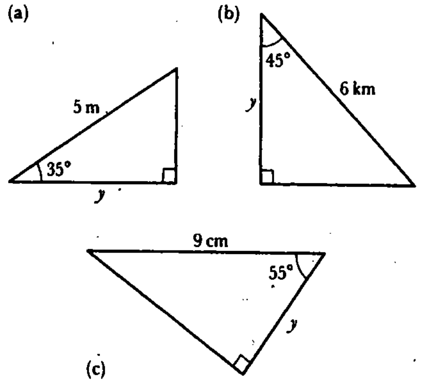


Fig. 12.9

3 Find the value of z in each of the triangles in Fig. 12.10.

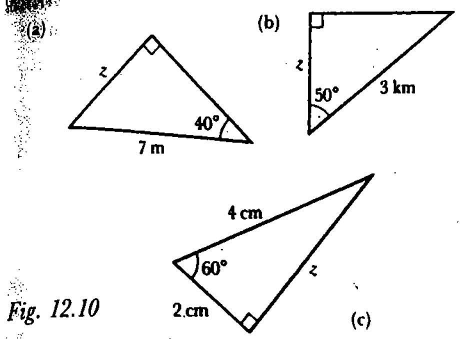


Fig. 12.10

4 A ladder, 5 m long, leans against a wall so that it makes an angle of 60° with the horizontal ground. Calculate how far up the wall the ladder reaches.

5 A village is 10 km on a bearing 050° from a point O. Calculate how far the village is north of O.

6 A diagonal of a square is 20 cm long. How long is each side?

7 The vertical angle of a cone is 70° and its slant height is 11 cm. Calculate the height of the cone.

8 A rhombus of side 10 cm has obtuse angles of 110°. Sketch the rhombus, showing its diagonals and as many angles as possible. Hence calculate the lengths of the diagonals of the rhombus.

Using sine and cosine tables

Four-figure sine and cosine tables are given on pages 281 and 282. These are used in much the same way as tangent tables.

Notice the following:

1 In the sine table, as angles increase from 0° to 90°, their sines *increase* from 0 to 1.

2 In the cosine table, as angles increase from 0° to 90°, their cosines *decrease* from 1 to 0.

Exercise 12c (Oral or written)

Use the tables on pages 281 and 282 to find the value of the following.

1 sin 56°	**2** sin 80°	**3** sin 5°
4 cos 41°	**5** cos 78°	**6** cos 12°
7 cos 74°	**8** sin 16°	**9** sin 38°
10 cos 52°	**11** sin 21°	**12** cos 69°
13 sin 43,5°	**14** sin 60,8°	**15** sin 14,2°
16 cos 19,6°	**17** cos 80,8°	**18** cos 33,3°
19 sin 45° 12′	**20** sin 25° 54′	**21** sin 81° 24′
22 cos 30° 30′	**23** cos 9° 48′	**24** cos 56° 6′
25 cos 54,7°	**26** sin 35,3°	**27** sin 28,6°
28 cos 61,4°	**29** cos 66° 24′	**30** sin 23° 36′

Example 5

Use tables to find the angle (a) whose sine is $\frac{2}{7}$, (b) whose cosine is 0,4475.

(a) Let the angle be A, then $\sin A = \frac{2}{7}$.
Express $\frac{2}{7}$ as a decimal correct to 4 d.p.
$\frac{2}{7} = 0{,}28571\ldots = 0{,}2857$ to 4 d.p.
$\sin A = 0{,}2857$
Looking within the sine table entries, 0,2857 is opposite 16° and under 0,6°. Thus A = 16,6° to the nearest 0,1°.

(b) Let $\cos B = 0{,}4475$.
In the table, cos 63° 24′ = 0,4478
Comparing the decimal fractions:
4478 − 4475 = 3
Thus 3 needs to be subtracted from 4478 to give the required 4475. In the cosine table, when differences are subtracted the corresponding minutes are *added*. In the differences column the value 3 can be found in the 1′ column. Thus, add 1′ to the 63° 24′ to give:
B = 63° 25′

Notes:

1 In the sine table, the differences are added as usual.

2 In the cosine table, because cosines of acute angles decrease as the angles increase, the differences are subtracted.

Exercise 12d (Oral or written)

Use tables to find the angles whose (a) sines, (b) cosines are as follows.

1 0,5878	**2** 0,7986	**3** 0,3584
4 0,6018	**5** 0,5299	**6** 0,4067
7 0,8339	**8** 0,1685	**9** 0,5165
10 0,5990	**11** 0,7302	**12** 0,0732
13 0,4949	**14** 0,5555	**15** 0,2000
16 $\frac{2}{3}$	**17** $\frac{9}{25}$	**18** $\frac{4}{7}$
19 0,2212	**20** 0,5673	**21** 0,6309

Example 6

A wire 12 m long goes from the top of a 6-metre pole to a point on a vertical wall 10 m above the ground. What is the angle between the wire and the wall? (Assume that the wire is stretched tight.)

Fig. 12.11 represents the data of the question. θ is the required angle.

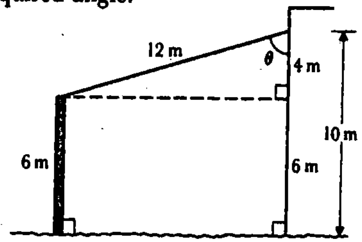


Fig. 12.11

Adding the construction line shown dotted in Fig. 12.11,

$\cos\theta = \frac{4}{12} = \frac{1}{3} = 0{,}3333$

$\theta = 70° \; 28'$

Example 7

Calculate the length of the hypotenuse of the triangle in Fig. 12.12.

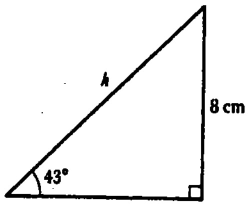


Fig. 12.12

In Fig. 12.12,

$$\sin 43° = \frac{8}{h}$$

$\Leftrightarrow h \times \sin 43° = 8$

$$\Leftrightarrow \quad h = \frac{8}{\sin 43°}\text{ cm} = \frac{8}{0{,}6820}\text{ cm}$$

From reciprocal tables, $\frac{1}{0{,}6820} = 1{,}466$.

Hence, $h = 8 \times 1{,}466$ cm
$= 11{,}728$ cm
$= 11{,}7$ cm to 3 s.f.

Example 8

A car travels 120 m along a straight road which is inclined at 8° to the horizontal. Calculate the vertical distance through which the car rises.

Fig. 12.13 is a sketch of the road.

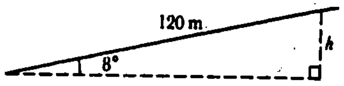


Fig. 12.13

In Fig. 12.13, h is the vertical distance.

$$\sin 8° = \frac{h}{120}$$

$\Leftrightarrow \quad h = 120 \times \sin 8°$ m
$= 120 \times 0{,}1392$ m
$= 16{,}704$ m
$= 16{,}7$ to 3 s.f.

Exercise 12e

Give all calculated lengths correct to 2 significant figures. Give all calculated angles correct to the nearest 0,1°.

1 Calculate the lengths a, b, c, d, e, f, g, h in Fig. 12.14, all lengths being in cm.

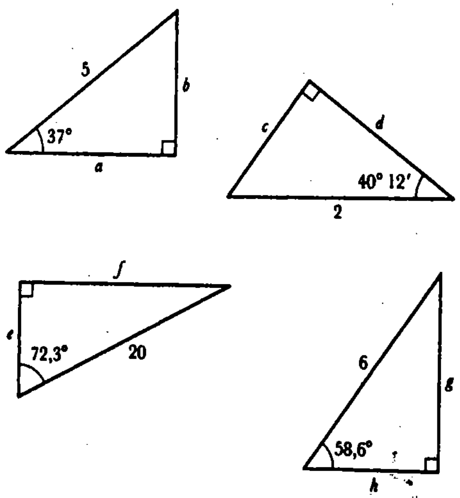
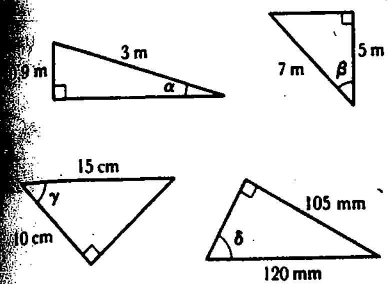


Fig. 12.14

...lculate the angles $\alpha, \beta, \gamma, \delta$ in Fig. 12.15.



...12.15

Calculate the length of the hypotenuse in each of the triangles in Fig. 12.16, all lengths being in cm.

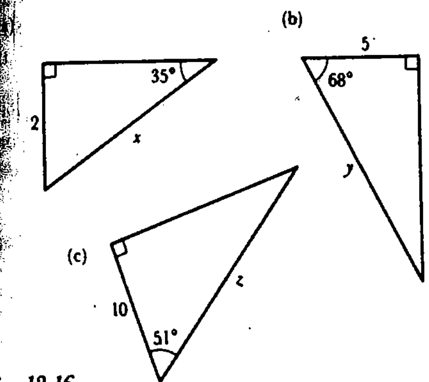
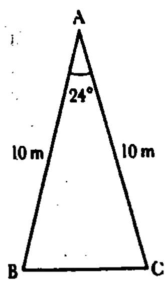
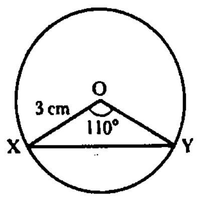


Fig. 12.16

4 Make suitable construction lines, then calculate the lengths BC, XY and PQ in Fig. 12.17.





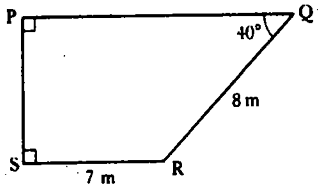


Fig. 12.17

5 Calculate the angles $\alpha, \beta, \gamma, \delta$ in Fig. 12.18.

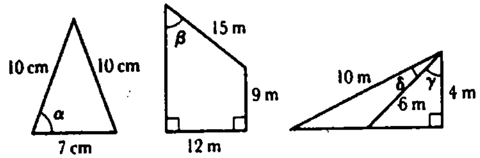


Fig. 12.18

6 A tightly stretched wire goes from a point on horizontal ground to the top of a vertical pole. If the wire is 8 m long and is inclined at 68° to the horizontal, calculate the height of the pole.

7 A point P is 40 km from Q on a bearing 061°. Calculate the distance that P is (a) north of Q, (b) east of Q.

8 Fig. 12.19 is a side view of a table which is supported by legs inclined at θ to the horizontal.

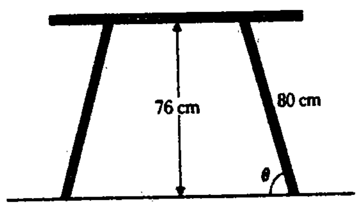


Fig. 12.19

If the table is 76 cm high and each leg is 80 cm long, calculate the value of θ.

9 The roof of a hut is made from sheets of corrugated iron of length 2 m inclined at 18° to the horizontal. (See Fig. 12.20.)

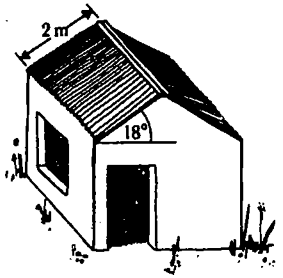


Fig. 12.20

Calculate the width of the hut.

10 A stone rolls 300 m down a slope. As it falls, it drops 120 m vertically.
Calculate the angle of the slope (α in Fig. 12.21).

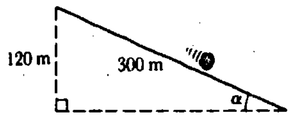


Fig. 12.21

11 A stone is suspended from a point P by a piece of string 50 cm long. It swings back and forward.

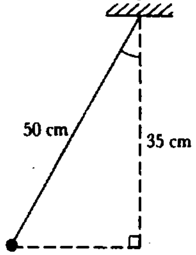


Fig. 12.22

Calculate the angle the string makes with the vertical when the stone is 35 cm vertically below P.

12 An aeroplane is flying at a height of 200 m. Its angle of elevation to an observer on the ground is 23°.

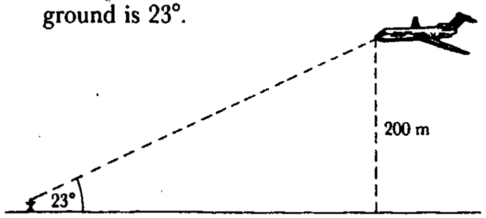


Fig. 12.23

Calculate the distance of the aeroplane from the observer.

13 Fig. 12.24 shows some workers using a board to slide loads from a platform on to a lorry.

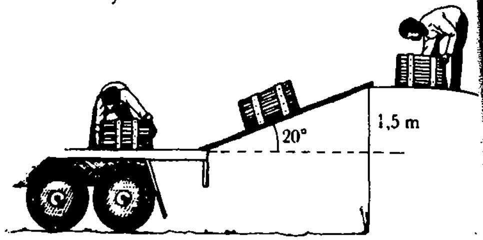


Fig. 12.24

The platform is 1,5 m higher than the lorry. It is found that the best position for the board is when it is inclined at 20° to the horizontal. Calculate the length of the board.

14 A 5-metre plank rests on a wall 2 m high, so that 1,5 m of the plank projects beyond the wall.

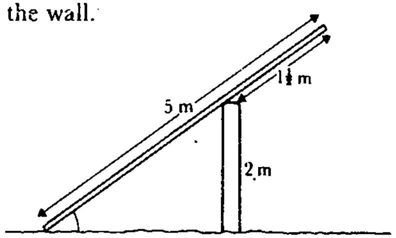


Fig. 12.25

(a) What angle does the plank make with the wall? (b) How high is the end of the plank above the ground?

15 The arms of a pair of compasses are 10 cm long and the angle between them is 35°. Calculate the radius of the circle that the compasses will draw.

Solving right-angled triangles (Summary)

Sines, cosines and tangents of angles are known as **trigonometrical** ratios. **Trigonometry** means the measurement of lengths and angles.

Right-angled triangles can be solved by using Pythagoras' theorem and the trigonometrical ratios.

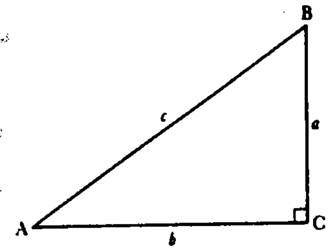


Fig. 12.26

In Fig. 12.26, △ABC is any triangle right-angled at C.

1 $c^2 = a^2 + b^2$

2 $\sin A = \frac{a}{c}$, $\sin B = \frac{b}{c}$

3 $\cos A = \frac{b}{c}$, $\cos B = \frac{a}{c}$

4 $\tan A = \frac{a}{b}$, $\tan B = \frac{b}{a}$

Exercise 12f

1 From Fig. 12.27, write down the trigonometrical ratios of (a) $\sin\theta$, (b) $\cos\theta$, (c) $\tan\theta$ in as many ways as possible in terms of a, b, c, d and e.

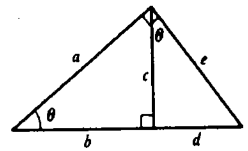


Fig. 12.27

2 Find, to the nearest cm, the length of the shadow of a 1-metre vertical stick when the elevation of the sun is 33°.

3 In Fig. 12.28, calculate α and β. Hence find the size of AB̂C.

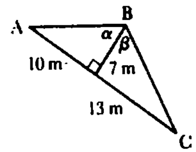


Fig. 12.28

4 In Fig. 12.29, find the size of QR̂S.

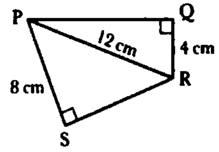


Fig. 12.29

5 A girl walks 800 m on a bearing of 129°. Calculate how far (a) east, (b) south she is from her starting point.

6 An equilateral triangle has three sides of length 2 m. Calculate the height of the triangle (a) using Pythagoras' theorem, (b) using a trigonometrical ratio.

7 In Fig. 12.30, (a) find HL, (b) hence find LN̂H.

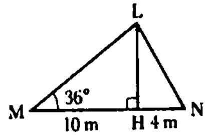


Fig. 12.30

8 A regular pentagon is drawn so that its vertices lie on the circumference of a circle of radius 4,5 cm. Find the length of a side of the pentagon to the nearest mm.

9 A rhombus of side 5 cm has acute angles of 84°. Find the lengths of the diagonals of the rhombus.

10 A rectangular table has sides 2 m and 1,2 m. It is pushed into the corner of a room so that one of the long sides makes 20° with a wall. Fig. 12.31 shows a plan of the corner.

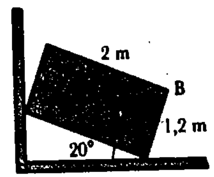


Fig. 12.31

Find the distance of the corner B from each wall.

Chapter 13

Scale drawing (4) Solids, plans, elevations

Freehand sketches of solids (Revision)

Exercise 13a (Revision)

Do not use a ruler in this exercise. All drawings should be freehand.

1 Fig. 13.1 shows four steps in drawing a cuboid.

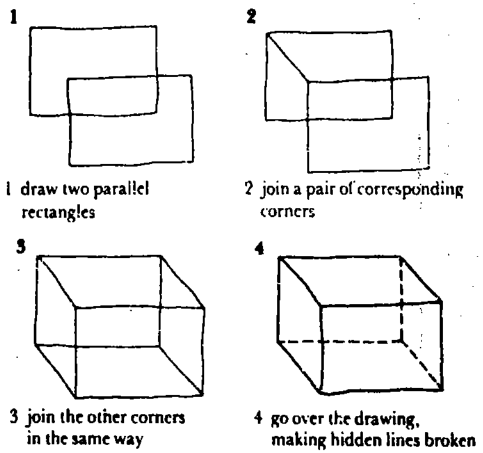


Fig. 13.1

Use the method of Fig. 13.1 to draw some cuboids. Practise until you can draw a good freehand cuboid.

2 Make freehand copies of the cuboids shown in Fig. 13.2.

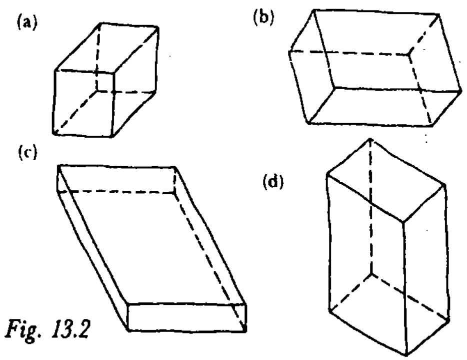


Fig. 13.2

3 Fig. 13.3 shows four steps in drawing a square-based pyramid.

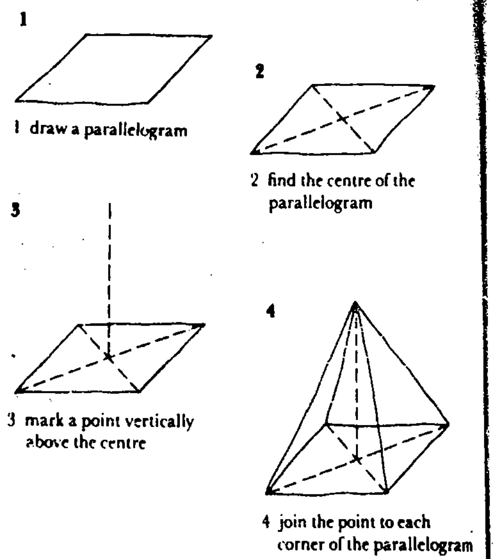


Fig. 13.3

Use the method of Fig. 13.3 to draw some square-based pyramids. Practise until you can draw a good pyramid.

4 Make some copies of Fig. 13.4.

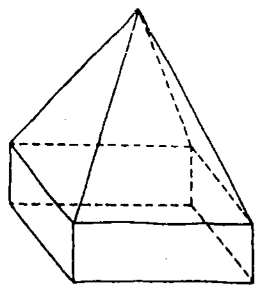


Fig. 13.4

Fig. 13.5 shows four steps in sketching a cylinder.

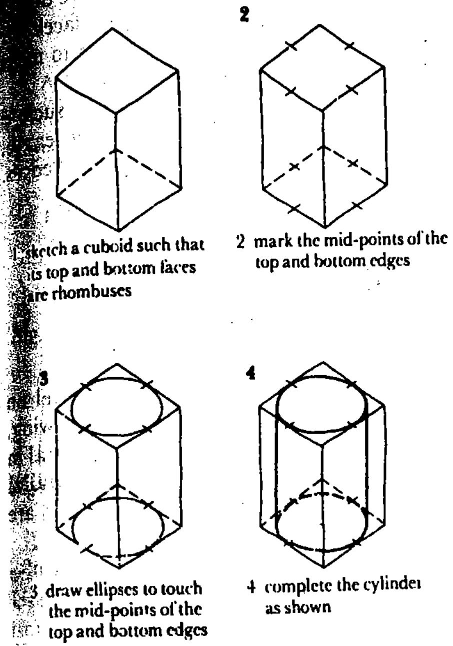


Fig. 13.5

Use the method of Fig. 13.5 to draw some cylinders. Practise until you can draw cylinders without drawing the surrounding cuboids.

6 Make freehand copies of the cylinders in Fig. 13.6.

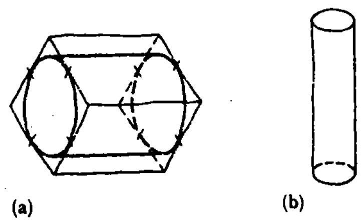


Fig. 13.6

7 Fig. 13.7 shows three steps in drawing a cone.

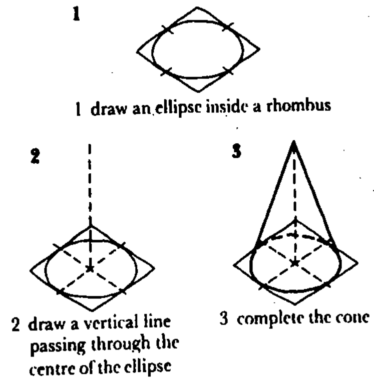


Fig. 13.7

Use the method of Fig. 13.7 to draw some cones. Practise until you can draw cones without drawing the starting rhombus.

8 Make freehand copies of the solids in Fig. 13.8.

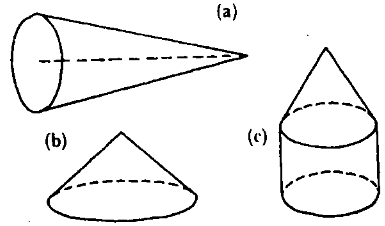


Fig. 13.8

9 Fig. 13.9 shows a method of sketching a triangular prism.

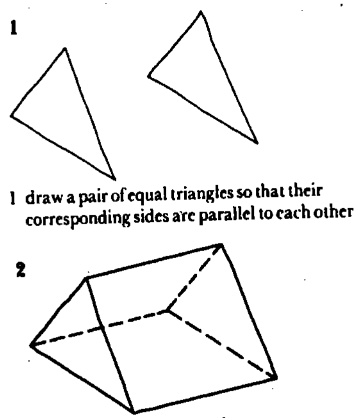


Fig. 13.9

Use the method of Fig. 13.9 to sketch some triangular prisms.

10 Make freehand copies of the solids in Fig. 13.10.

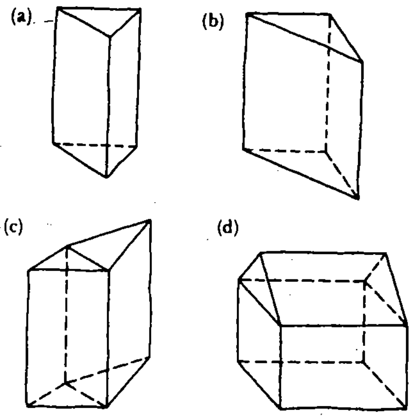


Fig. 13.10

Exercise 13a shows some of the difficulties of drawing solids on plain paper. This is because solids have **3 dimensions**, length, breadth and height, whereas drawing paper has only 2 dimensions, length and breadth. The remainder of this chapter discusses two ways of making accurate drawings of solids:

1 **Parallel projection,** where attempts are made to draw the whole solid.

2 **Orthogonal projection,** where the solid is split up and its parts are drawn separately.

Parallel projection

Fig. 13.11 shows a cuboid 4 cm long, 3 cm wide and 2 cm high.

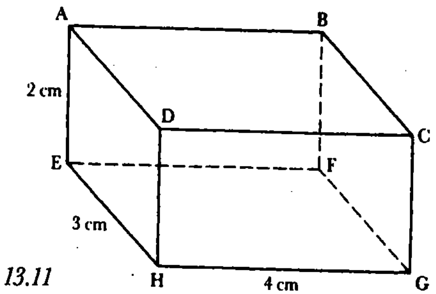


Fig. 13.11

Each face of a cuboid is in the shape of a rectangle. However, in Fig. 13.11, only the front and back faces, DCGH and ABFE, are drawn as full size rectangles. The other faces are drawn as parallelograms. This helps to make the drawing look as if it has depth.

In Fig. 13.11, all of the 3 cm edges, such as EH, are *shortened* to about $\frac{2}{3}$ of their true length. They are also drawn at 45° to the vertical edges.

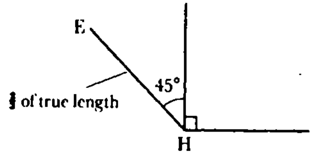


Fig. 13.12

Notice that all lines which are parallel on the cuboid are also parallel on the drawing. Vertical lines on the cuboid are also vertical in the drawing. Such drawings are called **parallel projections**. (In technical drawing, they are also called *oblique projections*.)

Fig. 13.13 shows a parallel projection of a square based pyramid.

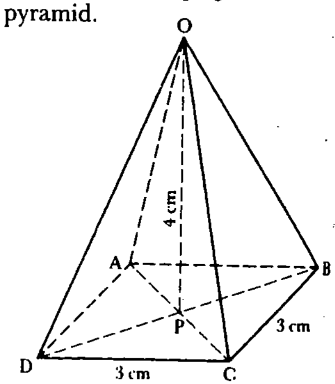


Fig. 13.13

In this figure, only OP, CD and AB are drawn to scale; all the other lines have been shortened.

Fig. 13.14 shows a different parallel projection of the 4 cm by 3 cm by 2 cm cuboid. In this figure, each edge is drawn to the correct length and *all* the faces are represented by parallelograms. This is a special parallel projection called an *isometric* projection. In an isometric projection, all horizontal and vertical edges appear correctly to scale; lines representing horizontal edges meet vertical lines at 60°.

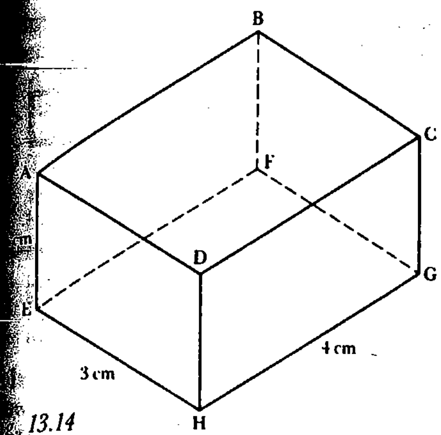


Fig. 13.14

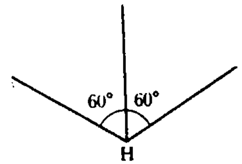


Fig. 13.15

For quick and accurate drawings, use a 45° set square when drawing parallel projections. Use a 60°/30° set square when drawing isometric projections. Always make a freehand sketch *before* making an accurate drawing.

Exercise 13b

1 Fig. 13.16 shows some common objects drawn in parallel projection.

(a) State what each object is.

(b) Which of the objects have been drawn in isometric projection?

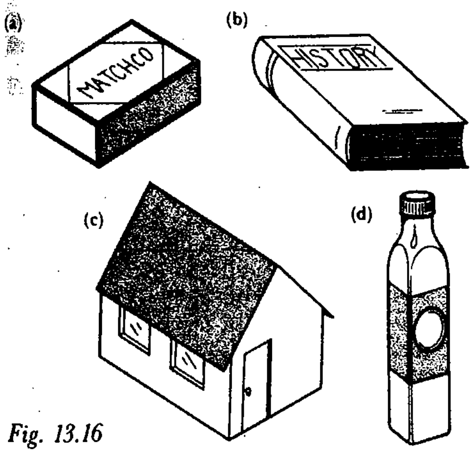


Fig. 13.16

2 Draw a cube of edge 3 cm in full-size parallel projection.

3 Draw a cube of edge 4 cm in full-size isometric projection.

4 Draw a full-size parallel projection of a cuboid 5 cm long, 4 cm wide and 3 cm high.

5 Draw the cuboid of question 4 in isometric projection.

6 A building is in the shape of a cuboid 10 m long, 4 m wide and 6 m high. Use a scale of 1 cm to 2 m to draw an isometric projection of the building.

7 Draw a full-size parallel projection of a pyramid 8 cm high on a square base of side 5 cm.

8 A triangular prism is 8 cm long and has a cross-section in the shape of an equilateral triangle of side 4 cm. Draw the prism full size in parallel projection such that (a) a rectangular face is horizontal, (b) a triangular face is horizontal.

Parallel projections give a good idea of the shape of the solids they represent. However, since many lengths and angles are changed, only a few useful measurements can be taken from them. In practice, engineers use a different kind of projection to represent solids.

Orthogonal projection

Fig. 13.17 is an isometric projection of a matchbox.

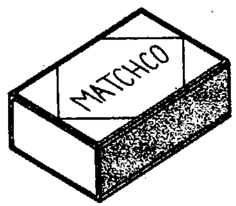


Fig. 13.17

This picture of the matchbox can be broken down into three main parts:

(a) the **plan**: the shape of the matchbox when looking at it from a point vertically above it;

(b) the **front elevation**: the shape of the matchbox when looking at it directly in front;
(c) the **side elevation**: the shape of the matchbox from the side.

Fig. 13.18 shows the directions of view for the plan, front elevation and side elevation of the matchbox.

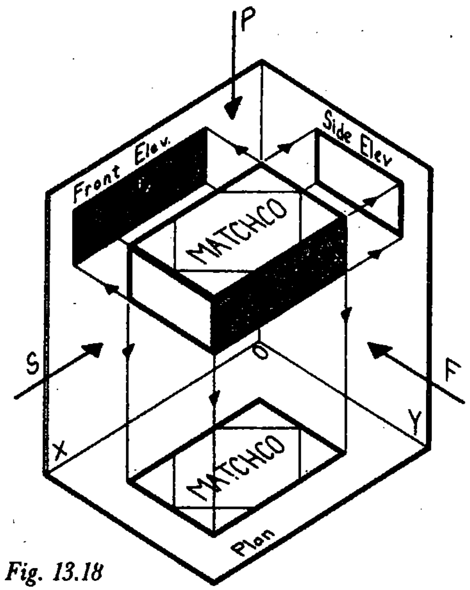


Fig. 13.18

Imagine that the matchbox is placed parallel to three planes as shown. Each plane is perpendicular to the other two.

The *plan* is the shape seen from arrow P. It is shown on the plane below the matchbox.

The *front elevation* is the shape seen from arrow F. It is shown on the plane behind the matchbox.

The *side elevation* is the shape seen from S. It is shown on the plane on the other side of the matchbox.

The plan and elevations can be seen properly by cutting the planes along OY and flattening them out. This is shown in Fig. 13.19.

Fig. 13.19 is called the **orthogonal projection** of the matchbox. An orthogonal projection of a solid contains its plan and one or more elevations. Each is shown separately and each is drawn accurately to scale.

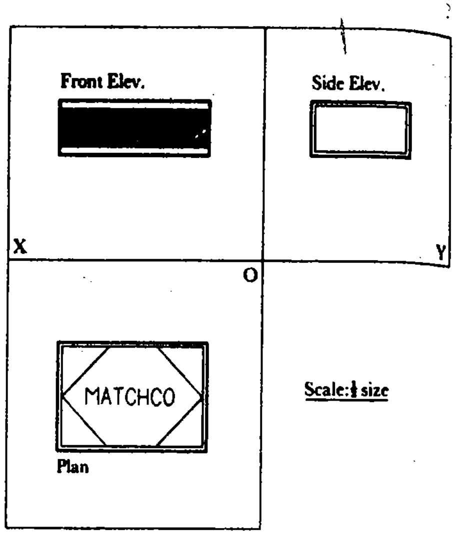


Fig. 13.19

Fig. 13.20 shows the isometric and orthogonal projections of a simple building.

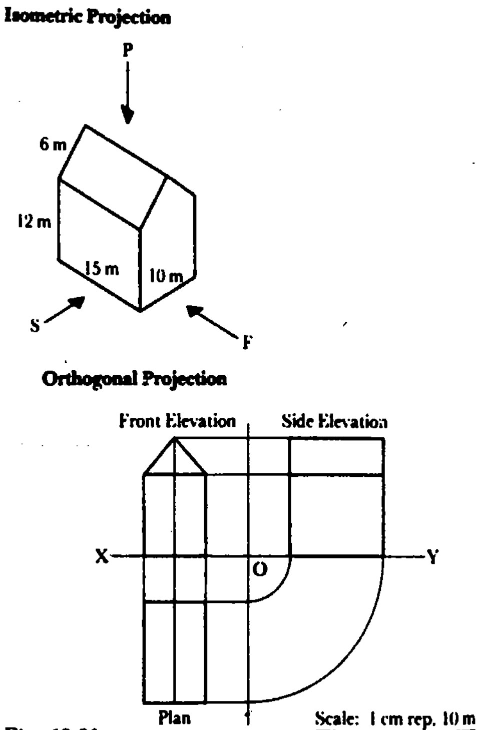


Fig. 13.20

Notice the following about the orthogonal projection:

(a) A **ground line**, XOY, is drawn. The elevations normally stand on the ground line.
(b) There are thin lines connecting the plan and elevations. These are construction lines. They are used to transfer measurements from one part of the drawing to another.
(c) Each part of the drawing is drawn accurately to scale.

In practice, begin by drawing whichever view is the easiest to construct. Obtain the other views from the first drawing using measurements where necessary. In Fig. 13.20 the order of working might be as follows:

1 Construct a ground line XOY.
2 Draw the front elevation accurately to scale.
3 Construct the plan. Draw thin vertical lines through the corners of the front elevation. Use these to get the width and the central line of the plan. The length, 15 m, is found by measurement, using the same scale as before.
4 Construct the side elevation on line XOY. Draw horizontal lines from the front elevation and arcs, centre O, from the plan. Use these lines and arcs to draw the side elevation.
5 Label each view and state the scale.

Example 1

Construct the plan and front and side elevations of a cone of height 4 cm *and base diameter* 3 cm. *Hence find the slant height of the cone.*

First, make a sketch of the cone.

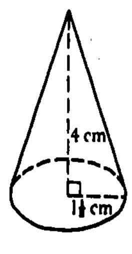


Fig. 13.21

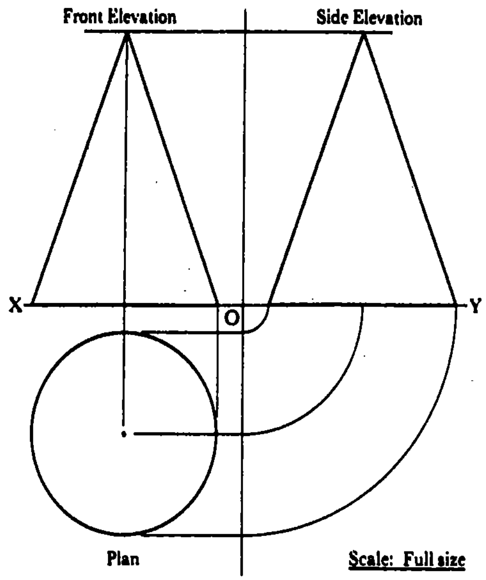


Fig. 13.22

Draw the plan and elevations.

The order of working was as follows:
1 Base line XOY.
2 Plan: a circle of radius 1,5 cm. Note the point at the centre to represent the vertex of the cone.
3 Front elevation: isosceles triangle of height 4 cm.
4 Side elevation: same as front elevation.

By measurement, slant height = 4,3 cm.

Exercise 13c

1 Name the solids given by the plans and front elevations in Fig. 13.23 overleaf.

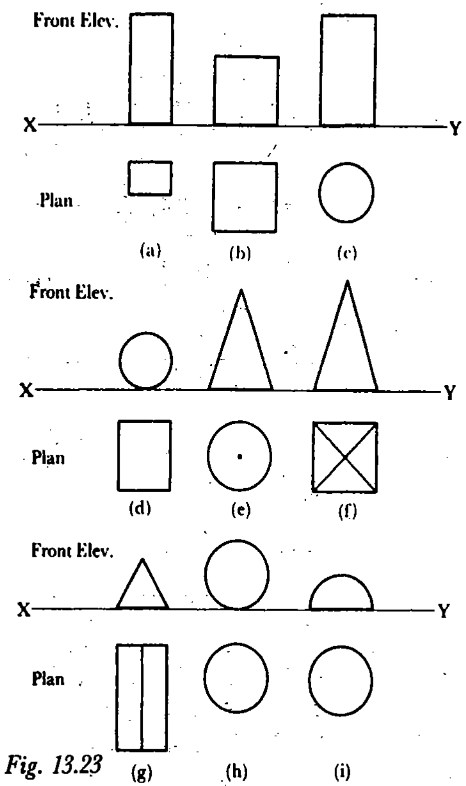


Fig. 13.23

2 Draw the plan and the front and side elevations of the cuboids sketched in Fig. 13.24. All dimensions are in cm.

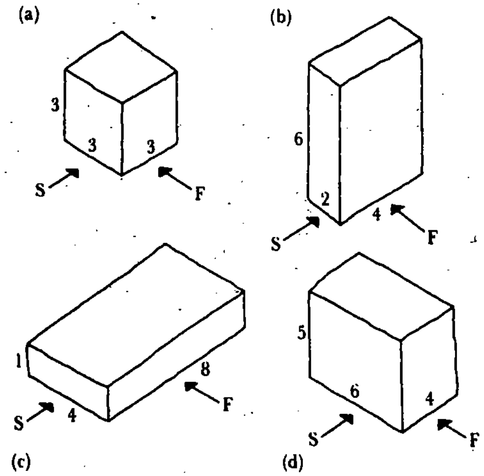


Fig. 13.24

3 Draw the plan and the front and side elevations of the cylinders sketched in Fig. 13.25. All dimensions are in cm.

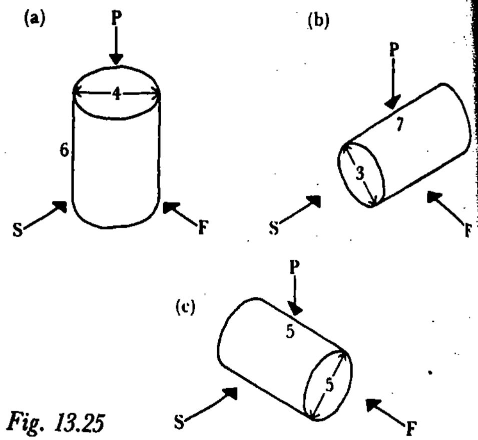


Fig. 13.25

4 Draw the plan and the front and side elevations of the solids sketched in Fig. 13.26. All dimensions are in cm.

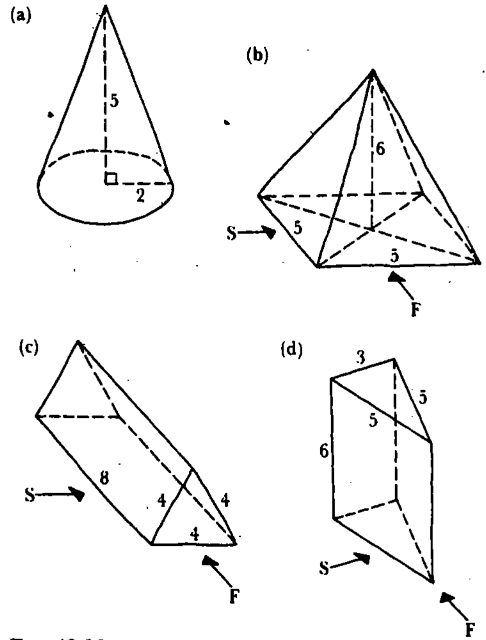


Fig. 13.26

Draw the plan and front and side elevations of a sphere of radius 3 cm.

A coin is in the shape of a cylindrical disc, 30 mm in diameter and 2 mm thick. Make a sketch of the coin showing its measurements. Use a scale of 1 cm represents 4 mm and draw the plan and front elevation of the coin.

A cuboid is 7,5 cm long, 5 cm wide and 3,5 cm high. Sketch the cuboid showing its measurements. Draw the plan and front and side elevations (a) with the 7,5 cm side parallel to the ground line (XY), (b) with the 5 cm side parallel to the ground line.

A triangular prism is such that every edge is 6 cm long. Its front elevation is an equilateral triangle. Sketch the prism showing its measurements. Draw its plan and side elevation.

A cone is 6 cm high and rests on its base of diameter 4 cm. Sketch the cone showing its measurements. Draw the plan and front elevation of the cone. Hence measure the slant height and vertical angle of the cone.

10 Fig. 13.27 shows some building designs made from simple shapes. All dimensions are in metres. Choose a suitable scale and draw the plan and front and end elevations of each building.

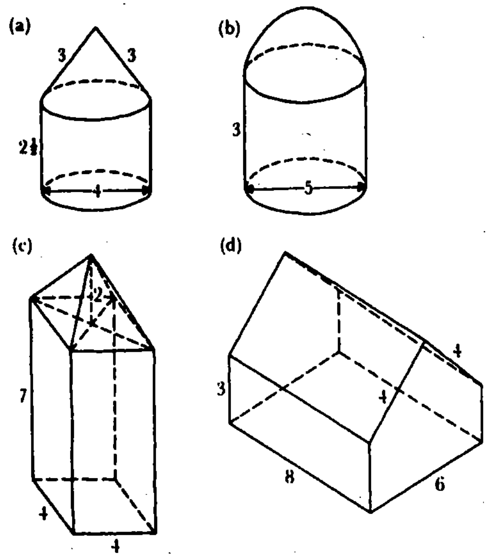


Fig. 13.27

Chapter 14

Circle geometry (1)

Arcs and chords

A **chord** of a circle is a straight line joining any two points on its circumference. A **diameter** is a chord which passes through the centre of the circle. A chord which is not a diameter divides the circumference into two **arcs** of different sizes, a **major arc** and a **minor arc** (Fig. 14.1).

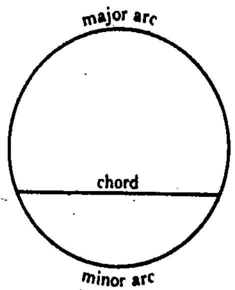


Fig. 14.1

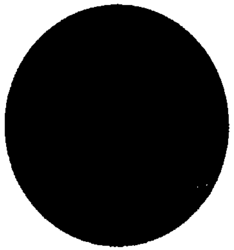


Fig. 14.2

The chord also divides the circle into two **segments** of different sizes, a **major segment** and a **minor segment** (Fig. 14.2).

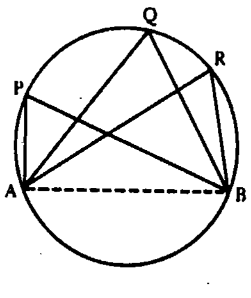


Fig. 14.3

In Fig. 14.3, P, Q and R are points on the circumference of a circle. $A\hat{P}B$, $A\hat{Q}B$, $A\hat{R}B$ are angles **subtended** at the circumference by the chord AB or by the minor arc AB. $A\hat{P}B$, $A\hat{Q}B$, $A\hat{R}B$ are all **angles in the same** major **segment** APQRB.

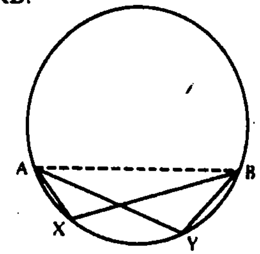


Fig. 14.4

Similarly, in Fig. 14.4, $A\hat{X}B$ and $A\hat{Y}B$ are angles subtended by the chord AB or by the major arc AB in the minor segment AXYB.

Theorem

The angle which an arc of a circle subtends at the centre is twice that which it subtends at any point on the remaining part of the circumference.

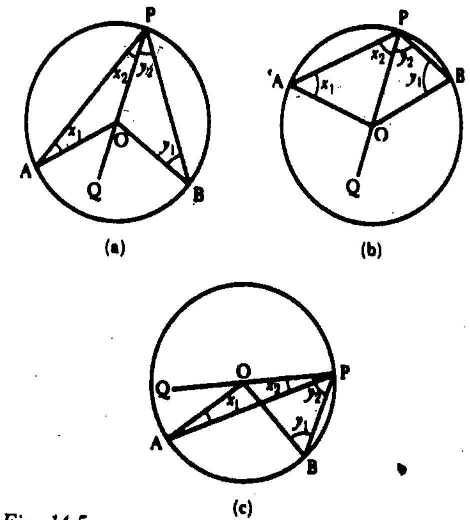


Fig. 14.5

Given: a circle APB with centre O.

To prove: $A\hat{O}B = 2 \times A\hat{P}B$.

Construction: Join PO and produce it to any point Q.

Proof:

With the lettering of Fig. 14.5,

OA = OB (*radii*)

$\therefore x_1 = x_2$ (*base angles of isos.* Δ)

$A\hat{O}Q = x_1 + x_2$ (*ext. angle of* ΔAOP)

$\therefore A\hat{O}Q = 2x_2$ ($x_1 = x_2$)

Similarly, $B\hat{O}Q = 2y_2$

In Fig. 14.5(a) $A\hat{O}B$ / In Fig. 14.5(b) reflex $A\hat{O}B$ $= A\hat{O}Q + B\hat{O}Q$

$= 2x_2 + 2y_2$

$= 2(x_2 + y_2)$

$= 2 \times A\hat{P}B$

In Fig. 14.5(c) $A\hat{O}B = B\hat{O}Q - A\hat{O}Q$

$= 2y_2 - 2x_2$

$= 2(y_2 - x_2)$

$= 2 \times A\hat{P}B$

In every case, $A\hat{O}B = 2 \times A\hat{P}B$

Example 1

In Fig. 14.6, O *is the centre of circle* WXYZ. *If* $X\hat{W}Z = 33°$, *find* $X\hat{O}Z$ *and* $X\hat{Y}Z$.

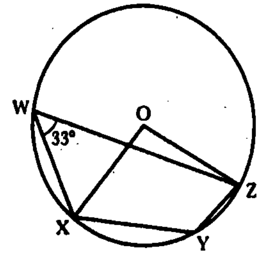


Fig. 14.6

$X\hat{O}Z = 2 \times 33°$ (*angle at centre = 2 × angle at circumference*)

$= 66°$

reflex $X\hat{O}Z = 360° - 66°$ (*angles at a point*)

$= 294°$

$X\hat{Y}Z = \frac{1}{2}$ of $294°$ (*angle at centre = 2 × angle at circumference*)

$= 147°$

Exercise 14a

1 Find the lettered angles in Fig. 14.7. (O is the centre of each circle.)

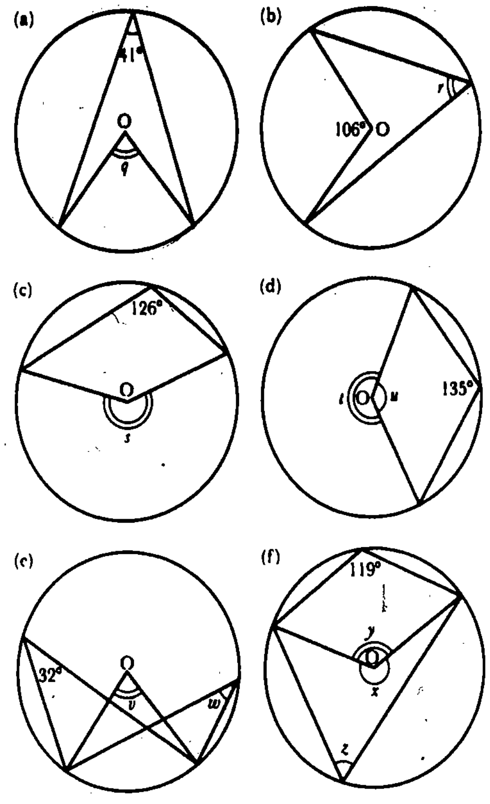


Fig. 14.7

2 In Fig. 14.8, P, R and S are points on a circle centre O. If $P\hat{O}R = 150°$, calculate $P\hat{S}R$.

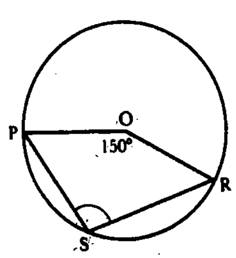


Fig. 14.8

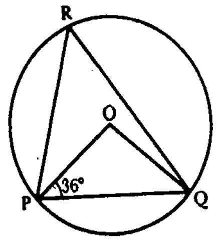


Fig. 14.9

3 In Fig. 14.9, P, Q and R are points on a circle, centre O. If $O\hat{P}Q = 36°$, what is the size of $P\hat{R}Q$?

4 Find the lettered angles in Fig. 14.10. A construction is needed in part (d). (O is the centre of each circle.)

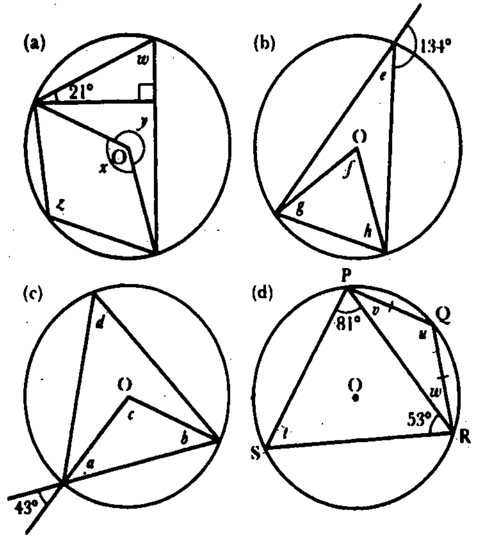


Fig. 14.10

5 In Fig. 14.11, L, M, N are points on a circle, centre O. $N\hat{M}O = a$ and $N\hat{L}O = b$. Find the obtuse $M\hat{O}L$ in terms of a and b.

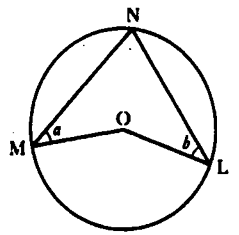


Fig. 14.11

Theorem

Angles in the same segment of a circle are equal.

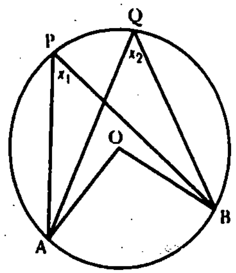


Fig. 14.12

Given: P and Q are any points on the major arc of circle APQB.

To prove: $A\hat{P}B = A\hat{Q}B$.

Construction: Join A and B to O, the centre of the circle.

Proof:

With the lettering of Fig. 14.12,

$A\hat{O}B = 2x_1$ (*angle at centre = 2 × angle at circumference*)

$A\hat{O}B = 2x_2$ (*same reason*)

$\therefore x_1 = x_2$ $(= \frac{1}{2}A\hat{O}B)$

$\therefore A\hat{P}B = A\hat{Q}B$

Since P and Q are *any* points on the major arc, all angles in the major segment are equal to each other. The theorem is also true for angles in the minor segment (Fig. 14.13).

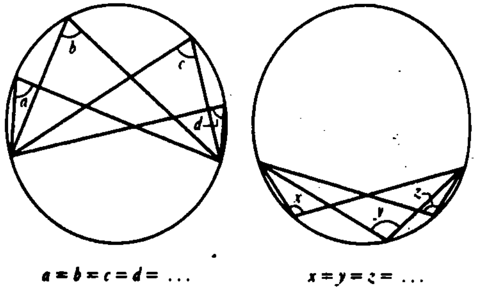


Fig. 14.13

Theorem

The angle in a semicircle is a right angle.

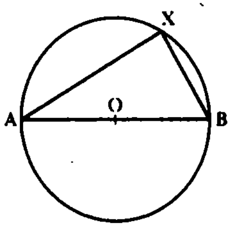


Fig. 14.14

Given: AB is a diameter of a circle, centre O. X is any point on the circumference of the circle.

To prove: $A\hat{X}B = 90°$

Proof: $A\hat{O}B = 2A\hat{X}B$ (*angle at centre = 2 × angle at circumference*)

but $A\hat{O}B = 180°$ (*straight angle*)

$\therefore 2A\hat{X}B = 180°$

$\therefore A\hat{X}B = 90°$

Example 2

In Fig. 14.15, PQ *is a diameter of circle* PMQN, *centre* O. *If* $P\hat{Q}M = 63°$, *find* $Q\hat{N}M$.

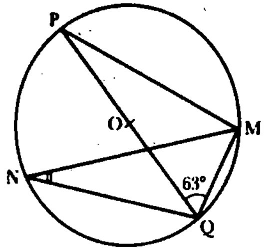


Fig. 14.15

In $\triangle QPM$, $P\hat{M}Q = 90°$ (*angle in semicircle*)

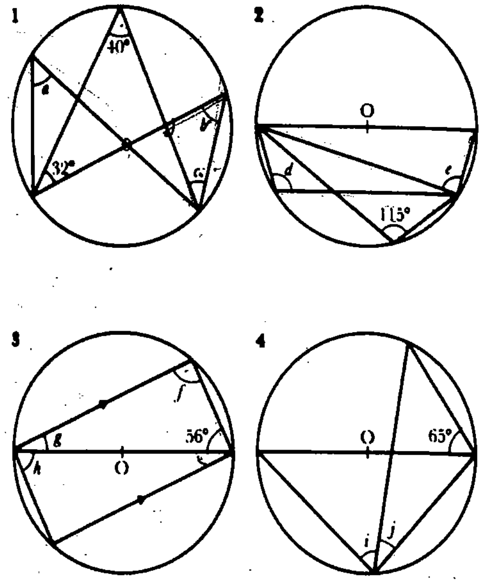
$\therefore Q\hat{P}M = 180° - 90° - 63°$ (*angle sum of* △)

$= 27°$

$\therefore Q\hat{N}M = 27°$ (*in same segment as* $Q\hat{P}M$)

Exercise 14b

Find the lettered angles in each of the following. Where a point O is given it is the centre of the circle.



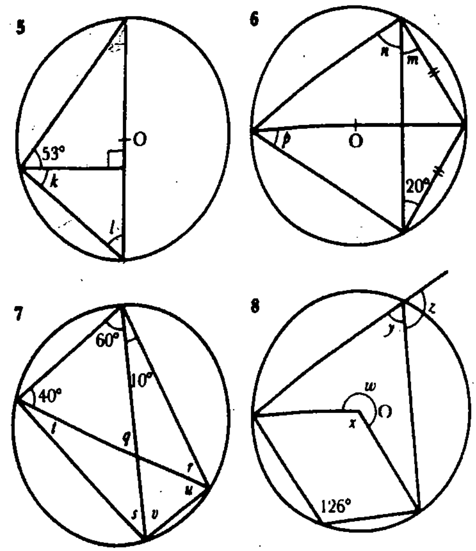


Fig. 14.16

Cyclic quadrilaterals

The vertices of a **cyclic quadrilateral** lie on the circumference of a circle. In Fig. 14.17, ABCD and PQRS are cyclic quadrilaterals.

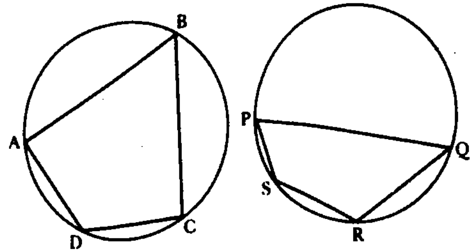


Fig. 14.17

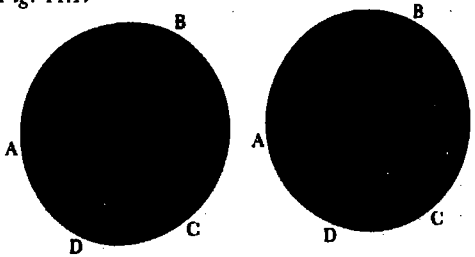


Fig. 14.18

Opposite angles of a cyclic quadrilateral lie in opposite segments of a circle (Fig. 14.18).

Theorem

The opposite angles of a cyclic quadrilateral are supplementary.

or

Angles in opposite segments are supplementary.

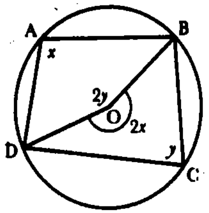


Fig. 14.19

Given: a cyclic quadrilateral ABCD.
To prove: $B\hat{A}D + B\hat{C}D = 180°$ (Note: the sum of supplementary angles is 180°.)
Construction: Join B and D to the centre O of circle ABCD.
Proof:
With the lettering of Fig. 14.19,

$B\hat{O}D = 2y$ (*angle at centre = 2 × angle at circumference*)
reflex $B\hat{O}D = 2x$ (*same reason*)
$\therefore 2x + 2y = 360°$ (*angles at a point*)
$\therefore x + y = 180°$
$\therefore B\hat{A}D + B\hat{C}D = 180°$

It follows from the previous theorem that **the exterior angle of a cyclic quadrilateral is equal to the interior opposite angle.**

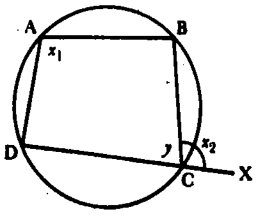


Fig. 14.20

In Fig. 14.20,

$x_1 + y = 180°$ (*opp. angles of cycl. quad.*)
$x_2 + y = 180°$ (*straight angle*)
$\therefore x_1 = x_2$ (= 180° − *y*)
$\therefore B\hat{C}X = B\hat{A}D$

Example 3

In Fig. 14.21, CE *is a diameter of circle* ABCDE. *If* $A\hat{B}C = 127°$, *find* $A\hat{C}E$.

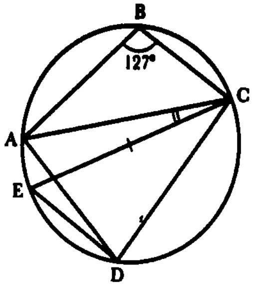


Fig. 14.21

$A\hat{D}C = 180° - 127$ (*opp. angles of cycl. quad.*)
$= 53°$
$E\hat{D}C = 90°$ (*angle in semicircle*)
$\therefore E\hat{D}A = 90° - 53°$
$= 37°$
$\therefore A\hat{C}E = 37°$ (*in same segment as* $E\hat{D}A$)

Example 4

In Fig. 14.22, P, Q, R, S *are points on a circle centre* O. QP *is produced to* X. *If* $X\hat{P}S = 77°$ *and* $P\hat{S}O = 68°$, *find* $P\hat{Q}O$.

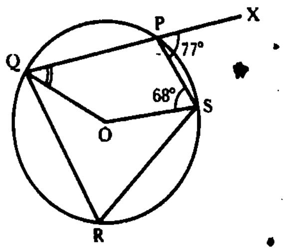


Fig. 14.22

$Q\hat{R}S = 77°$ (= *ext. angle of cycl. quad.*)
$\therefore Q\hat{O}S = 2 \times 77°$ (*angle at centre = 2 × angle at circumference*)
$= 154°$
$Q\hat{P}S = 180° - 77°$ (*straight angle*)
$= 103°$
In quad. PQOS,
$P\hat{Q}O = 360° - 154° - 103° - 68°$ (*angle sum of quad.*)
$\therefore P\hat{Q}O = 35°$

ercise 14c

Use Fig. 14.23 to answer the following.
(a) (i) Which arc subtends $A\hat{B}E$?
(ii) Write down an angle equal to $A\hat{B}E$.
(b) (i) Which arc subtends $B\hat{E}C$?
(ii) Write down two angles equal to $B\hat{E}C$.
(c) (i) Which arc subtends $E\hat{A}C$?
(ii) Write down an angle equal to $E\hat{A}C$.

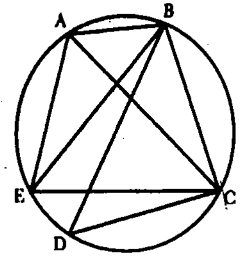
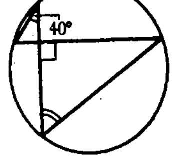


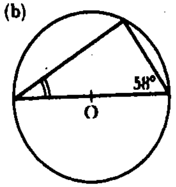
Fig. 14.23

2 Find the marked angle in each of the following. Where a point O is given it is the centre of the circle. (*Hint:* make a careful copy of each figure and write in the angles as they are found. Some constructions may be necessary.)

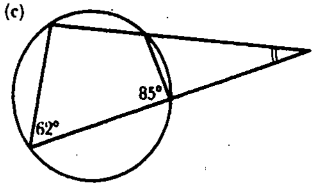
(a)



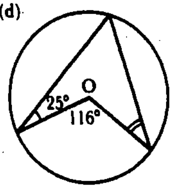
(b)



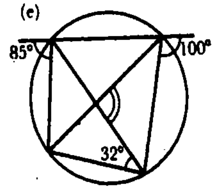
(c)



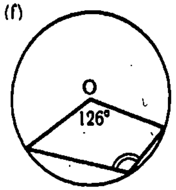
(d)



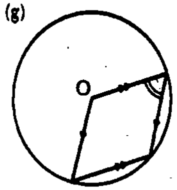
(e)



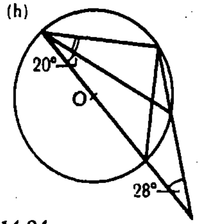
(f)



(g)



(h)



(i)

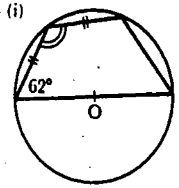


Fig. 14.24

3 In Fig. 14.25, AB is a diameter of semicircle ABCD. If $A\hat{B}D = 16°$, calculate $B\hat{C}D$. (*Hint:* Join CA.)

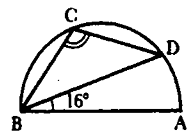


Fig. 14.25

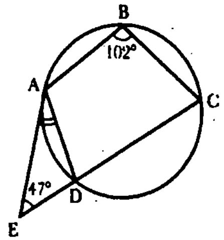


Fig. 14.26

4 In Fig. 14.26, A, B, C, D are points on a circle such that $A\hat{B}C = 102°$. CD is produced to E so that $A\hat{E}D = 47°$. Calculate $E\hat{A}D$.

5 In Fig. 14.27, ABCD is a cyclic quadrilateral such that AB || DC. If $A\hat{C}D = 35°$ and $D\hat{B}C = 72°$, calculate the sizes of the angles of the trapezium.

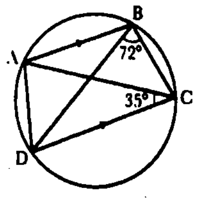


Fig. 14.27

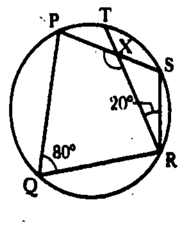


Fig. 14.28

6 In Fig. 14.28, $P\hat{Q}R = 80°$ and $S\hat{R}T = 20°$ What size is $P\hat{X}R$?

7 In Fig. 14.29, O is the centre of circle MLY. If $O\hat{L}Y = 50°$ and $O\hat{M}Y = 15°$, calculate $M\hat{O}L$.

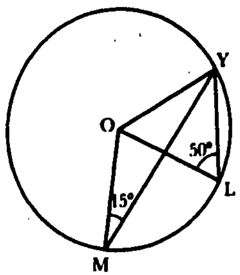


Fig. 14.29

8 In Fig. 14.30, PQRS is a cyclic quadrilateral such that $Q\hat{P}R = 18°$ and $R\hat{Q}S = 42°$. If $P\hat{S}R = 78°$, calculate the angles of $\triangle PQS$. What kind of triangle is $\triangle PQS$?

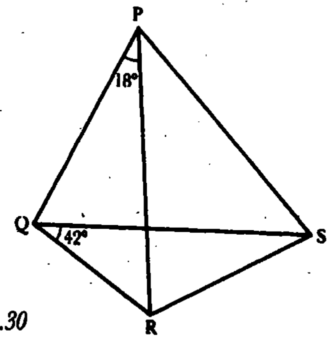


Fig. 14.30

9 In Fig. 14.31 calculate the value of x giving a reason for each step in your answer.

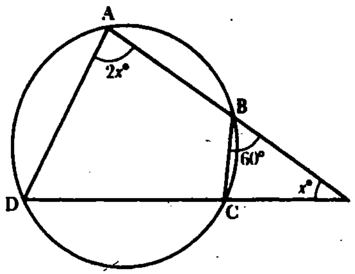


Fig. 14.31

10 In Fig. 14.32 P, Q, R, S are points on a circle such that $P\hat{Q}S = P\hat{R}Q$. (a) Prove that PS = PQ. (b) Hence, if SQ is a diameter of the circle and QR is produced to Z, determine $S\hat{Q}P$ and $S\hat{R}Z$, giving reasons.

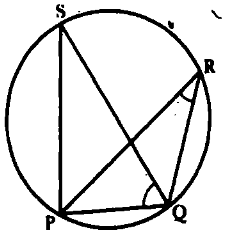


Fig. 14.32

Chapter 15

Sets (3)

Complement of a set

In Fig. 15.1, A is a subset of the universal set $\mathscr{E}$, $A \subseteq \mathscr{E}$.

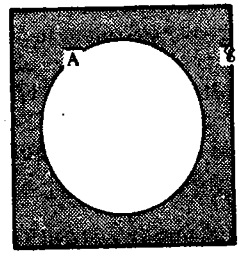


Fig. 15.1

The **complement** of A is the set which contains all those elements of $\mathscr{E}$ which are *not* members of A. A′ is short for the complement of A. In Fig. 15.1, the shaded region represents A′.

Example 1

If $\mathscr{E} = \{1; 2; 3; 4; 5\}$, $A = \{1; 3\}$ *and* $B = \{3; 4\}$ *find (a)* A′, *(b)* B′, *(c)* $(A \cap B)'$, *(d)* $(A \cup B)'$.

(a) $A' = \{2; 4; 5\}$
(b) $B' = \{1; 2; 5\}$
(c) $A \cap B = \{3\}$
$(A \cap B)' = \{1; 2; 4; 5\}$
(d) $A \cup B = \{1; 3; 4\}$
$(A \cup B)' = \{2; 5\}$

Example 2

$\mathscr{E}$ is the set of all cars, R is the set of all red cars and F is the set of all cars with four doors. Show on a Venn diagram the set of all red cars which do not have four doors.

F = {cars with four doors}
F′ = {cars which do *not* have four doors}
$R \cap F'$ = {red cars which do not have four doors}

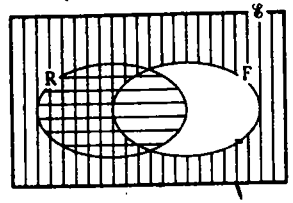


Fig. 15.2

In Fig. 15.2, the region with horizontal shading represents R. The region with vertical shading represents F′ The region which is cross-shaded represents $R \cap F'$, the set of all red cars which do not have four doors.

Exercise 15a

1 Given $\mathscr{E} = \{1; 2; 3; 4; 6; 8\}$, write down the complements of the following sets.
(a) {1; 2; 3} (b) {4; 6; 8}
(c) {2; 4} (d) {1; 3; 6; 8}
(e) {6} (f) {1; 2; 3; 4; 6}
(g) ∅ (h) $\mathscr{E}$
(i) $\{1; 2; 3; 6\} \cup \{1; 2; 8\}$
(j) $\{1; 2; 3; 6\} \cap \{1; 2; 8\}$

2 If $\mathscr{E} = \{a; b; c; d; e\}$, $A = \{a; c; e\}$ and $B = \{b; e\}$, list the members of the following sets.
(a) A′ (b) B′ (c) $A' \cup B$
(d) $A \cup B'$ (e) $A' \cap B$ (f) $A \cap B'$
(g) $A' \cup A$ (h) $B' \cap B$ (i) $(A \cup B)'$
(j) $A' \cup B'$ (k) $(A \cap B)'$ (l) $A' \cap B'$

3 Compare the answers to parts (i) and (l) of question 2. What do you notice?

4 Compare the answers to parts (j) and (k) of question 2. What do you notice?

5 If $\mathscr{E} = \{1; 2; 3; 4; 5; \ldots; 10\}$, $A = \{1; 2; 5; 7\}$, $B = \{1; 3; 6; 7\}$, write down the sets A′, B′, $(A \cap B)'$ and $(A \cup B)'$.

6 Using the sets of question 5, show that $A' \cup B' = (A \cap B)'$ and $A' \cap B' = (A \cup B)'$.

7 Use Venn diagrams to illustrate the results of question 6.

8 If $\mathscr{E}$ = {days of the week}, S = {words which contain the letter s} and N = {words which contain six letters}
(a) list the members of the sets S, N, S′, N′;
(b) list the members of the set (i) (S ∪ N)′, (ii) (S ∩ N)′;
(c) hence, without further working, list the members of the sets (i) S′ ∩ N′, (ii) S′ ∪ N′.

9 Use a Venn diagram to show that if A ⊂ B then B′ ⊂ A′, and vice versa.

10 If $\mathscr{E}$ = {all teachers}, M = {mathematics teachers} and W = {teachers who are women}, show on a Venn diagram the set of all mathematics teachers who are men.

Sets (General revision)

Example 3

$\mathscr{E}$ = {animals}, H = {animals with horns}, W = {wild animals}. Show on a Venn diagram the set of wild animals that do not have horns.

H = {animals with horns}
H′ = {animals which do *not* have horns}
W ∩ H′ = {wild animals which do not have horns}

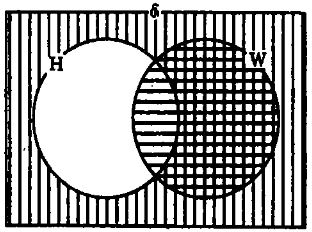


Fig. 15.3

In Fig. 15.3, the region with horizontal shading represents W. The region with vertical shading represents H′. The region which is cross-shaded represents W ∩ H′, the set of wild animals which do not have horns.

Example 4

Given that x is an integer, list the members of $\{x: x > -4\} \cap \{x: x \leqslant 1\}$.

If x is an integer, then
$\{x: x > -4\} = \{-3; -2; -1; 0; 1; 2; 3; \ldots\}$
and
$\{x: x \leqslant 1\} = \{\ldots -4; -3; -2; -1; 0; 1\}$
Hence $\{x: x > -4\} \cap \{x: x \leqslant 1\}$
$= \{-3; -2; -1; 0; 1\}$

In Example 4, $\{x: x > -4\}$ and $\{x: x \leqslant 1\}$ are examples of sets written in **set builder notation**. The expression $\{x: x > -4\} \cap \{x: x \leqslant 1\}$ can be shortened to $\{x: -4 < x \leqslant 1$, x is an integer$\}$, i.e. 'the set of values x, such that x is greater than −4 and less than or equal to 1, where x is an integer'. The expression $-4 < x \leqslant 1$ is a **range of values** of x.

Example 5

A survey of 100 families showed that 32 had a TV set and 51 had a gas cooker. If 40 had neither, how many had both?

Let $\mathscr{E}$ = {all families in the survey}, T = {families with TV sets}, G = {families with gas cookers}. It is required to find n(T ∩ G). In Fig. 15.2, x = n(T ∩ G).

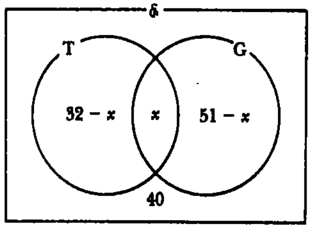


Fig. 15.4

Since n(T) = 32, n(T ∩ G′) = 32 − x.
Similarly n(G ∩ T′) = 51 − x
and n(T′ ∩ G′) = 40.
The totals for the regions must add up to the number of families in the universal set, n($\mathscr{E}$):

$$x + (32 - x) + (51 - x) + 40 = 100$$
$$123 - x = 100$$
$$x = 23$$

23 families had both a TV set and a gas cooker.

Table 15.1 contains the symbols and language of sets.

Table 15.1

Symbols	Meaning
P = {a; b; c}	P is the set a, b, c
Q = {1; 2; 3; ...}	... means 'and so on'
A = {x: x is a natural number} B = {(x; y): $y = mx + c$} C = {x: $a < x < b$}	Sets A, B, C are various examples of set-builder notation: sets A and C give values of x; set B is a set of points (x; y)
∅ or { }	the empty set
$\mathscr{E}$ or $\mathscr{U}$	the universal set
∈	is a member of
∉	is *not* a member of
A ⊂ B	A is a proper subset of B
A ⊆ B	A is a subset of B
A ⊇ B	A contains B
⊄, ⊈, ⊉	the negations of ⊂, ⊆, ⊇
A ∪ B	union of A and B
A ∩ B	intersection of A and B
n(A)	number of elements in set A
A′	complement of set A

Exercise 15b

1 If $\mathscr{E}$ = {f; a; c; t; o; r; i; s; e}, P = {r; a; t; i; o} and Q = {s; e; t} write down the members of the following sets.
(a) P′ (b) Q′
(c) (P ∩ Q)′ (d) (P ∪ Q)′

2 Use the sets of question 1 to demonstrate that P′ ∪ Q′ = (P ∩ Q)′ and P′ ∩ Q′ = (P ∪ Q)′. Illustrate these results by suitable shading on a Venn diagram.

3 If $\mathscr{E}$ = {c; h; i; c; k; e; n}, P = {n; i; c; e} and Q = {h; e; n} list the elements of the following:
(a) P ∩ Q (b) P ∪ Q
(c) (P ∪ Q)′ (d) (P ∩ Q)′
(e) P′ ∩ Q (f) P ∪ Q′

4 If $\mathscr{E}$ = {integers}, list the members of the following sets using ... where appropriate.
(a) $\{x: x \leqslant 9\}$ (b) $\{x: x \geqslant -3\}$
(c) $\{x: x > 5\}$ (d) $\{x: x < 0\}$
(e) $\{x: x - 4 = 0\}$ (f) $\{x: 2x + 3 = 15\}$
(g) $\{x: -6 \leqslant x < 2\}$
(h) $\{x: -8\frac{1}{2} < x < -1\frac{1}{4}\}$

5 If $\mathscr{E} = \{x: 1 \leqslant x \leqslant 10$, x is an integer$\}$, A = {x: x is a perfect square} and B = {x: x is a factor of 20},
(a) find n(A′), (b) find n(A ∪ B)′,
(c) list the members of the set B′,
(d) list the members of the set A ∩ B′.

6 If $\mathscr{E}$ = {1; 2; 3; 4; 5; ...; 20} list the members of the following sets.
(a) $\{x: x$ is a square number, $x \in \mathscr{E}\}$
(b) $\{x: x + 2 > 15, x \in \mathscr{E}\}$
(c) $\{x: x$ is a factor of 40, $x \in \mathscr{E}\}$
(d) $\{(x; y): y = 3x + 1, x \in \mathscr{E}, y \in \mathscr{E}\}$
(e) $\{(x; y): y = 2x^2 - 1, x \in \mathscr{E}, y \in \mathscr{E}\}$
(f) $\{(x; y): y > 17 + x, x \in \mathscr{E}, y \in \mathscr{E}\}$

7 $\mathscr{E}$ = {countries of Africa}, N = {countries which lie wholly or partly north of the equator}, S = {countries which lie wholly or partly south of the equator}. If n($\mathscr{E}$) = 47, n(N) = 32 and n(S) = 21, through how many countries of Africa does the equator pass? [What is n(N ∩ S)?]

8 In a choir of 38 students, 22 are girls. 17 of the students are at least 160 cm tall. If 14 of the girls are less than 160 cm, how many of the boys are also less than 160 cm?

9 In a class, $2x$ students are less than 15 years old, x students are over 13 years old and 17 students are 14 years old. Show this information on a Venn diagram. If there are 37 students in the class, find x.

10 In the Venn diagram of Fig. 15.5 overleaf, $\mathscr{E}$ = {people in a village}, X = {cattle owners}, Y = {car owners}. The letters p, q, r, s represent the numbers of people in the subsets shown.

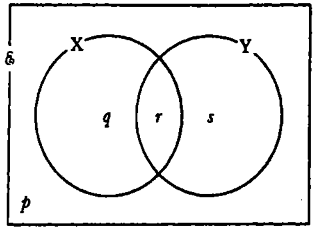


Fig. 15.5

If $n(\mathscr{E}) = 259$, $n(X) = 43$ and $n(Y) = 32$,

(a) express q in terms of r;
(b) express s in terms of r;
(c) state the greatest possible value of r;
(d) find the smallest possible value of p;
(e) find p, q and r if $s = 6$.

Venn diagrams with more than two subsets

Venn diagrams can be drawn to represent any number of sets, though they become very complicated if the number is large. Fig. 15.6 is a Venn diagram for three subsets of a universal set.

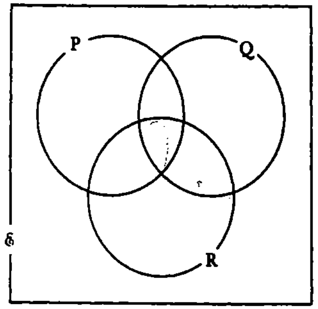


Fig. 15.6

Diagrams such as Fig. 15.6 can be used to simplify problems which at first sight appear to be difficult.

Example 6

In a group of students, 18 play football, 19 play basketball and 16 play volleyball. 6 play football only, 9 play basketball only, 5 play football and basketball only and 2 play basketball and volleyball only. How many play (a) all three games, (b) football and volleyball only, (c) volleyball only? (d) If 8 play no games at all, how many students are there altogether?

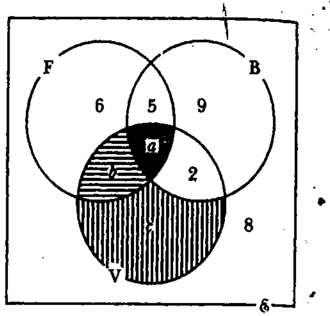


Fig. 15.7

In the Venn diagram of Fig. 15.7, $\mathscr{E}$ = {all students}, F = {football players}, B = {basketball players}, V = {volleyball players}.

The 6 who play football only are represented by the region of F which does not lie within B or V; see Fig. 15.7. Similarly, the given values 9, 5 and 2 are written in the appropriate regions in Fig. 15.7.

(a) Those who play all three games are represented by the dotted region. This region lies in F, B and V, i.e. the set $F \cap B \cap V$.
From the first sentence of the question,

$$n(B) = 19$$

Also, from Fig. 15.7,

$$n(B) = 9 + 5 + 2 + a$$
$$\text{Hence, } 19 = 9 + 5 + 2 + a$$
$$\Leftrightarrow \quad a = 3$$

3 students play all three games.

(b) Those who play football and volleyball only are represented by the region with horizontal shading. This region lies within F and V but not within B, i.e. the set $F \cap V \cap B'$.
From the first sentence of the question,

$$n(F) = 18$$

Also, from Fig. 15.7,

$$n(F) = 6 + 5 + a + b$$
$$= 6 + 5 + 3 + b \text{ (since } a = 3)$$
$$\text{Hence } 18 = 6 + 5 + 3 + b$$
$$\Leftrightarrow \quad b = 4$$

4 students play football and volleyball only.

(c) The region with vertical shading represents those playing volleyball only, i.e. the set $V \cap (F \cup B)'$.

$$\text{Hence } 16 = 2 + a + b + c$$
$$= 2 + 3 + 4 + c \text{ (since } a = 3, b = 4)$$
$$\Leftrightarrow \quad c = 7$$

7 students play volleyball only.

(d) The total number of students

$$= 6 + 5 + 9 + 2 + a + b + c + 8$$
$$= 6 + 5 + 9 + 2 + 3 + 4 + 7 + 8$$
$$= 44$$

Example 7

In a group of 10 university students, 6 are taking mathematics, 5 philosophy and 7 economics. 3 take maths and philosophy, 2 philosophy and economics, and 4 economics and maths. Each student is taking at least one of these subjects. How many of the students are taking all three subjects?

Let M = {students taking mathematics},
P = {students taking philosophy},
E = {students taking economics}.

Note that since each student is taking at least one of the subjects, $M \cup P \cup E = \mathscr{E}$. Fig. 15.8 is the appropriate Venn diagram. The centre section represents $M \cap P \cap E$, showing x students taking all three subjects.

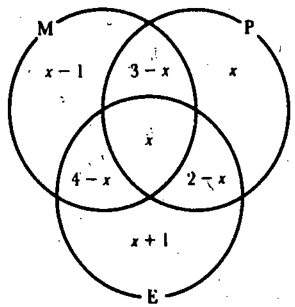


Fig. 15.8

Altogether, 3 students take maths and philosophy, i.e. $n(M \cap T) = 3$. Hence $(3 - x)$ students take maths and philosophy only. $3 - x$ is written in the appropriate region of Fig. 15.8.

Similarly, since $n(P \cap E) = 2$ and $n(E \cap M) = 4$, $2 - x$ and $4 - x$ are written in the appropriate regions of Fig. 15.8.

Since $n(M) = 6$, the number taking maths only is

$$6 - (3 - x) - x - (4 - x)$$
$$= 6 - 3 + x - x - 4 + x$$
$$= x - 1$$

Similarly, the number taking philosophy only is x and the number taking economics only is $x + 1$. The values $x - 1$, x and $x + 1$ are written in the appropriate regions of Fig. 15.8.

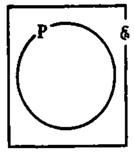
As there are 10 students, $n(M \cup P \cup E) = 10$.

$$\text{Hence} \quad (x - 1) + (3 - x) + x + (4 - x) + x + (2 - x) + (x + 1) = 10$$
$$\Leftrightarrow \quad x + 9 = 10$$
$$\Leftrightarrow \quad x = 1$$

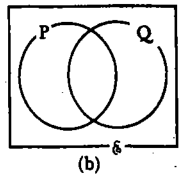
1 student takes all three subjects.

Exercise 15c

1

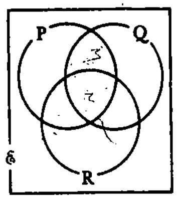


(a)



(b)

Fig. 15.9



(c)

(a) On a copy of Fig. 15.9(a) shade set P′.
(b) On a copy of Fig. 15.9(b) shade set $P' \cap Q$.
(c) On a copy of Fig. 15.9(c) shade set $(P' \cap Q) \cup R$.

2 For each of the following make a freehand copy of Fig. 15.6 on page 138 and shade the given set.

(a) $P \cup Q \cup R$
(b) $P \cap Q \cap R$
(c) $(P \cup Q) \cap R$
(d) $(P \cap Q) \cup R$
(e) $P \cup (Q \cap R)$
(f) $P \cap Q' \cap R$
(g) $P' \cup (Q \cap R)$
(h) $Q' \cap (P \cup R)$

(i) $(P \cup Q) \cap (R \cup Q')$
(j) $(P \cap R) \cup (Q \cap R')$

3 (a) Make a freehand copy of the Venn diagram in Fig. 15.10.

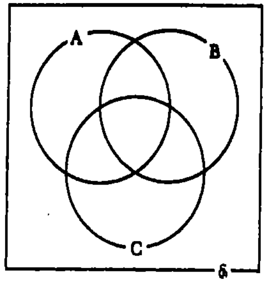


Fig. 15.10

(b) Fill in the members of the sets given that $A = \{p; q; r; s\}$ $B = \{q; r; s; t; u\}$, $C = \{p; r; u; y\}$.
(c) Hence or otherwise list the members of (i) $(A \cup B) \cap C$, (ii) $(A \cap B) \cup C$.

4 The numbers of elements of each region of the Venn diagram of Fig. 15.11 are as shown. If $n(ℰ) = 200$ find (a) x, (b) $n(P \cup R)$, (c) $n(P \cup Q \cup R)$, (d) $n(Q \cap R)$.

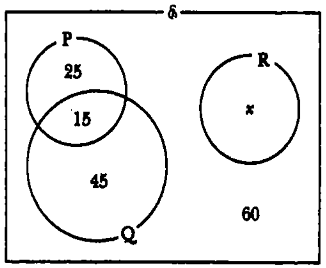


Fig. 15.11

5 The numbers of elements in each region of the Venn diagram of Fig. 15.12 are as shown.

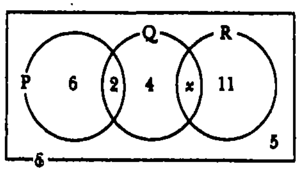


Fig. 15.12

If $n(ℰ) = 36$, find
(a) x (b) $n(Q)$
(c) $n(P \cup R)$ (d) $n(P \cap R)$
(e) $n(R' \cap Q')$ (f) $n(P' \cup Q)$

6 In the Venn diagram of Fig. 15.13, the numbers of elements are as shown.

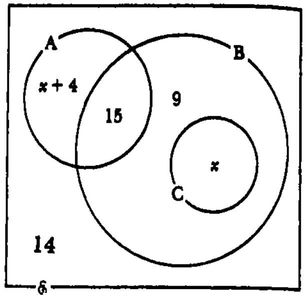


Fig. 15.13

If $n(ℰ) = 9x$, find
(a) x (b) $n(A)$
(c) $n(A \cup B)$ (d) $n(C')$
(e) $n(A \cap B')$ (f) $n(A \cup C)'$

7 All the pupils in a class of 35 play at least one of the games volleyball, netball, hockey. 10 play volleyball only, 5 play netball only and 3 play hockey only. If 2 play all three games and equal numbers play 2 games, how many altogether play volleyball?

8 20% of the people in a village own a dog, 30% own goats and 40% own cattle, 5% own both a dog and goats, 4% own both a dog and cattle and 3% own both goats and cattle. If 1% own all three, what percentage of the village own none of the three?

9 150 students were asked what they were doing last night between 8 pm and 9 pm. 50 said they watched TV only; 60 said they listened to the radio only; 5 said they read a book only. 20 watched TV and listened to the radio; 15 watched TV and read a book; 10 listened to the radio and read a book. x students did all three and x students did none of these things.
Illustrate the information on a clearly labelled Venn diagram. Find x and hence find the total number who read a book at some time between 8 pm and 9 pm.

10 In a school, the students can speak at least one of the languages Ndebele, Shona or Tonga.
Form 3A contains 38 students of which 2 can speak all three languages, 4 speak Ndebele and Shona only, 1 speaks Shona and Tonga only and no student speaks Ndebele and Tonga only. There are x students who speak Ndebele only, $5x$ who speak Shona only and $x - 4$ who speak Tonga only.
Illustrate this information on a clearly labelled Venn diagram, showing the number in each separate region. Find x and hence find the total number of Shona speakers.

Reasoning

Logic and sets

The simplest and most common form of logical argument is when a conclusion can be deduced from two statements or **premises**.

Example 8
What can be deduced from the following premises?
(1) Mary is a sprinter.
(2) All sprinters are healthy.

The conclusion is that Mary is healthy.
The data in the example can be represented in symbols and on a Venn diagram. If m = Mary, S = {sprinters}, H = {healthy people}, ℰ = {all people}, then
premise (1): $m \in S$
premise (2): $S \subseteq H$
conclusion: $m \in H$
It can be seen how the conclusion follows if the premises are arranged in a chain linked by the common term, S:

$m \in S$; $S \subseteq H \Rightarrow m \in H$

The main conclusion is found by excluding the middle term.
Fig. 15.14 is a Venn diagram showing the relationship between the premises.

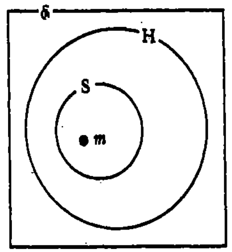


Fig. 15.14

Example 9
Some girls are clever. Some girls wear glasses. Can it be deduced from these statements that some clever girls wear glasses?

Let ℰ = {all girls}, C = {clever girls}, G = {girls wearing glasses}. Fig. 15.15 shows that there are four ways of arranging C and G within ℰ.

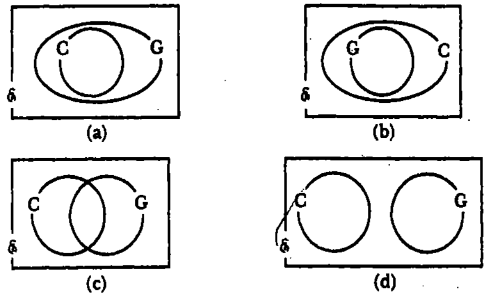


Fig. 15.15

Each part of Fig. 15.15 correctly illustrates the premises of the question. Part (d) shows that there is not necessarily an intersection of C with G. Therefore it cannot be deduced that some clever girls wear glasses.

Language and sets

Words such as 'not', 'all', 'some', 'either . . . or' can be represented on Venn diagrams. Read the following examples carefully.

'not': complement

premise: Cattle owners are not poor.
ℰ = {all people}, C = {cattle owners}, P = {poor people}, P′ = {people who are not poor}

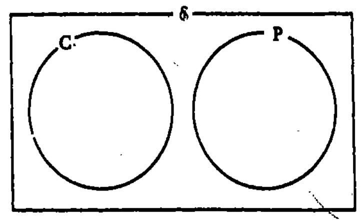


Fig. 15.16

$C \subseteq P' \Leftrightarrow P \subseteq C'$

conclusion: Poor people are not cattle owners.

'all': subset

premise: All houses have doors.
$\mathscr{E}$ = {all things}, H = {houses}, D = {things with doors}

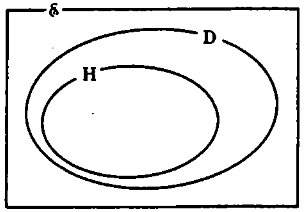


Fig. 15.17

$H \subset D \Leftrightarrow D' \cap H = \emptyset$

conclusion: There is no house which does not have a door.

'some': intersection

premise: Some snakes are friendly.
$\mathscr{E}$ = {all animals}, S = {snakes}, F = {friendly animals}

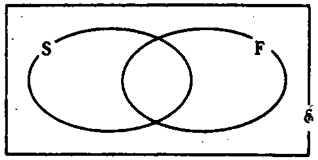


Fig. 15.18

$S \cup F \neq \emptyset, S \cap F' \neq \emptyset$

conclusion: Some snakes are not friendly.

'either ... or': union

premise: Students in 3B either do economics or geography.
$\mathscr{E}$ = {students in 3B}, E = {students of economics}, G = {students of geography}

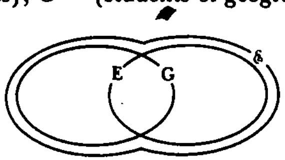


Fig. 15.19

$E \cup G = \mathscr{E} \Leftrightarrow E' \subseteq G$
$\Leftrightarrow G' \subseteq E$

conclusions:
(1) Students who don't do economics do geography. (2) Students who don't do geography do economics. (3) Some students may do both. (That is $E \cap G \neq \emptyset$; this will normally be the case unless there is some special reason for the two subsets to be disjoint.)

Example 10

Given $\mathscr{E}$ = {all quadrilaterals}, P = {all parallelograms}, R = {rectangles}, S = {squares} use a suitable Venn diagram to write the following statements in set notation: (a) there are some parallelograms which are not rectangles; (b) quadrilaterals which are not parallelograms or not rectangles are among those which are not squares.

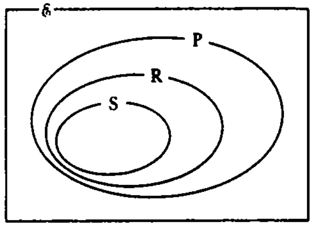


Fig. 15.20

Fig. 15.20 is a Venn diagram showing the relationship between P, R and S.

(a) $P \cap R' \neq \emptyset$
(b) $P' \cup R' \subset S'$, or since $P' \subset R'$, it would be correct to write $R' \subset S'$.

Exercise 15d

1 If L = {all living beings}, H = {human beings}, A = {Africans}, it follows that $H \subseteq L$ and $A \subseteq H$. What other conclusion can be drawn?

2 What can be deduced from the following premises?
(1) All squares are rectangles.
(2) All rectangles are parallelograms.

3 If $\mathscr{E}$ = {all cars}, M = {modern cars}, S = {cars with seat belts}, H = {cars with hazard lights}, express the following premises in set language.
(1) All modern cars have seat belts.
(2) All cars with hazard lights are modern.
Hence draw a conclusion from the premises.

If all pilots are courageous, all astronauts are pilots and all people who have landed on the moon are astronauts, does it follow that everyone who has landed on the moon is courageous? Illustrate your answer on a Venn diagram.

What, if anything, can be deduced from the following statements: all houses have windows; all shops have windows; all restaurants are shops?

Draw one or more Venn diagrams which illustrate the statements, (1) all bicycles have two wheels, (2) all bicycles have handlebars. Can it be deduced that all two-wheeled vehicles have handlebars?

$\mathscr{E}$ = {all people}, Y = {young people}, S = {successful people} and H = {happy people}.
(a) Which symbols represent the following sets?
(i) {old people} (ii) {unsuccessful people}
(iii) {young happy people}
(iv) {unsuccessful old people}
(b) If $S' \subseteq H'$ and $H' \subseteq Y'$ write in words the conclusion from the two statements.

8 Use a Venn diagram to show that if $A \subseteq B$ then $B' \subseteq A'$ and vice versa.
Hence solve this problem: If $\mathscr{E}$ = {pupils in a school}, B = {boys}, F = {pupils who play football} and W = {pupils who do woodwork}, what conclusion can be drawn from the following statements: all boys play football; no girl does woodwork?

9 Draw a Venn diagram to illustrate the premise that some isosceles triangles are right-angled. What conclusions follow from your diagram?

10 (a) Use a Venn diagram to show that if $C \subseteq G$, then $C \cap G' = \emptyset$.
(b) What conclusions can be drawn from the statement that there are no farmers who own cattle and who do not own goats?

11 (a) Show that if $F \subseteq H$ then $F' \cup H = \mathscr{E}$.
(b) Hence draw a conclusion from the premise that everybody either does not like food or is happy.

12 (a) Draw a Venn diagram showing the relationship between the following sets:

$\mathscr{E}$ = {people}, W = {women}, M = {mothers}, G = {grandmothers}.

(b) Write the following statements in set notation:
(i) There are some people who are mothers but who are not grandmothers.
(ii) People who are not women or who are not mothers are among those who are not grandmothers.

Chapter 16

Graphs (1) Algebraic graphs

Linear functions (Revision)

Example 1

Draw a graph of $y = 3 - 2x$ for values of x from -2 to $+4$. Use the graph (a) to find y when $x = -1{,}4$ (b) to solve the equation $3 - 2x = -4$.

Calculate the values of y which correspond to the whole number values of x.
When $x = -2, y = 3 - 2(-2) = 3 + 4 = +7$ and so on. Table 16.1 gives the values of y for values of x from -2 to $+4$.

Table 16.1

x	-2	-1	0	$+1$	$+2$	$+3$	$+4$
y	$+7$	$+5$	$+3$	$+1$	-1	-3	-5

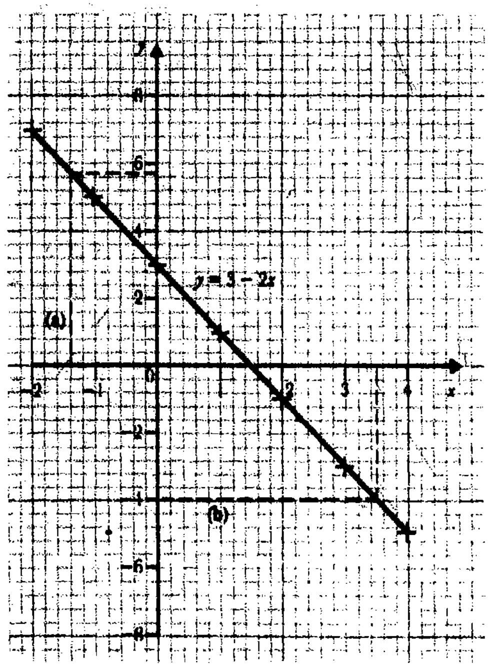


Fig. 16.1

Fig. 16.1 is the graph of the equation $y = 3 - 2x$.
(a) From the graph, $y = 5{,}8$ when $x = -1{,}4$.
(b) Since $y = 3 - 2x$, to solve the equation $3 - 2x = -4$, means to find the value of x when $y = -4$. From the graph, $x = 3{,}5$.

In the equation $y = 3 - 2x$, y is a **function** of x, i.e. the value of y depends on the value of x. $3 - 2x$ is the function of x. The line in Fig. 16.1 is the graph of the equation $y = 3 - 2x$, *or*, the **graph of the function** $3 - 2x$. A **linear function** of x is one which contains terms in x of power 1 only. The graph of a linear function is always a straight line.

When plotting the graph of a linear function, two points are sufficient to determine the line. However, in practice it is advisable to plot *three* points. If the three points lie in a straight line it is likely that the working is correct.

Exercise 16a

1 Table 16.2 gives corresponding values of x and y for the equation $y = 3x - 2$.

Table 16.2

x	-1	2	5
y	-5	4	13

(a) Choose suitable scales and draw the graph of $y = 3x - 2$.
(b) Find the value of y when $x = 3{,}5$.
(c) Find the value of x when $y = 5{,}5$.
(d) Write down the coordinates of the points where the line cuts the axes.

2 (a) Given that $y = 5 - 3x$, copy and complete Table 16.3.

Table 16.3

x	-3	0	$+3$
y			

(b) Hence draw the graph of the function $-3x$ for values of x from -3 to $+3$.
(c) Use the graph to solve the equation $3x = \frac{1}{2}$.
(a) Draw the graphs of $y = x$, $y = x + 2$, $= 5 - x$ for values of x from 0 to $+5$ *within the same axes.*
(b) What do you notice about the 3 lines?
(a) Make two copies of Table 16.4.

Table 16.4

x	-4	0	$+2$
y			

(b) Complete the tables for $y = 1 - 2x$ and $= x + 7$ and draw the graphs of these equations within the same axes.
(c) Use the graphs to solve the equations simultaneously.
Solve the following pairs of simultaneous equations graphically.
(a) $y = 3x$, $y = x + 1$ (b) $3x - y = 12$, $4x + 2y = 1$
(*Hint:* In (b), first make y the subject of each equation.)

Quadratic functions

A **quadratic function** of x is an expression in x in which 2 is the highest power of x. For example, $3x^2 - 5x + 1$ is a quadratic function of x.

Example 2

Draw the graph of $y = x^2 + 2x - 3$ for values of x from -4 to $+2$. Use the graph to find (a) the value of y when $x = 1{,}5$, (b) the values of x when $y = 1$, (c) the minimum value of $x^2 + 2x - 3$, (d) the value of x when y is least.

Table 16.5 is the necessary table of values. Values of y are obtained by adding the values of x^2, $2x$ and -3.

Table 16.5

x	-4	-3	-2	-1	0	1	2
x^2	16	9	4	1	0	1	4
$+2x$	-8	-6	-4	-2	0	2	4
-3	-3	-3	-3	-3	-3	-3	-3
y	5	0	-3	-4	-3	0	5

Fig. 16.2 is the graph of $y = x^2 + 2x - 3$.

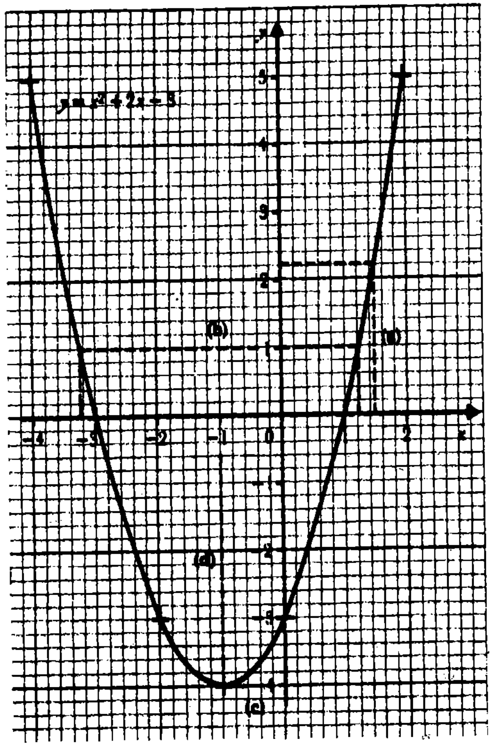


Fig. 16.2

From the graph,
(a) $y = 2{,}25$ when $x = 1{,}5$
(b) $x = 3{,}25$ or $1{,}25$ when $y = 1$
(c) -4 is the minimum value of $x^2 + 2x - 3$
(d) $x = -1$ when y has its lowest value.

Notice that the graph of a quadratic function is a curved line. The shape of the curve is called a **parabola**. It takes a lot of practice to be able to draw a smooth curve. When drawing a parabola it is usually helpful to position your hand inside the curve, drawing from left to right as in Fig. 16.3. In many cases this will mean turning the graph paper round.

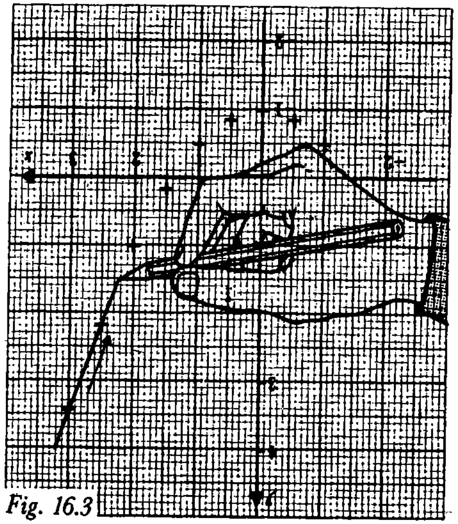


Fig. 16.3

Example 3

Fig. 16.4 is the graph of the function $2 - 3x - 2x^2$.

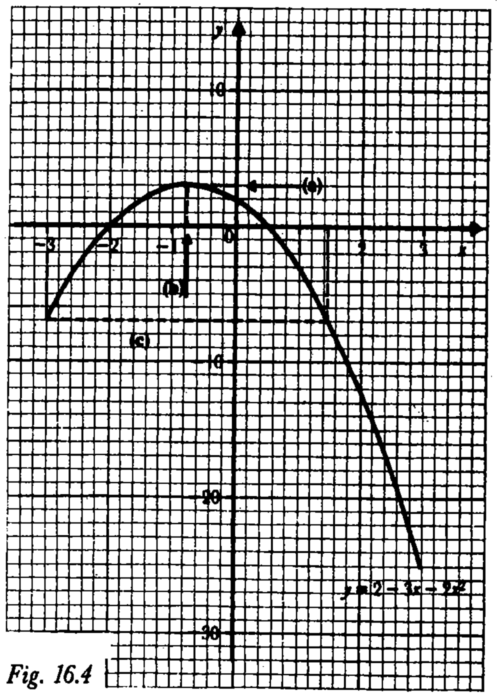


Fig. 16.4

(a) What is the maximum value of $2 - 3x - 2x^2$?
(b) For what value of x is y greatest?
(c) Find the values of x when $y = -7$.
(d) For what range of values of x is y positive?

(a) See construction (a) on Fig. 16.4. The highest point on the curve corresponds to $y = 3{,}1$ (approximately). Since $y = 2 - 3x - 2x^2$, the greatest value of $2 - 3x - 2x^2$ is 3,1 (approximately).
(b) See construction (b) on Fig. 16.4. y is greatest when $x = 0{,}75$.
(c) See construction (c) on Fig. 16.4, when $y = -7$, $x = -3$ or $+1{,}5$.
(d) y is positive for all parts of the curve above the x-axis. y is positive when $x > -2$ and $x < 0{,}5$.
More neatly, y is positive for the range $-2 < x < 0{,}5$.

Some of the answers in Example 3 are only approximate. The graphs in this chapter are drawn to a small scale to save space. A bigger scale will give more accurate results.

Notice that the parabola in Fig. 16.4 is upside-down with respect to that of Fig. 16.2. When the coefficient of x^2 is positive, the parabola is ∪-shaped.
When the coefficient of x^2 is negative, the parabola is ∩-shaped.

Exercise 16b

1 In Fig. 16.2,
(a) find the value of y when $x = -2{,}5$,
(b) find the values of x when $y = 3$,
(c) find the range of values of x for which y is negative?

2 In Fig. 16.4, (a) find the value of y when $x = 2{,}8$, (b) find the values of x when $y = -3$, (c) find the values of x when $y = 2$.

3 Fig. 16.5 is the graph of $y = x^2 - 5x + 8$.
(a) What is the minimum value of $x^2 - 5x + 8$?
(b) For what value of x is y least?
(c) Find the value of y when $x = 4{,}5$.
(d) Find the values of x when $y = 4{,}6$.

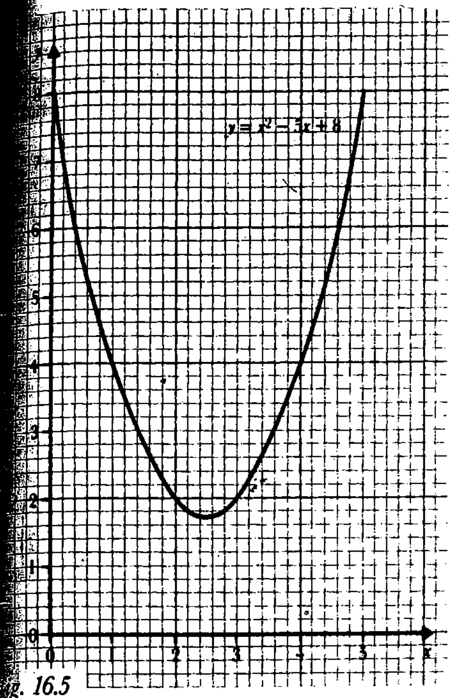


Fig. 16.5

4 Fig. 16.6 is the graph of the function $y = 3x - x^2$.

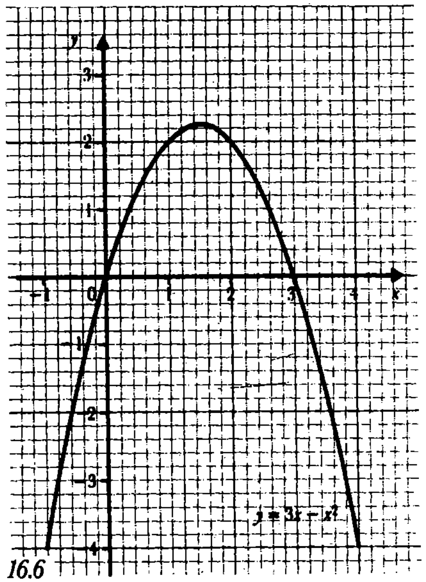


Fig. 16.6

(a) What is the greatest value of y?
(b) For what value of x is $3x - x^2$ a maximum?
(c) Find the values of x when $y = -3$.
(d) For what range of values of x is $3x - x^2$ positive?

5 Table 16.6 is a table of values for $y = x^2 + 3x - 1$ from $x = -4$ to $x = 1$.

Table 16.6

x	−4	−3	−2	−1	0	1
y	3	−1	−3	−3	−1	3

(a) Use a scale of 2 cm to 1 unit on both axes to draw the graph of $y = x^2 + 3x - 1$. Use the graph to find (b) the minimum value of y, (c) the value of y when $x = -3{,}5$, (d) the values of x when $y = 3$.

6 Table 16.7 is a table of values for $y = x - x^2$ from $x = -2$ to $x = 3$.

Table 16.7

x	−2	−1	0	1	2	3
y	−6	−2	0	0	−2	−6

Choose a suitable scale and draw the graph of $y = x - x^2$. Use the graph to (a) state the range of values of x for which y is positive, (b) find the value of x for which $x - x^2$ is a maximum, (c) find the maximum value of y, (d) find the value of y when $x = 2{,}2$, (e) find the values of x when $y = -5$.

7 (a) Given that $y = x^2 - 2x - 1$, copy and complete Table 16.8.

Table 16.8

x	−2	−1	0	1	2	3	4
x^2	4	1	0	1	4		
$-2x$	4	2	0	−2	−4		
-1	−1	−1	−1	−1	−1	−1	−1
y	−7	2		−2			

(b) Use a scale of 2 cm to 1 unit on both axes to draw the graph of $y = x^2 - 2x - 1$. Use your graph to find (c) the value of y when $x = 1{,}3$, (d) the values of x when $y = 6$, (e) the minimum value of y.

8 (a) Given that $y = 4 + 3x - x^2$, copy and complete Table 16.9.

Table 16.9

x	-1	0	1	2	3	4
4	4	4	4	4	4	4
$+3x$	-3		3		9	
$-x^2$	-1		-1		-9	
y	0		6		4	

(b) Choose a suitable scale and then draw the graph of $y = 4 + 3x - x^2$. Use your graph to (c) state the range of values of x for which $4 + 3x - x^2$ is positive, (d) find the value of y when $x = -0,6$, (e) find the values of x when $y = 3$, (f) find the maximum value of y.

9 Using a scale of 2 cm to 1 unit on both axes, draw the graph of $y = x^2$ for values of x from -3 to $+3$. Read off the values of y when $x =$ (a) 1,5, (b) $-0,7$, (c) 2,3, (d) $-1,3$. Find the values of x when $y =$ (e) 5, (f) 8, (g) 1,9, (h) 0,8.

10 (a) Draw the graph of the function $x^2 - 3x$ from $x = -1$ to $x = +4$ (b) Compare your graph with that of Fig. 16.6. What is the main difference between the two graphs? (c) For what range of values of x is $x^2 - 3x$ negative? (d) What is the minimum value of $x^2 - 3x$?

11 (a) Draw the graph of $y = x^2 - 2x$ from $x = -2$ to $x = +4$. (b) Compare the graph with that of question 7. What is the main difference between the two graphs? (c) What is the minimum value of y? (d) Find the values of x when $y = 7$.

12 (a) Draw the graph of $y = 5 + 4x - x^2$ from $x = -1$ to $x = +5$. Use your graph to find (b) the value of y when $x = 1,7$, (c) the values of x when $y = 4$, (d) the range of values of x for which $y > 7$.

Functional notation

Any algebraic expression which involves the variable x (and no other variable) is called a **function of x**. Its value depends on the value of x. The symbol $f(x)$ is short for *function of x*. $f(2)$ would be the same function with 2 written for x. Likewise $f(-1)$ is the same function with -1 substituted for x.

Example 4

If $f(x) = 5x^3 - 3x^2 + x - 4$, find $f(2)$ and $f(-1)$.

$$f(x) = 5x^3 - 3x^2 + x - 4$$
$$f(2) = 5 \times 2^3 - 3 \times 2^2 + 2 - 4$$
$$= 40 - 12 + 2 - 4 = 26$$
$$f(-1) = 5 \times (-1)^3 - 3 \times (-1)^2 + (-1) - 4$$
$$= -5 - 3 - 1 - 4 = -13$$

Example 5

If $f(x) = \dfrac{3(x-2)}{x^2 + 3x + 2}$, find

(a) $f(0)$, (b) $f(2)$, (c) $f(-3)$, (d) $f(x + 2)$.

(a) $f(0) = \dfrac{3(-2)}{2} = -3$

(b) $f(2) = \dfrac{3 \times 0}{4 + 6 + 2} = \dfrac{0}{12} = 0$

(c) $f(-3) = \dfrac{3(-3-2)}{9 - 9 + 2} = \dfrac{-15}{2} = -7\frac{1}{2}$

(d) $f(x + 2) = \dfrac{3(x + 2 - 2)}{(x + 2)^2 + 3(x + 2) + 2}$
$= \dfrac{3x}{x^2 + 7x + 12}$

Exercise 16c

1 If $f(x) = x^2$, find the following. Express your answers as simply as possible.
(a) $f(0)$ (b) $f(1)$ (c) $f(3)$
(d) $f(-2)$ (e) $f(-3)$ (f) $f(0,5)$
(g) $f(2x)$ (h) $f(x + 1)$ (i) $f(-3d)$

2 If $f(x) = 4x^2 - 1$, find the values of the following.
(a) $f(0)$ (b) $f(1)$ (c) $f(\frac{1}{2})$
(d) $f(\frac{1}{4})$ (e) $f(-2)$ (f) $f(x - 2)$

3 If $f(x) = 2^x$, find:
(a) $f(1)$ (b) $f(3)$ (c) $f(-1)$
(d) $f(0)$ (e) $f(-2)$ (f) $f(\frac{1}{2})$

4 If $f(x) = \dfrac{x^2 - x - 6}{2x - 5}$, find the values of:
(a) $f(0)$ (b) $f(1)$ (c) $f(2)$
(d) $f(-1)$ (e) $f(-2)$ (f) $f(3)$

Chapter 17

Indices and logarithms (2)

Standard form (Revision)

Table 17.1 shows that fractions such as one tenth, one hundredth, ... can be expressed as negative powers of 10.

Table 17.1

number	power of 10
1 000	10^3
100	10^2
10	10^1
1	10^0
0,1	10^{-1}
0,01	10^{-2}
0,001	10^{-3}

A number in the form $A \times 10^n$, where A is a number between 1 and 10 and n is a positive or negative integer, is said to be in **standard form**. For example $3,753 \times 10^2$ and $8,5 \times 10^{-3}$ are numbers in standard form.

Example 1

Express the following in standard form.
(a) 234 (b) 4 231 (c) 0,23 (d) 0,031

(a) $234 = 2,34 \times 100$
$= 2,34 \times 10^2$

(b) $4\,231 = 4,231 \times 1\,000$
$= 4,231 \times 10^3$

(c) $0,23 = 2,3 \times 0,1$
$= 2,3 \times 10^{-1}$

(d) $0,031 = 3,1 \times 0,01$
$= 3,1 \times 10^{-2}$

Exercise 17a (Oral revision)

Express the following in standard form.

1 32,4	2 0,471
3 3 472 000	4 0,000 613 1
5 4 576	6 51 720
7 0,043 81	8 0,000 000 231
9 623 000 000	10 0,003 471
11 8,04	12 510
13 6 500	14 24,12
15 0,027 9	16 0,000 193
17 0,102	18 12 800
19 0,007 43	20 500,8

Logarithms of numbers less than 1

In Chapter 8 it was shown that the base 10 logarithm of a number is the power to which 10 is raised to give that number. Tables were used to find the logarithms of numbers. For example,

$$37 = 3,7 \times 10$$
$$= 10^{0,5682} \times 10^1$$
$$= 10^{0,5682+1}$$
$$= 10^{1,5682}$$
$$\log 37 = 1,5682$$

The logarithms of numbers less than 1 are found by using negative powers of 10. For example,

$$0,037 = 3,7 \times 0,01$$
$$= 10^{0,5682} \times 10^{-2}$$
$$= 10^{0,5682+(-2)}$$
$$= 10^{-2+0,5682}$$
$$\log 0,037 = -2 + 0,5682$$

In practice, the logarithm of 0,037 is written as $\bar{2},5682$ and said as 'bar 2 point 5682'.

Notice that $\bar{2},5682$ is *not* the same as $-2,5682$:

$$\bar{2},5682 \text{ means } -2 + 0,5682,$$
$$-2,5682 \text{ means } -2 + (-0,5682).$$

The logarithm $\bar{2},5682$ has a negative integer ($\bar{2}$) and positive fraction (0,5682). The integer part of a logarithm can be found by expressing the given number in standard form. For example,

$\log 0,041\,32 = \log (4,132 \times 10^{-2})$
the integer is $\bar{2}$
$\log 0,00087 = \log (8,7 \times 10^{-4})$
the integer is $\bar{4}$

Exercise 17b

1 Write down the integer parts of the logarithms of the following.

(a) 0,064 (b) 0,000 68
(c) 0,75 (d) 0,005 9
(e) 0,000 8 (f) 0,000 02
(g) 0,001 5 (h) 0,802
(i) 0,000 004 (j) 0,011
(k) 0,000 047 (l) 0,000 009

2 Use tables to write down the logarithms of the numbers in question 1.

3 Write down the logarithms of the following.

(a) 5,192 (b) 0,051 92
(c) 0,519 2 (d) 0,000 519 2
(e) 0,038 62 (f) 0,000 197
(g) 0,650 4 (h) 0,000 000 32
(i) 0,8 (j) 0,000 025 14
(k) 0,800 4 (l) 0,008 4

4 Use antilog tables to write down the numbers whose logarithms are as follows.

(a) 0,3645 (b) $\bar{3},3645$ (c) $\bar{1},3645$
(d) $\bar{6},3645$ (e) $\bar{2},4997$ (f) $\bar{4},8939$
(g) $\bar{1},7892$ (h) $\bar{3},5739$ (i) $\bar{4},0612$
(j) $\bar{1},7323$ (k) $\bar{5},8226$ (l) $\bar{2},5008$

Multiplication and division

When calculations involve numbers less than 1, the laws of indices can be used as before.

As in Chapter 8, the following calculations may be computed on a calculator. 8-figure calculator outcomes are given as a check.

Example 2
Evaluate
(*a*) 0,768 5 × 0,034 15, (*b*) 0,768 5 ÷ 341,5

(a) 0,768 5 × 0,034 15
$= (7,685 \times 10^{-1}) \times (3,415 \times 10^{-2})$
$= (10^{0,8857} \times 10^{-1}) \times (10^{0,5334} \times 10^{-2})$ (from log tables)
$= 10^{0,8857+0,5334} \times 10^{(-1)+(-2)}$
$= 10^{1,4191} \times 10^{-3}$
$= 10^{0,4191} \times 10^{1} \times 10^{-3}$
$= 2,625 \times 10^{1+(-3)}$ (from antilog tables)
$= 2,625 \times 10^{-2}$
$= 0,02625$ (= 0.0262442)

Rough check: 0,8 × 0,03 = 0,024

(b) 0,768 5 ÷ 341,5
$= (7,685 \times 10^{-1}) \div (3,415 \times 10^{2})$
$= (10^{0,8857} \times 10^{-1}) \div (10^{0,5334} \times 10^{2})$
$= 10^{0,8857-0,5334} \times 10^{(-1)-(2)}$
$= 10^{0,3523} \times 10^{-3}$
$= 2,251 \times 10^{-3} = 0,002\,251$
(= 0.002250366)

Rough check: 0,8 ÷ 300 = 0,008 ÷ 3 ≃ 0,002 7

The method used in Example 2 is lengthy. It may lead to errors in horizontal addition and subtraction. It is more useful to set the work out in table form as in Chapter 8:

(a)

No	Log
0,768 5	$\bar{1},8857$
0,034 15	$\bar{2},5334$
0,026 25	$\bar{2},4191$

(b)

No	Log
0,768 5	$\bar{1},8857$
341,5	2,5334
0,002 251	$\bar{3},3523$

When setting the work out in tables, notice the following:

1. The fractional parts of the logarithms are positive. These are added and subtracted in the usual way.
2. The integral parts of the logarithms may be positive or negative. These must be added and subtracted as directed numbers.

The examples in Table 17.2 opposite show how to simplify logarithms with 'bar' notation. Read each example carefully.

Table 17.2

	simplify	working	result
(a)	$\bar{3},2 + \bar{5},4$	$\bar{3} + 0,2 + \bar{5} + 0,4 = \bar{8} + 0,6$	$\bar{3},2 + \bar{5},4 = \bar{8},6$
(b)	$\bar{3},7 + \bar{5},8$	$\bar{3} + 0,7 + \bar{5} + 0,8 = \bar{8} + 1,5$	$\bar{3},7 + \bar{5},8 = \bar{7},5$
(c)	$\bar{2},9 + 5,6$	$\bar{2} + 0,9 + 5 + 0,6 = 3 + 1,5$	$\bar{2},9 + 5,6 = 4,5$
(d)	$3,5 - 5,1$	$3 + 0,5 - (5 + 0,1) = \bar{2} + 0,4$	$3,5 - 5,1 = \bar{2},4$
(e)	$3,4 - 6,9$	$2 + 1,4 - (6 + 0,9) = \bar{4} + 0,5$	$3,4 - 6,9 = \bar{4},5$
(f)	$\bar{5},7 - \bar{2},3$	$\bar{5} + 0,7 - (\bar{2} + 0,3) = \bar{3} + 0,4$	$\bar{5},7 - \bar{2},3 = \bar{3},4$
(g)	$\bar{2},8 - \bar{6},1$	$\bar{2} + 0,8 - (\bar{6} + 0,1) = 4 + 0,7$	$\bar{2},8 - \bar{6},1 = 4,7$
(h)	$\bar{5},3 - \bar{2},7$	$\bar{6} + 1,3 - (\bar{2} + 0,7) = \bar{4} + 0,6$	$\bar{5},3 - \bar{2},7 = \bar{4},6$

Notice example (h). In this case 0,7 cannot be subtracted from 0,3. 1 is added to 0,3, making it 1,3. To keep the value of the logarithm the same, 1 is subtracted from 5, making it 6.

Exercise 17c

Simplify the following, expressing the answers in 'bar' notation where necessary.

1 $\bar{2},6 + \bar{3},1$ **2** $\bar{1},5 + \bar{2},5$ **3** $\bar{2},8 + \bar{1},3$
4 $\bar{5},4 - 1,1$ **5** $\bar{5},4 - \bar{1},2$ **6** $\bar{5},7 - 2,5$
7 $\bar{3},8 - \bar{5},2$ **8** $\bar{3},4 - \bar{4},2$ **9** $\bar{3},5 - \bar{1},7$
10 $\bar{4},8 + 3,7$ **11** $\bar{2},5 - \bar{4},8$ **12** $\bar{4},6 + 1,8$
13 $\bar{2},6 - \bar{1},8$ **14** $\bar{3},2 - \bar{3},8$ **15** $\bar{2},0 - 1,7$
16 $2,6 + \bar{3},4$ **17** $\bar{3},4 + \bar{3},8$ **18** $\bar{3},3 - \bar{5},5$

Example 3
Evaluate 0,082 4 × 6,51.

No	Log
0,082 4	$\bar{2},9159$
6,51	0,8136
0,536 4	$\bar{1},7295$

0,082 4 × 6,51 = 0,536 4 (= 0.536424).
Check: 0,08 × 7 = 0,56

Example 4
Evaluate 6,802 ÷ 0,094.

No	Log
6,802	0,8326
0,094	$\bar{2},9731$
72,36	1,8595

6,802 ÷ 0,094 = 72,36 (= 72.361702).
Check: 7 ÷ 0,09 = 700 ÷ 9 ≃ 78

Notice in Example 4, $\frac{6,082}{0,094} = \frac{680,2}{9,4}$.

Bar notation could have been avoided by multiplying the numerator and denominator by 100.

Example 5
Evaluate 57,9 × 0,002 8 × 0,6.

No	Log
57,9	1,7627
0,002 8	$\bar{3},4472$
0,6	$\bar{1},7782$
0,097 29	$\bar{2},9881$

57,9 × 0,002 8 × 0,6 = 0,097 29
(= 0.097272)
Check: 60 × 0,003 × 0,6 = 0,18 × 0,6 = 0,108

Example 6

Evaluate $\dfrac{4,762 \times 0,007\,853}{0,012\,9}$.

No	Log
4,762	0,6778
0,007 853	$\bar{3}$,8951
numerator	$\bar{2}$,5729
0,012 9	$\bar{2}$,1106
2,899	0,4623

$\dfrac{4,762 \times 0,007\,853}{0,012\,9} = 2,899$ (= 2.8989136 [calculator])

Check: $\dfrac{4,8 \times 0,008}{0,012} = \dfrac{48 \times 8}{120} = 3,2$

Exercise 17d

Evaluate the following. You may use a calculator if you wish. Use rough calculation to check each result.

1 3,925 × 0,031 75 **2** 0,764 2 × 0,350 7
3 0,673 5 × 0,928 **4** 0,093 5 × 8,672
5 0,342 6 × 0,193 8 **6** 0,067 2 × 0,098 53
7 0,569 2 ÷ 0,094 3 **8** 29,57 ÷ 119,8
9 9,43 ÷ 56,92 **10** 5,673 ÷ 98,42
11 0,52 ÷ 0,092 35 **12** 0,766 2 ÷ 9,325
13 8,686 × 0,507 2 **14** 0,838 4 ÷ 0,900 6
15 3,925 ÷ 0,031 71
16 0,296 × 0,008 2 × 5,437
17 $\dfrac{9,23 \times 87,6}{991,7}$ **18** $\dfrac{0,096\,1 \times 4,873}{0,834\,5}$
19 $\dfrac{0,254\,3}{0,085\,72}$ **20** 0,964 2 × 0,424 3
21 $\dfrac{2,647 \times 0,009\,21}{0,057\,38}$
22 0,264 × 0,008 35 × 10,4
23 0,348 × 0,088 6 × 3,94
24 $\dfrac{0,628\,3 \times 17,85}{8,347}$

Powers and roots of numbers less than 1

A scientific calculator may be advantageous when finding roots of numbers.

Example 7

Evaluate (*a*) $(0,085)^3$, (*b*) $\sqrt[4]{0,000\,7}$.

(a) $0,085 = 8,5 \times 10^{-2}$
$(0,085)^3 = (8,5 \times 10^{-2})^3$
$= (10^{0,9294} \times 10^{-2})^3$ (from log tables)
$= 10^{0,929 \times 3} \times 10^{(-2) \times 3}$
$= 10^{2,7882} \times 10^{-6}$
$= 10^{0,7882} \times 10^2 \times 10^{-6}$
$= 6,141 \times 10^{-4}$ (from antilog tables)
$= 0,000\,614\,1$
(= 0.000614125 [calculator])

(b) $0,000\,7 = 7 \times 10^{-4}$
$\sqrt[4]{0,000\,7} = (7 \times 10^{-4})^{\frac{1}{4}}$
$= (10^{0,8451} \times 10^{-4})^{\frac{1}{4}}$
$= 10^{0,8451 \div 4} \times 10^{(-4) \div 4}$
$= 10^{0,2113} \times 10^{-1}$
$= 1,627 \times 10^{-1}$
$= 0,1627$
(= 0.1626576* [calculator])

(*obtained by pressing the [√] key twice).

As before, it is advisable to set out logarithm work in table form:

(a)

No	Log
0,085	$\bar{2}$,9294
$(0,085)^3$	$\bar{2}$,9294 × 3
0,000 614 1	$\bar{4}$,7882

(b)

No	Log
0,000 7	$\bar{4}$,8451
$\sqrt[4]{(0,000\,7)}$	$\bar{4}$,8451 ÷ 4
0,162 7	$\bar{1}$,2113

In (a), each part of the logarithm is multiplied separately:

integer: $\bar{2} \times 3 \quad = \bar{6}$
fraction: $0,9294 \times 3 = 2,7882$
adding: $\bar{2},9294 \times 3 = \bar{4},7882$

In (b), each part of the logarithm is divided separately:

integer: $\bar{4} \div 4 \quad = \bar{1}$
fraction: $0,8451 \div 4 = 0,2113$
adding: $\bar{4},8451 \div 4 = \bar{1},2113$

Table 17.3 gives examples of multiplication division of logarithms with bar notation.

Table 17.3

simplify	working	result
$\bar{2},7 \times 4$	$= 4(\bar{2} + 0,7)$	
	$= \bar{8} + 2,8$	
	$= \bar{6} + 0,8$	$= \bar{6},8$
$\bar{6},9 \div 3$	$= 3\overline{)\bar{6} + 0,9}$	
	$\bar{2} + 0,3$	$= \bar{2},3$
$\bar{4},7 \div 3$	$= 3\overline{)\bar{4} + 0,7}$	
	$= 3\overline{)\bar{6} + 2,7}$	
	$= \bar{2} + 0,9$	$= \bar{2},9$

Notice in example (c) that $\bar{4}$ cannot be divided exactly by 3. Replace $\bar{4}$ with $\bar{6} + 2$. The negative part of the logarithm is divisible by 3.

Exercise 17e

Simplify the following, expressing the answers in bar notation.

1 $\bar{4}$,3 × 2 **2** $\bar{3}$,6 ÷ 3 **3** $\bar{1}$,9 × 3
4 $\bar{6}$,8 ÷ 2 **5** $\bar{3}$,8 × 3 **6** $\bar{5}$,8 ÷ 3
7 $\bar{2}$,5 ÷ 5 **8** $\bar{2}$,6 × 4 **9** $\bar{3}$,2 ÷ 4
10 $\bar{3}$,8 ÷ 4 **11** $\bar{6}$,5 ÷ 5 **12** $\bar{7}$,2 ÷ 4

Example 8

Evaluate $0,610\,4^3$.

No	Log
$0,610\,4^3$	$\bar{1}$,7856 × 3
0,227 4	= $\bar{1}$,3568

$0,610\,4^3 = 0,227\,4$ (= 0.2274278 [calculator])
Check: $0,6^3 = 0,216$

Example 9

Evaluate $\sqrt[3]{0,361\,2}$.

No	Log
$\sqrt[3]{0,361\,2}$	$\bar{1}$,5577 ÷ 3
	= ($\bar{3}$ + 2,5577) ÷ 3
0,7122	= $\bar{1}$,8526

$\sqrt[3]{0,361\,2} = 0,712\,2$ (= 0.7121682 [calculator])
Check: $0,7^3 = 0,343$

Notice, in Example 9, it is easier to check the cube of the answer, 0,7, than to guess the cube root of the given number, 0,361 2.

Example 10

Evaluate $(5,375 \times 10^{-6})^3$.

No	Log
$(5,375 \times 10^{-6})^3$	$\bar{6}$,7304 × 3
$1,553 \times 10^{-16}$	= $\overline{16}$,191 2

$(5,375 \times 10^{-6})^3 = 1,553 \times 10^{-16}$
(= 1.5528711 × 10^{-16} [calculator])
Check: $(5 \times 10^{-6})^3 = 125 \times 10^{-18}$
$= 1,25 \times 10^{-16}$

Notice, in Example 10, that since the given number is in standard form, the result is also given in standard form.

Example 11

Evaluate $\sqrt[5]{\dfrac{6,231}{242,7}}$.

No	Log
6,231	0,7946
242,7	2,3850
	$\bar{2}$,4096
$\sqrt[5]{\dfrac{6,231}{242,7}}$	$\bar{2}$,4096 ÷ 5
	= ($\bar{5}$ + 3,4096) ÷ 5
0,4807	= $\bar{1}$,6819

$\sqrt[5]{\dfrac{6,231}{242,7}} = 0,480\,7$ (= 0.4807259 [calculator])

Check: $\frac{6}{240} = \frac{1}{40} = 0,025$
also $(0,4807)^5 \simeq (0,5)^5 \simeq 0,03$

Exercise 17f

Evaluate the following using either tables or an appropriate calculator. Use rough calculations, where possible, to check the results.

1 $0,692\,7^2$ **2** $0,493\,4^3$ **3** $0,542\,5^2$
4 $0,215^4$ **5** $0,672\,5^3$ **6** $0,294\,5^4$
7 $\sqrt[3]{0,388\,7}$ **8** $\sqrt[3]{0,038\,87}$
9 $\sqrt{0,267\,3}$ **10** $\sqrt[4]{0,063\,57}$
11 $\sqrt[4]{0,061\,3}$ **12** $\sqrt[10]{0,065\,68}$
13 $0,926\,7^3$ **14** $\sqrt[3]{0,066\,42}$
15 $\sqrt[4]{0,054\,57}$ **16** $0,846\,2^2$
17 $\sqrt[5]{0,008\,91}$ **18** $0,998\,7^{10}$

19 $(0{,}657 \times 0{,}83)^4$ **20** $(0{,}624 \div 0{,}098\,35)^2$

21 $(3{,}47 \div 8{,}74)^3$ **22** $\sqrt[3]{\frac{349}{2\,608}}$

23 $\sqrt[3]{0{,}285\,3^5}$ **24** $(8{,}462 \times 0{,}086\,77)^5$

25 $\sqrt{0{,}682\,1 \times 0{,}005\,924}$

26 $\sqrt[5]{\dfrac{26{,}7}{9{,}562}}$ **27** $\dfrac{1{,}487^3 - 1}{1{,}487^3 + 1}$

28 $(4{,}291 \times 10^5)^4$ **29** $(3{,}172 \times 10^{-3})^6$

30 $\sqrt{6{,}231 \times 10^{-5}}$

Theory of logarithms

In Chapter 8 it was shown that the statements $100 = 10^2$ and $\log 100 = 2$ are two ways of writing the same thing. The second statement is written in full as $\log_{10} 100 = 2$, i.e. 'the logarithm to the base 10 of 100 is 2'.

Logarithms can be in bases other than 10. For example, since $8 = 2^3$, $\log_2 8 = 3$. The logarithm to the base 2 of 8 is 3.

In general,

if $N = a^x$, then $\log_a N = x$.

The logarithm to a given base of a number is the power to which the base must be raised to make the number.

Example 12

Evaluate (*a*) $\log_2 32$, (*b*) $\log_9 27$, (*c*) $\log_5 0{,}04$.

(a) Let $N = \log_2 32$
then $2^N = 32 = 2^5$
hence $N = 5$

(b) Let $x = \log_9 27$
then $9^x = 27$
i.e. $(3^2)^x = 3^3$
$3^{2x} = 3^3$
hence $2x = 3$
and $x = 1\frac{1}{2}$

(c) Let $r = \log_5 0{,}04$
then $5^r = 0{,}04 = \frac{1}{25} = 5^{-2}$
hence $r = -2$

Exercise 17g

Evaluate the following logarithms.

1 $\log_2 4$ **2** $\log_{10} 1\,000$ **3** $\log_5 25$
4 $\log_3 81$ **5** $\log_{12} 144$ **6** $\log_6 216$
7 $\log_7 \frac{1}{7}$ **8** $\log_4 64$ **9** $\log_4 8$
10 $\log_8 4$ **11** $\log_{25} 0{,}2$ **12** $\log_{49} 7$
13 $\log_{1\,000} 10$ **14** $\log_4 \frac{1}{8}$ **15** $\log_{16} 0{,}25$
16 $\log_{100} 0{,}001$ **17** $\log_9 \frac{1}{27}$ **18** $\log_8 0{,}0625$
19 $\log_{1,2} 1{,}728$ **20** $\log_{0,2} 25$

The three basic laws of indices can now be given in their equivalent logarithmic form.

Laws of indices

1 $x^a \times x^b = x^{a+b}$
2 $x^a \div x^b = x^{a-b}$
3 $(x^a)^b = x^{ab}$

Laws of logarithms

1 $\log(MN) = \log M + \log N$
2 $\log\left(\dfrac{M}{N}\right) = \log M - \log N$
3 $\log(M^p) = p \log M$

Example 13

Prove that $\log (MN) = \log M + \log N$.

Let $\log_x M = a$ and $\log_x N = b$, where x is any base.

Then $M = x^a$ and $N = x^b$
$MN = x^a \times x^b$
$= x^{a+b}$

$\log_x (MN) = a + b = \log_x M + \log_x N$
Hence $\log (MN) = \log M + \log N$ is true for any base.

Example 14

Given that $\log 2 = 0{,}30103$ *and* $\log 3 = 0{,}47712$, *calculate, without using any tables.* (*a*) $\log 5$, (*b*) $\log 6$, (*c*) $\log 0{,}9$.

(a) $\log 5 = \log \frac{10}{2}$
$= \log 10 - \log 2$
$= 1 - 0{,}30103$
$= 0{,}69897$

(b) $\log 6 = \log (2 \times 3)$
$= \log 2 + \log 3$
$= 0{,}30103 + 0{,}477\,12$
$= 0{,}77815$

(c) $\log 0{,}9 = \log \frac{9}{10}$
$= \log 9 - \log 10$
$= \log 3^2 - 1$
$= 2 \log 3 - 1$
$= 2 \times 0{,}47712 - 1$
$= 0{,}95424 - 1$
$= 0{,}04566$

Example 15

Simplify the following as far as possible.
(*a*) $\log 8 + \log 5$ (*b*) $2 - \log 4$ (*c*) $\log 9 \div \log 3$
(*d*) $\frac{2}{5} \log 32$

(a) $\log 8 + \log 5 = \log (8 \times 5)$
$= \log 40$

(b) $2 - \log 4 = \log 100 - \log 4$
$= \log \frac{100}{4}$
$= \log 25$

(c) $\log 9 \div \log 3 = \dfrac{\log 9}{\log 3} = \dfrac{\log 3^2}{\log 3}$
$= \dfrac{2 \log 3}{\log 3}$
$= 2$

(d) $\frac{2}{5} \log 32 = \log 32^{\frac{2}{5}}$
$= \log (\sqrt[5]{32})^2 = \log 2^2$
$= \log 4$

Exercise 17h

Assume that all the logarithms are to base 10.

1 Given that $\log 2 = 0{,}3010$, $\log 3 = 0{,}4771$ and $\log 7 = 0{,}8451$, evaluate the following.
(a) $\log 5$ (b) $\log 8$ (c) $\log 49$
(d) $\log 14$ (e) $\log 35$ (f) $\log 42$

2 Given that $\log 2 = 0{,}30101$, evaluate $\log 16$ without using tables.

3 Express the following as logarithms of single numbers.
(a) $\log 3 + \log 4$ (b) $\log 15 - \log 3$
(c) $3 \log 5$ (d) $\frac{1}{2} \log 25$
(e) $\log 81 - \log 3$ (f) $1 + \log 3$
(g) $1 - \log 5$ (h) $2 \log 3 + \log 6$
(i) $2 - 2 \log 5$ (j) $\frac{3}{4} \log 16$

4 Evaluate, without any tables,
$3 \log 2 + \log 20 - \log 1{,}6$

5 Solve the following for x.
(a) $\log_{10} x = 3$ (b) $\log_{10} x = -2$

6 Simplify the following without using tables.
(a) $\dfrac{\log 4}{\log 2}$ (b) $\dfrac{\log 16}{\log 8}$
(c) $\dfrac{\log 243}{\log 3}$ (d) $\dfrac{\log \sqrt{5}}{\log 5}$
(e) $\dfrac{\log \sqrt{3}}{\log 9}$ (f) $\dfrac{\log 0{,}2}{\log 25}$

7 Simplify the following.
(a) $\log 8 - \log 4$ (b) $\log 8 \div \log 4$
(c) $\log 8 + \log 4$ (d) $\dfrac{\log 8 - \log 4}{\log 4 - \log 2}$

8 Express the following equations in index form.
(a) $\log x = b$
(b) $\log x + \log y = 1$
(c) $\log x + 1 = 0$
(d) $\log x + 2 \log y = 3$
(e) $\log x - \log y = \log z$

9 If $\log x - \log (2x - 1) = 1$, find x.

10 Use the method of Example 13 to prove
(a) $\log \left(\dfrac{M}{N}\right) = \log M - \log N$,
(b) $\log (M^p) = p \log M$.

Exercise 17i (General practice)

Evaluate the following, using tables or a calculator. Give each answer correct to 3 s.f.

1 $42{,}87 \times 23{,}82 \times 1{,}27$

2 $\dfrac{7{,}143 \times 821{,}5}{0{,}001\,4}$

3 $5{,}024^3$ **4** $\sqrt[3]{26{,}71}$

5 $\sqrt[4]{82{,}64 \times 137{,}5}$ **6** $\sqrt[3]{0{,}056\,78}$

7 $\sqrt{0{,}000\,562}$ **8** $\sqrt[4]{(384{,}5)^3}$

9 $\sqrt[5]{(16{,}84)^3}$ **10** $0{,}076\,34^{\frac{1}{4}}$

11 $\dfrac{0{,}062\,95}{0{,}081\,83}$ **12** $(0{,}267 \times 0{,}934)^3$

13 $\left(\dfrac{0{,}009\,213}{0{,}076\,46}\right)^2$ **14** $\dfrac{0{,}387\,2 \times 0{,}092\,16}{0{,}056\,57}$

15 $\dfrac{0{,}358\,1 \times 0{,}028\,47}{0{,}009\,418 \times 3{,}219}$ **16** $6{,}28 \times \sqrt{\dfrac{304}{981}}$

17 $\dfrac{0{,}678\,4^2}{0{,}921\,6^3}$ **18** $\dfrac{13{,}87^2 \times 2{,}95}{4\,009}$

19 $\dfrac{15{,}47 + 0{,}085\,2}{(254)^{\frac{1}{4}}}$

20 $\sqrt[3]{69{,}5^2 - 30{,}5^2}$ **21** $\sqrt[3]{8{,}3^3 + 1{,}7^3}$

Chapter 18

Quadratic expressions (2)

Factorising quadratic expressions

Example 1 (Revision)
Factorise $a^2 - 17a + 42$.

1st step: $a^2 - 17a + 42 = (a \quad)(a \quad)$
2nd step: Find two numbers such that their product is $+42$ and their sum is -17. Since the middle term is negative, consider negative factors only.

	factors of $+42$	*sum of factors*
(a)	-42 and -1	-43
(b)	-21 and -2	-23
(c)	-14 and -3	-17
(d)	-7 and -6	-13

Of these, only (c) gives the required result.
Hence $a^2 - 17a + 42 = (a - 14)(a - 3)$
Check: $(a - 14)(a - 3) = a^2 - 14a - 3a + 42$
$= a^2 - 17a + 42$

alternative method:
1st step: Find the product of the first and last terms:
$a^2 \times (+42) = +42a^2$
2nd step: Find two terms such that their product is $+42a^2$ and their sum is $-17a$ (the middle term).

	factors of $+42a^2$	*sum of factors*
(a)	$-42a$ and $-a$	$-43a$
(b)	$-21a$ and $-2a$	$-23a$
(c)	$-14a$ and $-3a$	$-17a$
(d)	$-7a$ and $-6a$	$-13a$

Of these, only $-14a + (-3a) = -17a$.
3rd step: Replace $-17a$ with $-14a - 3a$ in the given expression.
Factorise by grouping.
$a^2 - 17a + 42 = a^2 - 14a - 3a + 42$
$= a(a - 14) - 3(a - 14)$
$= (a - 14)(a - 3)$

Example 2
Factorise $x^2 + 8x - 20$.

By the alternative method:
1st step: Find the product of the first and last terms.

$x^2 \times (-20) = -20x^2$

2nd step: Find two terms such that their product is $-20x^2$ and their sum is $+8x$.

	factors of $-20x^2$	*sum of factors*
(a)	$-20x$ and $+x$	$-19x$
(b)	$+20x$ and $-x$	$+19x$
(c)	$-10x$ and $+2x$	$-8x$
(d)	$+10x$ and $-2x$	$+8x$
(e)	$-5x$ and $+4x$	$-x$
(f)	$+5x$ and $-4x$	$+x$

Of these only (d) gives the required result.
3rd step: Replace $+8x$ with $+10x - 2x$ in the given expression.
Factorise by grouping.
$x^2 + 8x - 20 = x^2 + 10x - 2x - 20$
$= x(x + 10) - 2(x + 10)$
$= (x + 10)(x - 2)$

Exercise 18a (Revision)
Factorise the following.

1 $x^2 + 9x + 14$ **2** $y^2 - 12x + 35$
3 $z^2 - 3z + 2$ **4** $a^2 + 9x - 22$
5 $d^2 - 3d - 40$ **6** $h^2 + 3h - 40$
7 $n^2 - 18n + 32$ **8** $s^2 + 9s + 20$
9 $k^2 + 11k - 26$ **10** $b^2 - 11b + 18$
11 $a^2 + 14a + 45$ **12** $a^2 - 8a - 33$
13 $x^2 - x - 42$ **14** $x^2 + 13x - 48$
15 $x^2 + x - 6$ **16** $t^2 - t - 12$
17 $x^2 - 27x + 50$ **18** $x^2 + 39x + 38$
19 $r^2 - 25$ **20** $a^2 - 14a + 49$

More difficult quadratic expressions

In the quadratic expressions of Exercise 18a, the coefficient of x^2, a^2, . . ., etc., was always 1. However, there are many quadratic expressions which are not so simple. For example, $2a^2 + 7a - 15$, where the coefficient of a^2 is 2, and $2a^2 + 7ab - 15b^2$ which contains more than one letter. Factors of such expressions are found by trial and error as before. Read the following examples carefully. These use the alternative method, shown in Examples 1 and 2, leading to factorisation by grouping. This method is recommended when factorising more difficult quadratic expressions.

Example 3
Factorise $2a^2 + 7a - 15$.

1st step: Find the product of the first and last terms.
$2a^2 \times (-15) = -30a^2$
2nd step: Find two terms such that their product is $-30a^2$ and their sum is $+7a$ (the middle term).

	factors of $-30a^2$	*sum of factors*
(a)	$-30a$ and $+a$	$-29a$
(b)	$+30a$ and $-a$	$+29a$
(c)	$-15a$ and $+2a$	$-13a$
(d)	$+15a$ and $-2a$	$+13a$
(e)	$-10a$ and $+3a$	$-7a$
(f)	$+10a$ and $-3a$	$+7a$
(g)	$-6a$ and $+5a$	$-a$
(h)	$+6a$ and $-5a$	$+a$

Of these, only (f) gives the required result.
3rd step: Replace $+7a$ in the given expression with $+10a - 3a$.
Factorise by grouping.
$2a^2 + 7a - 15 = 2a^2 + 10a - 3a - 15$
$= 2a(a + 5) - 3(a + 5)$
$= (a + 5)(2a - 3)$

In Example 3, all 8 possibilities were written down. In practice there is no need to do this. The method can be shortened as follows:
1 Notice in the second step that the sum of the two terms is positive. It is necessary to consider only those factors in which the positive term is greater numerically than the negative term.
2 Stop when the required result is reached.

Example 4
Factorise $7 - 22x + 3x^2$.

1st step: $7 \times (+3x^2) = 21x^2$
2nd step: Find two terms such that their sum is $-22x$ and their product is $+21x^2$. Since the middle term is negative, consider negative factors only.

factors of $+21x^2$	*sum of factors*
$-21x$ and $-x$	$-22x$(stop)

3rd step: Replace $-22x$ with $-21x - x$ in the given expression.
Factorise by grouping.
$7 - 22x + 3x^2 = 7 - 21x - x + 3x^2$
$= 7(1 - 3x) - x(1 - 3x)$
$= (1 - 3x)(7 - x)$

Example 5
Factorise $8x^2 - 14x - 9$.

1st step: $8x^2 \times (-9) = -72x^2$
2nd step:

factors of $-72x^2$	*sum of factors*
$-72x$ and $+x$	$-71x$
$-36x$ and $+2x$	$-34x$
$-18x$ and $+4x$	$-14x$(stop)

3rd step:
$8x^2 - 14x - 9 = 8x^2 - 18x + 4x - 9$
$= 2x(4x - 9) + 1(4x - 9)$
$= (4x - 9)(2x + 1)$

Notice, in Example 5, that the middle term is negative. It was necessary to consider only those factors in which the negative term was greater numerically than the positive term.

Example 6
Factorise $6a^2 + 15a + 9$.

3 is a common factor. First take out the common factor.
$6a^2 + 15a + 9 = 3(2a^2 + 5a + 3)$
$2a^2 \times (+3) = 6a^2$

factors of $+6a^2$	*sum of factors*
$+6a$ and $+a$	$+7a$
$+3a$ and $+2a$	$+5a$ (stop)

$6a^2 + 15a + 9 = 3(2a^2 + 5a + 3)$
$= 3(2a^2 + 3a + 2a + 3)$
$= 3[a(2a + 3) + 1(2a + 3)]$
$= 3(2a + 3)(a + 1)$

Example 7

Factorise

(*a*) $2a^2 + 7ab - 15b^2$,
(*b*) $2a^2b^2 + 7ab - 15$.

Notice that these two examples have the same coefficients as the expression in Example 3.

(a) $2a^2 + 7ab - 15b^2 = (a + 5b)(2a - 3b)$
(b) $2a^2b^2 + 7ab - 15 = (ab + 5)(2ab - 3)$

Use multiplication to check the solutions to Example 7.

Exercise 18b

Factorise the following.

1 $a^2 + 8a + 15$ **2** $b^2 - 7b + 10$
3 $c^2 + 4c - 21$ **4** $d^2 - 5d - 14$
5 $e^2 + 2e - 8$ **6** $w^2 + 5w + 6$
7 $x^2 + 5x - 6$ **8** $y^2 - 5y + 6$
9 $z^2 - 5z - 6$ **10** $2d^2 + 3d + 1$
11 $2e^2 - 3e + 1$ **12** $2f^2 - f - 1$
13 $a^2 + 7a + 10$ **14** $a^2 + 7ab + 10b^2$
15 $a^2b^2 + 7ab + 10$ **16** $x^2 - 2xy - 15y^2$
17 $m^2 + 10m - 24$ **18** $n^2 - 10n - 24$
19 $u^2 - 10u + 24$ **20** $v^2 - 11v + 24$
21 $m^2 + 4m - 21$ **22** $m^2 + 4mn - 21n^2$
23 $m^2n^2 + 4mn - 21$ **24** $3a^2 - 4a + 1$
25 $3b^2 + b - 2$ **26** $2x^2 + 5x - 3$
27 $2y^2 - 5y - 3$ **28** $2z^2 - 5z + 3$
29 $1 + 3m + 2m^2$ **30** $15 - 2n - n^2$
31 $1 - 2u - 8u^2$ **32** $u^2 + 2uv - 8v^2$
33 $a^2 + 5ab - 36b^2$ **34** $a^2 + 9ab - 36b^2$
35 $a^2 + 16ab - 36b^2$ **36** $2b^2 - 10b + 12$
37 $c^2 - 4c - 77$ **38** $77 - 4d - d^2$
39 $3e^2 + 3e - 18$ **40** $3f^2 + 2f - 1$
41 $a^2 + 4ab + 3b^2$ **42** $1 + 4x + 3x^2$
43 $2g^2 - 5g + 2$ **44** $2h^2 + 5h + 3$
45 $3h^2 + 7hk + 2k^2$ **46** $12x^2 - 13x - 14$
47 $a^2 + 25a - 150$ **48** $b^2 + 25b + 150$
49 $3c^2 - 11c + 6$ **50** $3d^2 + 7d - 6$
51 $5e^2 - 9e - 2$ **52** $7f^2 + 10f + 3$
53 $35 - 12a + a^2$ **54** $35 - 2b - b^2$
55 $35 + 36c + c^2$ **56** $35 + 30d - 5d^2$
57 $3a^2 + 5ab + 2b^2$ **58** $3m^2 + 5mn - 2n^2$
59 $3u^2 + 7uv + 2v^2$ **60** $6n^2 - 7n - 3$
61 $7v^2 + 22v + 3$ **62** $4y^2 - 12y + 5$
63 $2h^2 - 15h - 27$ **64** $2k^2 - 15k + 27$
65 $x^2y^2 - xy - 30$ **66** $2u^2v^2 + uv - 6$
67 $5 - 7a - 6a^2$ **68** $10p^2 - 41p - 45$
69 $10q^2 - 43q + 45$ **70** $8a^2 - 17a + 9$
71 $8b^2 - 18b + 9$ **72** $8c^2 - 21c - 9$
73 $8d^2 - 22d + 9$ **74** $8e^2 - 49e + 75$
75 $8f^2 - 50f + 75$ **76** $12a^2b^2 + 11ab - 5$
77 $12m^2 - 4mn - 5n^2$
78 $12t^2 - 11t + 2$
79 $12x^2y^2 - 11xy - 1$
80 $24p^2 + pq - 23q^2$

Quadratic equations

If the product of two numbers is 0, then one of the numbers (or possibly both of them) must be 0. For example,

$3 \times 0 = 0$, $0 \times 5 = 0$ and $0 \times 0 = 0$

In general, if $a \times b = 0$,
then *either* $a = 0$
or $b = 0$

Example 8

Solve the equation $(x - 2)(x + 7) = 0$.

If $(x - 2)(x + 7) = 0$
then either $x - 2 = 0$ or $x + 7 = 0$
$\Leftrightarrow$ $x = 2$ or -7

Example 9

Solve the equation $a(a + 3) = 0$.

If $a(a + 3) = 0$
then either $a = 0$ or $a + 3 = 0$
$\Leftrightarrow$ $a = 0$ or -3

Example 10

Solve the equation $(m - 5)^2 = 0$.

If $(m - 5)^2 = 0$
then $(m - 5)(m - 5) = 0$
$\Leftrightarrow$ $m - 5 = 0$ twice
$\Leftrightarrow$ $m = 5$ twice.

Example 11

Solve the equation $d(d - 4)(d + 6)^2 = 0$.

If $d(d - 4)(d + 6)^2 = 0$, then any *one* of the four factors of the LHS may be 0,
i.e. $d = 0$ or $d - 4 = 0$ or $d + 6 = 0$ twice.
$\Leftrightarrow$ $d = 0, 4$, or -6 twice.

The solutions of the equations in Examples 8, 9, 10, 11 are called the **roots** of the equations. Notice that Examples 10 and 11 contain repeated roots. The meaning of these will be explained on page 164.

Exercise 18c

Solve the following equations.

1 $(a - 3)(a + 5) = 0$
2 $(b - 2)(b - 1) = 0$
3 $(x + 2)(x + 6) = 0$
4 $y(y - 5) = 0$
5 $(m - 3)^2(m - 4) = 0$
6 $(n - 5)(n + 3)^2 = 0$
7 $u(u + 5)(u + 1) = 0$
8 $(v - 7)(v + 5)(v - 3) = 0$
9 $x^2(x + 3) = 0$ **10** $y^2(y - 4)^2 = 0$
11 $5(a + 2)(a - 4) = 0$
12 $4b(b + 6)^2 = 0$
13 $3d^2(d - 7) = 0$ **14** $6m^2(m + 3)^2 = 0$
15 $(6 - n)(4 + n) = 0$
16 $(5 + u)(3 - u) = 0$
17 $v(v - 2)(v + 2) = 0$
18 $x^2(x + 5)(x - 5) = 0$
19 $y^2(3 + y) = 0$
20 $a(2 - a)^2(1 + a) = 0$

Example 12

Solve the equation $(3a + 2)(2a - 7) = 0$.

If $(3a + 2)(2a - 7) = 0$, then
either $3a + 2 = 0$ or $2a - 7 = 0$
$\Leftrightarrow$ $3a = -2$ or $2a = 7$
$\Leftrightarrow$ $a = -\frac{2}{3}$ or $a = \frac{7}{2}$
i.e. $a = -\frac{2}{3}$ or $3\frac{1}{2}$

Check: By substitution:
If $a = -\frac{2}{3}$, $(3a + 2)(2a - 7)$
$= (-2 + 2)(-1\frac{1}{3} - 7)$
$= 0 \times (-8\frac{1}{3}) = 0$
If $a = 3\frac{1}{2}$, $(3a + 2)(2a - 7)$
$= (10\frac{1}{2} + 2)(7 - 7)$
$= 12\frac{1}{2} \times 0 = 0$

Exercise 18d

Solve the following equations. Check the results by substitution.

1 $(d - 5)(3d - 2) = 0$
2 $(2m - 1)(m + 4) = 0$
3 $(a + 3)(5a + 2) = 0$
4 $(4x + 3)(3x + 1) = 0$
5 $(2y - 7)(y + 2) = 0$
6 $(4b - 12)(b - 5) = 0$
7 $(4h - 1)(2h + 3) = 0$
8 $(5 - d)(5 - 2d) = 0$
9 $(5 + 3m)(2 - 5m) = 0$
10 $(3n + 7)(4n - 1) = 0$
11 $(3a + 10)(3a - 12) = 0$
12 $(4b - 3)^2 = 0$
13 $(2c + 1)^2 = 0$
14 $(1 - 2d)(2 + 3d) = 0$
15 $(3e + 5)^2 = 0$
16 $m(3m - 4)(7 - 2m) = 0$
17 $(9 + 3n)(5 - 7n) = 0$
18 $(11 - 4x)^2 = 0$
19 $3y(2y + 9)(5 - y) = 0$
20 $5a^2(15 + 4a)^2 = 0$

A **quadratic equation** is one in which 2 is the highest power of the letter (or letters) in the equation. For example, $m^2 - 5m - 14 = 0$ is a quadratic equation. In this example, the LHS of the equation can be factorised:

$(m + 2)(m - 7) = 0.$

The equation is now like many of those in Exercises 18c and 18d.

Example 13

Solve the equation $4y^2 + 5y - 21 = 0$.

$4y^2 + 5y - 21 = 0$
$\Leftrightarrow$ $(y + 3)(4y - 7) = 0$
$\Leftrightarrow$ either $y + 3 = 0$ or $4y - 7 = 0$
$\Leftrightarrow$ $y = -3$ or $4y = 7$
$\Leftrightarrow$ $y = -3$ or $y = \frac{7}{4}$
i.e. $y = -3$ or $1\frac{3}{4}$

Check: By substitution:
If $y = -3$, $4y^2 + 5y - 21$
$= 36 - 15 - 21 = 0$
If $y = 1\frac{3}{4}$, $4y^2 + 5y - 21$
$= 4 \times \frac{7}{4} \times \frac{7}{4} + 5 \times \frac{7}{4} - 21$
$= \frac{49}{4} + \frac{35}{4} - 21 = 0$

Example 14

Solve the equation $a^2 - 3a = 0$.

$a^2 - 3a = 0$
$\Leftrightarrow$ $a(a - 3) = 0$
$\Leftrightarrow$ either $a = 0$ or $a - 3 = 0$
i.e. $a = 0$ or 3

Example 15
Solve the equation $m^2 = 16$.

Rearrange the equation.

If $m^2 = 16$

then $m^2 - 16 = 0$

Factorising (difference of two squares):

$(m-4)(m+4) = 0$

either $m - 4 = 0$ or $m + 4 = 0$

i.e. $m = +4$ or $m = -4$

i.e. $m = \pm 4$

Example 16
Solve the equation $2x^2 = 3x + 5$.

Rearrange the equation to give a quadratic expression on the LHS and 0 (zero) on the RHS.

$2x^2 = 3x + 5$

$\Leftrightarrow \quad 2x^2 - 3x - 5 = 0$

$\Leftrightarrow \quad (2x-5)(x+1) = 0$

either $2x - 5 = 0$ or $x + 1 = 0$

$\Leftrightarrow \quad 2x = 5$ or $x = -1$

$\Leftrightarrow \quad x = \frac{5}{2}$ or $x = -1$

i.e. $x = 2\frac{1}{2}$ or -1

Notice, in Examples 15 and 16, that where necessary the given equation should be rearranged to give a quadratic expression on the left-hand side and zero on the other side.

Exercise 18e
Solve the following quadratic equations.

1 $a^2 - 3a + 2 = 0$
2 $b^2 + 5b + 6 = 0$
3 $c^2 - c - 2 = 0$
4 $d^2 + 2d - 3 = 0$
5 $e^2 - 7e + 10 = 0$
6 $m^2 - 4m = 0$
7 $n^2 + 5n = 0$
8 $p^2 + 7p + 12 = 0$
9 $q^2 + 2q - 8 = 0$
10 $x^2 - 2x + 1 = 0$
11 $y^2 - 5y + 4 = 0$
12 $a^2 - 9a = 0$
13 $b^2 - 9 = 0$
14 $c^2 = 25$
15 $u^2 - 8u - 9 = 0$
16 $v^2 + 2v - 35 = 0$
17 $x^2 - 6x + 9 = 0$
18 $y^2 + 8y + 16 = 0$
19 $z^2 - 4z = 0$
20 $z^2 - 4 = 0$
21 $h^2 - 15h + 54 = 0$
22 $k^2 - 15k - 54 = 0$
23 $2m^2 - 5m = 0$
24 $2m^2 - 5m + 3 = 0$
25 $2m^2 - 5m - 3 = 0$
26 $3n^2 + n = 0$
27 $a^2 + a = 90$
28 $b^2 - b = 72$
29 $3x^2 + 4x + 1 = 0$
30 $9h^2 = 6h - 1$
31 $16k^2 + 8k + 1 = 0$
32 $2c^2 + 5c + 3 = 0$
33 $3d^2 - 5d - 2 = 0$
34 $4e^2 - 20e + 25 = 0$
35 $9f^2 + 12f + 4 = 0$
36 $4a^2 - 11a = 3$
37 $b^2 + 7b = 44$
38 $7m^2 = 3m$
39 $5n^2 + 2n = 0$
40 $2p^2 - 11p + 5 = 0$
41 $5q^2 + 11q + 2 = 0$
42 $25z^2 = 9$
43 $6y^2 = y + 1$
44 $6h^2 + 13h - 5 = 0$
45 $16t^2 = 49$
46 $4r^2 - 49 = 0$
47 $8s^2 + 14s = 15$
48 $6x^2 = 7x + 20$
49 $12y^2 + y - 35 = 0$
50 $63z = 49 + 18z^2$

Equations with non-rational roots

If $m^2 = 16$, then $m = \pm 4$. (See Example 15.)

Hence, if $(x-3)^2 = 16$

then $x - 3 = \pm 4$

and $x = 3 \pm 4$

$= 7$ or -1 $\quad (3 + 4 = 7$, $3 - 4 = -1)$

Example 17
Solve the equation $(x+3)^2 = 7$.

If $(x+3)^2 = 7$

then $x + 3 = \pm\sqrt{7}$

$x = -3 \pm \sqrt{7}$

$\sqrt{7}$ is a non-rational number, the roots equation $(x+3)^2 = 7$ are non-rational. roots may be found approximately by ing $\pm\sqrt{7}$ as $\pm 2{,}65$.

ercise 18f
the following equations. If an equation non-rational roots, leave the answer in the given in Example 17.

1 $(x-2)^2 = 9$	2 $(x-7)^2 = 4$
3 $(x+3)^2 = 4$	4 $(x+2)^2 = 25$
5 $(x-1)^2 = 2$	6 $(x+4)^2 = 3$
7 $(x-3)^2 = 5$	8 $(x+2)^2 = 2$
9 $(x-2)^2 = \frac{1}{4}$	10 $(x-6)^2 = 36$
11 $(x-4)^2 = 10$	12 $(x+5)^2 = \frac{1}{9}$
13 $(x+3)^2 = 49$	14 $(x-1)^2 = 7$
15 $(x-8)^2 = 3$	16 $(x-1)^2 = \frac{9}{25}$
17 $(x+1)^2 = 2\frac{1}{4}$	18 $(x+7)^2 = 6$
19 $(x+\frac{1}{3})^2 = \frac{4}{9}$	20 $(x+9)^2 = 3$
21 $(x-6)^2 = 5$	22 $(x-2\frac{1}{2})^2 = 6\frac{1}{4}$
23 $(x+10)^2 = 8$	24 $(x-6)^2 = 2\frac{7}{9}$

Completing the square

Example 18
What must be added to $x^2 + 6x$ *to make the expression a perfect square?*

Suppose $x^2 + 6x + k$ is a perfect square and that it is equal to $(x+a)^2$.

i.e. Let $x^2 + 6x + k = (x+a)^2$

then $x^2 + 6x + k = x^2 + 2ax + a^2$

By comparing coefficients of x,

$2a = 6$

$a = 3$

By comparing the constant terms,

$k = a^2$

$k = 3^2 = 9$

Therefore 9 must be added to the expression.

Check: $x^2 + 6x + 9 = (x+3)^2$

In practice, the quantity to be added is **the square of half of the coefficient of** x (or whatever letter is involved). In Example 18, the coefficient of x is 6, half of 6 is 3 and the square of 3 is 9. Hence 9 must be added.

Example 19
What must be added to $d^2 - 5d$ *to make it into a perfect square? Factorise the result.*

The coefficient of d is -5. Half of -5 is $-\frac{5}{2}$.

$(-\frac{5}{2})^2 = +\frac{25}{4}$

$\frac{25}{4}$ must be added.

$d^2 - 5d + \frac{25}{4}$ is a perfect square.

$d^2 - 5d + \frac{25}{4} = (d - \frac{5}{2})^2$

$= (d - 2\frac{1}{2})^2$

Example 20
Add a term to $n^2 + 1\frac{1}{2}n$ *to make the expression a perfect square. Express the result as the square of a bracketed expression.*

The coefficient of n is $+1\frac{1}{2}$.

Half of $+1\frac{1}{2} = \frac{1}{2} \times \frac{3}{2} = \frac{3}{4}$. $(\frac{3}{4})^2 = \frac{9}{16}$.

$\frac{9}{16}$ must be added.

$n^2 + 1\frac{1}{2}n + \frac{9}{16} = (n + \frac{3}{4})^2$

Examples 19 and 20 may be checked by squaring out the bracket in the final result.

Exercise 18g
In the following, add a term making the given expression into a perfect square. Write the result as the square of a bracketed expression.

1 $a^2 + 8a$	2 $b^2 + 10b$	3 $c^2 - 4c$
4 $d^2 - 6d$	5 $x^2 + 5x$	6 $y^2 - 3y$
7 $z^2 - 7z$	8 $m^2 + 2m$	9 $n^2 - n$
10 $u^2 - \frac{1}{2}u$	11 $v^2 + \frac{1}{4}v$	12 $h^2 + \frac{2}{3}h$
13 $k^2 - 1\frac{1}{3}k$	14 $g^2 - 4g^2$	15 $a^2 + \frac{3}{8}a$
16 $b^2 - \frac{4}{5}b$	17 $c^2 - 1\frac{1}{2}c$	18 $m^2 - 8m$
19 $m^2 - 8mn$	20 $a^2 - 6ad$	21 $x^2 + 10xy$
22 $m^2 + 3mn$	23 $u^2 - 1\frac{3}{5}u$	24 $v^2 - \frac{3}{4}v$

Example 21
Solve the equation $x^2 - 8x + 3 = 0$.

The LHS does not factorise, so the equation is rearranged making the LHS a perfect square.

$x^2 - 8x + 3 = 0$.

Subtract 3 from both sides.

$x^2 - 8x = -3$.

Add 16 to both sides.

$x^2 - 8x + 16 = -3 + 16$

$\Leftrightarrow \quad (x-4)^2 = 13$

$\Leftrightarrow \quad x - 4 = \pm\sqrt{13}$

$\Leftrightarrow \quad x = 4 \pm \sqrt{13}$

Example 22
Solve the equation $a^2 + 3a - 2 = 0$.

The LHS does not factorise.
$a^2 + 3a - 2 = 0$
$\Leftrightarrow \quad a^2 + 3a = 2.$
Add to both sides the square of $\frac{3}{2}$.
$a^2 + 3a + (\frac{3}{2})^2 = 2 + \frac{9}{4}$
$= \dfrac{8 + 9}{4}$
$\Leftrightarrow \quad (a + \frac{3}{2})^2 = \frac{17}{4}$
$\Leftrightarrow \quad a + \frac{3}{2} = \pm \sqrt{\frac{17}{4}}$
$= \pm \dfrac{\sqrt{17}}{2}$
$\Leftrightarrow \quad a = -\frac{3}{2} \pm \dfrac{\sqrt{17}}{2}$
$= \dfrac{-3 \pm \sqrt{17}}{2}$

If the LHS of the equation factorises, use the method of factorisation rather than completing the square. For example,
$x^2 - 7x + 10 = 0$
$\Leftrightarrow \quad (x - 5)(x - 2) = 0$
$\Leftrightarrow \quad x = 5 \text{ or } 2$
rather than
$x^2 - 7x + 10 = 0$
$\Leftrightarrow \quad x^2 - 7x = -10$
$\Leftrightarrow \quad x^2 - 7x + (\frac{7}{2})^2 = -10 + \frac{49}{4}$
$= \dfrac{-40 + 49}{4}$
$\Leftrightarrow \quad (x - \frac{7}{2})^2 = \frac{9}{4}$
$\Leftrightarrow \quad x - \frac{7}{2} = \pm \frac{3}{2}$
$\Leftrightarrow \quad x = \frac{7}{2} \pm \frac{3}{2}$
$= \frac{10}{2} \text{ or } \frac{4}{2}$
$= 5 \text{ or } 2$

Exercise 18h
Solve the following equations. Factorise where possible. Otherwise, solve by completing the square, leaving the answers in the form of Examples 21 and 22.

1 $a^2 + 4a - 21 = 0$
2 $b^2 - b - 12 = 0$
3 $c^2 - 4c - 2 = 0$
4 $d^2 + 2d - 2 = 0$
5 $n^2 + 4n + 4 = 0$
6 $p^2 - 10p + 15 = 0$
7 $q^2 + 10q + 22 = 0$
8 $t^2 - 6t + 9 = 0$
9 $m^2 + 6m + 7 = 0$
10 $y^2 - 3y + 1 = 0$
11 $z^2 - 5z + 6 = 0$
12 $h^2 + 5h + 4 = 0$
13 $k^2 - 5k + 2 = 0$
14 $g^2 + 5g + 2 = 0$
15 $x^2 - 8x - 1 = 0$
16 $a^2 - a - 1 = 0$
17 $b^2 + b - 3 = 0$
18 $y^2 + 7y - 30 = 0$
19 $m^2 - 7m + 11 = 0$
20 $x^2 + 3x - 2 = 0$
21 $x^2 - 10x + 25 = 0$
22 $v^2 + 9v + 19 = 0$
23 $n^2 - 12n + 1 = 0$
24 $u^2 - 14u - 3 = 0$

Graphical solution of quadratic equations

Example 23
Solve the equation $x^2 - 2x - 3 = 0$ graphically.

$x^2 - 2x - 3$ is a function of x. Let $y = x^2 - 2x - 3$ and find the values of x when $y = 0$.
Table 18.1 is a table of values for the function $x^2 - 2x - 3$.

Table 18.1

x	-2	-1	0	1	2	3	4	5
x^2	4	1	0	1	4	9	16	25
$-2x$	4	2	0	-2	-4	-6	-8	-10
-3	-3	-3	-3	-3	-3	-3	-3	-3
$x^2 - 2x - 3$	5	0	-3	-4	-3	0	5	12

Fig. 18.1 is the graph of the function $x^2 - 2x - 3$.

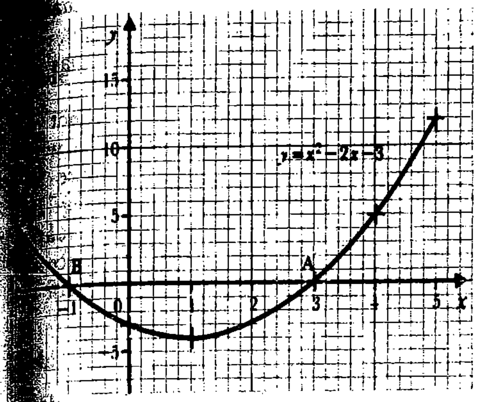


Fig. 18.1

We know that $y = x^2 - 2x - 3$, so the solutions of $x^2 - 2x - 3 = 0$ are the values of x where $y = 0$. The place at which $y = 0$ is the x-axis. In Fig 18.1, the curve cuts the x-axis at points A and B.
At A, $x = 3$
At B, $x = -1$
Hence $x^2 - 2x - 3 = 0$ when $x = 3$ or -1.
Check: By factorisation:
$x^2 - 2x - 3 = 0$
$(x - 3)(x + 1) = 0$
$x = 3 \text{ or } -1$

For the equation given in Example 23, the method of factorisation is much easier than the graphical method. However, when the expression cannot be factorised, the graphical method can be used to give approximate roots as in Example 24.

Example 24
Find the roots of the equation $3x^2 + x - 7 = 0$.

$3x^2 + x - 7$ does not factorise.
Let $y = 3x^2 + x - 7$ and construct a table of values (Table 18.2).
Fig. 18.2 is the graph of $y = 3x^2 + x - 7$.

Table 18.2

x	-3	-2	-1	0	1	2	3
$3x^2$	27	12	3	0	3	12	27
$+x$	-3	-2	-1	0	1	2	3
-7	-7	-7	-7	-7	-7	-7	-7
y	17	3	-5	-7	-3	7	23

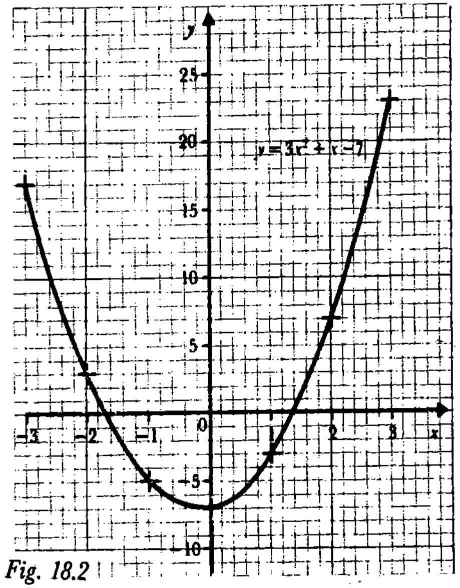


Fig. 18.2

From the graph, $y = 0$ when
$x \approx 1{,}4$
or $x \approx -1{,}7$

Hence the approximate roots of the equation are 1,4 and $-1{,}7$.

The accuracy of the results depends on the scale used to draw the graph. With the scale in Fig. 18.2, results are correct to one decimal place only. In practice a larger scale should be used, such as 2 cm to one unit on the x-axis and 2 cm to 2 units on the y-axis.

Example 25
Draw a graph to find the roots of the equation $4x^2 - 20x + 25 = 0$.

Let $y = 4x^2 - 20x + 25$ and construct a table of values (Table 18.3).

Table 18.3

x	0	1	2	3	4	5
$4x^2$	0	4	16	36	64	100
$-20x$	0	-20	-40	-60	-80	-100
$+25$	25	25	25	25	25	25
y	25	9	1	1	9	25

Fig. 18.3 is the graph of $y = 4x^4 - 20x + 25$.

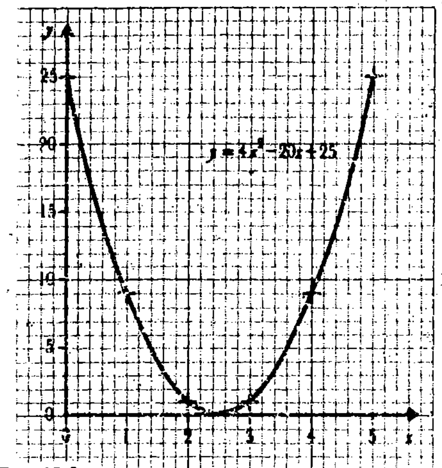


Fig. 18.3

The curve does not *cut* the x-axis. It appears to *touch* the x-axis where $x = 2{,}5$.
This result can be checked by factorisation:

$4x^2 - 20x + 25 = 0$
$(2x - 5)(2x - 5) = 0$
i.e. $(2x - 5)^2 = 0$
$\Leftrightarrow \quad x = 2\frac{1}{2}$ (twice)

When the curve touches the x-axis, the roots are said to be **coincident.**

The curve of a quadratic function is usually in one of three positions with respect to the x-axis (Fig. 18.4).

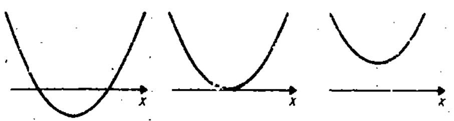


Fig. 18.4 (a) (b) (c)

In Fig. 18.4(a) the curve cuts the x-axis at two clear points. These two points give the roots of the related equation.

In Fig. 18.4(b), the two points are coincident, i.e. the points are so close together that the curve touches the x-axis at one point. This corresponds to an equation which has one repeated root.

In Fig. 18.4(c) the curve does not cut the x-axis. The roots of an equation which gives a curve in such a position are said to be **imaginary.** For example, in Fig. 16.5 on page 147, the roots of the equation $x^2 - 5x + 8 = 0$ are imaginary since the graph of $y = x^2 - 5x + 8$ does not cut the x-axis.

Exercise 18i

1 Use Fig. 16.2 on page 145 to write down the roots of the equation $x^2 + 2x - 3 = 0$. Check the result by factorisation.
2 Use Fig. 16.4 on page 146 to write down the roots of the equation $2 - 3x - 2x^2 = 0$. Use factorisation to check the result.
3 Use Fig. 16.6 on page 147 to write down the roots of the equation $3x - x^2 = 0$. Check the result by factorisation.
4 Each curve in Fig. 18.5 corresponds to a quadratic equation. Find the roots of these equations where possible.

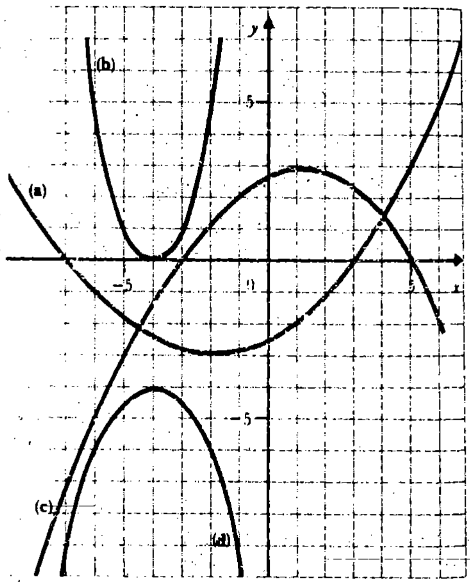


Fig. 18.5

Table 18.4 is a table of values for $y = x^2 + x - 8$ from $x = -4$ to $x = +3$.

Table 18.4

x	-4	-3	-2	-1	0	1	2	3
y	4	-2	-6	-8	-8	-6	-2	4

Use Table 18.4 to solve the equation $x^2 + x - 8 = 0$ graphically.

Table 18.5 is a table of values for $y = 3x^2 + 10x + 6 = 0$ from $x = -4$ to $x = +1$.

Table 18.5

x	-4	-3	-2	-1	0	1
y	14	3	-2	-1	6	19

Use Table 18.5 to solve the equation $3x^2 + 10x + 6 = 0$.

7 (a) Given that $y = x^2 + 3x - 2$, copy and complete Table 18.6.

Table 18.6

x	-5	-4	-3	-2	-1	0	1	2
x^2	25	16	9	4				
$+3x$	-15	-12	-9	-6				
-2	-2	-2	-2	-2				
y	8	2	-2	-4				

(b) Hence draw a graph to find the roots of the equation $x^2 + 3x - 2 = 0$.

8 (a) Given that $y = 4x^2 - 12x + 9$, copy and complete Table 18.7.

Table 18.7

x	-1	0	1	2	3	4
$4x^2$	4			16		64
$-12x$	12			-24		-48
$+9$	9			9		9
y	25			1		25

(b) Hence draw a graph and find the roots of the equation $4x^2 - 12x + 9 = 0$.

9 (a) Draw the graph of the function $x^2 + 2x - 2$ from $x = -4$ to $x = +2$.
(b) Hence find the approximate roots of the equation $x^2 + 2x - 2 = 0$.

10 (a) Draw the graph of the function $11 + 8x - 2x^2$ from $x = -2$ to $x = +6$.
(b) Hence find the approximate roots of the equation $2x^2 - 8x - 11 = 0$.

The formula for solving quadratic equations

The general form of a quadratic equation is $ax^2 + bx + c = 0$. The roots of this are found by completing the square.

$$ax^2 + bx + c = 0$$

$$\Leftrightarrow \quad x^2 + \frac{b}{a}x + \frac{c}{a} = 0$$

$$\Leftrightarrow \quad x^2 + \frac{b}{a}x = -\frac{c}{a}$$

$$\Leftrightarrow \quad x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 = -\frac{c}{a} + \left(\frac{b}{2a}\right)^2$$

$$= -\frac{c}{a} + \frac{b^2}{4a^2}$$

$$\Leftrightarrow \quad \left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$$

$$\Leftrightarrow \quad x + \frac{b}{2a} = \pm\sqrt{\frac{b^2 - 4ac}{4a^2}}$$

$$= \pm\frac{\sqrt{b^2 - 4ac}}{2a}$$

$$\Leftrightarrow \quad x = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}$$

$$\Leftrightarrow \quad \boldsymbol{x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}}$$

Example 26

Find, correct to 2 decimal places, the roots of the equation $3x^2 - 5x - 7 = 0$.

Comparing $3x^2 - 5x - 7 = 0$ with $ax^2 + bx + c = 0$: $a = 3$, $b = -5$, $c = -7$.

$$x = \frac{-(-5) \pm \sqrt{(-5)^2 - 4 \times 3 \times (-7)}}{2 \times 3}$$

$$= \frac{5 \pm \sqrt{25 + 84}}{6}$$

$$= \frac{5 \pm \sqrt{109}}{6}$$

$$= \frac{5 \pm 10{,}44}{6}$$

$$= \frac{15{,}44}{6} \text{ or } \frac{-5{,}44}{6}$$

$$\simeq 2{,}57 \text{ or } -0{,}91$$

Exercise 18j

Use the formula to solve the following equations. Give the roots correct to 2 decimal places where necessary. Use factorisation to check the results of the first ten.

1 $x^2 + 5x + 6 = 0$
2 $x^2 - 5x + 4 = 0$
3 $x^2 - 4x - 5 = 0$
4 $2x^2 + 5x + 3 = 0$
5 $3x^2 - 4x + 1 = 0$
6 $3x^2 - 5x - 2 = 0$
7 $5x^2 - 3x - 2 = 0$
8 $4x^2 + 7x - 2 = 0$
9 $6x^2 + 13x + 6 = 0$
10 $3x^2 - 13x - 10 = 0$
11 $x^2 + 3x + 1 = 0$
12 $x^2 - 2x - 4 = 0$
13 $2x^2 + 7x - 3 = 0$
14 $3x^2 - 5x - 3 = 0$
15 $5x^2 - 6x - 3 = 0$
16 $5x^2 + 8x - 2 = 0$
17 $3x^2 + 7x + 3 = 0$
18 $3x^2 - 12x + 10 = 0$
19 $3x^2 - 8x + 2 = 0$
20 $5x^2 + 3x - 3 = 0$

Word problems leading to quadratic equations

Example 27

Find two numbers whose difference is 5 and whose product is 266.

Let the smaller number be x.
Then the larger number is $x + 5$.
Their product is $x(x + 5)$.

Hence $x(x + 5) = 266$
$x^2 + 5x - 266 = 0$
$(x - 14)(x + 19) = 0$
$x = 14$ or -19

The other number is $14 + 5$ or $-19 + 5$ i.e. 19 or -14.
The two numbers are 14 and 19, or -19 and -14.
Check: $14 \times 19 = 266$ and $-19 \times -14 = 266$

Example 28

A man is 4 times older than his son. 5 years ago the product of their ages was 175. Find their present ages.

Let the son's age be x years.
Then the father's age is $4x$ years.
5 years ago, the son's age was $(x - 5)$ years and the father's age was $(4x - 5)$ years.
The product of their ages was $(x - 5)(4x - 5)$.

Hence $(x - 5)(4x - 5) = 175$
$4x^2 - 25x + 25 = 175$
$4x^2 - 25x - 150 = 0$
$(4x + 15)(x - 10) = 0$
$x = 10$ or $-\frac{15}{4}$

However, $-\frac{15}{4}$ is not sensible for an age (ages cannot be negative). Therefore the son is 10 years old and the father is 40 years old.
Check: $(40 - 5)(10 - 5) = 35 \times 5 = 175$

Notice in Example 28 that one of the roots does not give a sensible answer. Disregard any root which is not sensible in terms of the given question.

Exercise 18k

In each problem, form a quadratic equation and then solve it. *Note:* in this exercise all the quadratic equations can be solved by factorisation.

1 Find two numbers which differ by 4, and whose product is 45.

The width of a classroom is 4 metres less than the length. Its area is 45 m². Find the dimensions of the classroom.

Two numbers have a difference of 3. The sum of their squares is 89. Find the numbers.

Two square rooms have a total floor area of 89 m². One room is 3 m longer each way than the other. Find the dimensions of the two rooms.

A girl is 6 years younger than her oldest brother. The product of their ages is 135. Find their ages.

The ages of two sisters are 11 and 8 years. In how many years' time will the product of their ages be 208?

Find the number which when added to its square makes 90.

Twice the square of a certain whole number added to 3 times the number makes 90. Find the number.

The area of a rectangle is 60 cm². The length is 11 cm more than the width. Find the width.

A rectangular plot measures 12 m by 5 m. A path of constant width runs along one side and one end. If the total area of the plot and the path is 120 m², find the width of the path. (*Hint:* Make a sketch and let the width of the path be x m.)

11 A rectangular piece of cardboard measures 17 cm by 14 cm. Strips of equal width are cut off one side and one end. The area of the remaining piece is 108 cm². Find the width of the strips removed.

12 A man is 37 years old and his son's age is 8. How many years ago was the product of their ages 96?

13 A certain number is subtracted from 18 and from 13. The product of the two numbers obtained is 66. Find the first number.

14 Find two consecutive numbers whose product is 156.

15 Find two consecutive even numbers whose product is 224.

16 Find two consecutive odd numbers whose product is 195.

17 The square of a certain number is 22 less than 13 times the original number. Find the number.

18 A woman is 3 times as old as her daughter. 8 years ago the product of their ages was 112. Find their present ages.

19 Twice a certain whole number subtracted from 3 times the square of the number leaves 133. Find the number.

20 The base of a triangle is 3 cm longer than its corresponding height. If the area is 44 cm², find the length of its base.

Chapter 19

Geometrical transformations (2)

Congruencies (Revision)

A **transformation** is the name given to a change in position or dimensions (or both) of a shape. The **image** of a shape is the figure which results after a transformation. If the image has the same dimensions as the original shape, the transformation is called a **congruency** (see Chapter 3). There are three basic congruencies: **translations, reflections** and **rotations.**

Example 1

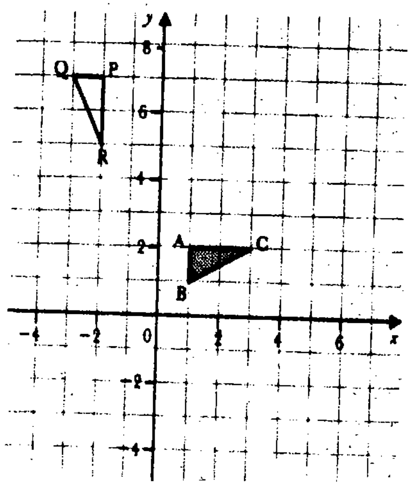


Fig. 19.1

Use Fig. 19.1 to answer the following.
(a) △ ABC is translated so that the image of C is the point (7; − 3). *Find the images of A and B.*
(b) Construct the reflection of △ ABC in the line y = x.
(c) Describe completely the single transformation that maps △ ABC onto △ PQR.

(a) Translation is movement in a straight li without turning. If C is translated to the p (7; − 3), then points A and B must be translat by the same amount in the same direction shown in Fig. 19.2.

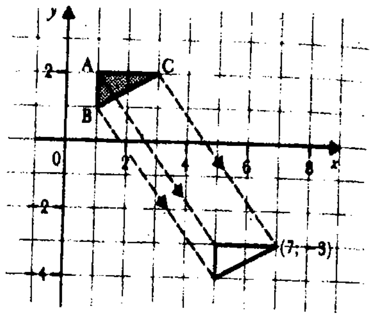


Fig. 19.2

From Fig. 19.2, the images of A and B (5; − 3) and (5; − 4) respectively.
(b) In Fig. 19.3, *m* is the line $y = -x$.

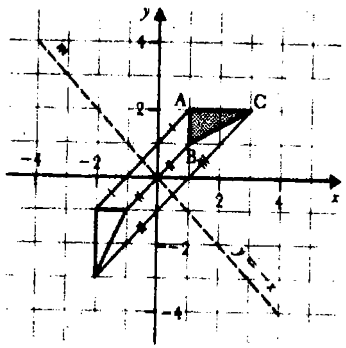
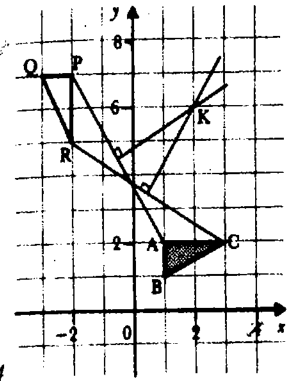


Fig. 19.3

The image of A is found by drawing a perp dicular from A to *m* and producing it by equal length on the other side of *m*. Simila for B and C.

...paring the given triangles ABC and ...PQR appears to have been rotated ...pect to △ABC. This suggests that the ...mation is a rotation. Fig. 19.4 shows ...find the centre of rotation and hence its



...19.4

...a pair of corresponding points such as A ...P. Construct the perpendicular bisector of ...Repeat for another pair of corresponding ...nts such as C and R. The perpendicular ...ctors meet at K(2; 6) which is the centre of ...ation.
...e single transformation is a clockwise ...ation of 90° about the point (2; 6).

...e letters T, M, R are often used to represent ...nslations, reflections and rotations ...spectively. For example:
T(A) is the image of point A after translation T, M(5; − 2) is the image of point (5; − 2) after a reflection M, R(ABC) is the image of △ABC after rotation R.

Example 2

Q is a quadrilateral with vertices at (1; 1), (2; − 1), (4; 0), (3; 1).
(a) M is a reflection in the line y = 2. Find the coordinates of the vertices of quadrilateral M(Q).
(b) S is a transformation which maps Q onto quadrilateral S(Q) *with vertices at* (− 5; − 3), − 7; − 3), (− 8; − 2), (− 6; − 1). *Draw* S(Q) *and describe* S *fully.*

Fig. 19.5 shows how parts (a) and (b) may be answered by drawing a suitable diagram.

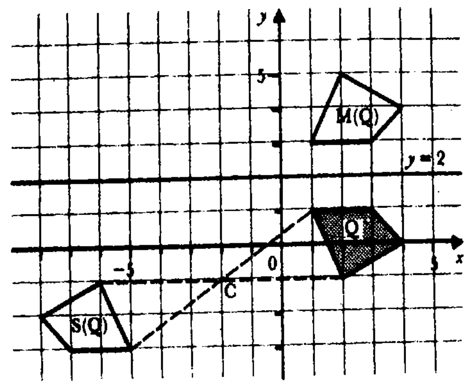


Fig. 19.5

(a) In Fig. 19.5, M(Q) is the reflection of Q in line $y = 2$. M(Q) has coordinates (1; 3), (2; 5), (4; 4), (3; 3).
(b) S(Q) is fully inverted (upside down) with respect to Q. This suggests a rotation of 180°. Corresponding points are joined in order to locate the centre of rotation. S is a rotation of 180° about the point (− 2; − 1).
Notice that if Q is transformed to S(Q) by S, Q is said to be **mapped** onto S(Q) by S.

Exercise 19a

Use graph paper to answer the questions in this exercise.

1

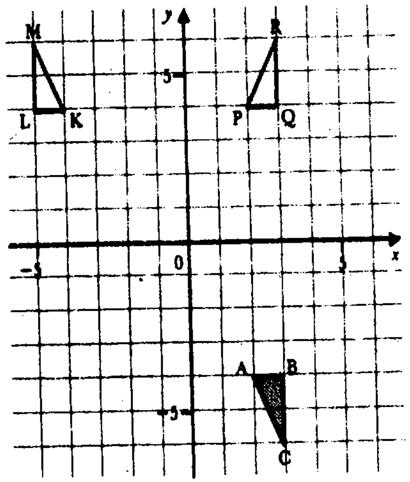


Fig. 19.6

In Fig. 19.6, describe completely the single transformation which maps
(a) $\triangle ABC$ onto $\triangle PQR$,
(b) $\triangle ABC$ onto $\triangle KLM$,
(c) $\triangle KLM$ onto $\triangle PQR$.

2 Make a copy of Fig. 19.7 and draw the image of ABCD under
(a) a clockwise rotation of 90° about the origin,
(b) a reflection in the line $y = -x$.

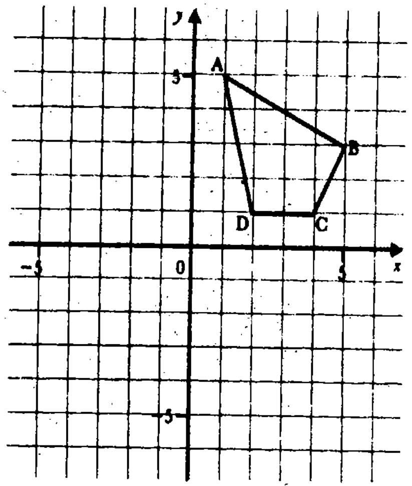


Fig. 19.7

3 (a) Using a scale of 1 cm to represent 1 unit on each axis, draw x and y axes for $-4 \leqslant x \leqslant 10$ and $-4 \leqslant y \leqslant 12$. Draw a triangle with vertices $(-1; 1)$, $(-1; 10)$, $(-4; 7)$ and label it F.
(b) A transformation R maps triangle F onto the triangle R(F) which has vertices $(0; -2)$, $(9; -2)$ and $(6; 1)$. Draw the triangle R(F) and fully describe the transformation R.
(c) M is a reflection in the line $y = x$. Find, by drawing, the coordinates of the vertices of the triangle M(F).

4 Quadrilaterals ABCD, $A_1B_1C_1D_1$, $A_2B_2C_2D_2$ and line m are as shown in Fig. 19.8.
(a) $A_1B_1C_1D_1$ is the image of ABCD under a reflection. Write down the equation of the mirror line.

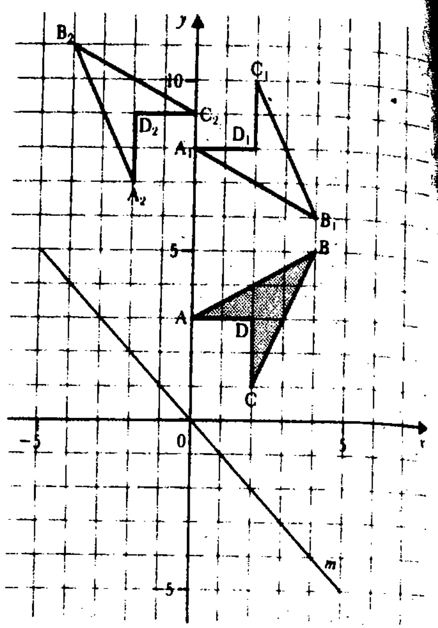


Fig. 19.8

(b) $A_2B_2C_2D_2$ is the image of ABCD under an anticlockwise rotation. Write down (i) the angle of rotation, (ii) the coordinates of the centre of rotation.
(c) $A_3B_3C_3D_3$ (not shown in Fig. 19.8) is the image of ABCD under a translation. If B_3 has coordinates $(8; 1)$, what are the coordinates of C_3?
(d) $A_4B_4C_4D_4$ (not shown in Fig. 19.8) is the image of ABCD under reflection in line m. Write down (i) the equation of m, (ii) the coordinates of D_4.

5 M is a reflection in the line $x = 2$ and R is a clockwise rotation of 90° about the origin. A is the point $(1; 2)$, B is $(4; 6)$ and C is $(7; 1)$.
(a) Find the coordinates of (i) M(A), (ii) R(B).
(b) Find the coordinates of the point D if $M(D) = C$.

Enlargement

Enlargement is a transformation in which a shape is magnified (made larger) or diminished

(smaller). Fig. 19.9 shows some enlargements of a quadrilateral ABCD.

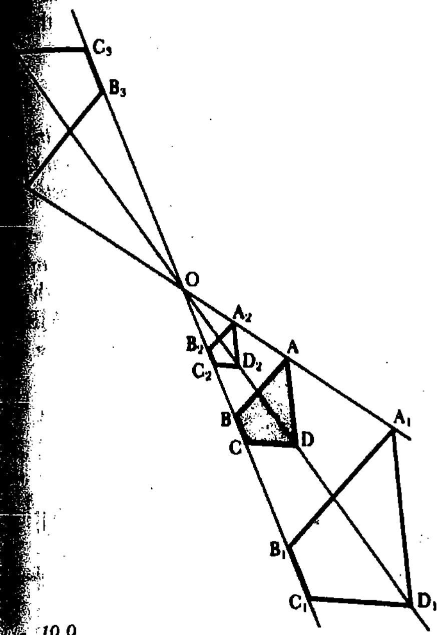


Fig. 19.9

Note the following:

1 ABCD, $A_1B_1C_1D_1$, $A_2B_2C_2D_2$, $A_3B_3C_3D_3$ are **similar** to each other, i.e. corresponding angles are equal and corresponding sides are in the same ratio.

2 O is the **centre of enlargement**.

3 Comparing ABCD with $A_1B_1C_1D_1$,

$$\frac{OA_1}{OA} = \frac{OB_1}{OB} = \frac{OC_1}{OC} = \frac{OD_1}{OD} = 2$$

Any one of the above ratios gives the **scale factor** of the enlargement. In this case the scale factor is 2 and the enlargement is bigger than the original shape.

4 $$\frac{A_1B_1}{AB} = \frac{B_1C_1}{BC} = \frac{C_1D_1}{CD} = \frac{D_1A_1}{DA} = 2$$

The ratio of the lengths of corresponding sides is also equal to the scale factor.

5 Comparing $A_2B_2C_2D_2$ with ABCD, the scale factor is $\frac{1}{2}$ and the enlargement is smaller than the original shape.

6 Comparing $A_3B_3C_3D_3$ with ABCD, the scale factor is $-1\frac{1}{2}$. A negative scale factor indicates that the image is on the other side of the centre of enlargement from the original shape.

7 In an enlargement of the plane, every point is transformed except the centre of enlargement. The centre of enlargement remains **invariant** or untransformed.

Example 3

Triangle $P(1; 6)$, $Q(5; 4)$, $R(3; 2)$ *is mapped onto triangle* $P'(-2; -6)$, $Q'(-4; -5)$, $R'(-3; -4)$ *by an enlargement E. By drawing the triangles on graph paper find (a) the coordinates of the centre of enlargement, (b) the scale factor of E.*

Fig. 19.10 shows $\triangle$s PQR and P'Q'R'.

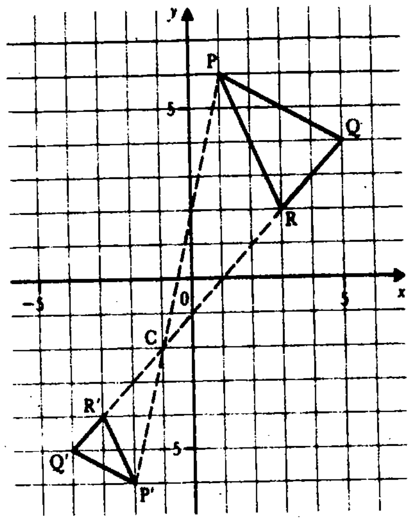


Fig. 19.10

(a) The lines PP' and QQ' intersect at C, the centre of enlargement.
C has coordinates $(-1; -2)$.

(b) Scale factor $= \dfrac{CR'}{CR} = \dfrac{-2}{4} = -\dfrac{1}{2}$

Notice that CR' is negative with respect to the direction of CR.

Shear

Fig. 19.11 shows a way of transforming the shape of a book by pushing its top surface.

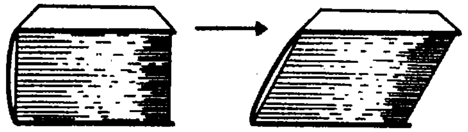


Fig. 19.11

This kind of transformation is called a **shear**. Fig. 19.12 shows two shears of a unit square OABC.

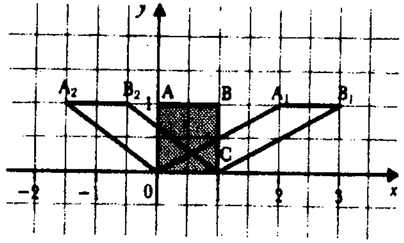


Fig. 19.12

Note the following:

1 One line is fixed, or invariant. All other points move parallel to the invariant line. In Fig. 19.12 the x-axis is invariant but any other line could have been chosen.

2 The points on both sides of the invariant line move by an amount proportional to their distances from the line.

3 Comparing OA_1B_1C with OABC, **shear factor** =

$$\frac{\text{distance moved by any point}}{\text{distance of that point from the invariant line}}$$

$$= \frac{AA_1}{AO} \text{ or } \frac{BB_1}{BC} = 2$$

In this case the shear factor is 2.

4 Comparing $OA_2B_2C_2$ with OABC,

$$\text{shear factor} = \frac{AA_2}{AO} \text{ or } \frac{BB_2}{BC} = -1\tfrac{1}{2}$$

The shear factor is negative since the shear is in the negative direction of the x-axis.

5 OABC, OA_1B_1C and OA_2B_2C have equal areas since they are all parallelograms of equal height on the same base. Area is an invariant property of any shape that is sheared.

Example 4

Find the coordinates of the image of triangle A(− 1; 1), B(3; 2), C(1; − 2) *after a shear (a) of factor* 1 *with the x-axis invariant, (b) of factor* − 2 *with the y-axis invariant.*

Fig. 19.13 shows △ABC and (a) its image $\triangle A_1B_1C_1$ after a shear of factor 1 with the x-axis invariant, and (b) its image $\triangle A_2B_2C_2$ with the y-axis invariant. In each case the arrows show the movement of the shear.

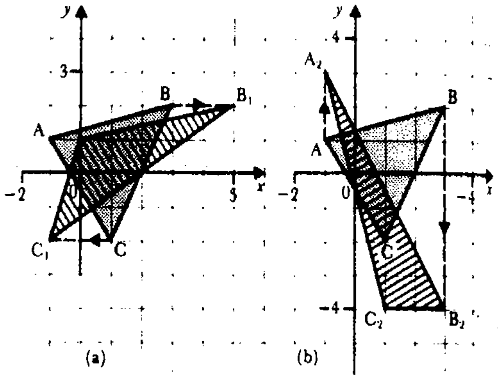
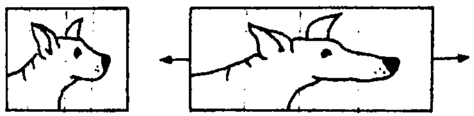


Fig. 19.13

(a) The image of △ABC is $A_1(0; 1)$, $B_1(5; 3)$, $C_1(-1; -2)$.
(b) The image of △ABC is $A_2(-1; 3)$, $B_2(3; -4)$, $C_2(1; -4)$.

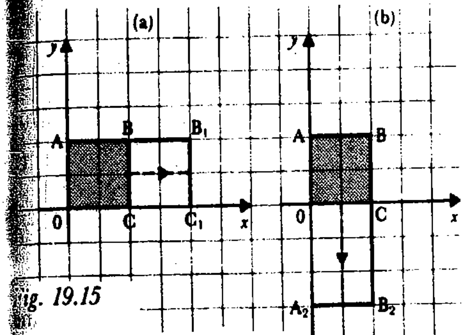
Stretch

In Fig. 19.14(a) an animal has been drawn on a rubber sheet, such as a piece of a car inner tube.



g. 19.14

rts (b) and (c) of Fig. 19.14 show what ppens if the rubber sheet is stretched one y or another. This kind of transformation is lled a **stretch**. Fig. 19.15 shows two stretches a unit square OABC.



ig. 19.15

ote the following:

1 In each case one line is fixed and the unit square is stretched in a direction perpendicular to that line.

2 In Fig. 19.15(a),
Stretch factor =

$$\frac{\text{distance of image of point from fixed line}}{\text{distance of original point from the fixed line}}$$

$$= \frac{AB_1}{AB} \text{ or } \frac{OC_1}{OC} = 2$$

This transformation is a **one-way stretch** of factor 2 in the direction of the x-axis with the y-axis invariant.

3 In Fig. 19.15(b), the transformation is a one-way stretch of factor $-1\tfrac{1}{2}$ in the direction of the y-axis with the x-axis invariant.

It is possible to combine two one-way stretches to give a **two-way stretch.**

Fig. 19.16 shows the effect on a unit square of a two-way stretch of factors 2 and $-1\tfrac{1}{2}$ in the x and y directions respectively.

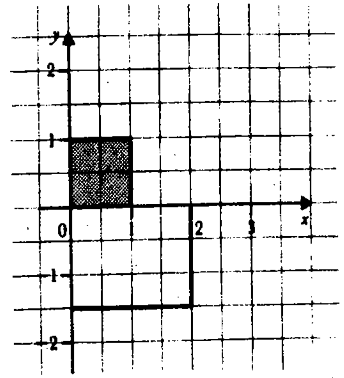


Fig. 19.16

Example 5

R is a rhombus with vertices at (1; 4), (2; 1), (3; 4) *and* (2; 7). *S is a stretch of factor* 3 *in the x-direction with the y-axis invariant. (a) Find the coordinates of* S(R). *(b) What kind of quadrilateral is* S(R)*?*

Fig. 19.17 shows R and S(R).

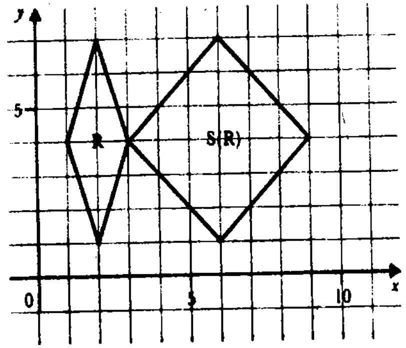


Fig. 19.17

(a) A stretch factor of 3 in the x-direction has the effect of multiplying each x-coordinate of R by 3. S(R) has coordinates (3; 4), (6; 1), (9; 4) and (6; 7).
(b) S(R) is a square.

Exercise 19b

Questions 1–4 are suitable for class discussion. Unless told otherwise, use graph paper to answer the remaining questions.

1 Fig. 19.18 contains some enlargements of a triangle T and a rectangle R. In every case the origin is the centre of enlargement.

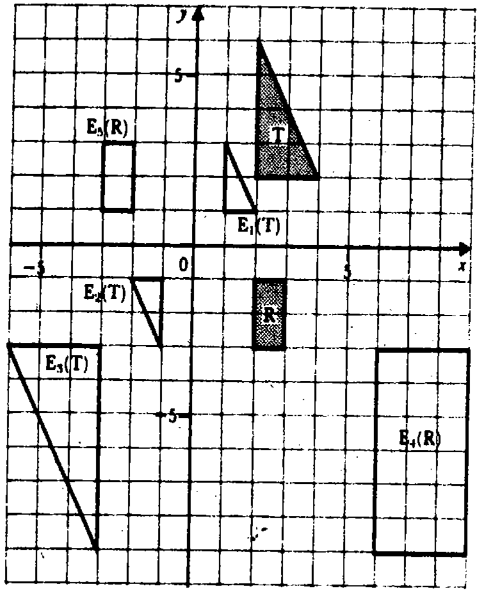


Fig. 19.18

State the scale factors of E_1, E_2, E_3, E_4 and E_5.

2 Each part of Fig. 19.19 contains a unit square and its image under a shear. Each image has been shaded.

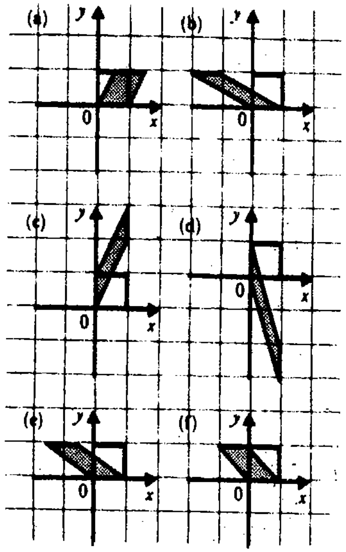


Fig. 19.19

Describe each shear fully, giving the she[ar] factor, the direction and the equation [of] the invariant line.

3 Each part of Fig. 19.20 contains a uni[t] square and its image after a one-way stretch. Each image has been shaded.

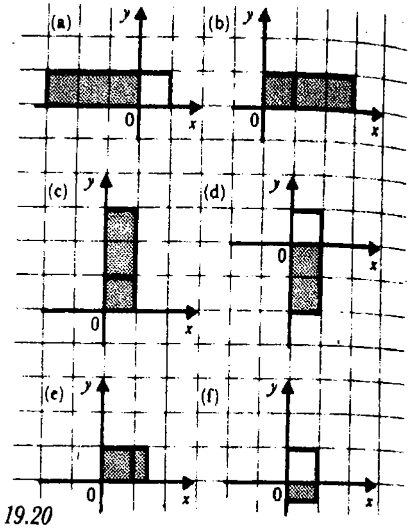


Fig. 19.20

Describe each stretch fully, giving the stretch factor, its direction and the equation of the invariant line.

4 Each part of Fig. 19.21 contains a unit square and its image after a two-way stretch. Each image has been shaded. Describe each transformation fully.

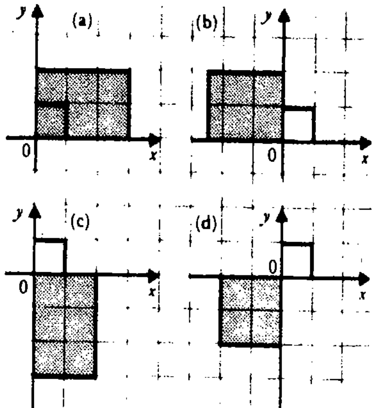


Fig. 19.21

quadrilateral has vertices P(3; 4), Q(1; 5), (− 1; 4), S(1; 3). It is enlarged to a quadrilateral with vertices (0; 1), (4; − 1), (8; 1), (4; 3).

(a) Choose a suitable scale and draw PQRS and its enlargement.

(b) What kind of quadrilateral is PQRS?

(c) If (0; 1) is the image of P, state the scale factor and the coordinates of the centre of enlargement.

(d) On the other hand, if (8; 1) is the image of P, state the scale factor and the position of the centre of enlargement.

E is an enlargement of factor $1\frac{1}{2}$ with the origin as centre. H is a shear of factor $1\frac{1}{2}$ in the x-direction with the x-axis invariant. A is the point (− 2; 6) and B is (4; 2). Find the coordinates of

(a) E(A), (b) E(B),

(c) H(A), (d) H(B),

and hence state the length of H(AB).

For each part of this question, trace the letter K and points P_1 and P_2 onto a sheet of plain paper.

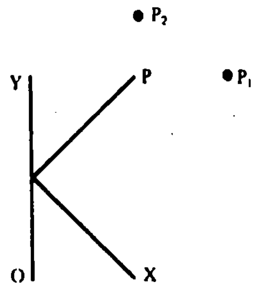


Fig. 19.22

Draw the image of K after

(a) a shear which maps P onto P_1 with OX invariant,

(b) a one-way stretch which maps P onto P_1 with OY invariant,

(c) a shear which maps P onto P_2 with OY invariant,

(d) a one-way stretch which maps P onto P_2 with OX invariant.

8 Rectangle ABCD in Fig. 19.23 is given a shear of factor $\frac{1}{2}$ in the x-direction with the x-axis invariant.

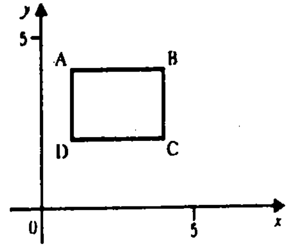


Fig. 19.23

(a) Find the coordinates of image A′B′C′D′.

(b) What kind of quadrilateral is A′B′C′D′?

9 Parallelogram ABCD in Fig. 19.24 is given a shear H of factor 2 in the x-direction with the x-axis invariant.

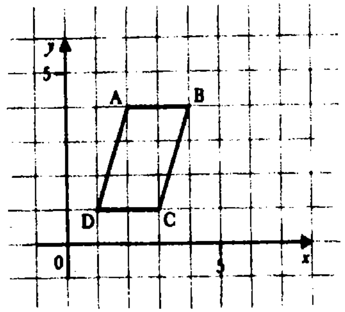


Fig. 19.24

(a) Find the coordinates of H(ABCD).

(b) What kind of shape is H(ABCD)?

10 Parallellogram ABCD of Fig. 19.24 is given a one-way stretch S of factor 2 in the x-direction with the y-axis invariant.

(a) Find the coordinates of S(ABCD).

(b) What kind of shape is S(ABCD)?

Combined transformations

Example 6

H is a shear of factor 2 in the x-direction with the line $y = 0$ invariant. M is a reflection in the line $x = -1$. T is a triangle with vertices at (0; 2), (1; 0) and (2; 1). Find the vertices of the final image of T if it is first sheared by H and then transformed by M.

The line $y = 0$ is the x-axis. Fig. 19.25 shows T, its image H(T) after shearing by H and its final image MH(T), where MH(T) is the reflection of H(T) in the line $x = -1$.

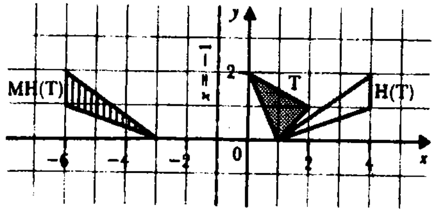


Fig. 19.25

From Fig. 19.25, the final image of T has its vertices at $(-6; 2)$, $(-3; 0)$ and $(-6; 1)$.

In Example 6 notice that MH(T) means that transformation H is done *before* transformation M.

$$MH(T) = M[H(T)]$$

The order in which the transformations are done is usually important. In general MH $\neq$ HM. For example, with the data of Example 6, Fig. 19.26 shows that the outcome HM(T) is different from that of MH(T).

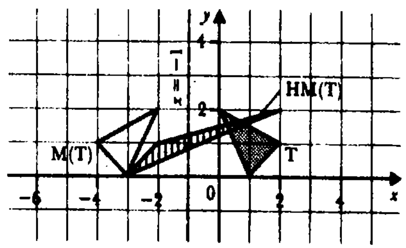


Fig. 19.26

Exercise 19c

Use graph paper to answer the questions in this exercise.

Questions 1, 2, 3, 4 all refer to Fig. 19.27.

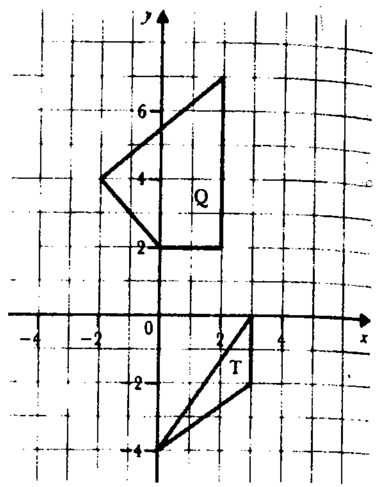


Fig. 19.27

1 Triangle T is first rotated anticlockwise through 90° about the origin, then sheared by factor 1 in the y-direction with the y-axis invariant. Find the coordinates of the vertices of T's final image.

2 U(T) represents a translation of triangle T such that the image of its obtuse-angled vertex is the point (0; 2). E is an enlargement of scale factor $1\frac{1}{2}$, centre the origin. Find the coordinates of the vertices of the triangle given by EU(T).

3 Quadrilateral Q is rotated through 180° about the point (0; 2). The result is then enlarged by scale factor -2 with the origin as centre. Find the coordinates of the vertices of the final image of Q.

4 S is a one-way stretch of factor $1\frac{1}{2}$ in the x-direction with $x = 0$ invariant. M is a reflection in the line $x = 1$. Find the coordinates of the vertices of (a) MS(Q), (b) SM(Q).

5 T is a translation which would transform the origin to the point (3; 1). R is a clockwise rotation of 90° about the origin. A is the point $(-4; 1)$, B is $(3; -1)$ and C is $(5; -2)$. Find the coordinates of

(a) T(A), (b) R(B),
(c) RT(A), (d) TR(B),
(e) the point D, if RT(D) = C.

M is a reflection in the line $x = y$. H is a shear of factor 2 in the x-direction with the line $y = 0$ invariant. If P is the point (3; 2), find the coordinates of the following.
(a) M(P) (b) H(P) (c) HM(P)
(d) MH(P) (e) MM(P) (f) HH(P)

$\triangle$ABC has vertices A(2; 1), B(6; 4), C(5; 6). $\triangle$PQR has vertices $P(-7; 1)$, $Q(-3; 2)$, $R(-4; -4)$. Given that $\triangle$ABC can be mapped onto $\triangle$PQR by a rotation of $\theta°$ about A followed by a reflection in a line m, find (a) θ, (b) the equation of m.

In Fig. 19.28, the triangle A can be mapped onto the triangle B by a reflection in the y-axis followed by a reflection in a second line.
(a) Find the equation of the second mirror line.
(b) Describe fully a single transformation that will map A onto B.

In Fig. 19.28, triangle C is the image of triangle A under a transformation given by ME(A) where E is an enlargement with the centre as origin and M is a reflection.
(a) State the scale factor of E.
(b) State the equation of the corresponding line of reflection.

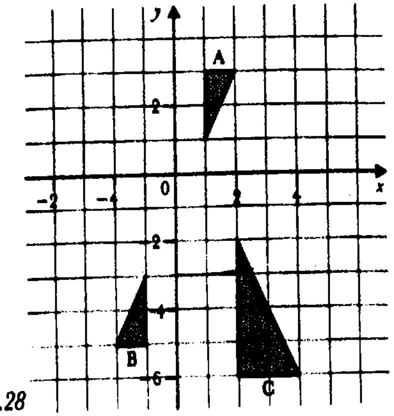


Fig. 19.28

(c) B can be mapped onto C by a reflection in the y-axis followed by an enlargement. State the centre of this enlargement.

10 The cartesian plane is first sheared by a factor -2 in the x-direction with the x-axis invariant and then reflected in the line $y = -x$. P is the point $(a; b)$ and Q is its image after the combined transformation described above.
(a) Express the coordinates of Q in terms of a and b.
(b) Hence find the coordinates of P if Q is the point $(-5; 8)$.

Chapter 20

Irrational numbers

Rational and irrational numbers

Numbers such as 8; $4\frac{1}{2}$; $\frac{1}{5}$; 0,211; $\sqrt{\frac{49}{16}}$; $0,\dot{3}$ can be expressed as exact fractions or ratios:

$$\frac{8}{1}, \frac{9}{2}, \frac{1}{5}, \frac{211}{1\,000}, \frac{7}{4}, \frac{1}{3}.$$

Such numbers are called **rational numbers**.

Numbers which cannot be written as exact fractions are called **irrational numbers**. $\sqrt{7}$ is an example of an irrational number. $\sqrt{7} = 2{,}645\,751\ \ldots$, the decimals extending without end and without recurring.

π is another example of an irrational number. $\pi = 3{,}141\,592\ \ldots$, again extending forever without repetition. The fraction $\frac{22}{7}$ is often used for the value of π. However, $\frac{22}{7}$ is a rational number and is only an approximate value of π.

All recurring decimals are rational numbers. Read the following example carefully.

Example 1

Express $3{,}\dot{1}\dot{7}$ *as a rational number.*

Let $n = 3{,}\dot{1}\dot{7}$

i.e. $n = 3{,}\ 17\ 17\ 17\ \ldots\ \ldots$ (1)

Multiply both sides by 100

$100n = 317{,}17\ 17\ 17\ \ldots\ \ldots$ (2)

Subtract (1) from (2),

$99n = (317{,}17\ 17\ \ldots) - (3{,}17\ 17\ \ldots)$

$99n = 314$

$n = \frac{314}{99}$

Thus $3{,}\dot{1}\dot{7} = \frac{314}{99}$, a rational number.

An irrational number extends forever *and* is non-recurring.

Exercise 20a

1 Which of the following are rational and which are irrational?

(a) 9 (b) $\frac{1}{8}$ (c) $\sqrt{9}$ (d) 0,9

(e) $2\frac{2}{3}$ (f) $5\frac{3}{4}$ (g) $\frac{11}{16}$ (h) 0,815

(i) $\sqrt{16}$ (j) $\sqrt{17}$ (k) $\sqrt{10}$

(l) $\sqrt{100}$ (m) $\frac{22}{7}$ (n) 3,142

(o) π (p) $\sqrt{8}$ (q) $\sqrt{49}$

(r) $\sqrt{4{,}9}$ (s) $4{,}9^2$ (t) $\frac{1}{4{,}9}$

(u) $0{,}\dot{6}$ (v) $0{,}\dot{2}$ (w) $0{,}\dot{7}$

(x) $0{,}8\dot{3}$ (y) $\sqrt{2\frac{1}{4}}$ (z) $\sqrt{5}$

2 Express the ollowing recurring decimals as rational numbers.

(a) $8{,}\dot{3}$ (b) $6{,}\dot{6}$ (c) $4{,}\dot{7}$

(d) $3{,}\dot{1}\dot{9}$ (e) $3{,}\dot{2}\dot{8}$ (f) $1{,}\dot{6}\dot{1}$

Surds

Many square roots are irrational:

$\sqrt{3} = 1{,}732\,592\ldots$ and $\sqrt{28} = 5{,}291\,502\ldots$.

Irrational numbers of this kind are called **surds**.

Exercise 20b (Discussion)

By putting $m = 9$ and $n = 4$, find which of the following pairs of expressions are equal.

1 $\sqrt{mn}$, $\sqrt{m} \times \sqrt{n}$ **2** $\sqrt{m+n}$, $\sqrt{m} + \sqrt{n}$

3 $\sqrt{\frac{m}{n}}$, $\frac{\sqrt{m}}{\sqrt{n}}$ **4** $\sqrt{m-n}$, $\sqrt{m} - \sqrt{n}$

5 $2\sqrt{m}$, $\sqrt{2m}$ **6** $3\sqrt{n}$, $\sqrt{9n}$

Exercise 20b demonstrates the fact that

$$\sqrt{mn} = \sqrt{m} \times \sqrt{n} \quad \text{and} \quad \sqrt{\frac{m}{n}} = \frac{\sqrt{m}}{\sqrt{n}}.$$

Use the above facts when simplifying surds.

[Exa]mple 2

[Simpl]ify (a) $\sqrt{45}$, *(b)* $\sqrt{162}$, *(c)* $\sqrt{x^2y}$.

[(a)] $\sqrt{45} = \sqrt{9 \times 5} = \sqrt{9} \times \sqrt{5} = 3\sqrt{5}$

[(b)] $\sqrt{162} = \sqrt{81 \times 2} = \sqrt{81} \times \sqrt{2} = 9\sqrt{2}$

[(c)] $\sqrt{x^2y} = \sqrt{x^2} \times \sqrt{y} = x\sqrt{y}$

[Exe]rcise 20c

[Sim]plify the following by making the number [und]er the square root sign as small as possible.

[1] $\sqrt{20}$	**2** $\sqrt{32}$	**3** $\sqrt{48}$
[4] $\sqrt{75}$	**5** $\sqrt{72}$	**6** $\sqrt{24}$
[7] $\sqrt{63}$	**8** $\sqrt{54}$	**9** $\sqrt{200}$
[10] $\sqrt{84}$	**11** $\sqrt{99}$	**12** $\sqrt{150}$
[13] $\sqrt{98}$	**14** $\sqrt{288}$	**15** $\sqrt{147}$

[Ex]ample 3

[Ex]press the following as the square root of a single [n]umber. (a) $2\sqrt{5}$ *(b)* $7\sqrt{3}$

(a) $2\sqrt{5} = \sqrt{4} \times \sqrt{5} = \sqrt{4 \times 5} = \sqrt{20}$

(b) $7\sqrt{3} = \sqrt{49} \times \sqrt{3} = \sqrt{49 \times 3} = \sqrt{147}$

[N]otice that Example 3 is the reverse process of Example 2.

Exercise 20d

Express each of the following as the square root of a single number.

1 $2\sqrt{3}$	**2** $3\sqrt{2}$	**3** $2\sqrt{2}$	**4** $3\sqrt{3}$
5 $5\sqrt{2}$	**6** $3\sqrt{5}$	**7** $2\sqrt{7}$	**8** $4\sqrt{6}$
9 $6\sqrt{3}$	**10** $5\sqrt{5}$	**11** $10\sqrt{3}$	**12** $3\sqrt{10}$
13 $2\sqrt{11}$	**14** $3\sqrt{8}$	**15** $5\sqrt{7}$	

Multiplication of surds

When two or more surds are to be multiplied together, they should first be simplified if possible. Then whole numbers should be taken with whole numbers, and surds with surds.

Example 4

Simplify (a) $\sqrt{27} \times \sqrt{50}$,

(b) $\sqrt{12} \times 3\sqrt{60} \times \sqrt{45}$, *(c)* $(2\sqrt{5})^2$.

(a) $\sqrt{27} \times \sqrt{50} = \sqrt{9 \times 3} \times \sqrt{25 \times 2}$

$= 3\sqrt{3} \times 5\sqrt{2} = 15\sqrt{6}$

(b) $\sqrt{12} \times 3\sqrt{60} \times \sqrt{45}$

$= \sqrt{4 \times 3} \times 3\sqrt{4 \times 15} \times \sqrt{9 \times 5}$

$= 2\sqrt{3} \times 3 \times 2\sqrt{15} \times 3\sqrt{5}$

$= 36\sqrt{3 \times 15 \times 5}$

$= 36\sqrt{15 \times 15}$

$= 36 \times 15 = 540$

(c) $(2\sqrt{5})^2 = 2\sqrt{5} \times 2\sqrt{5} = 4 \times 5 = 20$

[i.e. $(2\sqrt{5})^2 = 2^2 \times (\sqrt{5})^2 = 4 \times 5 = 20$]

It is sometimes possible to pair off surds to give a simpler result.

Example 5

Simplify the following.

(a) $\sqrt{2} \times \sqrt{3} \times \sqrt{5} \times \sqrt{12} \times \sqrt{45} \times \sqrt{50}$,

(b) $\sqrt{3} \times \sqrt{6}$.

(a) $\sqrt{2} \times \sqrt{3} \times \sqrt{5} \times \sqrt{12} \times \sqrt{45} \times \sqrt{50}$

$= \sqrt{2 \times 3 \times 5 \times 12 \times 45 \times 50}$

$= \sqrt{(2 \times 50) \times (3 \times 12) \times (5 \times 45)}$

$= \sqrt{100 \times 36 \times 225}$

$= 10 \times 6 \times 15$

$= 900$

(b) $\sqrt{3} \times \sqrt{6} = \sqrt{3 \times 6} = \sqrt{18}$

$= \sqrt{9 \times 2} = 3\sqrt{2}$

or $\sqrt{3} \times \sqrt{6} = \sqrt{3} \times \sqrt{3 \times 2}$

$= \sqrt{3} \times \sqrt{3} \times \sqrt{2} = 3\sqrt{2}$

Exercise 20e

Simplify the following.

1 $\sqrt{5} \times \sqrt{10}$	**2** $\sqrt{8} \times \sqrt{2}$
3 $\sqrt{2} \times \sqrt{6} \times \sqrt{3}$	**4** $\sqrt{30} \times \sqrt{5}$
5 $\sqrt{12} \times \sqrt{3}$	**6** $(4\sqrt{3})^2$
7 $(\sqrt{2})^3$	**8** $\sqrt{15} \times \sqrt{12}$
9 $\sqrt{32} \times \sqrt{12}$	**10** $(\sqrt{3})^5$

11 $(2\sqrt{7})^2$

12 $\sqrt{10} \times 3\sqrt{2} \times \sqrt{20}$

13 $\sqrt{5} \times \sqrt{24} \times \sqrt{30}$

14 $(2\sqrt{3})^3$

15 $\sqrt{6} \times \sqrt{8} \times \sqrt{10} \times \sqrt{12}$

Division of surds

If a fraction has a surd in the denominator, it is usually best to **rationalise the denominator.** To rationalise the denominator means to make the denominator into a rational number, usually a whole number. This is done by multiplying the numerator and denominator of the fraction by a surd which makes the denominator rational.

Example 6

Rationalise the denominators of the following.

(*a*) $\frac{6}{\sqrt{3}}$ (*b*) $\frac{7}{\sqrt{18}}$ (*c*) $\frac{5}{\sqrt{5}}$

(a) $\frac{6}{\sqrt{3}} = \frac{6}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}} = \frac{6 \times \sqrt{3}}{\sqrt{3} \times \sqrt{3}} = \frac{6\sqrt{3}}{3}$

$= 2\sqrt{3}$

(b) $\frac{7}{\sqrt{18}} = \frac{7}{3\sqrt{2}} = \frac{7}{3\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}} = \frac{7\sqrt{2}}{3 \times 2}$

$= \frac{7\sqrt{2}}{6}$

(c) $\frac{5}{\sqrt{5}} = \frac{5 \times \sqrt{5}}{\sqrt{5} \times \sqrt{5}} = \sqrt{5}$

Notice in parts (a) and (b) of Example 6, multiplication of the given fractions by $\frac{\sqrt{3}}{\sqrt{3}}$ and $\frac{\sqrt{2}}{\sqrt{2}}$ is equivalent to multiplication by 1. Hence the value of the given fraction is not changed.

The fact that $\frac{\sqrt{m}}{\sqrt{n}} = \sqrt{\frac{m}{n}}$ can also be used to simplify fractions which contain surds.

Example 7

Simplify the following.

(*a*) $\frac{\sqrt{18}}{\sqrt{2}}$ (*b*) $\frac{\sqrt{5}}{\sqrt{2}}$ (*c*) $\sqrt{\frac{16}{7}}$

(a) $\frac{\sqrt{18}}{\sqrt{2}} = \sqrt{\frac{18}{2}} = \sqrt{9} = 3$

(b) $\frac{\sqrt{5}}{\sqrt{2}} = \sqrt{\frac{5}{2}} = \sqrt{2\frac{1}{2}}$ or $\sqrt{2{,}5}$ or $\frac{1}{2}\sqrt{10}$

(c) $\sqrt{\frac{16}{7}} = \frac{\sqrt{16}}{\sqrt{7}} = \frac{4}{\sqrt{7}} = \frac{4 \times \sqrt{7}}{\sqrt{7} \times \sqrt{7}} = \frac{4\sqrt{7}}{7}$

Exercise 20f

Simplify the following by rationalising the denominators.

1 $\frac{2}{\sqrt{2}}$ 2 $\frac{6}{\sqrt{2}}$ 3 $\frac{10}{\sqrt{5}}$ 4 $\frac{4}{\sqrt{8}}$

5 $\frac{21}{\sqrt{6}}$ 6 $\frac{15}{\sqrt{3}}$ 7 $\sqrt{\frac{4}{5}}$ 8 $\sqrt{\frac{9}{7}}$

9 $\frac{21}{\sqrt{7}}$ 10 $\frac{2\sqrt{3}}{\sqrt{6}}$ 11 $\frac{8}{\sqrt{18}}$ 12 $\sqrt{\frac{12}{50}}$

13 $\frac{3\sqrt{2}}{\sqrt{10}}$ 14 $\frac{30}{\sqrt{75}}$ 15 $\frac{30}{\sqrt{72}}$

Example 8

Simplify $3\sqrt{50} - 5\sqrt{32} + 4\sqrt{8}$.

$3\sqrt{50} - 5\sqrt{32} + 4\sqrt{8}$

$= 3\sqrt{25 \times 2} - 5\sqrt{16 \times 2} + 4\sqrt{4 \times 2}$

$= 3 \times 5\sqrt{2} - 5 \times 4\sqrt{2} + 4 \times 2\sqrt{2}$

$= 15\sqrt{2} - 20\sqrt{2} + 8\sqrt{2}$

$= 3\sqrt{2}$

Example 9

Simplify $\frac{5\sqrt{7} \times 2\sqrt{3}}{\sqrt{45} \times \sqrt{21}}$

$\frac{5\sqrt{7} \times 2\sqrt{3}}{\sqrt{45} \times \sqrt{21}} = \frac{5\sqrt{7} \times 2\sqrt{3}}{3\sqrt{5} \times \sqrt{3} \times \sqrt{7}}$

$= \frac{5 \times 2}{3\sqrt{5}} = \frac{2\sqrt{5}}{3}$

[illegible]ise 20g

[illegible]fy the following.

[illegible]$12} + \sqrt{3}$

[illegible]$2} - \sqrt{18}$

[illegible]$175} - 4\sqrt{7}$

[illegible]$45} + 3\sqrt{20} - 8\sqrt{5}$

[illegible]$99} - \sqrt{44} - \sqrt{11}$

[illegible]$\sqrt{8} - 3\sqrt{32} + 4\sqrt{50}$

[illegible]$\sqrt{150} - \sqrt{96} - 2\sqrt{24}$

[illegible]$\sqrt{54} + \sqrt{24} - \sqrt{216}$

[illegible]$\sqrt{28} - 5\sqrt{63} + 4\sqrt{216}$

[illegible] $\frac{\sqrt{18} \times \sqrt{20} \times \sqrt{24}}{\sqrt{8} \times \sqrt{30}}$

[illegible] $\frac{\sqrt{3} \times \sqrt{8} \times \sqrt{39}}{\sqrt{24} \times \sqrt{26}}$

[illegible] $\sqrt{3} + \frac{1}{\sqrt{3}} - \frac{1}{\sqrt{27}}$

[illegible] $2\sqrt{2} - \frac{3}{\sqrt{2}} + \frac{4}{\sqrt{8}}$

[illegible] $\frac{12}{\sqrt{24} - \sqrt{6}}$

15 $\frac{4}{\sqrt{18} + \sqrt{2}}$

Evaluation of expressions with surds

When evaluating a fraction containing a surd, it is advisable to rationalise its denominator. For example:

$\frac{2}{\sqrt{3}} = \frac{3{,}464}{3} \simeq 1{,}154\ldots$

The last step involves division by 1,732. This requires the use of tables or a calculator. However, it is much simpler to rationalise the denominator of the fraction:

$\frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3} \simeq \frac{2 \times 1{,}732}{3} = \frac{3{,}464}{3} \simeq 1{,}154\ldots$

Notice that all the steps can be done mentally.

Example 10

Without using tables, evaluate $\sqrt{\frac{2}{3}}(\sqrt{0{,}54} + \sqrt{6})$.

$\sqrt{\frac{2}{3}}(\sqrt{0{,}54} + \sqrt{6})$

$= \sqrt{\frac{2}{3}} \times \sqrt{0{,}54} + \sqrt{\frac{2}{3}} \times \sqrt{6}$

$= \sqrt{\frac{2}{3} \times 0{,}54} + \sqrt{\frac{2}{3} \times 6}$

$= \sqrt{0{,}36} + \sqrt{4}$

$= 0{,}6 + 2 = 2{,}6$

Notice that when evaluating expressions of this kind, it is usual to take the positive square root only.

Exercise 20h

1 Given that $\sqrt{2} = 1{,}414$ and $\sqrt{3} = 1{,}732$, evaluate the following correct to 3 s.f. without the use of tables or calculator.

(a) $\frac{1}{\sqrt{3}}$ (b) $\frac{1}{\sqrt{2}}$ (c) $\frac{2}{\sqrt{2}}$

(d) $\frac{3}{\sqrt{3}}$ (e) $\frac{10}{\sqrt{2}}$ (f) $\frac{6}{\sqrt{3}}$

2 If $\sqrt{5} = 2{,}236$, evaluate $\sqrt{\frac{7 \times 23 - 1}{8}}$ correct to 2 d.p.

3 Evaluate $\sqrt{3^2 \times 0{,}25 \times 6}$ correct to 2 d.p., given that $\sqrt{2} = 1{,}414$.

4 $\sqrt{1{,}225} = 1{,}107$, $\sqrt{12{,}25} = 3{,}5$ and $\sqrt{100} = 10$. Evaluate $\sqrt{1\ 225}$.

5 Without using tables, evaluate the following.

(a) $\frac{\sqrt{8}}{\sqrt{50} - \sqrt{2}}$

(b) $(\sqrt{19} + \sqrt{11})(\sqrt{19} - \sqrt{11})$

(c) $(5\sqrt{2{,}5} - \sqrt{10})\sqrt{0{,}4}$

(d) $(\sqrt{0{,}6} - \sqrt{15})^2$

π

The problem of finding the value of π has occupied mathematicians through the ages. The most famous attempt to find π was by Archimedes, around 250 BC. His method was as follows.

In Fig. 20.1, squares are drawn inside and outside a circle of radius r.

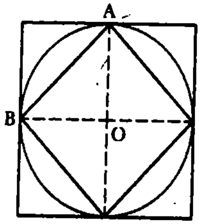


Fig. 20.1

Area of circle $= \pi r^2$

In Fig. 20.1, the area of the circle is about halfway between the area of the inner square and that of the outer square.

Area of inner square $= 4 \times \Delta AOB$
$= 4 \times \frac{1}{2}r^2 = 2r^2$

Area of outer square $= 8 \times \Delta AOB$
$= 8 \times \frac{1}{2}r^2 = 4r^2$

Thus the area of the circle lies between $2r^2$ and $4r^2$. Thus the value of π lies between 2 and 4, probably around 3.

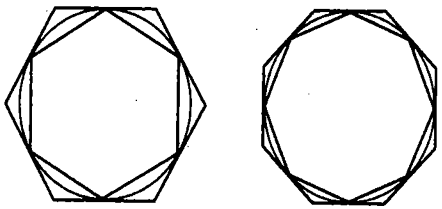


Fig. 20.2

Archimedes worked in this way, using regular polygons with more and more sides.

In Fig. 20.2 it can be seen that the greater the number of sides of the polygons, the closer their area is to that of the circle. Using polygons of 96 sides, Archimedes showed that the value of π was between $3\frac{10}{71}$ and $3\frac{1}{7}$. Both of these values are correct to 2 decimal places.

Exercise 20i

An experiment to find the value of π.

(a) Collect some tins and bottles of various diameters.

(b) Measure the diameter, d, of each object. (An easy way is to place the object on a ruler, then take readings at opposite ends of a diameter.)

(c) Use a piece of string or a strip of paper to measure the circumference, C, of each object.

(d) Make a table of values of d and C.

(e) Draw a graph of d (on the horizontal axis) against C (on the vertical axis). Your graph should look like the sketch in Fig. 20.3.

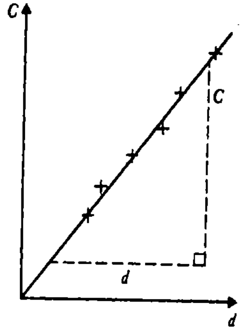


Fig. 20.3

(f) Construct a suitable triangle as in Fig. 20.3. Measure the height of the triangle, C, and the base of the triangle, d. Hence find the value of $\frac{C}{d}$ correct to 2 d.p.

(*Note:* $C = 2\pi r = \pi d$; hence $\pi = \frac{C}{d}$.)

Trigonometrical ratios of 45°, 30°, 60°

Tan, sin and cos of 45°

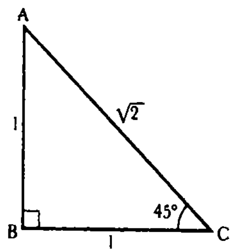


Fig. 20.4

In Fig. 20.4, ΔABC is right-angled at B and AB = BC = 1 unit.

$AC^2 = 1^2 + 1^2$ (*Pythagoras' theorem*)
$= 2$

$AC = \sqrt{2}$ units

Since AB = BC, $\hat{A} = \hat{C}$ (*isosceles* Δ)

$\hat{A} + \hat{C} = 90°$ (*sum of angles of* Δ)

$\hat{A} = \hat{C} = 45°$

$\tan 45° = \frac{1}{1} = 1$

$\sin 45° = \frac{1}{\sqrt{2}}$

$\cos 45° = \frac{1}{\sqrt{2}}$

Any triangle with angles of 45°, 45°, 90° has sides whose lengths are in the ratio $1 : 1 : \sqrt{2}$.

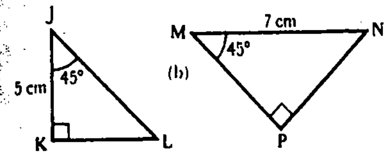


Fig. 20.5

For example, in Fig. 20.5(a) JK = 5 cm therefore KL = 5 cm and JL = $5\sqrt{2}$ cm.

In Fig. 20.5(b), MN = 7 cm.

$\cos 45° = \frac{MP}{7} = \frac{1}{\sqrt{2}}$

$MP = \frac{7}{\sqrt{2}}$ cm

Similarly $NP = \frac{7}{\sqrt{2}}$ cm

Hence MP and NP are found by dividing MN by $\sqrt{2}$.

Tan, sin and cos of 60° and 30°

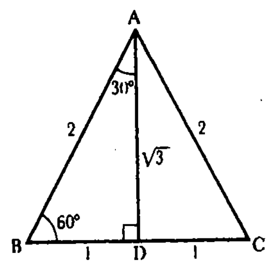


Fig. 20.6

In Fig. 20.6, ABC is an equilateral triangle with sides of 2 units in length. AD is an altitude.

BD = DC = 1 unit (AD *bisects* BC)

In ΔABD,

$AB^2 = AD^2 - BD^2$ (*Pythagoras' theorem*)

$2^2 = AD^2 + 1^2$

$AD^2 = 2^2 - 1^2 = 4 - 1 = 3$

$\Rightarrow AD = \sqrt{3}$ units

Since $\hat{B} = 60°$ (*equilateral* Δ)

$\tan 60° = \frac{\sqrt{3}}{1} = \sqrt{3}$

$\sin 60° = \frac{\sqrt{3}}{2}$

$\cos 60° = \frac{1}{2}$

Since $B\hat{A}D = 30°$ (*sum of angles of* ΔABD)

$\tan 30° = \frac{1}{\sqrt{3}}$

$\sin 30° = \frac{1}{2}$

$\cos 30° = \frac{\sqrt{3}}{2}$

Any triangle with angles of 30°, 60°, 90° has sides whose lengths are in the ratio $1 : \sqrt{3} : 2$.

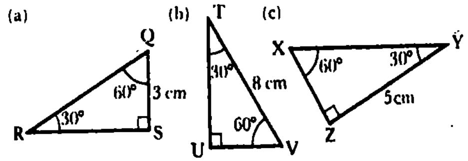


Fig. 20.7

For example, in Fig. 20.7,

(a) If QS = 3 cm, then QR = 6 cm and RS = $3\sqrt{3}$ cm

(b) If TV = 8 cm, then UV = 4 cm and RS = $4\sqrt{3}$ cm

(c) $\tan 30° = \frac{XZ}{YZ}$

$\Rightarrow \frac{1}{\sqrt{3}} = \frac{XZ}{5}$

$\Leftrightarrow XZ = \frac{5}{\sqrt{3}}$ cm

$$\sin 30° = \frac{XZ}{XY} = \frac{1}{2}$$

$$\Rightarrow \quad XY = 2 \times XZ$$

$$= 2 \times \frac{5}{\sqrt{3}}\text{ cm} = \frac{10}{\sqrt{3}}\text{ cm}$$

Notice that XZ : ZY : XY

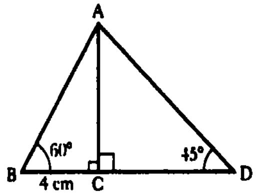
$$= \frac{5}{\sqrt{3}} : 5 : \frac{10}{\sqrt{3}}$$

$$= 5 : 5\sqrt{3} : 10$$

$$= 1 : \sqrt{3} : 2$$

Example 11

In Fig. 20.8, if BC = 4 cm, *find* AD.

Fig. 20.8 

Either: Using the ratios of the sides of the Δs:

In ΔABC with angles 30°, 60°, 90°,

if BC = 4 cm, then AC = $4\sqrt{3}$ cm.

In ΔACD with angles of 45°, 45°, 90°,

if AC = $4\sqrt{3}$ cm, then AD = $4\sqrt{3}$ cm $\times \sqrt{2}$

$= 4\sqrt{6}$ cm

Or: Using trigonometrical ratios of 60° and 45°:

$$\text{In } \Delta ABC, \tan 60° = \frac{AC}{BC}$$

$$\Rightarrow \quad \sqrt{3} = \frac{AC}{4}$$

$$\Leftrightarrow \quad AC = 4\sqrt{3}\text{ cm}$$

$$\text{In } \Delta ACD, \sin 45° = \frac{AC}{AD}$$

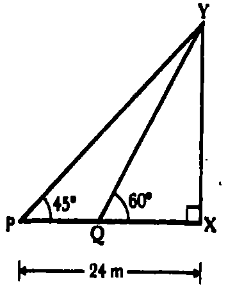
$$\Rightarrow \quad \frac{1}{\sqrt{2}} = \frac{4\sqrt{3}}{AD}$$

$$\Leftrightarrow \quad AD = 4\sqrt{3} \times \sqrt{2}\text{ cm}$$

$$= 4\sqrt{6}\text{ cm}$$

Example 12

In Fig. 20.9, if PX = 24 m, *find* PQ.

Fig. 20.9 

Either: Using the ratios of the sides of the Δs:

In ΔPXY with angles of 45°, 45°, 90°,

if PX = 24 m, then XY = 24 m

In ΔQXY with angles of 30°, 60°, 90°

if XY = 24 m, then QX = $\frac{24}{\sqrt{3}}$ m

$$PQ = 24\text{ m} - \frac{24}{\sqrt{3}}\text{ m}$$

$$= 24\left(1 - \frac{1}{\sqrt{3}}\right)\text{ m}$$

Or: Using the trigonometrical ratios of 45° and 60°:

$$\text{In } \Delta PXY, \tan 45° = \frac{XY}{PX}$$

$$\Rightarrow \quad 1 = \frac{XY}{24}$$

$$\Leftrightarrow \quad XY = 24\text{ m}$$

$$\text{In } \Delta QXY, \tan 60° = \frac{XY}{QX}$$

$$\Rightarrow \quad \sqrt{3} = \frac{24}{QX}$$

$$\Leftrightarrow \quad QX = \frac{24}{\sqrt{3}}$$

Hence PQ = $24\left(1 - \frac{1}{\sqrt{3}}\right)$ m (as before)

Example 13

In Fig. 20.10, if AB = 6 cm, *calculate x and y. (Leave the answers in surd form with rational denominators.)*

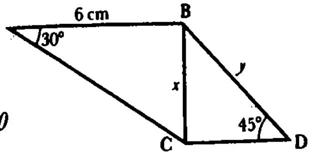


Fig. 20.10

In ΔABC,

$$\tan 30° = \frac{x}{6}$$

$$\Rightarrow \quad \frac{x}{6} = \frac{1}{\sqrt{3}}$$

$$\Leftrightarrow \quad x = \frac{6}{\sqrt{3}}\text{ cm} = \frac{6\sqrt{3}}{3}\text{ cm}$$

$$= 2\sqrt{3}\text{ cm}$$

In ΔBCD,

$$\sin 45° = \frac{x}{y}$$

$$\frac{1}{\sqrt{2}} = \frac{2\sqrt{3}}{y}$$

$$\Leftrightarrow \quad y = 2\sqrt{3} \times \sqrt{2}\text{ cm}$$

$$= 2\sqrt{6}\text{ cm}$$

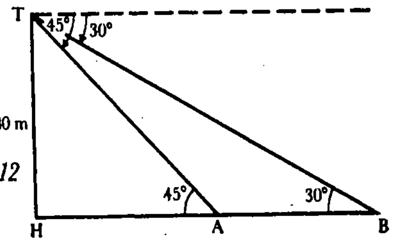
Exercise 20j

In each part of Fig. 20.11 on page 186, calculate the lengths marked x and y. All dimensions are in cm. Give answers in surd form with rational denominators.

Example 14

From the top of a tower 80 m *high two boats are seen in a direction due south. The angles of depression of the boats from the top of the tower are* 45° *and* 30°. *Find the distance between the boats.*

First, draw a sketch. In Fig. 20.12 HT represents the tower; A and B are the positions of the boats.

Fig. 20.12 

AB is the distance between the boats.

$$AB = HB - HA$$

In ΔTHB,

$$\tan 30° = \frac{80}{HB}$$

$$\Rightarrow \quad \frac{1}{\sqrt{3}} = \frac{80}{HB}$$

$$\Leftrightarrow \quad HB = 80\sqrt{3}\text{ m}$$

In ΔTHA,

$$\tan 45° = \frac{80}{HA}$$

$$\Rightarrow \quad 1 = \frac{80}{HA}$$

$$\Leftrightarrow \quad HA = 80\text{ m}$$

Hence AB = $80\sqrt{3}$ − 80 m

$= 80(\sqrt{3} - 1)$ m

Notice the importance of drawing a sketch.

Exercise 20k

Draw a sketch in each question. Leave the answers in surd form with rational denominators.

1 From the top of a tower, the angle of depression of a car is 30°. If the tower is 20 m high, how far is the car from the foot of the tower?
2 The angle of elevation of X from Y is 30°. If XY = 40 m, how high is X above Y?
3 In ΔABC, AB = 6 cm, $\hat{A}$ = 90° and $\hat{B}$ = 30°. AD is an altitude. Find CD.
4 In the isosceles triangle ACB, AB = AC = 4 cm, B$\hat{A}$C = 30° and CN is an altitude. Find BN.
5 XYZ is an isosceles triangle with XY = XZ = 6 cm and Y$\hat{X}$Z = 120°. Calculate the length of YZ.
6 A cone has a circular base, a perpendicular height of 21 cm, and a semi-vertical angle of 30°. Calculate the slant height of the cone. Find the area of its base. (Take π to be $\frac{22}{7}$.)
7 When the angle of elevation of the sun is 30°, the shadow of a vertical tower is 20 m longer than when the elevation of the sun is 60°. Find the height of the tower.

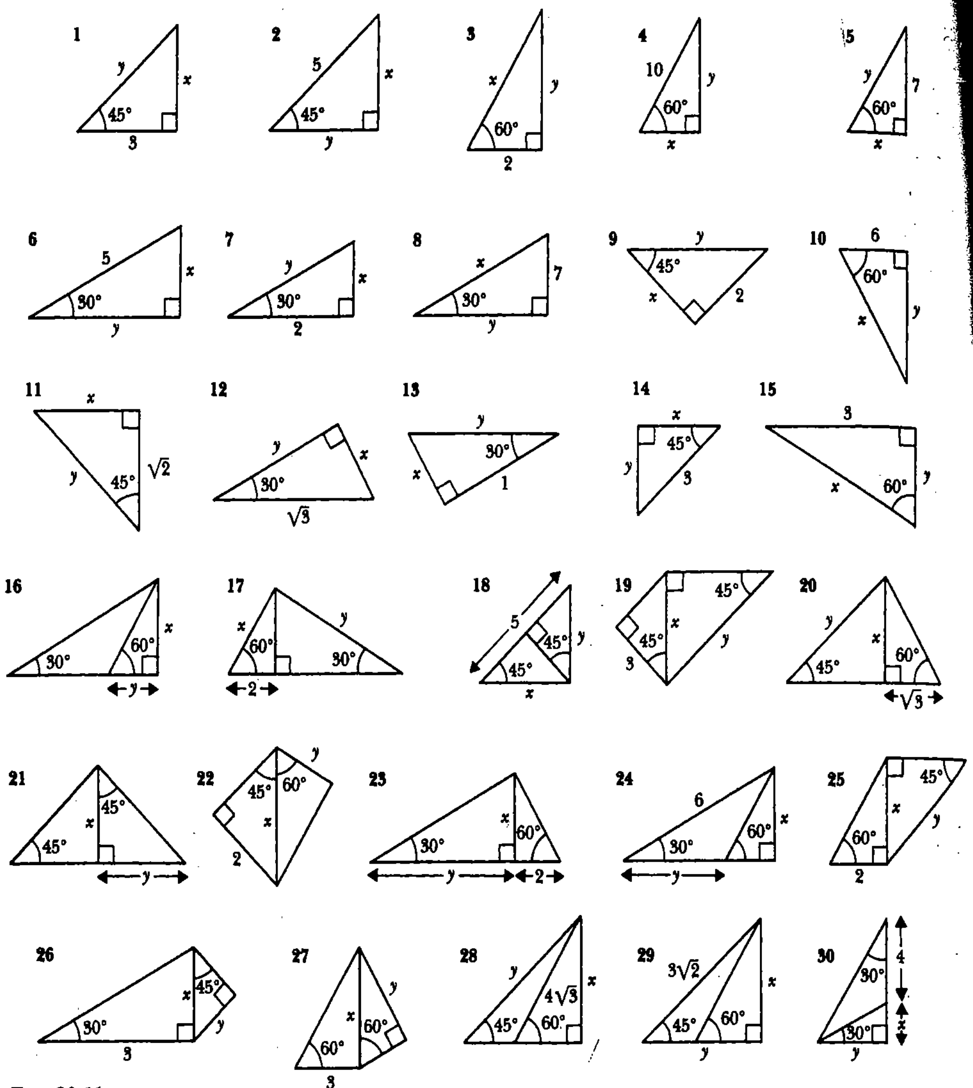


Fig. 20.11

8 Two huts and a radio mast are on level ground such that one hut is due east of the mast and the other is due west of it. From the top of the mast the angles of depression of the huts are 60° and 45° respectively. If the mast is 150 m high, find the distance between the huts.

9 A and B are two points on level ground, both due south of a flag-post. The angle of elevation of the top of the flag-post is 60° from A and 45° from B. If A is 20 m from the foot of the flag-post, find AB.

10 The top of a building 24 m high is observed from the top and from the bottom of a tree (which is vertical). The angles of elevation are found to be 45° and 60° respectively. By a suitable calculation find the height of the tree.

vision exercises and tests

hapters 11–20

vision exercise 5

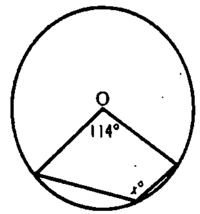
Simplify the following. (Do not use tables or a calculator.)

(a) $1\frac{3}{4} + 3\frac{1}{8} \times 1\frac{2}{3} + \frac{1}{12}$ (b) $\frac{7,6 \times 4,5}{2,85}$

(c) 135% of \$2,20 (d) 0,48 ÷ 0,001 6

A long straight road makes an angle of 17° with the horizontal. Two posts are 1 km apart on the road. Calculate in metres (a) the horizontal distance, (b) the difference in height, between the posts.

In Fig. R11, O is the centre of the circle. Find the value of x.



g. R11

4 The number of elements in each region of the Venn diagram of Fig. R12 are as shown.

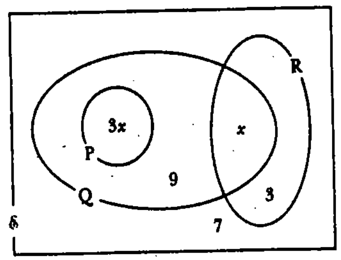


Fig. R12

If n(ℰ) = 43, find
(a) x, (b) n(Q), (c) n(R′), (d) n(P′).

5 (a) Prepare a table for the graph of $y = x^2 + 3x - 4$ for values of x from -6 to $+3$.
(b) Use a scale of 1 cm to 1 unit on both axes and draw the graph.
(c) Use the graph to
  (i) find the least value of y;
  (ii) find the values of x when $y = 1$;
  (iii) state the range of values of x for which y is negative.

6 Fig. R13 shows quadrilateral Q and its images A and B.

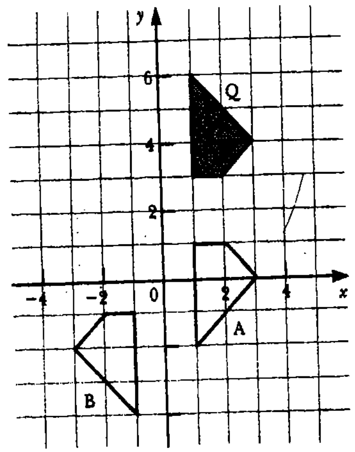


Fig. R13

Describe completely the single transformation which maps Q onto (a) A, (b) B.

7 Write the following numbers in standard form.
(a) 5 000 (b) 0,038
(c) 75,48 (d) 0,000 025

8 Expand (a) $(\frac{1}{2}x + 3)(3x + 2)$
(b) $(2t + 3u)(2t - 3u)$

9 Express each of the following as the square root of a single number.
(a) $2\sqrt{5}$ (b) $5\sqrt{3}$ (c) $3\sqrt{11}$ (d) $10\sqrt{7}$

10 A cuboid is 8 cm long, 4 cm wide and 5 cm high. Sketch the cuboid showing its measurements. Draw the plan and front and side elevations of the cuboid.

Revision test 5

1 Express 40 cm as a percentage of 8 m.
A 5% B 8% C 10% D 20% E 32%

2 In Fig. R14, O is the centre of the circle and $A\hat{B}C = 140°$. Find y.

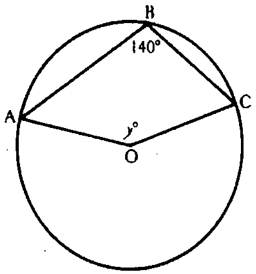


Fig. R14

A 20° B 70° C 80° D 240° E 280°

3 Express 0,007 8 in standard form.
A $7,8 \times 10^{-3}$ B $7,8 \times 10^{-2}$
C $7,8 \times 10^{1}$ D $7,8 \times 10^{2}$
E $7,8 \times 10^{3}$

4 $(2x + 3)$ is a factor of $6x^2 + x - 12$. The other factor is
A $(x + 6)$ B $(2x - 3)$ C $(3x + 4)$
D $(3x - 4)$ E $(4x - 9)$

5 The quadrilateral Q in Fig. R13 is reflected in the line $x = 0$. Which one of the following is *not* an image of one of its vertices?
A (− 1; 3) B (− 1; 6) C (− 2; 3)
D (− 2; 4) E (− 3; 4)

6 A village P is 20 km from a town Q on a bearing of 032°. Calculate the distance that P is (a) north, (b) east of Q.

7 If $\mathscr{E}$ = {all towns}, L = {large towns}, W = {towns with wide streets}, T = {towns with traffic lights}, express the following statements in set language.
(a) All large towns have wide streets.
(b) All towns with traffic lights are large.
(c) Harare has traffic lights.
Hence draw a conclusion from the given statements.

8 Find x if
(a) $(x - 3)^2 = 1\frac{7}{9}$, (b) $(x - 2)^2 = 11$,
(c) $(x + 5)^2 = 8$.
(Use square-root tables where necessary.)

9 Without using tables, evaluate the following.
(a) $\dfrac{\sqrt{75}}{\sqrt{48} - \sqrt{3}}$
(b) $(\sqrt{7} + \sqrt{3})(\sqrt{7} - \sqrt{3})$
(c) $\sqrt{0,2}\,(\sqrt{80} - \sqrt{5})$

10 Given Fig. R13, find the coordinates of the vertices of quadrilateral Q after (a) an anti-clockwise rotation of 90° about the point (2; 1), (b) a reflection in the line $y = x$.

Revision exercise 6

1 By selling an article for $31,51, a trader makes a profit of 15%.
(a) Calculate the cost price of the article.
(b) Calculate the selling price if the trader wishes to make a profit of 35%.

2 By drawing and measurement, find approximately (a) the value of sin 37°, (b) the angle whose cosine is $\frac{7}{10}$.

3 In Fig. R15, AB is a diameter of the circle. Calculate the sizes of the following angles.
(a) $A\hat{B}P$ (b) $A\hat{B}Z$ (c) $A\hat{Y}Z$
(d) $B\hat{Y}Z$ (e) $A\hat{B}Y$ (f) $Y\hat{Z}B$

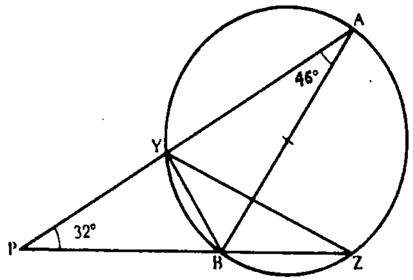


Fig. R15

For each of the following, make a copy of Fig. R16 and shade the given set.

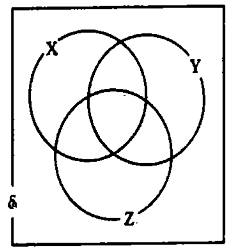


Fig. R16

(a) $X \cup (Y \cap Z')$
(b) $(X \cup Y) \cap Z'$
(c) $(X \cap Y \cap Z)' \cap (X \cup Y)$
(d) $(X \cup Y') \cap (Y \cup Z')$

Copy and complete Table R1 for the graph of $y = 3x^2 - 5x + 6$.

Table R1

x	− 2	− 1	0	1	2	3
y	28		6		8	

Draw the graph of $y = 3x^2 - 5x + 6$, using Table R1.
(a) Find the value of x for which y is least.
(b) Find the value of y when $x = -1,6$.

6 Use logarithm tables to calculate the following. Round answers to 3 s.f.
(a) $163 \times 0,002\,07$
(b) $\dfrac{(0,538)^2}{2,655}$
(c) $0,770\,8 \div 0,000\,39$
(d) $\sqrt[3]{0,080\,05}$

7 Factorise the following.
(a) $a^2 + 7a + 6$
(b) $2b^3 - 22b^2 + 56b$
(c) $15c^2 + 31c + 10$
(d) $8d^2 + 37d - 15$

8 A triangle F has vertices at (1; 0), (1; 2) and (3; 4). H is a shear of factor − 2 in the x-direction with the x-axis invariant. R is a rotation of 90° clockwise about the origin. Find the vertices of the following triangles.
(a) H(F) (b) R(F)
(c) RH(F) (d) HR(F)

9 A pendulum consists of a mass hanging at the end of a string 18 cm long. Find the vertical height through which the mass rises and falls as the pendulum swings through 30° on each side of the vertical. (Use the value 1,7 for $\sqrt{3}$.)

10 Fig. R17 is a sketch of a solid made from a cube and a cone. If edge of cube = height of cone = 5 cm, draw a full size plan and front elevation (F) of the solid.

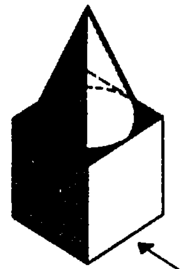


Fig. R17

Revision test 6

1 $\frac{1}{5}$ of the people in a village have been inoculated against measles. If 50 people have been inoculated, the number of people in the village is
A 10 B 150 C 250 D 300 E 500

Fig. R18 is a Venn diagram showing the elements $p, q, r, s, \ldots, z$ arranged within sets A, B, C, $\mathscr{E}$. Use Fig. R18 to answer questions 2 and 3.

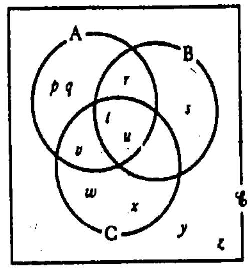


Fig. R18

2 What is $n(A \cup B)'$?
A 2 B 3 C 4 D 7 E 8

3 Which one of the following gives the members of the set $A' \cap B \cap C$?
A $\emptyset$ B $\{s\}$ C $\{t; u\}$
D $\{y; z\}$ E $\{w; x\}$

4 In Fig. R19, Q = X(P). Describe the transformation X completely.

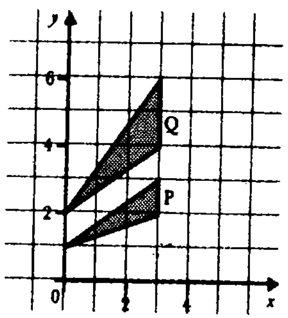


Fig. R19

A stretch, factor 2, y-direction, $y = 0$ invariant
B stretch, factor 1, y-direction, $y = 0$ invariant
C translation, 1 unit in y-direction
D shear, factor 2, y-direction, $x = 0$ invariant
E shear, factor 1, y-direction, $x = 0$ invariant

5 The first digit of the square root of 79 is
A 2 B 4 C 7 D 8 E 9

6 By drawing and measuring, find approximately (a) the value of cos 44°, (b) the size of the angle whose sine is $\frac{4}{5}$.

7 Solve the following quadratic equations.
(a) $x^2 - 4x = 5$
(b) $y^2 - 10y + 16 = 0$
(c) $3x^2 + 14x + 8 = 0$
(d) $2x^2 - 11x - 21 = 0$

8 On page 145, Fig. 16.2 is a graph of $y = x^2 + 2x - 3$. Use the graph to find the range of values of x for which $y \leqslant 0$.

9 Use logarithm tables or a calculator to find (a) $(0{,}650\ 2)^4$, (b) $\sqrt[4]{(0{,}650\ 2)}$ correct to 3 s.f.

10 W, X, Y and Z are points on th[e] circumference of a circle such that WZ is [a] diameter, $X\hat{W}Y = 36°$ and $Y\hat{W}Z = 29°$. Calculate (a) $X\hat{Y}Z$, (b) $W\hat{X}Y$.

Revision exercise 7

1 Bricks have a mass of 1,75 kg each. How many bricks are there in $5\frac{1}{4}$ tonnes?

2 In $\triangle ABC$, $\hat{A} = 38°$, $\hat{B} = 90°$ and AC = 9 cm.
Calculate (a) $\hat{C}$, (b) AB, (c) BC.

3 In Fig. R20, AB is a diameter and $A\hat{C}D = 27°$.

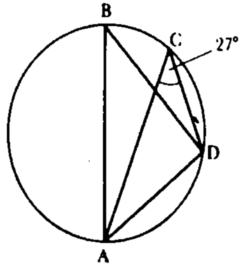


Fig. R20

Find (a) $A\hat{B}D$, (b) $B\hat{A}D$.

4 100 people in a survey drink at least one of the following every day: tea, coffee, water. Two people drink coffee only, 17 people drink tea and coffee and 23 people drink coffee and water. If 71 people do not drink coffee, how many drink all three?

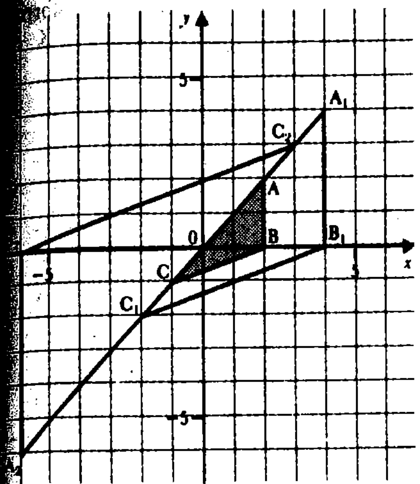
5 Use Fig. 16.4 on page 146 to state the range of values of x for which $2 - 3x - 2x^2$ is decreasing.

6 Simplify $\frac{3}{4}\log 81$, expressing the answer in terms of log 3.

7 Solve the following equations.
(a) $x^2 - 5x - 14 = 0$
(b) $2a^2 + 11a + 5 = 0$
(c) $6t^2 - 5t - 4 = 0$
(d) $x^2 + \frac{3}{2}x - 1 = 0$

8 Rationalise the denominators of the following.
(a) $\frac{3}{\sqrt{3}}$ (b) $\sqrt{\frac{7}{5}}$
(c) $\frac{3\sqrt{2}}{\sqrt{3}}$ (d) $\frac{18}{\sqrt{12}}$

In Fig. R21, $\triangle A_1B_1C_1$ and $\triangle A_2B_2C_2$ [are] enlargements of $\triangle ABC$. Describe the [en]largements fully.



R21

(b) If triangle ABC is given a shear of factor 3 in the x-direction with the line $y = 0$ invariant, state the coordinates of its image A′B′C′.

The front elevation of a triangular prism is an equilateral triangle of side 5 cm. If the prism is 7 cm long, draw its plan and side elevation. (Make a sketch first.)

Revision test 7

1 In Fig. R22, sin P =
A $\frac{p}{q}$ B $\frac{q}{p}$ C $\frac{r}{q}$ D $\frac{p}{r}$ E $\frac{r}{p}$

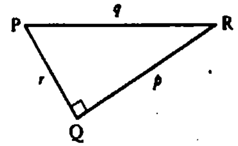


Fig. R22

2 In Fig. R22, which of the following statements is (are) true?
I $\sin \hat{P} = \cos \hat{R}$ II $\cos \hat{P} = \sin \hat{R}$
III $\tan \hat{P} = \tan \hat{R}$
A none of them B I and II only
C II and III only D I and III only
E all of them

3 The antilog of $\bar{2}{,}3869$ is
A 0,002 437 B 0,024 37
C 0,243 7 D 2,437
E 24,37

4 Find the roots of the equation $x^2 + 12x - 28 = 0$. The greater of the two roots is
A −14 B −2 C 2 D 7 E 14

5 A diagonal of a rectangle is 4 cm long and makes an angle of 60° with one side. The length, in cm, of the longest side of the rectangle is
A $2\sqrt{2}$ B $2\sqrt{3}$ C 4
D $4\sqrt{2}$ E $4\sqrt{3}$

6 (a) In a test a student scored 51 marks out of 80. Express this result as a percentage to the nearest whole per cent.
(b) In an examination, $\frac{1}{5}$ of the candidates failed. It is known that 240 candidates passed. How many candidates took the examination?

7 In Fig. R23, obtain an equation in x. Hence find the value of x.

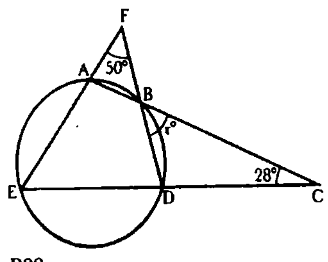


Fig. R23

8 A 192-page geography book contains writing, pictures and maps. 40 pages have writing only, 5 pages have pictures only and 12 pages have maps only. n pages have pictures and maps only, $3n$ pages have writing and maps only and $5n$ pages have writing and pictures only.

If 18 pages contain all three, find n and hence find the total number of pages in the book which have maps on them.

9 Given that $\log 2 = 0{,}301\,03$ and $\log 3 = 0{,}477\,12$, find the value of the following without using tables.
(a) $\log 1{,}5$ (b) $\log 50$ (c) $\log 36$

10 Triangle ABC in Fig. R21 on page 191 is given a one-way stretch of factor -2 in the y-direction with the line $y = 0$ invariant. State the coordinates of its image A″B″C″.

Revision exercise 8

1 (a) What percentage of 8 is 12?
(b) The original cost of a ball pen is 23c. By the time the pen reaches the shop its price has increased by 200%. What is the price in the shop?

2 The diagonals of a rectangle are 10 cm long and intersect at an angle of 120°. Make a sketch of the rectangle. Hence use trigonometry to calculate the length and breadth of the rectangle.

3 L, M, N, P are points on the circumference of a circle, centre O. LN and PM intersect at X. $N\hat{L}M = 35°$ and $L\hat{X}P = 98°$. Calculate (a) $L\hat{N}P$, (b) $L\hat{O}P$.

4 If $\mathscr{E} = \{x: 1 \leqslant x \leqslant 20, x \text{ is an integer}\}$, $P = \{x: x \text{ is a factor of } 36\}$ and $Q = \{x: x \text{ is an odd number}\}$,
(a) list the members of set P,
(b) list the members of set Q′,
(c) find n(P′), (d) find n(P ∪ Q).

5 Table R2 gives corresponding values of x and y for the relation $y = 2x^2 - 5x - 6$.

Table R2

x	-2	-1	0	1	2	3	4
y	12	1	-6	-9	-8	-3	6

(a) Use scales of 2 cm to 1 unit on the x-axis and 1 cm to 1 unit on the y-axis and draw the graph of $y = 2x^2 - 5x - 6$.
(b) Use your graph to find (i) the least value of y,
(ii) the solutions of $2x^2 - 5x - 6 = 0$.

6 Solve the equation $2x^2 + 6x + 1 = 0$,
(a) by completing the square,
(b) by formula.

7 Factorise the following.
(a) $x^2 - 3x - 4$ (b) $6y^2 + 11y +$
(c) $3a^2 - 11a + 8$ (d) $2d^2 + 7d - 15$

8 Describe fully the single transformation which maps quadrilateral K onto quadrilateral K′ in Fig. R24.

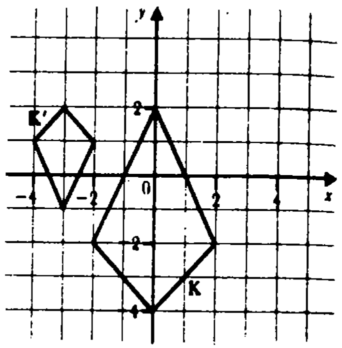


Fig. R24

9 In Fig. R25, $A\hat{B}D = A\hat{E}C = 90°$ and AB = 10 cm.

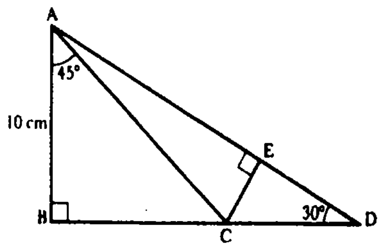


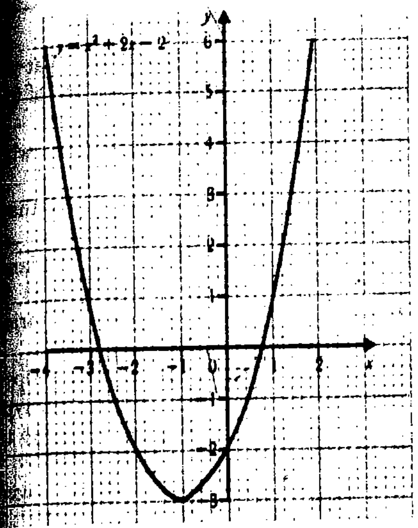
Fig. R25

(a) Calculate BC, AC, CD, CE and $C\hat{A}D$.
(b) Hence write down a numerical expression for sin 15°. (Leave answers in terms of $\sqrt{2}$ and $\sqrt{3}$ where necessary.)

10 A cone is 7 cm high and rests on a base of diameter 6 cm.
(a) Make a sketch of the cone showing its measurements.
(b) Draw the plan and front elevation of the cone.
(c) Hence measure the slant height and vertical angle of the cone.

...sion test 8

...R26 is the graph of the function ...$2x - 2$. Use the graph to answer questions ...3.



... R26

What is the value of $x^2 + 2x - 2$ when $x = 1{,}8$?
A $-3{,}2$ B $-2{,}4$ C $1{,}2$ D $1{,}8$ E $4{,}8$

What is the lowest value of $x^2 + 2x - 2$?
A -4 B -3 C -2 D -1 E 0

For what range of values of x is $x^2 + 2x - 2$ increasing?
A $x > -3$ B $x > -2{,}7$ C $x > -1$
D $x < 0{,}7$ E $x < -1$

4 In Fig. R27, O is the centre of the circle, $W\hat{X}Y = 80°$ and $W\hat{X}Z = 45°$. Calculate $Y\hat{X}Z$.

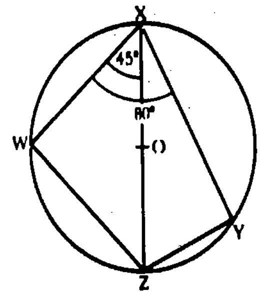


Fig. R27

A 35° B 45° C 55° D 80° E 125°

5 If $\sin A = \frac{4}{5}$, then $\tan A =$
A $\frac{2}{5}$ B $\frac{3}{5}$ C $\frac{3}{4}$ D 1 E $\frac{4}{3}$

6 A salary of \$9 000 was increased by 20%. A year later the new salary was increased by 10%. Calculate the present salary.

7 A ladder 6 m long leans against a vertical wall and makes an angle of 80° with the horizontal ground. Calculate, to 2 s.f., how far up the wall the ladder reaches.

8 (a) Make a copy of the Venn diagram in Fig. R28.

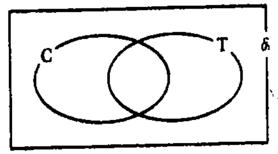


Fig. R28

(b) On your diagram, enter the members of the sets given that
$\mathscr{E} = \{c; o; m; p; u; t; e; r\}$, $T = \{t; e; r; m\}$ and $C = \{c; r; o; p\}$.
(c) List the elements of the following sets.
(i) T′ (ii) C′ (iii) (C ∪ T)′ (iv) C ∪ T′

9 In Fig. R24 on page 192, K′ = RE(K) where E is an enlargement with the origin as centre and R is a rotation.
(a) State the scale factor of E.
(b) Describe R fully.

10

Fig. R29

Fig. R29 is the plan of a triangular prism resting so that face PQST is horizontal.
(a) How many edges has the prism?
(b) Which of those edges are horizontal?
(c) Are there any vertical edges?

Non-routine problems (2)

1 Fig. Q10 shows some cubes arranged on a table. In Fig. Q10(a) the cubes are in columns; in Fig. Q10(b), they are in rows.

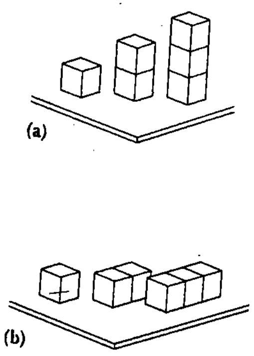


Fig. Q10

In both cases it would be possible to see a number of faces of the cubes. These can be called *visible faces*. Copy and complete Table Q3.

Table Q3

number of cubes	number of visible faces when cubes are in columns	number of visible faces when cubes are in rows
1	5	5
2	9	8
3	13	11
4		
n		

2 Triangles are to be drawn such that their sides are each an integral number of units in length.
(a) Describe all those triangles which have perimeters of 12 units.
(b) What is the smallest perimeter that is possible?
(c) Is it possible to have such a triangle with a perimeter of 4 units?
(d) Investigate triangles of various perimeters.

3 In a certain village there are two groups of people: X-types and Y-types. X-types always tell the truth; Y-types always tell lies.
Here is a discussion between three villagers, A, B and C:
A says, 'All three of us are X-types'.
B says, 'That is correct'.
C says, 'No, that is not correct'.
In the same village, three other people, D, E and F, happen to meet:
D says, 'All three of us are Y-types'.
E says, 'Only one of us is an X-type'.
F doesn't say anything.
From the above information identify A, B, C, D, E, F as X-types or Y-types.

4 A garage manager receives the following bill:

22 tyres: \$*29,3*,

(oil having made the first and last digits illegible). All the tyres are the same and each costs more than \$25. What are the missing digits in the bill and how much does each tyre cost?

5 In a village there are exactly 10% more boys than girls, 15% more women than men and 20% more children than adults. How many boys, girls, men and women are there?
(There are less than 6 000 people in the village. Boys and girls count as children; men and women count as adults.)

You have a card divided into 16 spaces as in Fig. Q11:

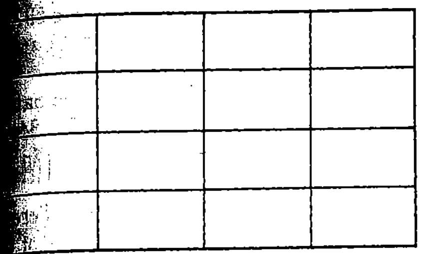


Fig. Q11

You have plenty of stamps of value \$1, \$2, \$3, \$4 and \$5. What is the greatest value you can stick on the card if you cannot place stamps of the same value in a straight line (either horizontally, vertically or diagonally)?

A rectangular bar of soap is such that its top is of area 30 cm^2, its side is 24 cm^2 and its front is 20 cm^2. Find the length, breadth and height of the bar of soap.

In the following addition, each letter represents a distinct digit:

$$\begin{array}{r} \text{SEND} \\ +\ \text{MORE} \\ \hline \text{MONEY} \end{array}$$

Find what each letter stands for.

9 This book was written during 1991. Using mathematical symbols and the digits 1, 9, 9, 1 once each, *in that order*, it is possible to produce other numbers:

$1 + 99^1 = 100$
$(19 - 9) \times 1 = 10$

(a) Make calculations of this kind which produce the numbers 0, 1, 2, 3, 4, 5, 6, 7, 8, 9.
(b) Repeat for the year in which you are using this book.
(c) Extend the problem to other numbers. Challenge your friends to make calculations to fit a number of your choice, e.g. 4444.

10 Fig. Q12 shows a 5 × 5 × 5 cube which has been painted black.

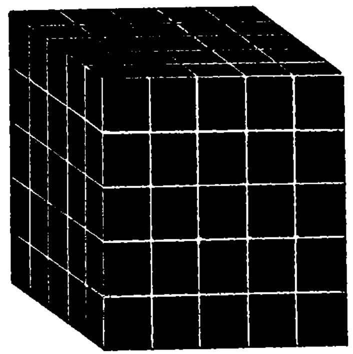


Fig. Q12

If the cube is carefully cut through the surface lines by planes parallel to its faces, there will be 125 unit cubes.
Consider the faces of the small cubes:
(a) How many will have 0 black faces?
(b) How many will have only 1 black face?
(c) How many will have 2 black faces?
(d) How many will have 3 black faces?
(e) Generalise for an $n \times n \times n$ cube, i.e. find a rule that will predict the number of black faces on the unit cubes which can be cut from a black $n \times n \times n$ cube.

11 Assume that you have 12 matches each of unit length.
(a) It is possible to place the matches in various ways so as to make polygons which have integral areas. For example, Fig. Q13 shows a square of area 9 square units and a rectangle of area 8 square units.

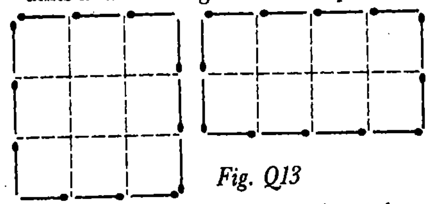


Fig. Q13

Using all 12 matches, make polygons whose areas are 1, 2, 3, 4, 5, 6, 7 square units. (In every case, there is more than one possible solution.)
(b) What is the greatest area that can be enclosed by the 12 matches? (This area may not be integral.)
(c) Investigate the above problems for other numbers of matches.
Note: in all cases, the whole length of the match must be used.

12 Investigate chains formed by summing the cubes of the digits of a given number, e.g. starting with 23:

$\boxed{23} \xrightarrow{2^3+3^3} \boxed{35} \xrightarrow{3^3+5^3} \boxed{152} \xrightarrow{1^3+5^3+2^3}$

13 Draw some noughts and crosses. Join each cross to each nought. Fig. Q14 gives two examples.

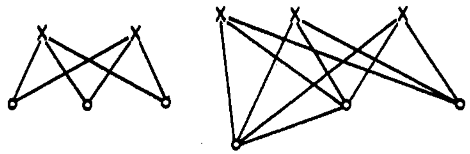


Fig. Q14

Place the noughts and crosses such that they *maximise* the number of intersections of the lines.

(a) Copy and extend Table Q4.

Table Q4

number of				
noughts	**crosses**	**lines**	**intersections**	**regions**
3	2	6	3	5
3	3	9	9	13
2	2			
3	4			
etc.	etc.			
m	*n*			

(b) Find a rule which gives the numb[er] of lines, intersections and regions for [*m*] noughts and *n* crosses.

14 A '7-number' is any number whose digit sum is divisible by 7. For example 597 is [a] 7-number because 5 + 9 + 7 = 21 and 2[1] is divisible by 7.

(a) The sequence of 7-numbers in natural ascending order begins 7, 16, 25, 34, ... Write down the next ten numbers in this sequence.

(b) What is the maximum difference between two consecutive members of this sequence?

(c) Investigate '*n*-numbers'.

15 (a) What is the sum of the marked angles in Fig. Q15?

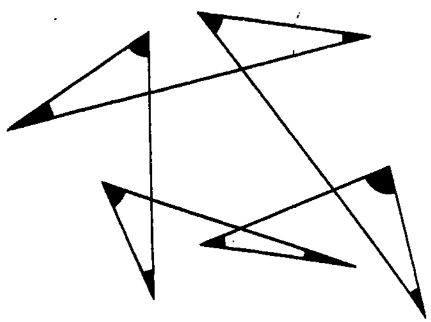


Fig. Q15

(b) Investigate shapes with different numbers of exterior triangles.

[Cha]pter 21

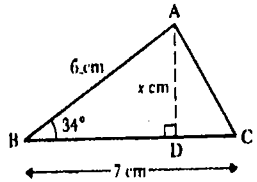
[M]ensuration of plane shapes

[Usi]ng trigonometry in area [pro]blems

[Exa]mple 1

[Find] the area of $\triangle ABC$ *to the nearest* cm^2 *if* [AB] = 6 cm, BC = 7 cm *and* $\hat{B} = 34°$.

[Let] the height of the triangle be *x* cm.



[Fig.] 21.1

[I]n $\triangle ABD$, $\frac{x}{6} = \sin 34°$

$x = 6 \sin 34°$
$= 6 \times 0{,}5592$
$= 3{,}3552$

[Ar]ea of $\triangle ABC = \frac{1}{2} \times BC \times AD$
$= \frac{1}{2} \times 7 \times 3{,}3552\ cm^2$
$= \frac{1}{2} \times 23{,}4864\ cm^2$
$= 11{,}7432\ cm^2$
$= 12\ cm^2$ to the nearest cm^2.

Example 2

Calculate the area of parallelogram PQRS *if* QR = 5 cm, RS = 6 cm, $Q\hat{R}S = 118°$.

In Fig. 21.2, QD is the height of the parallelogram.

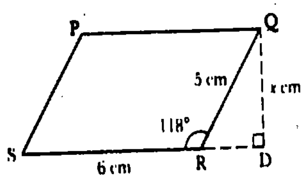


Fig. 21.2

Let QD be *x* cm.
In $\triangle QRD$, $Q\hat{R}D = 180° - 118° = 62°$

$\frac{x}{5} = \sin 62°$
$x = 5 \sin 62°$
$= 5 \times 0{,}8829$
$= 4{,}4145$

Area of PQRS $= SR \times QD$
$= 6 \times 4{,}4145\ cm^2$
$= 26{,}487\ cm^2$
$= 26\ cm^2$ to the nearest cm^2.

In Examples 1 and 2, since the data of the questions are given in whole numbers of cm and degrees, it is suitable to give the results to the nearest whole number of cm^2.

Example 3

Calculate the area of the trapezium in Fig. 21.3.

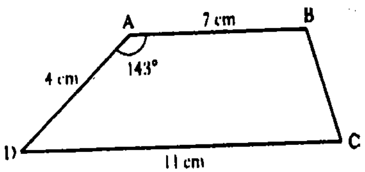


Fig. 21.3

Construct the height AP of the trapezium as in Fig. 21.4.

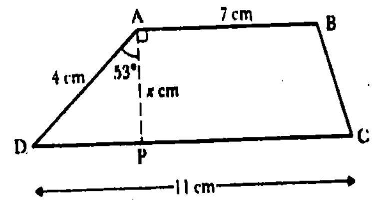


Fig. 21.4

In $\triangle ADP$, $D\hat{A}P = 143° - 90° = 53°$

$$\frac{x}{4} = \cos 53°$$

$$x = 4 \cos 53°$$
$$= 4 \times 0{,}6018$$
$$= 2{,}4072$$

Area of ABCD $= \frac{1}{2}(AB + DC) \times AP$
$$= \tfrac{1}{2}(7 + 11) \times 2{,}4072 \text{ cm}^2$$
$$= \tfrac{1}{2} \times 18 \times 2{,}4072 \text{ cm}^2$$
$$= 9 \times 2{,}4072 \text{ cm}^2$$
$$= 21{,}6648 \text{ cm}^2$$
$$= 22 \text{ cm}^2 \text{ to 2 s.f.}$$

Notice in Examples 1, 2 and 3 that rounding off is only done at the *last* stage of the working. Do not round off at an earlier stage.

Exercise 21a

Give all answers in this exercise to a suitable degree of accuracy.

1 Find the areas of the triangles in Fig. 21.5. All dimensions are in cm.

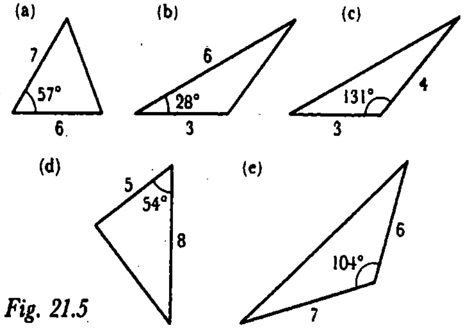


Fig. 21.5

2 Find the areas of the parallelograms in Fig. 21.6. All dimensions are in cm.

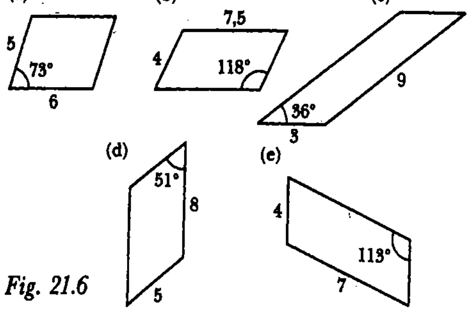


Fig. 21.6

3 Find the areas of the trapeziums in Fig. 21.7. All dimensions are in metres.

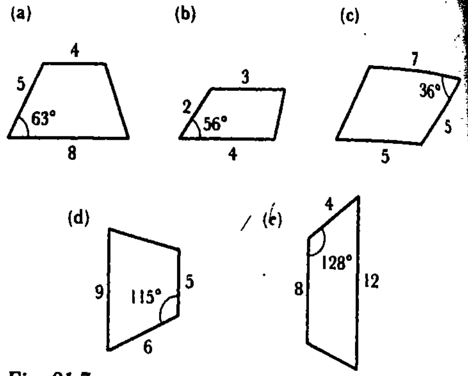


Fig. 21.7

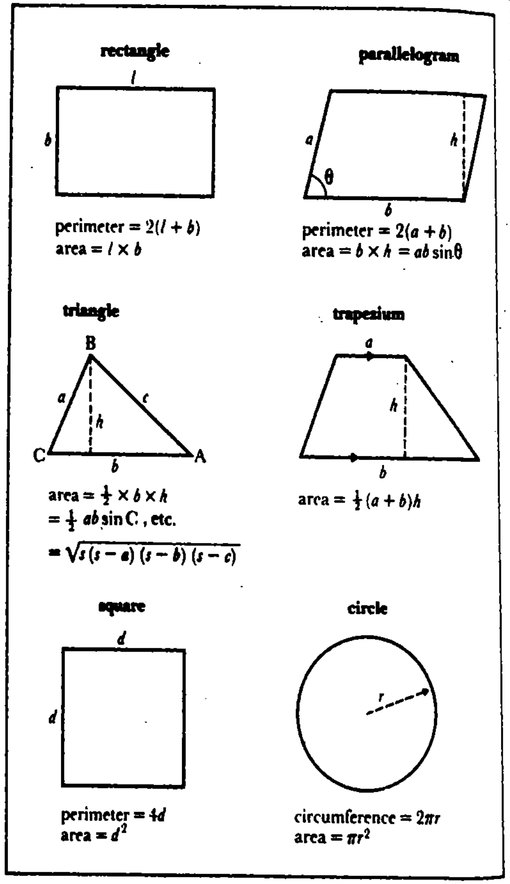


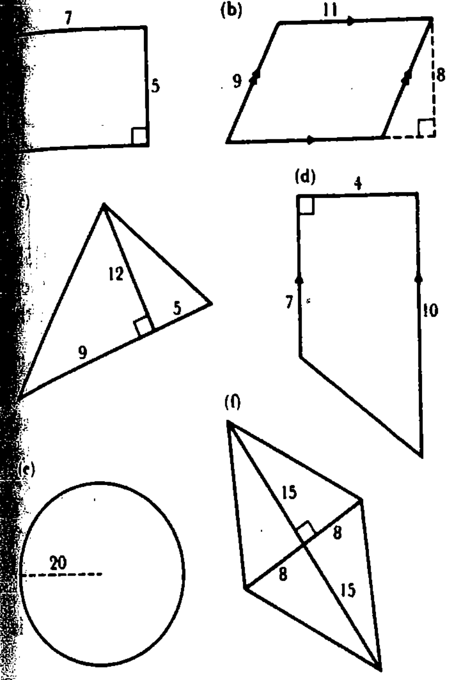
Fig. 21.8

…rimeter and area

…21.8 on page 198 gives the formulae for …perimeters and areas of common shapes …dy found in earlier books of this course.

…rcise 21b

…Calculate the area and perimeter of each …shape in Fig. 21.9. All dimensions are in …cm. Use the value 3,12 for π.



…ig. 21.9

2 In Fig. 21.10, ADFJ is a rectangle and all dimensions are in cm.

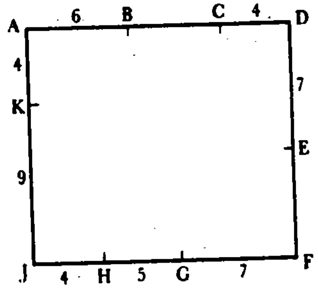


Fig. 21.10

Calculate the areas of the following.
(a) $\triangle ABK$ (b) $\triangle DKE$
(c) $\triangle ABG$ (d) $\triangle HEG$
(e) trapezium CDFJ
(f) parallelogram CDHJ
(g) trapezium ADGH (h) $\triangle AGE$
(i) quadrilateral BEHK (j) $\triangle CGK$

3 A rectangle 9 cm long is equal in area to a square which has a perimeter of 24 cm. Find the width of the rectangle.

4 Find the length of the side of a square which is equal in area to a rectangle measuring 45 cm by 5 cm.

5 What is the area of the shape in Fig. 21.11?

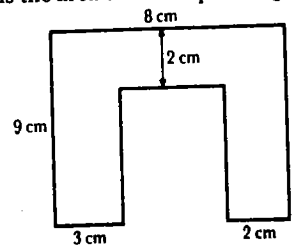


Fig. 21.11

6 Calculate the shaded area in Fig. 21.12. Use the value $3\frac{1}{7}$ for π.

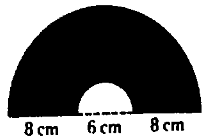


Fig. 21.12

7 Fig. 21.13 shows two concentric circles with centre O and radius r and R respectively. If $R = 3r$, express the area of the shaded part in terms of π and r.

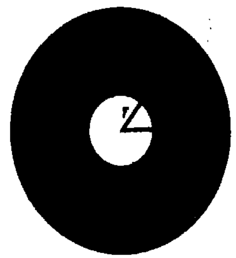


Fig. 21.13

8 The parallel sides of a trapezium are 11 cm and 13 cm. If the area of the trapezium is 84 cm², calculate the distance between the parallel sides.

9 A triangle is equal in area to a rectangle which measures 10 cm by 9 cm. If the base of the triangle is 12 cm long, find its altitude.

10 A bicycle wheel has a diameter of 63 cm. Calculate how many times the wheel turns round in travelling 19,8 km. Use the value $3\frac{1}{7}$ for π.

11 Fig. 21.14 shows the altitudes QT and PM of $\triangle PQR$. If QR = 8 cm, PR = 7 cm and QT = 4 cm, what is PM?

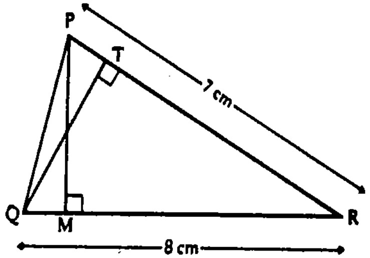


Fig. 21.14

12 BM, CN are altitudes of $\triangle ABC$. If AB = 3,5 cm, AC = 3,2 cm and BM = 2,1 cm, calculate the area of $\triangle ABC$ and hence find CN.

13 ABCD is a parellelogram and AM, AN are the perpendiculars from A to BC, CD respectively. If AB = 6,3 cm, AD = 4,9 cm and AN = 4,2 cm, calculate the area of the parallelogram and hence find AM.

14 The area of a square is R cm^2. Write down an expression for half its perimeter in terms of R.

15 The length of a rectangle is three times its width. If the perimeter is 72 cm, calculate the width of the rectangle.

16 The area of a trapezium is 14,7 cm^2. If the parallel sides are 5,3 cm and 3,1 cm long, find the perpendicular distance between them.

17 In Fig. 21.15, PQRS is a parallelogram, HSR is a straight line and $H\hat{P}Q = 90°$. If HQ = 10 cm and PQ = 6 cm, what is the area of the parallelogram?

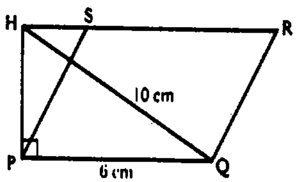


Fig. 21.15

18 Calculate the area of each shape in Fig. 21.16 correct to 2 s.f. All dimensions are in metres.

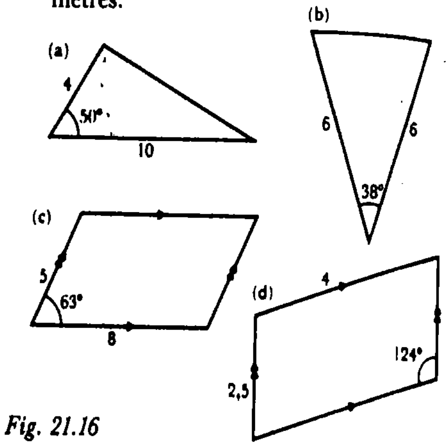


Fig. 21.16

19 PQRS is a trapezium in which PQ‖SR, PQ = 16 cm, QR = 12 cm, RS = 8 cm and $P\hat{Q}R = 30°$ (Fig. 21.17). Calculate the area of PQRS.

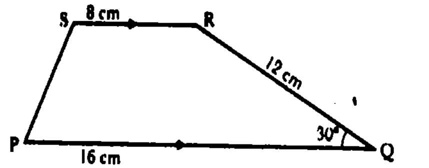


Fig. 21.17

20 The area of a parallelogram ABCD is 43 cm^2. AB = 7 cm, BC = 9 cm and $A\hat{B}C < 90°$.
Calculate $A\hat{B}C$ to the nearest degree.

Parallelograms and triangles between the same parallels

In Fig. 21.18 opposite, UV and XY are parallel lines. Parallelograms ABCD and EFGH have their bases on XY and their opposite sides on UV. Triangles JKL and MNO are drawn with their bases on XY and their opposite vertices on UV. The four shapes lie between the same parallels UV and XY.

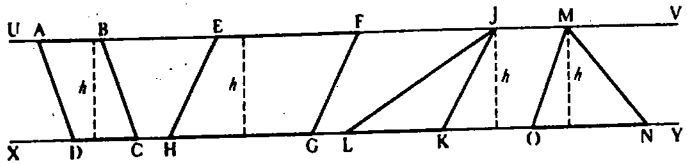


Fig. 21.18

Notice that the altitudes of the four shapes are all equal (h). The altitude is the distance between the two parallel lines.

Theorem

Parallelograms on the same base and between the same parallels are equal in area.

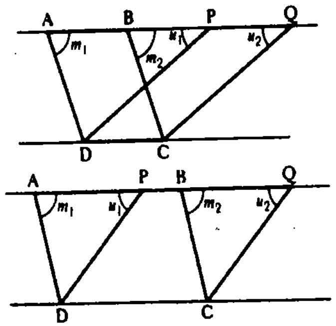


Fig. 21.19

Given: Parallelograms ABCD, PQCD on the same base DC and between the same parallels AQ, DC.

To prove: ABCD = PQCD

Proof:

In $\triangle$s APD, BQC,

$m_1 = m_2$ (*corr. angles*)

$u_1 = u_2$ (*corr. angles*)

AD = BC (*opp. sides of* ‖^{gm})

Hence $\triangle APD \equiv \triangle BQC$ (*AAS*)

$\Rightarrow$ quad AQCD − $\triangle$APD

= quad AQCD − $\triangle$BQC

Hence PQCD = ABCD

Theorem

Triangles on the same base and between the same parallels are equal in area.

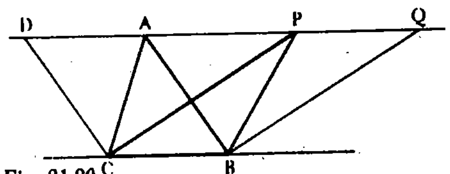


Fig. 21.20

Given: $\triangle$s ABC, PBC on the same base BC and between the same parallels AP and CD.

To prove: $\triangle ABC = \triangle PBC$

Construction: Draw CD‖BA and BQ‖CP to complete parallelograms ABCD and PCBQ.

Proof:

$\triangle ABC = \frac{1}{2}$‖^{gm} DABC (*diag. bisects* ‖^{gm})

$= \frac{1}{2}$‖^{gm} PQBC (*same base, same* ‖*s*)

$= \triangle PBC$

Example 4

In Fig. 21.21, ABCD *is any quadrilateral and* E *is a point on* CD *such that* AE‖BC. *Prove that quadrilateral* ABED *and* $\triangle$ACD *have equal areas.*

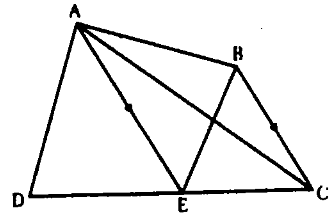


Fig. 21.21

quad ABED = quad ABCD − $\triangle$BCE

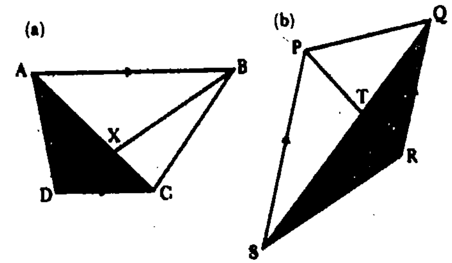
$\triangle$ACD = quad ABCD − $\triangle$BCA

But $\triangle BCE = \triangle BCA$ (*same base, same* ‖*s*)

$\therefore$ quad ABED = $\triangle$ACD

Exercise 21c

1 Name a triangle equal in area to the shaded triangle in each part of Fig. 21.22.



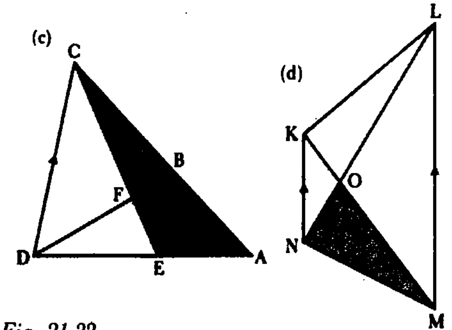


Fig. 21.22

2 In Fig. 21.23, ST∥PQ. Name a triangle equal in area to △PQS.

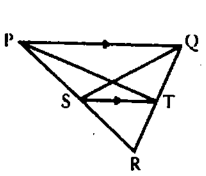


Fig. 21.23

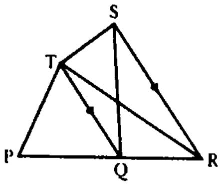


Fig. 21.24

3 In Fig. 21.24, name a quadrilateral equal in area to △PTR.

4 In the rhombus KTMP, XY∥TM, RS∥PM and XT = TS (Fig. 21.25). Name the parallelogram equal in area to RSMP.

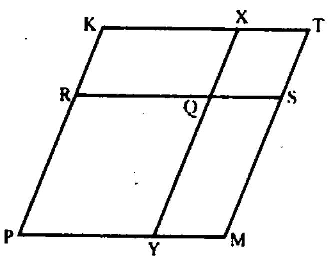


Fig. 21.25

5 In the trapezium in Fig. 21.26, X and Y are points on SR such that SX = YR. Prove that trapezium PQXS is equal in area to trapezium PQRY.

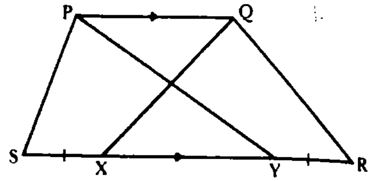


Fig. 21.26

6 In Fig. 21.27, ABCD and ABEC are parallelograms. EBF, DAF are straight lines. Prove that (a) △BAF = △ADC, (b) area of quad FACE = area of quad ADEB.

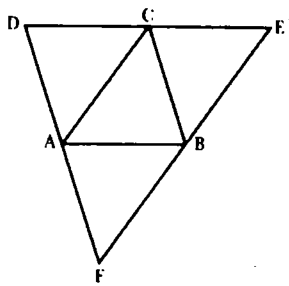


Fig. 21.27

7 Draw a rectangle measuring 10 cm by 8 cm. Construct a rhombus, equal in area to the rectangle, with sides of 10 cm. Measure the shorter diagonal of the rhombus.

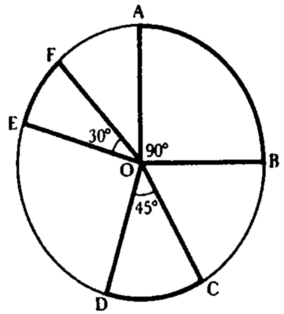
8 Draw a rectangle of altitude 5 cm with a base of length 7 cm. On the same base construct a parallelogram of equal area, with an angle of 70°. Measure the lengths of the diagonals of the parallelogram.

9 Draw a parallelogram with sides of 6 cm and 8 cm and an angle of 81°. Construct a rhombus with sides of 8 cm, equal in area to the parallelogram. Measure the shorter diagonal of the rhombus.

10 Construct a scalene triangle with sides of length 7, 8 and 6 cm. Construct a right-angled triangle equal in area to the scalene triangle such that one of the sides containing the right angle is 8 cm long. Measure the hypotenuse of the right-angled triangle.

cs and sectors of circles

ngth of arc



. 21.28

Fig. 21.28, the arc AB subtends an angle of at O, the centre of the circle. The whole cumference subtends 360° at O. Therefore length of arc AB is $\frac{90}{360}$ or $\frac{1}{4}$ of the circumference of the circle. Similarly arc CD is $\frac{45}{360}$ or f the circumference and arc EF is $\frac{30}{360}$ or $\frac{1}{12}$ of circumference. It can be seen that the length an arc of a circle is proportional to the angle hich the arc subtends at the centre.

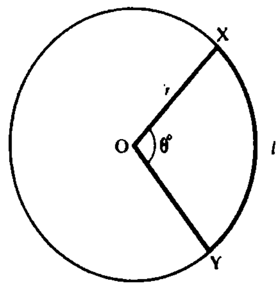


Fig. 21.29

In Fig. 21.29, arc XY subtends an angle of $\theta°$ at O. The circumference of the circle is $2\pi r$. Therefore, in Fig. 21.29, the length, l, of the arc XY is given as

$$l = \frac{\theta}{360} \times 2\pi r$$

Example 5

An arc subtends an angle of 105° *at the centre of a circle of radius* 6 cm. *Find the length of the arc.*

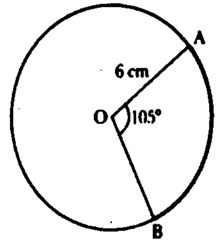


Fig. 21.30

$$\text{arc AB} = \frac{105}{360} \times 2\pi \times 6 \text{ cm}$$
$$= \frac{105}{360} \times 2 \times \frac{22}{7} \times 6 \text{ cm}$$
$$= 11 \text{ cm}$$

Example 6

Calculate the perimeter of a sector of a circle of radius 7 cm, *the angle of the sector being* 108°.

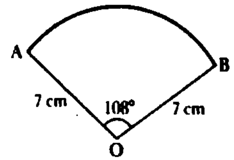


Fig. 21.31

$$\text{arc AB} = \frac{108}{360} \times 2 \times \frac{22}{7} \times 7 \text{ cm}$$
$$= 13{,}2 \text{ cm}$$

perimeter of sector AOB

$$= (7 + 7 + 13{,}2) \text{ cm} = 27{,}2 \text{ cm}$$

Example 7

What angle does an arc 6,6 cm *in length subtend at the centre of a circle of radius* 14 cm?

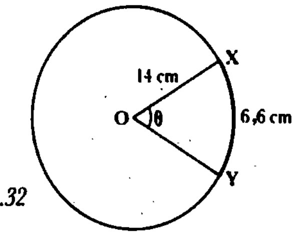


Fig. 21.32

$$\text{arc XY} = \frac{\theta}{360} \times 2\pi \times 14 \text{ cm}$$

$$6{,}6 = \frac{\theta}{360} \times 2 \times \frac{22}{7} \times 14$$

$$\theta = \frac{6{,}6 \times 360 \times 7}{2 \times 22 \times 14}$$

$$= 27$$

The arc subtends an angle of 27°.

9 A triangle is equal in ar... which measures 10 cm ... of the triangle is 12 cm...
10 A bicycle wheel ha... Calculate how m... round in travel... $3\frac{1}{7}$ for π.
11 Fig. 21.14 ... of $\triangle$PQ... QT = ...

$x = \frac{144}{360} \times 2\pi \times 5$
$= \frac{18}{45} \times 10\pi$
$= 4\pi$

The arc is 4π cm long.

Exercise 21d

Where necessary, use the value $3\frac{1}{7}$ for π.

1 In Fig. 21.34, each circle is of radius 6 cm. Express the length of the arcs, $l, m, n, \ldots, z$ in terms of π.

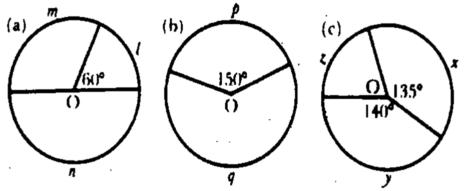


Fig. 21.34

2 Complete Table 21.1 for arcs of circles. Make a rough sketch in each case.

3 In terms of π, what is the length of an arc which subtends an angle of 30° at the centre of a circle of radius $3\frac{1}{2}$ cm?

4 What is the length of an arc which subtends an angle of 60° at the centre of a circle of radius $\frac{1}{2}$ m?

5 What angle does an arc 5,5 cm in length subtend at the centre of a circle of diameter 7 cm?

Table 21.1

	radius	angle at centre	length of arc
(a)	7 cm	90°	
(b)	35 m	72°	
(c)	4,2 cm	120°	
(d)	5,6 m	135°	
(e)	14 m		11 m
(f)	21 cm		22 cm
(g)		150°	330 cm
(h)		108°	132 cm

6 An arc of length 28 cm subtends an angle of 24° at the centre of a circle. In the same circle, what angle does an arc of length 35 cm subtend?

7 In Fig. 21.35, O is the centre of the circle.

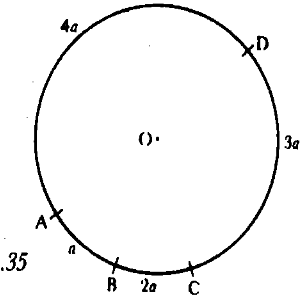


Fig. 21.35

Find the sizes of the following.
(a) A$\hat{Q}$B (b) A$\hat{D}$B (c) A$\hat{O}$D (d) A$\hat{C}$D
(e) B$\hat{C}$D (f) B$\hat{D}$C (g) C$\hat{B}$D (h) C$\hat{A}$D

8 In Fig. 21.36, 4 pencils are held together in a 'square' by an elastic band.

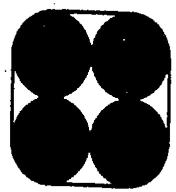


Fig. 21.36

If the pencils are of diameter 7 mm, what is the length of the band in this position?

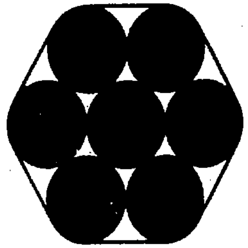


Fig. 21.37

The elastic band in question 8 is used to hold 7 of the same pencils as shown in Fig. 21.37. What is the length of the band in this position?

A piece of wire 22 cm long is bent into an arc of a circle of radius 4 cm. What angle does the wire subtend at the centre of the circle?

What angle does an arc of 10 cm subtend at the centre of a circle of radius 10 cm? Give the answer to the nearest 0.1°.

In Fig. 21.38, the radius of the circle is 10 cm. Calculate the length of the minor arc XY.

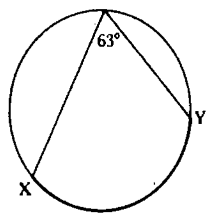


Fig. 21.38

13 The minute-hand of a clock is 6 cm long. How far does the end of the hand travel in 35 minutes?

14 A piece of string is wound tightly round a cylinder for 20 complete turns. The length of the string is found to be 3,96 m. Calculate the diameter of the cylinder in cm.

15 Fig. 21.39 shows a cross-section of a sheet of corrugated iron.

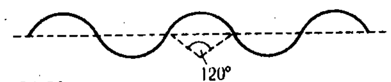


Fig. 21.39

The sheet is a series of arcs of radius 10 cm, each arc subtending 120° at its centre. If there are 14 such arcs in one sheet, how wide would the sheet be if flattened out?

16 In a circle of radius 6 cm a chord is drawn 3 cm from the centre. (a) Calculate the angle subtended by the chord at the centre of the circle. (b) Hence find the length of the minor arc cut off by the chord.

17 Fig. 21.40 shows a circular wire clip of radius 7 mm with a gap of 7 mm between the ends. Calculate the total length of wire in the clip to the nearest mm, given that about 4 mm are used altogether in making the turnovers at the ends.

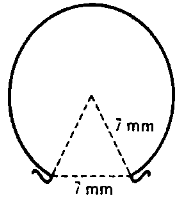


Fig. 21.40

18 Two friction wheels are of diameter 20 mm and 200 mm respectively. They touch at P and rotate without slipping (Fig. 21.41).

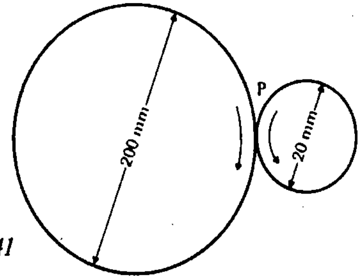


Fig. 21.41

Calculate the number of turns made by the small wheel when the large wheel rotates through 60°.

19 Water is taken from a well 11 m deep in a bucket, the rope winding onto a drum 35 cm in diameter.

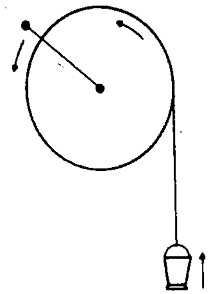


Fig. 21.42

(a) Through what angle does the handle turn in winding up 1 metre of rope? (b) How many revolutions of the handle does it take to bring the bucket up from the bottom? (c) If the arm of the handle is 42 cm long, how far does the hand of the winder travel when bringing the bucket up from the bottom?

20 In Fig. 21.43, AB is a chord of a circle of radius 10 cm, M is the mid-point of AB and MN ⊥ AB.

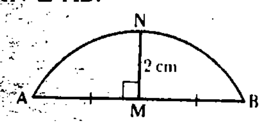


Fig. 21.43

If MN = 2 cm, calculate (a) AB, (b) the angle that arc ANB subtends at the centre of the circle, (c) the difference in length between AB and arc ANB to the nearest mm.

Area of sector

In Fig. 21.44, the area of sector AOB is $\frac{90}{360}$ or $\frac{1}{4}$ of the area of the whole circle. The area of sector COD is $\frac{45}{360}$ or $\frac{1}{8}$ of the whole circle and sector EOF is $\frac{30}{360}$ or $\frac{1}{12}$ of the whole circle. The area of a sector of a circle is proportional to the angle of the sector.

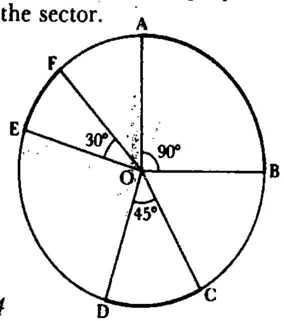


Fig. 21.44

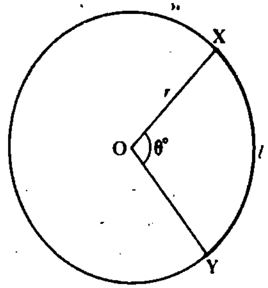


Fig. 21.45

In Fig. 21.45, the angle of the sector is $\theta°$. The area of the whole circle is πr^2. Therefore:

Area of sector XOY = $\frac{\theta}{360} \times \pi r^2$.

Example 9

A sector of 80° is removed from a circle of radius 12 cm. What area of the circle is left?

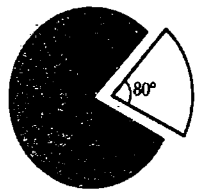


Fig. 21.46

Angle of sector left = 360° − 80° = 280°

Area of sector left $= \frac{280}{360} \times \pi \times 12^2$ cm^2

$= \frac{7}{9} \times \frac{22}{7} \times 12 \times 12$ cm^2

$= 352$ cm^2

Example 10

Calculate the area of the shaded segment of the circle shown in Fig. 21.47.

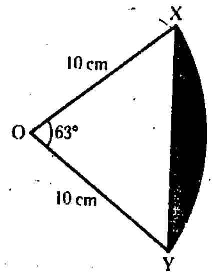


Fig. 21.47

Area of segment

= area of sector XOY − area of △XOY

$= \frac{63}{360} \times \pi \times 10^2 - \frac{1}{2} \times 10 \times 10 \times \sin 63°$ cm^2

$= \frac{7}{40} \times \frac{22}{7} \times 100 - \frac{1}{2} \times 100 \times 0{,}891\,0$ cm^2

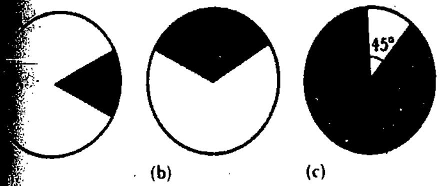
$= 55 - 44{,}55$ cm^2

$= 10{,}45$ cm^2

Exercise 21e

Take π to be $3\frac{1}{7}$ unless told otherwise.

1 In Fig. 21.48, each circle is of radius 6 cm. Calculate the areas of the shaded sectors in terms of π.



21.48

Complete Table 21.2 for areas of sectors of circles. Make a rough sketch in each case.

Table 21.2

	radius	angle of sector	area of sector
(a)	7 cm	90°	
(b)	6 cm	70°	
(c)	35 cm	144°	
(d)	14 m		462 m^2
(e)	2 cm		2,2 cm^2
(f)		140°	99 m^2

Calculate the area of a sector of a circle which subtends an angle of 45° at the centre of the circle, radius 14 cm.

4 The arc of a circle of radius 20 cm subtends an angle of 120° at the centre. Use the value 3,142 for π to calculate the area of the sector correct to the nearest cm^2.

5 The area of circle PQR with centre O is 72 cm^2. What is the area of sector POQ if $P\hat{O}Q = 40°$?

6 A pie chart is divided into four sectors as shown in Fig. 21.49. Each sector represents a percentage of the whole. The two larger sectors are equal and each represents x%. What is the angle subtended by one of those larger sectors?

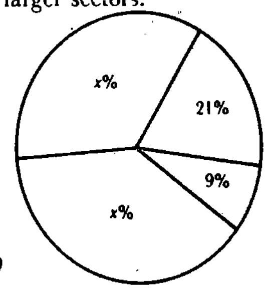


Fig. 21.49

7 Calculate the shaded parts in Fig. 21.50. All dimensions are in cm and all arcs are circular.

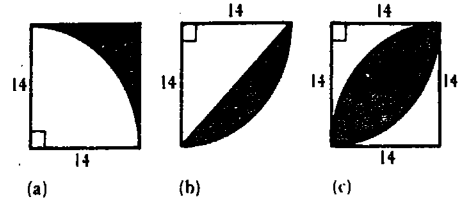


Fig. 21.50

8 Calculate the area of the shaded segment of the circle shown in Fig. 21.51.

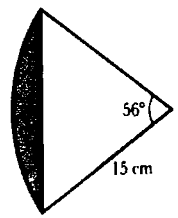


Fig. 21.51

9 In Fig. 21.52, ABCD is a rhombus with dimensions as shown. BXD is a circular arc, centre A. Calculate the area of the shaded section to the nearest cm^2.

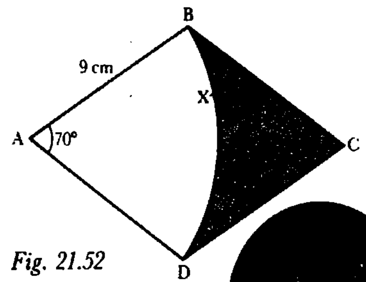


Fig. 21.52

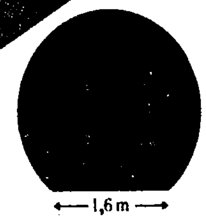


Fig. 21.53

10 Fig. 21.53 shows the cross-section of a tunnel. It is in the shape of a major segment of a circle of radius 1 m on a chord of length 1,6 m. Calculate (a) the angle subtended at the centre of the circle by the major arc correct to the nearest 0,1°, (b) the area of the cross-section of the tunnel, to 2 d.p.

Chapter 22

Simultaneous equations (2)

Simultaneous linear equations (Revision)

Example 1

Solve the simultaneous equations
$2x + 5y = 1,\ 3x - 2y = 30.$

$$\begin{aligned} 2x + 5y &= 1 \quad (1)\\ 3x - 2y &= 30 \quad (2)\\ (1)\times 3:\quad 6x + 15y &= 3\\ (2)\times 2:\quad 6x - 4y &= 60\\ \text{Subtract:}\quad 19y &= -57\\ \Leftrightarrow\quad y &= -3 \end{aligned}$$

Substitute -3 for y in (1)

$$\begin{aligned} 2x + 5(-3) &= 1\\ 2x - 15 &= 1\\ 2x &= 16\\ \Leftrightarrow\quad x &= 8 \end{aligned}$$

$x = 8$ and $y = -3$

Check: $2x + 5y = 16 - 15 = 1$
$3x - 2y = 24 - (-6) = 30$

Exercise 22a (Revision)

Solve the following pairs of simultaneous equations.

1 $2x - y = 8$, $3x + y = 17$

2 $2x + 3y = 10$, $2x + y = 2$

3 $a + 2b = 13$, $2a - 3b = 5$

4 $2m - n = 5$, $3m + 2n = -24$

5 $4x + 6y = 21$, $7x - 3y = 3$

6 $12x + 12y = -7$, $4x - 3y = 7$

7 $2a + 5b = 2$, $3a + 2b = 25$

8 $7r + 3s = -3$, $6r + 9s = 0$

9 $5x - 19 = 2y$, $3y + 18 = 4x$

10 $4a = 1 - 3b$, $4b = -1 - 6a$

Example 2

Solve the equations $\frac{1}{2}x + \frac{1}{3}y = 4,\ \frac{1}{4}y - \frac{1}{3}x = \frac{1}{6}.$

First simplify the equations by clearing fractions. Then solve in the usual way.

$$\begin{aligned} \tfrac{1}{2}x + \tfrac{1}{3}y &= 4 \quad (1)\\ \tfrac{1}{4}y - \tfrac{1}{3}x &= \tfrac{1}{6} \quad (2)\\ (1)\times 6:\quad 3x + 2y &= 24 \quad (3)\\ (2)\times 12:\quad 3y - 4x &= 2 \quad (4)\\ (3)\times 3:\quad 9x + 6y &= 72\\ (4)\times 2:\quad -8x + 6y &= 4\\ \text{Subtract:}\quad 17x &= 68\\ \Leftrightarrow\quad x &= 4 \end{aligned}$$

Substitute 4 for x in (3)

$$\begin{aligned} 12 + 2y &= 24\\ 2y &= 12\\ \Leftrightarrow\quad y &= 6 \end{aligned}$$

$x = 4$ and $y = 6$

Check: $\frac{1}{2}x + \frac{1}{3}y = 2 + 2 = 4$
$\frac{1}{4}y - \frac{1}{3}x = 1\frac{1}{2} - 1\frac{1}{3} = \frac{1}{6}$

Example 3

Solve the equations $\frac{2}{x} - \frac{1}{y} = 3,\ \frac{4}{x} + \frac{3}{y} = 16.$

Instead of using x and y as the unknowns, let the unknowns be $\left(\frac{1}{x}\right)$ and $\left(\frac{1}{y}\right)$.

$$\frac{2}{x} - \frac{1}{y} = 3$$

$$\frac{4}{x} + \frac{3}{y} = 16$$

$$2\left(\frac{1}{x}\right) - 1\left(\frac{1}{y}\right) = 3 \quad (1)$$

$$4\left(\frac{1}{x}\right) + 3\left(\frac{1}{y}\right) = 16 \quad (2)$$

$$(1)\times 3:\quad 6\left(\frac{1}{x}\right) - 3\left(\frac{1}{y}\right) = 9 \quad (3)$$

Add equations (2) and (3):

$$\text{Add:}\quad 10\left(\frac{1}{x}\right) = 25$$

$$\frac{1}{x} = \frac{25}{10} = \frac{5}{2}$$

$$\Leftrightarrow\quad x = \frac{10}{25} = \frac{2}{5}$$

Substitute $\frac{5}{2}$ for $\frac{1}{x}$ in (1)

$$5 - 1\left(\frac{1}{y}\right) = 3$$

$$\frac{1}{y} = 2$$

$$\Leftrightarrow\quad y = \frac{1}{2}$$

$x = \frac{2}{5}$ and $y = \frac{1}{2}$

Check: $\frac{2}{x} - \frac{1}{y} = \frac{2\times 5}{2} - \frac{1\times 2}{1} = 3$

$\frac{4}{x} + \frac{3}{y} = \frac{4\times 5}{2} + \frac{3\times 2}{1} = 16$

Example 4

Solve the equations
$2x - 3y + 2 = x + 2y - 5 = 3x + y.$

If three expressions are equal to each other, they can be grouped into three equations. For example, if $a = b = c$, then $a = b$, $a = c$ and $b = c$.

To solve the simultaneous equations, make two different equations from the given equality:

$$2x - 3y + 2 = 3x + y$$
$$x + 2y - 5 = 3x + y$$

Rearrange to give the unknowns on one side in alphabetical order.

$$\begin{aligned} -x - 4y &= -2 \quad (1)\\ -2x + y &= 5 \quad (2)\\ (1)\times 2:\quad -2x - 8y &= -4 \quad (3)\\ (2) - (3):\quad 9y &= 9\\ \Leftrightarrow\quad y &= 1 \end{aligned}$$

Substitute 1 for y in (1)

$$\begin{aligned} -x - 4 &= -2\\ -x &= 2\\ \Leftrightarrow\quad x &= -2 \end{aligned}$$

$x = -2$ and $y = 1$

Check: $2x - 3y + 2 = -4 - 3 + 2 = -5$
$x + 2y - 5 = -2 + 2 - 5 = -5$
$3x + y = -6 + 1 = -5$

Example 5

Find x and y if $3^{2x-y} = 1$ and $\frac{16^x}{4} = 8^{3x-y}$.

Express each equation as a linear equation.
Express $3^{2x-y} = 1$ in powers of 3.

$$3^{2x-y} = 3^0$$
$$\Leftrightarrow\quad 2x - y = 0 \quad (1)$$

Express $\frac{16^x}{4} = 8^{3x-y}$ in powers of 2.

$$\begin{aligned} \frac{2^{4x}}{2^2} &= 2^{3(3x-y)}\\ 2^{4x-2} &= 2^{9x-3y}\\ \Rightarrow\quad 4x - 2 &= 9x - 3y\\ \Leftrightarrow\quad 5x - 3y &= -2 \quad (2)\\ 2x - y &= 0 \quad (1)\\ 5x - 3y &= -2 \quad (2) \end{aligned}$$

From (1), $y = 2x$
Substitute $2x$ for y in (2)

$$\begin{aligned} 5x - 3(2x) &= -2\\ \Leftrightarrow\quad 5x - 6x &= -2\\ \Leftrightarrow\quad -x &= -2\\ \Leftrightarrow\quad x &= 2\\ \Leftrightarrow\quad y &= 2\times 2 = 4 \end{aligned}$$

$x = 2$ and $y = 4$

Check: $3^{2x-y} = 3^{4-4} = 3^0 = 1$

$\frac{16^x}{4} = \frac{16^2}{4} = 4\times 16 = 64$

$8^{3x-y} = 8^{6-4} = 8^2 = 64$

Exercise 22b

Solve the following pairs of equations.

1 $x - \frac{y}{2} = 1$, $\frac{x}{2} + \frac{y}{3} = 2\frac{5}{6}$

2 $x + \frac{y}{2} = \frac{1}{2}$, $\frac{x}{2} - \frac{y}{6} = 1\frac{1}{2}$

3 $3(x + y) = 7(y - x)$, $5(3x - y) = x + 3$

4 $7(a + b) = b - a$, $4(3a + 2b) = b - 8$

5 $f - 2g + 3 = 2f - 3g + 2 = 1$

6 $\frac{3}{a} + \frac{5}{b} = \frac{9}{a} - \frac{5}{b} = 1$

7 $1{,}5x - 0{,}7y = 0{,}1$, $0{,}3x + 1{,}1y = 2{,}5$

8 $2{,}3m + 1{,}8n = 5{,}1$
$0{,}9m + 2{,}4n = 0{,}3$

9 $\frac{2}{e} - \frac{3}{f} = 1$
$\frac{8}{e} + \frac{9}{f} = \frac{1}{2}$

10 $\frac{5x}{8} - \frac{y}{2} = \frac{1}{4}$
$\frac{2x}{3} - \frac{3y}{5} = \frac{2}{15}$

11 $\frac{1}{3}(m - 3n) = 2$
$\frac{m + n}{4} = \frac{1}{2}$

12 $3(3f + 2g) = 5 - f$
$4g + 5 = 2(g - 5f)$

13 $3(2x - y) = x + y + 5$
$5(3x - 2y) = 2(x - y) + 1$

14 $2a + 3b - 1 = 3a + b + 7 = a + 2b$

15 $2{,}32x + 1{,}44y = 15{,}6$
$4{,}8x - 1{,}92y = 2{,}88$

16 $\frac{4s}{3} + \frac{3t}{2} = 4$
$\frac{s}{2} + \frac{t}{4} + 1 = 0$

17 $\frac{3}{c} - \frac{4}{d} = \frac{1}{3}$
$\frac{2}{c} - \frac{5}{d} = 1$

18 $3p - 5q - 4 = 5p + 8 = 2p + q + 7$

19 $2^{x+2y} = 1{,}3^{2x+y} = 27$

20 $\frac{27^x}{81^{x+2y}} = 9,\ x + 4y = 0$

Word problems leading to simultaneous equations

Example 6

A two-digit number is such that the sum of its digits is 11. The number is 27 greater than the number obtained by interchanging the digits. Find the number.

Let the digits be x and y, where x is the tens digit.
Then the number is $10x + y$.
From the first sentence of the question,
$x + y = 11$ (1)
The number obtained by interchanging the digits is $10y + x$.
Hence, from the second sentence of the question,

$10x + y - (10y + x) = 27$
$\Leftrightarrow\ 10x + y - 10y - x = 27$
$\Leftrightarrow\ 9x - 9y = 27$
$\Leftrightarrow\ x - y = 3$ (2)
But, $x + y = 11$ (1)

Adding (1) and (2),
$2x = 14$
$\Leftrightarrow\ x = 7$
Subtract (2) from (1)
$-2y = -8$
$\Leftrightarrow\ y = 4$

The number is 74.
Check: $7 + 4 = 11$ (1st sentence)
$74 - 47 = 27$ (2nd sentence)

Example 7

A motorist travels for 30 km *at* x km/h *and for* 90 km *at* y km/h *and takes* $2\frac{1}{2}$ hours *for the journey. If the speeds are interchanged the journey takes* $2\frac{1}{6}$ hours. *Find* x *and* y.

Time taken to travel 30 km at x km/h $= \frac{30}{x}$ h

Time taken to travel 90 km at y km/h $= \frac{90}{y}$ h

Hence, from the first sentence of the question,

$$\frac{30}{x} + \frac{90}{y} = 2\tfrac{1}{2} \qquad (1)$$

Similarly, from the 2nd sentence of the question,

$$\frac{30}{y} + \frac{90}{x} = 2\tfrac{1}{6} \qquad (2)$$

Writing (1) and (2) with $\frac{1}{x}$ and $\frac{1}{y}$ as the unknowns,

$$30\left(\frac{1}{x}\right) + 90\left(\frac{1}{y}\right) = 2\tfrac{1}{2} \qquad (3)$$

$$90\left(\frac{1}{x}\right) + 30\left(\frac{1}{y}\right) = 2\tfrac{1}{6} \qquad (4)$$

$$\times 3:\quad 90\left(\frac{1}{x}\right) + 270\left(\frac{1}{y}\right) = 7\tfrac{1}{2} \qquad (5)$$

$$-\ (5):\quad -240\left(\frac{1}{y}\right) = -5\tfrac{1}{3} = -\tfrac{16}{3}$$

$$\frac{1}{y} = \frac{16}{3 \times 240} = \frac{1}{45}$$

$$\Leftrightarrow\ y = 45$$

bstitute $\frac{1}{45}$ for $\frac{1}{y}$ in (3)

$$30\left(\frac{1}{x}\right) + 2 = 2\tfrac{1}{2}$$

$$30\left(\frac{1}{x}\right) = \tfrac{1}{2}$$

$$\frac{1}{x} = \frac{1}{2 \times 30} = \frac{1}{60}$$

$$x = 60$$

$= 60$ and $y = 45$
heck: $\frac{30}{60} + \frac{90}{45} = \frac{1}{2} + 2 = 2\frac{1}{2}$ (1st sentence)
$\frac{30}{45} + \frac{90}{60} = \frac{2}{3} + 1\frac{1}{2} = 2\frac{1}{6}$ (2nd sentence)

xercise 22c

1 7 cups and 8 plates cost \$9,40. 8 cups and 7 plates cost \$9,65. Calculate the cost of a cup and of a plate.

2 Four knives and six forks cost \$2,06; six knives and five forks cost \$2,49. Find the cost of (a) a knife, (b) a fork.

3 Half of A's money plus one-fifth of B's make \$1. Two-thirds of A's money plus two-fifths of B's make \$1,50. How much money has each?

4 Divide 75 into two parts so that one part is $\frac{2}{3}$ of the other.

5 A woman cycles for x hours at 12 km/h and y hours at 18 km/h. Altogether she cycles 78 km in 5 hours. Find x and y.

6 Fig. 22.1 shows an equilateral triangle with the lengths of its sides given in terms of a and b.
Find a and b and hence find the length of the sides of the triangle.

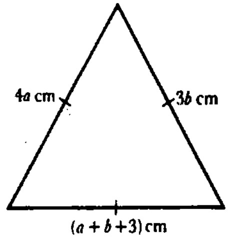


Fig. 22.1

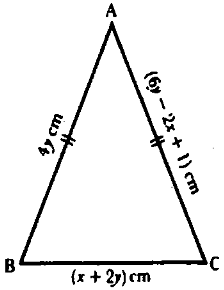


Fig. 22.2

7 The perimeter of the isosceles triangle in Fig. 22.2 is 28 cm.
Find x and y and hence state the lengths of the sides of the triangle.

8 If in Fig. 22.2, $\hat{A} = 2\frac{1}{4}n^\circ + \frac{1}{2}m^\circ$, $\hat{B} = 2n^\circ - \frac{2}{5}m^\circ$, $\hat{C} = 2m^\circ$, find m, n and the sizes of $\hat{A}$ and $\hat{B}$.

9 The difference between the digits of a two-digit number is 1. The number itself is 1 more than 5 times the sum of its digits. If the units digit is greater than the tens digit, find the number.

10 In a two-digit number, the sum of the digits is 8. The difference between this number and the number with the digits reversed is 54. What is the number?

11 In a positive number of two digits, the sum of the digits is 15. If the digits are interchanged, the number is increased by 9. Find the number.

12 A man's age and his son's add to 45 years. Five years ago the man was 6 times as old as his son. How old was the man when the son was born?

13 Chipo's and Tsitsi's ages add up to 29. 7 years ago Chipo was twice as old as Tsitsi. Find their present ages.

14 A girl travels 10 km in 50 min if she runs for 8 km and walks for 2 km. If she runs 4 km and walks 6 km, her time is 1 h 15 min. Find her running and walking speeds.

15 If 1 is added to both numerator and denominator of a fraction, the fraction becomes $\frac{1}{2}$. If 8 is added to both, the fraction becomes $\frac{2}{3}$. What is the fraction?

Chapter 23

General arithmetic (3) Ratio, rate, proportion

Ratio (Revision)

Example 1

If 57 : 95 = 12 : x, *evaluate* x.

If 57 : 95 = 12 : x,

then $\frac{57}{95} = \frac{12}{x}$

Clearing fractions,

$57x = 12 \times 95$

$$x = \frac{12 \times 95}{57} = \frac{95 \times 4}{19} = 5 \times 4 = 20$$

Example 2

The selling price of a second-hand car is originally \$8 190. *The dealer reduces the price in the ratio* 11 : 13. *What is the new selling price of the car?*

Let the new selling price be \$$x$.

Then, $\frac{x}{8\,190} = \frac{11}{13}$

$$\Leftrightarrow \quad x = \frac{11 \times 8\,190}{13}$$

$$= 11 \times 630 = 6\,930$$

The new price of the car is \$6 930.

Exercise 23a (Revision)

1 Express the following ratios in their simplest forms.
   (a) 14 : 21 (b) 25 to 15
   (c) 12 kg to 30 kg (d) 75 cm : 1 m
   (e) \$1,25 : 75c (f) $\frac{3}{4} : \frac{1}{2}$
   (g) $1\frac{1}{2}$ to $3\frac{1}{2}$ (h) 50 min to $1\frac{1}{4}$ h
   (i) 1 litre : 350 mℓ (j) 75 mm^2 to 4,5 cm^2

2 Complete the following ratios.
   (a) 9 : 24 = 3 : □
   (b) 4 : 9 = □ : 63
   (c) 25 : □ = 5 : 6
   (d) □ : 13 = 15 : 39
   (e) 12 : □ = 3 : 7
   (f) 24 : 8 = □ : 7
   (g) 15 : 20 = 18 : □
   (h) $\frac{30}{6} = \frac{\square}{9}$ (i) $\frac{\square}{8} = \frac{21}{24}$ (j) $\frac{4}{7} = \frac{20}{\square}$

3 Find the result of *increasing* the following quantities in the given ratios.
   (a) 16c in the ratio 7 : 4
   (b) 21 m in the ratio 10 : 7
   (c) 45 s in the ratio 5 : 3
   (d) 75 cm in the ratio 9 : 5
   (e) 91c in the ratio 9 : 7

4 Find the result of *decreasing* the following quantities in the given ratios.
   (a) \$1,25 in the ratio 3 : 5
   (b) 84 km in the ratio 6 : 7
   (c) $4\frac{1}{2}$ weeks in the ratio 5 : 9
   (d) 1 h 3 min in the ratio 7 : 9
   (e) 1,35 m^2 in the ratio 5 : 9

5 If 135 : x = 3 : 5, evaluate x.

6 Evaluate x if 45 : 72 = 40 : x.

7 A metal is composed of copper and zinc in the ratio 3 : 2 by volume. Find the volume of a piece of the metal which contains 42 cm^3 of copper.

8 As a result of inflation, a trader increases all prices in the ratio 19 : 17. What will be the new price of a watch marked at \$35,70?

9 When meat is cooked, its mass reduces in the ratio 7 : 11. A piece of uncooked meat has a mass of 2,53 kg. What mass is lost when the meat is cooked?

10 A year ago the daily sales of a newspaper averaged 16 320. Now the average daily sales are 28 560. Express the ratio of present sales to last year's sales in its simplest terms.

mparison of ratios

ratio can be expressed in the form n : 1, ere n is a whole number, a fraction or a cimal. This form is useful when comparing ios.

ample 3

ress the ratio \$5,70 : \$1,90 *in the form* n : 1.

$\frac{,70}{90} = \frac{5,7}{1,9} = \frac{57}{19} = \frac{57 \div 19}{19 \div 19} = \frac{3}{1}$

e ratio is 3 : 1.

ample 4

nd which ratio is the greater, 7 : 13 *or* 8 : 15.

$= \frac{7 \div 13}{13 \div 13} = \frac{0,538\ldots}{1}$

$= \frac{8 \div 15}{15 \div 15} = \frac{0,533\ldots}{1}$

o 3 d.p.,

13 = 0,538 : 1

15 = 0,533 : 1

e first ratio is greater.

e next example gives an alternative method r comparing ratios.

Example 5

Find which ratio is greater, 9 : 16 *or* 7 : 12.

The problem is to compare the ratio $\frac{9}{16}$ with the ratio $\frac{7}{12}$.

The LCM of 16 and 12 is 48.

$\frac{9}{16} = \frac{9 \times 3}{16 \times 3} = \frac{27}{48}$

$\frac{7}{12} = \frac{7 \times 4}{12 \times 4} = \frac{28}{48}$

i.e. 9 : 16 = 27 : 48

and 7 : 12 = 28 : 48

The second ratio is greater than the first.

For maps and plans, the scale is often given as a ratio in the form 1 : n. For example, if the scale is 5 cm to 1 km, 5 cm on the map represents 1 km on the ground.

5 cm : 1 km = 5 cm : 100 000 cm
= 1 : 20 000

The scale of the map is 1 : 20 000.

Notice that a scale of 1 : 2 is greater than a scale of 1 : 10, since $\frac{1}{2}$ is greater than $\frac{1}{10}$.

Example 6

Express the ratio 8 : 13 *in the form* 1 : n.

$8 : 13 = \frac{8}{8} : \frac{13}{8}$
$= 1 : 1{,}625$

Example 7

A plan is made of a school. It is found that the length of the laboratory, 15,6 *m, is represented on the plan by a line* 7,8 cm *long. Find the scale of the plan in the form* 1 : n.

$$\frac{\text{line on plan}}{\text{length on ground}} = \frac{7{,}8 \text{ cm}}{15{,}6 \text{ m}} = \frac{7{,}8 \text{ cm}}{1\,560 \text{ cm}} = \frac{78}{15\,600} = \frac{1}{200}$$

The scale of the plan is 1 : 200.

Exercise 23b

Give results correct to 3 s.f. where necessary.

1 Express the following ratios in the form n : 1.
   (a) 3 : 8 (b) 7 : 5
   (c) 14 to 3 (d) 4 to 7
   (e) \$29 to \$4 (f) 6 m to 8 m
   (g) 2,4 m : 1,5 m (h) 25 cm : 11 cm
   (i) 80c to \$2 (j) \$4,25 : \$2,50

2 Express the following ratios in the form 1 : n.
   (a) 4 to 11 (b) 9 : 13
   (c) 7 : 2 (d) 8 to 3
   (e) 2,5 m : 4 m (f) 1,5 g to 48 g
   (g) 0,5 cm to 6,6 m (h) 2,5 kg : 0,7 kg
   (i) 8 cm to 633,6 m (j) 5 cm : 1 km

3 Complete the following.
   (a) 15 : 4 = □ : 1
   (b) 8 : 5 = 1 : □

(c) 1 : 4,5 = 3 : □
(d) 6,2 : 1 = □ : 5
(e) 4 : 1 = 9 : □
(f) □ : 3,6 = 4 : 1
(g) □ : 1,6 = 1 : 4,8
(h) 2 cm : □ = 1 : 396
(i) □ : 36 kg = 1 : 1 800
(j) □ : 1 km = 1 : 4 000

4 Find which of the following pairs of ratios is greater.
(a) 18 : 5, 11 : 3
(b) 11 : 6, 13 : 7
(c) 5 : 8, 4 : 7
(d) 15 : 7, 13 : 6
(e) 1 : 7, 1 : 8
(f) 7 : 15, 8 : 17
(g) 17 : 6, 20 : 7
(h) 11 m : 13 m, 7 g : 8 g
(i) 1,5 m : 40 cm, $6\frac{1}{2}$ s : $1\frac{3}{4}$ s
(j) $1,70 : 90c, $3 : $1,60

5 Express the following scales in the form 1 : *n*.
(a) 1 cm represents 10 m
(b) 10 cm represents 1 km
(c) 4 cm represents 600 m
(d) 5 cm to 1,5 km
(e) 20 cm represents 5 km

6 Eggs in one shop are priced at 71c for 5 and in another at $1,65 a dozen. Which is cheaper?

7 A house covers a rectangle of ground, 15,7 m by 12,3 m. On the plan of the house, the length of the rectangle is 78,5 cm. What is the scale of the plan in the form 1 : *n*? Find the width of the house on the plan.

8 A map is drawn on a scale of 1 cm to 5 km.
(a) Find the scale of the map in the form 1 : *n*.
(b) Find, in centimetres, the distance on the map between Kadoma and Kwekwe (72 km).

9 A boy cycles 16 km in an hour and a girl runs 4,4 m in a second. Which is faster?

10 The mass of 47 cm³ of mercury is 638,73 g. The mass of 1 cm³ of water is 1 g. Find the ratio of the masses of equal volumes of mercury and water in the form *n* : 1. (This ratio is called the *specific gravity* of mercury.)

Rate

Ratios compare quantities which are of the *same kind.* For example 4 kg : 7 kg. The units may be different, e.g. 1 cm : 5 km, but the quantities are of the same kind, since it is always possible to express km in cm.

Quantities of *different* kinds may be connected in the form of a **rate**. The following are some examples of rates.

(a) A workman is paid $10,80 for an 8-hour day. His rate of pay is $1,35 per hour.
(b) A cyclist travels 28 km in 2 hours. Her rate is 14 km per hour. In this case, the rate is called **speed**.
(c) A 4-metre beam of uniform cross-section and mass 120 kg has a mass of 30 kg per metre. This rate gives the mass per unit length.
(d) A piece of metal has a volume of 20 cm³ and a mass of 180 g. Its **density** is 9 g/cm³. The density of gases, liquids and solids is the rate giving the mass per unit volume.
(e) A town of 32 000 people has an area of 40 km². The **population density** of the town is 800 people/km². Population density is a rate giving the average number of people per unit of area.

Example 8
Find, in km/h, *the rate at which a car travels if it goes* $38\frac{1}{2}$ km *in* 35 min.

In 35 min the car goes $38\frac{1}{2}$ km.

In 1 min the car goes $\frac{38\frac{1}{2}}{35}$ km

In 60 min the car goes $\frac{38\frac{1}{2}}{35} \times 60$ km

$= \frac{77 \times 60}{2 \times 35}$ km

$= \frac{11 \times 60}{2 \times 5}$ km

$= 66$ km

The rate (speed) of the car is 66 km/h.

Example 9
A village is roughly square in shape. Its perimeter is 6 km. If the population density of the village is 1 200 people/km², find the approximate population of the village.

Perimeter of village ≃ 6 km
Side of square ≃ $\frac{6}{4}$ km = $1\frac{1}{2}$ km
Area of village ≃ $1\frac{1}{2}$ km × $1\frac{1}{2}$ km
= $2\frac{1}{4}$ km²

Population density is 1 200 people/km²,
i.e. 1 km² contains 1 200 people.
Hence $2\frac{1}{4}$ km² contains 1 200 × $2\frac{1}{4}$ people

$= \frac{1\,200 \times 9}{4}$ people

= 300 × 9 people
= 2 700 people

Population of village ≃ 2 700 people.

Exercise 23c

1 A workman is paid $58 for a 40-hour week. Calculate his hourly rate of pay.

2 A car travels 153 km in $2\frac{1}{4}$ h. Calculate its average speed in km/h.

3 A steel beam 5,2 m long has a mass of 137,8 kg. Find its mass in kg/m.

4 If 42 cm² of sea water has a mass of 43,26 g, find its density in g/cm³.

5 A town has an area of 24 km² and a population of 31 000. Calculate the population density of the town per km² correct to 2 s.f.

6 A shop has a sale and reduces all its prices at the rate of 15c in the $. Find the sale price of an article marked at $7,40.

7 A car uses petrol at the rate of 1 litre for every 6,5 km travelled. How many litres does it use on a journey of 117 km?

8 A cask has a capacity of 20 litres. It contains wine of density 0,88 kg/litre. What is the mass of the wine?

9 A village is roughly in the shape of a rectangle $1\frac{1}{3}$ km by $1\frac{1}{4}$ km. What is its total population if the average density is 570 people per km²?

10 I travelled at 60 km/h and took 2 hours for a certain journey. How long would it have taken me if I had travelled at 50 km/h?

11 A woman is paid $22,62 for working $14\frac{1}{2}$ hours. Find her rate of pay per hour.

12 A bridge is 220 m long and has a mass of 11 220 tonnes. Find its mass in t/m.

13 A city has an area of 612 hectares and a population of 94 860. Calculate its population density in people/hectare.

14 Each week a man works from 8.00 a.m. to 12.30 p.m. on six days and from 2.00 p.m. to 5.30 p.m. on four days. His rate of pay is 96c per hour. What is his total wage?

15 If I can ride a bicycle at a rate of 5 m/s, how long will it take me to ride a distance of 12 km at the same rate?

16 A car uses petrol at the rate of 1 litre for every 11 km. If the price of petrol is 39c per litre, find the cost of the petrol for a journey of 891 km.

17 In 1991 a factory produced 9 324 bicycles. Allowing 2 weeks for holidays and a further 100 days for weekends, find the rate of production in bicyles per day.

18 In a town with a population of 53 280 there were 562 deaths in one year. Find the death rate per 1 000 persons correct to 3 s.f.

19 In the town of question 18, 613 babies were born in the same year. Find the birth rate per 1 000 persons. (3 s.f.)

20 A car journey takes $4\frac{1}{2}$ hours when the car travels at an average speed of 120 km/h. Calculate the time taken for the same journey when the average speed is reduced to 108 km/h. [Camb]

Proportional division

To divide a quantity into two parts which are in the ratio 2 : 5, first split it into 7 equal shares (since 2 + 5 = 7). The required parts will then be 2 and 5 of these equal shares, i.e. $\frac{2}{7}$ and $\frac{5}{7}$ of the original quantity.

Example 10
Divide $11,70 *between Bob and Sipho so that their shares are in the ratio* 8 : 5.

8 + 5 = 13

$\frac{1}{13}$ of \$11,70 = $\frac{1170}{13}$ cents = 90c

Bob's share = 8 × 90c = 720c = \$7,20
Sipho's share = 5 × 90c = 450c = \$4,50
Check: \$7,20 + \$4,50 = \$11,70

Example 11
Divide 299 into 3 parts in the ratio $\frac{1}{3} : \frac{5}{6} : \frac{3}{4}$.

$\frac{1}{3} : \frac{5}{6} : \frac{3}{4}$ = 4 : 10 : 9 (multiplying each by 12)
4 + 10 + 9 = 23
$\frac{1}{23}$ of 299 = $\frac{299}{23}$ = 13
$\frac{4}{23}$ of 299 = 4 × 13 = 52
$\frac{10}{23}$ of 299 = 10 × 13 = 130
$\frac{9}{23}$ of 299 = 9 × 13 = 117
Check: 52 + 130 + 117 = 299

Example 12
X, Y, Z *share* \$85 *so that for every* \$1 *that* X *gets,* Y *gets* \$3 *and for every* \$2 *that* Y *gets,* Z *gets* \$3. *Find* Y's *share.*

If X has 1 share, then Y has 3 shares. Z gets $1\frac{1}{2}$ times as much as Y. Hence Z gets $4\frac{1}{2}$ shares. They receive money in the ratio
1 : 3 : $4\frac{1}{2}$ = 2 : 6 : 9.
2 + 6 + 9 = 17
$\frac{1}{17}$ of \$85 = $\frac{85}{17}$ = \$5
Y gets $\frac{6}{17}$ of \$85
$\frac{6}{17}$ of \$85 = \$5 × 6
= \$30
Y's share is \$30.

Example 13
A *and* B *invest money in a business.* A *invests* \$5 250 *for* 4 months *and* B *invests* \$9 000 *for* 3 months. *How should they share the first year's profits of* \$2 720?

A's investment = \$5 250 × 4 'Dollar-months'
= 21 000 Dollar-months
B's investment = \$9 000 × 3 'Dollar-months'
= 27 000 Dollar-months
Their investments are in the ratio
21 000 : 27 000 = 7 : 9
7 + 9 = 16
Thus A should get $\frac{7}{16}$ of the profits and B should get $\frac{9}{16}$ of the profits.

$\frac{1}{16}$ of \$2 720 = \$170
$\frac{7}{16}$ of \$2 720 = 7 × \$170 = \$1 190
$\frac{9}{16}$ of \$2 720 = 9 × \$170 = \$1 530
A gets \$1 190 and B gets \$1 530.
Check: \$1 190 + \$1 530 = \$2 720

Notice in Example 13 that a new unit, a 'Dollar-month', is used. The investment of each person is the product of the amount of money invested and the amount of time for which it was invested. The shares should be in the ratio of this product.

Exercise 23d

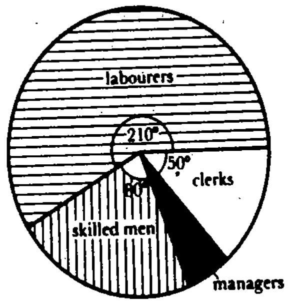
1 Divide the following quantities in the given ratios.
(a) 98c, 5 : 9 (b) \$5,76, 1 : 3 : 5
(c) 96 m, 3 : 4 : 5 (d) 56 kg, 5 : 3
(e) 153, 5 : 2 : 6 : 4 (f) \$33,76, 2 : 5 : 9
(g) 10,8 kg, 11 : 7 (h) 22,95 m, 5 : 12
(i) \$5,39, $1\frac{1}{3} : \frac{1}{2} : 2\frac{1}{4}$ (j) 28,6 kg, 4 : 5 : 6 : 7
2 Ann, Ben and Kudzai are aged 12, 14 and 20 respectively. They share \$6,90 in the ratio of their ages. How much does each get?
3 The cost \$2 320 of producing a machine arises from cost of materials, labour and overheads in the ratio 7 : 9 : 2. Calculate the cost of labour for producing 32 such machines.
4 X, Y, Z share \$68 so that for every \$1 that Z gets, Y gets \$2 and for every \$3 that Y gets, X gets \$4. How much does Y get?
5 If \$82 is divided among Rudo, Zodwa and Charles so that Rudo's share is $\frac{2}{3}$ of Zodwa's and Zodwa's is $\frac{3}{4}$ of Charles's share, how much does Charles get?
6 Two builders share 11 tonnes of bricks so that one has $1\frac{1}{2}$ times as many as the other. The total cost is \$227,50. How much does each pay?
7 Three people share 5 992 Dollars so that the first gets twice as much as the second and the second gets twice as much as the third. How much does the first get?
8 A, B and C share \$7,56 so that A has $2\frac{1}{2}$ times as much as C, and B has $3\frac{1}{2}$ times as much as C. Find their shares.

...pter 24

...atistics (3) Graphs

...erpretation of statistical graphs

...ample 1
...ie chart in Fig. 24.1 shows the divisions of the ...force of a factory.



...g. 24.1

(a) In its simplest terms, what fraction of the workforce are skilled men? (b) What percentage of the workforce are labourers? (c) Give the ratio of managers to clerks in its simplest terms. (d) If the factory employs 108 people, how many are managers?

(a) The angle of the sector representing skilled men is 80°.

$\frac{80}{360}$ of the workforce are skilled men.

$\frac{80}{360} = \frac{8}{36} = \frac{2}{9}$
$\frac{2}{9}$ of the workforce are skilled men.

(b) Fraction of the workforce which are

labourers = $\frac{210}{360}$

Percentage of the workforce which are labourers = $\frac{210}{360}$ × 100%

= $\frac{7 \times 100}{12}$% = $58\frac{1}{3}$%

(c) Angle representing managers
= 360° − (50 + 80 + 210)°
= 360° − 340° = 20°
Ratio of managers to clerks
= 20 : 50
= 2 : 5

(d) Number of managers = $\frac{20}{360}$ of 108
= $\frac{1}{18}$ × 108
= 6

It is not usual for pie charts to show the sizes of sectors as in Example 1. Where no angles are given, use a protractor if necessary.

Example 2
The bar chart in Fig. 24.2 shows the results of a spelling test.

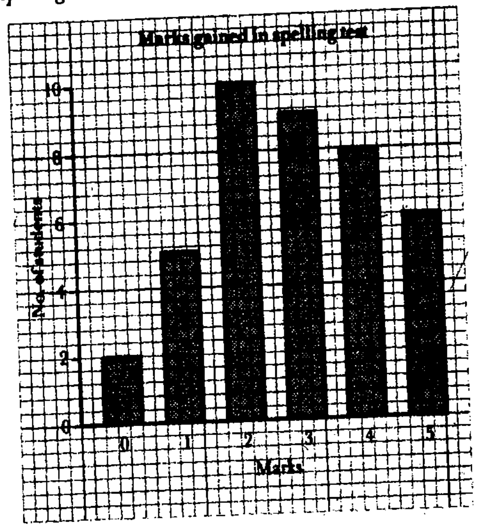


Fig. 24.2

(a) What was the range of marks? (b) How many students took the test? (c) What was the mode? (d) What was the median?

(a) Lowest mark scored = 0
Highest mark scored = 5
The range of marks is from 0 to 5.

(b) From the bars in the graph, it can be seen that 2 students scored 0 marks, 5 students scored 1 mark, and so on.

Number of students who took the test
$= 2 + 5 + 10 + 9 + 8 + 6 = 40$

(c) The mode is the score which occurred most often.
Most students scored 2 marks.
Modal score = 2 marks
(*Note:* the mode corresponds with the highest bar in the bar chart.)

(d) The median is the score obtained by the middle student, when the marks are arranged in order.
There are 40 students. The median is the average of the marks obtained by the 20th and 21st student.
The first two students scored 0 marks, the next 5 students scored 2 marks, and so on. Counting on in this way it can be seen that the 20th and 21st students both scored 3 marks.
Median score = 3 marks.

Exercise 24a

1 Fig. 24.3 shows the temperature of a patient recorded every hour.

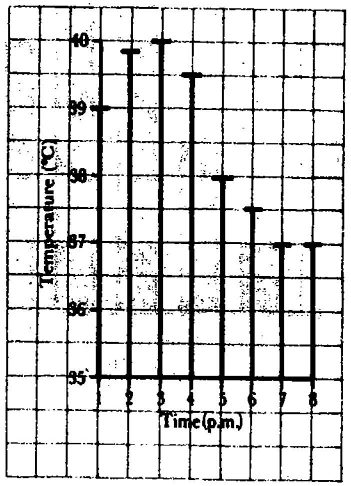


Fig. 24.3

(a) At what time was the highest temperature recorded?

(b) What was the patient's temperature at 6 p.m.?

(c) During which hour did the temperature rise the most?

(d) During which hour did the temperature fall the most?

2 Fig. 24.4 shows the mark distribution for a class test. The marks range from 3 to 9.

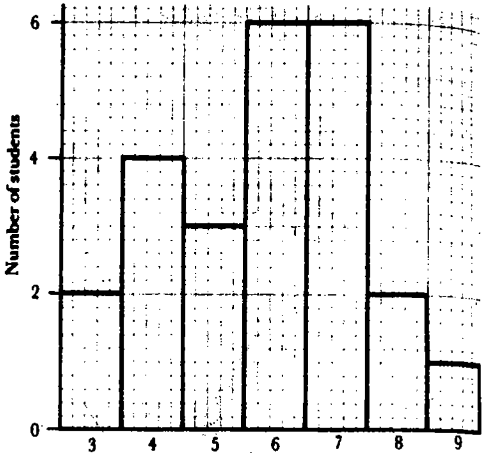


Fig. 24.4

(a) How many students took the test?

(b) What was the median mark for the test?

3 The pie chart in Fig. 24.5 represents 24 hours in the life of a student.

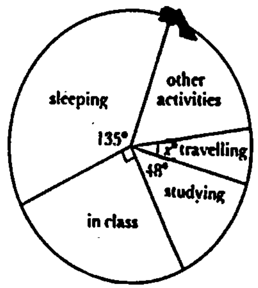


Fig. 24.5

(a) What fraction of the time is spent sleeping?

(b) What percentage of the time is spent studying?

(c) How much time is spent studying?

(d) If 1 h 20 min is spent travelling, calculate the value of x.

The examination results of a class are given by the bar chart in Fig. 24.6.

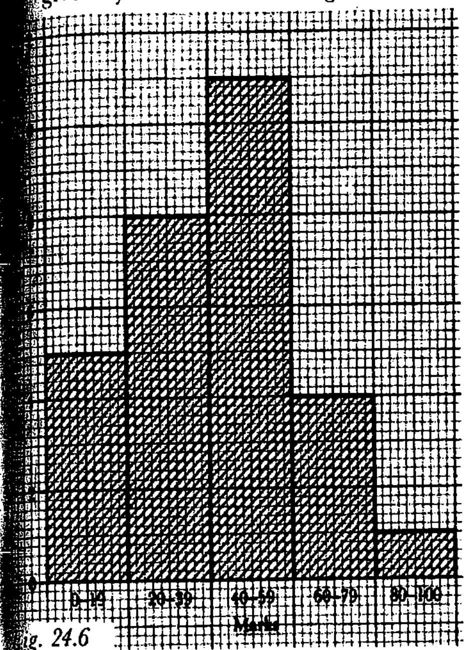


Fig. 24.6

(a) How many pupils took the examination?

(b) If the pass mark is 40, how many pupils passed the examination?

5 Fig. 24.7 shows the numbers of times the words *a*, *and*, *in*, *it*, *to* appear in a paragraph in a book.

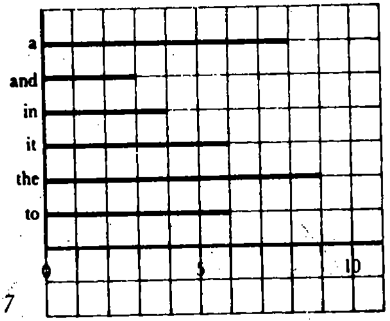


Fig. 24. 7

(a) How many times does the word *in* appear?

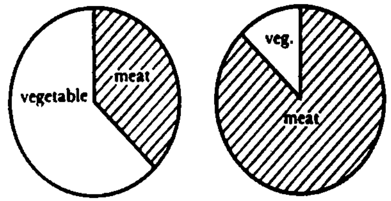
(b) Which word appears 8 times?

(c) Which two words appear the same number of times?

(d) Which is the most frequent word?

(e) What is the ratio of the numbers of appearances of the most frequent word to that of the least frequent word?

6 Fig. 24.8(a) shows the proportion, by mass, of meat to vegetables in a stew. Fig. 24.8(b) shows the relative costs of the meat and vegetables in the same stew.



(a) stew (by mass) (b) stew (by cost)

Fig. 24.8

(a) Use a protractor to find the ratio of the masses of meat to vegetables in the stew.

(b) How many grammes of meat are there in 2 kg of stew?

(c) Find the ratio of the costs of meat to vegetables in the stew.

(d) If it costs $7,44 to make the stew, how much money was spent on vegetables?

7 Fig. 24.9 shows the distribution of marks in a test.

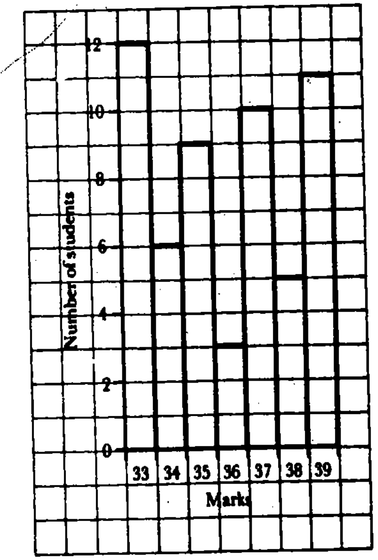


Fig. 24.9

(a) What is the range of the scores?
(b) How many students took the test?
(c) What is the modal score?
(d) What is the median score?

8 Fig. 24.10 shows the distribution of children in Classes 1–6 of a Primary School.

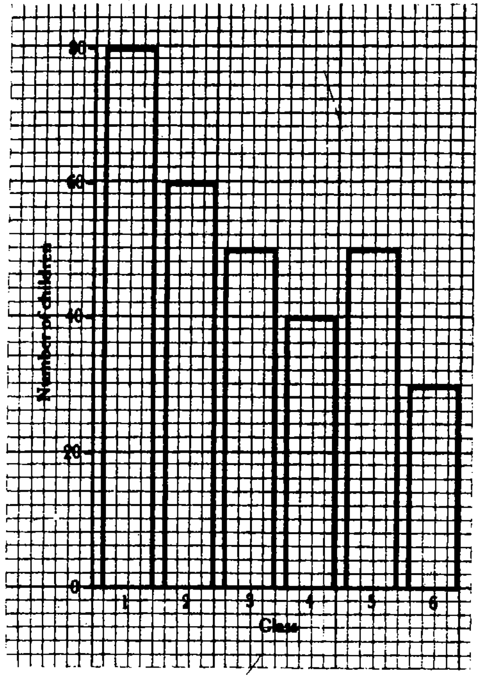


Fig. 24.10

(a) Which two classes have the same number of children?
(b) Which class has 40 children in it?
(c) How many children are in the school?
(d) At the beginning of the year, each child in classes 5 and 6 is given 4 exercise books and each child in the other classes is given 3 exercise books. How many exercise books are given out?

9 The pie chart in Fig. 24.11 shows the proportion of money that four people, A, B, C, D, invest in a business. The pie chart also shows how the yearly profit is divided.
(a) If B invests $4 800 how much did each of the others invest?
(b) In the first year the total profits were $6 432. How much did A receive?

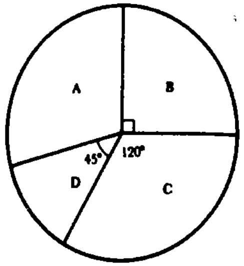


Fig. 24.11

10 Fig. 24.12 is a bar chart comparing the cost of a 1 carat diamond with the cost of 1 ounce of gold in five-year intervals from 1960 to 1980.

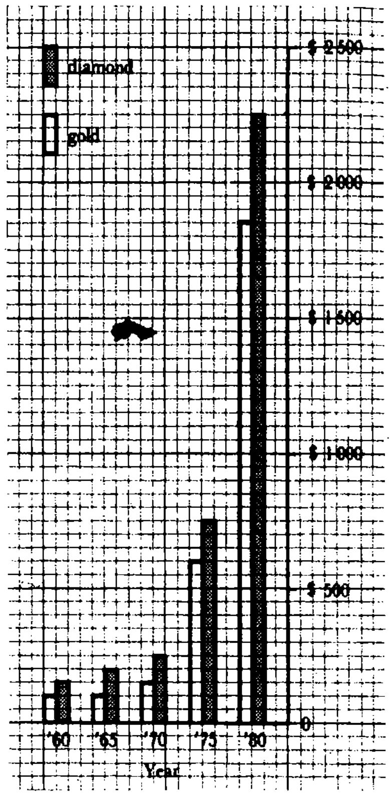


Fig. 24.12

(a) What was the cost of a 1 carat diamond in 1970?
(b) What was the cost of 1 ounce of gold in 1975?

(c) In 1980, what was the difference between the cost of a 1 carat diamond and the cost of 1 ounce of gold?
(d) In which year was the cost of a 1 carat diamond double that of 1 ounce of gold?
(e) What is the ratio of the cost of diamonds in 1970 to the cost in 1980?

rawing statistical graphs

ctogram

ample 3

e numbers of pupils in a secondary school in Harare the years 1988 to 1992 are as given in Table 24.1. aw a pictogram to represent the data.

ble 24.1

1988	1989	1990	1991	1992
120	520	680	720	720

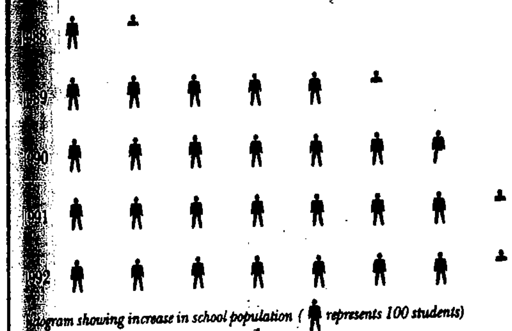


Fig. 24.13

Notes:
1 A pictogram represents data in the form of pictures.
2 A pictogram, although providing an easily understood graph of the data, is seldom accurate. It is difficult to represent fractions correctly.

Bar chart

Example 4

Table 24.2 shows the numbers of pupils in each Form 3 class in a school. Represent this information on a bar chart.

Table 24.2

Class	3_1	3_2	3_3	3_4
No of pupils	40	30	35	36

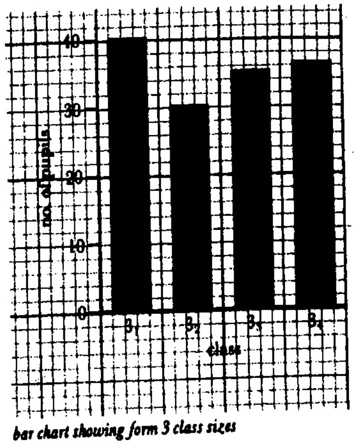


Fig. 24.14

Notes:
1 Data are represented by a series of bars. The bars are usually vertical but may sometimes be horizontal.
2 The height of the bars is proportional to the frequency of the data. The width of the bars is of no significance.

Pie chart

Example 5

A communal farmer has 30 head of cattle, 20 goats, 29 chickens and 11 sheep. Illustrate this distribution of livestock on a pie chart.

Each set of livestock is represented by a sector of a circle. To calculate the angles of the sectors, first find the total number of livestock. Express the number of livestock in each set as a fraction of the total, then find that fraction of 360°. The working is shown in Table 24.3 overleaf.

Table 24.3

Set of livestock	Number of livestock	Angle of sector
cattle	30	$\frac{30}{90} \times 360° = 120°$
goats	20	$\frac{20}{90} \times 360° = 80°$
chickens	29	$\frac{29}{90} \times 360° = 116°$
sheep	11	$\frac{11}{90} \times 360° = 44°$
total	90	360°

The angles calculated in Table 24.3 are then used to draw the pie chart in Fig. 24.15.

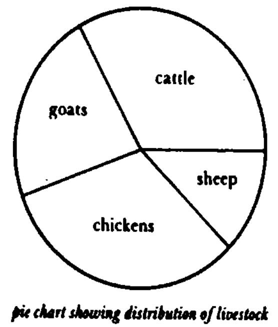


Fig. 24.15

Notes:

1 The angles of a pie chart should be drawn to the correct sizes. However, it is not necessary to enter the sizes of the angles on the graph.

2 Each sector should be labelled showing the information which relates to that sector.

3 As with all graphs, the pie chart should have a title.

Frequency polygon

Example 6

Table 24.4 shows the frequency distribution of examination marks for a class of 40 pupils.

Table 24.4

Marks	Frequency
30–39	4
40–49	3
50–59	12
60–69	10
70–79	6
80–89	5
	40 (total)

Represent the information in Table 24.4 on (a) a bar chart, (b) a frequency polygon.

(a) The information in Table 24.4 is grouped into **classes** or **intervals**. If every single mark were recorded, it would be difficult to draw an appropriate graph. Fig 24.16 shows a suitable bar chart.

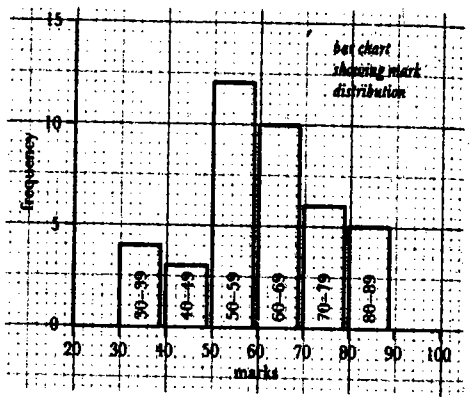


Fig. 24.16

(b) A **frequency polygon** is drawn by plotting the frequencies at the mid-points of the class intervals. Successive points are joined by straight lines. In this case the mid-points of the intervals correspond to examination marks of $34\frac{1}{2}$, $44\frac{1}{2}$, $54\frac{1}{2}$, ... and so on. Notice in Fig. 24.17 that the end points are joined to readings of $24\frac{1}{2}$ and $94\frac{1}{2}$ on the base line (since the frequencies of these readings are both zero).

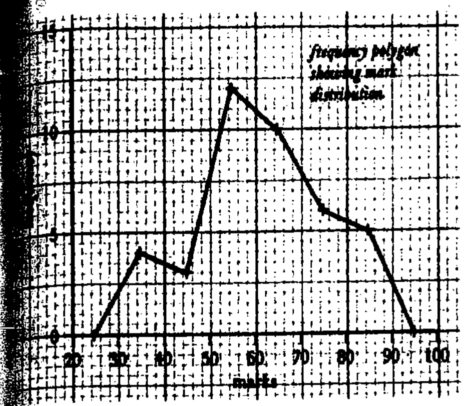


Fig. 24.17

Notes:

1 A frequency polygon can only be drawn when it is possible to represent the data on the horizontal axis by a continuous scale.

2 With grouped data, it is essential to define the limits of the groups very carefully so that the central values can be found.

Exercise 24b

1 Two farmers, Mr Zita and Mr Miti, doubled their tomato production in 1992 compared to 1991. They showed their production in the form of pictograms as in Fig. 24.18.

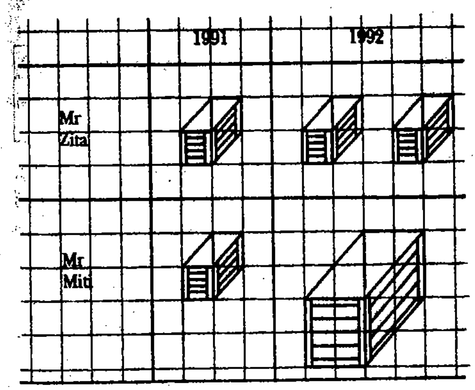


Fig. 24.18

Which of the two farmers used the most reasonable method of showing that his crop had doubled, and why?

2 In a class of 36 students, 9 cycle to school, 15 walk to school, 7 use buses and 5 come by car. Show this information on (a) a pictogram, (b) a bar chart, (c) a pie chart.

3 Table 24.5 gives the ages of some students who were born in April.

Table 24.5

Age (years)	13	14	15	16	17
Number of students	8	5	6	3	8

Show this information in a bar chart.

4 A trader buys 80 mangoes. She finds that 8 are bad and a further 22 are unripe. Show by means of a pie chart the proportions of bad mangoes, unripe mangoes and mangoes which are ready to eat.

5 Table 24.6 shows the mark distribution obained by a class in a test marked out of 24.

Table 24.6

Score	0–4	5–9	10–14	15–19
Frequency	4	5	15	6

Represent the data on (a) a bar chart, (b) a frequency polygon.

6 In a class of 30 pupils 15 drink Cola-Cola, 10 drink Fanta and 5 drink Pepsi-Cola. Construct a pie chart to show the data.

7 In one day a post office handled 50 parcels. Their masses to the nearest kg are given in Table 24.7.

Table 24.7

Mass (kg)	1	2	3	5	9
Number of parcels	14	20	12	3	1

(a) Represent this information in a suitable graph. (b) What is the modal mass?

8 A box of ballpoint pens contains 50 pens. 100 such boxes were examined for faulty pens (i.e. pens which would not write properly). Table 24.8 shows the results of the examination.

Table 24.8

Number of faulty pens	0	1	2	3	4	5	6
Number of boxes	7	10	12	20	19	18	14

(a) Draw a bar chart to show the information in the table.
(b) State the modal number of faulty pens.

9 Table 24.9 gives the budget estimation of a Local Government Authority for a year.

Table 24.9

Item	Amount ($)
agriculture	105 000
development	120 000
education	165 000
health	90 000
water supply	45 000
others	75 000

Represent this budget by means of a pie chart.

10 Table 24.10 is a frequency distribution table giving examination marks out of 100 for a class of 35 students.

Table 24.10

Marks	Frequency
30–39	3
40–49	8
50–59	14
60–69	6
70–79	4

Construct a frequency polygon to illustrate this information.

11 Table 24.11 gives the number of hours per week allotted to each subject taught at a technical school.

Table 24.11

Subject	Maths	Tech. drawing	Wood-work	Mech.	Science	Metal-work
Hours/wk	5	4	2	2	4	3

Present this information on a pie chart.

12 A mathematical set should contain 5 instruments: 2 set squares, 1 protractor, 1 ruler and 1 pair of compasses. A teacher checked the mathematical sets of 40 students and found that some of their instruments were missing. The teacher's results are given in Table 24.12.

Table 24.12

Number of missing instruments	0	1	2	3	4	5
Number of mathematical sets	16	10	7	4	3	0

(a) Represent the information in the table by an appropriate diagram.
(b) Find the mode and the median.

13 The approximate mean monthly rainfall for Harare is given in Table 24.13.

Table 24.13

Month	J	F	M	A	M	J	J	A	S	O	N	D
Rainfall (mm)	190	230	150	40	20	10	0	10	30	30	110	120

Show this information on a suitable graph.

14 The following is a list of marks obtained by a group of students in a test marked out of 100.

20	80	88	25	0	15	2	60	3
55	60	59	57	54	51	62	63	70
77	43	55	44	49	81	82	35	36

(a) Make a table showing the frequency distribution for the data grouped in intervals of 10 marks: 0–9, 10–19, ..., 90–99.
(b) Draw a frequency polygon which illustrates the data in your table.

Chapter 25

Similarity (2) Areas and volumes of similar shapes

Areas of similar shapes

Fig. 25.1 represents two similar rectangles, one 7 cm × 3 cm, the other 28 cm × 12 cm.

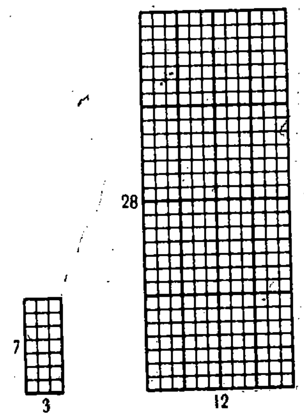


Fig. 25.1

Scale factor of the big rectangle to the small rectangle = ratio of their corresponding sides

$$= \frac{28 \text{ cm}}{7 \text{ cm}} \text{ or } \frac{12 \text{ cm}}{3 \text{ cm}}$$

= 4 in both cases

Area factor of the big rectangle to the small rectangle = ratio of their areas

$$= \frac{28 \times 12 \text{ cm}^2}{7 \times 3 \text{ cm}^2} = 4 \times 4$$

$$= 4^2$$

Hence the area factor is the *square* of the scale factor of the rectangles. $4^2 = 16$. Notice in Fig. 25.1 that the small rectangle fits 16 times into the big rectangle.

Fig. 25.2 represents two circles with radii 5 cm and 3 cm.

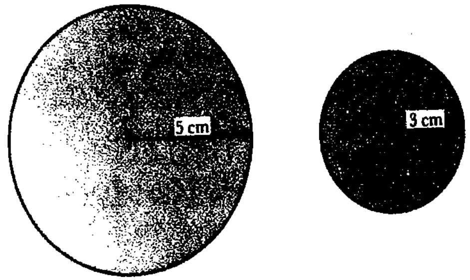


Fig. 25.2

Scale factor of the small circle to the big circle = ratio of the radii

$$= \frac{3 \text{ cm}}{5 \text{ cm}} = \frac{3}{5}$$

Area factor of the small circle to the big circle = ratio of their areas

$$= \frac{\pi \times 3^2 \text{ cm}^2}{\pi \times 5^2 \text{ cm}^2} = \frac{3^2}{5^2}$$

$$= \left(\frac{3}{5}\right)^2 \text{ or } \frac{9}{25}$$

Again, the ratio of the areas is the square of the scale factor of the given circles.

In general, the ratio of the areas of two similar figures is the square of the scale factor of the two figures.

Example 1

A map is drawn to a scale of 1 : 5 000. *On the map, a village has an area of* 6 cm². *Find the true area of the village in hectares.* (1 ha = 10 000 m²)

Scale factor = 5 000
Area factor = $(5\ 000)^2$
= 25 000 000

Area of village $= 25\,000\,000 \times 6\ \text{cm}^2$

$$= \frac{25\,000\,000 \times 6}{100 \times 100}\ \text{m}^2$$

$$= \frac{25\,000\,000 \times 6}{10\,000 \times 100 \times 100}\ \text{ha}$$

$$= \frac{25 \times 6}{100}\ \text{ha} = 1{,}5\ \text{ha}$$

Example 2

A woman uses 5 m^2 *and* 3,2 m^2 *of cloth when making similar dresses for herself and her daughter respectively. If the woman is* 165 cm *tall, how tall is the daughter?*

Assume that the woman and daughter are similar in build.

Area factor of daughter's dress to woman's dress

$$= \frac{3{,}2\ \text{m}^2}{5\ \text{m}^2} = \frac{32}{50} = \frac{16}{25} = \left(\frac{4}{5}\right)^2$$

Scale factor = square root of area factor = $\frac{4}{5}$

Height of woman = 165 cm
Height of daughter = $\frac{4}{5}$ of 165 cm

$$= \frac{4 \times 165}{5}\ \text{cm}$$

$= 4 \times 33$ cm
$= 132$ cm

Exercise 25a

1 Two similar rectangles have corresponding sides in the ratio 10 : 3. Find the ratio of their areas.
2 Two similar triangles have corresponding sides of length 4 cm and 7 cm. Find the ratio of their areas.
3 Two similar hexagons have corresponding sides of 2 cm and 5 cm. (a) Find the ratio of their areas. (b) If the area of the larger hexagon is 150 cm^2, find the area of the smaller one.
4 The ratio of the area of two circles is $\frac{4}{9}$. (a) Find the ratio of their radii. (b) If the smaller circle has a radius of 12 cm, find the radius of the larger one.
5 A map of Harare is drawn to a scale 1 : 50 000. On the map the airport covers an area of 8 cm^2. Find the true area of the airport in hectares.
6 A sports stadium covers an area of 6 hectares. Find the area in cm^2 of the sports stadium when drawn on a map of scale 1 : 5 000.
7 A photograph measuring 8 cm by 10 cm costs 68c. What will be the cost of an enlargement measuring 20 cm by 25 cm?
8 Two square floor tiles are made of the same material. One costs 68c and its edge is 30 cm long. Find the cost of the other if its edge is 50 cm long.
9 Two rectangular flags are similar in shape. Their areas are 5 m^2 and 0,8 m^2. If the height of the larger flag is 180 cm, find the height of the smaller flag.
10 The area of the windscreen of a bus is 1,21 m^2. In a photograph of the bus, the windscreen is a rectangle 12 cm by 3 cm. Find the length and breadth of the real windscreen.

Volumes of similar solids

Fig. 25.3 represents two similar cuboids, one 5 cm × 2 cm × 1 cm, the other 15 cm × 6 cm × 3 cm.

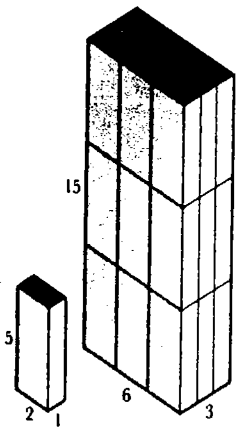


Fig. 25.3

Scale factor of the big cuboid to the small cuboid = ratio of their corresponding edges

$$= \frac{15\ \text{cm}}{5\ \text{cm}}\ \text{or}\ \frac{6\ \text{cm}}{2\ \text{cm}}\ \text{or}\ \frac{3\ \text{cm}}{1\ \text{cm}}$$

= 3 in each case

Volume factor of the big cuboid to the small cuboid = ratio of their volumes

$$= \frac{15 \times 6 \times 3\ \text{cm}^3}{5 \times 2 \times 1\ \text{cm}^3} = 27$$

$= 3^3$

Hence the volume factor is the *cube* of the scale factor of the cuboids. In Fig. 25.3 it should be possible to see that the small cuboid will fit 27 times into the big cuboid.

Fig. 25.4 represents two cylinders, one of height $2h$ and radius $2r$, the other of height h and radius r.

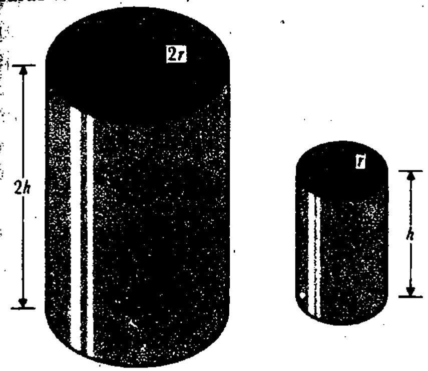


Fig. 25.4

Since the heights and radii are in the same ratio, the cylinders are similar.

Scale factor of the small cylinder to the big cylinder = ratio of corresponding lengths

$$= \frac{h}{2h}\ \text{or}\ \frac{r}{2r}$$

$= \frac{1}{2}$ in both cases

Volume factor of the small cylinder to the big cylinder = ratio of their volumes

$$= \frac{\pi r^2 h}{\pi (2r)^2 2h} = \frac{1}{8} = \left(\tfrac{1}{2}\right)^3$$

Again, the ratio of the volumes is the cube of the scale factor of the two shapes.

In general, the ratio of the volumes of similar solids is the cube of the scale factor of the two solids.

Example 3

Two pots, similar in shape, are respectively 21 cm *and* 14 cm *high. If the smaller pot holds* 1,2 litres, *find the capacity of the larger one.*

Scale factor $= \frac{21}{14} = \frac{3}{2}$

Thus, volume factor $= \left(\frac{3}{2}\right)^3 = \frac{27}{8}$

The smaller pot holds 1,2 litres
The larger pot holds $1{,}2 \times \frac{27}{8}$ litres
$= 0{,}15 \times 27$ litres
$= 4{,}05$ litres

Example 4

Two heaps of rice are of similar shape and contain 128 kg *and* 250 kg *of rice respectively. If the height of the bigger heap is* 70 cm, *find the height of the smaller one.*

Mass is proportional to volume, thus,
ratio of volumes = ratio of masses

$$= \frac{128}{250} = \frac{64}{125} = \frac{4^3}{5^3} = \left(\frac{4}{5}\right)^3$$

Scale factor = $\frac{4}{5}$
Height of bigger heap = 70 cm
Height of smaller heap = $\frac{4}{5}$ of 70 cm
$= 4 \times 14$ cm
$= 56$ cm

Exercise 25b

1 Two similar cups have heights in the ratio 2 : 3. Find the ratio of their capacities.
2 Two similar blocks have corresponding edges of length 10 cm and 20 cm. Find the ratio of their masses.
3 A soap bubble 4 cm in diameter is blown out until its diameter is 8 cm. By what ratio has the volume of air in the bubble increased?

4 Two metal bolts are similar in shape and have diameters of 5 mm and 15 mm. (a) Find the ratio of their masses. (b) If the smaller bolt's mass is 12 g, find the mass of the larger bolt.
5 Two similar buckets hold $13\frac{1}{2}$ litres and 4 litres respectively. (a) Find the ratio of their heights. (b) If the larger bucket is of height 36 cm, find the height of the smaller bucket.
6 Two similar pots have heights of 16 cm and 10 cm. If the smaller pot holds 0,75 litres, find the capacity of the larger pot.
7 A sports trophy is in the shape of a cup 30 cm high. The winners are each given copies of the cup, $7\frac{1}{2}$ cm high. If one of the copies holds 100 mℓ, find the capacity of the trophy in litres.
8 A tin of beans costs 35c. How much would a similar tin, 3 times the height and diameter, full of beans cost?
9 A pencil manufacturer makes a giant model pencil, 3 m long, as a factory symbol. If a real pencil is 18 cm long and has a volume of 9 cm^3, find the volume in m^3 of the giant model.
10 A builder makes a scale model of a real house. The volumes of air in the scale model and the real house are 27 500 cm^3 and 220 m^3 respectively. If the height of the door in the real house is 2,4 m, find the height of the door in the scale model.

Example 5

Two similar tins contain 960 g and 405 g of margarine respectively. If the area of the base of the larger tin is 120 cm^2, find the area of the base of the smaller tin.

$$\text{Ratio of volumes} = \text{ratio of masses} = \frac{960}{405} = \frac{64}{27} = \frac{4^3}{3^3} = \left(\frac{4}{3}\right)^3$$

Thus, scale factor = $\frac{4}{3}$

and the area factor = $\left(\frac{4}{3}\right)^2 = \frac{16}{9}$

Area of larger base = 120 cm^2
Area of smaller base = $\frac{9}{16}$ of 120 cm^2
= $67\frac{1}{2}$ cm^2

Notice in Example 5 that the scale factor must be found before the areas can be compared.

Exercise 25c (Mixed practice)

1 A metal tray measuring 40 cm by 30 cm costs $1,92. What should be the price of a tray which is similar but 50 cm long?
2 Two plastic cups are similar in shape and their heights are 7,5 cm and 12,5 cm. If the plastic needed to make the first cost 54c, find the cost for the second.
3 In question 2, if the second cup holds $1\frac{7}{8}$ litres, find the capacity of the first cup in mℓ.
4 A cylindrical oil drum 70 cm long is made of sheet metal which costs $5,75. Find the cost of the metal for a similar oil drum 84 cm long.
5 A statue stands on a base of area 1,08 m^2. A scale model of the statue has a base of area 300 cm^2. Find the mass of the statue (in tonnes) if the scale model is of mass 12,5 kg.
6 A railway engine is of mass 72 tonnes and is 11 m long. An exact scale model is made of it and is 44 cm long. Find the mass of the model.
7 In question 6, if the tanks of the model hold 0,8 litres of water, find the capacity of the tanks of the railway engine.
8 A garden has an area of 3 025 m^2, and it is represented on a map by an area of 16 cm^2. (a) Find the scale of the map. (b) Find the true length of a wall which is represented on the map by a line 2,8 cm long.
9 In a scale drawing of a school grounds, a path 120 cm wide is shown to be 15 mm wide.
(a) Find the scale of the drawing.
(b) Find the area of the school grounds if the corresponding area on the plan is 2 025 cm^2.
10 A model car is an exact copy of a real one. The windscreen of the model measures 35 cm by 10 cm and the real car has a windscreen of area 0,315 m^2. If the mass of the model is 25 kg, find the mass of the real car.

Chapter 26

Probability (1)

Experimental probability

A farmer asks, 'Will it rain this month?'
The answer to the farmer's question depends on three things: the month, the place where the farmer is, and what has happened in the past in that month at that place. Table 26.1 gives some answers to the question for different places and months.

Table 26.1
Will it rain this month?

place	month	answer to question
Chirundu	January	yes
Hwange	July	no
Mutare	March	yes
Gwanda	September	maybe

Is it possible to give a more accurate answer to a farmer in Gwanda in September? It is known that an average of 10 mm of rain falls in Gwanda in September. However, this is an average found by keeping records over 12 years. The actual rainfall for Gwanda in September over the 12 years was as follows.

18 mm	0 mm	17 mm	9 mm
11 mm	22 mm	14 mm	0 mm
16 mm	0 mm	7 mm	6 mm

From the above data it can be seen that rain fell in 9 of the 12 months of September. If future years follow the pattern of the past it is likely that Gwanda will get rain in 9 out of the next 12 Septembers. We say that the **probability** of rain falling in Gwanda in September is $\frac{9}{12}$ (or $\frac{3}{4}$ or 0,75). This probability can never be exact. However, it is the best measure we can give from the data we have. The number $\frac{9}{12}$ is based on experimental records. It is an example of **experimental probability**.

Example 1

A girl writes down the numbers of male and female children of her mother and father. She also writes down the numbers of male and female children of her parents' brothers and sisters. Her results are shown in Table 26.2.

Table 26.2

	number of children	
	male	female
mother and father	2	5
mother's brothers	6	8
mother's sisters	4	8
father's brothers	5	8
father's sisters	7	7
Totals	24	36

(a) Find the experimental probability that when the girl has children of her own, her first born will be a girl.
(b) If the girl eventually has 10 children, how many are likely to be male?

(a) In the girl's family there is a total of 60 children. 36 of these are female. If the girl's own children follow the pattern of her family, then the experimental probability that her first born will be a girl is $\frac{36}{60} = \frac{3}{5}$.
(b) Following the family pattern, $\frac{3}{5}$ of the girl's children will be female and $\frac{2}{5}$ will be male.
Number of male children that the girl is likely to have = $\frac{2}{5}$ of 10 = 4.

Notice that the results in Example 1 are based on experimental probability. Thus we are using the past to predict the future. Events can easily turn out differently. The answers in Example 1 are no more than calculated guesses.

Exercise 26a (Class discussion)

1 A woman has 4 children. They are all boys. The children of the rest of her family are equally divided between males and females. What is the woman's next child likely to be, a boy or a girl?

2 A soothsayer throws some chicken bones on the ground. From the pattern of the bones, she says that rain will fall next week. Is this a good method? Does it always work? Compare this method with the use of rainfall records. Can rainfall records always tell us when rain will fall?

Exercise 26b (Experiments)

1 A bottle top rests in either a cup or a cap position. See Fig. 26.1.

Fig. 26.1 cup position cap position

Find a bottle top and drop it 100 times on your desk. Count the number of times it lands in a cup position and the number of times it lands in a cap position. Record your results in a tally table:

Table 26.3

cup position	卌 II
cap position	卌 IIII

Complete the following.
(a) the bottle top landed in the cup position ... times out of 100 throws. The experimental probability that a bottle top will land in the cup position is ....

Compare your results with other people's results. Give some reasons why your results may be different.

2 A drawing pin either rests with its point up or its point down. See Fig. 26.2.

Fig. 26.2 point up point down

(a) Repeat the experiment in question 1 using a drawing pin.
(b) Find the experimental probability that the drawing pin will land point up if dropped on the floor.
(c) Compare your results with other people.

3 A coin, when tossed, either lands with its head up or its tail up.
(a) Repeat experiment 1 using a coin.
(b) Find the experimental probability that if a coin is tossed it will land tails up.
(c) Compare your results with other people.

4 Cut four pieces of string so that three are 15 cm long and one is 10 cm long. Hold all four pieces in your hand so that all four lengths look the same (Fig. 26.3).

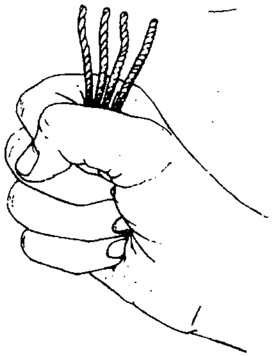


Fig. 26.3

Ask a friend to choose one piece. Write down whether a long piece or the shorter piece is chosen. Put the string back, mix them up and ask someone else. Repeat 20 times.

Find the experimental probability that someone will choose the short piece of string.

5 Draw a circle of radius 5 cm on a large sheet of paper. Get 10 paper clips. Hold the paper clips about 30 cm above the centre of the circle and drop them onto the paper.

Count the number of paper clips inside the circle and the number outside the circle. (If a paper clip falls on the circumference, count it as being inside the circle.)

In Fig. 26.4 there are 6 paper clips inside the circle and 4 outside. The results can be recorded as shown in Table 26.4.

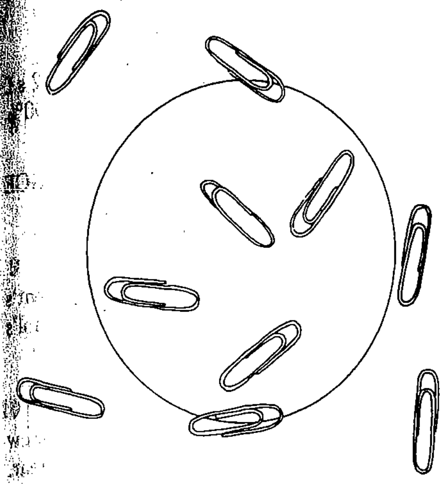


Fig. 26.4

Table 26.4

number of paper clips inside	6
number of paper clips outside	4

Repeat 10 times. Find the experimental probability that a paper clip dropped from a height of 30 cm will stay within 5 cm of the point where it falls.

6 Write down the numbers of male and female children in your family. Follow the method of Example 1. Hence find the experimental probability that your first-born child will be a boy.

7 Make a survey of the first 10 vehicles that pass your school gates. How many were cars? Use your result to predict how many of the next 10 vehicles will be cars. Check your prediction on the next 10 vehicles.

8 Open this book at any page. Read the right-hand page number. Write down whether the page number includes a 5 or not.

Repeat 50 times, recording your results as shown in Table 26.5.

Table 26.5

page number has a 5 in it	
page number has no 5s in it	

Find the experimental probability that if this book is opened anywhere, the right-hand page number will have a 5 in it.

9 Number the faces of a hexagonal pencil from 1 to 6 (Fig. 26.5).

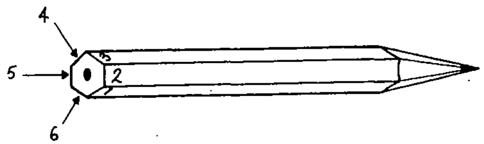


Fig. 26.5

Roll the pencil across your desk. Make a note of the number on the top face when it stops. Repeat 100 times.

How many times did you roll a number 4? What is the experimental probability of rolling a number 4?

Probability as a fraction

Probability is a measure of the likelihood of a **required outcome** happening. It is usually given as a fraction:

$$\text{probability} = \frac{\text{number of required outcomes}}{\text{number of possible outcomes}}$$

In Example 1, the required outcomes were female children and the possible outcomes were both male and female children. Thus, probability of having a female child

$$= \frac{\text{number of female children}}{\text{number of male and female children}}$$

$$= \frac{36}{60} = \frac{3}{5} = 0{,}6$$

If we are completely sure that something will happen, the probability is 1. For example, if today is Tuesday, the probability that tomorrow is Wednesday is 1.

If we are sure that something *cannot* happen, the probability is 0. For example, the probability of rolling a 7 on the pencil in Fig. 26.5 is

0, because there is no number 7 on the pencil.

If the probability of something happening is x, then the probability of it *not* happening is $1 - x$. For example, if the probability of it raining next month is $\frac{9}{12}$, then the probability of it *not* raining is $\frac{3}{12}$.

Example 2

It is known that out of every 1 000 new cars, 50 develop a mechanical fault in the first 3 months. What is the probability of buying a car that will develop a mechanical fault within 3 months?

Number of cars developing faults = 50
Number of cars altogether = 1 000
Probability of buying a faulty car = $\frac{50}{1000} = \frac{1}{20}$

Example 3

A market trader has 100 oranges for sale. 4 of them are bad. What is the probability that an orange chosen at random is good?
['At random' means 'without carefully choosing'.]

Either:
4 out of 100 oranges are bad,
thus 96 out of 100 oranges are good.
Probability of getting a good orange
$= \frac{96}{100} = \frac{24}{25}$
or:
Probability of getting a bad orange $= \frac{4}{100} = \frac{1}{25}$
thus,
probability of getting a good orange $= 1 - \frac{1}{25}$
$= \frac{24}{25}$

Example 4

City School enters candidates for the GCE. The results for the years 1988 to 1992 are given in Table 26.6.

Table 26.6

year	1988	1989	1990	1991	1992
number of candidates	86	93	102	117	116
number gaining GCE passes	51	56	57	65	70

(a) Find the school's success rate as a percentage.
(b) What is the approximate probability of a student at City School getting a GCE pass?

(a) Total number of passes
$= 51 + 56 + 57 + 65 + 70$
$= 299$

Total number of candidates
$= 86 + 93 + 102 + 117 + 116$
$= 514$

Success rate as a fraction $= \frac{299}{514} = 0{,}58$ to 2 s.f.
Success rate as a percentage $= 0{,}58 \times 100\%$
$= 58\%$

(b) The probability of a student getting a GCE pass $\simeq 0{,}58$
$= 0{,}6$ to 1 s.f.

In part (b) it is assumed that the student's chances of success are the same as the school's success rate.

Exercise 26c

1 Statistics show that 4 out of every 100 new radios break down within the first year. What is the probability of buying a radio which does not break down in the first year?

2 It has rained on 5th June 18 times in the last 20 years. What is the probability that it will rain on 5th June next year?

3 The midday temperatures during a week were 26 °C, 26 °C, 27 °C, 27 °C, 26 °C, 27 °C, 27 °C. What is the probability that the midday temperature on the next day will be (a) 2 °C, (b) 35 °C, (c) 26 °C, (d) 27 °C?

4 A matchbox contains 15 used sticks and 25 unused sticks.
(a) How many sticks are in the box altogether?
(b) What is the probability that a stick chosen at random is unused?

5 A statistical survey shows that 4 out of every 10 women wear a size 16 dress. What is the probability that a woman chosen at random does not wear a size 16 dress?

6 An advertisement says, '7 out of every 10 people prefer Red Ring margarine.' 50 people were asked which margarine they preferred. If the advertisement is true, approximately how many people will say Red Ring?

A trader has 100 mangoes for sale. 20 of them are unripe. Another 5 of them are bad. If a mango is picked at random, find the probability that it is (a) unripe, (b) bad, (c) neither unripe nor bad?
If 20 of the mangoes were chosen at random, how many would you expect to be (d) unripe, (e) bad.

8 It is known that 1 in 40 of the light bulbs sold by a certain trader is faulty. If one bulb is taken at random from a large number, what is the probability of it being a good one?

9 Nda and Ebenezer play table tennis together. They have already played 10 games and Nda has won 9 of them. What is the probability that he will win the 11th game?

10 Given the data of question 9, Ebenezer wins the 11th and 12th games. What is the probability that he will win the 13th game?

11 A crate contains 15 bottles of Coke and 9 bottles of Sprite. If I choose a bottle at random, what is the probability that it is (a) Coke, (b) Sprite, (c) either a Coke or a Sprite, (d) neither Coke nor Sprite?

12 20 cards are numbered from 1 to 20. A card is chosen at random. What is the probability that it does *not* have the digit 1 in its number?

13 Table 26.7 shows the numbers of pupils getting a place in secondary school from Northside Primary School for the years 1988 to 1992.

Table 26.7

	1988	**1989**	**1990**	**1991**	**1992**
number of pupils leaving Northside Primary School	54	58	60	63	65
number of pupils gaining a secondary school place	25	26	31	28	34

(a) Find Northside Primary School's success rate as a percentage.
(b) Find the approximate probability that a pupil chosen at random from Northside Primary School will gain a secondary school place.

14 Table 26.8 gives the results of a traffic survey on a city road one morning. The table shows the total number of vehicles per hour and the numbers of those that were cars and lorries.

Table 26.8

	total number of vehicles	**number of cars**	**number of lorries**
0800–0900	46	28	3
0900–1000	37	14	10
1000–1100	32	13	14
1100–1200	35	20	12

(a) For the whole morning, find the total number of vehicles, cars and lorries.
(b) Find the percentage of the vehicles that were cars.
(c) Find the percentage of the vehicles that were lorries.
(d) Find the probability that the next vehicle to come along the road is a car.
(e) Of the next 20 vehicles on the road, how many would you expect to be lorries?

Theoretical probability

It is possible to calculate probabilities without doing experiments or keeping records. Consider the following.

Coin tossing

If a coin is tossed, there are only two possible outcomes: a head or a tail. Each is equally likely. The probability of getting a head is $\frac{1}{2}$. The probability of getting a tail is $\frac{1}{2}$. Since these values can be calculated without throwing any coins they are called **theoretical probabilities**.

Die throwing

When a die like that in Fig. 26.6 is thrown, any one of six numbers will come out on top.

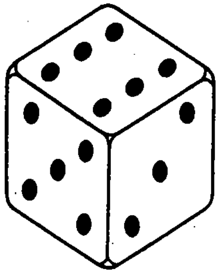


Fig. 26.6

Each number is equally likely. Hence the theoretical probability of getting a particular number is $\frac{1}{6}$. For example, the theoretical probability of throwing a 4 is $\frac{1}{6}$.

Theoretical probability is based on the fact that the coin and the die are fair. Each outcome is equally likely.

Example 5

A die is thrown. Find the probability that the outcome is divisible by 3.

There are 6 possible outcomes: 1; 2; 3; 4; 5; 6
Two of these are divisible by 3: 3 and 6

$$\text{Probability} = \frac{\text{number of required outcomes}}{\text{number of possible outcomes}} = \frac{2}{6} = \frac{1}{3}$$

When a die is thrown there is a $\frac{1}{3}$ probability that the outcome will be divisible by 3.

Probability can also be defined in terms of the language of sets:

$$p(R) = \frac{n(R)}{n(\mathscr{E})}$$

where p(R) is the probability of a required outcome happening,
R = {required outcomes} and $\mathscr{E}$ is the universal set, {possible outcomes}. In Example 5:

$\mathscr{E}$ = {possible outcomes} = {1; 2; 3; 4; 5; 6}
R = {required outcomes} = {3; 6}

$$p(R) = \frac{n(R)}{n(\mathscr{E})} = \frac{2}{6} = \frac{1}{3}$$

Example 6

Three coins are thrown. What is the probability of getting 2 heads and 1 tail?

$\mathscr{E}$ = {HHH, HHT, HTH, HTT, THH, THT, TTH, TTT}
R = {HHT, HTH, THH}

$$p(R) = \frac{n(R)}{n(\mathscr{E})} = \frac{3}{8}$$

Example 7

A card is picked at random from a pack of 52 playing cards. What is the probability that it is a 7?*

Number of possible outcomes = 52
Number of required outcomes = 4(7♣, 7♢, 7♡, 7♠)

Probability of picking a 7 = $\frac{4}{52} = \frac{1}{13}$

*A pack of playing cards contains 52 cards in 4 suits: clubs (♣), diamonds (♢), hearts (♡), spades (♠).
There are 13 cards in each suit: A; 2; 3; 4; 5; 6; 7; 8; 9; 10; J; Q; K.
Clubs and spades are black; diamonds and hearts are red.

Exercise 26d

1 A fair 6-sided die is thrown. Find the probability of getting
(a) a 2 (b) a 5 (c) a 6
(d) a 0 (e) a 1 (f) a 7
(g) either a 1, 2, 3, 4, 5 or 6
(h) an odd number
(i) a number divisible by 7
(j) either a 1, 2 or 5
(k) a prime number
(l) a square number
(m) a number greater than 2
(n) a number less than 6

2 A card is picked at random from a pack of 52 playing cards. Find the probability of picking
(a) the 9♠ (b) the 2♣
(c) the J♡ (d) a 3♢
(e) a Queen (Q) (f) an Ace (A)
(g) a diamond (h) a black card
(i) a black 8 (j) a red King
(k) either a 4 or a 5
(l) a black diamond

3 Two coins are tossed together. What is the probability of getting (a) a head and a tail, (b) two tails?

A fair coin was tossed 3 times. Each time it came up heads. What is the probability that it will come up heads next time?

A letter is chosen at random from the alphabet. Find the probability that it is
(a) P (b) either M or N
(c) a vowel (d) either X, Y or Z
(e) one of the letters of the word MATHEMATICS
(f) one of the letters of the word PROBABILITY

6 A bag contains 2 white balls and 3 red balls. A ball is picked at random. What is the probability that it is (a) white, (b) red?

7 A school contains 750 boys and 450 girls. A student is chosen at random. What is the probability that a girl is chosen?

8 5 000 tickets are sold in a state lottery. What is my probability of getting first prize if I buy 4 tickets?

9 A boy is playing Ludo with a die. He needs a 1 to win. What is the probability that he wins on his next throw?

10 Table 26.9 shows the numbers of pupils in each age group in a class.

Table 26.9

age (years)	13	14	15
number of pupils	9	26	5

(a) How many pupils are in the class?
(b) What is the probability that a pupil chosen at random from the class will be (i) 13 years old, (ii) over 13 years old, (iii) 14 years old or less?

11 It is known that 7 out of 10 girls in a school do not wear a necklace. What is the probability that a girl chosen at random from the school is wearing a necklace?

12 Two dice are thrown at the same time and their total is noted.
(a) Copy and complete Table 26.10.

Table 26.10

first die

	+	1	2	3	4	5	6
	1	2	3	4	5	6	7
	2	3	4	5	6	7	8
second	3	4	5	6			
die	4						
	5						
	6						

(b) How many possible outcomes are there?
(c) How many of these outcomes give a total of 5 for the two dice?
(d) What is the probability of getting a total of 5?
(e) Find the probability that the total for the two dice is
(i) 2 (ii) 4 (iii) 6 (iv) 7
(v) 7 or 11 (vi) a prime number

Chapter 27

Inequalities (2)

Inequalities in one variable

Graphical representation

$x > -2$ is an inequality in one variable, x. The inequality can be represented by a simple line graph such as that in Fig. 27.1.

Fig. 27.1

Similarly, Fig. 27.2 is a graph of the set of values given by $x \leqslant 3$.

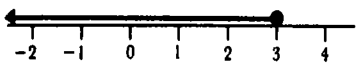


Fig. 27.2

If $x > -2$ and $x \leqslant 3$ are **simultaneous inequalities**, then the values of x must satisfy *both* $x > -2$ *and* $x \leqslant 3$. The graphs in Figs. 27.1 and 27.2 can be combined to illustrate the solution set of the simultaneous inequalities as shown in Fig. 27.3.

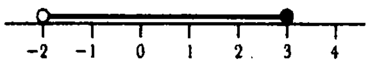


Fig. 27.3

Note the following:

1 The symbols ● and ○ show whether or not a value is included in the graph. For example, in Fig. 27.3 the 3 is included (●) and the -2 is not included (○).

2 $(-2) < x \leqslant 3$ is a short and convenient way of writing the simultaneous inequalities $x > -2$ and $x \leqslant 3$.

Example 1

Illustrate on a single number line the solution set of the simultaneous inequalities $x \geqslant 1$, $-3 < x < 5$.

Fig. 27.4 shows the two given inequalities.

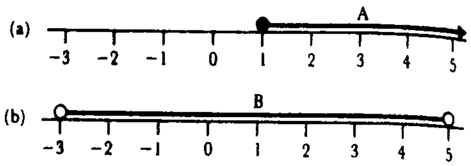


Fig. 27.4

Fig. 27.4(a) shows the set A = $\{x: x \geqslant 1\}$.
Fig. 27.4(b) shows the set B = $\{x: -3 < x < 5\}$.
If x belongs to both sets then $x \in A \cap B$ where $A \cap B = \{x: 1 \leqslant x < 5\}$.
Fig. 27.5 shows the required solution set, $A \cap B$.

Fig. 27.5

Exercise 27a

1 Fig. 27.6 contains the graphs of six inequalities. Express each inequality in the form $a * x * b$ where a and b are numbers and $*$ may be $<$ or $\leqslant$.

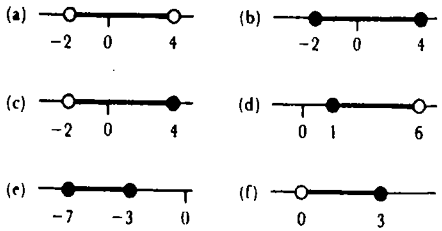


Fig. 27.6

Illustrate each of the following inequalities on a number line.
(a) $-4 \leqslant x \leqslant 1$ (b) $-1 < x < 4$
(c) $-5 \leqslant x < -2$ (d) $0 \leqslant x < 3$
(e) $-1 < x < 1$ (f) $7 \leqslant x \leqslant 8$

(a) Show the solution sets of $x \leqslant 5$ and $\leqslant x \leqslant 7$ on separate number lines.
(b) Hence show the solution set of the simultaneous inequalities $x \leqslant 5$ and $\leqslant x \leqslant 7$ on a single number line.

Illustrate on the number line the solution set of the simultaneous inequalities $x \leqslant 3$, $-2 < x < 6$. [Camb]

On the number line, illustrate the set $\{-5 < x < 2\} \cap \{-3 \leqslant x \leqslant 3\}$

Solution by calculation

Example 2

Find the values of x which are multiples of 4 and which satisfy both of the following inequalities:
$2x + 3 < 70$, $x \geqslant 24$.

$2x + 3 < 70$ and $x \geqslant 24$ are simultaneous inequalities.
If $2x + 3 < 70$
then $2x < 67$
$\Leftrightarrow \quad x < 33\frac{1}{2}$
Also $x \geqslant 24$
Hence x lies in the range $24 \leqslant x < 33\frac{1}{2}$.
The values of x which lie within that range *and* which are multiples of 4 are 24; 28; 32.

Example 3

List the integer values of x which satisfy $3x - 7 < 24 \leqslant 5x - 8$.

Expressing the inequalities in two parts,
$3x - 7 < 24$ (1)
and $24 \leqslant 5x - 8$ (2)
From (1), $3x < 31$
$x < 10\frac{1}{3}$
From (2), $32 \leqslant 5x$
$x \geqslant 6\frac{2}{5}$
Hence, combining both inequalities,
$6\frac{2}{5} \leqslant x < 10\frac{1}{3}$.
Fig. 27.7 represents the combined inequality.
From Fig. 27.7, integer values of x satisfying both parts of the inequality are 7, 8, 9 and 10.

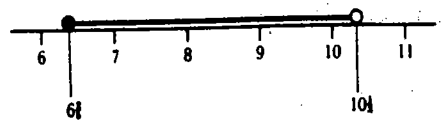


Fig. 27.7

Exercise 27b

1 Express each of the following pairs of simultaneous inequalities in the form $a * x * b$ where a and b are numbers and $*$ may be $<$ or $\leqslant$.
(a) $x \geqslant 3$, $2x - 3 \leqslant 15$
(b) $25 > 1 - 6x$, $1 > 3x + 7$
(c) $2x - 7 < 3 < 27 + 4x$
(d) $3x + 8 \leqslant 0 \leqslant 21 + 4x$
(e) $5x - 36 < -1 \leqslant 2x - 1$

2 State the integer values of x which are members of the following sets.
(a) $\{x: 2 \leqslant x < 9\}$ (b) $\{x: 1\frac{1}{2} < x < 7\frac{3}{4}\}$
(c) $\{x: -7\frac{3}{4} < x < -1\frac{1}{2}\}$
(d) $\{x: -2\frac{1}{4} \leqslant x \leqslant 3\frac{1}{2}\}$

3 If $6 < x < 7$, which of the following could be values of x?
(a) $+\sqrt{36}$ (b) 2π (c) $\frac{1}{2} \times 14$
(d) $\frac{1}{0{,}15}$ (e) $\frac{(-5)^2}{4}$ (f) $\tan 81°$

4 If $6x < 2 - 3x$ and $x - 7 < 3x$, what single range of values of x satisfies both inequalities?

5 What is the range of values of x for which $3(1 - x) < 3$ and $3(1 - x) \geqslant 0$ are both satisfied?

6 y is such that $4y - 7 \leqslant 3y \leqslant 5y + 8$. Express this inequality in the form $a \leqslant y \leqslant b$ where a and b are both integers.

7 Find an integer value of x such that $3x + 5 < 1 < 2x + 6$.

8 List the integer values of x which satisfy $3x - 4 < 27 \leqslant 4x - 5$. [Camb]

9 Express the inequality $3x - 2 < 10 + x < 2 + 5x$ in the form $a < x < b$ where a and b are numbers. Hence find the perfect square which satisfies the given inequality.

10 List the integer values of x, where x is prime, satisfying both the inequalities: $x > 18$, $3x + 2 < 93$. [Camb]

Inequalities in two variables

$(x; y)$ represents any point on the cartesian plane which has coordinates x and y. In Fig. 27.8 the unshaded region represents the set of points given by $\{(x; y): x \geqslant 1 \text{ and } y < 2\}$.

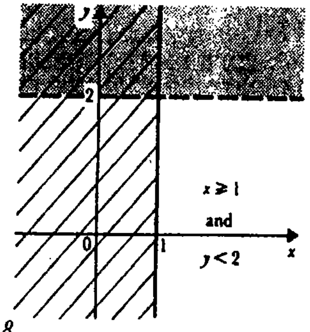


Fig. 27.8

In Fig. 27.8,

1 $x \geqslant 1$ is the set of all points to the right of the boundary line $x = 1$. The line $x = 1$ is drawn solid to show that the points on the line are included. The region to the left of the line is shaded to show that it is *not* required.

2 $y < 2$ is the set of all points below the boundary line $y = 2$. The line $y = 2$ is drawn broken to show that the points on the line are *not* included. The region above the line is shaded to show that it is *not* required.

Example 4

Show on a graph the region which contains the set of points $\{(x; y): 2x + y < 3\}$.

First make y the subject of the given inequality:

$y < 3 - 2x$

The line $y = 3 - 2x$ is the boundary between the required region and the set of points which are not required.

If $y = 3 - 2x$, then

when $x = 0, y = 3$ and

when $y = 0, x = 1\frac{1}{2}$

Since the points on the line $y = 3 - 2x$ are not included, a broken line is drawn through the points $(0; 3)$ and $(1\frac{1}{2}; 0)$.

Since $y < 3 - 2x$, the points *below* the line $y = 3 - 2x$ are required.

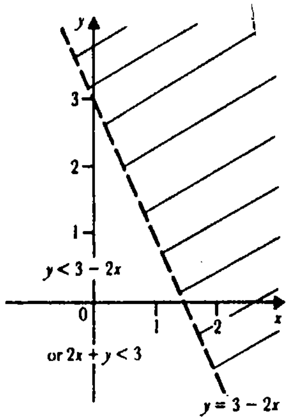


Fig. 27.9

In Fig. 27.9 the region above the line is shaded to show that it is not required.

Note the following:

1 y is made the subject of the inequality in order to determine the required region.

2 The boundary lines cross the axes at the points where $x = 0$ and $y = 0$. It is usually most convenient to draw boundary lines through such points since their coordinates are easily calculated.

3 A boundary line may be solid or broken, depending on whether the inequality is included or not.

4 The region which is *not* required is shaded.

Example 5

Show on a graph the region which contains the solution set of the simultaneous inequalities $2x + 3y < 6, y - 2x \leqslant 2, y \geqslant 0$.

Consider the first inequality, $2x + 3y < 6$. It may be rewritten as

$3y < 6 - 2x$

$\Leftrightarrow \quad y < \frac{1}{3}(6 - 2x)$

$y = \frac{1}{3}(6 - 2x)$ is a boundary line.

when $x = 0, y = 2$

when $y = 0, x = 3$

Points below the broken line through $(0; 2)$ and $(3; 0)$ satisfy the inequality $2x + 3y < 6$. Similarly, points on and below the solid line through $(0; 2)$ and $(-1; 0)$ satisfy the inequality $y - 2x \leqslant 2$.

Likewise, points on and above the x-axis (i.e. the line $y = 0$) satisfy the inequality $y \geqslant 0$.

The solution set is the unshaded triangular region shown in Fig. 27.10.

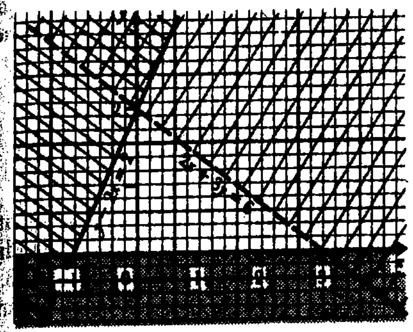


Fig. 27.10

Example 6

Solve graphically the simultaneous inequalities $4x + 3y < 12, y \geqslant 0, x > 0$ *for integral values of x and y.*

In Fig. 27.11,

$4x + 3y = 12$ (broken),

$y = 0$ (solid),

$x = 0$ (broken)

are the boundary lines. The solution set lies within the unshaded region.

In Fig. 27.11 the solution set is shown by the five points marked by spots. The solution set is $\{(1; 0), (1; 1), (1; 2), (2; 0), (2; 1)\}$.

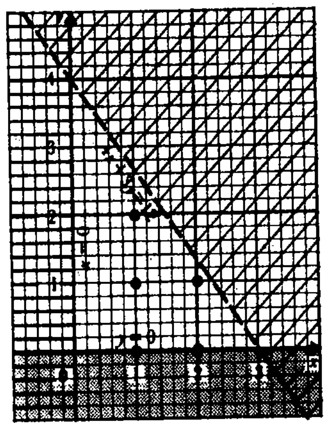


Fig. 27.11

Exercise 27c

1 In Fig. 27.12 the lines m, $x + y = 2$ and $x + 2y = 5$ are the boundaries of the unshaded region which contains the solution set of three simultaneous inequalities.

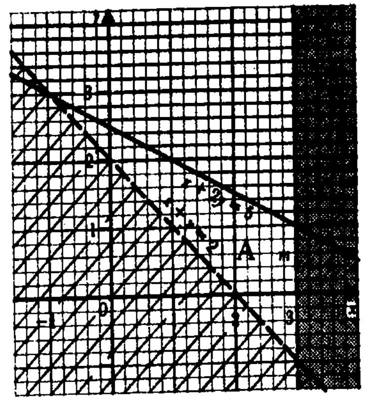


Fig. 27.12

(a) What is the equation of the line m?

(b) Write down the three inequalities which define the unshaded region, A.

(c) Write down the members of the solution set, given that it contains integral values of x and y only.

2 Using graph paper, draw the regions defined by each of the following. [Use solid and broken lines as explained earlier; leave each required region unshaded.]

(a) $y \geqslant 0, y < 3x, x + y \leqslant 4$

(b) $x \geqslant -3, y \leqslant 2, x - y < 2$

(c) $y \leqslant 5, x - y \leqslant 1, 4x + 3y \geqslant 12$

(d) $x \geqslant 0, y \geqslant 0, x + y < 6, y - x < 2$

(e) $y < 3, x < 4, 2x + y + 2 \geqslant 0,$
$x - y - 2 \geqslant 0$

3 Solve each of the following graphically for integral values of x and y.

(a) $y \geqslant 0, x - y \geqslant 1, 3x + 4y < 12$

(b) $y \geqslant 1, y - x < 5, 2x + y \leqslant 0$

(c) $y > -2, x > 0, 2x + y < 4$

(d) $x + y \leqslant 2, x - y \leqslant 2, 2x + y \geqslant 2$

(e) $y \geqslant 0, y \leqslant 4, 4x + 3y > 0, 5x + 2y < 10$

Chapter 28

Graphs (2) Travel graphs

Distance–time graphs

Reading graphs

Example 1

Fig. 28.1 is a graph representing the journeys of a pedestrian X and a motorist Y. X walks steadily towards a village. Y drives to the village, stays for a while and then returns.

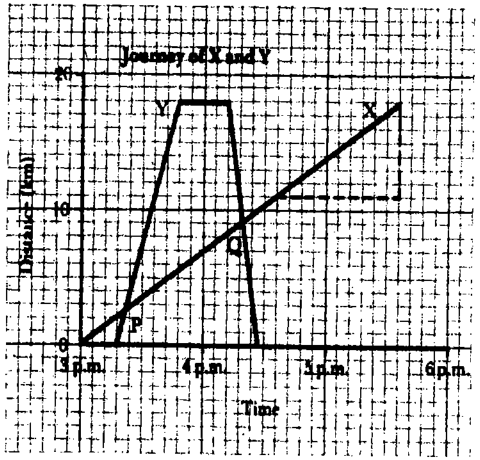


Fig. 28.1

(*a*) *What was X's average walking speed?*
(*b*) *How many minutes did Y stay in the village?*
(*c*) *At what speed did Y drive back from the village?*
(*d*) *How far did X walk between the two times that Y passed him?*

(a) *Either:*

$$\text{X's average speed} = \frac{\text{total distance travelled}}{\text{total time taken}}$$

$$= \frac{18\text{ km}}{2{,}6\text{ h}} \simeq 6{,}9\text{ km/h}$$

or:

Draw a horizontal line 1 hour long at a convenient place. Read off the corresponding vertical line. (See the dotted lines in Fig. 28.1.) The vertical line represents the distance travelled in 1 hour, and is 7 km.
Hence X's speed $\simeq$ 7 km/h.

(b) On the time-axis, 1 small square represents $\frac{1}{10}$ hour (6 min). In Fig. 28.1 the time that Y stays in the village is shown by the horizontal line, 4 small squares long.
$\therefore$ Time that Y stays in village

$$\simeq 4 \times 6\text{ min} = 24\text{ min}$$

(c)

$$\text{Y's return speed} = \frac{\text{distance travelled}}{\text{time taken}}$$

$$\simeq \frac{18}{0{,}25}\text{ km/h}$$

$$= 18 \div \tfrac{1}{4}\text{ km/h}$$

$$= 18 \times 4\text{ km/h}$$

$$= 72\text{ km/h}$$

(d) In Fig. 28.1 the points P and Q represent the times and positions of X and Y when Y passed X.
At Q, distance travelled by X $\simeq$ 9 km
At P, distance travelled by X $\simeq$ $2\frac{1}{2}$ km
Distance travelled by X between P and Q

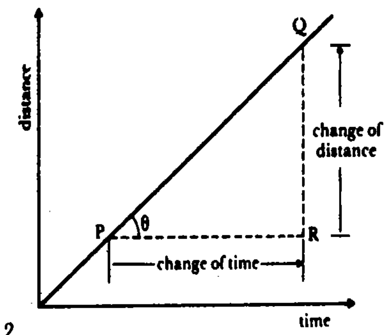
$$\simeq (9 - 2\tfrac{1}{2})\text{ km}$$

$$= 6\tfrac{1}{2}\text{ km}$$

Notice the following points:

1 In travel graphs, time is always given on the horizontal axis.
2 Answers obtained from graphs are not usually exact. However, accuracy can be improved by drawing graphs to a larger scale.
3 Speed is the rate of change of distance with time.

the sketch graph of Fig. 28.2, PQR is any ht-angled triangle drawn on the travel ph. Between P and Q the time changes m P to R and the distance changes from R Q.



g. 28.2

$$\text{eed along PQ} = \frac{\text{Change of distance}}{\text{change of time}}$$

$$= \frac{RQ}{PR}$$

otice also, $\tan\theta = \frac{RQ}{PR}$

he value of $\tan\theta$ is called the **gradient** of the ne PQ. In a time–distance graph, the gradient a line always gives a measure of the speed n that part of the graph. Since θ can be taken t any point on the line, any convenient right-ngled triangle can be used to find the gradient as in Example 1, part (a)).

Exercise 28a

Most of the questions in this Exercise are suitable for class discussion.

1 Use Fig. 28.1 to answer the following.
(a) How far had X walked when Y started towards the village?
(b) At what speed did Y drive towards the village?
(c) What was the time between Y leaving the village and X arriving at the village?
(d) How far had X walked when Y completed his journey?

2 Fig. 28.3 is a distance–time graph showing the distance covered in 2 hours at a speed of 48 km/h.

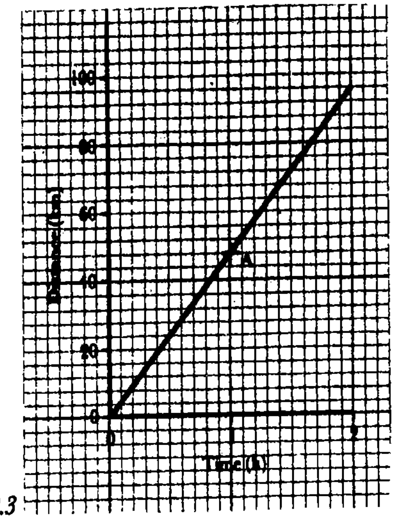


Fig. 28.3

(a) What is the time and distance covered at point A?
(b) To the nearest km, what distance is covered in (i) 1 h 42 min, (ii) $\frac{3}{4}$ h?
(c) Find the time it takes to travel (i) 60 km, (ii) 34 km.

3 Fig. 28.4 is the travel graph of a cyclist who stopped once on a journey of 15 km.

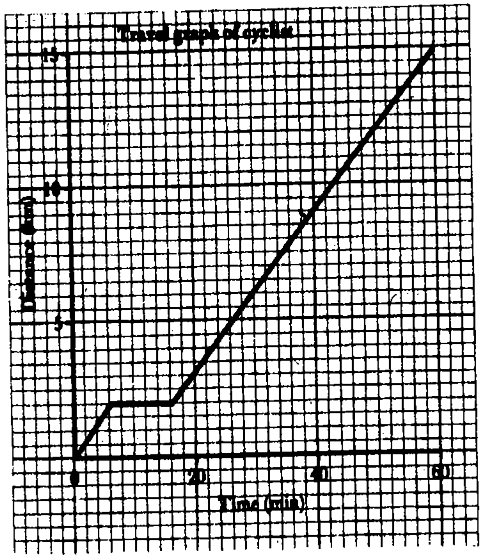


Fig. 28.4

(a) How long did the cyclist stop for?
(b) How far had the cyclist travelled after 48 min?
(c) What was the average speed for the whole journey?
(d) Neglecting the stop, what was the average *cycling* speed?
(e) How long would the journey have taken without the stop?

4 Fig. 28.5 shows the journey of a motorist. (Turn this page through 90° anti-clockwise to see Fig. 28.5 properly.)

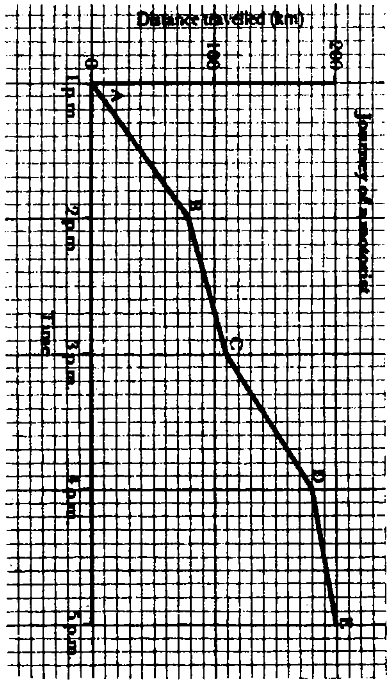


Fig. 28.5

(a) Did the motorist stop at any time?
(b) At what time had the motorist completed half the distance?
(c) How far had the motorist travelled by 3.30 p.m.?
(d) What was the average speed for the *whole* journey?
(e) What was the speed between stages A and B of the journey?
(f) What was the speed between stages B and C of the journey?
(g) What was the speed between stages C and D of the journey?
(h) What was the speed between stages D and E of the journey?

5 Two soldiers, A and B, march backwards and forwards outside the gate of a military barracks. They cross in front of the gate. Fig. 28.6 is a graph of their movements.

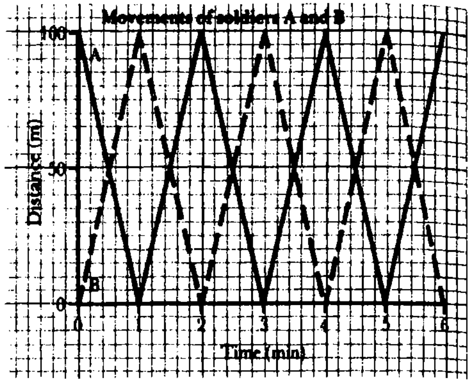


Fig. 28.6

(a) What is the greatest distance between a soldier and the gate?
(b) How far does each soldier march in 1 min?
(c) Calculate their marching speed in km/h.
(d) After ½ hour, how many times will they have passed each other?

6 Fig. 28.7 shows the outcome of a 100 m race between A and B.

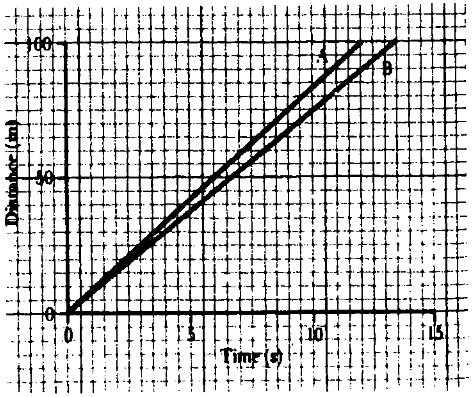


Fig. 28.7

(a) Who won the race?
(b) What distance did he win by?
(c) How far apart were the runners after 6 seconds?
(d) What was A's speed (in m/s)?

7 Fig. 28.8 shows the journeys of a lorry and a car.

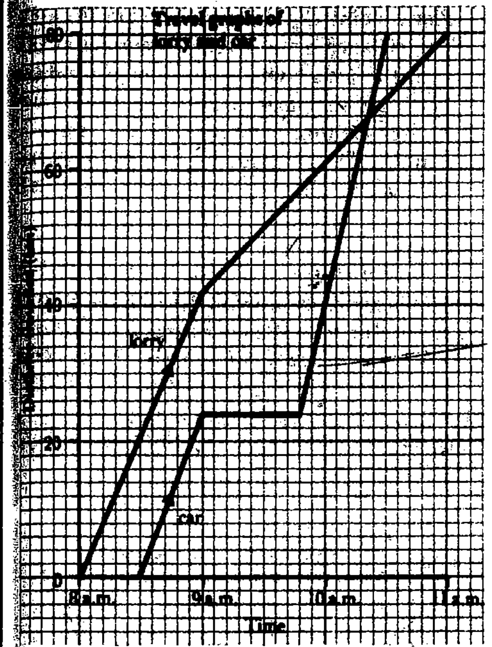


Fig. 28.8

(a) It was necessary to change the car's wheel. How long did this take?
(b) What was the greatest distance between the car and the lorry?
(c) When did the lorry change its speed?
(d) At what time did the car pass the lorry?
(e) How far had they travelled when this happened?

8 Use Fig. 28.8 to answer the following.
(a) What was the lorry's average speed for the whole journey?
(b) What was the speed of the lorry (i) before, (ii) after changing speed?
(c) What was the car's average speed for the whole journey?
(d) What was the speed of the car after the wheel was changed?

9 Fig. 28.9 shows part of a travel graph for a cyclist and a motorist.

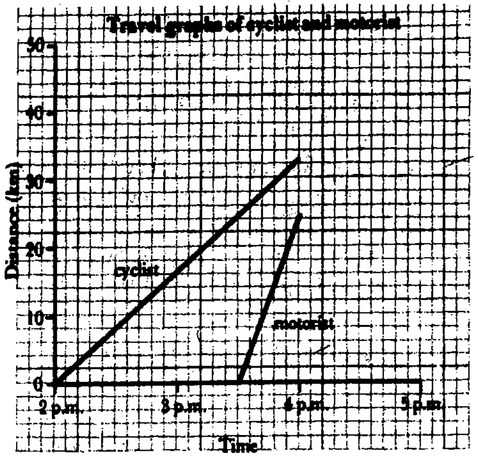


Fig. 28.9

(a) Find the speed of the cyclist.
(b) Find the speed of the motorist.
(c) Use the graph to estimate (i) the time, (ii) the distance when the motorist overtakes the cyclist. (Assume that they both continue at the same speed.)

10 Fig. 28.10 shows the graphs of a pedestrian who walked and of a motorist who drove to a village and back.

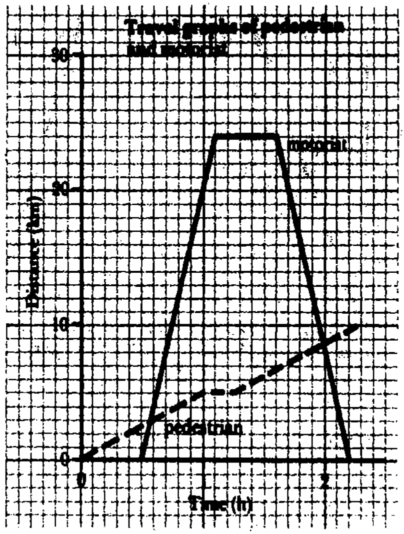


Fig. 28.10

(a) How long did the motorist stay in the village?
(b) What was the motorist's average *driving* speed?
(c) The pedestrian rested for a while. How many minutes?
(d) What was the pedestrian's average *walking* speed?
(e) How far did the pedestrian walk between the times that the motorist passed her?
(f) What was the greatest distance between the pedestrian and the motorist?

Drawing distance–time graphs

Example 2

A cyclist leaves home at 0900 and rides at a steady 12 km/h to a place 20 km away. She spends 45 minutes there, then returns at 16 km/h. At what time does she get home again?

Fig. 28.11 is a graph of the cyclist's journey.

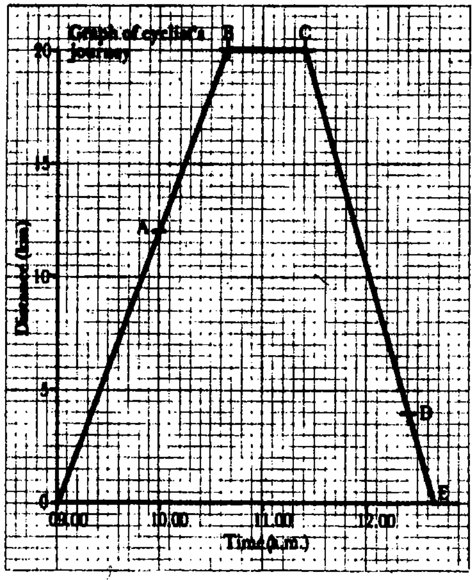


Fig. 28.11

Method:

1 Choose suitable scales, place time on the horizontal axis and mark 0900 at the origin.
2 In one hour the cyclist travels 12 km. Plot the point A at (1000; 12 km). Join the origin to A and produce it to B (20 km from home).
3 Mark a point C at the same horizontal level but 45 minutes beyond B. C represents the starting point for the journey home. (*Note:* between B and C time increases, but distance stays the same.)
4 Plot a point D 1 hour and 16 km from C. The distance is measured downwards as the cyclist returns home. Join CD and produce to cut the time axis at E.
5 E gives the time of arrival. This is approximately 1239 (each small square represents 6 min on this scale).

Example 3

A motorist starts from A *at* 1100 *and plans to arrive at* B, 100 km *away, at* 1300. *After* ½ hour *he has a puncture which takes* 18 min *to mend. How fast must he then travel so that he still arrives at* B *at* 1300?

Fig. 28.12 is the travel graph of the motorist's journey.

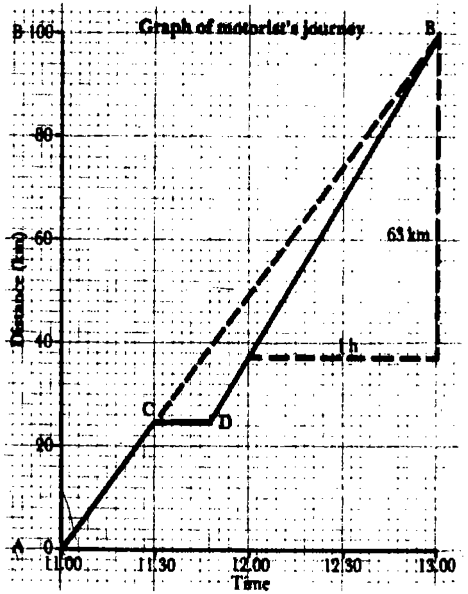


Fig. 28.12

ethod:

Choose suitable scales. Place 1100 at the origin A. Mark a point B at (1300; 100 km). AB represents the motorist's journey if he had not had a puncture.

Mark a point C on AB at 1130. Draw the line CD horizontally 1,2 cm long (representing 18 min on the scale in Fig. 21.12).

Join DB. Then ACDB represents the motorist's actual journey.

To find the speed between D and B, draw a horizontal line one hour long at any convenient place. Read off the corresponding vertical distance. (See the dotted lines to the right of DB in Fig. 28.12.) 63 km corresponds to 1 hour.

The motorist must travel at 63 km/h (approximately) to reach point B on time.

Exercise 28b

1 (a) Within the same axes, draw the graphs of the following world sprint records:

distance (m)	time (s)
100	9,9
200	19,7
400	43,8

Use scales of 2 cm to 10 s on the horizontal axis and 2 cm to 50 m on the vertical axis.
(b) Which record represents the fastest speed?

2 A car averages 68 km/h.
(a) Using scales of 2 cm to 10 min and 2 cm to 10 km, draw a distance-time graph from 0 to 30 min.
(b) Read off the distance covered in (i) 11 min, (ii) 25 min.
(c) Read off the time taken to travel (i) 10 km, (ii) 29 km.

3 A man sets out at 1000 to walk 25 km. He walks steadily at 6 km/h, but sits down for 12 min after each hour's walking. Draw a travel graph and hence find the time when he completes his journey.

4 At 1000 a girl starts walking to a town 8 km away. She walks at 6 km/h. She rests for ½ hour at the town and then returns at 16 km/h on a bicycle. Draw a travel graph and hence find the time when she gets home again.

5 Three cars, A, B, C, start one after the other in that order, at 5 min intervals, travelling at 90, 120, 150 km/h respectively. How long after the start of the race does B pass A, C pass A, C pass B?

6 Two men start at 0800 and travel towards one another from places 32 km apart. One cycles at 20 km/h and the other walks at 5 km/h.
Draw the graphs of their journeys within the same axes and hence find (a) the time when they pass each other, (b) the times when they are 5 km apart.

7 At 0900 a woman starts walking from Kadoma to Chegutu 32 km away at a steady 6 km/h. She sits down to rest for ½ hour at 1100. A bus which averages 30 km/h starts from Kadoma in the same direction at 1115. Draw travel graphs of the woman and the bus within the same axes. Hence find (a) the time, and (b) the distance from Kadoma, when the bus passes the woman.

8 Two men travel to a village 12 km away. The first walks steadily at 6 km/h without stopping. The second starts 30 min later and runs at 10 km/h, but takes a 30 min rest after 1 hour's running. Using scales of 2 cm to 30 min on the time axis and 1 cm to 1 km on the distance axis, draw a travel graph of their journeys. Hence find which man reaches the village first and by how many minutes.

9 X can run 100 m in 11,7 s and Y can run the same distance in 12,3 s. Use a graphical method to find how many metres start X should give Y in a 100-m race if they are to finish together.

10 Chido and Kudzai live 30 km apart. They arrange to meet at a point half-way between their houses at 1200. Chido starts at 1030 and cycles at 10 km/h. After 5 km he has a puncture which delays him for 10 min. Find graphically Chido's speed for the last 10 km if he arrived at the meeting point on time.

Speed–time graphs

Fig. 28.13 is a graph showing how the speed of a car varies with time over a short journey.

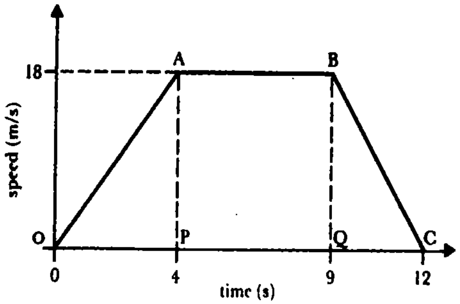


Fig. 28.13

The journey is in three stages, OA, AB, BC.

Stage OA

During the first 4 seconds the car speeds up, or **accelerates**, uniformly from rest, 0 m/s, to 18 m/s.

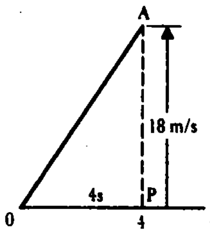


Fig. 28.14

The gradient of the graph during this stage gives the rate of change of speed, or **acceleration**, of the car.

Acceleration between O and A

$= \text{gradient of OA} = \dfrac{PA}{OP}$

$= \dfrac{18 \text{ m/s}}{4 \text{ s}}$

$= 4\frac{1}{2}$ m/s per second

The car is accelerating at $4\frac{1}{2}$ m/s per second. *m/s*^{2} is short for *m/s per second.*

Distance travelled during first stage

$= \text{average speed} \times \text{time}$

$= \dfrac{(0 + 18)}{2} \text{ m/s} \times 4 \text{ s}$

$= 36$ m

Alternatively, notice in Fig. 28.14,

area of $\triangle OAP = \frac{1}{2} \times 4 \text{ s} \times 18 \text{ m/s}$
$= 36$ m

Hence the area under the graph represents the distance travelled.

Stage AB

During the second stage the car travels with a constant speed of 18 m/s for 5 seconds.

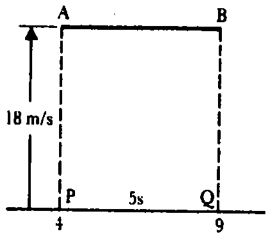


Fig. 28.15

The speed of the car does not change
$\Leftrightarrow$ acceleration between A and B = 0
Distance travelled during second stage
either $= \text{average speed} \times \text{time}$
$= 18 \text{ m/s} \times 5 \text{ s} = 90$ m
or $=$ area under AB
$= 18 \text{ m/s} \times 5 \text{ s} = 90$ m

Stage BC

During the final stage the car slows down, or **decelerates**, uniformly from 18 m/s to rest, 0 m/s.

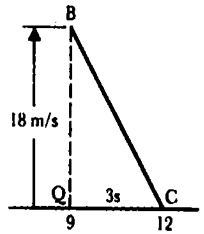


Fig. 28.16

he gradient of the graph during this final age gives the rate of change of speed. Since e gradient is negative there is a negative cceleration, or **deceleration**.

cceleration between B and C

$= \text{gradient of BC} = \dfrac{BQ}{QC}$

$= \dfrac{-18 \text{ m/s}}{3 \text{ s}}$

$= -6$ m/s per second

he car is decelerating at 6 m/s^2.
istance travelled during final stage
ither $= \text{average speed} \times \text{time}$

$= \dfrac{(18 + 0)}{2} \text{ m/s} \times 3 \text{ s}$

$= 27$ m
$=$ area under BC
$= \frac{1}{2} \times 3 \text{ s} \times 18 \text{ m/s}$
$= 27$ m

Notice the following:

1 **Acceleration** is the rate of change of speed with time. The gradient of a speed–time graph gives the acceleration of the object under consideration. **Deceleration** is the decrease of speed with time.
2 The area under a speed–time graph represents the distance travelled by the object under consideration.

Example 4
During a journey, a car accelerates uniformly for 40 seconds. *Its speed, v* km/h, *is given at 10-second intervals in Table 28.1.*

Table 28.1

t (s)	0	10	20	30	40
v (km/h)	31	44	57	70	83

Find (a) its acceleration in km/h per second, *(b) the distance travelled in* km *during the whole* 40 seconds.

Fig. 28.17 is a graph of the data in Table 28.1.

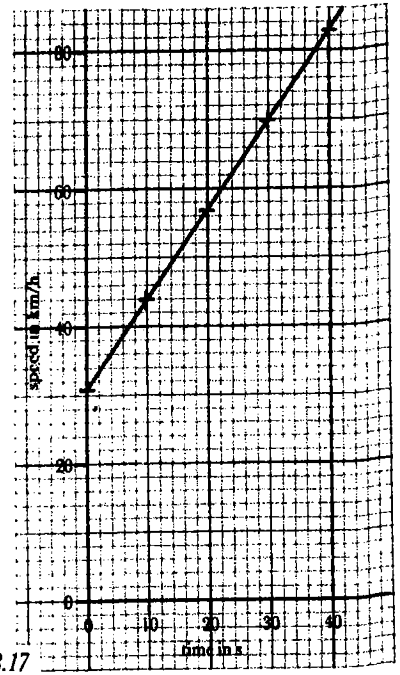


Fig. 28.17

(a) Acceleration = gradient of the graph

$= \dfrac{(83 - 31) \text{ km/h}}{40 \text{ s}}$

$= \dfrac{52 \text{ km/h}}{40 \text{ s}}$

$= 1{,}3$ km/h per second

(b) The area under the graph represents the distance travelled. However, since the speed scale is in km/h, the time scale must be expressed in hours in order to give an outcome in km.
Area under graph

$= \frac{1}{2}(31 + 83) \times \dfrac{40}{60 \times 60} \text{km}$

$= \dfrac{114 \times 40}{2 \times 60 \times 60} \text{km}$

$= \dfrac{19}{30}$ km

Notes:
1 In part (a) the acceleration may be expressed in mixed units of time.

2 In part (b) it is necessary for units of time to be the same when finding distance.

Example 5

Fig. 28.18 is the speed–time graph of a car journey.

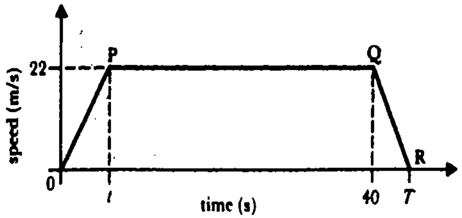


Fig. 28.18

The car starts from rest and accelerates at $2\frac{3}{4}$ m/s² for t seconds until its speed is 22 m/s. It then travels at this speed until, 40 seconds after starting, its brakes bring it uniformly to rest. The total journey is 847 m long and takes T seconds.
Calculate (a) the value of t, (b) the distance travelled during the first t seconds, (c) the value of T, (d) the final deceleration.

(a) Initial acceleration = gradient of OP

$$\text{Hence } 2\tfrac{3}{4} = \frac{22}{t}$$

$$\Leftrightarrow \quad t = \frac{22}{2\frac{3}{4}} = 8$$

(b) Distance travelled during first t seconds
= area under OP
= $\frac{1}{2} \times 8 \times 22$ m
= 88 m

(c) Total distance travelled
= area of trapezium OPQR
= $\frac{1}{2}(PQ + T) \times 22$ metres
= $(32 + T)$ 11 metres
Hence $847 = (32 + T)\,11$

$$\Leftrightarrow 32 + T = \frac{847}{11} = 77$$

$$\Leftrightarrow \quad T = 77 - 32 = 45$$

(d) Acceleration during last stage
= gradient of QR

$$= \frac{-22}{T - 40} = \frac{-22}{5}\text{ m/s}^2$$

$$= -4{,}4\text{ m/s}^2$$

The final deceleration is 4,4 m/s².

Example 6

In Fig. 28.19, OABC is the speed–time graph of a journey.

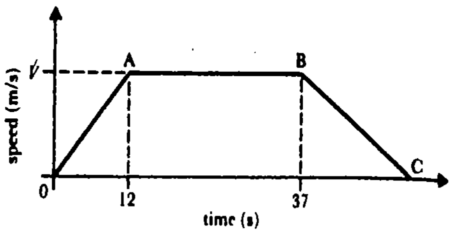


Fig. 28.19

(a) If 868 m is covered in the first 37 seconds, calculate the top speed V. (b) If the final deceleration is $3\frac{1}{2}$ m/s², calculate the total time for the journey. (c) Find the average speed for the whole journey.

(a) Distance travelled in the first 37 seconds
= area under OAB
= $\frac{1}{2}(AB + 37) \times V$ metres
where AB = 37 − 12 = 25
Hence $868 = \frac{1}{2}(25 + 37)V$
$868 = \frac{1}{2} \times 62V$

$$\Leftrightarrow \quad V = \frac{868}{31} = 28$$

The top speed is 28 m/s.

(b) If the deceleration from B to C takes t seconds, then

$$\frac{V}{t} = 3\tfrac{1}{2}$$

$$\frac{28}{t} = 3\tfrac{1}{2}$$

$$\Leftrightarrow \quad t = \frac{28}{3\frac{1}{2}} = 8$$

Total time taken = 37 s + 8 s = 45 s

(c) Average speed for the whole journey

$$= \frac{\text{total distance travelled}}{\text{total time taken}}$$

$$= \frac{\frac{1}{2}(45 + 25)28}{45}\text{ m/s}$$

$$= \frac{35 \times 28}{45}\text{ m/s} = \frac{7 \times 28}{9}\text{ m/s} = 21\tfrac{7}{9}\text{ m/s}$$

Exercise 28c

1 In your own words describe the journeys shown in the speed–time graphs in Fig. 28.20.

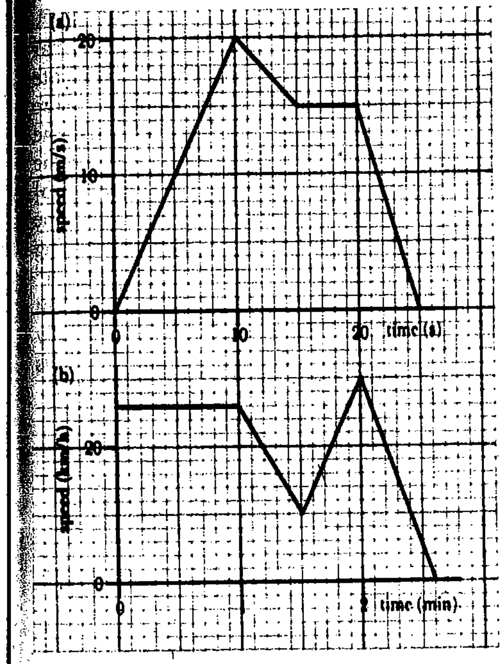


Fig. 28.20

2 Calculate the accelerations in each part of Fig. 28.20.

3 Find the total distance travelled for each journey in Fig. 28.20. (Answer in metres for part (a) in km for part (b).)

4 Table 28.2 gives the speeds, v m/s, of an object at 10-second intervals.

Table 28.2

t (s)	0	10	20	30	40
v (m/s)	9	16	23	30	37

Find (a) the acceleration and (b) the distance travelled throughout the 40 seconds.

5 Table 28.3 gives the speeds, V km/h, of a car at 5-second intervals.
Find (a) the acceleration of the car in km/h per second, (b) the distance travelled during the 30 seconds.

Table 28.3

t (s)	0	5	10	15	20	25	30
V (km/h)	55	63	71	79	87	95	103

6 Table 28.4 gives the speeds, v km/h, of a train at $\frac{1}{2}$-minute intervals.

Table 28.4

t (min)	0	$\frac{1}{2}$	1	$1\frac{1}{2}$	2	$2\frac{1}{2}$	3
v (km/h)	41	37	33	29	25	21	17

(a) Is the train's speed increasing or decreasing?
(b) Find the acceleration of the train in km/h per minute.
(c) Find the distance travelled during the 3 minutes (answer in km).

7 Fig. 28.21 is the speed–time graph of an object which accelerates from rest to a speed v m/s then decelerates to rest, taking 54 seconds altogether.

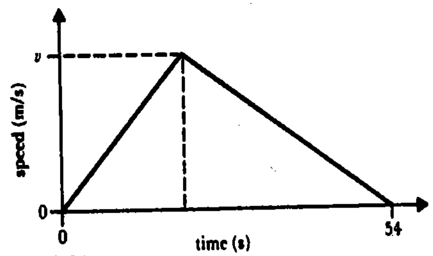


Fig. 28.21

(a) If the journey is 810 m, find the value of v.
(b) Hence find the deceleration, given that the initial acceleration is $1\frac{2}{3}$ m/s².

8 Fig. 28.22 is the graph of a car which starts from rest in 1st gear and then changes up to 2nd gear.

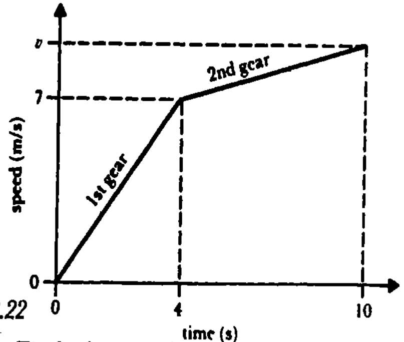


Fig. 28.22

(a) Find the acceleration and distance covered in 1st gear. (b) If the car travels 54 m in 2nd gear, find the value of v and the acceleration in that gear.

9 A train accelerates from rest at 5 km/h per minute until its speed is 60 km/h. It then travels at this speed for 17 min before decelerating at 15 km/h per minute until it comes to rest. Sketch this journey on a speed–time graph. Find the total time taken for the journey in minutes.

10 A car is travelling initially at 30 m/s. Between midday and 1205 it decelerates at 0,8 m/s per min. It then accelerates at 0,65 m/s per min for 12 min, after which it maintains a constant speed. Graphically or otherwise find the speed of the car at the following times.

(a) 1201 (b) 1204 (c) 1205
(d) 1207 (e) 1211 (f) 1219

11 Fig. 28.23 is the speed–time graph of an object which travels at a constant speed of 42 m/s for 4s and then slows down uniformly, coming to rest after a further 3s.

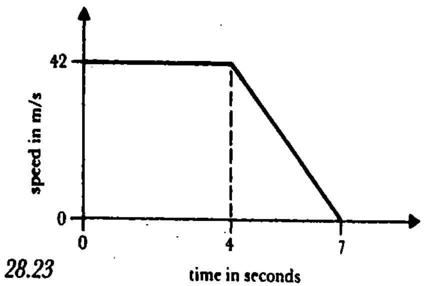


Fig. 28.23

Calculate (a) the speed of the object after 6 seconds, (b) the average speed of the object during the 7 seconds. [Camb]

12 Fig. 28.24 shows the speed–time graph of a car.

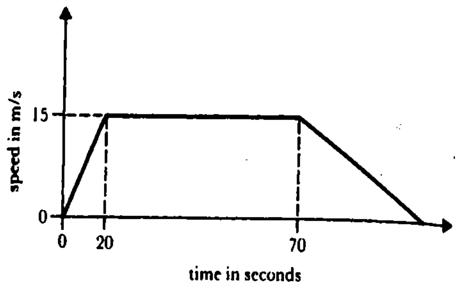


Fig. 28.24

(a) Calculate the acceleration of the car during the first 20 seconds. (b) Calculate the distance the car travels from rest before it begins to decelerate. (c) Given that the car decelerates at 0,5 m/s^2, calculate the total time taken for the journey. [Camb]

13 Fig. 28.25 is a travel graph showing the motion of an object which has a starting speed of 26 m/s. It decelerates at 4 m/s^2 for the first 3 seconds, travels at a constant speed for the next 5 seconds, and finally accelerates for 2 seconds until its speed is 37 m/s.

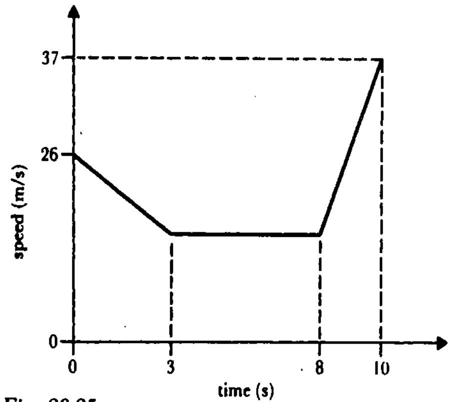


Fig. 28.25

Find (a) its speed after 7 seconds, (b) its speed after 9 seconds, (c) its average speed over the whole 10 seconds.

14 Fig. 28.26 is the speed–time graph of an electric train.

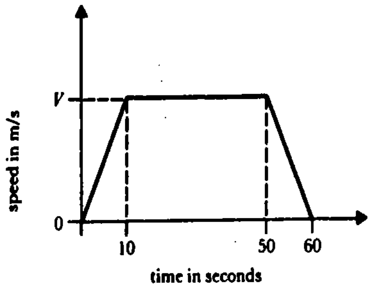


Fig. 28.26

Given that the total distance travelled in the 60 seconds is 700 metres, calculate
(a) the maximum speed V m/s,
(b) the acceleration of the train during the first 10 seconds,
(c) the distance travelled in the first 15 seconds. [Camb]

15 Fig. 28.27 is the speed–time graph of a car journey. The car starts from rest and for T seconds it accelerates at 2 m/s^2 until it reaches a speed of 16 m/s.

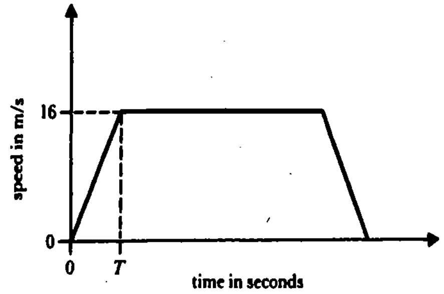


Fig. 28.27

(a) Find the value of T.
The car then travels at 16 m/s for 40 seconds, after which the driver applies the brakes and brings the car to rest in a further 10 seconds. Calculate (b) the distance travelled in the last 50 seconds, (c) the average speed of the car during the last 50 seconds. [Camb]

Chapter 29

Statistics (4) Averages

The **average** of a set of numbers is a number which is typical of the set as a whole and which is usually somewhere near the middle of the set. For example, if the average age of a class is 16 years 7 months, the ages of the students may range from about 15 years to 18 years. Also, the class would be in a secondary school, not in a primary school.

Mean (Revision)

There are many kinds of average. The **mean,** or **arithmetic mean**, is the most common kind. If there are n numbers in a set, then

$$\text{mean} = \frac{\text{sum of the numbers in the set}}{n}$$

Example 1

Calculate the mean of the following set of numbers.

176 174 178 181 174
175 179 180 177 182

Either:

$$\text{mean} = \frac{176 + 174 + 178 + \ldots + 182}{10}$$

$$= \frac{1\,776}{10} = 177{,}6$$

or:

Since the numbers range from 174 to 182, the mean will be somewhere between these values. Use 178 as a working mean and write two columns showing the positive and negative deviations from the working mean.

deviation from 178

+	−
0	2
3	4
1	4
2	3
4	1
10	14
	4

The result shows that the total deviation for the ten numbers is − 4 from a working mean of 178.

$$\text{true mean} = 178 + \left(\frac{-4}{10}\right)$$

$$= 178 - 0{,}4 = 177{,}6$$

The second of these two methods is recommended when given a large set of numbers which are of roughly the same size.

Example 2

5 children have an average age of 7 years 11 months. *If the youngest child is not included, the average age increases to* 8 years 4 months. *Find the age of the youngest child.*

Total age of all 5 children
= 5 × 7 yr 11 mo
= 35 yr 55 mo = 35 yr + 4 yr 7 mo
= 39 yr 7 mo

Total age of the 4 older children
= 4 × 8 yr 4 mo
= 32 yr 16 mo = 32 yr + 1 yr 4 mo
= 33 yr 4 mo

Age of youngest child
= 39 yr 7 mo − 33 yr 4 mo
= 6 yr 3 mo

Exercise 29a (Revision)

1 Calculate the mean of each of the following.
(a) 0; 4; 7; 7; 8; 8; 8; 9; 9; 10
(b) 10,5; 13,0; 13,9; 9,5; 11,3; 8,4
(c) 13; 31; 11; 44; 27; 26; 13; 34; 17
(d) \$10,33; \$6,25; 75c; \$3,59

2 (a) Given that the mean of 11; 17; 23; 37; 41; 51 is 30, write down the mean of 118; 178; 238; 378; 418; 518.
(b) Find and use a short method of calculating the mean of these numbers.
827 427 627 427 727 327
227 527 527 127 827 727

3 The following amounts of money are the profits made by a shopkeeper on 16 consecutive days.

\$6,29	\$6,35	\$6,27	\$6,21
\$6,40	\$6,30	\$6,38	\$6,26
\$6,28	\$6,32	\$6,30	\$6,36
\$6,43	\$6,25	\$6,33	\$6,23

Use a working mean of \$6,30 to calculate the mean profit per day.

4 Find x if the mean of the numbers 13, $2x$, 0, $5x$ and 11 is 9.

5 Calculate x if the mean of 30, x, 12, 40 and 10 is equal to x.

6 A mother has seven children. The mean age of the children is 13 years 2 months. If the mother's age is included, the mean age rises to 17 years 7 months. Calculate the age of the mother.

7 The average age of a class of 21 students is 14 years 3 months. If the oldest student is not counted, the average drops to 14 years 2 months. Calculate the age of the oldest student.

8 19 students have an average body-mass of 50,2 kg. A new student, of body-mass 44,2 kg joins the group. Calculate the new average body-mass of the students.

9 In an examination the pass mark was 45. The mean mark of eight students was 53, but two of them failed. What is the lowest possible mean mark for the six students who passed? (Assume that the marks are whole numbers.)

10 Kudzai scored 45% in the first paper of his mathematics examination and scored x% in the second paper (where x is a whole number). He was given a grade C for the subject which meant that the average of his marks on the two papers was greater than 48% but less than 52%. Find the possible values of x.

Average rates

When calculating the average of two or more rates, always find the totals of the quantities involved. Read Examples 3 and 4 carefully.

Example 3

For 3 days a trader made a profit at the rate of \$7,90 *per day. For the next 4 days her profit averaged* \$10,00 *per day. What was her average daily profit for the week?*

For the first 3 days:
profit = 3 × \$7,90 = \$23,70
For the next 4 days:
profit = 4 × \$10,00 = \$40,00
For all 7 days:
total profit = \$23,70 + \$40,0 = \$63,70

$$\text{Average daily profit} = \frac{\$63{,}70}{7} = \$9{,}10$$

Example 4

A man walked 12 km *at* 3 km/h *and cycled* 18 km *at* 9 km/h. *What was his average speed for the whole journey?*

$$\text{Average speed} = \frac{\text{total distance}}{\text{total time}}$$

Total distance = 12 km + 18 km = 30 km

$$\text{Time for 1st part of journey} = \frac{12}{3}\text{ h} = 4\text{ h}$$

$$\text{Time for 2nd part of journey} = \frac{18}{9}\text{ h} = 2\text{ h}$$

Total time = 4 h + 2 h = 6 h

$$\text{Average speed} = \frac{30}{6}\text{ km/h} = 5\text{ km/h}$$

Exercise 29b

1 On a journey, a motorist travels the first 40 km in $\frac{1}{2}$ hour, the next 34 km in 25 min and the last 7 km in 5 min. What is the average speed for the whole journey?

2 Danai lives 4 km from school. She walks 1 km at 6 km/h and travels the rest of the way by bus at 30 km/h. (a) How many minutes does the whole journey take? (b) What is her average speed in km/h?

3 Bob lives 5 km from school. He walks 1 km at 4 km/h and travels the rest of the way by bus at 16 km/h. What is his average speed for the whole distance?

4 A motorist averages 48 km/h for the first 30 km of a journey and 64 km/h for the next 120 km. What is the average speed for the whole journey?

5 For 4 weeks a man's average wage was \$61 per week. For the next 6 weeks his average wage was \$57 per week. What was his average weekly wage for the 10 weeks?

6 A factory employs 50 workers. 40 earn \$1,35/hour and 10 earn \$1,80/hour. What is the average hourly rate of pay?

7 A 3rd form contains three classes of 36, 33 and 31 students. In an examination the average marks for the classes were 65, 56 and 52 respectively. What was the average mark for the 3rd form altogether?

8 The road from A to B is 10 km uphill followed by 20 km downhill. A motorcyclist averages 36 km/h uphill and 90 km/h downhill. Calculate the average speed (a) from A to B, (b) from B to A.

9 A class contains 10 girls and 20 boys. The average height of the girls is 1,58 m and the average height of the boys is 1,67 m. Calculate the average height of the students in the class.

10 A lorry driver travelled 84 km between two towns. The first 60 km of road was untarred and the average speed over this part was 30 km/h. If the average speed for the whole journey was 36 km/h, calculate the average speed over the good part of the road.

11 A country's capital has a population of 600 000 and a population density of 1 200 people/km^2. The rest of the country has a population of 2,4 million and a density of 60 people/km^2. Calculate the population density of the whole country correct to 2 s.f.

12 The population density of country A of area x_1 km^2 is y_1 per km^2, and that of country B of area x_2 km^2 is y_2 per km^2. If X is the population density per km^2 when the two countries are merged into one, express X in terms of x_1, y_1, x_2, y_2.

Mixtures

Example 5

A shopkeeper sells rice in bags. 30 bags of rice costing \$40 per bag are mixed with 50 bags of another kind of rice costing \$35 per bag. If she sells the mixture at a gain of 20%, at what price does she sell a bag?

Cost of first 30 bags = 30 × \$40 = \$1 200
Cost of other 50 bags = 50 × \$35 = \$1 750
Total cost of 80 bags = \$1 200 + \$1 750
= \$2 950

Average cost of 1 bag = $\frac{\$2\,950}{80}$

Selling price per bag = 120% of $\frac{\$2\,950}{80}$

$= \frac{\$120 \times 2\,950}{100 \times 80}$

$= \frac{\$6 \times 295}{5 \times 8}$

$= \frac{\$3 \times 59}{4} = \frac{\$177}{4}$

= \$44,25

Example 6

In what proportion must teas costing \$11,60 and \$12,40 per kg be mixed to produce a tea costing \$12,10 per kg?

If the mixture costs \$12,10 per kg, then
on each kg at \$11,60, gain = 50c,
on each kg at \$12,40, loss = 30c.
Hence 3 kg of the first tea must be mixed with 5 kg of the second so that the gain on the first (3 × 50c) is exactly the same as the loss on the second (5 × 30c).
The required proportion is 3 : 5.

Exercise 29c

1 If 8 kg of coffee costing \$30,00 a kg is mixed with 12 kg of another kind of coffee costing \$33,00 a kg, what is the cost of the mixture per kg?

2 Equal quantities of sweets at two a cent and six a cent are mixed together. How many of the mixed sweets do you get for 1 cent?

3 Five bags of rice at \$2,28 per bag are mixed with four bags at \$1,50 and three bags at \$2. What is the mixture worth per bag?

4 Three kinds of tea at \$8,80, \$10,30 and \$10,70 per kg are mixed in the ratio 2 : 3 : 5. What is the mixture worth per kg?

5 Four ingredients costing \$3,20 per kg, \$2,40 per kg, \$1,60 per kg and 80c per kg are mixed so that their masses are in the ratio

4 : 1 : 3 : 2. Calculate the average cost per kg of the mixture.

6 In what proportion should teas at \$11,80 and \$12,40 per kg be mixed to obtain a mixture worth \$12,20 per kg?

7 In what ratio must two sorts of sugar, costing \$1,08 and 93c per kg respectively be mixed in order to produce a mixture worth 99c per kg?

8 A trader mixes 3 sacks of sugar costing \$90/sack with 7 sacks of sugar which cost \$70/sack. If she sells the mixture at \$95/sack, calculate her percentage profit.

9 A trader bought three kinds of nuts at \$1,00 per bag, 84c per bag and 60c per bag respectively. He mixed them in the ratio 3 : 5 : 4 respectively and sold the mixed nuts to make a profit of 25%. At what price per bag did he sell them?

10 Acid costing \$6,48 per litre is diluted with water in the ratio 8 parts of acid to 1 of water. The diluted acid is sold at \$7,20 per litre. Calculate the percentage profit. (Assume that the cost of water is negligible.)

Median and mode (Revision)

If a set of numbers is arranged in order of size, the middle term is called the **median**. If there is an even number of terms, the median is calculated as the arithmetic mean of the two middle terms.

Example 7

Find the median of (a) 15; 11; 8; 21; 17; 3; 8
(b) 3,8; 2,1; 4,4; 8,3; 9,2; 5,0.

(a) Arranging the numbers on rank order (i.e. from highest to lowest):
21; 17; 15; 11; 8; 8; 3
There are 7 numbers. The median is the 4th number, 11.
(b) Arranging the numbers from lowest to highest:
2,1; 3,8; 4,4; 5,0; 8,3; 9,2
There are 6 numbers. The median is the mean of the 3rd and 4th terms.

$$\text{Median} = \frac{4{,}4 + 5{,}0}{2} = 4{,}7$$

Notice the following:

1 The numbers must be arranged in order of size, either increasing or decreasing.

2 If a number is repeated (as 8 is in part (a)) it must be written down as many times as it appears.

3 There are as many numbers below the median as there are above it.

The **mode** of a set of numbers is the number which appears most often, i.e. the number with the greatest frequency. For example, 8 is the mode of the numbers in Example 7(a).

Example 8

21 students did an experiment to find the melting point of naphthalene. Table 29.1 shows their results.

Table 29.1

temperature (°C)	78	79	80	81	82	83	90
frequency	1	2	7	5	3	2	1

What was (a) the modal temperature, (b) the median temperature?

(a) 7 students recorded a temperature of 80 °C. This was the most frequent result.
Mode = 80 °C
(b) There were 21 students. The median is the 11th temperature. If the temperatures were written down in order, there would be one of 78 °C, two of 79 °C, seven of 80 °C, and so on. Since 1 + 2 + 7 = 10, the 11th temperature is one of the five 81 °Cs. Median = 81 °C.

Notice in part (b) that there was no need to write down all the temperatures in order. In many cases it is not practicable to write down all the data in order to pick out the median.

Exercise 29d

1 The body-masses of 8 policemen are as follows:
71 kg 58 kg 84 kg 60 kg
55 kg 72 kg 63 kg 80 kg
Find the median of their masses.

2 The scores obtained by 10 students in a test were 8; 3; 5; 2; 7; 5; 7; 3; 6; 5
What is the mode of these scores?

3 x, x, x, y represent four numbers. The mean of the numbers is 9 and their median is 11. Find y.

4 In Table 29.1, the student who recorded 90 °C clearly made a mistake. Disregard this result and find (a) the median, (b) the mean of the remaining 20 results.

5 Table 29.2 gives the ages of a group of students. (a) What is the mode of the data in the table?

Table 29.2

age in years	13	14	15	16	17
number of pupils	1	4	2	2	1

(b) What is the mean age of the students?

6 A group of students measured a certain line. They obtained the following results.

10,8 cm 10,8 cm 10,7 cm
10,9 cm 10,8 cm 10,9 cm
10,7 cm 11,8 cm 10,8 cm

(a) One of the results is clearly a mistake. Which one? Disregard the incorrect result and find (b) the mode, (c) the mean of the other results.

7 For the following set of numbers:
13; 14; 14; 15; 18; 18; 19; 19; 19; 21
(a) state the median, (b) state the mode, (c) calculate the mean.

8 The number of goals scored by a team in 9 netball matches are as follows:
3; 5; 7; 7; 8; 8; 8; 11; 15
Which of the following statements are true of these scores?
(a) The mean is greater than the mode.
(b) The mode and the median are equal.
(c) The mean, median and mode are all equal.

9 The handspans of students in a class are given to the nearest cm in Table 29.3.

Table 29.3

handspan (cm)	18	19	20	21	22	23
frequency	3	8	12	6	4	2

(a) State the modal handspan.
(b) How many students are in the class?
(c) Find the median handspan.
(d) Calculate the mean handspan correct to the nearest mm.

10 Table 29.4 shows the amounts of money (in \$) which some students have in their pockets. For this distribution, find, correct to the nearest cent, (a) the mean, (b) the median, (c) the mode.

Table 29.4

amount (\$)	1	2	3	4	5	6
no of students	1	3	2	5	1	4

11 A die is thrown 20 times with the following results:

5 1 4 6 4 4 1 5 2 1
1 4 6 5 3 1 3 6 6 2

(a) Make a frequency table of the results. Hence find (b) the mode, (c) the median of the results.

12 In a survey of the cost of a bar of soap, the following prices were found in ten shops:

80c 85c 75c 78c 85c
78c 85c 83c 82c 86c

Find (a) the median, (b) the modal price of a bar of soap. (c) Calculate the mean price correct to the nearest whole cent.

13 Table 29.5 gives the ages of a group of girls.

Table 29.5

ages (years)	13	14	15	16	17	18
frequency	3	0	3	2	2	5

(a) What is the modal age of the girls?
(b) What is the median age of the girls?
(c) Calculate the mean age of the girls.

14 Table 29.6 shows the number of pupils (f) scoring a given mark (x) in a test.

Table 29.6

x	2	3	4	5	6	7	8	9	10	11	12
f	3	8	7	10	13	16	15	15	6	2	5

(a) Find the mode. (b) Find the median. (c) Calculate the mean.

hapter 30

onsumer arithmetic (1)

ersonal income

ay

ere are three common ways that people get id. The first is to be employed; i.e. to work r an organisation which pays wages or salary. e second is to be self-employed; i.e. to trade rectly with other people by selling services or ods such as farm produce. A third way is to vest money in savings accounts which oduce interest. The examples which follow monstrate various methods of obtaining come.

ages and salary

xample 1

school clerk works from 0800 *to* 1200 *in the ornings and from* 1300 *to* 1630 *in the afternoons. the rate of pay is* \$2,40 *per hour, calculate (a) the eekly wage and hence (b) the annual pay of the clerk.*

(a) Number of hours worked per day
$= (12 - 8) + (16\frac{1}{2} - 13)$
$= 4 + 3\frac{1}{2}$
$= 7\frac{1}{2}$

Number of hours worked per week
$= 5 \times 7\frac{1}{2}$
$= 37\frac{1}{2}$

Weekly wage
= \$2,60 × 37,5
= \$97,50

(b) Annual pay = \$97,50 × 52
= \$5 070

Note in Example 1:
(a) A person who is paid by the hour is sometimes called a *wage earner.*
(b) A 5-day working week is assumed.
(c) Wage earners receive wages for every week in the year. Some of these weeks will be holidays.

Example 2

A lecturer's annual salary is \$24 106,20. *Express this as a monthly salary.*

Monthly salary = \$24 106,20 ÷ 12
= \$2 008,85

Note: Salary is a payment for a specific period of time (usually longer than a week). It is quite common to talk about *monthly salary* or *annual salary*. People on salaries are *not* paid by the hour.

Example 3

A woman trades in eggs and chickens. Here are her sales figures for a week:

day	*number sold* eggs	*number sold* chickens
Monday	30	4
Tuesday	9	0
Wednesday	44	2
Thursday	0	0
Friday	18	1
Saturday	53	11

Calculate her earnings if she sells eggs for 30c *each and chickens for* \$9 *each.*

Number of eggs sold
= 30 + 9 + 44 + 0 + 18 + 53
= 154

Income from eggs
= 30c × 154
= \$46,20

Number of chickens sold
= 4 + 0 + 2 + 0 + 1 + 11
= 18

Income from chickens
= \$9 × 18
= \$162

Total income = \$46,20 + \$162
= \$208,20

Commission (Revision)

Commission is a special payment given to encourage the selling of goods. It is usually calculated as a proportion of the value of the items sold. The proportion may be a percentage or a rate in the dollar.

Example 4
An insurance agent gets commission of 0,8% *on insurance policies sold. Find the commission for selling two policies worth* $20 000 *each and a mortgage endowment policy of* $25 000.

Total insurance sold
= 2 × $20 000 + $25 000
= $65 000
Commission = 0,8% of $65 000
= 8 × $65
= $520

Interest (Revision)

Interest is a payment given for saving money. It can also be the price paid for borrowing money. When interest is calculated on the basic sum of money saved (or borrowed) it is called **simple interest**.

Simple interest, I, can be calculated using the formula

$$I = \frac{PRT}{100}$$

where **P** is the **principal** (sum of money saved or borrowed), R is the annual **rate** of interest (given as a percentage) and T is the **time** (in years) for which the money is saved (or borrowed).

Example 5
Find how much $343,20 *amounts to in* 3 *years at* $12\frac{1}{2}\%$.

$$I = \frac{PRT}{100}$$

$$= \$\frac{343,2 \times 12\frac{1}{2} \times 3}{100}$$

$$= \$128,70$$

interest = $128,70
principal = $343,20
amount = $471,90

The amount comes to $471,90.

Note: The **amount** is the sum of the principal and the interest.

Exercise 30a

1 A skilled mechanic earns a total of $321,10 for a 38 hour week. What is this wage as an hourly rate?

2 A teacher has a monthly salary of $1391,45. Express this as an annual salary.

3 A clerk works from 8 a.m. to 4.30 p.m. on Mondays to Thursdays and from 8 a.m. to 4 p.m. on Fridays. He is given $\frac{1}{2}$-hour off each day for lunch. Calculate his weekly wage if he is paid $2,40 per hour.

4 A factory worker is paid $1,95/hour for a 40-hour week. Overtime is paid at the rate of $2,95/hour. Calculate how much the worker earns in a week when she works 51 hours. (*Note:* This means that she worked 40 hours at the regular rate and 11 hours at the overtime rate.)

5 A small co-operative grows maize products then sells them. It employs 7 farm workers and 2 shopkeepers. On average the farm workers work a 44-hour week and earn $1,60/hour; the shopkeepers work a 42-hour week and earn $1,25/hour.
Calculate (a) the average weekly wage of a farm worker and of a shopkeeper, (b) the annual wage bill for the co-operative, (c) how much the co-operative has left over in a year when maize sales amount to $33 964.
[Discuss the following with your teacher and your classmates:
Is it fair that the two categories of worker receive different wages?
How might the left-over money be used?]

6 A self-employed tyre repairer charges $8,25 for mending lorry tyres and $4,75 for car tyres. How much is he left with in a week when he repairs 33 lorry tyres and 48 car tyres but has to pay out $121,50 for materials?

7 A roadside trader sells fruit. She writes her prices on a notice board as shown in Fig. 30.1.

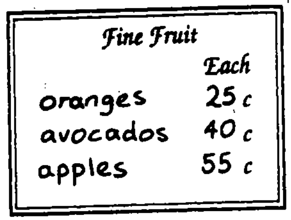


Fig. 30.1

What is her income if she sells 83 oranges, 26 avocados and 15 apples?

8 A mail order company gives a commission of 12c in the dollar on all orders. What is the commission on the following order?

dress	$94,99
skirt	$83,75
vest	$12,50
shirt	$39,99
clock	$83,50
calculator	$49,99

9 A novelist gets a royalty of $12\frac{1}{2}\%$ of the sales of her books. How much does she get in a year when 5 648 copies of her books sell for $14,99 each?
[*Note:* A *royalty* is the name given to the commission that authors get on the sales of their books. What do you think of this as a method of payment to authors?]

10 A salesman gets a basic hourly wage of $2,20 and a commission of 4,5% of the value of goods sold. How much does he get in a 40-hour week, when he sells $1648 worth of goods? [Why is it that some sales people are paid this way?]

11 Find the interest on the following.
(a) $820 for 4 years at 10%
(b) $160 for 6 years at 9%
(c) $787 for 3 years at 8%

12 Find what the following amount to.
(a) $500 at 10% for 3 years
(b) $2 130 at 9% for 5 years
(c) $432,50 at 8,25% for 2 years

Money transactions

Banking

Banks offer many services connected with money management. The most popular services include looking after money in a **savings account** or a **current account** and arranging **foreign exchange**. Banks can afford to offer these services because (a) they charge customers for them, and (b) they charge interest on loans.

Savings account

To encourage their customers to save, banks give interest on the money held in savings accounts. The interest is usually compound interest. It is paid to the account every half year, usually 30 June and 31 December.

Money is paid into the account using a **deposit form**. Fig. 30.2 is a typical deposit form showing how Ms Mapfumo deposited $51 on 19 March 1991.

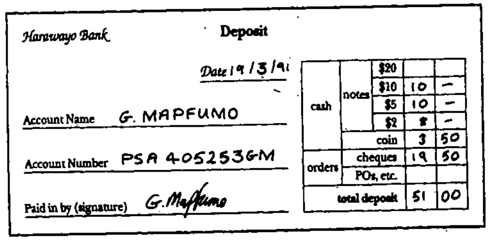


Fig. 30.2

Money is taken from the account by using a **withdrawal form**. Fig. 30.3 shows how Ms Mapfumo withdrew $76 on 28 April 1991.

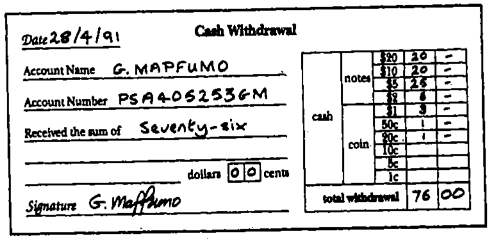


Fig. 30.3

Cheque Withdrawal

Date ______

Account Name ______

Account Number ______

Received the sum of (in words) ______ $ [|]

______ dollars [ ] cents

Signature ______

Fig. 30.4

Fig. 30.3 on page 259 is a **cash withdrawal** form. If money is required to be withdrawn as a cheque then a simpler form is used. Fig. 30.4 is an example of a **cheque withdrawal** form.

All the transactions are recorded in a **passbook** or **account book** which is the property of the saver. Fig. 30.5 shows Ms Mapfumo's passbook for the period 12 February to 24 July 1991.

Ms G MAPFUMO 19 Greenway Southerton HARARE		Ac No PSA405253G M	
Date	*Deposit*	*Withdrawal*	*Balance*
12 Feb 91			BF 221,15
12 Feb 91	CH - 39,00		260,15
19 Mar 91	CH 51,00		311,15
16 Apr 91	CH 16,83		327,98
28 Apr 91		CA 76,00	251,98
06 May 91	CH 23,02		275,00
15 May 91		CH 185,75	89,25
30 May 91		TRF 50,00	39,25
30 Jun 91	INT 11,29		50,54
14 Jul 91	CA 120,00		170,54
24 Jul 91		CA 80,00	CF 90,54

Fig. 30.5

The page in the savings book shown in Fig. 30.5 contains some abbreviations. Their meanings are as follows:

BF	Brought forward (from previous page)
CF	Carried forward (to next page)
CA	Cash
CH	Cheque
INT	Interest
TRF	Transfer (to or from another account)

Current account

Most customers simply want banks to keep their money safely and conveniently. Banks provide a **current account** service for this purpose. A current account handles money on behalf of the customer. Cash is paid into and out of the account and little or no interest is given.* Charges are often made for this service.* The current account service is the most popular service that banks offer.

* Whether or not interest is given and/or charges are made varies from bank to bank. In general no charges are made if a positive balance is kept in the current account.

Money may be paid into a current account in a number of ways. For example, many employees can arrange for their pay to be paid directly into their current accounts. On the other hand, money may be paid in using a deposit form similar to that used for savings accounts.

Current account holders are *not* given a passbook. Instead they are given (or may have to buy) a **cheque book**. Withdrawals are made from the current account by completing a page in the cheque book. This is known as 'writing a cheque'. Fig. 30.6 shows a typical page from a cheque book.

20/5/1991
To MR. B. CHITATE
$ 92-50
205972

HARAWAYO BANK 20/5/1991
Pay MR B. CHITATE
NINETY-TWO DOLLARS AND 50 CENTS $ 92-50
G Mapfumo
G. Mapfumo
205972 20 58 17 21941606

Fig. 30.6

In Fig. 30.6, the cheque is the main portion on the right. It shows that on 20 May 1991 Ms Mapfumo wrote a cheque for $92,50 in favour of Mr B Chitate. If Mr Chitate takes the cheque to a bank it will pay him the amount, provided the cheque is correctly written and that Ms Mapfumo has sufficient money in her current account. The small section on the left is a counterfoil. It is completed and kept by Ms Mapfumo as a record of the payment.

The numbers along the foot of the cheque show, from left to right, the cheque number, the bank branch number and the account number. They are written using a special style of numeral which computers can read.

The bank keeps a record of all the transactions made by the account holder. Every so often it sends a copy of the record to the customer. This record is called a **bank statement**. Fig. 30.7 below shows a typical bank statement.

Exercise 30b

1 Refer to Fig. 30.2, the deposit form.
   (a) How much cash was paid in on 19 March?
   (b) How many notes altogether were deposited? What were their values?

2 (a) Make a copy of a blank deposit form such as the one in Fig. 30.2.
   (b) Suppose you have the following to deposit:
   - 2 × $20 notes
   - 3 × $10 notes
   - 7 × $5 notes
   - 4 × $2 notes
   - $2,34 in coins
   - a cheque for $25,00
   - a cheque for $34,73

HARAWAYO BANK PTY LTD — **CURRENT ACCOUNT STATEMENT**

Harare Branch *Tel 190 31940*

Ac No 21941606

Ms G MAPFUMO
19 GREENWAY
SOUTHERTON
HARARE

31 MAY 91

DETAILS	PAYMENTS	RECEIPTS	DATE	BALANCE
			1991	
Balance forward			03 MAY	516,52
Cash	80,00		03 MAY	436,52
STO Baoc	97,30		07 MAY	339,22
Cash	50,00		08 MAY	289,22
BCHS	8,50		15 MAY	280,72
CH 205971	38,24		19 MAY	242,48
CH 205972	92,50		20 MAY	149,98
Counter credit	45,00		22 MAY	104,98
CH 205973	84,99		27 MAY	19,99
STO insurance	68,70		29 MAY	48,71 DR
TRF PSA405253MM		50,00	30 MAY	1,29
CH credit		509,14	31 MAY	510,43

Abbreviations
BCHS Bank Charges — CH Cheque — DR Overdrawn Balance — STO Standing Order — TRF Credit Transfer

Fig. 30.7

Complete the deposit form, showing the amounts and the final total in the appropriate spaces.

3 Refer to the withdrawal form in Fig. 30.3.
(a) What is Ms Mapfumo's account number? (b) How did Ms Mapfumo want her money to be paid?

4 You receive a bill for $80 from the City Council. From your Savings Account you withdraw $80 as a cheque made payable to the Council. Write a withdrawal form for the amount.

5 A clerk withdraws $623,50 in cash from the savings account of a 10-person Co-operative. The cash is to be used as wages for the people who work for the Co-operative.
(a) If the people are all paid equally, how much does each get?
(b) List the amounts of each cash denomination that the clerk might ask for.

6 Refer to the details in the passbook shown in Fig. 30.5.
(a) What was the interest for the period and when was it added?
(b) On which date did the account reach its highest point and for how many days did it stay at this amount?
(c) Find the total deposited during the period shown.
(d) Find the total withdrawn during the period shown.
(e) Find the difference between your answers for parts (c) and (d).
(f) Find the difference between the amount brought forward (BF) and the amount carried forward (CF).
(g) What do you notice about your last two answers?

7 Using the cheque shown in Fig. 30.6 as a model, write cheques to the following:
(a) Ms Mapfumo for $28,47,
(b) Capital Printers Pty for $500,
(c) your friend for $99,99,
(d) North East Coop for $280.

8 Refer to the current account statement in Fig. 30.7.
(a) What was the largest cheque that Ms Mapfumo wrote during the month?
(b) The largest amount that Ms Mapfumo received was her salary cheque. How much was it and when was it paid?
(c) What was the lowest amount showing in the balance column?
(d) What were the bank charges?
(e) Look at the transactions of 30 May in both Ms Mapfumo's Passbook and her Current Account. What did Ms Mapfumo do to clear her overdrawn balance? [Discuss why she did this.]
(f) Describe a quick way of checking that the final balance figure is correct. [See parts (e), (f), (g) of question 6.]

Exchange rates

An important service that banks provide is to exchange money from other currencies into dollars and vice versa. **Foreign exchange**, sometimes called *forex*, enables countries to trade with each other, despite their different currencies.

The value of currency varies from one country to another. However, the various currencies of the world are linked by agreed ratios, or **exchange rates**.

Table 30.1 shows the 1990 exchange rates in terms of foreign currency units per Zimbabwe dollar.

Table 30.1

country	monetary unit	Z$1 =
Botswana	Pula	P0,96
Germany	Deutsche Mark	DM0,90
Japan	Yen	64 yen
Mozambique	Metical	Me350
RSA	Rand	R1,20
Tanzania	Shilling	Sh65
UK	Pound	£0,30
USA	US Dollar	U$0,45
Zambia	Kwacha	K5,12

The above rates may be taken as guides only. Exchange rates vary from day to day. In Examples 6 and 7 the rates in Table 30.1 are used.

xample 6
change Z$50 *for (a) Pula, (b) Yen, (c) U$.*

Z$1 = P0,96
Z$50 = P0,96 × 50
= P48

Z$1 = 64 yen
Z$50 = 64 × 50 yen
= 3 200 yen

Z$1 = U$0,45
Z$50 = U$0,45 × 50
= U$22,50

xample 7
German tourist exchanges DM1 500 *into* Z$. *He nds* Z$1 058,20 *and converts the remaining* Z$ *o Zambian Kwacha. How many Kwacha does he t?*

rst, find how many Z$ the tourist gets:
DM0.90 = Z$1

$DM1 = Z\$\frac{1}{0.90}$

$M1\ 500 = Z\$\frac{1}{0.90} \times 1\ 500 = Z\$1\ 666{,}67$

ext, find the number of Z$ remaining:
$ remaining = Z$(1 666,67 − 1 058,20)
= Z$608,47
inally, convert Z$ to Kwacha:
Z$1 = K5,12
Z$608,47 = K5,12 × 608,47 = K3 115,37

ote:
(1) Unitary method was used in each case.
(2) In currency problems it is customary to round off to 2 decimal places.

Exercise 30c

Use the rates given in Table 30.1.
A calculator is recommended.

1 Find the equivalent of Z$60 in each of the following currencies.
(a) Deutsche Marks (b) Meticals
(c) Rands (d) Shillings
(e) US Dollars (f) Pounds

2 Convert the following amounts to U$.
(a) Z$100 (b) Z$20 (c) Z$70
(d) Z$24 (e) Z$7,50 (f) Z$67,43

3 Convert the following amounts to Z$.
(a) P240 (b) DM450 (c) R36
(d) U$100 (e) £75 (f) Sh400

4 How many Pounds would a trader get for Z$440?

5 A traveller pays U$20 Airport Tax. How much is this in Z$?

6 A visitor from Botswana changed P600 into Z$. She spent Z$420 then changed the remaining Z$ back to Pula. How many Pula did she get?

Depreciation and inflation

Many items, such as cars, clothes, electrical goods, lose value as time passes. This loss in value is called **depreciation**. Depreciation is usually given as a percentage of the item's value at the beginning of the year. For example, if a radio costing $100 depreciates by 20% per annum, then its value will be $80 at the end of the first year. At the end of the second year, its value will be $80 less 20% of $80 (or 80% of $80), i.e. $64.

Example 8
A freezer costing $1 700 *depreciates by* 25% *in its first year and* 20% *in its second year. Find its value after* 2 *years.*

1st year: Value of freezer	$1 700
− 25% depreciation	− 425
2nd year: Value of freezer	1 275
− 20% depreciation	− 255
Value after 2 years =	$1 020

Due to rising prices, money loses its value as time passes. Loss in value of money is called **inflation**. Inflation is a kind of depreciation. Inflation is usually given as the percentage increase in the cost of buying things from one year to the next. For example if the rate of inflation is 15% then a chair which cost $100 a year ago will now cost $115. Money has lost its value since it now costs more to buy the same thing.

Example 9

How long will it take for prices to double if the rate of inflation is 20% per annum?

Start with an original cost of 100 units:

initially,	cost = 100	
	rise = 20	
after 1 year,	cost = 120	
	rise = 24	(i.e. 20% of 120)
after 2 years,	cost = 144	
	rise = 28,8	(20% of 144)
after 3 years,	cost = 172,8	
	rise = 34,56	(20% of 172,8)
after 4 years,	cost = 207,36	

The cost after 4 years is a little more than double the original cost. Hence prices will double in just under 4 years.

Exercise 30d

Unless told otherwise, give answers correct to the nearest cent.

1 A car loses value each year by 20% of its value at the beginning of the year.
If a car costs $32 400, find its value 2 years later.

2 A new car costs $32 000. It depreciates by 25% in the first year, 20% in the 2nd year and 15% in each of the following years. Find the value of the car to the nearest $100 after 4 years.

3 A student pays $8 500 for a second-hand motor bike. The bike depreciates by 10% in the first year and, following some damage to the bike, by 30% in the 2nd year.
To the nearest $50, find how much the student can expect to get for the bike at the end of the second year.

4 A piece of land increases in value by 10% each year. By what percentage does its value increase over 3 years?
[*Hint:* Let the land have an initial value of 100 units.]

5 A building co-operative bought some land for $150 000. The land increased in value by 15% per annum.
What, to the nearest $100, was the land worth after 3 years?

[The last two questions are examples of **appreciation**: i.e. where items *increase in value* over time. Land, buildings and jewellery tend to appreciate with time.]

6 A mattress cost $400. Find the cost of buying the same kind of mattress in 2 years time if the rate of inflation is 15% per annum.

7 A new chair costs $100.
(a) Find the cost of buying the same kind of chair in 2 years time if the rates of inflation for the first and second years are 20% and 10% respectively.
(b) What would be the cost if the inflation rate was first 10%, followed by 20%?

8 Show that prices double in less than 3 years if the rate of inflation is 30% per annum. [*Hint:* Start with an initial cost of 100 units.]

9 A year ago a refrigerator cost $1 080 when new. Today a new one of the same kind costs $1 440.
(a) Calculate the percentage rate of inflation.
(b) If inflation stays at the same rate, what is the likely cost of the same kind of refrigerator next year?

10 The present cost of a radio is $96. If the rates of inflation for the next two years are 25% and 15% respectively, find the cost of buying the same kind of radio in 2 years time.

Data in newspapers

Newspapers often contain data which they present to readers in the form of tables, charts and graphs.

Data is most commonly given in tables. Figs. 30.8, 30.9, 30.10 give examples based on the foreign exchange, sport and commerce columns of a typical newspaper.

Notice that the exchange rates are given to four places of decimals.

Sports teams are often matched in leagues. The positions of the teams in the leagues are given in tables in newspapers. See Fig. 30.9.

CUSTOM FOREX RATES

The following exchange rates are used by the Department of Customs and Excise for converting foreign currency for the valuation of imported goods.

The rates may differ from those published by the Reserve Bank. This list was valid from April 13 to April 26 1991.

Country/currency	Rates/Z$
Australian dollar	0,5988
Botswana pula	0,9911
Burundi franc	76,8100
Chinese yuan	1,9263
EEC ecu	0,4395
French franc	3,1161
German mark	0,9128
Hong Kong dollar	4,0062
Indian rupee	7,7069
Japanese yen	64,1300
Kenya shilling	9,5282
Lesotho maloti	1,2269
Malawi kwacha	1,3278
Mozambican metical	323,3059
Nigerian naira	4,2332
RSA rand	1,2175
Swaziland lilongeni	1,2189
Tanzanian shilling	63,9667
Ugandan shilling	84,6900
UK pound	0,2851
USA dollar	0,5133
USSR rouble	0,3128
Zambian kwacha	5,1507

Fig. 30.8

Indoor Hockey
Women's First League

	P	W	D	L	F	A	Pts
Richwood Park	7	5	1	1	73	13	11
Scruples	7	5	0	2	78	16	10
Eagles	6	3	2	1	50	29	8
Alex	7	3	2	2	63	46	8
Raiders	7	3	0	4	47	58	6
Greendale	7	2	0	5	31	61	4
Postals	6	2	0	4	24	74	4
Zimbank	7	1	1	5	19	88	3

Fig. 30.9

The annual accounts of large companies are often published, either in advertisements or in the financial pages of the press. Fig. 30.10 is a typical statement for a large company which makes aerated drinking water.

Sometimes, newspapers use graphs and charts to give information.

SodaPop Pty Ltd
AUDITED RESULTS
FOR THE 52 WEEKS ENDED
31 DECEMBER 1991

	1991	1990	% Increase
Turnover	25 819 804	23 036 093	12,1
Taxation	2 512 575	2 355 293	6,7
Profit	919 144	874 821	5,1
Profit after taxation	1 593 431	1 480 472	7,6
Retaining profit bt f'ward	6 089 758	5 237 834	
Total capital	7 683 189	6 718 306	
Dividends: Ord shares	789 123	628 548	
Retained profit	6 894 066	6 089 758	

Fig. 30.10

Fig. 30.11 is a computer-drawn graph which shows the spot price of gold over a 24-hour period. Graphs like this are frequently published in the financial sections of newspapers.

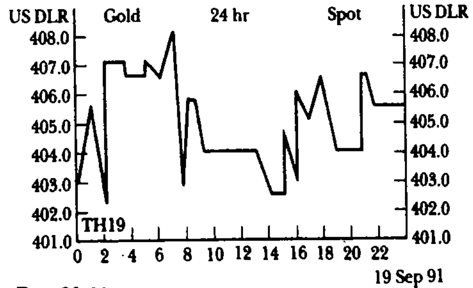


Fig. 30.11

Fig. 30.12 is a bar chart which was given in a newspaper. It shows the increase in secondary school enrolment in the first 10 years after Independence.

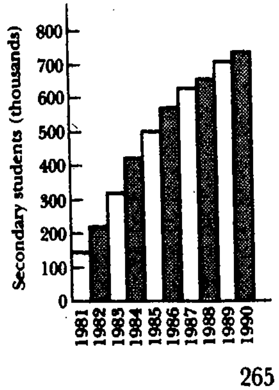


Fig. 30.12

Data in newspapers can often be informative. The following exercise should reveal some of the facts hidden in Figs. 30.8 to 30.12.

Exercise 30e

Refer to Figs. 30.8 to 30.12 when doing this exercise.

1 In Fig. 30.8 it can be seen that Kenya, Tanzania and Uganda each have shillings as their unit of currency. Which of the three shillings has the greater value?

2 Approximately how many Burundi francs are equivalent to 1 French franc?

3 By rounding off values to 1 significant figure, estimate how many Hong Kong dollars are equivalent to £1 (UK).

4 By rounding off values to 2 s.f., estimate how many Zambian kwacha are equivalent to 1 Malawian kwacha.

5 In the first seven months of 1990 Zimbabwe exported Z$100 000 000 worth of products to UK. How much forex (in pounds) did this create?

6 Use Fig. 30.9 to decide how many points a hockey team gets for winning, drawing and losing.

7 Which two hockey teams have not yet played each other?

8 How many matches have actually been played?

9 Which team has scored the greatest number of goals?

10 Greendale and Postals each have the same number of points. Suggest a reason why Greendale should be placed above Postals in the league.

11 Use Fig. 30.10 to find the actual increase in SodaPop's turnover from 1990 to 1991.

12 What amount will be carried forward to 1992 as 'retained profit'?

13 By what percentage did SodaPop's total capital increased from 1990 to 1991? [If you do not have a calculator, estimate the percentage increase.]

14 What percentage of the total capital was issued as share dividends in (a) 1990, and (b) 1991?
[Either use a calculator or estimate.]

15 What percentage of SodaPop's profits were taxed in (a) 1990, (b) 1991?
[Either use a calculator or estimate.]

16 The gold spot prices in Fig. 30.11 are given in US dollars per ounce.
On Thursday 19 September (a) what was the highest spot price and (b) when did it occur?

17 (a) What was the lowest spot price?
(b) When did it occur?

18 What was the spot price at 0400?

19 A gold dealer bought 200 ounces of gold at the beginning of the day and sold it at the close of the day. What was the dealer's profit on the transaction?
[Discuss whether you think that such profits are reasonable.]

20 The spot price remained steady at U$404 for two periods during the day. Estimate the length of each period.

21 Use Fig. 30.12 to estimate the numbers of students enrolled in secondary schools in (a) 1981, (b) 1990.

22 In which year did the secondary school enrolment figure pass the 600 000 mark?

23 How many years after independence did it take for the 1981 enrolment to treble?

24 If it took about 5 000 teachers to teach the students in 1981, approximately how many teachers were required in 1990?

25 Estimate the 1991 enrolment figure.

Revision exercises and tests

Chapters 21–30

Revision exercise 9

1 (a) Calculate the area of a rhombus whose diagonals are 10 cm and 9 cm.
(b) What angle does an arc of length 8,5 cm subtend at the centre of a circle of radius 8,5 cm?

2 The values of p and q are connected by the equation $y = x^2 + px + q$. When $x = -1$, $y = 0$ and when $x = -2$, $y = 0$. Find the values of p and q.

3 In a certain class, the ratio of boys to girls is 5 : 2. There are 12 girls in the class. How many students are in the class?

4 The pie chart in Fig. R30 shows the reading habits of students in a university.

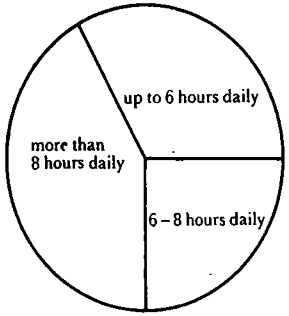


Fig. R30

What percentage of the students read for over 8 hours daily?

5 A gas cylinder is 75 cm long and holds 18 kg of liquid gas when full. How much gas will a similar cylinder, 50 cm long, hold when full?

6 A card is chosen at random from a pack of 52 cards. Find the probability that it is
(a) a six (b) a red six
(c) a club (d) a red club

7 (a) Show the solution sets of $3a - 7 < 12$ and $8 - 5a < 20$ on separate number lines.
(b) Hence show the solution set of the simultaneous inequalities $3a - 7 < 12$ and $8 - 5a < 20$ on a single number line.
(c) Write down the values of a which are even numbers.

8 A man took 40 min to run 8 km from town A to town B. He spent 25 min in town B. He then left town B in a taxi and arrived back at town A 10 min later. Represent this information on a graph using a scale of 1 cm to 5 min on the time axis and 1 cm to 1 km on the distance axis.

9 Using the graph drawn in question 8, or otherwise, find, in km/h,
(a) the average running speed of the man,
(b) the average speed of the taxi,
(c) the average speed of the man for the whole journey from A to B and back to A again (including the stop at B).

10 A motorist travelled the first 3 km of a journey at 80 km/h and the remaining distance of 8 km at 100 km/h. What is the average speed for the journey to the nearest km/h?

Revision test 9

1 The straight line $x + y = 2$ and the curve $x^2 + y^2 = 10$ intersect at the point (3; − 1). They also intersect at the point
A (− 3; 1) B (− 3; − 1) C (3; 1)
D (− 1; 3) E (1; − 3)

2 Increase $330 in the ratio 6 : 5.
A $180 B $275 C $360
D $390 E $396

3 Two similar cones have base diameters of 10 cm and 35 cm. The small cone is used to fill the big cone with rice. Approximately how many small cones will it take to fill the big cone?
A 4 B 7 C 12 D 16 E 43

4 A student doing question 3 in this test does not read the question and just picks one of the answers at random. What is the probability that it is the correct answer?
A 0 B $\frac{1}{5}$ C $\frac{1}{2}$ D $\frac{4}{5}$ E 1

5 What is the range of values of x shown in Fig. R31?

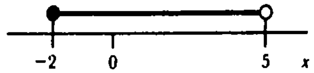


Fig. R31

A $-2 < x < 5$ B $5 < x < -2$
C $-2 \leqslant x < 5$ D $5 \leqslant x \leqslant -2$
E $-2 < x \leqslant 5$

6 The sides AB, CD of $\|^{gm}$ ABCD are produced to any points P and Q. Prove that Δs PCD and QAB are equal in area.

7 A Government spends its revenue as follows: 10% on Health Services, 30% on Education, 12% on Housing, 40% on Agriculture and 8% on other items. Draw a pie chart to show this information.

8 A motorist has an appointment in a town 130 km away at 1200. She starts at 10 a.m. and drives at 70 km/h. After 40 km she stops for $\frac{1}{2}$ hour for refreshment. By drawing a suitable graph, find her speed for the last 90 km if she is to reach the town on time.

9 Table R3 gives the numbers of pairs of shoes owned by 30 students in a class.

Table R3

number of pairs	1	2	3	4	5	6	7
number of students	10	6	5	3	3	2	1

(a) Find the mode, median and mean number of pairs of shoes for the class.
(b) If a student is picked at random from the class, what is the probability that he or she has more than 5 pairs of shoes?

10 Use the currency table in Fig 30.8 on page 265 to change the following amounts of money to Z\$. Give final answers to the nearest cent. If you do not have a calculator, round the given rates to 4 s.f. where necessary.
(a) 2 000 yuan (b) 500 ecu
(c) £60 (d) Sh250 (Kenya)

Revision exercise 10

1 ABCD is a trapezium with right angles at B and C. AB = 5 cm, DC = 12 cm and the area of the trapezium is 34 cm^2. Calculate A$\hat{D}$C.

2 Solve the following pairs of simultaneous equations.
(a) $y = \frac{1}{2}x$ $x + y = -6$
(b) $\frac{p}{3} + \frac{q}{5} = 4$ $\frac{p}{2} + \frac{3q}{2} + 12 = 0$

3 Divide 360° into four angles whose sizes are in the ratio 1 : 2 : 3 : 4.

4 The bar chart in Fig. R32 gives the estimated costs of providing rural health services for the years 1988 to 1990.

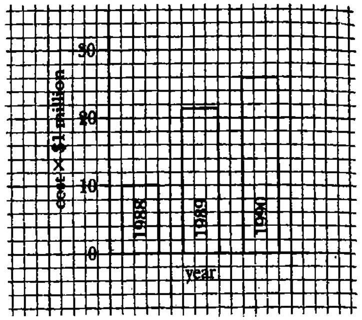


Fig. R32

(a) What was the cost in 1990?
(b) By how much did the 1990 cost exceed the 1988 cost?
(c) What was the average cost per year for the three years?

5 Two similarly shaped cooking pots are made from metal of the same thickness. They have capacities of 20 and 2,5 litres respectively. If the mass of the small pot is 1,5 kg when empty, what is the mass of the big pot when empty? (*Note:* The mass is proportional to the area of metal in the pot.)

6 There are x black balls and y white balls in a bag. A ball is picked at random.
(a) Write down an expression in x and y which gives the probability of picking a black ball.
(b) If there are 24 balls altogether, find how many are black if the probability of picking a white ball is $\frac{1}{6}$.

7 (a) On graph paper, draw the region defined by the three inequalities $y \geqslant 2$, $3x + y \geqslant 0$, $x + y < 3$. Leave the required region unshaded.
(b) Find the members of the solution set of the inequalities in part (a) given that it contains integral values of x and y only.

8 The performance of an aircraft during a flight was roughly as follows. It accelerated uniformly from rest for $\frac{1}{2}$ h until its speed was 800 km/h. After flying at this speed for 4 h it decelerated uniformly to rest in 24 min.
Draw a speed–time graph of the flight. Hence or otherwise find the distance travelled during the flight.

9 The mean of 12 numbers is 8,5. Two of the numbers are 9 and 13. What is the mean of the remaining ten numbers?

10 An article was bought for £26,50 in London. An import duty of 20% was paid on it in Harare. If £1 = \$3,50, what is the final cost of the article in dollars?

Revision test 10

1 What is the perimeter of a square of area $30\frac{1}{4}$ m^2?
A $5\frac{1}{2}$ m B $8\frac{1}{2}$ m C 11 m
D $15\frac{1}{2}$ m E 22 m

2 The average speed, in km/h, of a motorist who travels 72 km in 45 min is
A 45 B 54 C 64 D 88 E 96

3 The scale of a map is 1 : 2 500. How many m^2 does an area of 1 cm^2 on the map represent?
A 5 B 25 C 625 D 2 500 E 62 500

4 If x is an integer, what is the greatest value of x which satisfies $3x + 25 < 2 \leqslant x + 13$?
A -11 B -10 C -9 D -8 E -7

5 During a 'Sale', a large shop gives a $12\frac{1}{2}$% reduction on all marked prices. What will be the sale price of a table lamp marked at \$80?
A \$70 B \$72 C \$80 D \$88 E \$90

6 Solve the following pairs of simultaneous equations
(a) $3^x \times 3^y = 1$ (b) $x + y = 3$
$2^{2x-y} = 64$ $2x^2 + y^2 = 54$

7 Fig. R33 is a bar graph showing the numbers of cars produced by a factory during 6 weeks.

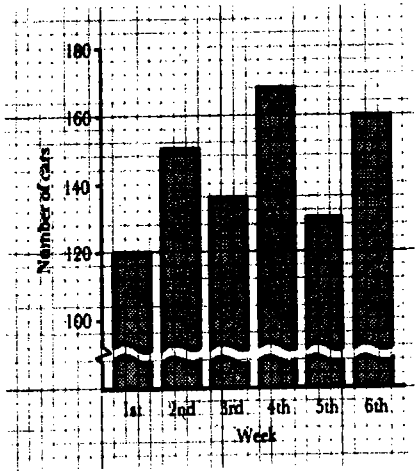


Fig. R33

(a) How many cars were produced altogether?
(b) Calculate the mean number of cars produced per week.

8 (a) A letter is chosen from the alphabet at random. What is the probability that it is contained in the word (i) MUTARE (ii) KWEKWE?
(b) 48% of the students of a school are girls. A student is picked at random. What is the probability that a boy is picked?

9 If $6(x + 1) > -4$ and $5 > 2(x - 3)$, what range of values of x and which integers satisfy both inequalities?

10 Ten pupils were asked to guess the number of grains of rice contained in a small tin. Their guesses were as follows:

100	90	50	60	80
25	40	100	60	100

Find the mean, median and mode of the guesses.

Revision exercise 11

1 In ΔABC a = 7,8 m, b = 8,5 m and $\hat{C}$ = 57°42′. Calculate the area of ΔABC.

2 Draw the graphs of the lines $y = 2x + 1$ and $2x + 2y = 7$ on the same axes. Find the coordinates of their point of intersection to the nearest decimal place.

3 Magnesium combines with oxygen in the ratio 3 : 2 by mass. How much magnesium would be needed to combine with 1,4 kg of oxygen? What would be the mass of the substance formed?

4 A 350 g packet of soap powder is of height 14 cm. Find the mass of soap powder contained in a similar packet 28 cm high.

5 Table R4 shows the result of a survey carried out to investigate the number of eggs in birds' nests.

Table R4

number of eggs	2	3	4	5	6	7
number of nests	4	6	24	50	12	4

(a) Find the modal and median number of eggs per nest.
(b) Calculate the mean number of eggs per nest.

6 A child finds a bird's nest with eggs in it. Use the data of Table R4 to estimate the probability that the nest contains
(a) 5 eggs,
(b) less than 5 eggs.

7 Choose a suitable graph to illustrate the data in Table R4.

8 (a) If $x - 6 \leq 1$ and $2x - 1 > 8$, what is the range of values of x which satisfies both inequalities?
(b) Sketch the graph of the range of values of x.
(c) Hence sketch another graph showing values of x which satisfy $x - 6 > 1$ and $2x - 1 \leq 8$.

9 Fig. R34 is the speed–time graph of a train journey.
Describe the four parts of the journey in your own words.

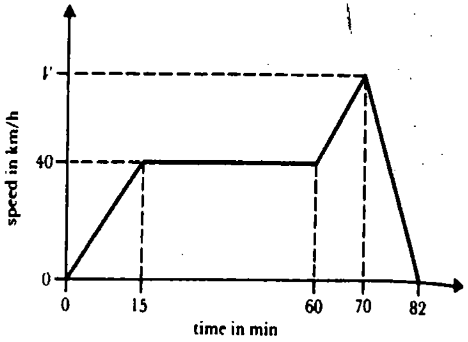


Fig. R34

10 Use Fig. R34 to answer the following.
(a) Find the initial acceleration in km/h².
(b) If the final deceleration is 270 km/h², find the maximum speed of the train, V km/h.
(c) Hence find the total distance travelled by the train.

Revision test 11

Fig. R35 is a pie chart showing the distribution of further education establishments in a country.

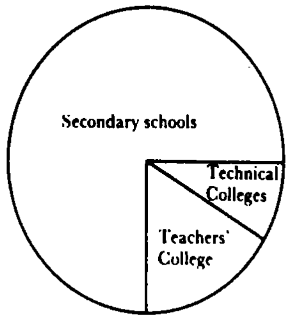


Fig. R35

Use Fig. R35 to answer questions 1 and 2.

1 If there are 14 Technical Colleges, how many Teachers' Colleges are there?
A 7 B 14 C 21 D 28 E 42

2 What is the ratio of Teachers' Colleges to Secondary Schools?
A 1 : 5 B 1 : 6 C 3 : 8
D 2 : 9 E 3 : 10

Fig. R36 is a graph showing the amount of water in a container during a period of ¼ hour. Use the graph to answer questions 3, 4, 5.

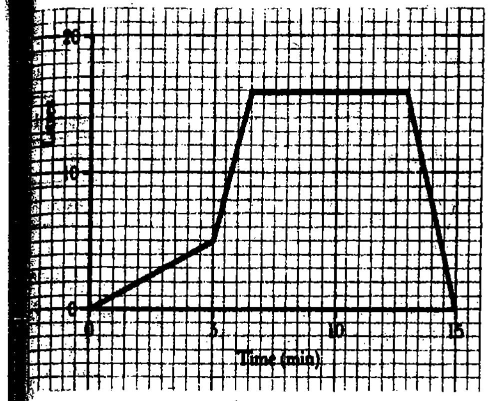


Fig. R36

3 For how long was the container being filled with water?
A 1½ min B 5 min C 6½ min
D 8 min E 11½ min

4 What was the greatest amount of water in the container?
A 10 ℓ B 13 ℓ C 16 ℓ
D 18 ℓ E 20 ℓ

5 At what rate did water pour out of the container?
A 1 ℓ/min B 2 ℓ/min C 4 ℓ/min
D 8 ℓ/min E 16 ℓ/min

6 The perimeter of a rectangular football pitch is 300 m. Its length is one and a half times its breadth. By solving two simultaneous equations find the length and breadth of the football pitch.

7 A town has a population density of 1 100 people per km². If the town covers an area of about 18 km², estimate its population to 1 s.f.

8 Two circular metal discs are of radius 9,9 cm and 13,2 cm respectively.
(a) Express the ratio of their areas in its simplest terms.
(b) The discs are melted down and recast as a single disc of the same thickness as before. Find the radius of this disc.

9 In Fig. R37 the lines k, m and n are the boundaries of the unshaded region which contains the solution set of three simultaneous inequalities.

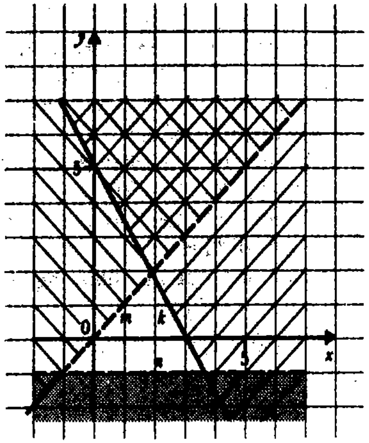


Fig. R37

(a) What are the equations of lines m and n?
(b) Given that the equation of k is $5x + 3y = 15$, write down the three inequalities that define the unshaded region.
(c) If the solution set contains integral values of x and y only, write down its members.

10 Beans costing 60c per packet are mixed with some other beans costing 90c per packet in the ratio 3 : 2. What is the cost of a packet of the mixture?

Revision exercise 12

1 Calculate the area of the major segment of a circle of radius 10 cm cut off by a chord of length 12 cm. (Take π to be 3,142.)

2 David bought 5 cups of beans and 4 cups of rice for $4,95. At the same market Chipo bought 4 cups of beans and 2 cups of rice for $3,06. Calculate the price of 1 cup of rice.

3 The scale of the plan of a building is 1 : 50.
(a) What length on the plan represents 12 m?
(b) What length on the building is represented by 9,6 cm?

4 The ratio of the areas of two similar rectangles is $\frac{8}{50}$. (a) Find the ratio of their lengths. (b) If the width of the smaller rectangle is 11 cm, find the width of the other rectangle.

5 A letter is chosen at random from the alphabet. Find the probability that it is
(a) X, (b) either X or Y,
(c) a consonant,
(d) one of the letters of the word CHAPTER,
(e) one of the letters of the word PARAGRAPH.

6 Sketch each of the following inequalities on a number line.
(a) $2 < x < 7$
(b) $-6 \leqslant x \leqslant 0$
(c) $-4 < x \leqslant -1$
(d) $-9 \leqslant x < 8$

7 In order to pass an examination consisting of four papers, a student must score a mean mark of 40% and not less than 33% in each paper. Table R5 shows the marks of three students.

Which student(s) passed the examination? Give reasons.

Table R5

	paper 1	paper 2	paper 3	paper 4
student A	30%	50%	60%	70%
student B	35%	36%	42%	44%
student C	60%	55%	38%	75%

Fig. R38 is a graph showing the journey made by Mr Muvuti between Rusape and Mutare. Use the graph to answer questions 8 and 9.

8 (a) How far is Mutare from Rusape?
(b) How long was Mr Muvuti in Mutare?
(c) Neglecting stops, how long did the journey take?
(d) How far from Mutare did Mr Muvuti stop on his way home?

9 (a) What was Mr Muvuti's average speed in the first hour?
(b) What was Mr Muvuti's average speed between Mutare and Rusape?

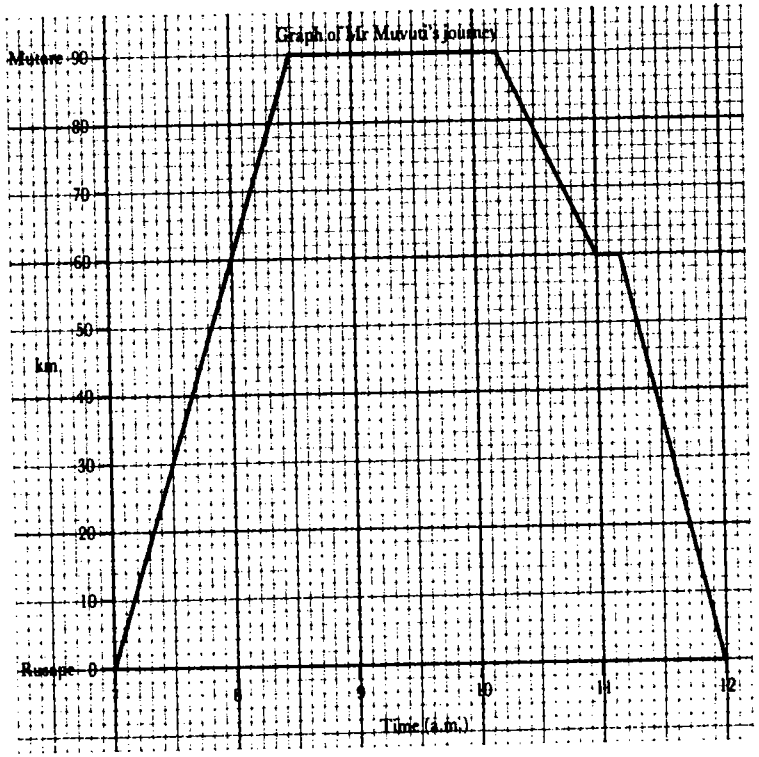


Fig. R38

10 (a) Make a blank Deposit Form like the one in Fig. 30.2 on page 259.
(b) Complete the deposit form, showing how a trader might enter the following amounts of money:
4 × $20 notes
13 × $10 notes
2 × $5 notes
16 × $2 notes
$13,54 in coins
a cheque for $61,99
a postal order for $22,50
(c) Enter the total amount of money in the correct place.

Revision test 12

1 Given that $3x + 7y = 1$ and $x - 7y = 19$, then $x + y =$
A -2 B -3 C 3 D 5 E 7

Table R6 shows the shoe sizes of some children. Use Table R6 to answer questions 2 and 3.

Table R6

shoe size	36	37	38	39	40
number of children	1	3	8	5	3

2 How many children are there?
A 1 B 8 C 20 D 24 E 40

3 Which size is the mode?
A 36 B 37 C 38 D 39 E 40

4 Three coins are tossed at the same time. List all the possible outcomes and hence decide the probability of getting two heads and a tail.
A $\frac{1}{3}$ B $\frac{1}{8}$ C $\frac{1}{4}$ D $\frac{3}{8}$ E $\frac{2}{3}$

5 A year ago a book cost $8. Today the same book costs $10. What is the rate of inflation?
A 2% B 7,5% C 12,5%
D 20% E 25%

6 Given Fig. R39, calculate (a) the area of $\|^{gm}$ ABCD, (b) AQ.

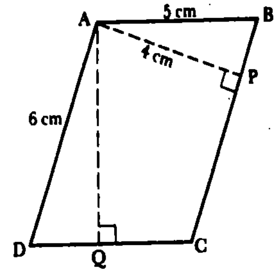


Fig. R39

7 A car uses petrol at the rate of 2 litres for every 21 km. How much petrol does the car use on a journey of 252 km?

8 Divide $135 in the ratio 2 : 3 : 4 : 6. Show your results in the form of a pie chart.

9 A road sign is in the shape of a metal triangle of height 70 cm and costs $11,76. How much will a similar road sign of height 1 m cost?

10 Express the inequality $x + 2 < 10 < x + 17$ in the form $a < x < b$ where a and b are integers. Sketch a line graph showing the inequality.

Non-routine problems (3)

1 (a) Given a 4 × 4 grid and 4 frogs, how many different ways can the frogs be placed in the grid so that there is never more than one frog in any row or column? Fig. Q16 gives one arrangement.

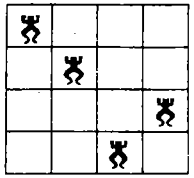


Fig. Q16

Note: reflections and rotations do not count as different.
(b) What if there were just 3 frogs? What about 2 × 2 grid with 2 frogs, a 3 × 3 grid with 3 frogs, and so on? How many arrangements for an $n \times n$ grid with n frogs? What if the frogs were placed in a rectangular grid?

2   *1 2 3 4 5 6 7 8 9 = 100*
Place arithmetical signs between the digits 1 to 9 so that their outcome is 100. You must not alter the order of the given digits.
[There are many solutions; try to find the one which uses least signs.]

3 Fig. Q17 is an example of an 8-point jewel pattern. It is created by joining every vertex of a regular octagon to every other vertex.

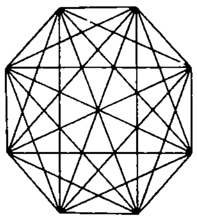


Fig. Q17

(a) How many lines are there joining the vertices?
(b) Describe the main difference between the patterns of jewels which have an odd number of points and those with an even number of points.

(c) How many lines will there be joining the vertices of an n-point jewel? [It may help if you investigate what happens in simpler cases. For example, Fig. Q18 shows 3-point, 4-point and 5-point jewels and their corresponding numbers of lines.]

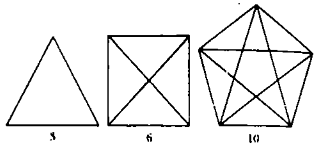


Fig. Q18

4 How many triangles (of any size) are there in the triangle in Fig. Q19?

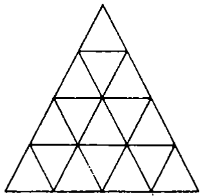


Fig. Q19

[It may help if you consider simpler cases and look for patterns.]

5 A woman had a basket of eggs for sale. She had 4 customers.
The 1st customer bought half her eggs and half an egg. The 2nd customer bought half of those remaining and half an egg. The 3rd customer bought half of those remaining and half an egg. The 4th customer bought half of those remaining and half an egg.
The woman was left with 8 eggs and did not break any when trading.
(a) How many eggs did she start with?
(b) How many eggs did each customer buy?

6 A die rests on a table.
Rose, who is sitting on one side of the table can see 3 faces and a total of 11 spots. Sam, who is on the opposite side of the table can see 3 faces and a total of 7 spots.
How many spots are there on the bottom face of the die?

7 Cut out three cardboard pieces as shown in Fig. Q20. Find at least eight ways of fitting all three pieces together to make a shape that has bilateral symmetry.

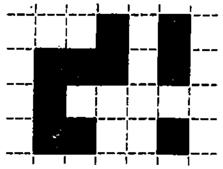


Fig. Q20

8 Many numbers can be written as the sum of consecutive integers, e.g.
12 = 3 + 4 + 5
13 = 6 + 7
In some cases, numbers can be expressed as two or more sums of consecutive integers:
9 = 2 + 3 + 4 *or* 4 + 5
15 = 7 + 8 *or* 4 + 5 + 6 *or* 1 + 2 + 3 + 4 + 5
Investigate for other numbers (e.g. can all numbers be represented in the same way? Are there any rules?).

9 [14] ⟶ [7] ⟶ [22] ⟶ [11] ⟶ [34] ⟶
The above number chain is formed according to the following rules:
(i) If the number is even, divide by 2.
(ii) If the number is odd, multiply by 3 and add 1.
Investigate what happens (a) if you continue the chain, (b) if you start with a different number, (c) if you change the rules.

10 Fig. Q21 shows a series of circles partitioned into regions by joining 1, 2, 3, 4, ... dots randomly placed on their circumferences.

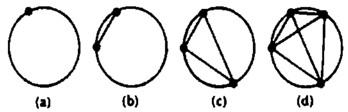


Fig. Q21

Find a rule which will give the number of (a) straight lines and (b) regions formed by n points on the circumference.

11 Fig. Q22 shows a '1–2–3 spiral'.

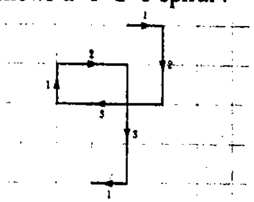


Fig. Q22

To draw the spiral, start with a line of 1 unit length, turn through 90° and draw a line of 2 units, turn through another 90° and draw a line of 3 units. Then repeat this cycle.
(a) Working on squared paper, continue the above spiral.
(b) Investigate other spirals (e.g. 1–2–4, 1–2–3–2–1, 2–4–1–3, 1–2–3–4–1). What do you notice? Try to classify your results.

12 The Braille alphabet is used by many blind people. It uses a system of raised dots based on a 3 × 2 rectangle. Fig. Q23 shows the basic rectangle and two of the letters.

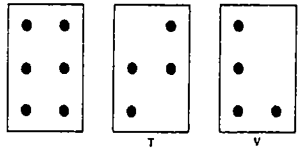


Fig. Q23

(a) How many different patterns can be made in this system? (b) Investigate for other sized rectangles.

13 In Fig. Q24 the square is of side 1 m and the four arcs are quarter circles. Find, in terms of π, the area of the shaded region.

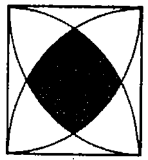


Fig. Q24

14 *abcd* is a four-figure number such that *abcd* = 4 × *dcba*. Find which digits the letters *a*, *b*, *c*, *d* represent.

15 Ten cars are parked in three streets. No two of the three streets have the same number of cars and no street is empty of cars.
Ephraim, Joseph and Ruth live one in each of the three streets. Each can see the cars in their own street, but not those in the other two. Ephraim phones Ruth, 'Is there an even or odd number of cars in your street?' Ruth's reply enables him to deduce the number of cars in the other two streets. How many cars are in Ephraim's street?

Mensuration tables and formulae, four-figure tables

SI units

Mass

The **gramme** is the basic unit of mass.

Unit	Abbreviation	Basic unit
1 kilogramme	1 kg	1 000 g
1 hectogramme	1 hg	100 g
1 decagramme	1 dag	10 g
1 gramme	1 g	1 g
1 decigramme	1 dg	0,1 g
1 centigramme	1 cg	0,01 g
1 milligramme	1 mg	0,001 g

The **tonne** (t) is used for large masses. The most common measures of mass are the milligramme, the gramme, the kilogramme and the tonne.

1 g = 1 000 mg
1 kg = 1 000 g = 1 000 000 mg
1 t = 1 000 kg = 1 000 000 g

Time

The **second** is the basic unit of time.

Unit	Abbreviation	Basic unit
1 second	1 s	1 s
1 minute	1 min	60 s
1 hour	1 h	3 600 s

Length

The **metre** is the basic unit of length.

Unit	Abbreviation	Basic unit
1 kilometre	1 km	1 000 m
1 hectometre	1 hm	100 m
1 decametre	1 dam	10 m
1 metre	1 m	1 m
1 decimetre	1 dm	0,1 m
1 centimetre	1 cm	0,01 m
1 millimetre	1 mm	0,001 m

The most common measures are the millimetre, the metre and the kilometre.

1 m = 1 000 mm
1 km = 1 000 m = 1 000 000 mm

Area

The **square metre** is the basic unit of area. Units of area are derived from units of length.

Unit	Abbreviation	Relation to other units of area
square millimetre	mm^2	
square centimetre	cm^2	$1\ cm^2 = 100\ mm^2$
square metre	m^2	$1\ m^2 = 10\ 000\ cm^2$
square kilometre	km^2	$1\ km^2 = 1\ 000\ 000\ m^2$
hectare (for land measure)	ha	$1\ ha = 10\ 000\ m^2$

Volume

The **cubic metre** is the basic unit of volume. Units of volume are derived from units of length.

Unit	Abbreviation	Relation to other units of volume
cubic millimetre	mm^3	
cubic centimetre	cm^3	$1\ cm^3 = 1\ 000\ mm^3$
cubic metre	m^3	$1\ m^3 = 1\ 000\ 000\ cm^3$

Capacity

The **litre** is the basic unit of capacity. 1 litre takes up the same space as 1 000 cm^3.

Unit	Abbreviation	Relation to other units of capacity	Relation to units of volume
millilitre	mℓ		$1\ m\ell = 1\ cm^3$
litre	ℓ	1 ℓ = 1 000 mℓ	$1\ \ell = 1\ 000\ cm^3$
kilolitre	kℓ	1 kℓ = 1 000 ℓ	$1\ k\ell = 1\ m^3$

The calendar

Remember this poem:

Thirty days have September,
April, June and November.
All the rest have thirty-one,
Excepting February alone;
This has twenty-eight days clear,
And twenty-nine in each Leap Year.

In a Leap Year, the date is divisible by 4. Thus 1984 was a Leap Year. Century year dates, such as 1900 and 2000, are Leap Years only if they are divisible by 400. Thus 1900 was not a Leap Year but 2000 will be a Leap Year.

Money

Some African currencies

Zimbabwe	100 cents (c)	= 1 Dollar ($)
Botswana	100 thebe (t)	= 1 Pula (P)
Kenya	100 cents (c)	= 1 Shilling (Sh)
Malawi	100 tambala (t)	= 1 Kwacha (K)
Mozambique	100 centavos (c)	= 1 Metical (M)
Nigeria	100 kobo (k)	= 1 Naira (₦)
Zambia	100 ngwee (n)	= 1 Kwacha (K)

Other currencies

Britain	100 pence (p) = 1 Pound (£)
USA	100 cents (¢) = 1 Dollar ($)

Exchange rates

At the time of going to press, $1 (Zimbabwe) was equivalent to the following.

US Dollar	$0,20
UK Pound	£0,11
Botswana Pula	P0,42
Kenya Shilling	Sh6,39
Mozambique Metical	M470
Zambia Kwacha	K30,14

Note: Exchange rates change from day to day. The above rates are only approximate.

Multiplication table

×	1	2	3	4	5	6	7	8	9	10
1	1	2	3	4	5	6	7	8	9	10
2	2	4	6	8	10	12	14	16	18	20
3	3	6	9	12	15	18	21	24	27	30
4	4	8	12	16	20	24	28	32	36	40
5	5	10	15	20	25	30	35	40	45	50
6	6	12	18	24	30	36	42	48	54	60
7	7	14	21	28	35	42	49	56	63	70
8	8	16	24	32	40	48	56	64	72	80
9	9	18	27	36	45	54	63	72	81	90
10	10	20	30	40	50	60	70	80	90	100

Divisibility tests

Any whole number is exactly divisible by

2 if its last digit is even
3 if the sum of its digits is divisible by 3
4 if its last two digits form a number divisible by 4
5 if its last digit is 5 or 0
6 if its last digit is even and the sum of its digits is divisible by 3
8 if its last three digits form a number divisible by 8
9 if the sum of its digits is divisible by 9
10 if its last digit is 0

Mensuration formulae

Plane shapes	Perimeter	Area
Square side s	$4s$	s^2
Rectangle length l, breadth b	$2(l + b)$	lb
Triangle base b, height h		$\frac{1}{2}bh$
Parallelogram base b, height h		bh
Trapezium height h, parallels a and b		$\frac{1}{2}(a + b)h$
Circle radius r	$2\pi r$	πr^2
Sector of circle radius r, angle θ	$2r + \frac{\theta}{360}2\pi r$	$\frac{\theta}{360}\pi r^2$

Solid shapes	Area	Volume
Cube edge s	$6s^2$	s^3
Cuboid length l, breadth b, height h	$2(lb + bh + lh)$	lbh
Prism		Ah
Cylinder radius r, height h	$2\pi rh + 2\pi r^2$	$\pi r^2 h$
Cone radius r, slant height l, height h	$\pi rl + \pi r^2$	$\frac{1}{3}\pi r^2 h$
Sphere radius r	$4\pi r^2$	$\frac{4}{3}\pi r^3$

Trigonometrical formulae

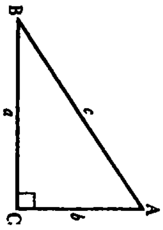


Fig. T1

In the right-angled triangle shown,
$c^2 = a^2 + b^2$ (*Pythagoras' theorem*)

$\tan B = \frac{b}{a}$ $\sin B = \frac{b}{c}$ $\cos B = \frac{a}{c}$

$\tan A = \frac{a}{b}$ $\sin A = \frac{a}{c}$ $\cos A = \frac{b}{c}$

Symbols

Symbol	Meaning
=	is equal to
≠	is not equal to
≃	is approximately equal to
≡	is identical or congruent to
⇔	is equivalent to
∝	is proportional to
>	is greater than
<	is less than
⩾	is greater than or equal to
⩽	is less than or equal to
°	degree (angle)
°C	degrees Celsius (temperature)
A, B, C,	points (geometry)
AB	the line through points A and B, *or* the distance between points A and B
△ABC	triangle ABC
∥gm ABCD	parallelogram ABCD
A$\hat{B}$C	angle ABC
⊥	is perpendicular to
∥	is parallel to
∴	therefore
π	pi
%	per cent
A= {p; q; r}	A is the set p, q, r
B = {1; 2; 3; ...}	B is the infinite set 1, 2, 3, etc
C = {x: x is an integer}	Set builder notation. C is the set of numbers x such that x is an integer
n(A)	number of elements in set A
∈	is an element of
∉	is not an element of
A′	complement of A
{} or ∅	the empty set
ℰ	the universal set
A ⊆ B	A is a subset of B
A ⊇ B	A contains B
⊈, ⊉	negations of ⊆ and ⊇
A ∪ B	union of A and B
A ∩ B	intersection of A and B

Logarithms

$x \rightarrow \log x$

x	0	1	2	3	4	5	6	7	8	9	Differences 1	2	3	4	5	6	7	8	9
10	0000	0043	0086	0128	0170	0212	0253	0294	0334	0374	4	8	12	17	21	25	29	33	37
11	0414	0453	0492	0531	0569	0607	0645	0682	0719	0755	4	8	11	15	19	23	26	30	34
12	0792	0828	0864	0899	0934	0969	1004	1038	1072	1106	3	7	10	14	17	21	24	28	31
13	1139	1173	1206	1239	1271	1303	1335	1367	1399	1430	3	6	10	13	16	19	23	26	29
14	1461	1492	1523	1553	1584	1614	1644	1673	1703	1732	3	6	9	12	15	18	21	24	27
15	1761	1790	1818	1847	1875	1903	1931	1959	1987	2014	3	6	8	11	14	17	20	22	25
16	2041	2068	2095	2122	2148	2175	2201	2227	2253	2279	3	5	8	11	13	16	18	21	24
17	2304	2330	2355	2380	2405	2430	2455	2480	2504	2529	2	5	7	10	12	15	17	20	22
18	2553	2577	2601	2625	2648	2672	2695	2718	2742	2765	2	5	7	9	12	14	16	19	21
19	2788	2810	2833	2856	2878	2900	2923	2945	2967	2989	2	4	7	9	11	13	16	18	20
20	3010	3032	3054	3075	3096	3118	3139	3160	3181	3201	2	4	6	8	11	13	15	17	19
21	3222	3243	3263	3284	3304	3324	3345	3365	3385	3404	2	4	6	8	10	12	14	16	18
22	3424	3444	3464	3483	3502	3522	3541	3560	3579	3598	2	4	6	8	10	12	14	15	17
23	3617	3636	3655	3674	3692	3711	3729	3747	3766	3784	2	4	6	7	9	11	13	15	17
24	3802	3820	3838	3856	3874	3892	3909	3927	3945	3962	2	4	5	7	9	11	12	14	16
25	3979	3997	4014	4031	4048	4065	4082	4099	4116	4133	2	3	5	7	9	10	12	14	15
26	4150	4166	4183	4200	4216	4232	4249	4265	4281	4298	2	3	5	7	8	10	11	13	15
27	4314	4330	4346	4362	4378	4393	4409	4425	4440	4456	2	3	5	6	8	9	11	13	14
28	4472	4487	4502	4518	4533	4548	4564	4579	4594	4609	2	3	5	6	8	9	11	12	14
29	4624	4639	4654	4669	4683	4698	4713	4728	4742	4757	1	3	4	6	7	9	10	12	13
30	4771	4786	4800	4814	4829	4843	4857	4871	4886	4900	1	3	4	6	7	9	10	11	13
31	4914	4928	4942	4955	4969	4983	4997	5011	5024	5038	1	3	4	6	7	8	10	11	12
32	5051	5065	5079	5092	5105	5119	5132	5145	5159	5172	1	3	4	5	7	8	9	11	12
33	5185	5198	5211	5224	5237	5250	5263	5276	5289	5302	1	3	4	5	6	8	9	10	12
34	5315	5328	5340	5353	5366	5378	5391	5403	5416	5428	1	3	4	5	6	8	9	10	11
35	5441	5453	5465	5478	5490	5502	5514	5527	5539	5551	1	2	4	5	6	7	9	10	11
36	5563	5575	5587	5599	5611	5623	5635	5647	5658	5670	1	2	4	5	6	7	8	10	11
37	5682	5694	5705	5717	5729	5740	5752	5763	5775	5786	1	2	3	5	6	7	8	9	10
38	5798	5809	5821	5832	5843	5855	5866	5877	5888	5899	1	2	3	5	6	7	8	9	10
39	5911	5922	5933	5944	5955	5966	5977	5988	5999	6010	1	2	3	4	5	7	8	9	10
40	6021	6031	6042	6053	6064	6075	6085	6096	6107	6117	1	2	3	4	5	6	8	9	10
41	6128	6138	6149	6160	6170	6180	6191	6201	6212	6222	1	2	3	4	5	6	7	8	9
42	6232	6243	6253	6263	6274	6284	6294	6304	6314	6325	1	2	3	4	5	6	7	8	9
43	6335	6345	6355	6365	6375	6385	6395	6405	6415	6425	1	2	3	4	5	6	7	8	9
44	6435	6444	6454	6464	6474	6484	6493	6503	6513	6522	1	2	3	4	5	6	7	8	9
45	6532	6542	6551	6561	6571	6580	6590	6599	6609	6618	1	2	3	4	5	6	7	8	9
46	6628	6637	6646	6656	6665	6675	6684	6693	6702	6712	1	2	3	4	5	6	7	7	8
47	6721	6730	6739	6749	6758	6767	6776	6785	6794	6803	1	2	3	4	5	5	6	7	8
48	6812	6821	6830	6839	6848	6857	6866	6875	6884	6893	1	2	3	4	4	5	6	7	8
49	6902	6911	6920	6928	6937	6946	6955	6964	6972	6981	1	2	3	4	4	5	6	7	8
50	6990	6998	7007	7016	7024	7033	7042	7050	7059	7067	1	2	3	3	4	5	6	7	8
51	7076	7084	7093	7101	7110	7118	7126	7135	7143	7152	1	2	3	3	4	5	6	7	8
52	7160	7168	7177	7185	7193	7202	7210	7218	7226	7235	1	2	2	3	4	5	6	7	7
53	7243	7251	7259	7267	7275	7284	7292	7300	7308	7316	1	2	2	3	4	5	6	6	7
54	7324	7332	7340	7348	7356	7364	7372	7380	7388	7396	1	2	2	3	4	5	6	6	7

x	0	1	2	3	4	5	6	7	8	9	Differences 1	2	3	4	5	6	7	8	9
55	7404	7412	7419	7427	7435	7443	7451	7459	7466	7474	1	2	2	3	4	5	5	6	7
56	7482	7490	7497	7505	7513	7520	7528	7536	7543	7551	1	2	2	3	4	5	5	6	7
57	7559	7566	7574	7582	7589	7597	7604	7612	7619	7627	1	2	2	3	4	5	5	6	7
58	7634	7642	7649	7657	7664	7672	7679	7686	7694	7701	1	1	2	3	4	4	5	6	7
59	7709	7716	7723	7731	7738	7745	7752	7760	7767	7774	1	1	2	3	4	4	5	6	7
60	7782	7789	7796	7803	7810	7818	7825	7832	7839	7846	1	1	2	3	4	4	5	6	6
61	7853	7860	7868	7875	7882	7889	7896	7903	7910	7917	1	1	2	3	4	4	5	6	6
62	7924	7931	7938	7945	7952	7959	7966	7973	7980	7987	1	1	2	3	3	4	5	6	6
63	7993	8000	8007	8014	8021	8028	8035	8041	8048	8055	1	1	2	3	3	4	5	5	6
64	8062	8069	8075	8082	8089	8096	8102	8109	8116	8122	1	1	2	3	3	4	5	5	6
65	8129	8136	8142	8149	8156	8162	8169	8176	8182	8189	1	1	2	3	3	4	5	5	6
66	8195	8202	8209	8215	8222	8228	8235	8241	8248	8254	1	1	2	3	3	4	5	5	6
67	8261	8267	8274	8280	8287	8293	8299	8306	8312	8319	1	1	2	3	3	4	5	5	6
68	8325	8331	8338	8344	8351	8357	8363	8370	8376	8382	1	1	2	3	3	4	4	5	6
69	8388	8395	8401	8407	8414	8420	8426	8432	8439	8445	1	1	2	2	3	4	4	5	6
70	8451	8457	8463	8470	8476	8482	8488	8494	8500	8506	1	1	2	2	3	4	4	5	6
71	8513	8519	8525	8531	8537	8543	8549	8555	8561	8567	1	1	2	2	3	4	4	5	5
72	8573	8579	8585	8591	8597	8603	8609	8615	8621	8627	1	1	2	2	3	4	4	5	5
73	8633	8639	8645	8651	8657	8663	8669	8675	8681	8686	1	1	2	2	3	4	4	5	5
74	8692	8698	8704	8710	8716	8722	8727	8733	8739	8745	1	1	2	2	3	4	4	5	5
75	8751	8756	8762	8768	8774	8779	8785	8791	8797	8802	1	1	2	2	3	3	4	5	5
76	8808	8814	8820	8825	8831	8837	8842	8848	8854	8859	1	1	2	2	3	3	4	5	5
77	8865	8871	8876	8882	8887	8893	8899	8904	8910	8915	1	1	2	2	3	3	4	4	5
78	8921	8927	8932	8938	8943	8949	8954	8960	8965	8971	1	1	2	2	3	3	4	4	5
79	8976	8982	8987	8993	8998	9004	9009	9015	9020	9025	1	1	2	2	3	3	4	4	5
80	9031	9036	9042	9047	9053	9058	9063	9069	9074	9079	1	1	2	2	3	3	4	4	5
81	9085	9090	9096	9101	9106	9112	9117	9122	9128	9133	1	1	2	2	3	3	4	4	5
82	9138	9143	9149	9154	9159	9165	9170	9175	9180	9186	1	1	2	2	3	3	4	4	5
83	9191	9196	9201	9206	9212	9217	9222	9227	9232	9238	1	1	2	2	3	3	4	4	5
84	9243	9248	9253	9258	9263	9269	9274	9279	9284	9289	1	1	2	2	3	3	4	4	5
85	9294	9299	9304	9309	9315	9320	9325	9330	9335	9340	1	1	2	2	3	3	4	4	5
86	9345	9350	9355	9360	9365	9370	9375	9380	9385	9390	1	1	2	2	3	3	4	4	5
87	9395	9400	9405	9410	9415	9420	9425	9430	9435	9440	0	1	1	2	2	3	3	4	4
88	9445	9450	9455	9460	9465	9469	9474	9479	9484	9489	0	1	1	2	2	3	3	4	4
89	9494	9499	9504	9509	9513	9518	9523	9528	9533	9538	0	1	1	2	2	3	3	4	4
90	9542	9547	9552	9557	9562	9566	9571	9576	9581	9586	0	1	1	2	2	3	3	4	4
91	9590	9595	9600	9605	9609	9614	9619	9624	9628	9633	0	1	1	2	2	3	3	4	4
92	9638	9643	9647	9653	9657	9661	9666	9671	9675	9680	0	1	1	2	2	3	3	4	4
93	9685	9689	9694	9699	9703	9708	9713	9717	9722	9727	0	1	1	2	2	3	3	4	4
94	9731	9736	9741	9745	9750	9754	9759	9763	9768	9773	0	1	1	2	2	3	3	4	4
95	9777	9782	9786	9791	9795	9800	9805	9809	9814	9818	0	1	1	2	2	3	3	4	4
96	9823	9827	9832	9836	9841	9845	9850	9854	9859	9863	0	1	1	2	2	3	3	4	4
97	9868	9872	9877	9881	9886	9890	9894	9899	9903	9908	0	1	1	2	2	3	3	4	4
98	9912	9917	9921	9926	9930	9934	9939	9943	9948	9952	0	1	1	2	2	3	3	4	4
99	9956	9961	9965	9969	9974	9978	9983	9987	9991	9996	0	1	1	2	2	3	3	3	4

Antilogarithms

$x \rightarrow 10^x$

x	0	1	2	3	4	5	6	7	8	9	Diff. 1	2	3	4	5	6	7	8	9
,00	1000	1002	1005	1007	1009	1012	1014	1016	1019	1021	0	0	1	1	1	1	2	2	2
,01	1023	1026	1028	1030	1033	1035	1038	1040	1042	1045	0	0	1	1	1	1	2	2	2
,02	1047	1050	1052	1054	1057	1059	1062	1064	1067	1069	0	0	1	1	1	1	2	2	2
,03	1072	1074	1076	1079	1081	1084	1086	1089	1091	1094	0	0	1	1	1	1	2	2	2
,04	1096	1099	1102	1104	1107	1109	1112	1114	1117	1119	0	1	1	1	1	2	2	2	2
,05	1122	1125	1127	1130	1132	1135	1138	1140	1143	1146	0	1	1	1	1	2	2	2	2
,06	1148	1151	1153	1156	1159	1161	1164	1167	1169	1172	0	1	1	1	1	2	2	2	2
,07	1175	1178	1180	1183	1186	1189	1191	1194	1197	1199	0	1	1	1	1	2	2	2	2
,08	1202	1205	1208	1211	1213	1216	1219	1222	1225	1227	0	1	1	1	1	2	2	2	3
,09	1230	1233	1236	1239	1242	1245	1247	1250	1253	1256	0	1	1	1	1	2	2	2	3
,10	1259	1262	1265	1268	1271	1274	1276	1279	1282	1285	0	1	1	1	1	2	2	2	3
,11	1288	1291	1294	1297	1300	1303	1306	1309	1312	1315	0	1	1	1	2	2	2	2	3
,12	1318	1321	1324	1327	1330	1334	1337	1340	1343	1346	0	1	1	1	2	2	2	2	3
,13	1349	1352	1355	1358	1361	1365	1368	1371	1374	1377	0	1	1	1	2	2	2	3	3
,14	1380	1384	1387	1390	1393	1396	1400	1403	1406	1409	0	1	1	1	2	2	2	3	3
,15	1413	1416	1419	1422	1426	1429	1432	1435	1439	1442	0	1	1	1	2	2	2	3	3
,16	1445	1449	1452	1455	1459	1462	1466	1469	1472	1476	0	1	1	1	2	2	2	3	3
,17	1479	1483	1486	1489	1493	1496	1500	1503	1507	1510	0	1	1	1	2	2	2	3	3
,18	1514	1517	1521	1524	1528	1531	1535	1538	1542	1545	0	1	1	1	2	2	2	3	3
,19	1549	1552	1556	1560	1563	1567	1570	1574	1578	1581	0	1	1	1	2	2	3	3	3
,20	1585	1589	1592	1596	1600	1603	1607	1611	1614	1618	0	1	1	1	2	2	3	3	3
,21	1622	1626	1629	1633	1637	1641	1644	1648	1652	1656	0	1	1	2	2	2	3	3	3
,22	1660	1663	1667	1671	1675	1679	1683	1687	1690	1694	0	1	1	2	2	2	3	3	3
,23	1698	1702	1706	1710	1714	1718	1722	1726	1730	1734	0	1	1	2	2	2	3	3	4
,24	1738	1742	1746	1750	1754	1758	1762	1766	1770	1774	0	1	1	2	2	2	3	3	4
,25	1778	1782	1786	1791	1795	1799	1803	1807	1811	1816	0	1	1	2	2	2	3	3	4
,26	1820	1824	1828	1832	1837	1841	1845	1849	1854	1858	0	1	1	2	2	3	3	3	4
,27	1862	1866	1871	1875	1879	1884	1888	1892	1897	1901	0	1	1	2	2	3	3	3	4
,28	1905	1910	1914	1919	1923	1928	1932	1936	1941	1945	0	1	1	2	2	3	3	4	4
,29	1950	1954	1959	1963	1968	1972	1977	1982	1986	1991	0	1	1	2	2	3	3	4	4
,30	1995	2000	2004	2009	2014	2018	2023	2028	2032	2037	0	1	1	2	2	3	3	4	4
,31	2042	2046	2051	2056	2061	2065	2070	2075	2080	2084	0	1	1	2	2	3	3	4	4
,32	2089	2094	2099	2104	2109	2113	2118	2123	2128	2133	0	1	1	2	2	3	3	4	4
,33	2138	2143	2148	2153	2158	2163	2168	2173	2178	2183	0	1	1	2	2	3	3	4	4
,34	2188	2193	2198	2203	2208	2213	2218	2223	2228	2234	1	1	2	2	3	3	4	4	5
,35	2239	2244	2249	2254	2259	2265	2270	2275	2280	2286	1	1	2	2	3	3	4	4	5
,36	2291	2296	2301	2307	2312	2317	2323	2328	2333	2339	1	1	2	2	3	3	4	4	5
,37	2344	2350	2355	2360	2366	2371	2377	2382	2388	2393	1	1	2	2	3	3	4	4	5
,38	2399	2404	2410	2415	2421	2427	2432	2438	2443	2449	1	1	2	2	3	3	4	4	5
,39	2455	2460	2466	2472	2477	2483	2489	2495	2500	2506	1	1	2	2	3	3	4	5	5
,40	2512	2518	2523	2529	2535	2541	2547	2553	2559	2564	1	1	2	2	3	4	4	5	5
,41	2570	2576	2582	2588	2594	2600	2606	2612	2618	2624	1	1	2	2	3	4	4	5	5
,42	2630	2636	2642	2649	2655	2661	2667	2673	2679	2685	1	1	2	2	3	4	4	5	6
,43	2692	2698	2704	2710	2716	2723	2729	2735	2742	2748	1	1	2	3	3	4	4	5	6
,44	2754	2761	2767	2773	2780	2786	2793	2799	2805	2812	1	1	2	3	3	4	4	5	6
,45	2818	2825	2831	2838	2844	2851	2858	2864	2871	2877	1	1	2	3	3	4	5	5	6
,46	2884	2891	2897	2904	2911	2917	2924	2931	2938	2944	1	1	2	3	3	4	5	5	6
,47	2951	2958	2965	2972	2979	2985	2992	2999	3006	3013	1	1	2	3	3	4	5	5	6
,48	3020	3027	3034	3041	3048	3055	3062	3069	3076	3083	1	1	2	3	4	4	5	6	6
,49	3090	3097	3105	3112	3119	3126	3133	3141	3148	3155	1	1	2	3	4	4	5	6	6
,50	3162	3170	3177	3184	3192	3199	3206	3214	3221	3228	1	1	2	3	4	4	5	6	7
,51	3236	3243	3251	3258	3266	3273	3281	3289	3296	3304	1	2	2	3	4	5	5	6	7
,52	3311	3319	3327	3334	3342	3350	3357	3365	3373	3381	1	2	2	3	4	5	5	6	7
,53	3388	3396	3404	3412	3420	3428	3436	3443	3451	3459	1	2	2	3	4	5	6	6	7
,54	3467	3475	3483	3491	3499	3508	3516	3524	3532	3540	1	2	2	3	4	5	6	6	7
,55	3548	3556	3565	3573	3581	3589	3597	3606	3614	3622	1	2	2	3	4	5	6	7	7
,56	3631	3639	3648	3656	3664	3673	3681	3690	3698	3707	1	2	3	3	4	5	6	7	8
,57	3715	3724	3733	3741	3750	3758	3767	3776	3784	3793	1	2	3	3	4	5	6	7	8
,58	3802	3811	3819	3828	3837	3846	3855	3864	3873	3882	1	2	3	4	4	5	6	7	8
,59	3890	3899	3908	3917	3926	3936	3945	3954	3963	3972	1	2	3	4	5	5	6	7	8
,60	3981	3990	3999	4009	4018	4027	4036	4046	4055	4064	1	2	3	4	5	6	6	7	8
,61	4074	4083	4093	4102	4111	4121	4130	4140	4150	4159	1	2	3	4	5	6	7	8	9
,62	4169	4178	4188	4198	4207	4217	4227	4236	4246	4256	1	2	3	4	5	6	7	8	9
,63	4266	4276	4285	4295	4305	4315	4325	4335	4345	4355	1	2	3	4	5	6	7	8	9
,64	4365	4375	4385	4395	4406	4416	4426	4436	4446	4457	1	2	3	4	5	6	7	8	9
,65	4467	4477	4487	4498	4508	4519	4529	4539	4550	4560	1	2	3	4	5	6	7	8	9
,66	4571	4581	4592	4603	4613	4624	4634	4645	4656	4667	1	2	3	4	5	6	7	9	10
,67	4677	4688	4699	4710	4721	4732	4742	4753	4764	4775	1	2	3	4	5	7	8	9	10
,68	4786	4797	4808	4819	4831	4842	4853	4864	4875	4887	1	2	3	4	6	7	8	9	10
,69	4898	4909	4920	4932	4943	4955	4966	4977	4989	5000	1	2	3	5	6	7	8	9	10
,70	5012	5023	5035	5047	5058	5070	5082	5093	5105	5117	1	2	4	5	6	7	8	9	11
,71	5129	5140	5152	5164	5176	5188	5200	5212	5224	5236	1	2	4	5	6	7	8	10	11
,72	5248	5260	5272	5284	5297	5309	5321	5333	5346	5358	1	2	4	5	6	7	9	10	11
,73	5370	5383	5395	5408	5420	5433	5445	5458	5470	5483	1	3	4	5	6	8	9	10	11
,74	5495	5508	5521	5534	5546	5559	5572	5585	5598	5610	1	3	4	5	6	8	9	10	12
,75	5623	5636	5649	5662	5675	5689	5702	5715	5728	5741	1	3	4	5	7	8	9	10	12
,76	5754	5768	5781	5794	5808	5821	5834	5848	5861	5875	1	3	4	5	7	8	9	11	12
,77	5888	5902	5916	5929	5943	5957	5970	5984	5998	6012	1	3	4	5	7	8	10	11	12
,78	6026	6039	6053	6067	6081	6095	6109	6124	6138	6152	1	3	4	6	7	8	10	11	13
,79	6166	6180	6194	6209	6223	6237	6252	6266	6281	6295	1	3	4	6	7	9	10	11	13
,80	6310	6324	6339	6353	6368	6383	6397	6412	6427	6442	1	3	4	6	7	9	10	12	13
,81	6457	6471	6486	6501	6516	6531	6546	6561	6577	6592	2	3	5	6	8	9	11	12	14
,82	6607	6622	6637	6653	6668	6683	6699	6714	6730	6745	2	3	5	6	8	9	11	12	14
,83	6761	6776	6792	6808	6823	6839	6855	6871	6887	6902	2	3	5	6	8	9	11	13	14
,84	6918	6934	6950	6966	6982	6998	7015	7031	7047	7063	2	3	5	6	8	10	11	13	15
,85	7079	7096	7112	7129	7145	7161	7178	7194	7211	7228	2	3	5	7	8	10	12	13	15
,86	7244	7261	7278	7295	7311	7328	7345	7362	7379	7396	2	3	5	7	8	10	12	13	15
,87	7413	7430	7447	7464	7482	7499	7516	7534	7551	7568	2	3	5	7	9	10	12	14	16
,88	7586	7603	7621	7638	7656	7674	7691	7709	7727	7745	2	4	5	7	9	11	12	14	16
,89	7762	7780	7798	7816	7834	7852	7870	7889	7907	7925	2	4	5	7	9	11	13	14	16
,90	7943	7962	7980	7998	8017	8035	8054	8072	8091	8110	2	4	6	7	9	11	13	15	17
,91	8128	8147	8166	8185	8204	8222	8241	8260	8279	8299	2	4	6	8	9	11	13	15	17
,92	8318	8337	8356	8375	8395	8414	8433	8453	8472	8492	2	4	6	8	10	12	14	15	17
,93	8511	8531	8551	8570	8590	8610	8630	8650	8670	8690	2	4	6	8	10	12	14	16	18
,94	8710	8730	8750	8770	8790	8810	8831	8851	8872	8892	2	4	6	8	10	12	14	16	18
,95	8913	8933	8954	8974	8995	9016	9036	9057	9078	9099	2	4	6	8	10	12	15	17	19
,96	9120	9141	9162	9183	9204	9226	9247	9268	9290	9311	2	4	6	8	11	13	15	17	19
,97	9333	9354	9376	9397	9419	9441	9462	9484	9506	9528	2	4	7	9	11	13	15	17	20
,98	9550	9572	9594	9616	9638	9661	9683	9705	9727	9750	2	4	7	9	11	13	16	18	20
,99	9772	9795	9817	9840	9863	9886	9908	9931	9954	9977	2	5	7	9	11	14	16	18	20

Sines of angles

$\theta \rightarrow \sin\theta$

θ	0′ 0,0°	6′ 0,1°	12′ 0,2°	18′ 0,3°	24′ 0,4°	30′ 0,5°	36′ 0,6°	42′ 0,7°	48′ 0,8°	54′ 0,9°	ADD Differences 1′	2′	3′	4′	5′
0°	0,0000	0017	0035	0052	0070	0087	0105	0122	0140	0157	3	6	9	12	15
1	0,0175	0192	0209	0227	0244	0262	0279	0297	0314	0332	3	6	9	12	15
2	0,0349	0366	0384	0401	0419	0436	0454	0471	0488	0506	3	6	9	12	15
3	0,0523	0541	0558	0576	0593	0610	0628	0645	0663	0680	3	6	9	12	15
4	0,0698	0715	0732	0750	0767	0785	0802	0819	0837	0854	3	6	9	12	15
5	0,0872	0889	0906	0924	0941	0958	0976	0993	1011	1028	3	6	9	12	14
6	0,1045	1063	1080	1097	1115	1132	1149	1167	1184	1201	3	6	9	12	14
7	0,1219	1236	1253	1271	1288	1305	1323	1340	1357	1374	3	6	9	12	14
8	0,1392	1409	1426	1444	1461	1478	1495	1513	1530	1547	3	6	9	12	14
9	0,1564	1582	1599	1616	1633	1650	1668	1685	1702	1719	3	6	9	12	14
10°	0,1736	1754	1771	1788	1805	1822	1840	1857	1874	1891	3	6	9	11	14
11	0,1908	1925	1942	1959	1977	1994	2011	2028	2045	2062	3	6	9	11	14
12	0,2079	2096	2113	2130	2147	2164	2181	2198	2215	2233	3	6	9	11	14
13	0,2250	2267	2284	2300	2317	2334	2351	2368	2385	2402	3	6	8	11	14
14	0,2419	2436	2453	2470	2487	2504	2521	2538	2554	2571	3	6	8	11	14
15	0,2588	2605	2622	2639	2656	2672	2689	2706	2723	2740	3	6	8	11	14
16	0,2756	2773	2790	2807	2823	2840	2857	2874	2890	2907	3	6	8	11	14
17	0,2924	2940	2957	2974	2990	3007	3024	3040	3057	3074	3	6	8	11	14
18	0,3090	3107	3123	3140	3156	3173	3190	3206	3223	3239	3	6	8	11	14
19	0,3256	3272	3289	3305	3322	3338	3355	3371	3387	3404	3	5	8	11	14
20°	0,3420	3437	3453	3469	3486	3502	3518	3535	3551	3567	3	5	8	11	14
21	0,3584	3600	3616	3633	3649	3665	3681	3697	3714	3730	3	5	8	11	14
22	0,3746	3762	3778	3795	3811	3827	3843	3859	3875	3891	3	5	8	11	14
23	0,3907	3923	3939	3955	3971	3987	4003	4019	4035	4051	3	5	8	11	14
24	0,4067	4083	4099	4115	4131	4147	4163	4179	4195	4210	3	5	8	11	13
25	0,4226	4242	4258	4274	4289	4305	4321	4337	4352	4368	3	5	8	11	13
26	0,4384	4399	4415	4431	4446	4462	4478	4493	4509	4524	3	5	8	10	13
27	0,4540	4555	4571	4586	4602	4617	4633	4648	4664	4679	3	5	8	10	13
28	0,4695	4710	4726	4741	4756	4772	4787	4802	4818	4833	3	5	8	10	13
29	0,4848	4863	4879	4893	4909	4924	4939	4955	4970	4985	3	5	8	10	13
30°	0,5000	5015	5030	5045	5060	5075	5090	5105	5120	5135	3	5	8	10	13
31	0,5150	5165	5180	5195	5210	5225	5240	5255	5270	5284	2	5	7	10	12
32	0,5299	5314	5329	5344	5358	5373	5388	5402	5417	5432	2	5	7	10	12
33	0,5446	5461	5476	5490	5505	5519	5534	5548	5563	5577	2	5	7	10	12
34	0,5592	5606	5621	5635	5650	5664	5678	5693	5707	5721	2	5	7	10	12
35	0,5736	5750	5764	5779	5793	5807	5821	5835	5850	5864	2	5	7	9	12
36	0,5878	5892	5906	5920	5934	5948	5962	5976	5990	6004	2	5	7	9	12
37	0,6018	6032	6046	6060	6074	6088	6101	6115	6129	6143	2	5	7	9	12
38	0,6157	6170	6184	6198	6211	6225	6239	6252	6266	6280	2	5	7	9	11
39	0,6293	6307	6320	6334	6347	6361	6374	6388	6401	6414	2	4	7	9	11
40°	0,6428	6441	6455	6468	6481	6494	6508	6521	6534	6547	2	4	7	9	11
41	0,6561	6574	6587	6600	6613	6626	6639	6652	6665	6678	2	4	7	9	11
42	0,6691	6704	6717	6730	6743	6756	6769	6782	6794	6807	2	4	6	9	11
43	0,6820	6833	6845	6858	6871	6834	6896	6909	6921	6934	2	4	6	8	11
44	0,6947	6959	6972	6984	6997	7009	7022	7034	7046	7059	2	4	6	8	10
45°	0,7081	7063	7096	7108	7120	7133	7145	7157	7169	7181	2	4	6	8	10
46	0,7193	7206	7218	7230	7242	7254	7266	7278	7290	7302	2	4	6	8	10
47	0,7314	7325	7337	7349	7361	7373	7385	7396	7408	7420	2	4	6	8	10
48	0,7431	7443	7455	7466	7478	7490	7501	7513	7524	7536	2	4	6	8	10
49	0,7547	7559	7570	7581	7593	7604	7615	7627	7638	7649	2	4	6	8	9
50°	0,7660	7672	7683	7694	7705	7716	7727	7738	7749	7760	2	4	6	7	9
51	0,7771	7782	7793	7804	7815	7826	7837	7848	7859	7869	2	4	5	7	9
52	0,7880	7891	7902	7912	7923	7934	7944	7955	7965	7976	2	4	5	7	9
53	0,7986	7997	8007	8018	8028	8039	8049	8059	8070	8080	2	3	5	7	9
54	0,8090	8100	8111	8121	8131	8141	8151	8161	8171	8181	2	3	5	7	8
55	0,8192	8202	8211	8221	8231	8241	8251	8261	8271	8281	2	3	5	7	8
56	0,8290	8300	8310	8320	8329	8339	8348	8358	8368	8377	2	3	5	6	8
57	0,8387	8396	8406	8415	8425	8434	8443	8453	8462	8471	2	3	5	6	8
58	0,8480	8490	8499	8508	8517	8526	8536	8545	8554	8563	2	3	5	6	8
59	0,8572	8581	8590	8599	8607	8616	8625	8634	8643	8652	1	3	4	6	7
60°	0,8660	8669	8678	8696	8695	8704	8712	8721	8729	8738	1	3	4	6	7
61	0,8746	8755	8763	8771	8780	8788	8796	8805	8813	8821	1	3	4	6	7
62	0,8829	8838	8846	8854	8862	8870	8878	8886	8894	8902	1	3	4	5	7
63	0,8910	8918	8926	8934	8942	8949	8957	8965	8973	8980	1	3	4	5	6
64	0,8988	8996	9003	9011	9018	9026	9033	9041	9048	9056	1	3	4	5	6
65	0,9063	9070	9078	9085	9092	9100	9107	9114	9121	9128	1	2	4	5	6
66	0,9135	9143	9150	9157	9164	9171	9178	9184	9191	9198	1	2	3	5	6
67	0,9205	9212	9219	9225	9232	9239	9245	9252	9259	9265	1	2	3	4	6
68	0,9272	9278	9285	9291	9298	9304	9311	9317	9323	9330	1	2	3	4	5
69	0,9336	9342	9348	9354	9361	9367	9373	9379	9385	9391	1	2	3	4	5
70°	0,9397	9403	9409	9415	9421	9426	9432	9438	9444	9449	1	2	3	4	5
71	0,9455	9461	9466	9472	9478	9483	9489	9494	9500	9505	1	2	3	4	5
72	0,9511	9516	9521	9527	9532	9537	9542	9548	9553	9558	1	2	3	3	4
73	0,9563	9568	9573	9578	9583	9588	9593	9598	9603	9608	1	2	2	3	4
74	0,9613	9617	9622	9627	9632	9636	9641	9646	9650	9655	1	2	2	3	4
75	0,9659	9664	9668	9673	9677	9681	9686	9690	9694	9699	1	1	2	3	4
76	0,9703	9707	9711	9715	9720	9724	9728	9732	9736	9740	1	1	2	3	3
77	0,9744	9748	9751	9755	9759	9763	9767	9770	9774	9778	1	1	2	3	3
78	0,9781	9785	9789	9792	9796	9799	9803	9806	9810	9813	1	1	2	2	3
79	0,9816	9820	9823	9826	9829	9833	9836	9839	9842	9845	1	1	2	2	3
80°	0,9848	9851	9854	9857	9860	9863	9866	9869	9871	9874	0	1	1	2	2
81	0,9877	9880	9882	9885	9888	9890	9893	9895	9898	9900	0	1	1	2	2
82	0,9903	9905	9907	9910	9912	9914	9917	9919	9921	9923	0	1	1	2	2
83	0,9925	9928	9930	9932	9934	9936	9938	9940	9942	9943	0	1	1	1	2
84	0,9945	9947	9949	9951	9952	9954	9956	9957	9959	9960	0	1	1	1	2
85	0,9962	9963	9965	9966	9968	9969	9971	9972	9973	9974	0	0	1	1	1
86	0,9976	9977	9978	9979	9980	9981	9982	9983	9984	9985	0	0	1	1	1
87	0,9986	9987	9988	9989	9990	9990	9991	9992	9993	9993	0	0	0	1	1
88	0,9994	9995	9996	9996	9996	9997	9997	9997	9998	9998	0	0	0	0	0
89	0,9998	9999	9999	9999	9999	1,000	1,000	1,000	1,000	1,000	0	0	0	0	0

Cosines of angles

θ → cos θ

θ	0' 0,0°	6' 0,1°	12' 0,2°	18' 0,3°	24' 0,4°	30' 0,5°	36' 0,6°	42' 0,7°	48' 0,8°	54' 0,9°	SUBTRACT Differences 1'	2'	3'	4'	5'
0°	1,000	1,000	1,000	1,000	1,000	1,000	,9999	9999	9999	9999	0	0	0	0	0
1	0,9998	9998	9998	9997	9997	9997	9996	9996	9995	9995	0	0	0	0	0
2	0,9994	9993	9993	9992	9991	9990	9990	9989	9988	9987	0	0	0	1	1
3	0,9986	9985	9984	9983	9982	9981	9980	9979	9978	9977	0	0	1	1	1
4	0,9976	9974	9973	9972	9971	9969	9968	9966	9965	9963	0	0	1	1	1
5	0,9962	9960	9959	9957	9956	9954	9952	9951	9949	9947	0	1	1	1	2
6	0,9945	9943	9942	9940	9938	9936	9934	9932	9930	9928	0	1	1	1	2
7	0,9925	9923	9921	9919	9917	9914	9912	9910	9907	9905	0	1	1	2	2
8	0,9903	9900	9898	9895	9893	9890	9888	9885	9882	9880	0	1	1	2	2
9	0,9877	9874	9871	9869	9866	9863	9860	9857	9854	9851	0	1	1	2	2
10°	0,9848	9845	9842	9839	9836	9833	9829	9826	9823	9820	1	1	2	2	3
11	0,9816	9813	9810	9806	9803	9799	9796	9792	9789	9785	1	1	2	2	3
12	0,9781	9778	9774	9770	9767	9763	9759	9755	9751	9748	1	1	2	3	3
13	0,9744	9740	9736	9732	9728	9724	9720	9715	9711	9707	1	1	2	3	3
14	0,9703	9699	9694	9690	9686	9681	9677	9673	9668	9664	1	1	2	3	4
15	0,9659	9655	9650	9646	9641	9636	9632	9627	9622	9617	1	2	2	3	4
16	0,9613	9606	9603	9598	9593	9588	9583	9578	9573	9568	1	2	2	3	4
17	0,9563	9558	9553	9548	9542	9537	9532	9527	9521	9516	1	2	3	3	4
18	0,9511	9505	9500	9494	9489	9483	9478	9472	9466	9461	1	2	3	4	5
19	0,9455	9449	9444	9438	9432	9426	9421	9415	9409	9403	1	2	3	4	5
20°	0,9397	9391	9385	9379	9373	9367	9361	9354	9348	9342	1	2	3	4	5
21	0,9336	9330	9323	9317	9311	9304	9298	9291	9285	9278	1	2	3	4	5
22	0,9272	9265	9259	9252	9245	9239	9232	9225	9219	9212	1	2	3	4	6
23	0,9205	9198	9191	9184	9178	9171	9164	9157	9150	9143	1	2	3	5	6
24	0,9135	9128	9121	9114	9107	9100	9092	9085	9078	9070	1	2	4	5	6
25	0,9063	9056	9048	9041	9033	9026	9018	9011	9003	8996	1	3	4	5	6
26	0,8988	8980	8973	8965	8957	8949	8942	8934	8926	8918	1	3	4	5	6
27	0,8910	8902	8894	8886	8878	8870	8862	8854	8846	8838	1	3	4	5	7
28	0,8829	8821	8813	8805	8796	8788	8780	8771	8763	8755	1	3	4	6	7
29	0,8746	8738	8729	8721	8712	8704	8695	8686	8678	8669	1	3	4	6	7
30°	0,8660	8652	8643	8634	8625	8616	8607	8599	8590	8581	1	3	4	6	7
31	0,8572	8563	8554	8545	8536	8526	8517	8508	8499	8490	2	3	5	6	8
32	0,8480	8471	8462	8453	8443	8434	8425	8415	8406	8396	2	3	5	6	8
33	0,8387	8377	8368	8358	8348	8339	8329	8320	8310	8300	2	3	5	6	8
34	0,8290	8281	8271	8261	8251	8241	8231	8221	8211	8202	2	3	5	7	8
35	0,8192	8181	8171	8161	8151	8141	8131	8121	8111	8100	2	3	5	7	8
36	0,8090	8080	8070	8059	8049	8039	8028	8018	8007	7997	2	3	5	7	9
37	0,7986	7976	7965	7955	7944	7934	7923	7912	7902	7891	2	4	5	7	9
38	0,7880	7869	7859	7848	7837	7826	7815	7804	7793	7782	2	4	5	7	9
39	0,7771	7760	7749	7738	7727	7716	7705	7694	7683	7672	2	4	6	7	9
40°	0,7660	7649	7638	7627	7615	7604	7593	7581	7570	7559	2	4	6	8	9
41	0,7547	7536	7524	7513	7501	7490	7478	7466	7455	7443	2	4	6	8	10
42	0,7431	7420	7408	7396	7385	7373	7361	7349	7337	7325	2	4	6	8	10
43	0,7314	7302	7290	7278	7266	7254	7242	7230	7218	7206	2	4	6	8	10
44	0,7193	7181	7169	7151	7145	7133	7120	7108	7096	7083	2	4	6	8	10

θ	0' 0,0°	6' 0,1°	12' 0,2°	18' 0,3°	24' 0,4°	30' 0,5°	36' 0,6°	42' 0,7°	48' 0,8°	54' 0,9°	SUBTRACT Differences 1'	2'	3'	4'	5'
45°	0,7071	7059	7046	7034	7022	7009	6997	6984	6972	6959	2	4	6	8	10
46	0,6947	6934	6921	6909	6896	6884	6871	6858	6845	6833	2	4	6	8	11
47	0,6820	6807	6794	6782	6769	6756	6743	6730	6717	6704	2	4	6	9	11
48	0,6691	6678	6665	6652	6639	6626	6613	6600	6587	6574	2	4	7	9	11
49	0,6561	6547	6534	6521	6508	6494	6481	6468	6455	6441	2	4	7	9	11
50°	0,6428	6414	6401	6388	6374	6361	6347	6334	6320	6307	2	4	7	9	11
51	0,6293	6280	6266	6252	6239	6225	6211	6198	6184	6170	2	5	7	9	11
52	0,6157	6143	6129	6115	6101	6088	6074	6060	6046	6032	2	5	7	9	12
53	0,6018	6004	5990	5976	5962	5948	5934	5920	5906	5892	2	5	7	9	12
54	0,5878	5864	5850	5835	5821	5807	5793	5779	5764	5750	2	5	7	9	12
55	0,5736	5721	5707	5693	5678	5664	5650	5635	5621	5606	2	5	7	10	12
56	0,5592	5577	5563	5548	5534	5519	5505	5490	5476	5461	2	5	7	10	12
57	0,5446	5432	5417	5402	5388	5373	5358	5344	5329	5314	2	5	7	10	12
58	0,5299	5284	5270	5255	5240	5225	5210	5195	5180	5165	2	5	7	10	12
59	0,5150	5135	5120	5105	5090	5075	5060	5045	5030	5015	3	5	8	10	13
60°	0,5000	4985	4970	4955	4939	4924	4909	4894	4879	4863	3	5	8	10	13
61	0,4848	4833	4818	4802	4787	4772	4756	4741	4726	4710	3	5	8	10	13
62	0,4695	4679	4664	4648	4633	4617	4602	4586	4571	4555	3	5	8	10	13
63	0,4540	4524	4509	4493	4478	4462	4446	4431	4415	4399	3	5	8	10	13
64	0,4384	4368	4352	4337	4321	4305	4289	4274	4258	4242	3	5	8	11	13
65	0,4226	4210	4195	4179	4163	4147	4131	4115	4099	4083	3	5	8	11	13
66	0,4067	4051	4035	4019	4003	3987	3971	3955	3939	3923	3	5	8	11	14
67	0,3907	3891	3875	3859	3843	3827	3811	3795	3778	3762	3	5	8	11	14
68	0,3746	3730	3714	3697	3681	3665	3649	3633	3616	3600	3	5	8	11	14
69	0,3584	3567	3551	3535	3518	3502	3486	3469	3453	3437	3	5	8	11	14
70°	0,3420	3404	3387	3371	3355	3338	3322	3305	3289	3272	3	5	8	11	14
71	0,3256	3239	3223	3206	3190	3173	3156	3140	3123	3107	3	6	8	11	14
72	0,3090	3074	3057	3040	3024	3007	2990	2974	2957	2940	3	6	8	11	14
73	0,2924	2907	2890	2874	2857	2840	2823	2807	2709	2773	3	6	8	11	14
74	0,2756	2740	2723	2706	2689	2672	2656	2639	2622	2605	3	6	8	11	14
75	0,2588	2571	2554	2538	2521	2504	2487	2470	2453	2436	3	6	8	11	14
76	0,2419	2402	2385	2368	2351	2334	2317	2300	2284	2267	3	6	8	11	14
77	0,2250	2233	2215	2198	2181	2164	2147	2130	2113	2096	3	6	9	11	14
78	0,2079	2062	2045	2028	2011	1994	1977	1959	1942	1925	3	6	9	11	14
79	0,1908	1891	1874	1857	1840	1822	1805	1788	1771	1754	3	6	9	11	14
80°	0,1736	1719	1702	1685	1668	1650	1633	1616	1599	1582	3	6	9	12	14
81	0,1564	1547	1530	1513	1495	1478	1461	1444	1426	1409	3	6	9	12	14
82	0,1392	1374	1357	1340	1323	1305	1288	1271	1253	1236	3	6	9	12	14
83	0,1219	1201	1184	1167	1149	1132	1115	1097	1080	1063	3	6	9	12	14
84	0,1045	1028	1011	0993	0976	0958	0941	0924	0906	0889	3	6	9	12	14
85	0,0872	0854	0837	0819	0802	0785	0767	0750	0732	0715	3	6	9	12	14
86	0,0698	0680	0663	0645	0628	0610	0593	0576	0558	0541	3	6	9	12	15
87	0,0523	0506	0488	0471	0454	0436	0419	0401	0384	0366	3	6	9	12	15
88	0,0349	0332	0314	0297	0279	0262	0244	0227	0209	0192	3	6	9	12	15
89	0,0175	0157	0140	0122	0105	0087	0070	0052	0035	0017	3	6	9	12	15

Tangents of angles

θ → tan θ

θ	0' 0,0°	6' 0,1°	12' 0,2°	18' 0,3°	24' 0,4°	30' 0,5°	36' 0,6°	42' 0,7°	48' 0,8°	54' 0,9°	ADD Differences 1'	2'	3'	4'	5'
0°	0,0000	0017	0035	0052	0070	0087	0105	0122	0140	0157	3	6	9	12	15
1	0,0175	0192	0209	0227	0244	0262	0279	0297	0314	0332	3	6	9	12	15
2	0,0349	0367	0384	0402	0419	0437	0454	0472	0489	0507	3	6	9	12	15
3	0,0524	0542	0559	0577	0594	0612	0629	0647	0664	0682	3	6	9	12	15
4	0,0699	0717	0734	0752	0769	0787	0805	0822	0840	0857	3	6	9	12	15
5	0,0875	0892	0910	0928	0945	0963	0981	0998	1016	1033	3	6	9	12	15
6	0,1051	1069	1086	1104	1122	1139	1157	1175	1192	1210	3	6	9	12	15
7	0,1228	1246	1263	1281	1299	1317	1334	1352	1370	1388	3	6	9	12	15
8	0,1405	1423	1441	1459	1477	1495	1512	1530	1548	1566	3	6	9	12	15
9	0,1585	1602	1620	1683	1655	1673	1691	1709	1727	1745	3	6	9	12	15
10°	0,1763	1781	1799	1817	1835	1853	1871	1890	1908	1926	3	6	9	12	15
11	0,1944	1962	1980	1998	2016	2035	2053	2071	2089	2107	3	6	9	12	15
12	0,2126	2144	2162	2180	2199	2217	2235	2254	2272	2290	3	6	9	12	15
13	0,2309	2327	2345	2364	2382	2401	2419	2438	2456	2475	3	6	9	12	15
14	0,2493	2512	2530	2549	2568	2586	2605	2623	2642	2661	3	6	9	12	16
15	0,2679	2698	2717	2736	2754	2773	2792	2811	2830	2849	3	6	9	13	16
16	0,2867	2886	2905	2924	2943	2962	2981	3000	3010	3038	3	6	9	13	16
17	0,3057	3076	3096	3115	3134	3153	3172	3191	3211	3230	3	6	10	13	16
18	0,3249	3269	3288	3307	3327	3346	3365	3385	3404	3424	3	6	10	13	16
19	0,3443	3463	3482	3502	3522	3541	3561	3581	3600	3620	3	7	10	13	16
20°	0,3640	3659	3679	3699	3719	3739	3759	3779	3799	3819	3	7	10	13	17
21	0,3839	3859	3879	3899	3919	3939	3959	3979	4000	4020	3	7	10	13	17
22	0,4040	4061	4081	4101	4122	4142	4163	4183	4204	4224	3	7	10	14	17
23	0,4245	4265	4286	4307	4327	4348	4369	4390	4411	4431	3	7	10	14	17
24	0,4452	4473	4494	4515	4536	4557	4578	4599	4621	4642	4	7	11	14	18
25	0,4663	4684	4706	4727	4748	4770	4791	4813	4834	4856	4	7	11	14	18
26	0,4877	4899	4921	4942	4964	4986	5008	5029	5051	5073	4	7	11	15	18
27	0,5095	5117	5139	5161	5184	5206	5228	5250	5272	5295	4	7	11	15	18
28	0,5317	5340	5362	5384	5407	5430	5452	5475	5498	5520	4	8	11	15	19
29	0,5543	5566	5589	5612	5635	5658	5681	5704	5727	5750	4	8	12	15	19
30°	0,5774	5797	5820	5844	5867	5890	5914	5938	5961	5985	4	8	12	16	20
31	0,6009	6032	6056	6080	6104	6128	6152	6176	6200	6224	4	8	12	16	20
32	0,6249	6273	6297	6322	6346	6371	6395	6420	6445	6469	4	8	12	16	20
33	0,6494	6519	6544	6569	6594	6619	6644	6669	6694	6720	4	8	13	17	21
34	0,6745	6771	6796	6822	6847	6873	6899	6924	6950	6976	4	9	13	17	21
35	0,7002	7028	7054	7080	7107	7133	7159	7186	7212	7239	4	9	13	18	22
36	0,7265	7292	7319	7346	7373	7400	7427	7454	7481	7508	5	9	14	18	23
37	0,7536	7563	7590	7618	7646	7673	7701	7729	7757	7785	5	9	14	18	23
38	0,7813	7841	7869	7898	7926	7954	7983	8012	8040	8069	5	9	14	19	24
39	0,8098	8127	8156	8185	8214	8243	8273	8302	8332	8361	5	10	15	20	24
40°	0,8391	8421	8451	8481	8511	8541	8571	8601	8632	8662	5	10	15	20	25
41	0,8693	8724	8754	8785	8816	8847	8878	8910	8941	8972	5	10	16	21	26
42	0,9004	9036	9067	9099	9131	9163	9195	9228	9260	9293	5	11	16	21	27
43	0,9325	9358	9391	9424	9457	9490	9523	9556	9590	9623	6	11	17	22	28
44	0,9657	9691	9725	9759	9793	9827	9861	9896	9930	9965	6	11	17	23	29

θ	0' 0,0°	6' 0,1°	12' 0,2°	18' 0,3°	24' 0,4°	30' 0,5°	36' 0,6°	42' 0,7°	48' 0,8°	54' 0,9°	ADD Differences 1'	2'	3'	4'	5'
45°	1,0000	0035	0070	0105	0141	0176	0212	0247	0283	0319	6	12	18	24	30
46	1,0355	0392	0428	0464	0501	0538	0575	0612	0649	0686	6	12	18	25	31
47	1,0742	0761	0799	0873	0875	0913	0951	0990	1028	1067	6	13	19	25	32
48	1,1106	1145	1184	1224	1263	1303	1343	1383	1423	1463	7	13	20	27	33
49	1,1504	1544	1585	1626	1667	1708	1750	1792	1833	1875	7	14	21	28	34
50°	1,1918	1960	2002	2045	2088	2131	2174	2218	2261	2305	7	14	22	29	36
51	1,2349	2393	2437	2484	2527	2572	2617	2662	2708	2753	8	15	23	30	38
52	1,2799	2846	2892	2938	2985	3032	3079	3127	3175	3222	8	16	24	31	39
53	1,3270	3319	3367	3416	3465	3514	3564	3613	3663	3713	8	16	25	33	41
54	1,3764	3814	3865	3916	3968	4019	4071	4124	4176	4229	9	17	26	34	43
55	1,4281	4335	4388	4442	4496	4550	4605	4659	4715	4770	9	18	27	36	45
56	1,4826	4882	4938	4994	5051	5108	5166	5224	5282	5340	10	19	29	38	48
57	1,5399	5458	5517	5577	5673	5697	5757	5818	5880	5941	10	20	30	40	50
58	1,6003	6066	6128	6191	6255	6319	6383	6447	6512	6577	11	21	32	43	53
59	1,6643	6709	6775	6842	6909	6977	7045	7113	7182	7251	11	23	34	45	56
60°	1,7321	7391	7461	7532	7603	7675	7747	7820	7893	7966	12	24	36	48	60
61	1,8040	8115	8190	8265	8341	8418	8495	8572	8650	8728	13	26	38	51	64
62	1,8807	8887	8967	9047	9128	9210	9292	9375	9458	9542	14	27	41	55	68
63	1,9626	9711	9797	9883	9970	0057	0145	0233	0323	0413	15	29	44	58	73
64	2,0503	0594	0686	0778	0872	0965	1060	1155	1251	1348	16	31	47	63	78
65	2,1445	1543	1642	1742	1842	1943	2045	2148	2251	2355	17	34	51	68	85
66	2,2460	2566	2673	2781	2889	2998	3109	3220	3332	3445	18	37	55	73	92
67	2,3559	3673	3789	3906	4023	4142	4262	4383	4504	4627	20	40	60	79	99
68	2,4751	4876	5002	5129	5257	5386	5517	5649	5782	5916	22	43	65	87	108
69	2,6051	6187	6325	6464	6605	6746	6889	7034	7179	7326	24	47	71	95	119
70°	2,7475	7625	7776	7929	8083	8239	8397	8556	8716	8878	26	52	78	104	131
71	2,9042	9208	9375	9544	9714	9887	0061	0237	0415	0595	29	58	87	116	145
72	3,0777	0961	1146	1334	1524	1716	1910	2106	2305	2506	32	64	96	129	161
73	3,2709	2914	3122	3332	3544	3759	3977	4197	4420	4646	36	72	108	144	180
74	3,4874	5105	5339	5576	5816	6059	6305	6554	6806	7062	41	81	122	163	204
75	3,7321	7583	7848	8118	8391	8667	8974	9232	9520	9812	46	83	139	186	232
76	4,0108	0408	0713	1022	1335	1653	1976	2303	2635	2972	differences unreliable				
77	4,3315	3662	4015	4374	4737	5107	5483	5864	6252	6645					
78	4,7046	7453	7867	8288	8716	9152	9594	0045	0504	0970					
79	5,1446	1929	2422	2924	3435	3955	4486	5026	5578	6140					
80°	5,6713	7297	7894	8502	9124	9785	0405	1066	1742	2432					
81	6,3138	3859	4596	5350	6122	6912	7720	8548	9395	0264					
82	7,1154	2066	3002	3962	4947	5958	6996	8062	9158	0285					
83	8,1443	2636	3863	5126	6427	7769	9152	0579	2052	3572					
84	9,514	9,677	9,845	10,02	10,20	10,39	10,58	10,78	10,99	11,20					
85	11,43	11,66	11,91	12,16	12,43	12,71	13,00	13,30	13,62	13,95					
86	14,30	14,67	15,06	15,46	15,89	16,35	16,83	17,34	17,89	18,46					
87	19,08	19,74	20,45	21,20	22,02	22,90	23,86	24,90	26,03	27,27					
88	28,64	30,14	31,82	33,69	35,80	38,19	40,92	44,07	47,74	52,08					
89	57,29	63,66	71,62	81,85	95,49	114,6	143,2	191,0	286,5	573,0					

Logarithms of sines

$\theta \rightarrow \log \sin \theta$

θ	0' 0,0°	6' 0,1°	12' 0,2°	18' 0,3°	24' 0,4°	30' 0,5°	36' 0,6°	42' 0,7°	48' 0,8°	54' 0,9°	ADD Differences 1'	2'	3'	4'	5'
0°	−∞	$\bar{3}$,2419	5429	7190	8439	9408	2,0200	0870	1450	1961					
1	$\bar{2}$,2419	2832	3210	3558	3880	4179	4459	4723	4971	5206					
2	$\bar{2}$,5428	5640	5842	6035	6220	6397	6567	6731	6889	7041					
3	$\bar{2}$,7188	7330	7468	7602	7731	7857	7979	8098	8213	8326					
4	$\bar{2}$,8436	8543	8647	8749	8849	8946	9042	9135	9226	9315	16	32	48	64	80
5	$\bar{2}$,9403	9489	9573	9655	9736	9816	9894	9970	$\bar{0}$046	$\bar{0}$120	13	26	39	52	65
6	$\bar{1}$,0192	0264	0334	0403	0472	0539	0605	0670	0734	0797	11	22	33	44	55
7	$\bar{1}$,0859	0920	0981	1040	1099	1157	1214	1271	1326	1381	10	19	29	38	48
8	$\bar{1}$,1436	1489	1542	1594	1646	1697	1747	1797	1847	1895	8	17	25	34	42
9	$\bar{1}$,1943	1991	2038	2085	2131	2176	2221	2266	2310	2353	8	15	23	30	38
10°	$\bar{1}$,2397	2439	2482	2524	2565	2606	2647	2687	2727	2767	7	14	20	27	34
11	$\bar{1}$,2806	2845	2883	2921	2959	2997	3034	3070	3107	3143	6	12	19	25	31
12	$\bar{1}$,3179	3214	3250	3284	3319	3353	3387	3421	3455	3488	6	11	17	23	28
13	$\bar{1}$,3521	3554	3586	3618	3650	3682	3713	3745	3775	3806	5	11	16	21	26
14	$\bar{1}$,3837	3867	3897	3927	3957	3986	4015	4044	4073	4102	5	10	15	20	24
15	$\bar{1}$,4130	4158	4186	4214	4242	4269	4296	4323	4350	4377	5	9	14	18	23
16	$\bar{1}$,4403	4430	4456	4482	4508	4533	4559	4584	4609	4634	4	9	13	17	21
17	$\bar{1}$,4659	4684	4709	4733	4757	4781	4805	4829	4853	4876	4	8	12	16	20
18	$\bar{1}$,4900	4923	4946	4969	4992	5015	5037	5060	5082	5104	4	8	11	15	19
19	$\bar{1}$,5126	5148	5170	5192	5213	5235	5256	5278	5299	5320	4	7	11	14	18
20°	$\bar{1}$,5341	5361	5382	5402	5423	5443	5463	5484	5504	5523	3	7	10	14	17
21	$\bar{1}$,5543	5563	5583	5602	5621	5641	5660	5679	5698	5717	3	6	10	13	16
22	$\bar{1}$,5736	5754	5773	5792	5810	5828	5847	5865	5883	5901	3	6	9	12	15
23	$\bar{1}$,5919	5937	5954	5972	5990	6007	6024	6042	6059	6076	3	6	9	12	15
24	$\bar{1}$,6093	6110	6127	6144	6161	6177	6194	6210	6227	6243	3	6	8	11	14
25	$\bar{1}$,6259	6276	6292	6308	6324	6340	6356	6371	6387	6403	3	5	8	11	13
26	$\bar{1}$,6418	6434	6449	6465	6480	6495	6510	6526	6541	6556	3	5	8	10	13
27	$\bar{1}$,6570	6585	6600	6615	6629	6644	6659	6673	6687	6702	2	5	7	10	12
28	$\bar{1}$,6716	6730	6744	6759	6773	6787	6801	6814	6828	6842	2	5	7	9	12
29	$\bar{1}$,6856	6869	6883	6896	6910	6923	6937	6950	6963	6977	2	4	7	9	11
30°	$\bar{1}$,6990	7003	7016	7029	7042	7055	7068	7080	7093	7106	2	4	6	9	11
31	$\bar{1}$,7118	7131	7144	7156	7168	7181	7193	7205	7218	7230	2	4	6	8	10
32	$\bar{1}$,7242	7254	7266	7278	7290	7302	7314	7326	7338	7349	2	4	6	8	10
33	$\bar{1}$,7361	7373	7384	7396	7407	7419	7430	7442	7453	7464	2	4	6	8	10
34	$\bar{1}$,7476	7487	7498	7509	7520	7531	7542	7553	7564	7575	2	4	6	7	9
35	$\bar{1}$,7586	7597	7607	7618	7629	7640	7650	7661	7671	7682	2	4	5	7	9
36	$\bar{1}$,7692	7703	7713	7723	7734	7744	7754	7764	7774	7785	2	3	5	7	9
37	$\bar{1}$,7795	7805	7815	7825	7835	7844	7854	7864	7874	7884	2	3	5	7	8
38	$\bar{1}$,7893	7903	7913	7922	7932	7941	7951	7960	7970	7979	2	3	5	6	8
39	$\bar{1}$,7989	7998	8007	8017	8026	8035	8044	8053	8063	8072	2	3	5	6	8
40°	$\bar{1}$,8081	8090	8099	8108	8117	8125	8134	8143	8152	8161	1	3	4	6	7
41	$\bar{1}$,8169	8178	8187	8195	8204	8213	8221	8230	8238	8247	1	3	4	6	7
42	$\bar{1}$,8255	8264	8272	8280	8289	8297	8305	8313	8322	8330	1	3	4	6	7
43	$\bar{1}$,8338	8346	8354	8362	8370	8378	8386	8394	8402	8410	1	3	4	5	7
44	$\bar{1}$,8418	8426	8433	8441	8449	8457	8464	8472	8480	8487	1	3	4	5	6

θ	0' 0,0°	6' 0,1°	12' 0,2°	18' 0,3°	24' 0,4°	30' 0,5°	36' 0,6°	42' 0,7°	48' 0,8°	54' 0,9°	ADD Differences 1'	2'	3'	4'	5'
45°	$\bar{1}$,8495	8502	8510	8517	8525	8532	8540	8547	8555	8562	1	2	4	5	6
46	$\bar{1}$,8569	8577	8584	8591	8598	8606	8613	8620	8627	8634	1	2	4	5	6
47	$\bar{1}$,8641	8648	8655	8662	8669	8676	8683	8690	8697	8704	1	2	3	5	6
48	$\bar{1}$,8711	8718	8724	8731	8738	8745	8751	8758	8765	8771	1	2	3	5	6
49	$\bar{1}$,8778	8784	8791	8797	8804	8810	8817	8823	8830	8836	1	2	3	4	5
50°	$\bar{1}$,8843	8849	8855	8862	8868	8874	8880	8887	8893	8899	1	2	3	4	5
51	$\bar{1}$,8905	8911	8917	8923	8929	8935	8941	8947	8953	8959	1	2	3	4	5
52	$\bar{1}$,8965	8971	8977	8983	8989	8995	9000	9006	9012	9018	1	2	3	4	5
53	$\bar{1}$,9023	9029	9035	9041	9046	9052	9057	9063	9069	9074	1	2	3	4	5
54	$\bar{1}$,9080	9085	9091	9096	9101	9107	9112	9118	9123	9128	1	2	3	4	5
55	$\bar{1}$,9134	9139	9144	9149	9155	9160	9165	9170	9175	9181	1	2	3	3	4
56	$\bar{1}$,9186	9191	9196	9201	9206	9211	9216	9221	9226	9231	1	2	3	3	4
57	$\bar{1}$,9236	9241	9246	9251	9255	9260	9265	9270	9275	9279	1	2	2	3	4
58	$\bar{1}$,9284	9289	9294	9298	9303	9308	9312	9317	9322	9326	1	2	2	3	4
59	$\bar{1}$,9331	9335	9340	9344	9349	9353	9358	9362	9367	9371	1	1	2	3	4
60°	$\bar{1}$,9375	9380	9384	9388	9393	9397	9401	9406	9410	9414	1	1	2	3	4
61	$\bar{1}$,9418	9422	9427	9431	9435	9439	9443	9447	9451	9455	1	1	2	3	3
62	$\bar{1}$,9459	9463	9467	9471	9475	9479	9483	9487	9491	9495	1	1	2	3	3
63	$\bar{1}$,9499	9503	9507	9510	9514	9518	9522	9525	9529	9533	1	1	2	3	3
64	$\bar{1}$,9537	9540	9544	9548	9551	9555	9558	9562	9566	9569	1	1	2	2	3
65	$\bar{1}$,9573	9576	9580	9583	9587	9590	9594	9597	9601	9604	1	1	2	2	3
66	$\bar{1}$,9607	9611	9614	9617	9621	9624	9627	9631	9634	9637	1	1	2	2	3
67	$\bar{1}$,9640	9643	9647	9650	9653	9656	9659	9662	9666	9669	1	1	2	2	3
68	$\bar{1}$,9672	9675	9678	9681	9684	9687	9690	9693	9696	9699	0	1	1	2	2
69	$\bar{1}$,9702	9704	9707	9710	9713	9716	9719	9722	9724	9727	0	1	1	2	2
70°	$\bar{1}$,9730	9733	9735	9738	9741	9743	9746	9749	9751	9754	0	1	1	2	2
71	$\bar{1}$,9757	9759	9762	9764	9767	9770	9772	9775	9777	9780	0	1	1	2	2
72	$\bar{1}$,9782	9785	9787	9789	9792	9794	9797	9799	9801	9804	0	1	1	2	2
73	$\bar{1}$,9806	9808	9811	9813	9815	9817	9820	9822	9824	9826	0	1	1	2	2
74	$\bar{1}$,9828	9831	9833	9835	9837	9839	9841	9843	9845	9847	0	1	1	1	2
75	$\bar{1}$,9849	9851	9853	9855	9857	9859	9861	9863	9865	9867	0	1	1	1	2
76	$\bar{1}$,9869	9871	9873	9875	9876	9878	9880	9882	9884	9885	0	1	1	1	2
77	$\bar{1}$,9887	9889	9891	9892	9894	9896	9897	9899	9901	9902	0	1	1	1	1
78	$\bar{1}$,9904	9906	9907	9909	9910	9912	9913	9915	9916	9918	0	1	1	1	1
79	$\bar{1}$,9919	9921	9922	9924	9925	9927	9928	9929	9931	9932	0	0	1	1	1
80°	$\bar{1}$,9934	9935	9936	9937	9939	9940	9941	9943	9944	9945	0	0	1	1	1
81	$\bar{1}$,9946	9947	0949	9950	9951	9952	9953	9954	9955	9956	0	0	1	1	1
82	$\bar{1}$,9958	9959	9960	9961	9962	9963	9964	9965	9966	9967	0	0	1	1	1
83	$\bar{1}$,9968	9968	9969	9970	9971	9972	9973	9974	9975	9975	0	0	1	1	1
84	$\bar{1}$,9976	9977	9978	9978	9979	9980	9981	9981	9982	9983	0	0	0	0	1
85	$\bar{1}$,9983	9984	9985	9985	9986	9987	9987	9988	9988	9989	0	0	0	0	0
86	$\bar{1}$,9989	9990	9990	9901	9991	9992	9992	9993	9993	9994	0	0	0	0	0
87	$\bar{1}$,9994	9994	9995	9995	9996	9996	9996	9996	9997	9997	0	0	0	0	0
88	$\bar{1}$,9997	9998	9998	9998	9998	9999	9999	9999	9999	9999	0	0	0	0	0
89	$\bar{1}$,9999	9999	0000	0000	0000	0000	0000	0000	0000	0000	0	0	0	0	0

Logarithms of cosines

$\theta \rightarrow \log \cos \theta$

θ	0' 0,0°	6' 0,1°	12' 0,2°	18' 0,3°	24' 0,4°	30' 0,5°	36' 0,6°	42' 0,7°	48' 0,8°	54' 0,9°	SUBTRACT Differences 1'	2'	3'	4'	5'
0°	0,0000	0000	0000	0000	0000	0000	0000	0000	0000,	$\bar{1}$,9999	0	0	0	0	0
1	$\bar{1}$,9999	9999	9999	9999	9999	9999	9998	9998	9998	9998	0	0	0	0	0
2	$\bar{1}$,9997	9997	9997	9996	9996	9996	9996	9995	9995	9994	0	0	0	0	0
3	$\bar{1}$,9994	9994	9993	9993	9992	9992	9991	9991	9990	9990	0	0	0	0	0
4	$\bar{1}$,9989	9989	9988	9988	9987	9987	9986	9985	9985	9984	0	0	0	0	0
5	$\bar{1}$,9983	9983	9982	9981	9981	9980	9979	9978	9978	9977	0	0	0	0	1
6	$\bar{1}$,9976	9975	9975	9974	9973	9972	9971	9970	9969	9968	0	0	0	1	1
7	$\bar{1}$,9968	9967	9966	9965	9964	9963	9962	9961	9960	9959	0	0	1	1	1
8	$\bar{1}$,9958	9956	9955	9954	9953	9952	9951	9950	9949	9947	0	0	1	1	1
9	$\bar{1}$,9946	9945	9944	9943	9941	9940	9939	9937	9936	9935	0	0	1	1	1
10°	$\bar{1}$,9934	9932	9931	9929	9928	9927	9925	9924	9922	9921	0	0	1	1	1
11	$\bar{1}$,9919	9918	9916	9915	9913	9912	9910	9909	9907	9906	0	1	1	1	1
12	$\bar{1}$,9904	9902	9901	9899	9897	9896	9894	9892	9891	9889	0	1	1	1	1
13	$\bar{1}$,9887	9885	9884	9882	9880	9878	9876	9875	9873	9871	0	1	1	1	2
14	$\bar{1}$,9869	9867	9865	9863	9861	9859	9857	9855	9853	9851	0	1	1	1	2
15	$\bar{1}$,9849	9847	9845	9843	9841	9839	9837	9835	9833	9831	0	1	1	1	2
16	$\bar{1}$,9828	9826	9824	9822	9820	9817	9815	9813	9811	9808	0	1	1	2	2
17	$\bar{1}$,9806	9804	9801	9799	9797	9794	9792	9789	9787	9785	0	1	1	2	2
18	$\bar{1}$,9782	9780	9777	9775	9772	9770	9767	9764	9762	9759	0	1	1	2	2
19	$\bar{1}$,9757	9754	9751	9749	9746	9743	9741	9738	9735	9733	0	1	1	2	2
20°	$\bar{1}$,9730	9727	9724	9722	9719	9716	9713	9710	9707	9704	0	1	1	2	2
21	$\bar{1}$,9702	9699	9696	9693	9690	9687	9684	9681	9678	9675	0	1	1	2	2
22	$\bar{1}$,9672	9669	9666	9662	9659	9656	9653	9650	9647	9643	1	1	2	2	3
23	$\bar{1}$,9640	9637	9634	9631	9627	9624	9621	9617	9614	9611	1	1	2	2	3
24	$\bar{1}$,9607	9604	9601	9597	9594	9590	9587	9583	9580	9576	1	1	2	2	3
25	$\bar{1}$,9573	9569	9566	9562	9558	9555	9551	9548	9544	9540	1	1	2	2	3
26	$\bar{1}$,9537	9533	9529	9525	9522	9518	9514	9510	9507	9503	1	1	2	3	3
27	$\bar{1}$,9499	9495	9491	9487	9483	9479	9475	9471	9467	9463	1	1	2	3	3
28	$\bar{1}$,9459	9455	9451	9447	9443	9439	9435	9431	9427	9422	1	1	2	3	3
29	$\bar{1}$,9418	9414	9410	9406	9401	9397	9393	9388	9384	9380	1	1	2	3	4
30°	$\bar{1}$,9375	9371	9367	9362	9358	9353	9349	9344	9340	9335	1	1	2	3	4
31	$\bar{1}$,9331	9326	9322	9317	9312	9308	9303	9298	9294	9289	1	2	2	3	4
32	$\bar{1}$,9284	9279	9275	9270	9265	9260	9255	9251	9246	9241	1	2	2	3	4
33	$\bar{1}$,9236	9231	9226	9221	9216	9211	9206	9201	9196	9191	1	2	3	3	4
34	$\bar{1}$,9186	9181	9175	9170	9165	9160	9155	9149	9144	9139	1	2	3	3	4
35	$\bar{1}$,9134	9128	9123	9118	9112	9107	9101	9096	9091	9085	1	2	3	4	5
36	$\bar{1}$,9080	9074	9069	9063	9057	9052	9046	9041	9035	9029	1	2	3	4	5
37	$\bar{1}$,9023	9018	9012	9006	9000	8995	8989	8983	8977	8971	1	2	3	4	5
38	$\bar{1}$,8965	8959	8953	8947	8941	8935	8929	8923	8917	8911	1	2	3	4	5
39	$\bar{1}$,8905	8899	8893	8887	8880	8874	8868	8862	8855	8849	1	2	3	4	5
40°	$\bar{1}$,8843	8836	8830	8823	8817	8810	8804	8797	8791	8784	1	2	3	4	5
41	$\bar{1}$,8778	8771	8765	8758	8751	8745	8738	8731	8724	8718	1	2	3	5	6
42	$\bar{1}$,8711	8704	8697	8690	8683	8676	8669	8662	8655	8648	1	2	3	5	6
43	$\bar{1}$,8641	8634	8627	8620	8613	8606	8598	8591	8584	8577	1	2	4	5	6
44	$\bar{1}$,8569	8562	8555	8547	8540	8532	8525	8517	8510	8502	1	2	4	5	6

θ	0' 0,0°	6' 0,1°	12' 0,2°	18' 0,3°	24' 0,4°	30' 0,5°	36' 0,6°	42' 0,7°	48' 0,8°	54' 0,9°	SUBTRACT Differences 1'	2'	3'	4'	5'
45°	$\bar{1}$,8495	8487	8480	8472	8464	8457	8449	8441	8433	8426	1	3	4	5	6
46	$\bar{1}$,8418	8410	8402	8394	8386	8378	8370	8362	8354	8346	1	3	4	5	7
47	$\bar{1}$,8338	8330	8322	8313	8305	8297	8289	8280	8272	8264	1	3	4	6	7
48	$\bar{1}$,8255	8247	8238	8230	8221	8213	8204	8195	8187	8178	1	3	4	6	7
49	$\bar{1}$,8169	8161	8152	8143	8134	8125	8117	8108	8099	8090	1	3	4	6	7
50°	$\bar{1}$,8081	8072	8063	8053	8044	8035	8026	8017	8007	7998	2	3	5	6	8
51	$\bar{1}$,7989	7979	7970	7960	7951	7941	7932	7922	7913	7903	2	3	5	6	8
52	$\bar{1}$,7893	7884	7874	7864	7854	7844	7835	7825	7815	7805	2	3	5	7	8
53	$\bar{1}$,7795	7785	7774	7764	7754	7744	7734	7723	7713	7703	2	3	5	7	9
54	$\bar{1}$,7692	7682	7671	7661	7650	7640	7629	7618	7607	7597	2	4	5	7	9
55	$\bar{1}$,7586	7575	7564	7553	7542	7531	7520	7509	7498	7487	2	4	6	7	9
56	$\bar{1}$,7476	7464	7453	7442	7430	7419	7407	7369	7384	7373	2	4	6	8	10
57	$\bar{1}$,7361	7349	7338	7326	7314	7302	7290	7278	7266	7254	2	4	6	8	10
58	$\bar{1}$,7242	7230	7218	7205	7193	7181	7168	7156	7144	7131	2	4	6	8	10
59	$\bar{1}$,7118	7106	7093	7080	7068	7055	7042	7029	7016	7003	2	4	6	9	11
60°	$\bar{1}$,6990	6977	6963	6950	6937	6923	6910	6896	6883	6869	2	4	7	9	11
61	$\bar{1}$,6856	6842	6828	6814	6801	6787	6773	6759	6744	6730	2	5	7	9	12
62	$\bar{1}$,6716	6702	6687	6673	6659	6644	6629	6615	6600	6585	2	5	7	10	12
63	$\bar{1}$,6570	6556	6541	6526	6510	6495	6480	6465	6449	6434	3	5	8	10	13
64	$\bar{1}$,6418	6403	6387	6371	6356	6340	6324	6308	6292	6276	3	5	8	11	13
65	$\bar{1}$,6259	6243	6227	6210	6194	6177	6161	6144	6127	6110	3	6	8	11	14
66	$\bar{1}$,6093	6076	6059	6042	6024	6007	5990	5972	5954	5937	3	6	9	12	15
67	$\bar{1}$,5919	5901	5883	5865	5847	5828	5810	5792	5773	5754	3	6	9	12	15
68	$\bar{1}$,5736	5717	5698	5679	5660	5641	5621	5602	5583	5563	3	6	10	13	16
69	$\bar{1}$,5543	5523	5504	5484	5463	5443	5423	5402	5382	5361	3	7	10	14	17
70°	$\bar{1}$,5341	5320	5299	5278	5256	5235	5213	5192	5170	5148	4	7	11	14	18
71	$\bar{1}$,5126	5104	5082	5060	5037	5015	4992	4969	4946	4923	4	8	11	15	19
72	$\bar{1}$,4900	4876	4853	4829	4805	4781	4757	4733	4709	4684	4	8	12	16	20
73	$\bar{1}$,4659	4634	4609	4584	4559	4533	4508	4482	4456	4430	4	9	13	17	21
74	$\bar{1}$,4403	4377	4350	4323	4296	4269	4242	4214	4186	4158	5	9	14	18	23
75	$\bar{1}$,4130	4102	4073	4044	4015	3986	3957	3927	3897	3867	5	10	15	20	24
76	$\bar{1}$,3837	3806	3775	3745	3713	3682	3650	3618	3586	3554	5	11	16	21	26
77	$\bar{1}$,3521	3488	3455	3421	3387	3353	3319	3284	3250	3214	6	11	17	23	28
78	$\bar{1}$,3179	3143	3107	3070	3034	2997	2959	2921	2883	2845	6	12	19	25	31
79	$\bar{1}$,2806	2767	2727	2687	2647	2606	2565	2524	2482	2439	7	14	20	27	34
80°	$\bar{1}$,2397	2353	2310	2266	2221	2176	2131	2085	2038	1991	8	15	23	30	38
81	$\bar{1}$,1943	1895	1847	1797	1747	1697	1646	1594	1542	1489	8	17	25	34	42
82	$\bar{1}$,1436	1381	1326	1271	1214	1157	1099	1040	0981	0920	10	19	29	38	48
83	$\bar{1}$,0859	0797	0734	0670	0605	0539	0472	0403	0334	0264	11	22	33	44	55
84	$\bar{1}$,0192	0120	0046	9970	9894	9816	9736	9655	9573	9489	13	26	39	52	65
85	$\bar{2}$,9403	9315	9226	9135	9042	8946	8849	8749	8647	8543	16	32	48	64	80
86	$\bar{2}$,8436	8326	8213	8098	7979	7857	7731	7602	7468	7330					
87	$\bar{2}$,7188	7041	6889	6731	6567	6397	6220	6035	5842	5640					
88	$\bar{2}$,5428	5206	4971	4723	4459	4179	3880	3558	3210	2832					
89	$\bar{2}$,2419	1961	1450	0870	0200	9408	8439	7190	5429	2419					

Logarithms of tangents

$\theta \to \log \tan \theta$

θ	0' 0,0°	6' 0,1°	12' 0,2°	18' 0,3°	24' 0,4°	30' 0,5°	36' 0,6°	42' 0,7°	48' 0,8°	54' 0,9°	ADD Differences 1'	2'	3'	4'	5'
0°	−∞	$\bar{3}$,2419	5429	7190	8439	9409	$\bar{2}$,0200	0870	1450	1962					
1	$\bar{2}$,2419	2833	3211	3559	3881	4181	4461	4725	4973	5208					
2	$\bar{2}$,5431	5643	5845	6038	6223	6401	6571	6736	6894	7046					
3	$\bar{2}$,7194	7337	7475	7609	7739	7865	7988	8107	8223	8336					
4	$\bar{2}$,8446	8554	8659	8762	8862	8960	9056	9150	9241	9331	16	32	48	64	81
5	$\bar{2}$,9420	9506	9591	9674	9756	9836	9915	9992	$\bar{0}$068	$\bar{0}$143	13	26	40	53	66
6	$\bar{1}$,0216	0289	0360	0430	0499	0567	0633	0699	0764	0828	11	22	34	45	56
7	$\bar{1}$,0891	0954	1015	1076	1135	1194	1252	1310	1367	1423	10	20	29	39	49
8	$\bar{1}$,1478	1533	1587	1640	1693	1745	1797	1848	1898	1948	9	17	26	35	43
9	$\bar{1}$,1997	2046	2094	2142	2189	2236	2282	2328	2374	2419	8	16	23	31	39
10°	$\bar{1}$,2463	2507	2551	2594	2637	2680	2722	2764	2805	2846	7	14	21	26	35
11	$\bar{1}$,2887	2927	2967	3006	3046	3085	3123	3162	3200	3237	6	13	19	26	32
12	$\bar{1}$,3275	3312	3349	3385	3422	3458	3493	3529	3564	3599	6	12	18	24	30
13	$\bar{1}$,3634	3668	3702	3736	3770	3804	3837	3870	3903	3935	6	11	17	22	28
14	$\bar{1}$,3968	4000	4032	4064	4095	4127	4185	4189	4220	4250	5	10	16	21	26
15	$\bar{1}$,4281	4311	4341	4371	4400	4430	4459	4488	4517	4546	5	10	15	20	25
16	$\bar{1}$,4575	4603	4632	4660	4688	4716	4744	4771	4799	4826	5	9	14	19	23
17	$\bar{1}$,4853	4880	4907	4934	4961	4987	5014	5040	5066	5092	4	9	13	18	22
18	$\bar{1}$,5118	5143	5169	5195	5220	5245	5270	5295	5320	5345	4	8	13	17	21
19	$\bar{1}$,5370	5394	5419	5443	5467	5491	5516	5539	5563	5587	4	8	12	16	20
20°	$\bar{1}$,5611	5634	5658	5681	5704	5727	5750	5773	5796	5819	4	8	12	15	19
21	$\bar{1}$,5842	5884	5887	5909	5932	5954	5976	5998	6020	6042	4	7	11	15	19
22	$\bar{1}$,6064	6068	6108	6129	6151	6172	6194	6215	6236	6257	4	7	11	14	18
23	$\bar{1}$,6279	6300	6321	6341	6362	6383	6404	6424	6445	6465	3	7	10	14	17
24	$\bar{1}$,6486	6506	6527	6547	6567	6587	6607	6627	6647	6667	3	7	10	13	17
25	$\bar{1}$,6687	6706	6726	6746	6765	6785	6804	6824	6843	6863	3	7	10	13	16
26	$\bar{1}$,6882	6901	6920	6939	6958	6977	6996	7015	7034	7053	3	6	9	13	16
27	$\bar{1}$,7072	7090	7109	7128	7146	7165	7183	7202	7220	7238	3	6	9	12	15
28	$\bar{1}$,7257	7275	7293	7311	7330	7348	7366	7384	7402	7420	3	6	9	12	15
29	$\bar{1}$,7438	7455	7473	7491	7509	7526	7544	7562	7579	7597	3	6	9	12	15
30°	$\bar{1}$,7614	7632	7649	7667	7684	7701	7719	7736	7753	7771	3	6	9	12	14
31	$\bar{1}$,7788	7805	7822	7839	7856	7873	7890	7907	7924	7941	3	6	9	11	14
32	$\bar{1}$,7958	7975	7992	8008	8025	8042	8059	8075	8092	8109	3	6	8	11	14
33	$\bar{1}$,8125	8142	8158	8175	8191	8208	8224	8241	8257	8274	3	5	8	11	14
34	$\bar{1}$,8290	8306	8323	8339	8355	8371	8388	8404	8420	8436	3	5	8	11	14
35	$\bar{1}$,8452	8468	8484	8501	8517	8533	8549	8565	8581	8597	3	5	8	11	13
36	$\bar{1}$,8613	8629	8644	8660	8676	8692	8708	8724	8740	8755	3	5	8	11	13
37	$\bar{1}$,8771	8787	8803	8818	8834	8850	8865	8881	8897	8912	3	5	8	10	13
38	$\bar{1}$,8928	8944	8959	8975	8990	9006	9022	9037	9053	9068	3	5	8	10	13
39	$\bar{1}$,9084	9099	9115	9310	9146	9161	9176	9192	9207	9223	3	5	8	10	13
40°	$\bar{1}$,9238	9254	9269	9284	9300	9315	9330	9346	9361	9376	3	5	8	10	13
41	$\bar{1}$,9392	9407	9422	9438	9453	9468	9483	9499	9514	9529	3	5	8	10	13
42	$\bar{1}$,9544	9560	9575	9590	9605	9621	9636	9651	9666	9681	3	5	8	10	13
43	$\bar{1}$,9697	9712	9727	9742	9757	9773	9788	9803	9818	9833	3	5	8	10	13
44	$\bar{1}$,9848	9864	9879	9894	9909	9924	9939	9955	9970	9985	3	5	8	10	13

θ	0' 0,0°	6' 0,1°	12' 0,2°	18' 0,3°	24' 0,4°	30' 0,5°	36' 0,6°	42' 0,7°	48' 0,8°	54' 0,9°	ADD Differences 1'	2'	3'	4'	5'
45°	0,0000	0051	0030	0045	0061	0076	0091	0106	0121	0136	3	5	8	10	13
46	0,0152	0167	0182	0197	0212	0228	0243	0258	0273	0288	3	5	8	10	13
47	0,0303	0319	0334	0349	0364	0379	0395	0410	0425	0440	3	5	8	10	13
48	0,0456	0471	0486	0501	0517	0532	0547	0562	0578	0593	3	5	8	10	13
49	0,0608	0624	0639	0654	0670	0685	0700	0716	0731	0746	3	5	8	10	13
50°	0,0762	0777	0793	0808	0824	0839	0854	0870	0885	0901	3	5	8	10	13
51	0,0916	0932	0947	0963	0978	0994	1010	1025	1041	1056	3	5	8	10	13
52	0,1072	1088	1103	1119	1135	1150	1166	1182	1197	1213	3	5	8	10	13
53	0,1229	1245	1260	1276	1292	1308	1324	1340	1356	1371	3	5	8	11	13
54	0,1387	1403	1419	1436	1451	1467	1483	1499	1516	1532	3	5	8	11	13
55	0,1548	1564	1580	1596	1612	1629	1645	1661	1677	1694	3	5	8	11	14
56	0,1710	1726	1743	1759	1776	1792	1809	1825	1842	1858	3	5	8	11	14
57	0,1875	1891	1908	1925	1941	1958	1975	1992	2008	2025	3	6	8	11	14
58	0,2042	2059	2076	2093	2110	2127	2144	2161	2178	2195	3	6	9	11	14
59	0,2212	2229	2247	2264	2281	2299	2316	2333	2351	2368	3	6	9	12	14
60°	0,2386	2403	2421	2438	2456	2474	2491	2509	2527	2545	3	6	9	12	15
61	0,2562	2580	2598	2616	2634	2652	2670	2689	2707	2725	3	6	9	12	15
62	0,2743	2762	2780	2798	2817	2835	2854	2872	2891	2910	3	6	9	12	15
63	0,2928	2947	2966	2985	3004	3023	3042	3061	3080	3099	3	6	9	13	16
64	0,3118	3137	3157	3176	3196	3215	3235	3254	3274	3294	3	6	10	13	16
65	0,3313	3333	3353	3373	3393	3413	3433	3453	3473	3494	3	7	10	13	17
66	0,3514	3535	3555	3576	3596	3617	3638	3659	3679	3700	3	7	10	14	17
67	0,3721	3743	3764	3785	3806	3828	3849	3871	3892	3914	4	7	11	14	18
68	0,3936	3958	3980	4002	4024	4046	4068	4091	4113	4136	4	7	11	15	19
69	0,4158	4181	4204	4227	4250	4273	4296	4319	4342	4366	4	8	12	15	19
70°	0,4389	4413	4437	4461	4484	4509	4533	4557	4581	4606	4	8	12	16	20
71	0,4630	4655	4680	4705	4730	4755	4780	4805	4831	4857	4	8	13	17	21
72	0,4882	4908	4934	4960	4986	5013	5039	5066	5093	5120	4	9	13	18	22
73	0,5147	5174	5201	5229	5256	5284	5312	5340	5368	5397	5	9	14	19	23
74	0,5425	5454	5483	5512	5541	5570	5600	5629	5659	5689	5	10	15	20	25
75	0,5719	5750	5780	5811	5842	5873	5905	5936	5968	6000	5	10	16	21	26
76	0,6032	6065	6097	6130	6163	6196	6230	6264	6298	6332	6	11	17	22	28
77	0,6366	6401	6436	6471	6507	6542	6578	6615	6651	6688	6	12	18	24	30
78	0,6725	6763	6800	6838	6877	6915	6954	6994	7033	7073	6	13	19	26	32
79	0,7113	7154	7195	7263	7278	7320	7363	7406	7449	7493	7	14	21	28	35
80°	0,7537	7581	7626	7672	7718	7754	7811	7858	7906	7954	8	16	23	31	39
81	0,8003	8052	8102	8152	8203	8255	8307	8360	8413	8467	9	17	26	35	43
82	0,8522	8577	8633	8690	8748	8806	8865	8924	8985	9046	10	20	29	39	49
83	0,9109	9172	9236	9301	9367	9433	9501	9570	9640	9711	11	22	34	45	56
84	0,9784	9857	9932	0008	0085	0164	0244	0326	0409	0494	13	26	40	53	66
85	1,0580	0669	0759	0850	0944	1040	1138	1238	1341	1446	16	32	48	64	81
86	1,1554	1664	1777	1893	2012	2135	2261	2391	2525	2663					
87	1,2806	2954	3106	3264	3429	3599	3777	3962	4155	4357					
88	1,4596	4792	5027	5275	5539	5819	6119	6441	6789	7167					
89	1,7581	8038	8550	9130	9800	$\bar{0}$591	$\bar{1}$561	$\bar{2}$810	$\bar{4}$571	$\bar{7}$581					

Squares

$x \to x^2$

x	0	1	2	3	4	5	6	7	8	9	Differences 1	2	3	4	5	6	7	8	9
10	1000	1020	1040	1061	1082	1103	1124	1145	1166	1188	2	4	6	8	10	13	15	17	19
11	1210	1232	1254	1277	1300	1323	1346	1369	1392	1416	2	5	7	9	11	14	16	18	21
12	1440	1464	1488	1513	1538	1563	1588	1613	1638	1664	2	5	7	10	12	15	17	20	22
13	1690	1716	1742	1769	1796	1823	1850	1877	1904	1932	3	5	8	11	13	16	19	22	24
14	1960	1988	2016	2045	2074	2103	2132	2161	2190	2220	3	6	9	12	14	17	20	23	26
15	2250	2280	2310	2341	2372	2403	2434	2465	2496	2528	3	6	9	12	15	19	22	25	28
16	2560	2592	2624	2657	2690	2723	2756	2789	2822	2856	3	7	10	13	16	20	23	26	30
17	2890	2924	2958	2993	3028	3063	3098	3133	3168	3204	3	7	10	14	17	21	24	28	31
18	3240	3276	3312	3349	3386	3423	3460	3497	3534	3572	4	7	11	15	18	22	26	30	33
19	3610	3648	3686	3725	3764	3803	3842	3881	3920	3960	4	8	12	16	19	23	27	31	35
20	4000	4040	4080	4121	4162	4203	4244	4285	4326	4368	4	8	12	16	20	25	29	33	37
21	4410	4452	4494	4537	4580	4623	4666	4709	4752	4796	4	9	13	17	21	26	30	34	39
22	4840	4884	4928	4973	5018	5063	5108	5153	5198	5244	4	9	13	18	22	27	31	36	40
23	5290	5336	5382	5429	5476	5523	5570	5617	5664	5712	5	9	14	19	23	28	33	38	42
24	5760	5808	5856	5905	5954	6003	6052	6101	6150	6200	5	10	15	20	24	29	34	39	44
25	6250	6300	6350	6401	6452	6503	6554	6605	6656	6708	5	10	15	20	25	31	36	41	46
26	6760	6812	6864	6917	6970	7023	7076	7129	7182	7236	5	11	16	21	26	32	37	42	48
27	7290	7344	7398	7453	7508	7563	7618	7673	7728	7784	5	11	16	22	27	33	38	44	49
28	7840	7896	7952	8009	8066	8123	8180	8237	8294	8352	6	11	17	23	28	34	40	46	51
29	8410	8468	8526	8585	8644	8703	8762	8821	8880	8940	6	12	18	24	29	35	41	47	53
30	9000	9060	9120	9181	9242	9303	9364	9425	9486	9548	6	12	18	24	30	37	43	49	55
31	9610	9672	9734	9797	9860	9923	9986	—	—	—	6	13	19	25	31	38	44	50	57
31	—	—	—	—	—	—	—	1005	1011	1018	1	1	2	3	3	4	5	5	6
32	1024	1030	1037	1043	1050	1056	1063	1069	1076	1082	1	1	2	3	3	4	5	5	6
33	1089	1096	1102	1109	1116	1122	1129	1136	1142	1149	1	1	2	3	3	4	5	5	6
34	1156	1163	1170	1176	1183	1190	1197	1204	1211	1218	1	1	2	3	3	4	5	6	6
35	1225	1232	1239	1246	1253	1260	1267	1274	1282	1289	1	1	2	3	4	4	5	6	6
36	1296	1303	1310	1318	1325	1332	1340	1347	1354	1362	1	1	2	3	4	4	5	6	7
37	1369	1376	1384	1391	1399	1406	1414	1421	1429	1436	1	2	2	3	4	5	5	6	7
38	1444	1452	1459	1467	1475	1482	1490	1498	1505	1513	1	2	2	3	4	5	5	6	7
39	1521	1529	1537	1544	1552	1560	1568	1576	1584	1592	1	2	2	3	4	5	6	6	7
40	1600	1608	1616	1624	1632	1640	1648	1656	1665	1673	1	2	2	3	4	5	6	6	7
41	1681	1689	1697	1706	1714	1722	1731	1739	1747	1756	1	2	2	3	4	5	6	7	7
42	1764	1772	1781	1789	1798	1806	1815	1823	1832	1840	1	2	3	3	4	5	6	7	8
43	1849	1858	1866	1875	1884	1892	1901	1910	1918	1927	1	2	3	3	4	5	6	7	8
44	1936	1945	1954	1962	1971	1980	1989	1998	2007	2016	1	2	3	4	5	5	6	7	8
45	2025	2034	2043	2052	2061	2070	2079	2088	2098	2107	1	2	3	4	5	5	6	7	8
46	2116	2125	2134	2144	2153	2162	2172	2181	2190	2200	1	2	3	4	5	6	7	7	8
47	2209	2218	2228	2237	2247	2256	2266	2275	2285	2294	1	2	3	4	5	6	7	8	9
48	2304	2314	2323	2333	2343	2352	2362	2372	2381	2391	1	2	3	4	5	6	7	8	9
49	2401	2411	2421	2430	2440	2450	2460	2470	2480	2490	1	2	3	4	5	6	7	8	9
50	2500	2510	2520	2530	2540	2550	2560	2570	2581	2591	1	2	3	4	5	6	7	8	9
51	2601	2611	2621	2632	2642	2652	2663	2673	2683	2694	1	2	3	4	5	6	7	8	9
52	2704	2714	2725	2735	2746	2756	2767	2777	2788	2798	1	2	3	4	5	6	7	8	9
53	2809	2820	2830	2841	2852	2862	2873	2884	2894	2905	1	2	3	4	5	6	7	9	10
54	2916	2927	2938	2948	2959	2970	2981	2992	3003	3014	1	2	3	4	5	7	8	9	10

x	0	1	2	3	4	5	6	7	8	9	Differences 1	2	3	4	5	6	7	8	9
55	3025	3036	3047	3058	3069	3080	3091	3102	3114	3125	1	2	3	4	6	7	8	9	10
56	3136	3147	3158	3170	3181	3192	3204	3215	3226	3238	1	2	3	5	6	7	9	9	10
57	3249	3260	3272	3283	3295	3306	3318	3329	3341	3352	1	2	3	5	6	7	8	9	10
58	3364	3376	3387	3399	3411	3422	3434	3446	3457	3469	1	2	4	5	6	7	8	9	10
59	3481	3493	3505	3516	3528	3540	3552	3564	3576	3588	1	2	4	5	6	7	8	10	11
60	3600	3612	3624	3636	3648	3660	3672	3684	3697	3709	1	2	4	5	6	7	8	10	11
61	3721	3733	3745	3758	3770	3782	3795	3807	3819	3832	1	2	4	5	6	7	9	10	11
62	3844	3856	3869	3881	3894	3906	3919	3931	3944	3956	1	2	4	5	6	7	9	10	11
63	3969	3982	3994	4007	4020	4032	4045	4058	4070	4083	1	3	4	5	6	8	9	10	11
64	4096	4109	4122	4134	4147	4160	4173	4186	4199	4212	1	3	4	5	6	8	9	10	12
65	4225	4238	4251	4264	4277	4290	4303	4316	4330	4343	1	3	4	5	7	8	9	10	12
66	4356	4369	4382	4396	4409	4422	4436	4449	4462	4476	1	3	4	5	7	8	9	11	12
67	4489	4502	4516	4529	4543	4556	4570	4583	4597	4610	1	3	4	5	7	8	9	11	12
68	4624	4638	4651	4665	4679	4692	4706	4720	4733	4747	1	3	4	5	7	8	10	11	12
69	4761	4775	4789	4802	4816	4830	4844	4858	4872	4886	1	3	4	6	7	8	10	11	13
70	4900	4914	4928	4942	4956	4970	4984	4998	5013	5027	1	3	4	6	7	8	10	11	13
71	5041	5055	5069	5084	5098	5112	5127	5141	5155	5170	1	3	4	6	7	9	10	11	13
72	5184	5198	5213	5227	5242	5256	5271	5285	5300	5314	1	3	4	6	7	9	10	11	13
73	5329	5344	5358	5373	5388	5402	5417	5432	5446	5461	1	3	4	6	7	9	10	12	13
74	5476	5491	5506	5520	5535	5550	5565	5580	5595	5610	1	3	4	6	7	9	10	12	13
75	5625	5640	5655	5670	5685	5700	5715	5730	5746	5761	2	3	5	6	8	9	11	12	14
76	5776	5791	5806	5822	5837	5852	5868	5883	5898	5914	2	3	5	6	8	9	11	12	14
77	5929	5944	5960	5975	5991	6006	6022	6037	6053	6068	2	3	5	6	8	9	11	12	14
78	6084	6100	6115	6131	6147	6162	6178	6194	6209	6225	2	3	5	6	8	9	11	13	14
79	6241	6257	6273	6288	6304	6320	6336	6352	6368	6384	2	3	5	6	8	10	11	13	14
80	6400	6416	6432	6448	6464	6480	6496	6512	6529	6545	2	3	5	6	8	10	11	13	14
81	6561	6577	6593	6610	6626	6642	6659	6675	6691	6708	2	3	5	7	8	10	11	13	15
82	6724	6740	6757	6773	6790	6806	6823	6839	6856	6872	2	3	5	7	8	10	12	13	15
83	6889	6906	6922	6939	6956	6972	6989	7006	7022	7039	2	3	5	7	8	10	12	13	15
84	7056	7073	7090	7106	7123	7140	7157	7174	7191	7208	2	3	5	7	8	10	12	14	15
85	7225	7242	7259	7276	7293	7310	7327	7344	7362	7379	2	3	5	7	9	10	12	14	15
86	7396	7413	7430	7448	7465	7482	7500	7517	7534	7552	2	3	5	7	9	10	12	14	16
87	7569	7586	7604	7621	7639	7656	7674	7691	7709	7726	2	3	5	7	9	10	12	14	16
88	7744	7762	7779	7797	7815	7832	7850	7868	7885	7903	2	4	5	7	9	11	12	14	16
89	7921	7939	7957	7974	7992	8010	8028	8046	8064	8082	2	4	5	7	9	11	13	14	16
90	8100	8118	8136	8154	8172	8190	8208	8226	8245	8263	2	4	5	7	9	11	13	14	16
91	8281	8299	8317	8336	8354	8372	8391	8409	8427	8446	2	4	5	7	9	11	13	15	16
92	8464	8482	8501	8519	8538	8556	8575	8593	8612	8630	2	4	6	7	9	11	13	15	17
93	8649	8668	8686	8705	8724	8742	8761	8780	8798	8817	2	4	6	7	9	11	13	15	17
94	8836	8855	8874	8892	8911	8930	8949	8968	8987	9006	2	4	6	8	9	11	13	15	17
95	9025	9044	9063	9082	9101	9120	9139	9158	9178	9197	2	4	6	8	10	11	13	15	17
96	9216	9235	9254	9274	9293	9312	9332	9351	9370	9390	2	4	6	8	10	12	14	15	17
97	9409	9428	9448	9467	9487	9506	9526	9545	9565	9584	2	4	6	8	10	12	14	16	18
98	9604	9624	9643	9663	9683	9702	9722	9742	9761	9781	2	4	6	8	10	12	14	16	18
99	9801	9821	9841	9860	9880	9900	9920	9940	9960	9980	2	4	6	8	10	12	14	16	18

Square roots from 1 to 10

x	0	1	2	3	4	5	6	7	8	9	Diff. 1	2	3	4	5	6	7	8	9
1,0	1,000	1,005	1,010	1,015	1,020	1,025	1,030	1,034	1,039	1,044	0	1	1	2	2	3	3	4	4
1,1	1,049	1,054	1,058	1,063	1,068	1,072	1,077	1,082	1,086	1,091	0	1	1	2	2	3	3	4	4
1,2	1,095	1,100	1,105	1,109	1,114	1,118	1,123	1,127	1,131	1,136	0	1	1	2	2	3	3	4	4
1,3	1,140	1,145	1,149	1,153	1,158	1,162	1,166	1,171	1,175	1,179	0	1	1	2	2	3	3	3	4
1,4	1,183	1,187	1,192	1,196	1,200	1,204	1,208	1,212	1,217	1,221	0	1	1	2	2	2	3	3	4
1,5	1,225	1,229	1,233	1,237	1,241	1,245	1,249	1,253	1,257	1,261	0	1	1	2	2	2	3	3	4
1,6	1,265	1,269	1,273	1,277	1,281	1,285	1,288	1,292	1,296	1,300	0	1	1	2	2	2	3	3	3
1,7	1,304	1,308	1,312	1,315	1,319	1,323	1,327	1,330	1,334	1,338	0	1	1	2	2	2	3	3	3
1,8	1,342	1,345	1,349	1,353	1,357	1,360	1,364	1,368	1,371	1,375	0	1	1	1	2	2	3	3	3
1,9	1,378	1,382	1,386	1,389	1,393	1,396	1,400	1,404	1,407	1,411	0	1	1	1	2	2	3	3	3
2,0	1,414	1,418	1,421	1,425	1,428	1,432	1,435	1,439	1,442	1,446	0	1	1	1	2	2	2	3	3
2,1	1,449	1,453	1,456	1,460	1,463	1,466	1,470	1,473	1,477	1,480	0	1	1	1	2	2	2	3	3
2,2	1,483	1,487	1,490	1,493	1,497	1,500	1,503	1,507	1,510	1,513	0	1	1	1	2	2	2	3	3
2,3	1,517	1,520	1,523	1,526	1,530	1,533	1,536	1,539	1,543	1,546	0	1	1	1	2	2	2	3	3
2,4	1,549	1,552	1,556	1,559	1,562	1,565	1,568	1,572	1,575	1,578	0	1	1	1	2	2	2	3	3
2,5	1,581	1,584	1,587	1,591	1,594	1,597	1,600	1,603	1,606	1,609	0	1	1	1	2	2	2	3	3
2,6	1,612	1,616	1,619	1,622	1,625	1,628	1,631	1,634	1,637	1,640	0	1	1	1	2	2	2	2	3
2,7	1,643	1,646	1,649	1,652	1,655	1,658	1,661	1,664	1,667	1,670	0	1	1	1	2	2	2	2	3
2,8	1,673	1,676	1,679	1,682	1,685	1,688	1,691	1,694	1,697	1,700	0	1	1	1	1	2	2	2	3
2,9	1,703	1,706	1,709	1,712	1,715	1,718	1,720	1,723	1,726	1,729	0	1	1	1	1	2	2	2	3
3,0	1,732	1,735	1,738	1,741	1,744	1,746	1,749	1,752	1,755	1,758	0	1	1	1	1	2	2	2	3
3,1	1,761	1,764	1,766	1,769	1,772	1,775	1,778	1,780	1,783	1,786	0	1	1	1	1	2	2	2	3
3,2	1,789	1,792	1,794	1,797	1,800	1,803	1,806	1,808	1,811	1,814	0	1	1	1	1	2	2	2	2
3,3	1,817	1,819	1,822	1,825	1,828	1,830	1,833	1,836	1,839	1,841	0	1	1	1	1	2	2	2	2
3,4	1,844	1,847	1,849	1,852	1,855	1,857	1,860	1,863	1,866	1,868	0	1	1	1	1	2	2	2	2
3,5	1,871	1,874	1,876	1,879	1,882	1,884	1,887	1,889	1,892	1,895	0	1	1	1	1	2	2	2	2
3,6	1,897	1,900	1,903	1,905	1,908	1,911	1,913	1,916	1,918	1,921	0	1	1	1	1	2	2	2	2
3,7	1,924	1,926	1,929	1,931	1,934	1,937	1,939	1,942	1,944	1,947	0	1	1	1	1	2	2	2	2
3,8	1,949	1,952	1,955	1,957	1,960	1,962	1,965	1,967	1,970	1,972	0	1	1	1	1	2	2	2	2
3,9	1,975	1,977	1,980	1,982	1,985	1,988	1,990	1,993	1,995	1,998	0	1	1	1	1	2	2	2	2
4,0	2,000	2,003	2,005	2,008	2,010	2,013	2,015	2,017	2,020	2,022	0	0	1	1	1	1	2	2	2
4,1	2,025	2,027	2,030	2,032	2,035	2,037	2,040	2,042	2,045	2,047	0	0	1	1	1	1	2	2	2
4,2	2,049	2,052	2,054	2,057	2,059	2,062	2,064	2,066	2,069	2,071	0	0	1	1	1	1	2	2	2
4,3	2,074	2,076	2,078	2,081	2,083	2,086	2,088	2,091	2,093	2,095	0	0	1	1	1	1	2	2	2
4,4	2,098	2,100	2,102	2,105	2,107	2,110	2,112	2,114	2,117	2,119	0	0	1	1	1	1	2	2	2
4,5	2,121	2,124	2,126	2,128	2,131	2,133	2,135	2,138	2,140	2,142	0	0	1	1	1	1	2	2	2
4,6	2,145	2,147	2,149	2,152	2,154	2,156	2,159	2,161	2,163	2,166	0	0	1	1	1	1	2	2	2
4,7	2,168	2,170	2,173	2,175	2,177	2,179	2,182	2,184	2,186	2,189	0	0	1	1	1	1	2	2	2
4,8	2,191	2,193	2,195	2,198	2,200	2,202	2,205	2,207	2,209	2,211	0	0	1	1	1	1	2	2	2
4,9	2,214	2,216	2,218	2,220	2,223	2,225	2,227	2,229	2,232	2,234	0	0	1	1	1	1	2	2	2
5,0	2,236	2,238	2,241	2,243	2,245	2,247	2,249	2,252	2,254	2,256	0	0	1	1	1	1	2	2	2
5,1	2,258	2,261	2,263	2,265	2,267	2,269	2,272	2,274	2,276	2,278	0	0	1	1	1	1	2	2	2
5,2	2,280	2,283	2,285	2,287	2,289	2,291	2,294	2,296	2,298	2,300	0	0	1	1	1	1	2	2	2
5,3	2,302	2,304	2,307	2,309	2,311	2,313	2,315	2,317	2,320	2,322	0	0	1	1	1	1	2	2	2
5,4	2,324	2,326	2,328	2,330	2,332	2,335	2,337	2,339	2,341	2,343	0	0	1	1	1	1	1	2	2
5,5	2,345	2,347	2,350	2,352	2,354	2,356	2,358	2,360	2,362	2,364	0	0	1	1	1	1	1	2	2
5,6	2,366	2,369	2,371	2,373	2,375	2,377	2,379	2,381	2,383	2,385	0	0	1	1	1	1	1	2	2
5,7	2,388	2,390	2,392	2,394	2,396	2,398	2,400	2,402	2,404	2,406	0	0	1	1	1	1	1	2	2
5,8	2,408	2,410	2,412	2,415	2,417	2,419	2,421	2,423	2,425	2,427	0	0	1	1	1	1	1	2	2
5,9	2,429	2,431	2,433	2,435	2,437	2,439	2,441	2,443	2,445	2,447	0	0	1	1	1	1	1	2	2
6,0	2,450	2,452	2,454	2,456	2,458	2,460	2,462	2,464	2,466	2,468	0	0	1	1	1	1	1	2	2
6,1	2,470	2,472	2,474	2,476	2,478	2,480	2,482	2,484	2,486	2,488	0	0	1	1	1	1	1	2	2
6,2	2,490	2,492	2,494	2,496	2,498	2,500	2,502	2,504	2,506	2,508	0	0	1	1	1	1	1	2	2
6,3	2,510	2,512	2,514	2,516	2,518	2,520	2,522	2,524	2,526	2,528	0	0	1	1	1	1	1	2	2
6,4	2,530	2,532	2,534	2,536	2,538	2,540	2,542	2,544	2,546	2,548	0	0	1	1	1	1	1	2	2
6,5	2,550	2,551	2,553	2,555	2,557	2,559	2,561	2,563	2,565	2,567	0	0	1	1	1	1	1	2	2
6,6	2,569	2,571	2,573	2,575	2,577	2,579	2,581	2,583	2,585	2,587	0	0	1	1	1	1	1	2	2
6,7	2,588	2,590	2,592	2,594	2,596	2,598	2,600	2,602	2,604	2,606	0	0	1	1	1	1	1	2	2
6,8	2,608	2,610	2,612	2,613	2,615	2,617	2,619	2,621	2,623	2,625	0	0	1	1	1	1	1	2	2
6,9	2,627	2,629	2,631	2,632	2,634	2,636	2,638	2,640	2,642	2,644	0	0	1	1	1	1	1	2	2
7,0	2,646	2,648	2,650	2,651	2,653	2,655	2,657	2,659	2,661	2,663	0	0	1	1	1	1	1	2	2
7,1	2,665	2,667	2,668	2,670	2,672	2,674	2,676	2,678	2,680	2,681	0	0	1	1	1	1	1	1	2
7,2	2,683	2,685	2,687	2,689	2,691	2,693	2,694	2,696	2,698	2,700	0	0	1	1	1	1	1	1	2
7,3	2,702	2,704	2,706	2,707	2,709	2,711	2,713	2,715	2,717	2,719	0	0	1	1	1	1	1	1	2
7,4	2,720	2,722	2,724	2,726	2,728	2,729	2,731	2,733	2,735	2,737	0	0	1	1	1	1	1	1	2
7,5	2,739	2,740	2,742	2,744	2,746	2,748	2,750	2,751	2,753	2,755	0	0	1	1	1	1	1	1	2
7,6	2,757	2,759	2,760	2,762	2,764	2,766	2,768	2,769	2,771	2,773	0	0	1	1	1	1	1	1	2
7,7	2,775	2,777	2,778	2,780	2,782	2,784	2,786	2,787	2,789	2,791	0	0	1	1	1	1	1	1	2
7,8	2,793	2,795	2,796	2,798	2,800	2,802	2,804	2,805	2,807	2,809	0	0	1	1	1	1	1	1	2
7,9	2,811	2,812	2,814	2,816	2,818	2,820	2,821	2,823	2,825	2,827	0	0	1	1	1	1	1	1	2
8,0	2,828	2,830	2,832	2,834	2,835	2,837	2,839	2,841	2,843	2,844	0	0	1	1	1	1	1	1	2
8,1	2,846	2,848	2,850	2,851	2,853	2,855	2,857	2,858	2,860	2,862	0	0	1	1	1	1	1	1	2
8,2	2,864	2,865	2,867	2,869	2,871	2,872	2,874	2,876	2,877	2,879	0	0	1	1	1	1	1	1	2
8,3	2,881	2,883	2,884	2,886	2,888	2,890	2,891	2,893	2,895	2,897	0	0	1	1	1	1	1	1	2
8,4	2,898	2,900	2,902	2,903	2,905	2,907	2,909	2,910	2,912	2,914	0	0	1	1	1	1	1	1	2
8,5	2,915	2,917	2,919	2,921	2,922	2,924	2,926	2,927	2,929	2,931	0	0	1	1	1	1	1	1	2
8,6	2,933	2,934	2,936	2,938	2,939	2,941	2,943	2,944	2,946	2,948	0	0	1	1	1	1	1	1	2
8,7	2,950	2,951	2,953	2,955	2,956	2,958	2,960	2,961	2,963	2,965	0	0	1	1	1	1	1	1	2
8,8	2,966	2,968	2,970	2,972	2,973	2,975	2,977	2,978	2,980	2,982	0	0	1	1	1	1	1	1	2
8,9	2,983	2,985	2,987	2,988	2,990	2,992	2,993	2,995	2,997	2,998	0	0	1	1	1	1	1	1	2
9,0	3,000	3,002	3,003	3,005	3,007	3,008	3,010	3,012	3,013	3,015	0	0	0	1	1	1	1	1	1
9,1	3,017	3,018	3,020	3,022	3,023	3,025	3,027	3,028	3,030	3,032	0	0	0	1	1	1	1	1	1
9,2	3,033	3,035	3,036	3,038	3,040	3,041	3,043	3,045	3,046	3,048	0	0	0	1	1	1	1	1	1
9,3	3,050	3,051	3,053	3,055	3,056	3,058	3,059	3,061	3,063	3,064	0	0	0	1	1	1	1	1	1
9,4	3,066	3,068	3,069	3,071	3,072	3,074	3,076	3,077	3,079	3,081	0	0	0	1	1	1	1	1	1
9,5	3,082	3,084	3,085	3,087	3,089	3,090	3,092	3,094	3,095	3,097	0	0	0	1	1	1	1	1	1
9,6	3,098	3,100	3,102	3,103	3,105	3,106	3,108	3,110	3,111	3,113	0	0	0	1	1	1	1	1	1
9,7	3,114	3,116	3,118	3,119	3,121	3,123	3,124	3,126	3,127	3,129	0	0	0	1	1	1	1	1	1
9,8	3,131	3,132	3,134	3,135	3,137	3,139	3,140	3,142	3,143	3,145	0	0	0	1	1	1	1	1	1
9,9	3,146	3,148	3,150	3,151	3,153	3,154	3,156	3,158	3,159	3,161	0	0	0	1	1	1	1	1	1

Square roots from 10 to 100

x	0	1	2	3	4	5	6	7	8	9	Diff. 1	2	3	4	5	6	7	8	9
10	3,162	3,178	3,194	3,209	3,225	3,240	3,256	3,271	3,286	3,302	2	3	5	6	8	9	11	12	14
11	3,317	3,332	3,347	3,362	3,376	3,391	3,406	3,421	3,435	3,450	1	3	4	6	7	9	10	12	13
12	3,464	3,479	3,493	3,507	3,521	3,536	3,550	3,564	3,578	3,592	1	3	4	6	7	8	10	11	13
13	3,606	3,619	3,633	3,647	3,661	3,674	3,688	3,701	3,715	3,728	1	3	4	5	7	8	10	11	12
14	3,742	3,755	3,768	3,782	3,795	3,808	3,821	3,834	3,847	3,860	1	3	4	5	7	8	9	11	12
15	3,873	3,886	3,899	3,912	3,924	3,937	3,950	3,962	3,975	3,988	1	3	4	5	6	8	9	10	11
16	4,000	4,012	4,025	4,037	4,050	4,062	4,074	4,087	4,099	4,111	1	2	4	5	6	7	9	10	11
17	4,123	4,135	4,147	4,159	4,171	4,183	4,195	4,207	4,219	4,231	1	2	4	5	6	7	8	10	11
18	4,243	4,254	4,266	4,278	4,290	4,301	4,313	4,324	4,336	4,347	1	2	3	5	6	7	8	9	10
19	4,359	4,370	4,382	4,393	4,405	4,416	4,427	4,438	4,450	4,461	1	2	3	5	6	7	8	9	10
20	4,472	4,483	4,494	4,506	4,517	4,528	4,539	4,550	4,561	4,572	1	2	3	4	6	7	8	9	10
21	4,583	4,594	4,604	4,615	4,626	4,637	4,648	4,658	4,669	4,680	1	2	3	4	5	6	8	9	10
22	4,690	4,701	4,712	4,722	4,733	4,743	4,754	4,765	4,775	4,785	1	2	3	4	5	6	7	8	9
23	4,796	4,806	4,817	4,827	4,837	4,848	4,858	4,868	4,879	4,889	1	2	3	4	5	6	7	8	9
24	4,899	4,909	4,919	4,930	4,940	4,950	4,960	4,970	4,980	4,990	1	2	3	4	5	6	7	8	9
25	5,000	5,010	5,020	5,030	5,040	5,050	5,060	5,070	5,079	5,089	1	2	3	4	5	6	7	8	9
26	5,099	5,109	5,119	5,128	5,138	5,148	5,158	5,167	5,177	5,187	1	2	3	4	5	6	7	8	9
27	5,196	5,206	5,215	5,225	5,235	5,244	5,254	5,263	5,273	5,282	1	2	3	4	5	6	7	8	9
28	5,292	5,301	5,310	5,320	5,329	5,339	5,348	5,357	5,367	5,376	1	2	3	4	5	6	7	7	8
29	5,385	5,394	5,404	5,413	5,422	5,431	5,441	5,450	5,459	5,468	1	2	3	4	5	5	6	7	8
30	5,477	5,486	5,495	5,505	5,514	5,523	5,532	5,541	5,550	5,559	1	2	3	4	4	5	6	7	8
31	5,568	5,577	5,586	5,595	5,604	5,612	5,621	5,630	5,639	5,648	1	2	3	3	4	5	6	7	8
32	5,657	5,666	5,675	5,683	5,692	5,701	5,710	5,718	5,727	5,736	1	2	3	3	4	5	6	7	8
33	5,745	5,753	5,762	5,771	5,779	5,788	5,797	5,805	5,814	5,822	1	2	3	3	4	5	6	7	8
34	5,831	5,840	5,848	5,857	5,865	5,874	5,882	5,891	5,899	5,908	1	2	3	3	4	5	6	7	8
35	5,916	5,925	5,933	5,941	5,950	5,958	5,967	5,975	5,983	5,992	1	2	2	3	4	5	6	7	8
36	6,000	6,008	6,017	6,025	6,033	6,042	6,050	6,058	6,066	6,075	1	2	2	3	4	5	6	7	7
37	6,083	6,091	6,099	6,107	6,116	6,124	6,132	6,140	6,148	6,156	1	2	2	3	4	5	6	7	7
38	6,164	6,173	6,181	6,189	6,197	6,205	6,213	6,221	6,229	6,237	1	2	2	3	4	5	6	6	7
39	6,245	6,253	6,261	6,269	6,277	6,285	6,293	6,301	6,309	6,317	1	2	2	3	4	5	6	6	7
40	6,325	6,332	6,340	6,348	6,356	6,364	6,372	6,380	6,387	6,395	1	2	2	3	4	5	6	6	7
41	6,403	6,411	6,419	6,427	6,434	6,442	6,450	6,458	6,465	6,473	1	2	2	3	4	5	5	6	7
42	6,481	6,488	6,496	6,504	6,512	6,519	6,527	6,535	6,542	6,550	1	2	2	3	4	5	5	6	7
43	6,557	6,565	6,573	6,580	6,588	6,595	6,603	6,611	6,618	6,626	1	2	2	3	4	5	5	6	7
44	6,633	6,641	6,648	6,656	6,663	6,671	6,678	6,686	6,693	6,701	1	2	2	3	4	5	5	6	7
45	6,708	6,716	6,723	6,731	6,738	6,745	6,753	6,760	6,768	6,775	1	1	2	3	4	4	5	6	7
46	6,782	6,790	6,797	6,804	6,812	6,819	6,826	6,834	6,841	6,848	1	1	2	3	4	4	5	6	7
47	6,856	6,863	6,870	6,878	6,885	6,892	6,899	6,907	6,914	6,921	1	1	2	3	4	4	5	6	7
48	6,928	6,935	6,943	6,950	6,957	6,964	6,971	6,979	6,986	6,993	1	1	2	3	4	4	5	6	6
49	7,000	7,007	7,014	7,021	7,029	7,036	7,043	7,050	7,057	7,064	1	1	2	3	4	4	5	6	6
50	7,071	7,078	7,085	7,092	7,099	7,106	7,113	7,120	7,127	7,134	1	1	2	3	4	4	5	6	6
51	7,141	7,148	7,155	7,162	7,169	7,176	7,183	7,190	7,197	7,204	1	1	2	3	4	4	5	6	6
52	7,211	7,218	7,225	7,232	7,239	7,246	7,253	7,259	7,266	7,273	1	1	2	3	3	4	5	6	6
53	7,280	7,287	7,294	7,301	7,308	7,314	7,321	7,328	7,335	7,342	1	1	2	3	3	4	5	5	6
54	7,349	7,355	7,362	7,369	7,376	7,382	7,389	7,396	7,403	7,410	1	1	2	3	3	4	5	5	6
55	7,416	7,423	7,430	7,436	7,443	7,450	7,457	7,463	7,470	7,477	1	1	2	3	3	4	5	5	6
56	7,483	7,490	7,497	7,503	7,510	7,517	7,523	7,530	7,537	7,543	1	1	2	3	3	4	5	5	6
57	7,550	7,556	7,563	7,570	7,576	7,583	7,589	7,596	7,603	7,609	1	1	2	3	3	4	5	5	6
58	7,616	7,622	7,629	7,635	7,642	7,649	7,655	7,662	7,668	7,675	1	1	2	3	3	4	5	5	6
59	7,681	7,688	7,694	7,701	7,707	7,714	7,720	7,727	7,733	7,740	1	1	2	3	3	4	4	5	6
60	7,746	7,752	7,759	7,765	7,772	7,778	7,785	7,791	7,797	7,804	1	1	2	3	3	4	4	5	6
61	7,810	7,817	7,823	7,829	7,836	7,842	7,849	7,855	7,861	7,868	1	1	2	3	3	4	4	5	6
62	7,874	7,880	7,887	7,893	7,899	7,906	7,912	7,918	7,925	7,931	1	1	2	3	3	4	4	5	6
63	7,937	7,944	7,950	7,956	7,962	7,969	7,975	7,981	7,987	7,994	1	1	2	3	3	4	4	5	6
64	8,000	8,006	8,012	8,019	8,025	8,031	8,037	8,044	8,050	8,056	1	1	2	2	3	4	4	5	6
65	8,062	8,068	8,075	8,081	8,087	8,093	8,099	8,106	8,112	8,118	1	1	2	2	3	4	4	5	6
66	8,124	8,130	8,136	8,142	8,149	8,155	8,161	8,167	8,173	8,179	1	1	2	2	3	4	4	5	5
67	8,185	8,191	8,198	8,204	8,210	8,216	8,222	8,228	8,234	8,240	1	1	2	2	3	4	4	5	5
68	8,246	8,252	8,258	8,264	8,270	8,276	8,283	8,289	8,295	8,301	1	1	2	2	3	4	4	5	5
69	8,307	8,313	8,319	8,325	8,331	8,337	8,343	8,349	8,355	8,361	1	1	2	2	3	4	4	5	5
70	8,367	8,373	8,379	8,385	8,390	8,396	8,402	8,408	8,414	8,420	1	1	2	2	3	4	4	5	5
71	8,426	8,432	8,438	8,444	8,450	8,456	8,462	8,468	8,473	8,479	1	1	2	2	3	4	4	5	5
72	8,485	8,491	8,497	8,503	8,509	8,515	8,521	8,526	8,532	8,538	1	1	2	2	3	3	4	5	5
73	8,544	8,550	8,556	8,562	8,567	8,573	8,579	8,585	8,591	8,597	1	1	2	2	3	3	4	5	5
74	8,602	8,608	8,614	8,620	8,626	8,631	8,637	8,643	8,649	8,654	1	1	2	2	3	3	4	5	5
75	8,660	8,666	8,672	8,678	8,683	8,689	8,695	8,701	8,706	8,712	1	1	2	2	3	3	4	5	5
76	8,718	8,724	8,729	8,735	8,741	8,746	8,752	8,758	8,764	8,769	1	1	2	2	3	3	4	5	5
77	8,775	8,781	8,786	8,792	8,798	8,803	8,809	8,815	8,820	8,826	1	1	2	2	3	3	4	4	5
78	8,832	8,837	8,843	8,849	8,854	8,860	8,866	8,871	8,877	8,883	1	1	2	2	3	3	4	4	5
79	8,888	8,894	8,899	8,905	8,911	8,916	8,922	8,927	8,933	8,939	1	1	2	2	3	3	4	4	5
80	8,944	8,950	8,955	8,961	8,967	8,972	8,978	8,983	8,989	8,994	1	1	2	2	3	3	4	4	5
81	9,000	9,006	9,011	9,017	9,022	9,028	9,033	9,039	9,044	9,050	1	1	2	2	3	3	4	4	5
82	9,055	9,061	9,066	9,072	9,077	9,083	9,088	9,094	9,099	9,105	1	1	2	2	3	3	4	4	5
83	9,110	9,116	9,121	9,127	9,132	9,138	9,143	9,149	9,154	9,160	1	1	2	2	3	3	4	4	5
84	9,165	9,171	9,176	9,182	9,187	9,192	9,198	9,203	9,209	9,214	1	1	2	2	3	3	4	4	5
85	9,220	9,225	9,230	9,236	9,241	9,247	9,252	9,257	9,263	9,268	1	1	2	2	3	3	4	4	5
86	9,274	9,279	9,284	9,290	9,295	9,301	9,306	9,311	9,317	9,322	1	1	2	2	3	3	4	4	5
87	9,327	9,333	9,338	9,343	9,349	9,354	9,359	9,365	9,370	9,375	1	1	2	2	3	3	4	4	5
88	9,381	9,386	9,391	9,397	9,402	9,407	9,413	9,418	9,423	9,429	1	1	2	2	3	3	4	4	5
89	9,434	9,439	9,445	9,450	9,455	9,460	9,466	9,471	9,476	9,482	1	1	2	2	3	3	4	4	5
90	9,487	9,492	9,497	9,503	9,508	9,513	9,518	9,524	9,529	9,534	1	1	2	2	3	3	4	4	5
91	9,539	9,545	9,550	9,555	9,560	9,566	9,571	9,576	9,581	9,586	1	1	2	2	3	3	4	4	5
92	9,592	9,597	9,602	9,607	9,613	9,618	9,623	9,628	9,633	9,638	1	1	2	2	3	3	4	4	5
93	9,644	9,649	9,654	9,659	9,664	9,670	9,675	9,680	9,685	9,690	1	1	2	2	3	3	4	4	5
94	9,695	9,701	9,706	9,711	9,716	9,721	9,726	9,731	9,737	9,742	1	1	2	2	3	3	4	4	5
95	9,747	9,752	9,757	9,762	9,767	9,772	9,778	9,783	9,788	9,793	1	1	2	2	3	3	4	4	5
96	9,798	9,803	9,808	9,813	9,818	9,823	9,829	9,834	9,839	9,844	1	1	2	2	3	3	4	4	5
97	9,849	9,854	9,859	9,864	9,869	9,874	9,879	9,884	9,889	9,894	1	1	1	2	3	3	4	4	5
98	9,900	9,905	9,910	9,915	9,920	9,925	9,930	9,935	9,940	9,945	0	1	1	2	2	3	3	4	4
99	9,950	9,955	9,960	9,965	9,970	9,975	9,980	9,985	9,990	9,995	0	1	1	2	2	3	3	4	4

Reciprocals

$x \to \frac{1}{x}$

x	0	1	2	3	4	5	6	7	8	9	SUBTRACT Differences 1	2	3	4	5	6	7	8	9
1,0	1,000	9901	9804	9709	9615	9524	9434	9346	9259	9174	9	18	28	37	46	55	64	74	83
1,1	0,9091	9009	8929	8850	8772	8696	8621	8547	8475	8403	8	15	23	31	38	46	53	61	69
1,2	0,8333	8264	8197	8130	8065	8000	7937	7874	7813	7752	7	13	20	26	33	39	46	52	59
1,3	0,7692	7634	7576	7519	7463	7407	7353	7299	7246	7194	6	11	17	22	28	33	39	44	50
1,4	0,7143	7092	7042	6993	6944	6897	6849	6803	6757	6711	5	10	14	19	24	29	33	38	43
1,5	0,6667	6623	6579	6536	6494	6452	6410	6369	6329	6289	4	8	13	17	21	25	29	33	38
1,6	0,6250	6211	6173	6135	6098	6061	6024	5988	5952	5917	4	7	11	15	18	22	26	29	33
1,7	0,5882	5848	5814	5780	5747	5714	5682	5650	5618	5587	3	7	10	13	16	20	23	26	29
1,8	0,5556	5525	5495	5464	5435	5405	5376	5348	5319	5291	3	6	9	12	15	17	20	23	26
1,9	0,5263	5236	5208	5181	5155	5128	5102	5076	5051	5025	3	5	8	11	13	16	18	21	24
2,0	0,5000	4975	4950	4926	4902	4878	4854	4831	4808	4785	2	5	7	10	12	14	17	19	21
2,1	0,4762	4739	4717	4695	4673	4651	4630	4608	4587	4566	2	4	7	9	11	13	15	17	19
2,2	0,4545	4525	4505	4484	4464	4444	4425	4405	4386	4367	2	4	6	8	10	12	14	16	18
2,3	0,4348	4329	4310	4292	4274	4255	4237	4219	4202	4184	2	4	5	7	9	11	13	14	16
2,4	0,4167	4149	4132	4115	4098	4082	4065	4049	4032	4016	2	3	5	7	8	10	12	13	15
2,5	0,4000	3984	3968	3953	3937	3922	3906	3891	3876	3861	2	3	5	6	8	9	11	12	14
2,6	0,3846	3831	3817	3802	3788	3774	3759	3745	3731	3717	1	3	4	6	7	9	10	11	13
2,7	0,3704	3690	3676	3663	3650	3636	3623	3610	3597	3584	1	3	4	5	7	8	9	11	12
2,8	0,3571	3559	3546	3534	3521	3509	3497	3484	3472	3460	1	2	4	5	6	7	9	10	11
2,9	0,3448	3436	3425	3413	3401	3390	3378	3367	3356	3344	1	2	3	5	6	7	8	9	10
3,0	0,3333	3322	3311	3300	3289	3279	3268	3257	3247	3236	1	2	3	4	5	6	8	9	10
3,1	0,3226	3215	3205	3195	3185	3175	3165	3155	3145	3135	1	2	3	4	5	6	7	8	9
3,2	0,3125	3115	3106	3096	3086	3077	3067	3058	3049	3040	1	2	3	4	5	6	7	8	9
3,3	0,3030	3021	3012	3003	2994	2985	2976	2967	2959	2950	1	2	3	4	4	5	6	7	8
3,4	0,2941	2933	2924	2915	2907	2899	2890	2882	2874	2865	1	2	3	3	4	5	6	7	8
3,5	0,2857	2849	2841	2833	2825	2817	2809	2801	2793	2786	1	2	2	3	4	5	6	6	7
3,6	0,2778	2770	2762	2755	2747	2740	2732	2725	2717	2710	1	2	2	3	4	5	5	6	7
3,7	0,2703	2695	2688	2681	2674	2667	2660	2653	2646	2639	1	1	2	3	4	4	5	6	6
3,8	0,2632	2625	2618	2611	2604	2597	2591	2584	2577	2571	1	1	2	3	3	4	5	5	6
3,9	0,2564	2558	2551	2545	2538	2532	2525	2519	2513	2506	1	1	2	3	3	4	4	5	6
4,0	0,2500	2494	2488	2481	2475	2469	2463	2457	2451	2445	1	1	2	2	3	4	4	5	5
4,1	0,2439	2433	2427	2421	2415	2410	2404	2398	2392	2387	1	1	2	2	3	3	4	5	5
4,2	0,2381	2375	2370	2364	2358	2353	2347	2342	2336	2331	1	1	2	2	3	3	4	4	5
4,3	0,2326	2320	2315	2309	2304	2299	2294	2288	2283	2278	1	1	2	2	3	3	4	4	5
4,4	0,2273	2268	2262	2257	2252	2247	2242	2237	2232	2227	1	1	2	2	3	3	4	4	5
4,5	0,2222	2217	2212	2208	2203	2198	2193	2188	2183	2179	0	1	1	2	2	3	3	4	4
4,6	0,2174	2169	2165	2160	2155	2151	2146	2141	2137	2132	0	1	1	2	2	3	3	4	4
4,7	0,2128	2123	2119	2114	2110	2105	2101	2096	2092	2088	0	1	1	2	2	3	3	4	4
4,8	0,2083	2079	2075	2070	2066	2062	2058	2053	2049	2045	0	1	1	2	2	3	3	3	4
4,9	0,2041	2037	2033	2028	2024	2020	2016	2012	2008	2004	0	1	1	2	2	2	3	3	4
5,0	0,2000	1996	1992	1998	1984	1980	1976	1972	1969	1965	0	1	1	2	2	2	3	3	4
5,1	0,1961	1957	1953	1949	1946	1942	1938	1934	1931	1927	0	1	1	2	2	2	3	3	3
5,2	0,1923	1919	1916	1912	1908	1905	1901	1898	1894	1890	0	1	1	1	2	2	3	3	3
5,3	0,1887	1883	1880	1876	1873	1869	1866	1862	1859	1855	0	1	1	1	2	2	2	3	3
5,4	0,1852	1848	1845	1842	1838	1835	1832	1828	1825	1821	0	1	1	1	2	2	2	3	3

x	0	1	2	3	4	5	6	7	8	9	SUBTRACT Differences 1	2	3	4	5	6	7	8	9
5,5	0,1818	1815	1812	1808	1805	1802	1799	1795	1792	1789	0	1	1	1	2	2	2	3	3
5,6	0,1786	1783	1779	1776	1773	1770	1767	1764	1761	1757	0	1	1	1	2	2	2	3	3
5,7	0,1754	1751	1748	1745	1742	1739	1736	1733	1730	1727	0	1	1	1	1	2	2	2	3
5,8	0,1724	1721	1718	1715	1712	1709	1706	1704	1701	1698	0	1	1	1	1	2	2	2	3
5,9	0,1695	1692	1689	1686	1584	1681	1678	1675	1672	1669	0	1	1	1	1	2	2	2	3
6,0	0,1667	1664	1661	1658	1656	1653	1650	1647	1645	1642	0	1	1	1	1	2	2	2	3
6,1	0,1639	1637	1634	1631	1629	1626	1623	1621	1618	1616	0	1	1	1	1	2	2	2	2
6,2	0,1613	1610	1608	1605	1603	1600	1597	1595	1592	1590	0	1	1	1	1	2	2	2	2
6,3	0,1587	1585	1582	1580	1577	1575	1572	1570	1567	1565	0	0	1	1	1	1	2	2	2
6,4	0,1562	1560	1558	1555	1553	1550	1548	1546	1543	1541	0	0	1	1	1	1	2	2	2
6,5	0,1538	1536	1534	1531	1529	1527	1524	1522	1520	1517	0	0	1	1	1	1	2	2	2
6,6	0,1515	1513	1511	1508	1506	1504	1502	1499	1497	1495	0	0	1	1	1	1	2	2	2
6,7	0,1493	1490	1488	1486	1484	1481	1479	1477	1475	1473	0	0	1	1	1	1	2	2	2
6,8	0,1471	1468	1466	1464	1462	1460	1458	1456	1453	1451	0	0	1	1	1	1	2	2	2
6,9	0,1449	1447	1445	1443	1441	1439	1437	1435	1433	1431	0	0	1	1	1	1	2	2	2
7,0	0,1429	1427	1425	1422	1420	1418	1416	1414	1412	1410	0	0	1	1	1	1	1	2	2
7,1	0,1408	1406	1404	1403	1401	1399	1397	1395	1393	1391	0	0	1	1	1	1	1	2	2
7,2	0,1389	1387	1385	1383	1381	1379	1377	1376	1374	1372	0	0	1	1	1	1	1	2	2
7,3	0,1370	1368	1366	1364	1362	1361	1359	1357	1355	1353	0	0	1	1	1	1	1	2	2
7,4	0,1351	1350	1348	1346	1344	1342	1340	1339	1337	1335	0	0	1	1	1	1	1	1	2
7,5	0,1333	1332	1330	1328	1326	1325	1323	1321	1319	1318	0	0	1	1	1	1	1	1	2
7,6	0,1316	1314	1312	1311	1309	1307	1305	1304	1302	1300	0	0	1	1	1	1	1	1	2
7,7	0,1299	1297	1295	1294	1292	1290	1289	1287	1285	1284	0	0	0	1	1	1	1	1	1
7,8	0,1282	1280	1279	1277	1276	1274	1272	1271	1269	1267	0	0	0	1	1	1	1	1	1
7,9	0,1266	1264	1263	1261	1259	1258	1256	1255	1253	1252	0	0	0	1	1	1	1	1	1
8,0	0,1250	1248	1247	1245	1244	1242	1241	1239	1238	1236	0	0	0	1	1	1	1	1	1
8,1	0,1235	1233	1232	1230	1229	1227	1225	1224	1222	1221	0	0	0	1	1	1	1	1	1
8,2	0,1220	1218	1217	1215	1214	1212	1211	1209	1208	1206	0	0	0	1	1	1	1	1	1
8,3	0,1205	1203	1202	1200	1199	1198	1196	1195	1193	1192	0	0	0	1	1	1	1	1	1
8,4	0,1190	1189	1188	1186	1185	1183	1182	1181	1179	1178	0	0	0	1	1	1	1	1	1
8,5	0,1176	1175	1174	1172	1171	1170	1168	1167	1166	1164	0	0	0	1	1	1	1	1	1
8,6	0,1163	1161	1160	1159	1157	1156	1155	1153	1152	1151	0	0	0	1	1	1	1	1	1
8,7	0,1149	1148	1147	1145	1144	1143	1142	1140	1139	1138	0	0	0	1	1	1	1	1	1
8,8	0,1136	1135	1134	1133	1131	1130	1129	1127	1126	1125	0	0	0	1	1	1	1	1	1
8,9	0,1124	1122	1121	1120	1119	1117	1116	1115	1114	1112	0	0	0	1	1	1	1	1	1
9,0	0,1111	1110	1109	1107	1106	1105	1104	1103	1101	1100	0	0	0	1	1	1	1	1	1
9,1	0,1099	1098	1096	1095	1094	1093	1092	1090	1089	1088	0	0	0	0	1	1	1	1	1
9,2	0,1087	1086	1085	1083	1082	1081	1080	1079	1078	1076	0	0	0	0	1	1	1	1	1
9,3	0,1075	1074	1073	1072	1071	1070	1068	1067	1066	1065	0	0	0	0	1	1	1	1	1
9,4	0,1064	1063	1062	1060	1059	1058	1057	1056	1055	1054	0	0	0	0	1	1	1	1	1
9,5	0,1053	1052	1050	1049	1048	1047	1046	1045	1044	1043	0	0	0	0	1	1	1	1	1
9,6	0,1042	1041	1039	1038	1037	1036	1035	1034	1033	1032	0	0	0	0	1	1	1	1	1
9,7	0,1031	1030	1029	1028	1027	1026	1025	1024	1022	1021	0	0	0	0	1	1	1	1	1
9,8	0,1020	1019	1018	1017	1016	1015	1014	1013	1012	1011	0	0	0	0	1	1	1	1	1
9,9	0,1010	1009	1008	1007	1006	1005	1004	1003	1002	1001	0	0	0	0	0	1	1	1	1

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