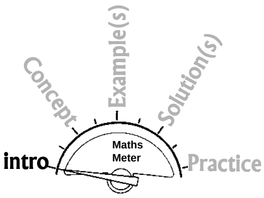
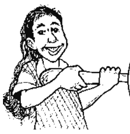

CONTENTS

1 Numbers



This chapter is going to remind students of some basic arithmetic they might have already met. Elementary work in arithmetic often creates problems for 'O' level candidates, hence, the need to revisit some basic concepts in arithmetic.

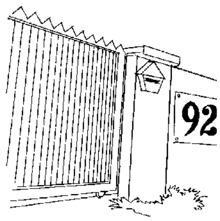
Basically this is a revision chapter of work hopefully covered previously.



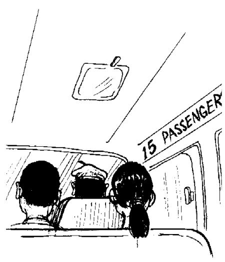
Syllabus Expectations

By the end of this chapter, students should be able to:

1. demonstrate familiarity with the concepts of odd, even, prime, natural, integer, rational and irrational numbers.
2. use the 4 arithmetic operations on these types of numbers.
3. find and use factors, multiples, HCF and LCM.
4. recognise the relationship among common fractions,decimal fractions and percentages and convert these from one form to the other.
5. apply the 4 arithmetic operations (+, –, ×, ÷) and rules of precedence on natural numbers, common fractions, decimal fractions, percentages, integers and directed numbers.
6. state and use place value i.e. write a number, in a certain base, in expanded form and vice versa.
7. convert numbers from one base to another.
8. add and subtract numbers in given bases.



ASSUMED KNOWLEDGE

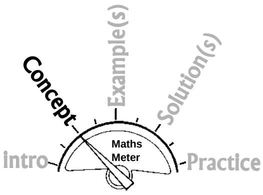


In order to tackle work in this chapter, it is assumed that pupils are able to:

- carry out the four arithmetic operations with ease.

- appreciate place value and different formats of numbers like wholes, simple fractions and decimals.

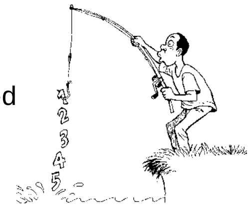
A. REAL NUMBERS



These are numbers which exist and are used in various aspects of life. The opposite of real numbers is *imaginary numbers* e.g. $\sqrt{-4}$. These are not going to be dealt with in this chapter.

Real numbers are thus classified as follows:

Natural Numbers



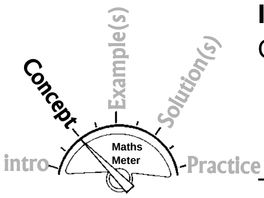
These are numbers which are used in everyday life by society. Thus numbers 1,2,3,4,5,... are natural. They are also called **counting numbers.**

Whole numbers

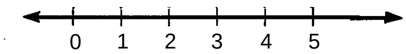
Natural numbers are whole numbers. Zero is also a whole number. Thus 0,1,2,3,4,5 ... are whole numbers.

Note that some whole numbers are even, odd and prime.
Why is 1 not a prime number?

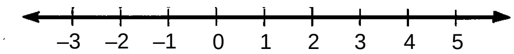
Integers



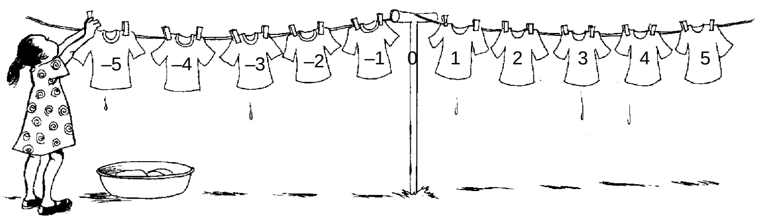
Consider the number line below.

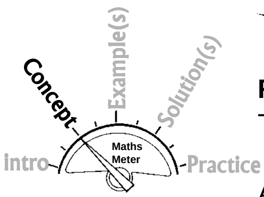


This number line can be extended to the left thus introducing negative numbers i.e.



Note that e.g. –5 is not a whole number but it is a <u>negative</u> whole number. It is the 5 part of –5 which is a whole number.



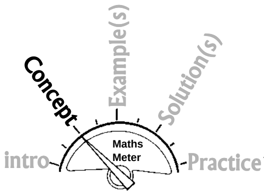


Rational numbers

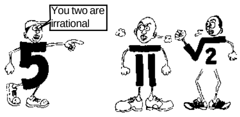
These are defined as $\frac{\text{integers}}{\text{naturals}}$

Any number which can be expressed in this form is a rational number. Do you notice that all the numbers discussed above are rational numbers?

Notice that the definition introduces fractions. Natural, whole numbers and integers are also rational numbers.



Irrational Numbers

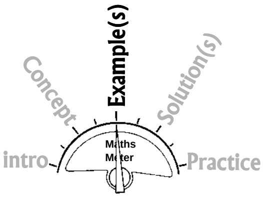


These are numbers which are not exact. Examples are pi (π) = 3,14286 . . . or $\sqrt{2}$ = 1,414 . . . In practical/real terms only approximate values of these can be obtained.

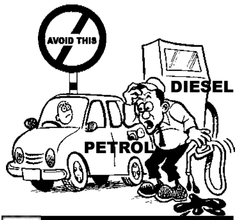
Also note that $\frac{22}{7}$ or $3\frac{1}{7}$ for π are only approximations.

Those irrational numbers whose square roots are not exact like $\sqrt{2}$ are called **surds**. These will be dealt with later.

Consider the following examples:



1. From the list of numbers –1,0,1,5,8,9 give all:
a) whole numbers.
b) integers which are not natural.
c) prime numbers.
d) integers which are not whole numbers.



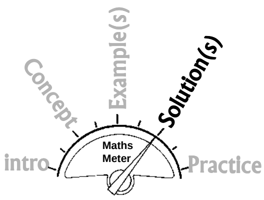
Common Error

0 is usually left out in (a) and (b)
1 and 9 are taken as prime numbers. They are not.

2. From the list –10, $-1\frac{3}{4}$, –3, $-\frac{22}{7}$, 0, 1, 5, $3\frac{1}{3}$, 7, 9
List all:
a) whole numbers.
b) natural numbers.
c) prime numbers.
d) rationals which are not integers.

3. Given natural numbers between 40 and 50, list all:
a) prime numbers.
b) odd numbers which are not prime.
c) even numbers divisible by 6.

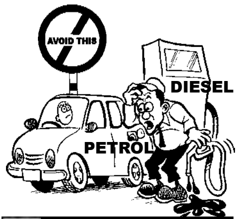
Solutions



1. a) 0,1,5,8,9 b) –1,0
c) 5 d) –1

2. a) Whole numbers = {0; 1; 5; 7; 9}
b) Natural numbers = {1; 5; 7; 9}
c) Prime numbers = {5; 7}
d) Not integers = $\{-1\frac{3}{4}; -\frac{22}{7}; 3\frac{1}{3}\}$

3. a) Prime numbers = {41; 43; 47}
b) Odd and not prime = {45; 49}
c) Even and divisible by 6 = {42; 48}

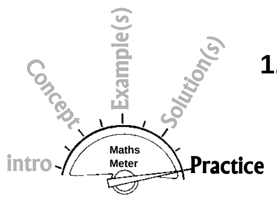


Common Error

2a) –10, –3 included. These are negative whole numbers.

2c) 1 and 9 included

3a) 45,49 included. These are not prime since they have other factors besides 1 and themselves.



1. List the elements of the set of:
a) Natural numbers.
b) Whole numbers.
c) Integers.

2. From the list of numbers 11; $-2\frac{2}{7}$; $\frac{5}{9}$; π ; $\frac{20}{3}$; 0 ; –19; 1; $\sqrt{3}$; 5 list:

a) all prime numbers.
b) whole numbers.
c) rational numbers which are not integers.
d) irrational numbers.
e) surds.
f) integers which are natural numbers.
g) integers which are not whole numbers.
h) whole numbers which are even numbers.

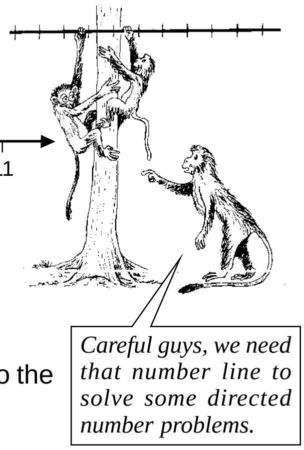
3. Is it true or false that:

a) 2 is not a prime number?
b) 5 is a multiple of 1?
c) 1 is a perfect number?
d) –2 is a whole number?
e) 0 is not a whole number?
f) 14 is a rational number?

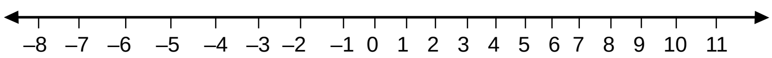
B. DIRECTED NUMBERS

These are numbers with signs before them e.g. –5 read as '*minus 5*' or +6 read as '*plus 6*'. ***Note that*** +6 is usually written as 6.

Consider the example below



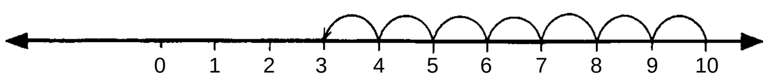
1. Using the line below, simplify the given problems.



a) 10 – 7 b) –5 + 2 c) 7 – 12 + 5

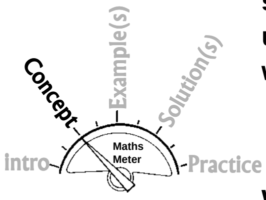
Solution

1 a) Begin at 10 on the number line and move 7 steps to the left (negatively). You get to 3, therefore 10 – 7 = 3. The diagram below illustrates this



b) Begin at –5 on the number line and move 2 steps to the right (positively). You will get to –3 therefore –5 + 2 = –3.

c) From 7, move 12 steps to the left to get to –5 and then from –5, move 5 steps to the right to get to 0.



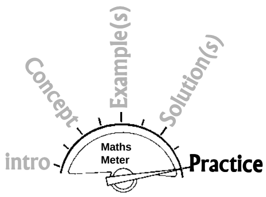
Thus the number line can be used to simplify any directed number problem. These results give us some idea of a general rule i.e. **when two numbers have different signs, ignore the signs and subtract the smaller number from the bigger number then use the sign of the bigger number on the answer. (Compare with a and b).**

a) 12 – 17 = –5 (sign of the bigger number 17)
b) –6 + 12 = +6 (sign of the bigger number 12)

When the signs are the same, ignore the signs and add the numbers and use the common sign on the answer.

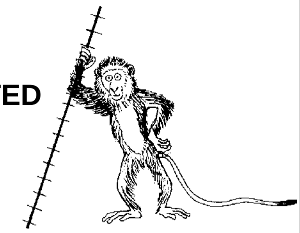
e.g. –5 – 2 = –7 10 + 11 = 21

(*Use the number line to check the answers above*).

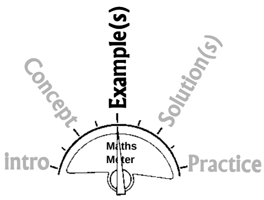


Simplify:

1. –13 –1	**2.** –1 + 13	**3.** –13 + 1
4. 1 – 13	**5.** –121 – 99	**6.** –121 + 99
7. 99 – 121	**8.** –99 + 121	**9.** –15 + 37
10. –15 – 37	**11.** – 6 + 9 – 10	**12.** 7 – 9 – 14
13. –13 + 99 + 13	**14.** 15 – 37 + 20	



C. MULTIPLICATION AND DIVISION WITH DIRECTED NUMBERS

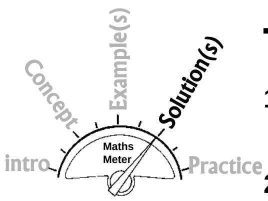


Can you use the number line to multiply?

Try the following examples with the number line given earlier on.

1. 2 × –5 **2.** –2 × –5

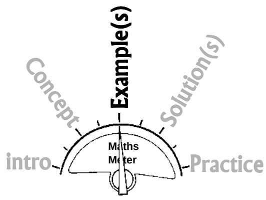
Solutions



1. 2 × –5 means 2 lots of –5 i.e. –5 –5 = –10. Thus from 0 move to the left *two* fives.

2. –2 × –5 means 2 lots of 5 but in which direction? The first minus on 2 says face to the left and the second minus on 5 says change direction to face to the right. So the answer will carry the sign you are facing. Thus –2 × –5 = +10.

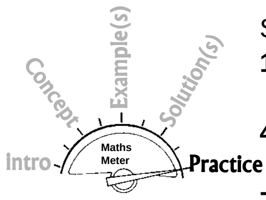
Since division is a way of multiplying, the results in similar circumstances have the same signs.



e.g. $2 \times -5 = -10$ and $-20 \div +4 = -5$] Multiplying and/or dividing numbers with different signs give a **negative** answer.

$-2 \times -5 = +10$ and $-20 \div -4 = +5$] Multiplying and/or dividing numbers with same signs give a **positive** answer.

Thus, there are four rules under directed numbers – two from combining numbers (+,–) and two from multiplying/dividing numbers. Revise these thoroughly and make sure you understand the correct situation to apply them. Make sure you are comfortable with the use of the rules before you continue to the next section.



Simplify:

1. -3×7	**2.** 5×-20	**3.** $-15 \div -3$
4. $-144 \div 12$	**5.** -7×-11	**6.** $154 \div -7$
7. -11×-12	**8.** $-108 \div -4$	**9.** -21×6
10. 5×-13	**11.** $77 \div -11$	**12.** $-308 \div -11$
13. $-3 \times -4 \times 7$	**14.** $-7 \times -5 \times -3$	

D. HCF and LCM

Hint

1 and the number itself are factors.

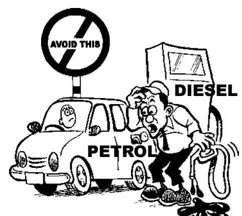
The number itself is the smallest multiple.

Remember factors of 12 (F_{12}) = {1;2;3;4;6;12}

Multiples of 12 (M_{12}) = {12;24;36;48; ...}

When dealing with more than one number, there may be common factors or common multiples. Consider 12 and 18.

Common factors (CF) are 1,2,3,6 therefore the Highest Common Factor (HCF) = 6.
Common multiples (CM) are 36, 72, 108, . . . the Lowest Common Multiple (LCM) = 36.



Common Error

The HCF may be skipped in one list

Consider the following examples

Hint

Make sure you have listed all the factors of each number

1. Finding the HCF of given numbers e.g. 18 and 24.

Method 1
Determine the largest factor common to both sets.
F_{18} = {1;2;3;6;9;18}
F_{24} = {1;2;3;4;6;8;12;24}
$\therefore$ HCF = 6

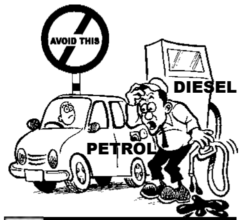
Tip

Express each number as a product of its prime factors. Begin with the lowest prime number (2).

Method 2 or

$$\begin{array}{r|r} 2 & 18 \\ 3 & 9 \\ 3 & 3 \\ & 1 \end{array} \qquad \begin{array}{r|r} 2 & 24 \\ 2 & 12 \\ 2 & 6 \\ 3 & 3 \\ & 1 \end{array}$$

$18 = 2 \times 3 \times 3$
$24 = 2 \times 2 \times 2 \times 3$
$HCF = 2 \times 3$
$= 6$



Common Error

The product is given using numbers which are not prime. Not all CFs are pulled down to the factors of the HCF.

Common Error

To multiply before there are no more CFs stage.

Tip

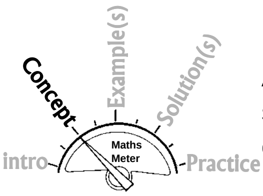
Pull out any common factors between the numbers until there are no more common factors.

Note that 2 and 3 are the common factors in the products.

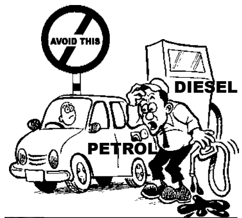
Method 3

$$\begin{array}{r|r|r} 2 & 18 & 24 \\ 3 & 9 & 12 \\ & 3 & 4 \end{array}$$ No more common factors

$\therefore HCF = 2 \times 3$
$= 6$



After studying these 3 methods, one sees that Method 1 appears simple when dealing with fairly small numbers. As the numbers get bigger, methods 2 and 3 become more convenient.
Method 3 gives the factors of the HCF straight away.
Practise applying each of these and choose the most convenient method for you.



Common Error

The LCM may be deep into the list. So candidates give up before getting there.

Common Error

Taking all factors for the LCM i.e. $2 \times 2 \times 2 \times 2 \times 3 \times 3 \times 3$. The answer is no longer the lowest/smallest.

Tip

List the multiples of each number until you meet the first common one. That is the LCM.

2. Find LCM of 18 and 24.

Method 1

$M_{18} = \{18;36;54;72;90;108\}$
$M_{24} = \{24;48;72; ...\}$
$\therefore LCM = 72$

Tip

Prime factorise as in method 2 for finding HCF. Take the biggest list of each factor to make the product of the LCM i.e. $2\times2\times2$ from 24 and 3×3 from 18.

Method 2

$18 = 2 \times 3 \times 3$
$24 = 2 \times 2 \times 2 \times 3$
$LCM = 2 \times 2 \times 2 \times 3 \times 3$
$\therefore LCM = 72$

Method 3

Prime factorise both numbers simultaneously.

$$\begin{array}{r|rr l} 2 & 18 & 24 & \text{both} \div 2 \\ 2 & 9 & 12 & 12 \div 2 \\ 2 & 9 & 6 & 6 \div 2 \\ 3 & 9 & 3 & \text{both} \div 3 \\ 3 & 3 & 1 & 3 \div 3 \\ & 1 & 1 & \end{array}$$

$\therefore LCM = 2 \times 2 \times 2 \times 3 \times 3$
$= 72$

3. Find the HCF of 24, 56 and 72.

Method 1

F_{24} = {1; 2; 3; 4; 6; <u>8</u>; 12; 24}

F_{56} = {1; 2; 4; 7; <u>8</u>; 14; 28; 56}

F_{12} = {1; 2; 3; 4; 6; <u>8</u>; 9; 12; 18; 24; 36; 72}

$\therefore$ HCF = 8

Method 2

24 $= 2 \times 2 \times 2 \times 3$

56 $= 2 \times 2 \times 2 \times 7$

72 $= 2 \times 2 \times 2 \times 3 \times 3$

$\therefore$ HCF $= 2 \times 2 \times 2$

$= 8$

Method 3

2	24	56	72
2	12	28	36
2	6	14	18
	3	7	9

HCF $= 2 \times 2 \times 2$

$= 8$

4. Find the LCM of 18, 60 and 90.

Method 1

M_{18} = {18; 36; 54; 72; 90; 108; 126; 144; 162; 180; ...}

M_{60} = {60; 120; 180;...}

M_{90} = {90; 180; ...}

$\therefore$ LCM = 180

Method 2

18 $= 2 \times 3 \times 3$

60 $= 2 \times 2 \times 3 \times 5$

90 $= 2 \times 3 \times 3 \times 5$

$\therefore$ LCM $= 2 \times 2 \times 3 \times 3 \times 5$

$= 180$

Method 3

2	18	60	90	All ÷ 2
2	9	30	45	Only 30 ÷ 2
3	9	15	45	All ÷ 3
3	3	5	15	3 and 15 ÷ 3
5	1	5	5	5 ÷ 5
	1	1	1	

$\therefore$ LCM $= 2 \times 2 \times 3 \times 3 \times 5$

$= 180$

Method 3 for LCM again seems most convenient. The idea is to end up with a chain of ones at the end of the division process. Thus use the smallest prime factor i.e. 2 until there is no number divisible by it, then move to the next prime number.

Notice that where the factor being used is not applicable, that number is carried down to the next line. Study the method carefully and try it in the next practice questions.

Find (i) HCF (ii) the LCM of:

1. 32 and 40. **2.** 18 and 90.

3. 6, 15 and 18. **4.** 24, 36 and 60.

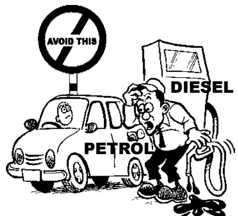
5. 30 and 45. **6.** 18, 24 and 84.

7. 18, 48 and 60. **8.** 78, 108 and 390.

9. 120 and 196. **10.** 84 and 147.

E. FRACTIONS

These fall under rational numbers and can be expressed in different forms. There are common fractions (often called simple fractions), decimal fractions, as well as percentages. Remember percentages are fractions.

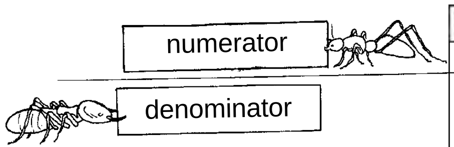


Common Errors

The division line (2) is usually ommitted or candidates think they can use the lines on the writing page as part (2). This is not acceptable.

Parts of a common fraction

Numerator ①
——————— ②
Denominator ③



A common fraction always has these three parts and hence you are expected to show the three parts when you are dealing with fractions.

Reducing fractions to their lowest terms or to their simplest form.

This is a very common requirement in examination questions.
e.g. $\frac{60}{72}$ is a common fraction but not in its lowest terms.
To reduce it to lowest terms we require common factors (or better still the HCF) between the numerator and the denominator.
Use the HCF or just the identified common factors to divide into the numerator and denominator until there are no more common factors.

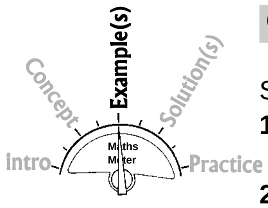
Solutions

e.g. $\frac{60}{72}$ by 6 $= \frac{10}{12}$ by 2 or $\frac{60}{72}$ by 12 $= \frac{5}{6}$

$= \frac{5}{6}$

Addition and subtraction of fractions

Fractions can be added or subtracted.



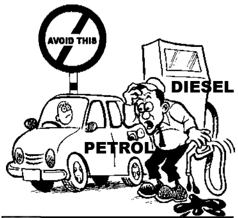
Consider the following examples:

Simplify

1. $\frac{3}{7} + \frac{7}{9}$

2. $\frac{7}{12} - \frac{5}{9}$

3. $\frac{1}{6} - \frac{2}{3} + \frac{7}{12}$



Common Errors

1. $\frac{7-5}{12-9} = \frac{2}{3}$

subtracting the numerator and denominators

2. $\frac{3+7}{7+9} = \frac{10}{16}$

adding the numerators and denominators

or $\frac{\cancel{3}^1}{\cancel{7}_1} + \frac{\cancel{7}^1}{\cancel{9}_3} = \frac{1+1}{1+3}$

$= \frac{2}{4}$

$= \frac{1}{2}$

Multiplying only the denominator by the appropriate factor but not the numerator

Tip

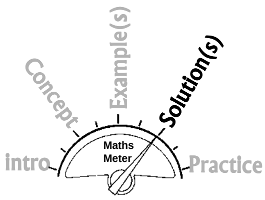
You cannot add or subtract fractions unless they have the same denominator. Think of the common denominator (LCM)

Solutions

1. $\frac{3}{7} + \frac{7}{9} = \frac{27+49}{63}$ or $\frac{27}{63} + \frac{49}{63} = \frac{76}{63}$

$= \frac{76}{63}$ $\qquad$ $= 1\frac{13}{63}$

$= 1\frac{13}{63}$



2. $\frac{7}{12} - \frac{5}{9} = \frac{21-20}{36}$ or $\frac{7}{12} \times \frac{3}{3} - \frac{5}{9} \times \frac{4}{4}$

$= \frac{1}{36}$

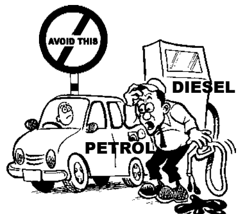
Make the denominators the same by multiplying by one in the form of $\frac{3}{3}$ and $\frac{4}{4}$ $\quad = \frac{21}{36} - \frac{20}{36}$

$= \frac{1}{36}$

3. $\frac{1}{6} - \frac{2}{3} + \frac{7}{12} = \frac{2}{12} - \frac{8}{12} + \frac{7}{12}$ or $\frac{1}{6} - \frac{2}{3} + \frac{7}{12}$

$= \frac{9}{12} - \frac{8}{12}$ $\qquad$ $= \frac{2-8+7}{12}$

$= \frac{1}{12}$ $\qquad$ $= \frac{1}{12}$



Common Errors

In 3

$\frac{2-15}{12}$ from $\frac{2-(8+7)}{12}$

making 7 negative also

$= \frac{-13}{12}$

$= -1\frac{1}{2}$

Remember directed numbers 2 − 8 = −6 then −6 + 7 = 1

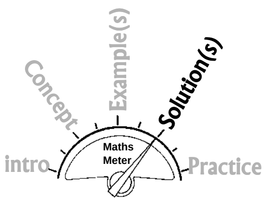
Thus in 1, 2 and 3

a) find the common denominator (LCM).

b) divide each denominator into the common denominator and multiply the result by the numerator.

c) simplify the numerator.

4. $4\frac{4}{5} - 2\frac{2}{3} + 1\frac{1}{2}$



Solution

Method 1

(Deal with whole numbers and fractions separately).

i.e. $3\,\frac{24-20+15}{30}$ $\quad$ Whole number 3 is from 4 – 2 + 1.

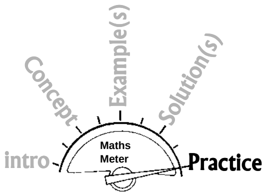
$= 3\frac{19}{30}$

Method 2 (Change to improper fractions first)

$$\frac{24}{5} - \frac{8}{3} + \frac{3}{2} = \frac{144 - 80 + 45}{30}$$

$$= \frac{109}{30}$$

$$= 3\frac{19}{30}$$



1. Reduce the following fractions to their lowest terms.

a) $\frac{5}{65}$ b) $\frac{77}{140}$ c) $\frac{60}{165}$

d) $\frac{39}{78}$ e) $\frac{100}{260}$ f) $\frac{140}{245}$

2. Simplify, giving answers in their lowest terms where necessary.

a) $\frac{3}{8} + \frac{1}{4}$ b) $\frac{3}{8} - \frac{5}{24}$

c) $1\frac{7}{18} + 2$ d) $1\frac{3}{16} - \frac{7}{12}$

e) $\frac{5}{6} + \frac{5}{8} + \frac{7}{16}$ f) $1\frac{1}{3} - \frac{4}{15} + \frac{7}{10}$

g) $\frac{5}{14} + 1\frac{3}{7} - \frac{5}{12}$ h) $3\frac{1}{5} - 4\frac{1}{2} + 2\frac{2}{3}$

i) $3\frac{1}{3} + 1\frac{1}{2} - 2\frac{5}{6}$ j) $2\frac{1}{2} - 1\frac{4}{5} + 1\frac{1}{2}$

Multiplication and division of fractions

Tip

Make sure all mixed numbers are expressed as improper fractions first then simplify by dividing by common factors.

Consider the following examples:

Simplify:

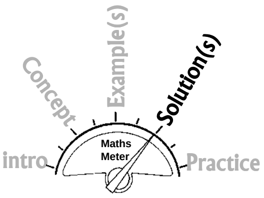
1. $5\frac{1}{4} \times \frac{6}{7}$ **2.** $1\frac{3}{4} \div 1\frac{1}{6}$

3. $2\frac{2}{3}$ of $1\frac{1}{2} \div 4\frac{4}{5}$

Solutions

1. $\frac{\cancel{21}^{3}}{\cancel{4}_{2}} \times \frac{\cancel{6}^{3}}{\cancel{7}_{1}} = \frac{9}{2}$

$$= 4\frac{1}{2}$$



2. Express mixed numbers as improper fractions first then change the division sign to a multiplication sign and invert the divisor.

$\frac{7}{4} \div \frac{7}{6} = \frac{7}{4} \times \frac{6}{7}$

$= \frac{3}{2}$

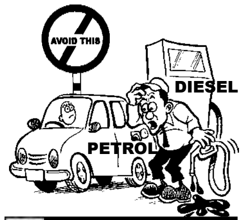
$= 1\frac{1}{2}$

Hint

'Of' means multiply.

3. $2\frac{2}{3}$ of $1\frac{1}{2} \div 4\frac{4}{5} = \frac{8}{3} \times \frac{3}{2} \times \frac{5}{24}$

$= \frac{5}{6}$



Common Errors

1. $\frac{7}{4} \times \frac{7}{6} = \frac{49}{24}$

$= 2\frac{1}{24}$

2. $\frac{4}{7} \times \frac{7}{6} = \frac{2}{3}$

All fractions are inverted.

3. $\frac{8}{3} \times \frac{3}{2} \times \frac{24}{5}$ No!

After a correct stage

$\frac{\cancel{8}^1}{\cancel{3}_1} \times \frac{\cancel{3}^1}{2} \times \frac{5}{\cancel{24}_3} = \frac{7}{6}$

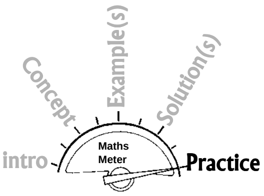
By adding numerators and denominators

1 + 1 + 5 = 7 and

1 + 2 + 3 = 6

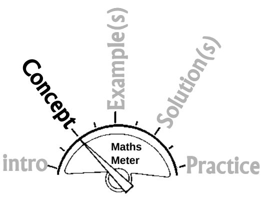
We are multiplying not adding.

Simplify, giving the answers in their simplest form where necessary.



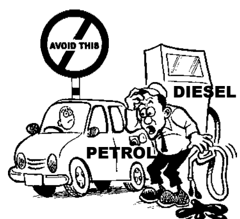
1. $6 \times 1\frac{3}{4}$
2. $1\frac{5}{17} \div 6$
3. $\frac{24}{5}$ of $\frac{4}{9}$
4. $4\frac{1}{2} \div 2\frac{5}{8}$
5. $\frac{27 \times 56 \times 35}{15 \times 63 \times 8}$
6. $\frac{1}{2} \div 4\frac{1}{6} \times \frac{5}{8}$
7. $2\frac{3}{4} \div 4\frac{2}{5} \times 1\frac{1}{3}$
8. $\frac{3}{4}$ of $1\frac{2}{5} \div (\frac{2}{3}$ of $1\frac{2}{5})$
9. $7\frac{1}{2} \times 2\frac{5}{8} \div 5\frac{5}{6}$
10. $3\frac{1}{5} \div (3\frac{3}{7} \div \frac{1}{5}) \times 1\frac{5}{7}$

Mixing Operations



So far we have met problems with mixed addition and subtraction as well as multiplication and division.
Addition and subtraction can be done in any order as long as you keep the correct signs for each number. e.g. 2 – 8 + 9 gives 3 using any of the following combinations: (2–8)+9 or (2+9)–8 or 2+(–8+9)

Division becomes multiplication when the divisor is inverted e.g. $4 \div \frac{3}{2} \times 3 = \frac{4}{1} \times \frac{2}{3} \times \frac{3}{1}$



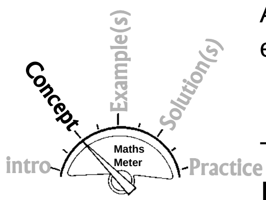
Common Error

$4 \div \frac{3}{2} \times 3 = 4 \times \frac{2}{3} \times \frac{1}{3}$

only $\frac{3}{2}$ is dividing

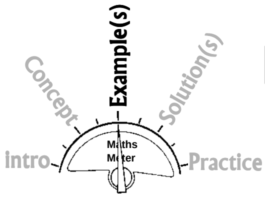
A problem may arise when addition and multiplication are mixed. e.g. $2 \times 3 + 5$. This can be viewed as meaning;

$(2 \times 3) + 5 = 6 + 5 = 11$ or $2 \times (3 + 5) = 2 \times 8 = 16$



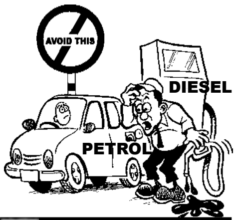
This brings us to the rules of precedence (**BODMAS**).

B	**O**	**D**	**M**	**A**	**S**
(Brackets)	(of)	(Division)	(Multiplication)	(Addition)	(Subtraction)



Consider the following examples:

1. $\dfrac{7+25}{21-5}$ **2.** $\frac{5}{9}\div\left(\frac{2}{3}-\frac{1}{6}\right)+\frac{3}{5}$ **3.** $\frac{3}{5}$ of $(1-2\frac{7}{8})+3$



Common Errors

$$\frac{7+25}{21-5}=\frac{1+5}{3-1}$$
$$=\frac{6}{2}$$
$$=3$$

Solutions

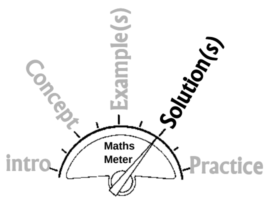
1. This should be viewed as $(7 + 25) \div (21 - 5)$ meaning the numerator and the denominator need to be simplified before dividing.

thus $\dfrac{7+25}{21-5}=\dfrac{32}{16}$

$= 2$

Hint

Each operation must be worked out as presented. Do not re-write the question rearranging the numbers to create the order of precedence.



2. $\frac{5}{9}\div\left(\frac{2}{3}-\frac{1}{6}\right)+\frac{3}{5}$

There are three operations in the problem (use **BODMAS**). Deal with them in their order.

$\frac{5}{9}\div\left(\frac{4-1}{6}\right)+\frac{3}{5} = \frac{5}{9}\div\frac{3}{6}+\frac{3}{5}$ brackets first

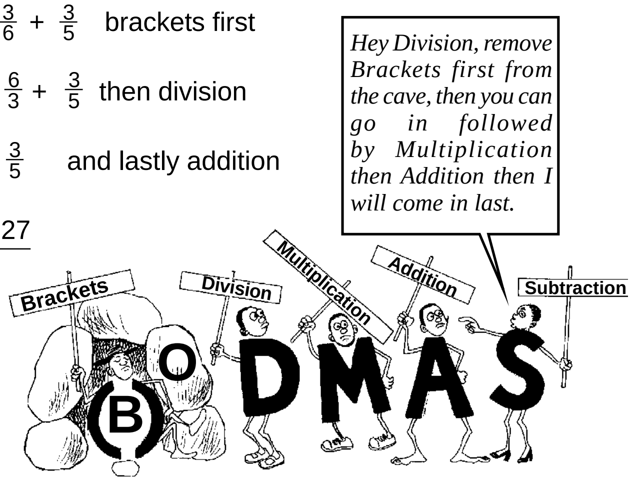
$= \frac{5}{9}\times\frac{6}{3}+\frac{3}{5}$ then division

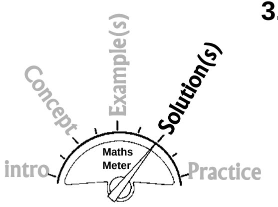
$= \frac{10}{9}+\frac{3}{5}$ and lastly addition

$= \dfrac{50+27}{45}$

$= \dfrac{77}{45}$

$= 1\frac{32}{45}$





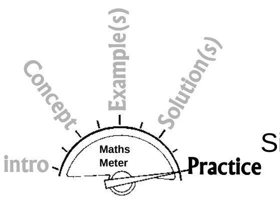
3. $\frac{3}{5}$ of $(1-2\frac{7}{8})+3 = \frac{3}{5}$ of $\left(\frac{8}{8}-\frac{23}{8}\right)+3$ (use **BODMAS**)

$= \frac{3}{\cancel{5}_1}\times\frac{-\cancel{15}^3}{8}+3$

$= \frac{-9}{8}+\frac{24}{8}$

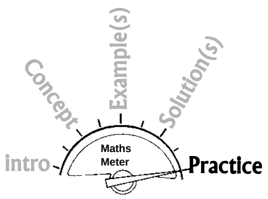
$= \frac{15}{8}$

$= 1\frac{7}{8}$



Simplify, giving the answers in their lowest terms where necessary.

1. $\dfrac{20+5}{20-5}$ **2.** $\frac{1}{5}$ of $35 + 7$



3. $30 \div \frac{6}{24} + 6$

4. $\frac{1}{5}$ of $(35 + 7)$

5. $\frac{2}{9} \div (\frac{2}{9} + \frac{1}{18}) - 1\frac{1}{8}$

6. $\frac{2}{3} \times \frac{6}{7} - \frac{5}{14} \div \frac{10}{21}$

7. $\frac{2}{3} \times (\frac{6}{7} - \frac{5}{14}) \div \frac{10}{21}$

8. $\frac{5}{7} \div \frac{3}{7}$ of $1\frac{5}{6}$

9. $(\frac{5}{7} \div \frac{3}{7})$ of $1\frac{5}{6}$

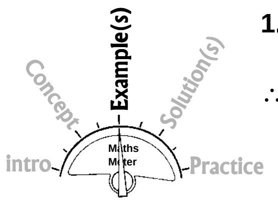
10. $2\frac{1}{3} - (4 \div 2\frac{2}{3}) + (\frac{3}{4} - \frac{2}{5})$ of $2\frac{1}{7}$

F. DECIMALS

Remember

- Simple fractions can be changed to decimals and vice versa

Consider the following examples



1. Change $\frac{3}{5}$ to a decimal fraction.

$\therefore$ $\frac{3}{5} = 0,6$ Divide denominator into numerator

0,6 is a good example of a **terminating decimal**; meaning one which ends.

Change the following decimals to proper or mixed fractions.

2. a) $0,16 = \frac{16}{100}$

$= \frac{4}{25}$

b) $2,25 = 2\frac{25}{100}$

$= 2\frac{1}{4}$

Some decimals are **non-terminating** (do not end)

3. $\frac{2}{7} = 0,2857143 \ldots$

Some decimals are **recurring**

4. $\frac{2}{3} = 0,666 \ldots$ In this case the 6 continues to appear (recurs)

The best way to write this decimal is $0,\dot{6}$.

In some cases more than one digit recurs

$\frac{1}{7} = 0,142857\ 142857\ 142857 \ldots$

$= 0,\dot{1}\dot{4}\dot{2}\dot{8}\dot{5}\dot{7}$, or $0,\dot{1}4285\dot{7}$ the dot/s stand for recurring.

This means $0,6 \neq 0,\dot{6}$

We can add, subtract, multiply and divide decimals in the same manner we can do with common fractions. Consider example 5(a to d)

Hint

Watch out for the layout.

5. a) $6,93 + 69,3$

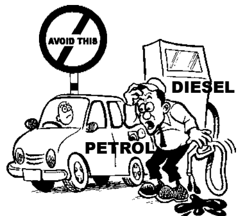
$$\begin{array}{r} 6,93 \\ +\ 69,3 \\ \hline 76,23 \\ \hline \end{array}$$

Maintain the place value of digits by making sure the commas are under each other.

b) $0,76 \times 0,25$

$$\begin{array}{r} 0,76 \\ \times\ 0,25 \\ \hline 1520 \\ 380 \\ \hline 0,1900 \\ \hline \end{array}$$

Multiply as 76×25



Common Errors

$\frac{2}{3} = 0,666 \ldots$

or $= 0,667$

rounding off when not told to.

Common Errors

a) 6,93 Wrong layout
69,3
1,386 Wrong decimal place!

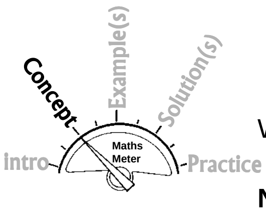
b) Wrong decimal place (Misconception from addition/subtraction)
19,00

Hint

Make the divisor a whole number first.

c) $0{,}18 \div 6 = \dfrac{0{,}18}{6}$
$= 0{,}03$

d) $1{,}8 \div 0{,}06 = \dfrac{180}{6}$
$= 30$



Common Errors

$\dfrac{0{,}18}{6} = 0{,}3$
One is not used first.

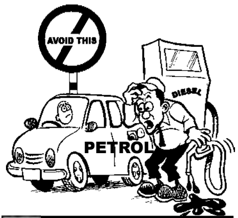
We can change decimals to percentages and vice versa.

NB: Decimals to percentages – simply multiply by 100.
Percentages to decimals – simply divide by 100.

Consider the following examples

Hint

To multiply by 100, simply move the comma 2 times to the right, one place for each zero.
To divide by 100, simply move the comma 2 times to the left.



Common Errors

Changing the decimal fraction to a common fraction then finding the percentage. This is a correct process but is prone to mistakes.

1. Express the following as percentages:

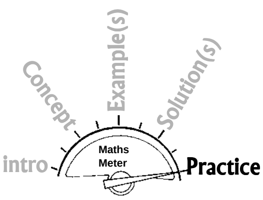
a) $0{,}85 = 0{,}85 \times 100$
$= 85\%$

b) $0{,}308 = 0{,}308 \times 100$
$= 30{,}8\%$

2. Expressing the following as decimals:

a) $0{,}2\% = \dfrac{0{,}2}{100}$
$= \dfrac{2}{1000}$
$= 0{,}002$

b) $65{,}6\% = \dfrac{65{,}6}{100}$
$= \dfrac{656}{1000}$
$= 0{,}656$



1. Simplify:

a) $27{,}64 - 7{,}856$.
b) $69{,}3 + 6{,}93 + 0{,}693$.
c) $2{,}4 \div 0{,}08$.
d) $57{,}3 \times 0{,}01$.
e) $0{,}024 \div 0{,}8$.
f) $(0{,}05)^2$.
g) $0{,}64 \div 8$.

2. Change the following to decimals:

a) $\frac{5}{8}$
b) $1\frac{7}{20}$
c) 3%
d) 2,5%
e) $33\frac{1}{3}\%$
f) $2\frac{4}{9}$

3. Change the following to decimals:

a) 1% b) 0,25% c) 45,7%

d) $\frac{7}{9}$ e) $5\frac{3}{11}$

4. Change the following to percentages:

a) $\frac{2}{3}$ b) $\frac{7}{5}$ c) $4\frac{7}{10}$

d) 0,27 e) 0,027 f) 0,0027

G. SEQUENCES

Many mathematical calculations would be made easier if our perception of number patterns was better. Some patterns are visual or geometric whilst others are numerical or algebraic.

A sequence is a set of numbers arranged in a particular order, for example 2, 4, 6, 8, 10, ... In this example 8 is the 4th term of this sequence. ***Notice that***, if a set of numbers, such as 2, 4, 6, 8, 10 are not given in any particular order such as 8, 6, 4, 2, 10 or 6, 10, 4, 2, 8., the numbers no longer form a sequence.

If u_1 is used to denote the first term of a sequence, u_2 the second and so on. A particular term of a sequence is generally referred to as the nth term and is denoted by u_n.
Note that n is a natural number.

A sequence may be defined by giving a connection between one term and the next. The connection can be given in words or as a formula.

Consider the examples below

1. The first term of a sequence is 2 and the next $(U_{n+1}) = 2u_n - 1$. Write down the next three terms of this sequence.

Solution

$u_1 = 2$
$u_2 = 2 \times 2 - 1 = 3$
$u_3 = 2 \times 3 - 1 = 5$
$u_4 = 2 \times 5 - 1 = 9$

The next three terms are 3, 5 and 9.

2. The first term of a sequence is 1 and the next term (u_{n+1}) is given by 2n + 1. Write down the next 3 terms of the sequence.

Solution

2. $u_1 = 1$
$u_2 = 2 \times 1 + 1 = 3$
$u_3 = 2 \times 3 + 1 = 7$
$u_4 = 2 \times 7 + 1 = 15$

Another way to define a sequence is to relate a term to its position in the sequence. i.e. to n, and give a formula for the n^{th} term. Suppose the nth term is 2n + 1, then the 3^{rd}, 4^{th} and 5^{th} term can be found by substituting n by 3, 4 and 5 respectively giving

3^{rd} term = 2(3) + 1 = 7
4^{th} term = 2(4) + 1 = 9
5^{th} term = 2(5) + 1 = 11

3. The nth term of a sequence, u_n, is given by the formula $u_n = 2^n + 1$. Find the third and fifth terms.

Solution

$u_n = 2^n + 1$
3rd term = $2^3 + 1 = 9$
5th term = $2^5 + 1 = 33$

In each question 1 to 5, the first two terms of a sequence are given and the rule for finding other terms are given. Write down the next three terms.

1. –2, 4, ... multiply by –2
2. 1, 3, ... add 2
3. –8, 4, ... divide by –2
4. 2, 4, ... add the previous two terms
5. 1, 0, ... multiply by 2 and subtract 2.

In each question 6 to 10, give the next two terms in each sequence and state in words the rule you used to find them.

6. 25, 22, 19, 16, ...
7. 1, 5, 9, 13, ...
8. 48, 24, 12, 6, ...
9. 2, 6, 18, 54, ...
10. 1, 3, 7, 13, ...

In questions 11 to 15, one of the terms of a sequence and the formula for finding the n^{th} term are given.

a) Write down the first four terms.

11. n^{th} term = 2(1+ n); 5^{th} term = 12
12. n^{th} term = 2^{n+1} ; 6^{th} term = 128
13. n^{th} term = $n^2 – n$; 7^{th} term = 42
14. n^{th} term = $(n-2)^2$; 5^{th} term = 9
15. n^{th} term = $2 \times 3^{n-1}$; 6^{th} term = 486

In questions 16 to 20, the nth term of the sequence is denoted by u_n and the first term is given. Give the next four terms.

16. $u_1 = 3$, $u_{n+1} = 2u_n$
17. $u_1 = 2$, $u_{n+1} = u_n + 3$
18. $u_1 = -2$, $u_{n+1} = u_n - 2$
19. $u_1 = -1$, $u_{n+1} = 2u_n - 1$
20. $u_1 = -3$, $u_{n+1} = -3u_n$

H. APPROXIMATIONS

Sometimes it is possible to give exact answers to questions. However, certain answers cannot be or need not be and are approximations.

Rounding off

Some calculations or measurements are required to a desirable degree of approximation and are rounded off.

Numbers can be rounded off to

- the nearest unit e.g. whole number, ten, thousand etc.
- the nearest fraction of a unit e.g. tenth, hundredth. etc.
- the desired number of decimal places e.g. 2d.p. etc.
- the desired number of significant figures e.g. 3s.f. etc.

If the digit following the one to be rounded off is less than 5, the digit remains unchanged. If the following digit is 5 or more the number is increased by one. Zeros are used to hold place values. Study the number below and its various possible approximations.

726, 4987 = 726 (to the nearest whole number)
4 is used to judge whether the 6 remains unchanged or is increased by one.

726, 4987 = 730 (to the nearest ten)
6 is the number used to round off the 2.

726, 4987 = 726,50 (to the nearest hundredth)
8 is the guide
or (to 2 d.p)
or (to 5 s.f)

Remember decimal places come after the comma. Thus 726,4987 has 4 decimal places.

726, 4987 = 700 (to 1s.f)

Common Errors

726,4987 = 7 (1s.f.)

Note that this number has 7 significant figures counting from the first 7 to the last 7. The first significant figure is 7, and is not changed by the 2 in rounding off.
The answer is 700, the zeros having been included to maintain the value of the number. Also in 0,07073, four significant figures exist. 7, 0, 7 and 3 i.e. start counting from the first non-zero digit. Thus

0,07073 = 0,07 (1s.f)
0,071 (2s.f)
0,0707 (3s.f)

Estimation

To estimate is to make an intelligent guess.
To make estimations easier, numbers in question are rounded off to 1 significant figure.

Consider the examples below:

Estimate **1.** $9{,}235 \times 18{,}767 \times 30{,}779$
$= 9 \times 20 \times 30$ (Round off to 1 s.f and multiply)
$= 5400$

2. $\dfrac{85{,}3 \times 9{,}7}{19{,}6} = \dfrac{90 \times 10}{20}$ (Round off to 1 s.f and multiply)

$= \dfrac{900}{20}$

$= 45$

3. $\sqrt{398} = \sqrt{400}$ (to 1 s.f)

$= 20$

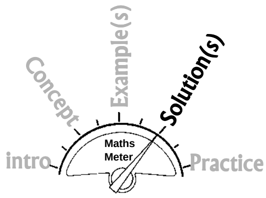
Limits of Accuracy

When a number is rounded to the nearest whole number, it means the rounded answer has a range of values which round off to it. e.g. Given that 7 is a result of rounding a number to the nearest whole number. This then means the number can be 6,5, 6,6, 6,7, ...7,4 (to one decimal place) the lower limit = 6,5
the higher limit = 7,4
Notice lower or upper bounds is also applicable.

Consider the following example:

1. A rectangle measures 6cm by 4cm, the sides having been measured to the nearest centimetre.
Find: a) the smallest possible length of the rectangle.
b) the biggest possible perimeter of the rectangle
c) the difference between the biggest and the smallest possible areas.

Solution



1. a) 5,5cm

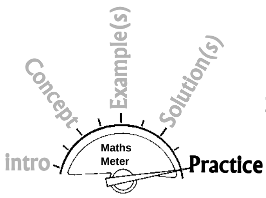
b) Biggest possible measurements are 6,4 and by 4,4cm

$\therefore$ Biggest possible perimeter $= 2(6,4 + 4,4)$
$= 2 \times 10,8$
$= 21,6\text{cm}$

c) Biggest possible area $= 6,4 \times 4,4)\text{cm}^2$
$= 28,16\text{cm}^2$

Smallest possible area $= (5,5 \times 3,5)\text{cm}^2$
$= 19,25\text{cm}^2$

$\therefore$ Difference $= (28,16 - 19,2)\text{cm}^2$
$= 8,91\text{cm}^2$



1. Estimate the values of the following by rounding off to 1 significant figure first.

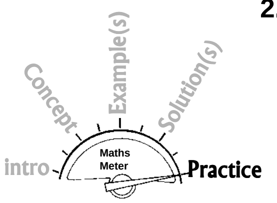
a) $7,3 + 9,7$ b) $9,3 \times 7,67$

c) $192 + 306$ d) $98 - 37$

e) $16,8 \div 0,98$ f) $\dfrac{38,8 \times 2,76}{18,7}$

g) $\sqrt{3,7}$ h) $\sqrt{411}$

i) $\sqrt{863}$ j) $\sqrt{96,9}$



2. The measurements of a rectangle, measured to the nearest cm, are $8\text{cm} \times 5\text{cm}$.

a) Find the smallest possible length.

b) Find the largest possible width.

c) Find the largest possible area of the rectangle.

3. The side of a square is given as 10cm, measured to the nearest centimetre. Find:

a) the smallest possible perimeter of the square.

b) the largest possible area of the square.

4. The sides of a triangle are given as 6cm, 9cm and 11cm, measured to the nearest cm. Find the largest possible perimeter of the triangle.

I. NUMBER BASES

Whilst our number system uses base 10 i.e. digits 0,1,2,3,4,5,6,7,8 and 9 other systems use different bases. Just think of time. Time uses base 60 and we cannot say 2hrs 70min. This is 3hrs 10min if we apply the base on the 70min.

This section introduces you to operations with bases 2,3,4,5,6,7,8,9 and 10.

Numbers in base ten or denary system

A number in base ten or **denary system** can be expanded using place value. For an example 5207 means 5 thousands + 2 hundreds + 0 tens + 7 units i.e. $5 \times 10^3 + 2 \times 10^2 + 0 \times 10^1 + 7 \times 10^0$.

Notice that we are using the powers of the base to expand this number. This fits very well with the elementary ideas of place value.

Hint

Remember any number to the power zero is equal to 1 e.g. $3^0 = 1$.

Th	H	T	U
5	2	0	7
10^3	10^2	10^1	10^0

As we move into other bases, this basic principle still holds. If we are in, say, base 3 we expand the number using powers of 3. But before we start expanding and doing other things let us study the digits used for each base.

Remember base 10 uses digits 0 to 9. It then follows that base 9 uses digits 0 to 8, base 8 uses digits 0 to 7 and so forth. Give the digits for base 2.

In fact, **the biggest digit in any given base is one less than the base**.

Notation in Number bases

1058_9 is a number in base 9.

Notice that the base is indicated as a small number below the last digit of the number. If the base is not indicated, this implies base 10. Normally base 10 is not indicated.

What is wrong with 5364_5 as a number? 6 is not a digit under base 5.

The bases can be written in words as well e.g. 5364_{seven}, 5364_{nine}, 5364_{ten}.

How is 20324_6 pronounced?

It is not twenty thousand - - -.

It is two, zero, three, two, four base six.

A very common form of examination question on bases is as follows:

'Re-write 3021_4 in powers of 4.

This means $3021_4 = 3 \times 4^3 + 0 \times 4^2 + 2 \times 4^1 + 1 \times 4^0$

The question wants the number in 'powers of 4' as shown above.

Common Errors

$3021_4 = 3 \times 4^3 + 0 \times 4^2 + 2 \times 4 + 1$

Powers don't exist on the last two terms.

Another form of the question asks the student to give the number being shown in the expansion.
e.g. $2 \times 5^4 + 3 \times 5^2 + 2 \times 5 + 4$
This is 20324_5
Notice in this answer, 0 must be inserted for the missing 5^3 and the base in use has to be shown. Notice also that the powers in the third and fourth parts have been deliberately left out of the question. The 5 of 2×5 has the power 1 and $4 = 4 \times 5^0$.

1. Expand the following using powers of the given base

a) $10\,748_{10}$ b) $70\,016_8$ c) 302_5

d) $1\,011_2$ e) $2\,022_3$ f) $1\,100_2$

g) $71\,064_8$ h) $3\,450_6$ I) $3\,405_7$

j) $11\,011_2$

2. Write down the number being shown in the expansion

a) $1 \times 2^3 + 1 \times 2$

b) $4 \times 6^4 + 4 \times 6^3 + 3 \times 6^2 + 2$

c) $3 \times 5^4 + 4 \times 5^2 + 2 \times 5 + 3$

d) $6 \times 8^3 + 4 \times 8^2 + 5$

e) $4 \times 7^5 + 3 \times 7^4 + 5 \times 7^2 + 2 \times 7$

f) $1 \times 2^4 + 1 \times 2^3 + 1$

g) $3 \times 4^3 + 1 \times 4^2 + 3$

h) $7 \times 9^5 + 4 \times 9^3 + 6 \times 9^2 + 8$

i) $1 \times 2^5 + 1 \times 2^3 + 1$ j) $2 \times 4^4 + 1 \times 4^3 + 3 \times 4$

Converting from one base to another

Numbers are very flexible. Remember, a fraction can be changed from a common fraction to a decimal fraction even to a percentage. It is the same with numbers in different bases. These numbers can be changed from one base to another.
Let us begin by changing an ordinary number (in base 10) to another base.

From base 10 to another base

Convert **1.** 37 to base 3 **2.** 5764 to base 7

Solutions

1.

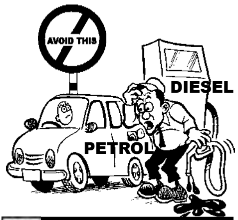
3	37
3	12 r 1
3	4 r 0
3	1 r 1
	0 r 1

$\therefore$ $37 = 1101_3$

2.

7	5764
7	823 r 3
7	117 r 4
7	16 r 5
7	2 r 2
	0 r 2

$\therefore$ $5\,764 = 22\,543_7$



Common Errors

Remainders are copied from the top to bottom such that 37=1011 and 5764=34522. This is wrong.

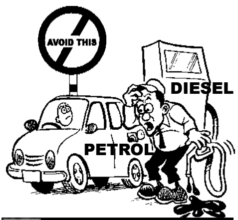
It is as simple as that! Continue dividing by the base you want to go to until you get zero. The remainders, **written from the bottom to the top,** form the answer.

From any base to base 10

Convert

3. $1\,011_2$ to base 10 **4.** 432_6 to base 10

Solutions



Common Errors

Mixing up methods i.e Method in question 1 applied in question 2.

Method 1

3. $2^3\ 2^2\ 2^1\ 2^0$

$\ \ 1\ \ 0\ \ 1\ \ 1$

$= 1\times2^3 + 0\times2^2 + 1\times2^1 + 1\times2^0$

$= 8 + 0 + 2 + 1$

$= 11$

4. $6^2\ 6^1\ 6^0$

$\ 4\ \ 3\ \ 2$

$= 4\times6^2 + 3\times6^1 + 2\times6^0$

$= 144 + 18 + 2$

$= 164$

Method 2

3. 1 0 1 1

$\underline{\times2}$

$2 + 0 = 2$

$\underline{\times2}$

$4 + 1 = 5$

$\underline{\times2}$

$10 + 1 = 11$

4. 4 3 2

$\underline{\times6}$

$24 + 3 = 27$

$\underline{\times\ 6}$

$162 + 2 = 164$

Method 1 is based on place value. It is a matter of expanding the number in powers of the given base then simplifying.

Method 2 is slightly more tricky. Here we are multiplying by the base, and adding on each digit before multiplying again, until the last digit is added.

It should be noted that questions 1 and 2 are very common in examination papers. These are very useful techniques to use if confronted with tricky problems. Consider the situation below.

Converting to a given base

Convert the following to the given base

5. 10011_2 to base 5 **6.** 432_5 to base 7

Since we are able to change from a given base to base 10 and from base 10 to any base, the above question can be tackled via base 10.

Solution

5. from base 2 to base 10

$2^4 \ 2^3 \ 2^2 \ 2^1 \ 2^0$

1 0 0 1 1

$= 16 + 0 + 0 + 2 + 1$

$= 19$

Then from base 10 to base 5

$$\begin{array}{r|ll} 5 & 19 & \\ 5 & 3 & r\ 4 \\ & 0 & r\ 3 \end{array}$$

$\therefore\ 10\ 011_2 = 34_5$

6. from base 5 to base 10

$5^2 \ 5^1 \ 5^0$

4 3 2

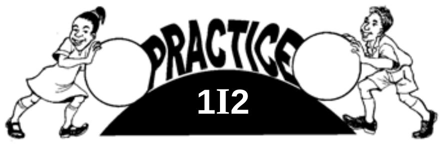
$= 100 + 15 + 2$

$= 117$

Then from base 10 to base 7

$$\begin{array}{r|ll} 7 & 117 & \\ 7 & 16 & r\ 5 \\ 7 & 2 & r\ 2 \\ & 0 & r\ 2 \end{array}$$

$\therefore\ 432_5 = 225_7$



1. Convert the following denary numbers (base 10) into the given base.

a) 102 to base 8 b) 18 to base 4

c) 77 to base 7 d) 39 to base 3

e) 147 to base 6 f) 83 to base 9

g) 230 to base 5 h) 111 to base 2

i) 111 to base 4 j) 147 to base 7

2. Convert the following numbers into denary (base 10)

a) 10011_2 b) 132_5

c) 132_7 d) 103_6

e) 333_4 f) 1011_3

g) 1011_9 h) 246_8

i) 1025_6 j) 102102_3

3. Convert the following numbers to the given base

a) 124_5 to base 2   b) 124_5 to base 6

c) 432_6 to base 8   d) 356_9 to base 3

e) 1213_4 to base 5   f) 10111_2 to base 4

g) 87_9 to base 7   h) 1202_3 to base 2

i) 5_6 to base 5   j) 20015_8 to base 9

4. Which of these numbers is even?
101_8 ; 101_3 ; 101_2 ; 101_5

5. Arrange the following numbers in order of size starting with the smallest.
57_8 ; 101_{10} ; 111_7 ; 211_3

Addition and subtraction

Have you ever thought of how addition and subtraction is done in base 10? Carefully study what is happening here.

a)
$$\begin{array}{r} 7\,5\,0\,9 \\ +\;8\;\;7\,9\,2 \\ \hline 16\;\;3\,0\,1 \\ \hline \end{array}$$

Do you notice that we are working in base 10? In addition, do you notice that the digits of the answer are remainders after using the base to divide each sum? We started with 9 + 2 = 11. We are saying how many tens (base) are there in 11? There is one ten with a remainder of 1. That is why we automatically say 'put down 1 and carry 1'. The one we put down is the remainder and the one we carry is the number of tens!

b)
$$\begin{array}{r} 8\;{}^{1}5\;\;9\;{}^{1}2 \\ -\;{}^{8}7\;\;7\;0^{1}\;9 \\ \hline 8\;\;8\;3 \\ \hline \end{array}$$

Look at the subtraction. Since 2 is smaller than 9, we give a ten (base) to the 2 to make it 12 and then subtract the 9 to get the 3. The method above is that of 'giving diagonally' i.e. adding the base to the top line whilst adding 1 to the bottom line.
Conversely you can borrow the base from the digit to the immediate left and not do anything to the bottom line.

i.e.
$$\begin{array}{r} \cancel{8}^{7}\;{}^{1}\cancel{5}\;\;{}^{8}\cancel{9}\;{}^{1}2 \\ -\;7\;\;7\;\;0\;\;9 \\ \hline 8\;\;8\;3 \\ \hline \end{array}$$

Consider these processes using other bases

1. $6\,786_9 + 2\,570_9$ **2.** $2\,403_6 - 542_6$

3. $1\,031_5 - 444_5$

Solutions

1.
$$\begin{array}{r} 6\ 7\ 8\ 6_9 \\ +\ 2\ 5\ 7\ 0_9 \\ \hline 10\ 4\ 6\ 6_9 \\ \hline \end{array}$$

In (1) 6 + 0 = 6 and the 6 was put down as it is because it is less than the base 9. Or one can view it as 9 into 6 = 0 remainder 6, so put down the 6 and carry nothing.

8 + 7=15, then 9 into 15 = 1 r 6. So put down 6 and carry 1.
7 + 5 + 1 =13, then 9 into 13 =1 r 4. So put down 4 and carry 1.
6 + 2 + 1 = 9, then 9 into 9 =1 r 0. So put down 0 and carry 1.
That is how the answer 10 466_9 was arrived at!

2.
$$\begin{array}{r} 2\ {}^{10}\cancel{4}\ {}^{6}\cancel{0}\ 3_6 \\ -\ {}^{1}\ \ {}^{6}\cancel{5}\ 4\ 2_6 \\ \hline 1\ \ 4\ \ 2\ 1_6 \\ \hline \end{array}$$

3 – 2 =1. Put down the 1.
Give the base on 0 to make it 6 and give 1 to 5 in the bottom line to make it 6 and then 6 – 4 = 2.
Give the base on 4 to make it 10 and give 1 to below 2 and then 10 – 6 = 4 and 2 – 1 = 1 to get 1421_6 as the answer.

3.
$$\begin{array}{r} 1\ {}^{5}\cancel{0}\ {}^{8}\cancel{3}\ {}^{6}\cancel{1}_5 \\ -\ 1\ \ \cancel{4}^{5}\ \cancel{4}^{5}\ \ 4_5 \\ \hline 3\ \ 2_5 \\ \hline \end{array}$$

The addition and subtraction process can have numbers in different bases.

Consider the following examples:

1. Simplify $1212_3 + 24_5$ giving the answer in base 3.

i.e. $5^1\ 5^0$
$2\ \ 4$
$= 10 + 4^5$
$= 14_{10}$

$$\begin{array}{r|l} 3 & 14 \\ 3 & 4\ \ r\ 2 \\ 3 & 1\ \ r\ 1 \\ & 0\ \ r\ 1 \end{array}$$

$\therefore\ 24_5 = 112_3$

Hint

Change 24_5 to base 3 first.

Common Errors

$$\begin{array}{r} 1\,212 \\ +24 \\ \hline 2\,020_3 \\ \hline \end{array}$$

The numbers are in different bases!

Thus
$$\begin{array}{r} 1\,212_3 \\ +\ \ 112_3 \\ \hline 2\,101_3 \\ \hline \end{array}$$

Hint

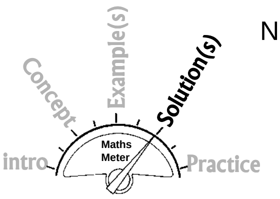
Change both numbers to base 10. Subtract then change the result to base 7.

2. Evaluate $45_8 - 1101_2$ giving the answer in base 7

i.e.

$$\begin{array}{ll} 8^1\ 8^0 & 2^3\ 2^2\ 2^1\ 2^0 \\ 4\quad 5 & 1\ \ 1\ \ 0\ \ 1 \\ = 32 + 5 & = 8 + 4 + 1 \\ = 37 & =13 \end{array}$$

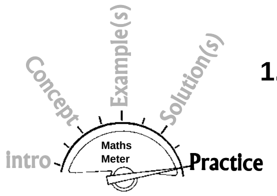
$\therefore\quad 37 - 13 = 24$



Now to base 7

$$\begin{array}{r|ll} 7 & 24 & \\ 7 & 3 & r\ 3 \\ & 0 & r\ 3 \end{array}$$

$\therefore\ 45_8 - 1101_2 = 33_7$



1. Simplify the following:

a) $11\,011_2 + 1\,011_2$ b) $312_5 - 134_5$

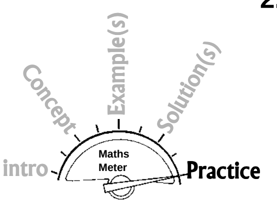
c) $5\,652_8 - 4\,776_8$ d) $3\,204_6 + 524_6$

e) $4\,015_7 + 3\,604_7$ f) $1\,012_3 - 221_3$

g) $11\,010_2 - 1\,011_2$ h) $2\,032_4 + 3\,313_4$

i) $71\,247_9 - 60\,358_9$ j) $71\,247_9 + 60\,358_9$

2. Evaluate the following giving the answers in the given base.



a) $333_{10} + 33_5$ in base 5 b) $57_8 + 57_9$ in base 8

c) $2222_3 - 222_4$ in base 5 d) $432_5 - 123_3$ in base 4

e) $345_{10} + 345_8$ in base 8 f) $345_{10} + 345_8$ in base 6

g) $321_4 + 12_3$ in base 3 h) $356_7 - 101_2$ in base 2

Multiplication and division in other bases (optional for enrichment)

Let us consider multiplication in base 10

1. $4\,758 \times 3 = 14\,274$

We are saying $3 \times 8 = 24$, put down 4 and carry 2. I hope you now appreciate where the 4 and 2 are coming from. The idea is to put down the remainder and carry the quotient.

2. $4\,758_9 \times 3_9 = 15\,486_9$

i.e. $8 \times 3 = 24$

$24 \div 9 = 2 \quad r\,6$

$5 \times 3 = 15$

$15 + 2 = 17$

$17 \div 9 = 1 \quad r\,8$

$7 \times 3 = 21$

$21 + 1 = 22$

$22 \div 9 = 2 \quad r\,4$

$4 \times 3 = 12$

$12 + 2 = 14$

$14 \div 9 = 1 \quad r\,5$

Answer is from here

3. $436_8 \times 315_8$

$$
\begin{array}{r}
4\,3\,6_8 \\
\times \quad 3\,1\,5_8 \\
\hline
1\,5\,3\,2\,0\,0 \\
4\,3\,6\,0 \\
+ \quad 2\,6\,2\,6 \\
\hline
1\,6\,2\,4\,0\,6_8 \\
\hline
\end{array}
$$

Please study these three lines in relation to the process described in 2. Make sure you understand how each digit is arrived at before the addition.
The addition is done as before.

How about division?

Remember in 2 above, $4\,758_9 \times 3_9 = 15\,486_9$

This means $15\,486_9 \div 3_9 = 4\,758_9$

and $15\,486_9 \div 4\,758_9 = 3_9$

How is this possible? Let us consider the first one

4.

$$
\begin{array}{r}
4\,758 \\
3\,\overline{)15\,486_9} \\
\underline{-13} \\
24 \\
\underline{-23} \\
18 \\
\underline{-16} \\
26 \\
\underline{26}
\end{array}
$$

Multiplication table for 3_9

3 x 1 = 3
3 x 2 = 6
3 x 3 = 10
3 x 4 = 13
3 x 5 = 16
3 x 6 = 20
3 x 7 = 23
3 x 8 = 26

Make sure you appreciate the multiplication table for 3, before going through the division process.

The multiplication table for 4758_9 is:

$4758 \times 1 = 4758$
$4758 \times 2 = 10627$
$4758 \times 3 = 15486$
4758×4 etc

This means to carry out the division correctly, multiplication tables for the divisor need to be established first. With practice one will be able to make an intelligent guess as to how many times the divisor will go into the part of the dividend under consideration.

Hint

The multiplication table must be correct for correct solutions to be found

Consider another example

5. $162406_8 \div 315_8$

```
        436
315 |162406
    -1464
      1400
     -1147
       2316
       2316
```

Multiplication table for 315_8

$315 \times 1 = 315$
$315 \times 2 = 632$
$315 \times 3 = 1147$
$315 \times 4 = 1464$
$315 \times 5 = 2001$
$315 \times 6 = 2316$
$315 \times 7 = 2633$

Evaluate the following giving the answers in the given base.

1. $1\,011_2 \times 11_2$

2. $32\,032_4 \div 122_4$

3. $2\,301_4 \times 22_4$

4. $24\,405_7 \div 34_7$

5. $435_6 \times 314_6$

6. $10\,001\,111_2 \div 1\,011_2$

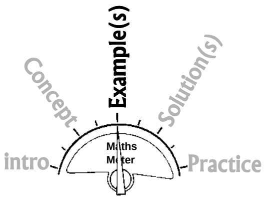
7. $5\,607_8 \times 246_8$

8. $221\,001_3 \div 222_3$

9. $4\,214_5 \times 343_5$

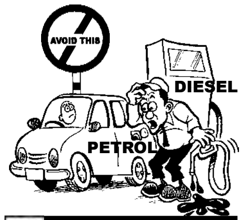
10. $4\,525_9 \div 48_9$

EXAM PRACTICE 1



Consider the examples below

1. a) Express 214_6 as a number in base 10.
b) Evaluate $23_5 - 32_5 + 33_5$ in base five.
c) Evaluate $73_8 + 1101_2$ in base eight.
d) Express $3^4 + 3^3 + 3$ as a number in base 3.
e) Given that $2n2$ is a number in base 5 and that $2n2_5 = 133_6$, find n.



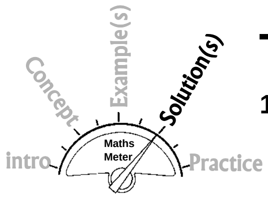
Common Error

$$\begin{array}{r|l} 6 & 214 \\ 6 & 35 \text{ r } 4 \\ 6 & 5 \text{ r } 5 \\ & 0 \text{ r } 5 \end{array}$$

$= \underline{554}$

(i) 23 – 120 = –42 by saying 23 – (32 + 33)
Note that the bracket makes 33 negative when it is not!

Solutions



1. a)

$$\begin{array}{l} 2\ 1\ 4 \\ \quad\ \times 6 \\ \quad 12 + 1 = 13 \\ \qquad\qquad \times 6 \\ \qquad\quad 78 + 4 = \underline{82} \end{array}$$

or

$$\begin{array}{l} 6^2\ \ 6^1\ \ 6^0 \\ 2\quad 1\quad 4 \\ = 72 + 6 + 4 \\ = 82 \end{array}$$

Hint
Follow directed number rules.

b)

$$\begin{array}{r} 23_5 \\ +\ 33_5 \\ \hline 111_5 \\ -\ 32_5 \\ \hline 24_5 \\ \hline \end{array}$$

or $23_5 - 32_5 = -4_5$
then $-4_5 + 33_5 = 24_5$

or $23_5 + 1_5$
$= 24_5$
$(-32_5 + 33_5 = 1_5)$
$= 1_5$

Hint
Change 1101_2 to base 8 first.

c) $2^3 2^2 2^1 2^0$
1 1 0 1
= 8 + 4 + 1 = 13

$$\begin{array}{r|l} 8 & 13 \\ 8 & 1 \text{ r } 5 \\ & \text{or} \quad 1 \end{array} \uparrow$$

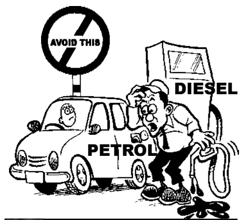
$$\therefore \quad \begin{array}{r} 73_8 \\ 15_8 \\ \hline 110_8 \\ \hline \end{array}$$

Hint
Remember place value!

d) $3^4 + 3^3 + 3 = 11\ 010_3$

Hint
Change both to base 10

e) $2n2_5 = 133_6$
$50 + 5n + 2 = 36 + 18 + 3$
$5n = 57 - 52$
$5n = 5$
$n = 1$



Common Error

Answer as 333 or 111
The 111 is derived either by disregarding the other two powers 3^2 and 3^0 or by simplifying $3^4 + 3^3 + 3 = 81 + 27 + 3$
*****
n = 3 by associating the respective digits without realising that the other digits are not the same also.

Hint
Make the divisor a whole number mathematically (i.e) $\frac{0{,}064}{0{,}8} = \frac{0{,}64}{8}$
Make sure you know how!

2. Giving the answer as a decimal fraction, find the exact value of
a) 0,064 ÷ 0,8.
b) $(0{,}05)^2 = 0{,}0025$ Remember this means $0{,}05 \times 0{,}05$

3. Giving the answer as a fraction in its lowers terms, find the value of
a) $\frac{1}{3} \div (\frac{1}{4} + \frac{1}{6})$

b) $(3\frac{1}{8} - 2\frac{1}{3}) \div (4\frac{1}{2} + 1\frac{5}{6})$

Remember BODMAS

3. a) $\frac{1}{3} \div (\frac{1}{4} + \frac{1}{6})$

$= \frac{1}{3} \div (\frac{3+2}{12})$

$= \frac{1}{3} \div \frac{5}{12}$

$= \frac{1}{3} \times \frac{12}{5}$

$= \frac{4}{5}$

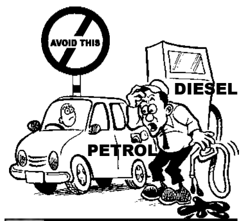
b) $(3\frac{1}{8} - 2\frac{1}{3}) \div (4\frac{1}{2} + 1\frac{5}{6})$

$= 1\frac{-5}{24} \div 5\frac{8}{6}$

$= \frac{19}{24} \div \frac{38}{6}$

$= \frac{\cancel{19}^{1}}{\cancel{24}_{4}} \times \frac{\cancel{6}^{1}}{\cancel{38}_{2}}$

$= \frac{1}{8}$



Common Errors

$(\frac{1}{4} + \frac{1}{6}) \div \frac{1}{3}$

i.e wanting to do brackets first. Remember the brackets have to be worked first whilst in their given correct position.

Now do the following:

1. Evaluate, giving your answer as a decimal.

a) 0,68 + 2,378 b) 0,03 × 0,7

c) 6,7 – 9,3 d) 0,046 – 0,273

e) 0,0408 ÷ 0,04

2. Giving your answers as a common fraction in its lowest terms find the value of:

a) $\frac{4+9}{8 \times 3}$ b) $\frac{5}{7}$ of $4\frac{1}{5}$ c) $3\frac{2}{5} \div 1\frac{7}{10} \times \frac{2}{5}$

3. Express:

a) $\frac{14}{25}$ as a percentage.

b) $\frac{7}{15}$ as a decimal.

c) 0,041 as a percentage.

d) 0,05% as a common fraction in its lowest terms.

4. a) Convert 3110_4 to a number in base 2.

b) Evaluate $432_5 - 143_5$ giving the answer in base ten.

5. a) Write down the number 3506_7 in expanded form.

b) Write down 1023_4 in powers of 4.

6. a) Evaluate $1202_3 + 63_8$ giving the answer in base 3.

b) Given that $24_n = 1110_2$ find the value of n.

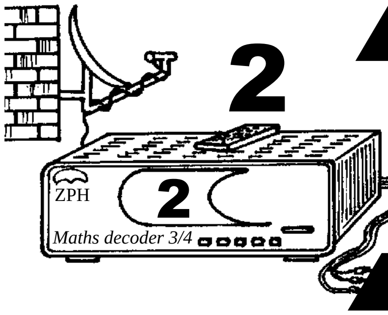
7. a) Evaluate the following, giving your answer in base 10.

$1_2 + 2_3 + 3_4 + 4_5 + 5_6 + 6_7$

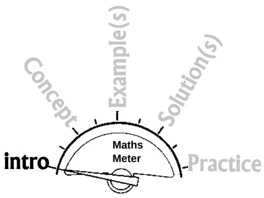
b) If $123_x = 38_{10}$. Find the base x.

8. a) Calculate $21_3 \times 21_3$ giving the answer in base 3.

b) Calculate $12133_5 \div 32_5$.



2 Algebraic Expressions and Change of subject of formulae



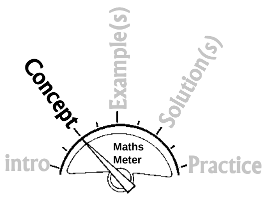
Algebra is a branch of mathematics which deals with letters as well as real numbers. A letter in this case is used to represent a number. An algebraic expression is a combination of numbers and letters using the four arithmetic processes, addition (+), subtraction (–), multiplication (×) and division (÷). Examples of such expressions are; $2y$ (number × letter)

$x + y$ (letter + letter)

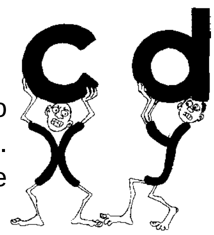
$\frac{3}{a}$ (number ÷ letter)

$3 - xy^2 + \frac{5}{y}$ $\left(\text{Number} - \text{letter} \times \text{letter}^2 + \frac{\text{number}}{\text{letter}}\right)$

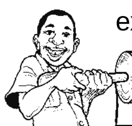
Expressions are made up of **terms**. A term is a combination of numbers and/or letters using multiplication and division. Terms are separated from one another by addition and/or subtraction signs. From the above examples of expressions, $2y$ is a term.



In $x + y$ the expression is made up of 2 terms $+x$ and $+y$.
In $\frac{3}{a}$ the expression has only one term.
In $3 - y^2 + \frac{5}{y}$ the expression is made up of 3 terms; 3, $-y^2$ and $+\frac{5}{y}$.
Other examples of single terms are $\frac{3x + y}{2}$ and $\frac{4x^2 - 3}{2x + y}$.



Notice the term always goes with its sign. The sign of a term is to its immediate left. Terms can either be like terms or unlike terms. The terms $2a$ and $-5a$ are like terms as they are in terms of the same letter (a).
$7r$, $-2y^2$ and $7p$ are all unlike terms because of the different letters.
In this chapter we are interested in manipulating linear and quadratic expressions in various ways.



Syllabus Expectations

By the end of this chapter, students should be able to:

1. manipulate basic arithmetic processes in letter symbols.
2. use the four arithmetic operations and rules of precedence to manipulate monomials and simple algebraic fractions.
3. substitute words and letters for numbers in algebraic expressions (including formulae).

4 find and use common factors, multiples, HCF's and LCM's of algebraic expressions.
expand expressions of the form $a(x + y)$
$(ax + by)(cx + d)$, $(ax + by)(cx + dy)$ etc where a, b, c, and d are rational numbers.

5 factorise expressions of the form:
$ax + bx + ay + by$,
$ka^2 - kb^2$,
$ax^2 + bx^2 + c$ where a, b, c and k are integers.

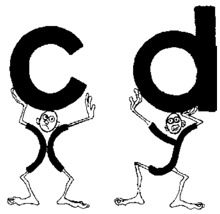
6 change the subject of a given formula, and substitute terms into a formula including those from other subjects (e.g science).

ASSUMED KNOWLEDGE

In order to tackle work in this chapter, it is assumed that students are able to:

- appreciate the meaning of $2x, ab, x^n$ (where n is an integer).
- manipulate directed numbers and/or expressions.
- manipulate fractions using the four arithmetic operations.
- apply the concept of BODMAS (rules of precedence).

A. LIKE AND UNLIKE TERMS



Remember ***like terms combine*** and ***unlike terms do not combine***.

Consider the following examples:

Simplify **1.** $5y + 3x + y - 7x + 3$
2. $7 - (x - 2y) + 3(5x - 2)$
3. $5x - (3 - x + 3y) - 7(5 - y)$

Hint
Identify like terms and combine them.

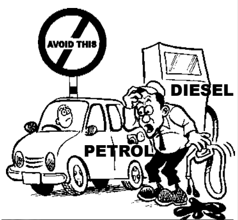
Solution

1. $5y + 3x + y - 7x + 3$
i.e $5y + y = 6y$
$3x - 7x = -4x$
$+ 3$ is on its own $= + 3$.
$\therefore = 6y - 4x + 3$ (No more like terms hence this is the final answer)

Hint
Remove brackets first.

2. $7 - (x - 2y) + 3(5x - 2)$
$= 7 - x + 2y + 15x - 6$ (Identify and combine like terms).
$= 1 + 14x + 2y$

3. $5x - 3 + x - 3y - 35 + 7y = 6x + 4y - 38$
The terms in the final answer can be given in any order.



Common Error
1) To remove the bracket in $7-(x-2y)$, the 7 is used ie $-7x + 14y$.
2) $7 - x - 2y$
3) Removing $+ 3(5x - 2) = 15x - 2$ is a common wrong practice. The 3 must be multiplied with all terms in the bracket.

PRACTICE 2A

Simplify

1. $3x + 5 + 3 + 2x$ **2.** $6 - 2x - 4 - 3x$

3. $x^2 + 5x - 7x - 2$ **4.** $8xy - 12xy + 10xy - 5xy$

5. $v^3 - 3v^2 + v^3 + 5v - 4v^2 - 8v$

6. $8m^2 + 24 + 10n - 15 - 12m^2 - 7n$

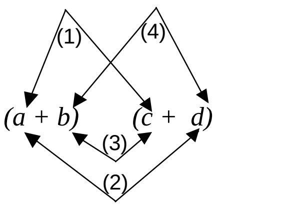
7. $7 - 3(4 - x)$ **8.** $3(4 - x) - 2(1 + 3x)$

9. $5y(y + 6) - (20y - y^2)$ **10.** $4a(b + a) - 2a(b - a)$

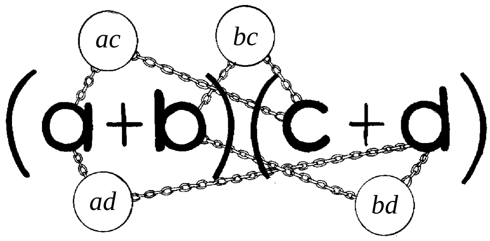
11. $a^2(a + 1) - a^2(a - 1)$ **12.** $x^2(5 - 2x) - x(7 - 3x^2)$

B. MULTIPLYING A BRACKET BY A BRACKET

The general rule here is to multiply each term in the second bracket by each term in the first bracket.
Below is an illustration of the usual steps to follow.



$= ac + ad + bc + bd$



Other important forms to remember at this level are:

$\mathbf{(a + b)^2 = a^2 + 2ab + b^2}$

$\mathbf{(a - b)^2 = a^2 - 2ab + b^2}$

$\mathbf{(a - b)(a + b) = a^2 - b^2}$

Do not forget to combine like terms if they are available.

Consider the following examples:

Expand and simplify:

1. $(x + 3)(x - 5)$ **2.** $(2y + 3)^2$

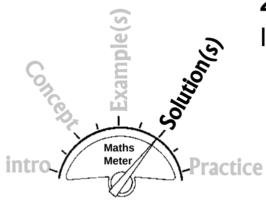
3. $(3x - 2)(3x + 2)$ **4.** $(4 - x)(7 - 5x)$

Solution

1. By splitting one bracket or following the method indicated by arrows as above

$x(x - 5) + 3(x - 5)$
$= x^2 - 5x + 3x - 15$
$= x^2 - 2x - 15$

$(x + 3)(x - 5)$
$= x^2 - 5x + 3x - 15$
$= x^2 - 2x - 15$



2. **Method 1** $(2y + 3)^2$

In short, (i) square the first term $(2y)^2 = 4y^2$

(ii) multiply the 2 terms and double the result.

$2y \times 3 \times 2 = 12y$

(iii) square the last term $(+3)^2 = +9$

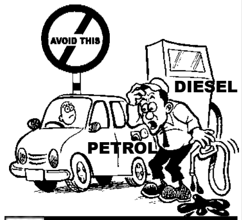
ie $4y^2 + 12y + 9$

Method 2

$(2y + 3)^2 = (2y + 3)(2y + 3)$ multiply as directed by arrows

$= 4y^2 + 6y + 6y + 9$

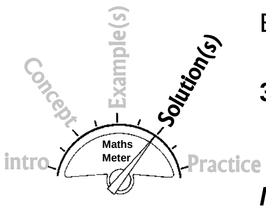
$= 4y^2 + 12y + 9$



Common Error

$(2y + 3)^2 = 4y^2 + 9$
ie Simply squaring the terms inside the bracket. This is wrong.

Expressions of the nature $(2y + 3)^2$ are **perfect square expressions.**



3. $(3x + 2)(3x – 2)$
$= 9x^2 – 6x + 6x – 4$
$= 9x^2 – 4$

or in short $(3x + 2)(3x – 2)$
$= (3x)^2 – (2)^2$
$= 9x^2 – 4$

Notice that, the brackets contain the same expressions only differing in signs!

Other examples are $(a + 2)(a – 2)$ and $(1 – 2a)(1 + 2a)$

Such expressions are called **difference of two squares**.

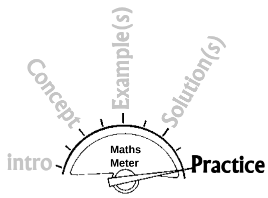
Hint

Notice that the letter in the brackets does not always have to be written first. Thus the instruction "expand" simply means remove brackets and combine like terms where they exist.

4. $(4 – x)(7 – 5x) = 28 – 20x – 7x + 5x^2$
$= 28 – 27x + 5x^2$

Expand

1. $(2x + 1)(x + 3)$	**2.** $(5x – 2)(x + 5)$
3. $(x – 1)(2x – 5)$	**4.** $(2x – 5)^2$
5. $(3x – 1)(2x – 5)$	**6.** $(7x + 2)(x – 6)$
7. $(5 – x)(5 + x)$	**8.** $(6 – 5y)^2$
9. $(3 – 2x)(1 + 6x)$	**10.** $(2x + 5y)(2x – 5y)$
11. $(2x – 7y)(3x – 11y)$	**12.** $(8x – 5y)^2$



C. FACTORS OF ALGEBRAIC EXPRESSIONS.

To factorise is to introduce brackets and state an expression as a product of its factors.

Consider the following examples:

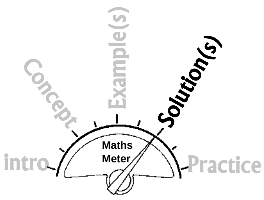
Hint

Check for common factors first.

1. **Given two terms:**

Factorise a) $5y^2 – 10y$ b) $a^2 – 16$ c) $5a^2 – 45b^2$

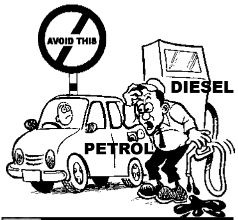
Solution



1. a) $5y^2 - 10y$
$5y(y - 2)$

5 and y are common. So pull out $5y$. Do you ***notice that*** the terms in the bracket are a result of dividing the terms by the common factor?

i.e $\dfrac{5y^2}{5y} = y$ $\quad \dfrac{-10y}{5y} = -2$



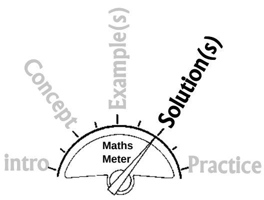
Common Error

The common factors are omitted in the final answer especially, the example (c) type of question. That 5 in (c) must be carried down to the answer.

b) $a^2 - 16$
i.e $= a^2 - 4^2$
$= (a + 4)(a - 4)$

There is no common factor, this is a difference of two squares. What is being squared is added in one bracket and that being subtracted in the other bracket.

Given 2 terms to factorise,
(i) check for common factors,
(ii) if no common factors exist, check for a difference of two squares.



c) $5a^2 - 45b^2$
$= 5(a^2 - 9b^2)$
$= 5(a - 3b)(a + 3b)$

There is a common factor 5. ***Notice*** the bracket needs to be factorised further since it is a difference of two squares.

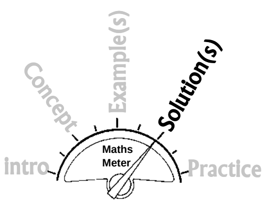
Hint

Group the terms into twos and factorise to create a common factor of the two terms so created.

2. **Given four terms:**

Factorise completely
a) $3p - 3q + pq - q^2$
b) $x^2 + xy - xy - y^2$
c) $ax + by - ay - bx$

Solution



2. a) $3p - 3q + pq - q^2$
$= 3(p - q) + q(p - q)$
$= (p - q)(3 + q)$

Common factor for the first 2 terms is 3 and for the next 2 terms is q. ***Do you notice*** the expression is now of 2 terms $3(p - q)$ and $+q(p - q)$? The common factor $(p - q)$ is now pulled out to give the answer.

The final stage is only possible when the brackets are the same in the earlier stage.

b) $x^2 + xy - xy - y^2 = x(x + y) - y(x + y)$
$= (x + y)(x - y)$

Common Error

In (b)
$x(x + y) - y(x - y)$
The second bracket is incorrect.

Hint

If the first 2 and second 2 terms do not have common factors re-arrange them.

c) $ax + by - ay - bx$

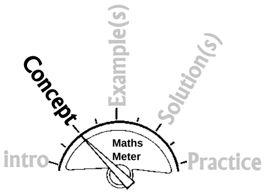
The first 2 terms have no common factor. There is need to re-arrange terms so that common factors are created.

i.e $ax - ay + by - bx$ or $ax - bx - ay + by$

Thus from $ax - ay + by - bx$

$= a(x - y) - b(-y + x)$ $\quad$ $(x - y)$ and $(-y + x)$

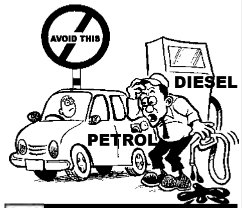
$= (x - y)(a - b)$ $\quad$ are the same.



or from $ax - bx - ay + by$

$= x(a - b) - y(a - b)$

$= (a - b)(x - y)$



Common Error

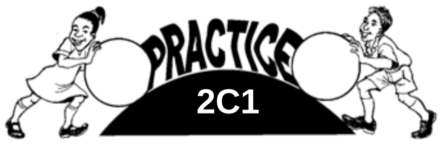
In (c) signs are mixed up during the re-arrangement. Remember a term has a sign to its immediate left. So it is shifted together with its sign. e.g rearranging $ax + by - ay - bx$ to $ax + bx - by + ay$ is wrong.

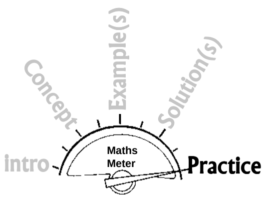
In general, **when given 4 terms** the following steps may be followed:

(i) The first two terms should have a common factor and the next two terms should have a common factor as well.

(ii) The brackets so created should contain the same expression. $x - y \neq y - x$. but $= x - y = -y + x$.

(iii) Re-arrange terms to create common factors, if necessary. Be careful to maintain original signs when rearranging.

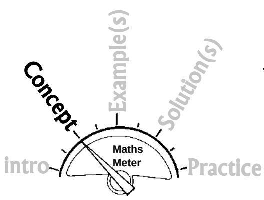




Factorise completely

1. $7x^3y^2 + 5x^2y^2$	**2.** $36 - 6x^2$
3. $x^2 - y^2$	**4.** $4 - x^2y^2$
5. $3x^2y + 3y$	**6.** $5x^2 - 45$
7. $45xy^2 - 9x^2y$	**8.** $x^2 - 1$
9. $2x^3 + 16xy$	**10.** $2x^2 - 72y^2$
11. $2 - 8y^2$	**12.** $2ax - 6a + cx - 3c$
13. $ax + bx - by - ay$	**14.** $2a^2 - 3h - 3a + 2ah$.
15. $ac - 6bd - 2bc + 3ad$	**16.** $vp - 12qu - 4up + 3qv$

The ability to identify the difference of two squares is very useful in evaluating cases like $135^2 - 35^2$. ***Note that*** squaring 135 without a calculator is not easy at all. Fortunately the problem is of a 'difference of two squares' type.



Thus $135^2 - 35^2 = (135 + 35)(135 - 35)$

$= 170 \times 100$

$= 17\,000$

The whole process becomes very simple once the difference of two squares is identified.

PRACTICE 2C2

Simplify without using a calculator or tables.

1. $67^2 - 33^2$ **2.** $100^2 - 99^2$

3. $(5{,}6)^2 - (3{,}6)^2$ **4.** $(5{,}9)^2 - (4{,}1)^2$

5. $48^2 - 52^2$ **6.** $575^2 - 425^2$

7. $1\ 000^2 - 999^2$ **8.** $469^2 - 431^2$

9. $(5{,}762)^2 - (4{,}238)^2$ **10.** $(12{,}87)^2 - 17{,}13)^2$

D. FACTORS OF QUADRATIC EXPRESSIONS

A quadratic expression has variables with a power not greater than two. Differences of two squares are quadratic expressions as well. The general format of a quadratic expression is $ax^2 + bx + c$, where a, b, and c are rational numbers.
Below is an illustration of the parts of a quadratic expression.

ax^2	$+ bx$	$+ c$
First term	Middle Term	Constant or Absolute Term

Note that a and b are coefficients of x^2 and x respectively while c is a constant.
Also ***note that*** the first term is so called because it is the term with the highest power. It doesn't have to be written first though.

e.g $5 - x - 2x^2$ $-2x^2$ is still the first term

Consider the following examples:

Factorise **1.** $x^2 + 6x + 5$ **2.** $x^2 + x - 6$

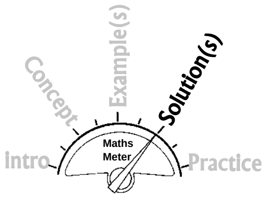
3. $x^2 - 2x - 15$ **4.** $x^2 - 22x + 72$

Solution

1. $x^2 + 6x + 5$

Method 1: Expand the middle term
Since the coefficient of x^2 is 1, expand $6x$, the middle term using factors of 5, the constant i.e $x^2 + 5x + x + 5$ or $x^2 + x + 5x + 5$ and factorise the four-term expression so created.

$x(x + 5) + 1\,(x + 5)$ or $x(x + 1) + 5(x + 1)$
$= (x + 5)\,(x + 1)$ or $(x + 1)\,(x + 5)$



Method 2: Trial and error

$x^2 + 6x + 5 = (x \quad)(x \quad)$

Step 1. Break x^2 into x and x and put these in each bracket to enable them to be multiplied to give x^2.

Step 2. Consider the factors of the constant (+5) which when added will give the coefficient of the middle term i.e $5 \times 1 = 5$, $+5 + +1 = +6$ and put these factors into the brackets as shown below.

$x^2 + 6x + 5$

$= (x + 5)(x + 1)$ or $(x + 1)(x + 5)$

2. $x^2 + x - 6$

Method 1: $= x^2 + 3x - 2x - 6$

$= x(x + 3) - 2(x + 3)$

$= (x + 3)(x - 2)$

Method 2: $= x^2 + x - 6$

$= (x + 3)(x - 2)$

Hint

Constant is –6 not 6 ∴ Factors of –6 which add up to 1 are –2 and 3.

3. $x^2 - 2x - 15 = (x - 5)(x + 3)$ using either of the two methods.

4. $x^2 - 22x + 72 = (x - 18)(x - 4)$

The factors of +72 required should both be negative so that they add up to –22.

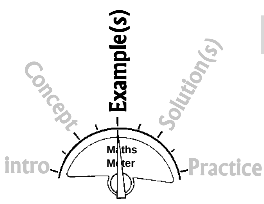
Factorise completely

Hint

A positive product is produced by multiplication of same sign numbers i.e.
$- \times -$ or
$+ \times +$.

1.	$x^2 + 7x + 10$	**2.**	$x^2 + 6x + 9$
3.	$x^2 + 4x - 12$	**4.**	$x^2 + 3x - 4$
5.	$x^2 - 3x - 4$	**6.**	$x^2 - 3x - 28$
7.	$x^2 - 11x + 28$	**8.**	$x^2 - 10x + 21$
9.	$x^2 - x - 30$	**10.**	$x^2 - 10x + 25$

What if the coefficient of x^2 is other than 1?



Consider the following examples:

Factorise **1.** $2x^2 + 7x + 3$ **2.** $8x^2 - 2x - 1$

3. $8 - x - 9x^2$ **4.** $3x^2 - 5xy - 2y^2$

Solution

1. $2x^2 + 7x + 3 = (2x + 1)(x + 3)$

As for **Method 1: Expand the middle term**

Step 1. Multiply coefficient of x^2 with the constant i.e
$^+2 \times {}^+3 = {}^+6$

Step 2. Expand middle term ^+7x using factors of the $^+6$.
i.e $2x^2 + 6x + x + 3$ and factorise
$= 2x(x + 3) + 1(x + 3)$
$= (x + 3)(2x + 1)$

Method 2: Trial and error

Here the coefficient of x^2 is a prime number. Simply put it into one of the brackets.

Step 1. Split $2x^2$ into $2x$ and x i.e $(2x \quad)(x \quad)$

Step 2. Consider numbers which multiply to give 3. i.e $^+1$ and $^+3$ or -1 and -3. ***Note that*** the middle term is positive so factors $^+1$ and $^+3$ are applicable here.

Step 3. Use trial and error method to place the $^+1$ and $^+3$ into the proper bracket i.e either $(2x+1)(x+3)$ or $(2x+3)(x+1)$
Thus $2x^2 + 7x + 3 = (2x + 1)(x + 3)$

2. $8x^2 - 2x - 1$. Here the coefficient of x^2 is not a prime number.

Method 1

Step 1. $8 \times -1 = -8$ and look for factors of -8 which add to -2.
i.e -4 and $+2$

Step 2. $8x^2 - 4x + 2x - 1$
$= 4x(2x - 1) + 1(2x - 1)$
$= (2x - 1)(4x + 1)$

Method 2: $8x^2 - 2x - 1$

either it is $(8x \quad)(x \quad)$ or $(4x \quad)(2x \quad)$,
by trial and error with factors of -1 i.e. $+1$ and -1,
one will arrive at $(4x + 1)(2x - 1)$.
This method requires a lot of practice to master.
Method 1 is more practical.

3. $8 - x - 9x^2$
Using method (1) $8 \times -9 = -72$ and $-9x + 8x = -x$.
$\therefore \; 8 + 8x - 9x - 9x^2$
$= 8(1 + x) - 9x(1 + x)$
$= (1 + x)(8 - 9x)$

4. $3x^2 - 5xy - 2y^2$

Step 1. Ignore the second letter (y) and factorise as $3x^2 - 5x - 2$
$= (3x + 1)(x - 2)$.

Step 2. Put the letter (y) back into the brackets to produce the answer.
$(3x + y)(x - 2y)$.

Hint

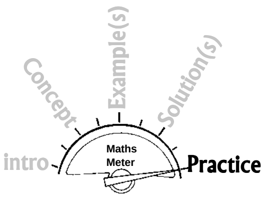
Do you see that Method (1) seems more friendly to beginners than Method 2? Make a wise choice of methods. Practice makes perfect.

OR Use Method (1)

$3x^2 - 6xy + xy - 2y^2$
$= 3x(x - 2y) + y(x - 2y)$
$= (x - 2y)(3x + y)$
$= (x - 2y)(3x + y)$

Step 1. Ignore the second letter y and expand $3x^2 - 5x - 2$ to $3x^2 - 6x + x - 2$

Step 2. Put back the letter y to get $3x^2 - 6xy + xy - 2y^2$ and factorise.



Factorise fully

1. $2x^2 + 9x + 4$

2. $3x^2 + x - 2$

3. $7x^2 - 23x + 6$

4. $4x^2 - 7x - 2$

5. $4x^2 + 12xy + 9y^2$

6. $6x^2 + xy - 5y^2$

7. $6 + 5x - 4x^2$

8. $3 - 2x - 8x^2$

9. $9x^2 - 30x + 25$

10. $8x^2 - 9x - 14$

11. $7x^2 - 5x - 2$

12. $5x^2 + 16x + 3$

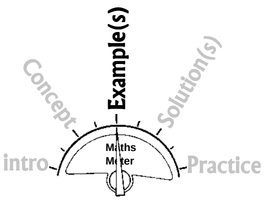
13. $22x^2 + 3x - 4$

14. $77 - 26x - 3x^2$

E. HCF AND LCM OF ALGEBRAIC EXPRESSIONS

Solutions to these problems rely on the ability to identify factors correctly.

Consider the following examples:



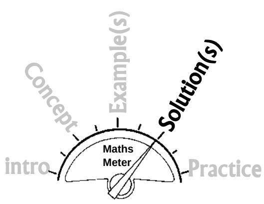
Find the HCF and LCM of

1. a^2b, a^3b^2 and a^2b^3

2. $3x + 3$, $2x + 2$ and $x^2 - 1$

3. $x^2 - 9$, $x^2 + 6x + 9$ and $2x^2 + 7x + 3$

Solutions



1. a and b are the common factors in the three terms, so use the a and b with the *lowest powers* for HCF and those with the *highest powers* for the LCM.

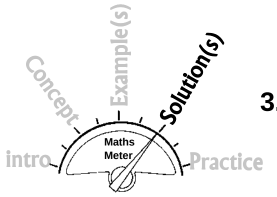
i.e a^2b
a^3b^2
a^2b^3

HCF = a^2b Pick out the lowest power of each common factor.

LCM = a^3b^3 Pick out the highest power of each factor.

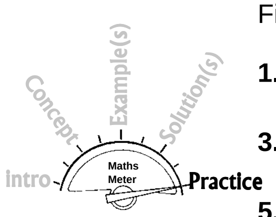
Hint
Make sure the expressions are fully factorised first.

2. $3x + 3 = 3(x + 1)$
$2x + 2 = 2(x + 1)$
$x^2 - 1 = (x - 1)(x + 1)$
$\therefore$ HCF $= (x + 1)$
LCM $= 2 \times 3(x - 1)(x + 1)$
$= 6(x - 1)(x + 1)$ No need to expand the brackets.



3. $x^2 - 9 = (x + 3)(x - 3)$
$x^2 + 6x + 9 = (x + 3)(x + 3) = (x + 3)^2$
$2x^2 + 7x + 3 = (2x + 1)(x + 3)$

HCF $= (x + 3)$
LCM $= (x + 3)^2(x - 3)(2x + 1)$



Find (i) the HCF and (ii) the LCM of:

1. xy^2 and x^2y^2 **2.** $2x^2y^2$, $6x^3y$ and $2xy^3$

3. $2x - 6$ and $3x - 9$ **4.** $2x^2 - 3x$ and $x^2 + 3x$

5. $3x^2 + 2x$ and $9x^2 - 4$ **6.** $2x^2 - 2x$ and $2x^2 - 2$

7. $x^2 - 4$, $x^2 - 4x + 4$ and $x^2 - 3x + 2$

8. $6x - 4$, $6x^2 - 4x$ and $18x^2 - 8$

9. $2x^2 + x - 3$, $4x^2 + 10x + 6$ and $6x^2 + 3x - 9$

10. $xy - 2y - 6x + 12$, $4 - x^2$ and $2 + 5x - 3x^2$

F. SIMPLIFYING ALGEBRAIC FRACTIONS

Reducing to lowest terms

Reduce the following to their simplest forms.

1. $\dfrac{2x + 8}{x^2 + x - 12}$ **2.** $\dfrac{x^2 - 3x}{x^3} \times \dfrac{x^2 - 4}{x^2 - x - 6}$

3. $\dfrac{2c^2 - 3cd + d^2}{c^3 - 2c^2d + cd^2} \div \dfrac{2cd + 3d^2}{c^2 - d^2} \times \dfrac{2c^2 + cd - 3d^2}{2c^2 + cd - d^2}$

Hint
First factorise each part of the fraction fully then identify common factors.

Solution

1. $\dfrac{2x + 8}{x^2 + x - 12} = \dfrac{2\cancel{(x + 4)}^{1}}{\cancel{(x + 4)}_{1}(x - 3)}$ [simplify using common factor $(x + 4)$]

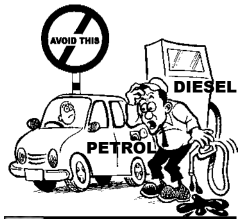
$= \dfrac{2}{x - 3}$

Hint

$\frac{x-3}{3-x} = -1$

2. $\dfrac{x^2-3x}{x^3} \times \dfrac{x^2-4}{6+x-x^2} = \dfrac{x\overset{-1}{\cancel{(x-3)}}}{\underset{x^2}{\cancel{x^3}}} \times \dfrac{(x-2)\overset{1}{\cancel{(x+2)}}}{\cancel{(3-x)}\underset{1}{\cancel{(x+2)}}}$

$= \dfrac{-(x-2)}{x^2}$

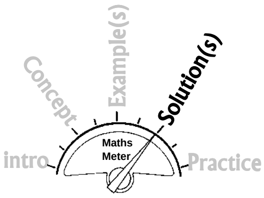


Common Error

Dividing further by x in $\dfrac{(x-2)}{x^2}$ to get

$\dfrac{1-2}{x} = \dfrac{-1}{x}$

Note that in $(x-2)$, x is not a factor but a term.



3. $\dfrac{2c^2-3cd+d^2}{c^3-2c^2d+cd^2} \div \dfrac{2cd+3d^2}{c^2-d^2} \times \dfrac{2c^2+cd-3d^2}{2c^2+cd-d^2}$

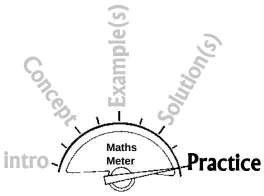
$\dfrac{\overset{1}{\cancel{(2c-d)}}\overset{1}{\cancel{(c-d)}}}{c\underset{1}{\cancel{(c-d)^2}}} \times \dfrac{\overset{1}{\cancel{(c-d)}}\overset{1}{\cancel{(c+d)}}}{cd\underset{1}{\cancel{(2c+3d)}}} \times \dfrac{\overset{1}{\cancel{(2c+3d)}}(c-d)}{\underset{1}{\cancel{(2c-d)}}\underset{1}{\cancel{(c+d)}}}$

Note: 1. The first denominator has been factorised twice i.e $c(c^2-2cd+d^2) = c(c-d)(c-d)$.

2. The second fraction has been inverted and multiplied because of the division sign.

The answer becomes $\dfrac{c-d}{cd}$

Do not make the error of dividing further.



Reduce the following to their simplest forms.

1. $\dfrac{10x^2y}{15x}$ **2.** $\dfrac{x^2+xz}{x^2+xy}$

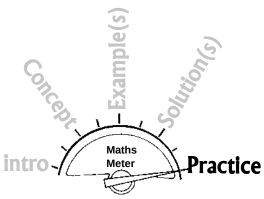
3. $\dfrac{p^2-q^2}{(p-q)^2}$ **4.** $\dfrac{6-a-a^2}{2a^2-a-6}$

5. $\dfrac{2bg^3-b^2g^2}{b^3g-2b^2g^2}$ **6.** $\dfrac{10x^2y^3}{9c^3d} \times \dfrac{18c^2d^3}{15xy^4}$

7. $\dfrac{f^2-9}{f^2+3f} \times \dfrac{f^2-4f}{f^2-7f+12}$ **8.** $\dfrac{3x^3}{xy-x^2} \div \dfrac{xy+x^2}{x^2-y^2}$

9. $\dfrac{v^2-v-6}{v^2-8v+16} \div \dfrac{v^2+v-12}{v^2-16}$

10. $\dfrac{m^2+mu+mn+un}{m^2+mu-mn-un} \times \dfrac{3n^2-3mn}{4m^2-4n^2}$



11. $\dfrac{15c^3d^4}{8ef^3} \div \dfrac{30d^2}{16e^3f^2} \times \dfrac{12ef}{20c^4d}$

12. $\dfrac{9-3d}{cd^2-cde} \times \dfrac{c^2-cd}{2d^2-6d} \div \dfrac{cd-c^2}{d^2-de}$

13. $\dfrac{a^2 - 9b^2}{a^2 - 2ab + b^2} \div \dfrac{a + 3b}{a^2 + 2ab} \times \dfrac{ab - a^2}{a^2 - ab - 6b^2}$

14. $\dfrac{(2x - y)^2 - (x - 2y)^2}{2x^2 + 3xy - 2y^2} \div \dfrac{5x^2 - 5xy}{2y^2 + 3xy - 2x^2}$

Adding and subtracting algebraic fractions

This is done in exactly the same way as in ordinary fractions. The following steps can help you to master the procedure.

Step 1. Find the common denominator (LCM of denominators).

Step 2. Create one fraction with the common denominator as the denominator.

Step 3. Simplify the numerator created.

Step 4. Factorise the numerator where possible.

Step 5. Check for common factors between the numerator and the denominator and simplify where possible.

Consider the following examples:

Simplify

1. $\dfrac{x}{3} + \dfrac{x - 1}{2}$

2. $\dfrac{5}{3x - 1} - \dfrac{3}{x + 3}$

3. $3 - \dfrac{x - 3y}{x - 1}$

4. $\dfrac{3}{x^2 - 5x + 4} + \dfrac{2}{x^2 - 3x - 4}$

5. $\dfrac{3x}{x^2 - y^2} - \dfrac{3y}{y^2 - x^2}$

Solution

1. $\dfrac{x}{3} + \dfrac{x - 1}{2}$

$= \dfrac{2x + 3(x - 1)}{6}$ Step 2, Step 1

$= \dfrac{2x + 3x - 3}{6}$

$= \dfrac{5x - 3}{6}$ Step 3

2. $\dfrac{5}{3x - 1} - \dfrac{3}{x + 3}$

$= \dfrac{5(x + 3) - 3(3x - 1)}{(3x - 1)(x + 3)}$

$= \dfrac{5x + 15 - 9x + 3}{(3x - 1)(x + 3)}$

$= \dfrac{-4x + 18}{(3x - 1)(x + 3)}$

$= \dfrac{-2(2x - 9)}{(3x - 1)(x + 3)}$

Hint

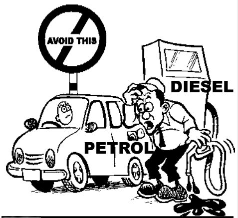
Note that 3 can be written as 3 so that the common denominator for 1 and (x + y) is x + y. 3 is a numerator.

3. $3 - \dfrac{x-3y}{x+y} = \dfrac{3(x+y)-1(x-3y)}{x+y}$

$= \dfrac{3x+3y-x+3y}{x+y}$

$= \dfrac{2x+6y}{x+y}$

$= \dfrac{2(x+3y)}{x+y}$



Common Error

$\dfrac{3x+3y-x-3y}{x+y}$
is left without further simplication.

Hint

In 2 and 3 factorisation of the numerator of the answer is necessary in case a common factor is created between the numerator and the denominator.

4. $\dfrac{3}{x^2-5x+4} + \dfrac{2}{x^2-3x-4}$ Factorise the denominators first.

$= \dfrac{3}{(x-4)(x-1)} + \dfrac{2}{(x-4)(x+1)}$ The different factors form the factors of the common denominator.

$= \dfrac{3(x+1)+2(x-1)}{(x-4)(x-1)(x+1)}$

$= \dfrac{3x+3+2x-2)}{(x-4)(x-1)(x+1)}$

$= \dfrac{5x+1}{(x-4)(x-1)(x+1)}$

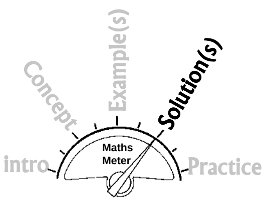
Hint

Do not expand the denominator.

5. $\dfrac{3x}{x^2-y^2} - \dfrac{3y}{y^2-x^2}$ Factorise the denominators.

$\dfrac{3x}{(x-y)(x+y)} - \dfrac{3y}{(y-x)(y+x)}$ Remember $y - x = -(x - y)$ Thus, replacing $y - x$ with $-(x - y)$ changes the sign between the fractions.

$\dfrac{3x}{(x-y)(x+y)} + \dfrac{3y}{(x-y)(y+x)}$



$= \dfrac{3x+3y}{(x-y)(x+y)}$

$= \dfrac{3\overset{1}{\cancel{(x+y)}}}{(x-y)\underset{1}{\cancel{(x+y)}}}$ A common factor $(x + y)$ exists between the numerator and the denominator.

$= \dfrac{3}{x-y}$

PRACTICE 2F2

Simplify

1. $\dfrac{3}{3x-1} - \dfrac{2}{2x-1}$

2. $2 + \dfrac{3b}{a-b}$

3. $\dfrac{5}{a-2b} + \dfrac{3}{2b-a}$

4. $6 - \dfrac{a-b}{a}$

5. $\dfrac{1}{x-2} - \dfrac{4}{x^2-4}$

6. $\dfrac{5b}{a-2b} + 2$

7. $\dfrac{y}{x^2-xy} - \dfrac{x}{xy-y^2}$

8. $\dfrac{3x+y}{x-2y} - 3$

9. $\dfrac{2xy}{3x^2+3y^2} + \dfrac{5xy}{2x^2+2y^2}$

10. $\dfrac{c}{c^2-a^2} - \dfrac{a}{ac-c^2}$

11. $\dfrac{1}{6u^2+18u} + \dfrac{1}{u^2-5u-24}$

12. $\dfrac{4c}{c^2-d^2} - \dfrac{4d}{d^2-c^2}$

13. $\dfrac{10}{2xy-3y^2} - \dfrac{15}{2x^2-3xy}$

14. $\dfrac{1-x^2}{x^2-4x+4} + \dfrac{x+2}{x-2}$

15. $\dfrac{3a+3}{a^2-a-20} + \dfrac{1}{a+4} - \dfrac{2}{a-5}$

16. $\dfrac{2}{2x-8} + \dfrac{2x-8}{x^2-8x+16}$

17. $\dfrac{2a}{a+5} - \dfrac{2a}{a-5} - \dfrac{100}{a^2-25}$

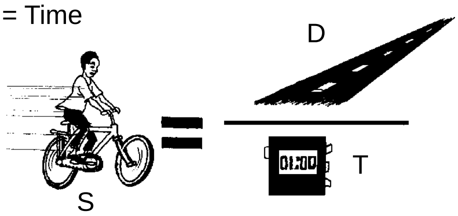
18. $\dfrac{x}{x+3} - \dfrac{x}{3-x} - \dfrac{18}{x^2-9}$

G. CHANGE OF SUBJECT OF FORMULAE

You should familiar with the following formulae:

1. Given that S = Speed D = Distance T = Time

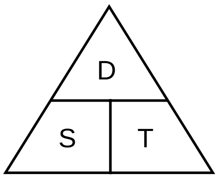
then $S = \dfrac{D}{T}$ $\quad D = ST \quad$ $T = \dfrac{D}{S}$

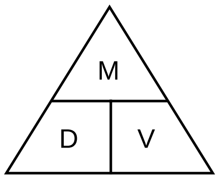


2. Given that D = Density M = Mass V = Volume

then $D = \dfrac{M}{V}$ $M = DV$ $V = \dfrac{M}{D}$

To easily memorise these formulae some people use these triangles.





The use of the triangle has limitations when you have more than three terms in the formula, for example

$v = u + at$

$u = v - at$

$a = \dfrac{v - u}{t}$

$t = \dfrac{v - u}{a}$

The subject of a formula is a single term in an algebraic formula which is expressed in terms of the others.

In $S = \dfrac{D}{T}$

S is the subject of the formula and in M = DV, M is the subject of the formula.

Similarly in $v = u + at$, v is the subject of the formula.

Consider the following examples:

Make x the subject of the formula

1. $a + x = c$

2. $mx - p = bx + w$

3. a) $\sqrt{x + a} = p$ b) find x when $p = -3$ and $a = 3$.

4. $\dfrac{a}{x^2} - b = q$

5. a) $\sqrt{\dfrac{a}{x^2} + v} = 3p$ b) find x when $a = 4$, $p = \frac{1}{3}$ and $v = -8$

Solution

Tip

Remember the change side change sign rule.

1. $a + x = c$

Transfer a to the RHS.

$x = c - a$

2. $mx - p = bx + w$

Group terms with x on one side of the equation, preferably to the left.

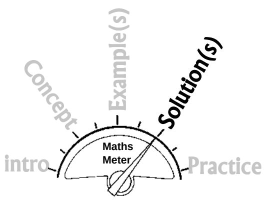
$mx - bx = w + p$

Factorise the left side.

$x(m - b) = w + p$

Divide both sides by $(m - b)$.

$$x = \frac{w + p}{m - b}$$



3. a) $\sqrt{x + a} = p$

Square both **sides** of the equation.

$(\sqrt{x + a})^2 = p^2$

$x + a = p^2$

$x = p^2 - a$

b) When $p = -3$ and $a = 3$

$x = (-3)^2 - 3$

$= 9 - 3$

$= 9 - 3$

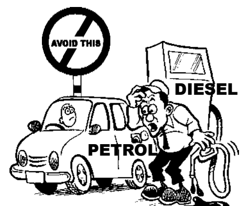
Tip

Remember the square of a number inside the square root sign is the number itself e.g.

$(\sqrt{4})^2 = 4$

$(\sqrt{3})^2 = 3$

$(\sqrt{x})^2 = x$



Common Error

If $\sqrt{x + a} = P$,

then $x + a = \sqrt{p}$ wrong

4. $\dfrac{a}{x^2} - b = q$

Transfer $-b$ to the RHS.

$\dfrac{a}{x^2} - q + b$

Multiply both sides by x^2

$a = x^2(q + b)$

Divide both sides by $(q + b)$.

$\dfrac{a}{q + b} = x^2$

Take square roots on both sides.

$\sqrt{\dfrac{a}{q + b}} = x$

Tip

You should always remember to find the square root of x^2 appreciating that

$\sqrt{x^2} = x$

5. a) $\sqrt{\frac{a}{x^2} + v} = 3p$

Square both sides.

$\frac{a}{x} + v = 9p^2$

Subtract v from both sides.

$\frac{a}{x^2} = 9p^2 - v$

Multiply both sides by x^2 to make x a numerator.

$a = x^2(9p^2 - v)$

Divide both sides by $(9p^2 - v)$

$\frac{a}{9p^2 - v} = x^2$

Find square roots of both sides

$\sqrt{\frac{a}{9p^2 - v}} = x$

b) When $a = 4$, $p = \frac{1}{3}$ and $v = -8$

$$x = \sqrt{\frac{4}{9(\frac{1}{3})^2 - (-8)}}$$

$$= \sqrt{\frac{4}{1+8}}$$

$$= \sqrt{\frac{4}{9}}$$

$$= \frac{2}{3}$$

It may be deduced that the general steps to be followed when making a letter/symbol the subject of the formula are:

1. Start by eliminating all the root signs and brackets where necessary.

2. Eliminate fractions if any.

3. Group like terms together, i.e, those which have the letter which should be made the subject of the formula.

4. Factorise like terms and divide both sides by the bracket.

5. Simplify as far as possible.

In this exercise a formula and a corresponding letter or two letters printed in brackets, after the formula are given.
Make that letter the subject of the formula and if more than one letter is bracketed, make each letter the subject of the formula in turn.

1. $A = \pi r^2$ (r)

2. $v = u + at$ (t)

3. $r^2 = u^2 + 2as$ (a)

4. $F = \dfrac{Gm}{r^2}$ (m)

5. $V = \dfrac{4\pi r^3}{3}$ (r)

6. $b = \dfrac{2x - 2}{6x + 3}$ (x)

7. $m^2 + 4qp = n + x$ (q)

8. $T = \sqrt{\dfrac{e}{g}}$ (g)

9. $\sqrt{\dfrac{p + q}{2m}} = 6x + 2v$ (p,m)

10. $\sqrt{\dfrac{3x^2 + m}{\frac{2}{v} + 1}} = x$ (v,m)

11. Make the letter in brackets the subject of the formula
a) $V = \pi r^2 h$ (r)

b) $A = 2\pi r^2 + 2\pi rh$ (h)

12. The formula for finding the total surface area in Fig 2.1 is
$A = 2\pi rh + \pi r^2 + \pi rl$

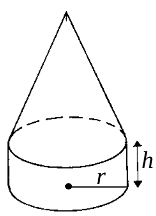
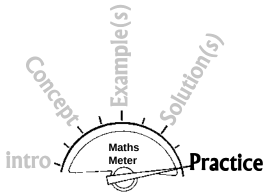


Fig. 2.1

a) Make h the subject of the formula.

b) Given that $r = 3$ $l = 7$, and $A = 132$, find h taking π to be $\frac{22}{7}$



13. The volume of a jug is given by the formula:
$V = \frac{1}{3}\pi h\,(2R^2 - r^2)$ where R is the radius of the top, r the radius of the bottom and h the vertical height of the jug.

a) Make R the subject of the formula.

b) Given that $\pi = 3{,}142$, $h = 17$, $R = 6$ and $r = 2$
Calculate v to the nearest whole number.

14 a) Make b the subject of the formula

$$\frac{x}{a} + \frac{y}{b} = 1$$

b) Find b given that $a = 750$, $y = \frac{13}{25}$ and $x = -30$

SUMMARY

1. Like terms combine, unlike terms do not e.g
$2x + 3x = 5x$ but $x + 2y = x + 2y$.

2. Simplify can mean
a) Remove brackets and combine like terms.
b) Combine fractions to a single one using the four arithmetic operations.

Common identities when removing brackets are:
(i) $(a + b)^2 = a^2 + 2ab + b^2$
(ii) $(a - b)^2 = a^2 - 2ab + b^2$
(iii) $(a - b)(a + b) = a^2 - b^2$

3. An expression in the form of $a^2 + b^2$ has no factors.

4. Factorise means the final stage should have brackets where all numbers or expressions are multiplying each other.
e.g $5(x + 2)(x + 2)$ is factorised.
$5(x + 2) - (x - 1)$ is not factorised because of the minus sign between $5(x + 2)$ and $(x - 1)$.

5. HCF has all common factors whilst,
LCM has every factor represented.

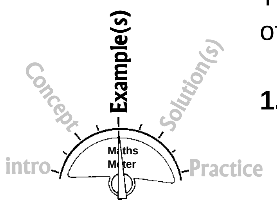
6. The HCF is normally used to reduce fractions to their lowest terms or simplest form. LCM is normally used as a common denominator so that a number of fractions can be combined into one, the lowest common one.

7. When changing the subject of a formula, eliminate root signs, brackets and fractions first, where need be.

8. When expressing a formula in terms of one letter:
(i) group the terms with the wanted letter on one side and the rest on the other side.

(ii) Factorise out the wanted letter and finally divide both sides with the bracket created.

EXAM PRACTICE 2



The following examples will help highlight key stages to take note of.

1. Chipo has $(3x + 2)$ sweets and Hazel has $(5 - 2x)$ sweets. Find in its simplest form, the number of sweets:
a) they have altogether
b) Chipo has more than Hazel.

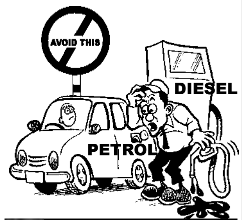
2. Simplify as far as possible
a) $\dfrac{45a^3b^2}{75a^5b}$ b) $\dfrac{x^2 + 4x + yx + 4y}{x^2 - y^2}$

3. Factorise completely a) $a^2(c - 3) - c + 3$
b) $px + qy - qx - py$

4. Find the HCF and LCM of $x^2 - 4$, $x^2 - 4x + 4$ and $2x^2 - 4x$.

5. Express as a single fraction a) $\dfrac{a}{3a - 1} + \dfrac{2}{3}$

b) $\dfrac{5}{3x - 3y} - \dfrac{1}{2x + 2y}$



Common Error

The requirement 'in its simplest form' is often ignored. This means the answer should be free of like terms.
$3x + 2 - 5 - 2x$ is common when $3x + 2 - 5 - 2x$ is expected.

Hint

Altogether implies addition

The question implies $3x + 2$ is bigger than $5 - 2x$.

Solution

1. a) $\therefore$ $3x + 2 + 5 - 2x$ Combine like terms.
$= x + 7$

Now by how much?
$3x + 2 - (5 - 2x) = 3x + 2 - 5 + 2x$
$= 5x - 3$

2. a) $\dfrac{45a^3b^2}{75a^5b} = \dfrac{3b}{5a^2}$ Divide both numerator and denominator by common factors or HCF $15a^3b$.

Hint

Some fractions may require factorisation of their parts first.

b) $\dfrac{x^2 + 4x + yx + 4y}{x^2 - y^2}$ Both parts need factorising first.

$$= \frac{x(x + 4) + y(x + 4)}{(x - y)\,(x + y)}$$

$$= \frac{(x + 4)\cancel{(x + y)}^{1}}{(x - y)\,\cancel{(x + y)}_{1}}$$

$$= \frac{x + 4}{x - y}$$

Common Error

$$\frac{\cancel{x^2}^{1} + 4x + yx + 4y}{\cancel{x^2}_{1} - y^2}$$

and $\dfrac{\cancel{x}^{1} + 4}{\cancel{x}_{1} - y}$

These x^2 and x are not factors. Remember factors are multipliers.

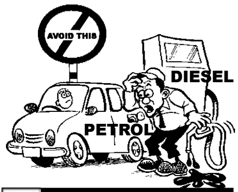
Hint

Do not expand the bracket instead create the same bracket with the last two terms.

3. a) $a^2(c-3) - c + 3 = a^2(c-3) - 1(c-3)$

$= (c-3)(a^2 - 1)$ second bracket is a difference of two squares.

$= (c-3)(a-1)(a+1)$



Common Error

Leaving the answer at $(c-3)(a^2-1)$ stage. (a^2-1) has factors! The question expects *complete* factorisation not partial. An equation $(c-3)(a-1)(a+1) = 0$ is formed and solved.

b) $px + qy - qx - py$

Re-arrange the terms before giving up!

$px - py - qx + qy = p(x-y) - q(x-y)$

$= (x-y)(p-q)$

Hint

Factorise the expressions first.

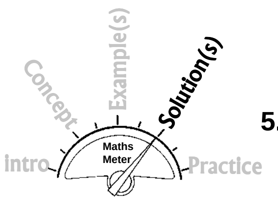
4. a) $x^2 - 4 = (x-2)(x+2)$

$x^2 - 4x + 4 = (x-2)^2$

$2x^2 - 4x = 2x(x-2)$

$\therefore$ HCF $= (x-2)$

LCM $= 2x(x-2)^2(x+2)$ Make sure all the factors are represented in the LCM and there is no need to expand.

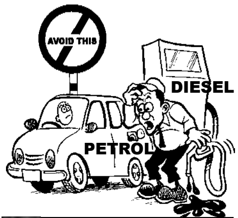


5. a) $\dfrac{a}{3a-1} + \dfrac{2}{3} = \dfrac{3a + 2(3a-1)}{3(3a-1)}$

$= \dfrac{3a + 6a - 2}{3(3a-1)}$

$= \dfrac{9a - 2}{3(3a-1)}$

Remember $9a$ is a term not a factor.
Thus the numerator has no common factors.



Common Error

$\dfrac{\cancel{9}^{3}a - 2}{\cancel{3}_{1}(3a-1)} = \dfrac{9a-2}{3a-1}$

Remember $9a$ is a term not a factor.

Hint

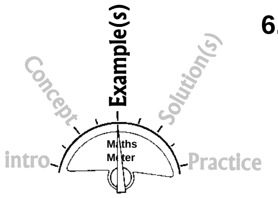
Do not multiply with the LCM.

b) $\dfrac{5}{3x-3y} - \dfrac{1}{2x+2y}$

$= \dfrac{5}{3(x+y)} - \dfrac{1}{2(x+y)}$ Factorise the denominators first.

$= \dfrac{10-3}{6(x+y)}$

$= \dfrac{7}{6(x+y)}$



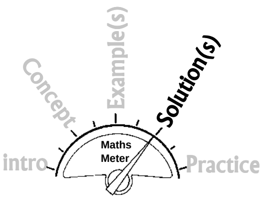
6. The total surface area of a solid cylinder with base radius r and height h is represented by:
$A = 2\pi r^2 + 2\pi rh$

a) Make h the subject of the formula.

b) Find h when A = 550, r = 3,5cm and $\pi = \frac{22}{7}$

Solution

a) $A = 2\pi r^2 + 2\pi rh$
$A - 2\pi r^2 = 2\pi rh$
$\therefore \quad h = \dfrac{A - 2\pi r^2}{2\pi r}$



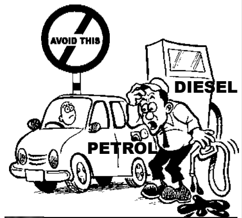
b) From $h = \dfrac{A - 2\pi r^2}{2\pi r}$

$$h = \frac{550 - 2 \times \frac{22}{7} \times \frac{7}{2} \times \frac{7}{2}}{2 \times \frac{22}{7} \times \frac{7}{2}}$$

$$= \frac{550 - 77}{22}$$

$$= \frac{473}{22}$$

$$= 21{,}5$$



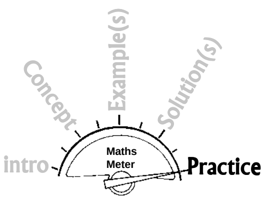
Common Error

$$\frac{550 - \cancel{2}^{1} \times \cancel{22}^{1} \times \frac{7}{2} \times \frac{\cancel{7}^{1}}{\cancel{2}}}{\cancel{2}_{1} \times \frac{\cancel{22}}{\cancel{7}} \times \frac{\cancel{7}^{1}}{\cancel{2}}}$$
$$= 550 - \frac{7}{2}$$

Now do the following:

1. Simplify a) $4(3 - x) - 5(7 - x)$

b) $\dfrac{(4 - q)^2 + 4 - q}{q^2 - 16}$



2. Factorise completely a) $3x^2 - 75$
b) $6x^2 + 2x - 4$
c) $pq + ps - qr - rs$

3. Find the HCF and LCM of:
$2x(x + 2)$, $6x^2(x - 1)(x + 2)$ and $4(x + 2)^2$

4. Express as a single fraction a) $\dfrac{a}{c^2 - ac} + \dfrac{c}{a^2 - ac}$

b) $y + 3 - \dfrac{y^2}{y + 1}$

5. Simplify as far as possible

a) $\dfrac{x^2 - 4y^2}{px + qx - 2py - 2qy}$ b) $\dfrac{ac}{bc - c^2} \div \dfrac{c^2}{c^2 - bc} \times \dfrac{bc + c^2}{c^2 + 2bc + b^2}$

6. The total Voltage (V) in a circuit is represented by V = IR – Ir
where I is the current,
R is the external resistance.
r is the internal resistance.

a) Make r subject of the formula.

b) Find R given that V = 100, I = 5A and r = 4.

7. For a given regular polygon with n sides, the size of each interior angle is x^0 where

$x = 180^0 - \dfrac{360^0}{n}$

a) Make n the subject of the formula.

b) Calculate n if x = 120.

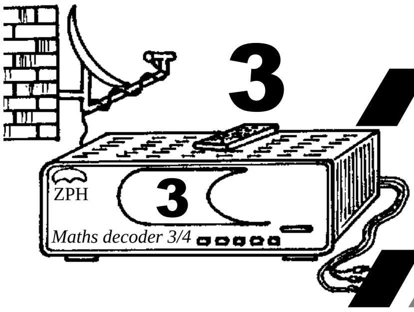
8. Make the letter in brackets the subject of the given formula.

a) $\dfrac{1}{u} + \dfrac{1}{v} = \dfrac{1}{f}$ (***v***)

b) $S = ut + at^2$ (***a***)

c) $y = \dfrac{4x - 3}{2l - 4}$ (***x***)

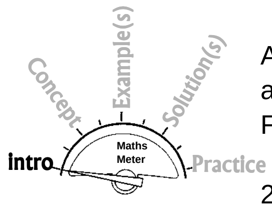
d) $q = \dfrac{p - 3r}{2p + r}$ (***r***)



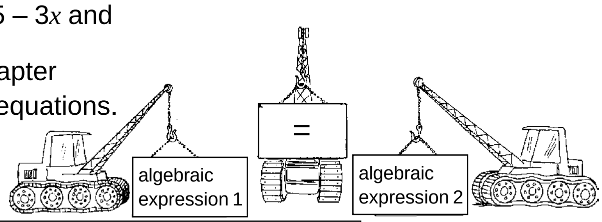
3 Solving Equations

$ax^2 + bx + c = 0$ $3x + 4y = 6$

$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ $\frac{y+1}{2} + \frac{2y-1}{3}$



An equation is created when two different algebraic expressions are equal.
For example $\frac{x-1}{2} = 3x$, $3 - x = 5 - 3x$ and $2x^2 - 9 = 0$ are equations. This chapter will focus on linear and quadratic equations.



Syllabus Expectations

By the end of this chapter, students should be able to:

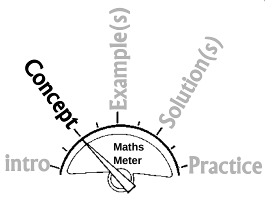
1. solve simple linear equations including those involving fractions.
2. solve simple linear simultaneous equations by substitution and elimination methods.
3. solve quadratic equations of the form $ax^2 + bx + c = 0$ by either factorisation or by formula.

ASSUMED KNOWLEDGE

In order to tackle work in this chapter. It is assumed that students are able to:

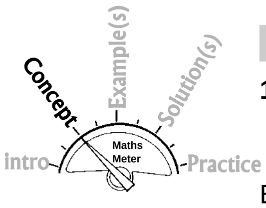
- identify like and unlike terms.
- expand expressions with brackets.
- factorise expressions up to quadratic expressions.

A. LINEAR EQUATIONS



With or without brackets

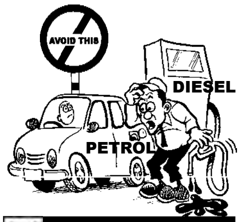
Linear equations are equations with variables to the power 1. Solving an equation is the process of finding the number being represented by the variables in the equation. This number should balance the equation when substituted into it. Sometimes the solution is simple.



Consider the following examples:

1. $x + 3 = 7$, clearly $x = 4$. Thus $x = 4$ is the solution to the equation $x + 3 = 7$.

Equations like a) $7 - 5x = 4x - 11$ or b) $3(2x - 1) - 2(x - 1) = 0$ are more demanding and require a more organised approach to solve them.



Common Errors

1) On changing side, the sign is not changed.
2) Brackets are removed by multiplying the first term only e.g. $6x - 1 - 2x - 1$. This is wrong, all terms inside the bracket must be multiplied by the factor outside.
3) $4x = 1$, $\therefore$ $x = 4$ is a common wrong solution.

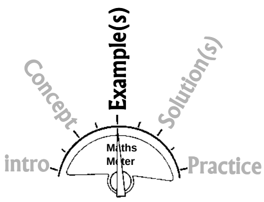
Study the steps below.

2. $7 - 5x = 4x - 11$

Step 1: Collect the terms with variables on one side of the equation, bearing in mind the 'change-side change-sign rule'.
i.e. $7 + 11 = 5x + 4x$

Step 2: Simplify each side
$18 = 9x$

Step 3: Divide both sides by 9 to make x the subject.
$x = 2$



3. $3(2x - 1) - 2(x - 1) = 0$

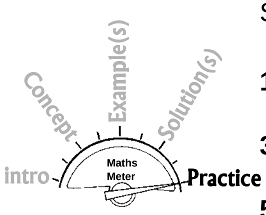
Step 1: Remove brackets.
$6x - 3 - 2x + 2 = 0$

Step 2: Group terms with variables on one side.
$6x - 2x = 3 - 2$
$4x = 1$

Step 3: Divide both sides by 4 to isolate the letter x.
$x = \frac{1}{4}$

Thus in all cases:

- Remove brackets first if they exist.
- Collect terms with variables on one side and the constants on the other side, maintaining the 'change side change sign' rule.
- Divide both sides by the coefficient of the letter, to make it the subject.



Solve for x

1. $5 + 2x = 7 - 5x$	**2.** $3 - 2x = 7 - 4x$
3. $2x - 7 = 8 - 3x$	**4.** $x - 9 = 15 - 2x$
5. $3x + 12 = 4x - 5$	**6.** $5x - 17 = 7 - 3x$
7. $12x + 21 = 41 - 4x + 12$	**8.** $14 - 30x - 12 = 10x - 18 - 45x$
9. $7(x - 1) + 2(3x + 5) = 29$	**10.** $8x - 3(5 - x) = 7$

11. $5(x + 5) + 2(x + 5) = –6$ **12.** $12 – x = 5(x + 3) – 3(x + 5)$

13. $4 + 5(2x + 3) = 8(3 –x)$ **14.** $4(7x – 9) – 7(4x + 3) = 6(9x – 1)$

15. $12 = –[–3\{4x –2(x – 2)\}]$ **16.** $3[2 – 4(x – 7)] = 26 – 8x$

B. EQUATIONS WITH FRACTIONS

The fractions can be common or decimal. Generally, get rid of these fractions by multiplying throughout by the common denominator (LCM).

Consider the following example:

1. Solve for x a) $2,4x – 3,8 = 15,4$

b) $\frac{x}{3} + \frac{2x}{5} = 4$

Solution

1. a) $2,4x – 3,8 = 15,4$ Multiply each term by 10.

$$24x – 38 = 154$$
$$24x = 192$$
$$x = \frac{192}{24}$$
$$x = 8$$

or

$$2,4x = 15,4 + 3,8$$
$$= 19,2$$
$$x = \frac{19,2}{2,4}$$
$$= \frac{192}{24}$$
$$x = 8$$

b) $\frac{x}{3} + \frac{2x}{5} = 4$ Multiply each term by 15

$15 \times \frac{x}{3} + 15 \times \frac{2x}{5} = 4 \times 15$ which is the LCM of 3 and 5.

$$5x + 6x = 60$$
$$11x = 60$$
$$\therefore \quad x = 5\tfrac{5}{11}$$

Sometimes brackets are needed when multiplying each term by the LCM. This is especially so when the numerator or denominator has more than one term.

2. Solve for c

a) $\frac{c + 1}{4} – \frac{c – 1}{3} = 1$ b) $\frac{2c + 3}{2c + 1} = \frac{c – 1}{c – 2}$

c) $\frac{3}{c^2 – 9} – \frac{4}{c^2 – 5c + 6} = 0$

Solution

2. a) $\dfrac{12(c+1)}{4} - \dfrac{12(c-1)}{3} = 1\times12$ Multiply each term by 12.

$$\begin{aligned} 3(c+1) - 4(c-1) &= 12 \\ 3c + 3 - 4c + 4 &= 12 \\ -c + 7 &= 12 \\ -c &= 5 \\ \therefore\quad c &= -5 \end{aligned}$$

Remove the brackets.

Tip

Do not end at $-c = 5$
Find c not $-c$.

b) $\dfrac{2c+3}{2c+5} = \dfrac{c-1}{c-2}$

Multiply each term by $(2c + 5)(c - 2)$ which is the LCM.

$$\begin{aligned} (c-2)(2c+3) &= (c-1)(2c+5) \\ 2c^2 - c - 6 &= 2c^2 + 3c - 5 \\ 2c^2 - 2c^2 - c - 3c &= 6 - 5 \\ -4c &= 1 \\ c &= -\tfrac{1}{4} \end{aligned}$$

Hint

Don't leave answer as $\frac{1}{-4}$ since the negative sign is still in the denominator. $-\frac{1}{4}$ or $\frac{-1}{4}$ are the accepted final versions.

c) $\dfrac{3}{c^2-9} - \dfrac{4}{c^2-5c+6} = 0$ Factorise the denominators first.

$\dfrac{3}{(c-3)(c+3)} - \dfrac{4}{(c-3)(c-2)} = 0$ Multiply each term by $(c-3)(c+3)(c-2)$.

$$\begin{aligned} 3(c-2) - 4(c+3) &= 0 \\ 3c - 6 - 4c - 12 &= 0 \\ -c &= 18 \\ \therefore\ c &= 18 \end{aligned}$$

Equations with fractions are often confused with expressions with fractions.

Below is a summary of the differences.

Concept	Expression	Equation
Example	$\dfrac{x}{2x-3} + \dfrac{3}{4x-2}$	$\dfrac{x}{2x-3} = \dfrac{3}{4x-2}$
Sign	No equal (=) sign either + or – between terms.	Always has an equal sign between terms or expressions.
Method of manipulation	Expressions are simplified or evaluated.	Equations are solved.
About LCM	Use the LCM to create a common denominator. Form a single fraction.	Multiply each term by the LCM to clear the fractions.

Consider the examples below and carefully compare the processes.

Expression

1. Simplify $\frac{3x-6}{4} - \frac{3}{4} + \frac{3x-4}{6}$

Step 1
Find Common Denominator (LCM) i.e. 12.

Step 2
Place the three fractions over the Common Denominator

i.e. $\frac{3(3x-6) - 3(3) + 2(3x-4)}{12}$

Step 3
Simplify the numerator maintaining the same Common Denominator at all stages.

i.e. $\frac{9x - 18 - 9 + 6x - 8}{12}$

$= \frac{15x - 35}{12}$

Equation

2. Solve $\frac{3x-6}{4} = \frac{3}{4} + \frac{3x-4}{6}$

Step 1
Find common denominator (i.e. 12).

Step 2
Multiply each term by the Common Denominator to clear the fractions.

i.e. $\frac{\overset{3}{\cancel{12}}(3x-6)}{\cancel{4}_1} = \frac{\overset{3}{\cancel{12}}(3)}{\cancel{4}_1} - \frac{\overset{2}{\cancel{12}}(3x-4)}{\cancel{6}_1}$

$9x - 18 = 9 - 6x + 8$

Step 3
Solve this equation in the usual manner.

i.e. $15x = 35$

$x = 2\frac{5}{15}$

$= 2\frac{1}{3}$

The important steps to ***note*** when solving equations with fractions are:

(i) Use the common denominator (LCM) to **multiply** each term in the equation.

(ii) The common denominator and the denominators disappear (are cancelled off) after the common denominator has been used.

(iii) The resultant equation (after multiplying throughout with the common denominator) may be linear or quadratic.

(iv) Make sure any algebraic denominator is fully factorised before finding the common denominator.

Tip

Always check if any denominator needs factorisation first i.e. the second one in this case.

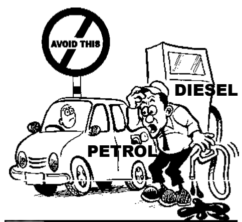
3. Solve $\frac{3}{x+3} - \frac{x-8}{x^2-9} = 1$

Solution

3. $\frac{3}{x+3} - \frac{x-8}{(x+3)(x-3)} = 1$

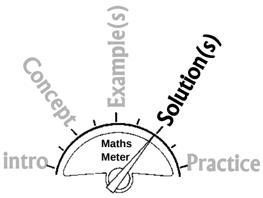
Multiply each term by $(x + 3)(x - 3)$

$3(x-3) - (x-8) = (x+3)(x-3)$



Common Errors

When one side is 1 as above or any whole number, the common denominator is not used to multiply it.
From the example above,
$(3(x-3) - (x-8) = 1$
Here the RHS has not been multiplied by the common denominator.
This is wrong.



Remove brackets

$3x - 9 - x + 8 = x^2 - 9$

Re-arrange the quadratic equation.

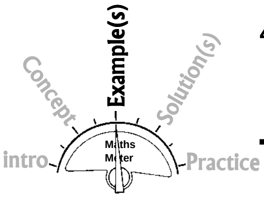
$x^2 - 2x - 8 = 0$

Solve by factorisation

$(x - 4)(x + 2) = 0$

either $x - 4 = 0$ or $x + 2 = 0$

$\therefore\ x = 4$ or -2



4. $\dfrac{x}{x+2} = \dfrac{4}{x-3}$

Solution

4. Since there is no denominator which needs factorisation, multiply through by the common denominator which is $(x+2)(x-3)$ and hence simplify

$x(x - 3) = 4(x + 2)$

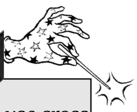
$x^2 - 3x = 4x + 8$

$x^2 - 7x - 8 = 0$

$(x - 8)(x + 1) = 0$

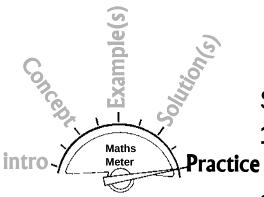
either $x - 8 = 0$ or $x + 1 = 0$

$\therefore\ x = 8$ or -1



Hint

You can also use cross multiplication.



Solve the following equations:

1. $2x - 2{,}8 = 4x - 1{,}6$

2. $3{,}1x - 55{,}9 = 12{,}3$

3. $7 - 2(x + 3{,}6) = 1{,}8$

4. $\dfrac{x}{4{,}2} + 3{,}6 = 5{,}4$

5. $\dfrac{x}{2} = 6$

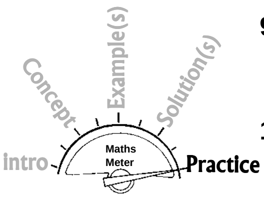
6. $5 - \dfrac{x}{3} = 0$

7. $\dfrac{y+4}{3} = 5$

8. $\dfrac{3y}{5} - \dfrac{1}{4} = 0$

9. $\dfrac{y}{3} = \dfrac{y}{2} + \dfrac{1}{3}$

10. $\dfrac{3y-1}{2} - \dfrac{3-y}{4} = \dfrac{3}{4}$



11. $\dfrac{y+1}{2} + \dfrac{2y+1}{3} = 2$

12. $\dfrac{3-y}{5} + 3 = \dfrac{2y+1}{10}$

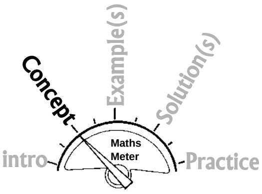
13. $\dfrac{7}{a-4} = \dfrac{2}{a-3}$

14. $\dfrac{4}{2a-5} - \dfrac{3}{a+2} = 0$

15. $\dfrac{3}{a+4} - \dfrac{a-6}{a^2-16} = 0$

16. $\dfrac{2}{a-3} = \dfrac{2a-1}{a^2-9} - \dfrac{3}{a+3}$

C. LINEAR SIMULTANEOUS EQUATIONS



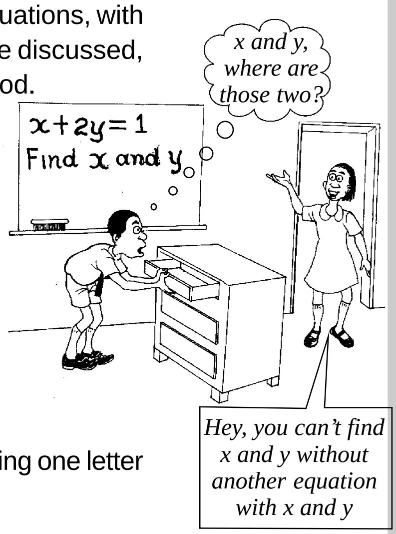
Simultaneous means at the same time. In this case we are going to deal with a system of equations with 2 variables (unknowns) like $x + 2y = 1$ and $x + y = 4$. In this situation it is not possible to solve the equations individually. Equations with two or more unknowns can not be solved individually.

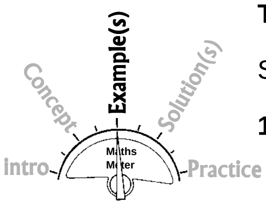
Thus:

(i) When only one unknown is involved – only one equation is needed to find that unknown.

(ii) When 2 unknowns are involved – 2 equations are needed to find the 2 unknowns.

This part of the chapter teaches you to solve linear equations, with two unknowns simultaneously. Two methods are to be discussed, i.e. the substitution method and the elimination method.





The substitution method

Solve the following simultaneously:

1. $x = 5 + 5y$
$2x + y = -1$

2. $10x + 7y = -1$
$2x + y = 5$

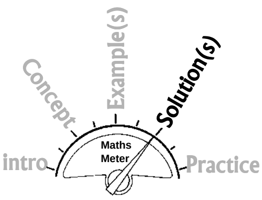
3. $3x + 4y = 6$
$2x + 3y = 5$

In general, the substitution method is achieved by making one letter subject of formula.

Solution

1. $x = 5 + 5y$ ...(i)
$2x + y = -1$...(ii)

x is already the subject of formula in the first equation.
Now, simply substitute the $5 + 5y$ for the x in the second equation.



$2(5 + 5y) + y = -1$ An equation with one unknown letter (y) has been created so we can go on to solve for y.

$10 + 10y + y = -1$
$11y = -11$
$y = -1$

Substitute -1 for y in the first equation.
$x = 5 + 5(-1)$
$x = 5 - 5$
$x = 0$

Snap check ☑: to check if your answer is correct, substitute the 2 values into each equation and see if both balance.

e.g. Equation 1 LHS $= 0$ LHS – Left Hand Side

RHS $= 5 + 5(-1)$ RHS – Right Hand Side

$= 0$

$\therefore$ RHS = LHS

Equation 2. LHS $= 2(0) + (-1)$

$= 0 - 1$

$= -1$

RHS $= -1$

$\therefore$ RHS = LHS

2. $10x + 7y = -1$ ...(i)

$2x + y = 5$ ...(ii)

Solution

2. Here we need to make one letter the subject and clearly, y from the second equation is more convenient.

i.e. $y = 5 - 2x$ Substitute $(5 - 2x)$ for y of the first equation.

$10x + 7(5 - 2x) = -1$ and solve for x.

$10x + 35 - 14x = -1$

$-4x = -36$

$x = 9$

Substitute 9 for x in the rearranged second equation.

$y = 5 - 2(9)$

$= 5 - 18$

$= -13$

3. $3x + 4y = 6$ ...(i)

$2x + 3y = 5$ ...(ii)

Solution

3. Making x the subject in the second equation.

$2x = 5 - 3y$

$$x = \frac{5 - 3y}{2}$$

Substituting $x = \dfrac{5 - 3y}{2}$ in (i)

$3(5 - 3y) + 8y = 12$

$15 - 9y + 8y = 12$

$-y = -3$

$y = 3$ Substitute into second equation.

$$x = \frac{5 - 3y}{2}$$

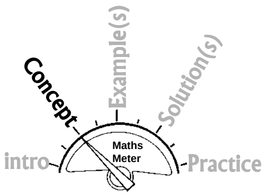
$$= \frac{5 - 9}{2}$$

$$x = \frac{-4}{2}$$

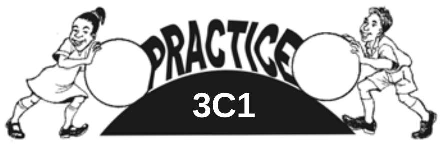
$$x = -2$$

Note in (3) any of the letters can be made the subject of the equation. In (1) and (2) it was clear which one was more convenient.

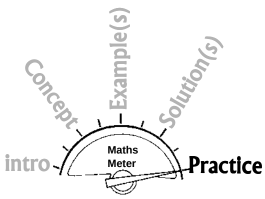
To summarise, the substitution method entails:



- Making one letter the subject of the formula.
- Substituting the expression of the letter into the other equation and solving.
- Substituting the solution into the rearranged equation to find the value of the other letter.



Use the substitution method to solve the following simultaneously.



1. $y = 3x - 11$
$3x + 2y = -4$

2. $5x - 3y = 21$
$x = 24 - 6y$

3. $8x + 15y = 11$
$4x - y = 31$

4. $2x + y = 17$
$x + y = 11$

5. $2x - y = 3$
$2x + 3y = 23$

6. $3x - 4y = -18$
$x - 4y = -30$

7. $x - y = 1$
$x + 2y = 20$

8. $2x - 7y = 6$
$4x + 7y = -30$

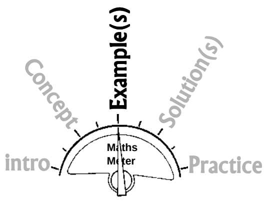
9. $3x + 2y = 7$
$3x - 2y = -1$

10. $2x + 3y = 6$
$6x - 4y = 5$

The elimination method

To eliminate is to 'get rid of'. The aim of this method is to get rid of one of the letters so the remaining unknown value can be found.

Consider the following examples



Solve simultaneously:

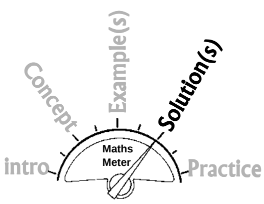
1. $2x - y = 3$
$2x + 3y = 23$

2. $2x - 7y = 6$
$4x + 7y = -3$

3. $10x + 7y = -1$
$2x + y = 5$

4. $3x + 4y = 6$
$2x + 3y = 5$

Solution



1. $2x - y = 3$...(i)
$2x + 3y = 23$...(ii)

In this case letter x can be eliminated right away because its coefficients are the same in both equations. The elimination can be achieved by subtracting the first equation from the second or vice versa.

Subtracting i.e. $2x - 2x = 0$

$3y - (-y) = 4y$ and $23 - 3 = 20$

$4y = 20$

$\therefore y = 5$

Substitute the 5 into either of the two equations, in this case, in (i)

$2x - (5) = 3$

$2x - 5 = 3$

$2x = 8$

$\therefore x = 4$

Note: All signs of the equation being subtracted from the other, change to the opposite signs.

2. $2x - 7y = 6$

$4x + 7y = -3$

Solution

2. In this case letter y can be eliminated right away by **adding** of the two equations.

i.e. $2x + 4x = 6x$, $-7y + (+7y) = 0$ and

$6 + (-3) = 3$

$6x = 3$

$\therefore x = \frac{1}{2}$

Substitute $\frac{1}{2}$ into the second equation.

$4(\frac{1}{2}) + 7y = -3$

$2 + 7y = -3$

$7y = -5$

$y = -\frac{5}{7}$

Tip

1. Elimination is only possible if coefficients of the targeted letter i.e. (letter to be eliminated) are the same.

*2.If these same coefficients have the same sign e.g. as in (a) **subtract** to eliminate*

*3.If these same coefficients have different signs e.g. as in (b) **Add** to eliminate.*

3. $10x + 7y = -1$ ...(i)

$2x + y = 5$ ...(ii)

Solution

3. Here no variable can be eliminated right away. Make a choice.

If the targeted variable to be eliminated is x:

multiply the second equation by 5 to obtain the same coefficient for x in both equations.

$10x + 5y = 25$ Now subtract this equation from the first.

$10x + 7y = -1$

$-(10x + 5y = 25)$

$2y = -26$

$y = -13$

Substitute $y = -13$ in equation (ii)

$2x + (-13) = 5$

$2x - 13 = 5$

$2x = 18$

$x = 9$

If targeted variable is y:

multiply the second equation by 7.

$10x + 7y = -1$ ...(i)

$2x + y = 5$ ...(ii)

$14x + 7y = 35$ ...(iii)

Subtract the first equation from the third.

$4x = 36$

$x = 9$

Substitute 9 in the second equation.

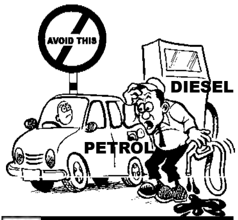
$2(9) + y = 5$

$y = 5 - 18$

$y = -13$

4. $3x + 4y = 6$ ...(i)

$2x + 3y = 5$ ...(ii)



Common Error

Students often forget to change the sign of each term when subtracting one equation from the other. Remember, always pause and think when a minus sign is involved.

Solution

4. Targeted variable is x.

Multiply the first equation by 2 and the second equation by 3.

$6x + 8y = 12$

$-(6x + 9y = 15)$ Subtract the equations.

$-y = -3$

$\therefore \quad y = 3$

$3x + 4(3) = 6$ Substitute in the first equation.

$3x = 6 - 12$

$3x = -6$

$x = -2$

Use the Elimination Method to solve the following, simultaneously.

1. $x + y = 7$
$2x - y = 8$

2. $3x + 4y = 7$
$3x - 4y = -1$

3. $4x - y = -8$
$3x - y = -6$

4. $4x - 3y = 5$
$2x + y = 5$

5. $2x + y = 8$
$5x - 2y = 2$

6. $2x - 3y = 5$
$5x - 3y = -4$

7. $3x - 2y = 3$
$7x - 5y = 6$

8. $5x - 3y = -33$
$x + 6y = 0$

9. $3x = 4 + 2y$
$2x + 3y = 7$

10. $9x + 10y = -3$
$14x = 7 - 15y$

Usually questions do not specify the method to use. So when confronted with simultaneous equations, the choice of the method is yours. Choose the more convenient method to the particular question.

Two other methods of solving simultaneous equations are:

(i) use of matrices – (see chapter on matrices)

(ii) graphical method (see chapter on functional graphs)

The next exercise will give you more practice on choosing the best method to a particular problem.

Where fractions are involved, clear them before you use any of the methods.

Use either of the two methods to solve the following simultaneously.

1. $2x - y = 8$
$x + y = 4$

2. $3x - 2y = -17$
$x - y = -8$

3. $4x - y = -8$
$3x - y = -5$

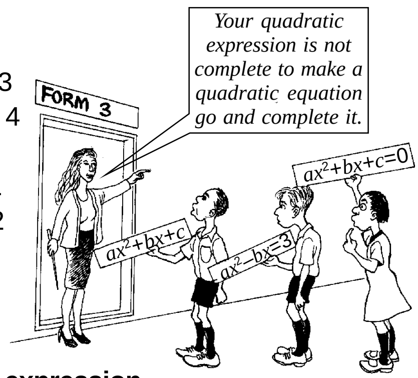
4. $3x + 2y = 9$
$2x + 3y = 8$

5. $5x + 6y = -2$
$3x - 8y = -7$

6. $6x + 5y = 2{,}3$
$10x + 10y = 4$

7. $x + 2y = 3$
$x + \frac{1}{2}y = 3$

8. $\frac{x}{3} - \frac{y}{2} = 1$
$\frac{1}{2}x + y = -2$



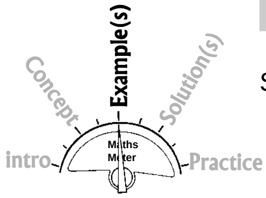
D. QUADRATIC EQUATIONS

A quadratic equation is formed when any quadratic **expression** is equated to zero. The general format is $ax^2 + bx + c = 0$ where $a \neq 0$.

This section will discuss two methods of solving quadratic equations. These are the factorisation and the formula methods.

The Factorisation Method

This method is based on the fact that when two or more numbers e.g. a and b multiply to give zero, then one of them must be zero. i.e. if $a \times b = 0$ then either $a = 0$ or $b = 0$ or both are equal to zero. As can be seen from $a \times b = 0$ the a and the b are factors of some expression. Thus, when given a quadratic equation, check if the expression factorises to create $a \times b$ first.



Consider the following examples

Solve: **1.** $x^2 + 7x - 8 = 0$ **2.** $2x^2 - 5x = 3$

3. $4x^2 = 6x$ **4.** $9x^2 - 4 = 0$

5. $x^2 - 6x + 9 = 0$ **6.** $\frac{1}{2y} = \frac{y}{2} + \frac{4}{3}$

Hint

Revisit concepts on factorisation if need be.

Solutions

1. $x^2 + 7x - 8 = 0$ The LHS can be factorised.

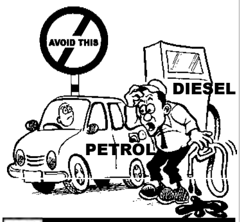
$(x + 8)(x - 1) = 0$

$a \times b$

$\therefore$ either $x + 8 = 0$ or $x - 1 = 0$ Solve each equation separately.

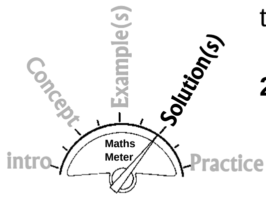
then $x = -8$ or $x = 1$

Note that the values of x, (-8 or $+1$) are often called the **roots** of the equation.



Common Errors

$3x^2 - 5x = 3$
$x(2x - 5)x = 3$ Wrong!



2. $2x^2 - 5x = 3$ Transfer 3 to the left to obtain the quadratic equation format.

$2x^2 - 5x - 3 = 0$

$(2x + 1)(x - 3) = 0$ Then factorise the LHS.

$\therefore$ either $2x + 1 = 0$ or $x - 3 = 0$

$x = -\frac{1}{2}$ or 3

3. $4x^2 = 6x$ Form a quadratic equation by grouping all terms on one side or transferring all terms to one side.

$4x^2 - 6x = 0$ Factorise the LHS.

$2x(2x - 3) = 0$

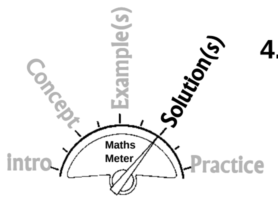
$\therefore$ $2x = 0$ or $2x - 3 = 0$

$x = 0$ or $1\frac{1}{2}$

Common Errors

Dividing both sides by x to $4x = 6$
$x = 1\frac{1}{2}$

One root is lost by so doing.



4. $9x^2 - 4 = 0$ LHS is a difference of 2 squares.

$(3x + 2)(3x - 2) = 0$

$3x + 2 = 0$ or $3x - 2 = 0$

$x = \frac{-2}{3}$ or $\frac{2}{3}$ Which can be shortened to $x = \pm\frac{2}{3}$ and said as 'plus or minus $\frac{2}{3}$.

Tip

Never ignore the 'twice' in this case.

5. $x^2 - 6x + 9 = 0$

$(x - 3)(x - 3) = 0$

$x - 3 = 0$ twice

$\therefore$ $x = 3$ twice

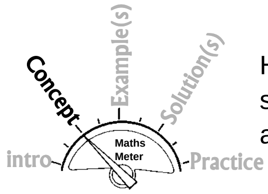
6. $\frac{1}{2y} = \frac{y}{2} + \frac{4}{3}$ Use the (LCM) to clear the fractions.

$3 = 3y^2 + 8y$

$3y^2 + 8y - 3 = 0$

$(3y - 1)(y + 3) = 0$

$\therefore$ $y = \frac{1}{3}$ or -3



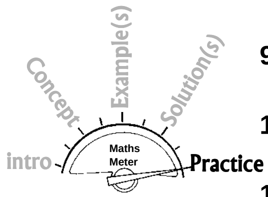
How do you know factorisation is possible? Generally, if the question simply asks you to solve and does not ask for approximations to the answer, then factorisation is possible.

Tip

– Make one side zero always.
– Keep the 'equals zero' at all stages.
– Check if factorisation is correct before proceeding.

Solve these quadratic equations.

1.	$x^2 + 5x + 6 = 0$	**2.**	$x^2 - 9 = 0$
3.	$3x^2 + 5x + 2 = 0$	**4.**	$8x^2 - 10x = 0$
5.	$6x^2 = 8x$	**6.**	$5x^2 = 11x - 2$
7.	$25 - 4x^2 = 0$	**8.**	$5x + 3 = 2x^2$
9.	$4x^2 - 12x + 9 = 0$	**10.**	$9x^2 + 6x + 1 = 0$
11.	$8x^2 - x = 7$	**12.**	$6x^2 = x + 12$
13.	$10x^2 - 27x - 9 = 0$	**14.**	$14x^2 + 5x - 24 = 0$
15.	$\frac{2}{x} - 3 = 5x$	**16.**	$\frac{y}{3} + \frac{4-3y}{y} = \frac{1}{y}$
17.	$\frac{x+4}{10} = \frac{x+1}{3x}$	**18.**	$\frac{1}{2y-4} + \frac{1}{2y} = \frac{2}{5}$



E. THE QUADRATIC FORMULA

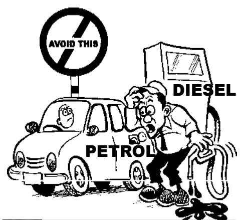
Some quadratic expressions do not factorise at all. Others are not easy to factorise! The quadratic formula helps us to solve such quadratic equations.

If $ax^2 + bx + c = 0$, the quadratic formula is

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Tip

The a, b and c in the $ax^2 + bx + c = 0$ are not interchangeable.



Common Errors

(i) $x = \frac{b \pm \sqrt{b^2 - 4ac}}{2}$
the minus before the letter b and the a in the denominator are usually omitted.

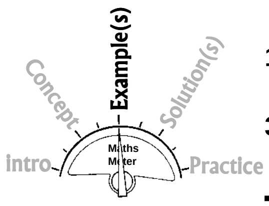
(ii) $x = -b \pm \frac{\sqrt{b^2 - 4ac}}{2a}$
The division line is under the root sign only.

Consider the following examples:

Solve these equations.

1. $x^2 + 7x - 8 = 0$ **2.** $2x^2 - 3x = 0$

3. $5 - x - 2x^2 = 0$



Solutions

1. $x^2 + 7x - 8 = 0$ Compare with $ax^2 + bx + c = 0$.
$a = 1$, $b = 7$ and $c = -8$

Substituting in the formula

$$\therefore\ x = \frac{-(7) \pm \sqrt{(7)^2 - 4(1)(-8)}}{2(1)}$$

$$= \frac{-7 \pm \sqrt{49 + 32}}{2}$$

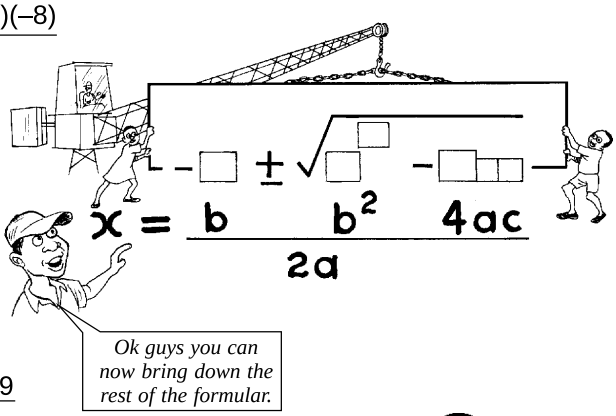
$$= \frac{-7 \pm \sqrt{81}}{2}$$

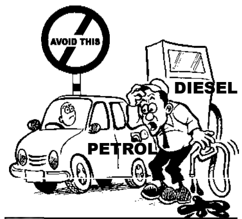
$$= \frac{-7 \pm 9}{2}$$

$$= \frac{-7 + 9}{2} \quad \text{or} \quad \frac{-7 - 9}{2}$$

$$= 1 \ \text{ or } -8$$

Compare with example (1) on page 69.





Common Errors

Substituting a negative, –b is simply taken as –3 instead of –(–3).

2. $2x^2 - 3x = 0$

$a = 2$, $b = -3$, $c = 0$

substituting in the formula

$$\therefore\ x = \frac{-(-3) \pm \sqrt{(-3)^2 - 4(2)(0)}}{2(2)}$$

$$= \frac{3 \pm \sqrt{9}}{4}$$

$$= \frac{3 + 3}{4} \quad \text{or} \quad \frac{3 - 3}{4}$$

$$= \frac{6}{4} \quad \text{or} \quad \frac{0}{4}$$

$$= 1\tfrac{1}{2} \ \text{ or } \ 0$$

Notice that the two examples above can be solved by the factorisation method as well.

3. $5 - x - 2x^2 = 0$

$a = -2$, $b = -1$, $c = 5$

Substituting into the formula

$$x = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(-2)(5)}}{2(-2)}$$

$$= \frac{1 \pm \sqrt{1 + 40}}{-4}$$

$$= \frac{1 \pm \sqrt{41}}{-4}$$

$$= \frac{1 + 6{,}403}{-4} \quad \text{or} \quad \frac{1 - 6{,}403}{-4}$$

$$= -1{,}85 \quad \text{or} \quad 1{,}35 \ \text{(to 3 s.f)}$$

Use the quadratic formula to solve the following equations. Give your answer to 2 decimal places where necessary.

1. $x^2 + 8x + 7 = 0$

2. $x^2 - 7x = 0$

3. $x^2 - 6x = -5$

4. $x^2 + 7x - 5 = 0$

5. $2x^2 - 8x - 7 = 0$

6. $5x^2 + 9x - 1 = 0$

7. $3x^2 + 2x - 6 = 0$

8. $4x^2 - x - 3 = 0$

9. $4x^2 + x - 3 = 0$

10. $9x^2 - 1 = 0$

11. $7x^2 + 6x + 1 = 0$

12. $3x^2 + x - 2 = 0$

13. $7 - 5x^2 = 0$

14. $3x^2 - 5 = 0$

15. $3 - 2x - x^2 = 0$

16. $5 + x - 2x^2 = 0$

The perfect square case

You should be familiar with the set of perfect numbers/squares i.e. 1, 4, 9, 16, 25, ...

Expressions like x^6, $(x + 2)^2$ are also perfect squares since x^6 can be expressed as $x^3 \times x^3$ and $(x + 2)^2$ can be expressed as $(x + 2)(x + 2)$. Some quadratic equations contain a perfect square expression on one side.

Consider the following examples:

Solve:

4. $(x + 3)^2 = 4$

5. $(2x - 3)^2 = 5$

6. $(x + 1)^2 = \frac{1}{9}$

Solutions

4. $(x + 3)^2 = 4$

It is not necessary to expand the LHS. Instead, take square roots of both sides since the LHS is a perfect square.

$x + 3 = \pm\sqrt{4}$ The square root of $(x + 3)^2$ is $x + 3$.

The $\pm$ is introduced on the RHS to obtain **two** roots of the quadratic equation.

$x + 3 = \pm 2$

$x = -3 + 2$ or $-3 - 2$

$= -1$ or -5

Tip

The side with the unknown is the one which should be the perfect square for this approach to work.

5. $(2x-3)^2 = 5$

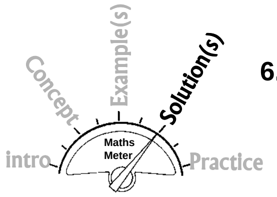
$2x - 3 = \pm\sqrt{5}$ Use tables or a calculator to simplify $\sqrt{5}$.

$2x = 3 \pm 2{,}236$

$\therefore\ x = \dfrac{5{,}236}{2}$ or $\dfrac{0{,}764}{2}$

$= 2{,}618$ or $0{,}382$

$= 2{,}62$ or $0{,}38$ (to 2 d.p)

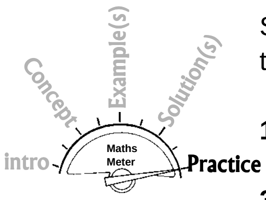


6. $(x+1)^2 = \frac{1}{9}$

$x + 1 = \pm\sqrt{\frac{1}{9}}$

$x = -1 \pm \frac{1}{3}$

$x = -\frac{2}{3}$ or $-1\frac{1}{3}$



Solve by taking the square roots on each side. Give the answers to 2 decimal places where necessary.

1. $(x+2)^2 = 1$

2. $(x-3)^2 = 4$

3. $(2x-1)^2 = 3$

4. $(7x-6)^2 = 121$

5. $(2x+5)^2 = 16$

6. $(5x+7)^2 = 25$

7. $(3x-5)^2 = 11$

8. $(x+6)^2 = 49$

9. $(6x-1)^2 = 6$

10. $(x-\frac{1}{7})^2 = \frac{9}{49}$

11. $(x-2)^2 = 1\frac{7}{9}$

12. $(x-\frac{2}{5})^2 = \frac{1}{25}$

13. $(2x-\frac{1}{9})^2 = \frac{4}{81}$

14. $3(3x-2)^2 = 27$

15. $2(2x+5)^2 = 72$

16. $5(9x+1)^2 = 10$

Tip

For the graphical method of solving quadratic equations, see the chapter on Functional Graphs in this book.

SUMMARY

1. Linear Equations have no unknowns with a power higher than 1 e.g. $5x + 3 = 7$ and $6y + x = 0$.

2. We use the 'change side change sign' rule when solving equations.

3. Common methods of solving linear simultaneous equations are:

Substitution method	**Elimination method**
⅄ Target a variable and isolate it. ⅄ Substitute the expression of the variable into the other equation and hence solve the equation. ⅄ Substitute value into the other rearranged equation to find the other unknown.	⅄ Target a variable and make sure coefficients of the targeted variable are the same in both equations. ⅄ Same sign – **subtract** Different signs – **add** and solve for remaining unknown. ⅄ Substitute in any of the two equations to find the unknown.

4. Always make a quick check on the values/solutions by substituting into one or both equations.

5. Quadratic Equations have zero on one side. Make sure they are in that form before proceeding.

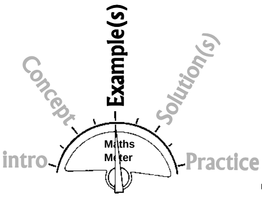
6. Use Factorisation and/or Quadratic Formula to solve quadratic equations.
Remember if $ax^2 + bx + c = 0$ then $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
is the quadratic formula.

7. Take the square roots for each side of a perfect square. Do not forget the $\pm$ signs on the other side.

EXAM PRACTICE 3

Tip

Be sure you understand the meaning of instructions like factorise, simplify and solve.



Consider the following examples

1. a) Given that $x + 2y = 5$ and $x^2 - 4y^2 = -15$, find the numerical value of $x - 2y$.

b) Hence or otherwise, find the numerical values of x and y.

2. Solve a) $7y^2 = 4y$

b) $3x^2 - 7x - 2 = 0$ giving the answer to 2 decimal places

c) $\frac{2x}{2} + 1 = \frac{x}{10}$

Solutions

1. a) Solving this question depends on the ability to see the difference of two squares in $x^2 - 4y^2$.

i.e. $x^2 - 4y^2 = (x^2 - (2y)^2$

$(x - 2y)(x + 2y) = -15$

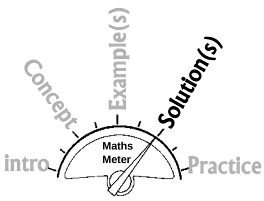
But $x + 2y = 5$

Replace $x + 2y$ with 5

So $5(x - 2y) = -15$

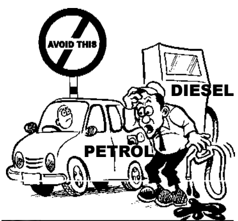
$\therefore\ x - 2y = \frac{-15}{5}$

$x - 2y = -3$



b) Using the elimination method on the two simultaneous equations:

$x - 2y = -3$...(i)	substituting for x to find y.
$\underline{x + 2y = 5}$ ...(ii)	$x - 2y = -3$
$2x \quad = 2$	$(1) - 2y = -3$
$\therefore\ x = 1$	$-2y = -4$
$y = 2$	$y = 2$



Common Errors

➤ Dividing through by y to $7y - 4 = 0$ hence losing one value of y.

2. a) $7y^2 = 4y$ Form a quadratic equation by transferrin $4y$ to the LHS.

$7y^2 - 4y = 0$

$y(7y - 4) = 0$

$\therefore\quad y = 0$ or $7y - 4 = 0$

$y = 0$ or $\frac{4}{7}$

Tip

The 2 d.p. given is usually a guide implying the expression cannot be factorised.

b) $3x^2 - 7x - 2 = 0$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2ac}$$

$$x = \frac{-(-7) \pm \sqrt{(-7)^2 - 4(3)(-2)}}{2(3)}$$

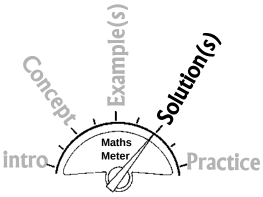
$$= \frac{7 \pm \sqrt{(49 + 24}}{6}$$

$$= \frac{7 \pm \sqrt{73}}{6}$$

$= 2{,}59$ or $-0{,}26$

Common Errors

In b(ii):
a) $-b$ taken as -7
b) Short division line.



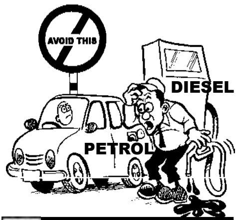
c) $\frac{2x}{5} + 1 = \frac{x}{10}$

Multiply each term by 10

$4x + 10 = x$

$3x = -10$

$x = -3\frac{1}{3}$



Common Errors

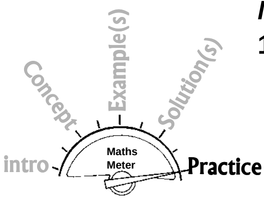
In 2c)

a) The 1 is not multiplied by the LCM 10.

b) At $3x = -10$

Common $\leftrightarrow$ wrong answers are $x = -3\frac{1}{10}$ and $x = \frac{-3}{10}$

Remember we are to divide by the coefficient of x.



Now do the following:

1. Solve:

a) $5 - (x - 3) - 2x = -1$

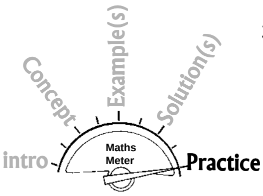
b) $3y - 2(2y - 7) = 2(3 + y) - 4$

c) $4 - \frac{2}{3}(x - 7) = \frac{4}{5}$

2. Solve:

a) $\dfrac{9}{2 - 3d} = \dfrac{4d}{18 - 27d}$ b) $\dfrac{x}{x - 3} = \dfrac{3}{x - 3} + 4$

c) $\dfrac{3}{6u + 18u} - \dfrac{7}{u^2 - 5u - 24} = 0$



3. Solve:

a) $\dfrac{y}{3} = \dfrac{12}{y}$ b) $3x^2 = 147$

c) $(y - \frac{1}{2})^2 = \frac{1}{4}$

4. Solve:

a) $8y^3 - 32y = 0$ b) $3x^2 - x - 2 = 0$

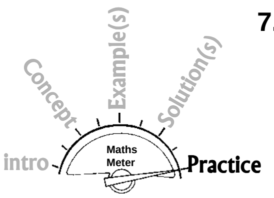
5. Solve, giving answers to 2 significant figures

a) $x^2 - 4x + 2 = 0$ b) $5 - 2x - x^2$

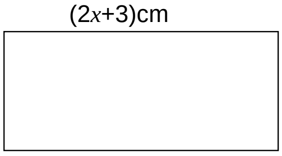
6. Solve the following simultaneous equations.

a) $2h + k = 0$
$2h + 2 = -3k$

b) $0,3x - 1,3y = 1,5$
$0,1x + 0,4y = 1$



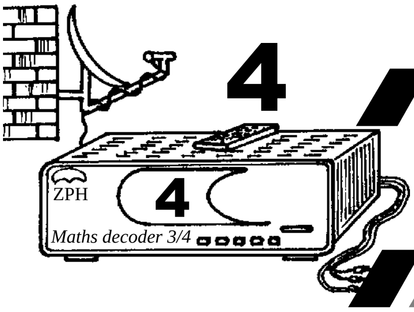
7.



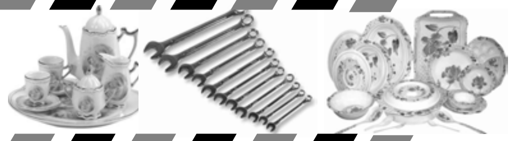
The diagram shows a rectangle measuring $(2x+3)$cm by $(x+2)$cm. Given that the area of the rectangle is 78cm^2,

a) form an equation for x and expres it in the form $ax^2 + bx + c = 0$ where a, b and c are integers.

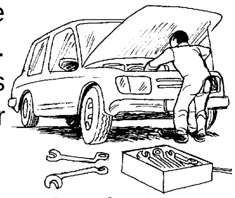
b) solve this equation hence find the length of the rectangle.



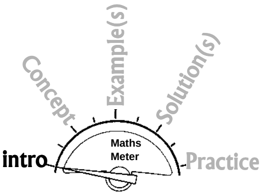
4 Sets



A set is a collection of objects or items with some common, defined characteristics. The characteristics must be able to be clearly defined and not subjective. For example, a set of seasons of the year is clearly described as {summer; autumn; winter; spring}. However, a set of beautiful ladies can not be clearly defined as beauty is subjective. Some common sets include: a tea set, a dinner set, a cutlery set etc.



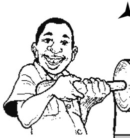
A mechanic uses a set of spanners



The following characteristics are associated with sets:

- they are all drawn from a universal set, a set which includes all the possible members.
- they may or may not have members called elements.
- sets can be listed or described as we shall discover in this chapter.
- symbols may be used to describe sets.
- any set may be defined by naming it e.g {set of even numbers}.

A set of cell phones



Syllabus Expectations

By the end of this chapter, students should be able to:

1. define types of sets by listing and describing.
2. use set builder notation to define a set.
3. describe sets using set notation.
4. combine sets and use Venn diagrams to describe them.
5. use sets and Venn diagrams to solve problems involving no more than three sets and the universal set.
6. solve simple problems using sets.

ASSUMED KNOWLEDGE

In order to tackle work in this chapter it is assumed that students are able to:

- apply inequality symbols.
- solve simple and simultaneous equations.

A. DEFINING SETS

A set may be defined in different ways as follows:

1. *by verbal description* e.g
Set A = {days of the week}.

This method uses words to fully describe situations. However where logic is concerned, this can be a long-winded process and also ambiguous.

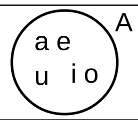
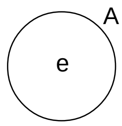
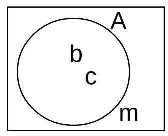
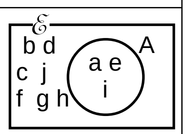
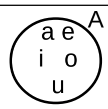
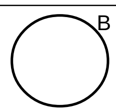
2. *by listing all the elements* (members of the sets) e.g
Set A = {Monday; Tuesday; Wednesday; Thursday; Friday; Saturday; Sunday}.
This is fine for small sets but rather cumbersome for bigger sets.

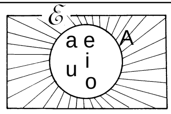
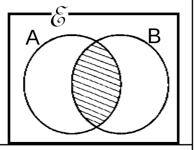
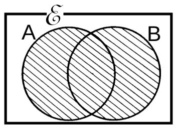
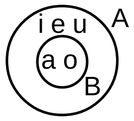
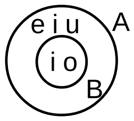
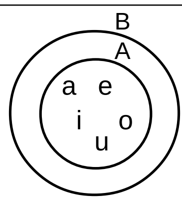
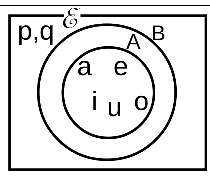
3. *by abstract mathematical symbols*
e.g Set A = {x:x is a weekday}.
This is shorthand.
Symbolism is an important tool in set description.

4. *by Venn diagrams* – This is a pictorial or diagrammatic representation of sets. It is a very useful tool. However, it displays elements or quantities not graphics (pictures).

Let us start by looking at common symbols used in sets in Table 4.1.

Table 4.1

Symbol	**Meaning**	**Illustration of how its used**	**Diagram**
{ . . .}	the set	A = {vowels}	
∈	is an element or member of	e∈A	
∉	is not an element of	m∉A	
$\mathscr{E}$	the universal set	$\mathscr{E}$ = {First ten Letters of the alphabet}	
n(A)	Number of elements in set A	n(A) = 5	
Ø or { }	null (empty) set	B = Ø	

A^{I}	complement of or not in A	$A^{I} = \varnothing$	
$\cap$	intersection or common elements or common area	$A \cap B$	
$\cup$	union of or elements in the given sets	$A \cup B$	
$\subset$	is a proper subset of	$B \subset A$	
$\supset$	A contains B	$A \supset B$	
$\subseteq$ $\supseteq$	Super subset or improper subset B⊇A means B has A as its super subset.	If every element of a set B is also an element of Set A, then A is a super subset of B and thus we deduce (i) $A \subseteq B$ (ii) hence $B \subseteq B$ (iii) also $\varnothing \subseteq B$	
$\not\subseteq$ or $\not\subset$	is not a subset of	$\{p,q\} \not\subset$ vowels.	
A ={x: $3 \leqslant x \leqslant 11$} x is an integer	A is the set of elements x such that x is an integer from 3 to 11 inclusive.	A = {x:x is an integer $3 \leqslant x \leqslant 11$	A = {3;4;5;6;7 8;9;10;11}

Hint

Revisit the chapter on inequalities and familiarise yourself with the following signs $\leqslant$, $<$, $>$, $\geqslant$ Remember x: means x is such that x is . . .

Hint

The Ø (null) set is a subset of any set and any set is a subset of itself.

Consider the following sets (Fig. 4.1)

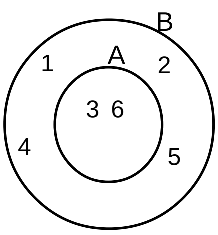


Fig. 4.1

From the set diagram the following can be deduced:

$A \subset B = \{3; 6\}$

$\varnothing \subseteq A$

$B \subseteq B = \{1; 2; 3; 4; 5; 6\}$

Also note that if $A \subseteq B$, then $B \supseteq A$.

1. Define the following terms as applied in mathematics:
a) set. b) subset. c) complement of a set.
d) set members. e) Venn diagrams.

2. Given the following:
$\mathscr{E}$ = {set of all small letters of the alphabet}.
A = {first 21 small letters of the alphabet}.
V = {set of vowels}.
state whether the following are true or false:
a) $V \subseteq A$
b) $n(A) = 21$
c) $n(V^{\prime}) = 16$
d) $V \cap A = \{a; e; i; o; u\}$
e) $V \supset A$
f) $w \notin \mathscr{E}$
g) $W \in A^{\prime}$
h) $V^{\prime} \cap A = \varnothing$

3. Given that A = {x:x is an integer $1 \leqslant x \leqslant 12$},
B = {prime numbers},
C = {perfect squares},
a) list members of set B.
b) list $B \cap C$.
c) represent the above information in a Venn Diagram.

4. If A = {2; 4; 6; 8; 10} B = {1; 2; 3; 4 ...15}. Find:
a) $A \cap B$ b) $n(A \cap B)$ c) $A \cup B$
d) $\{5; 7; 9\} \cap B$ e) $n(A^{\prime})$

B. TYPES OF SETS

Sets can be in diagram form. These diagrams are called **Venn diagrams**. Below are illustrations of each type using Venn diagrams.

Universal set: This is the most defining set which includes all elements under consideration. All other sets are defined from it and should naturally always be boxed within the universal set. (Fig. 4.2)

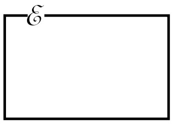
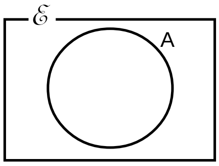
Its symbol is $\mathscr{E}$ or  in diagram form.

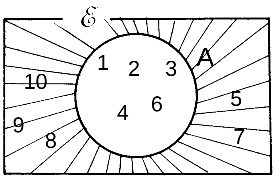
Fig. 4.2

Empty or null set: This is a set without any members (elements). (Fig 4.3)
Its symbol is
{ } or Ø or 

in diagram form.

Fig. 4.3

Complement of a set: This is a set of elements outside the defined set but within the universal set, for example, A^1 is the shaded space. (Fig 4.4)

A^{I} = 

If $\mathscr{E}$ = {1;2;3; . . .10}.
A = {1;2;3;4;6},
then A^{I} = {5;7;8;9;10}

Fig. 4.4

Subset
This is a set within a set (Fig 4.5)
$A \subset P$
$P \subset \mathscr{E}$
$A \subset \mathscr{E}$

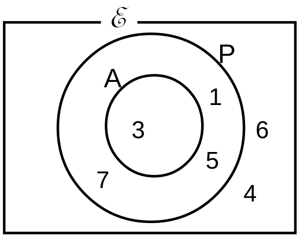


Fig. 4.5

Joint sets
These are two or more sets with some elements in common (Fig 4.6).

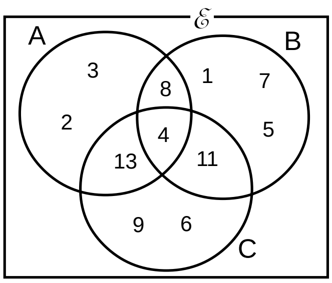
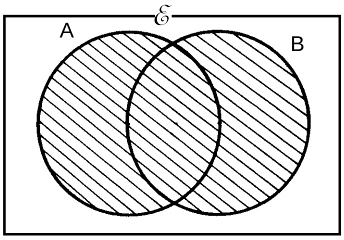


Fig. 4.6

Union sets

When two or more sets are joined all their members form the union set. (shaded space) (Fig 4.7)

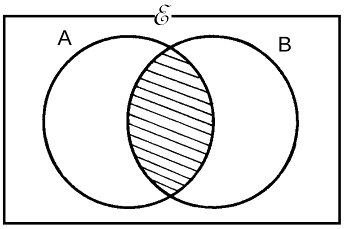


$A \cup B$

Fig. 4.7

Intersection Set

This is a set made of all elements which are common to both set A and set B (shaded space). (Fig 4.8)



$A \cap B$

Fig. 4.8

Disjoint or stand alone sets

This refers to two or more "stand alone" sets which have no elements in common i.e. $A \cap B = \varnothing$ as in (Fig 4.9).

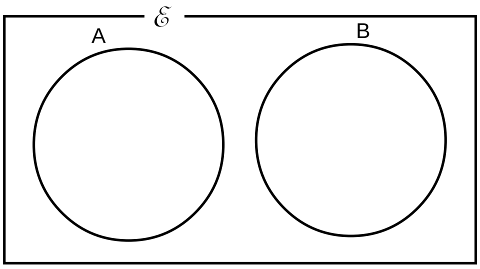


Fig. 4.9

Hint

Finite: having limits or a boundary.

*Infinite: having **no** limits or boundaries.*

Finite set: A finite set is a set which has a limited number of members when listed e.g. A = {0; 3; 6; 9 . .. . . . .60}

Infinite set: An infinite set is a set which has an endless number of members when listed e.g. A = {natural numbers}

Study the examples below carefully

1. In a Venn diagram (Fig. 4.10):

ℰ = {Students at a school}.
C = {Students who play chess}.
T = {Students who play tennis}.
V = {Students who play volleyball}.

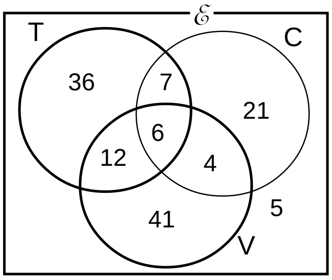


Fig. 4.10

a) How many students play volleyball?
b) How many students play both tennis and chess?
c) How many students play all the three games?
d) How many students play tennis and volleyball but not chess?
e) How many students are there at this school?

Solution

1. a) Volleyball $= 41 + 12 + 6 + 4$
$= 63$

b) $T \cap C$ $= 7 + 6$
$= 13$

c) All games $= (T \cap V) \cap C$
$= 6$

d) $T \cap V$ $= 12$

e) Total number of students $= 36 + 7 + 21 + 12 + 6 + 4 + 41 + 5$
$= 132$ students

2. List the elements of the following sets.

a) A = {x: x is a vowel}

b) B = {x: x is a multiple of 3, $1 \leqslant x \leqslant 25$ }

c) C = {(x; y): x is a cube of y, $1 \leqslant x \leqslant 70$, $1 \leqslant y \leqslant 6$}

Solution

2. a) Elements of A = a, e, i, o, u

b) Elements of B = 3, 6, 9, 12, 15, 18, 21, 24

c) Elements of C = (1;1) , (8;2), (27;3), (64;4)

1. a) Draw a Venn Diagram to represent the following:
(i) two disjoint sets.
(ii) a subset of a set.
(iii) the intersection of two sets.

b) Given the following two sets A and B.
A = {a; c; f; k}   B = {a; b; f; v; m}
Write down the two sets resulting from the following two operations:
(i) $A \cup B$
(ii) $A \cap B$

c) Use examples to explain the meaning of the following set expressions:
(i) $A \subset (A \cup B)$
(ii) $\{\ \} \subseteq A$

2. A and B are sets such that:
A = {1; 4; 5; 7; 8; 9; 10; 11;16}.
B = {2; 3; 5; 8}.
State whether the following statements are true or false.
a) $3 \in A$
b) $8 \in B$
c) $A \cap B$ = {5; 8}
d) $A \cup B$ = {1; 2; 3; 4; 7; 9; 10; 11; 16}
e) $B \subset A$
f) $A \not\subset B$

3. Given that A = {1; 4; 6; 7; 8} B = {1; 2; 5; 7; 9; 10} C ={1; 3;11}
D = {15; 16}. List the following sets:
a) $A \cup B$
b) $A \cap B$
c) $A \cap B \cap C$
d) $A \cap D$

4. a) Define a set.
b) Given that: A = {donkey; cow; horse}.
B = {bull; ox; dog; goat}.
C = {sheep; pig; rabbit; calf}.
D = {duck; cock; hen}.
E = {horse; cow; donkey; bull; ox; dog; goat}.

In terms of set notation show the relationship between:
(i) donkey and A.
(ii) donkey and B.

(iii) A and E.
(iv) C and E.
(v) {bull; ox} and B.
(vi) {bull; ox} and C.

5. Given that: P = {Maths; English; French; Science; Geography}

Q = {Physics; Chemistry; Biology}.

A = {English; French; Shona; Ndebele; Zulu}.

V = {Geography; Maths; History}.

State whether the following are true or false.
a) $V \subset P$
b) $V \not\subset P$
c) $(Q \cup A) = P \cup V$
d) Chemistry $\in Q$
e) Maths $\in V$
f) $\emptyset \cap P$ = English

6. Given that:
$\mathscr{E}$ = {x: 5 < x < 25, x integer}.
P = {6; 12; 18; 24}.
Q = {8; 12; 16; 20; 24}.
R = {10; 15; 20}.

a) Draw a Venn diagram to show all these sets in their correct relationships and showing all their elements.
b) List the elements:
$P \cap Q$
c) List the elements:
$Q \cup R$

7. Draw a large Venn diagram to show the relationship between the following sets:
$\mathscr{E}$ = {x: x is a whole number between 1 and 20 inclusive}.
A = {1; 3; 7; 12; 16; 20}.
B = {x: x is exactly divisible by 3}.
C = {x: x^2 is between 8 and 90}.

C. SHADING SETS IN VENN DIAGRAMS

At the beginning of this chapter we learnt that sets may be described or represented:
- by verbal description.
- by set notation.
- pictorially by Venn diagrams.

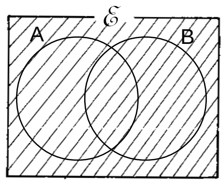
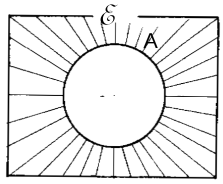
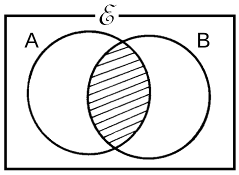
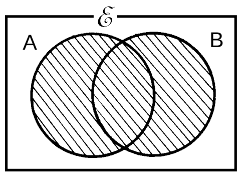
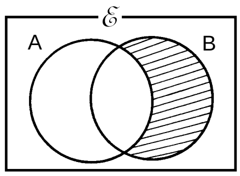
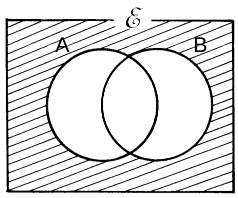
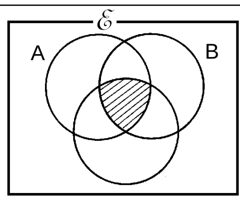
At this stage, it is important to appreciate the link between the three ways of defining sets. Reading across table 4.2 below will help you to understand the link between the three ways of describing sets.

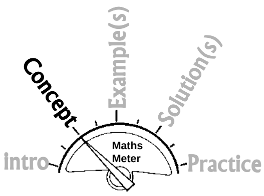
Hint

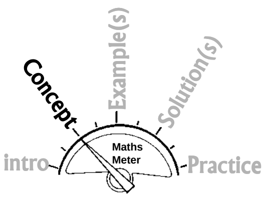
A key concept to be mastered in this part of the chapter is the ability to deduce the other way of representing sets once you are given one form e.g. from the shaded Venn diagram one should be able to write the set notation and vice versa.

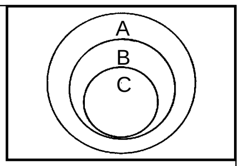
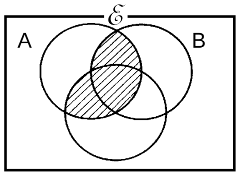
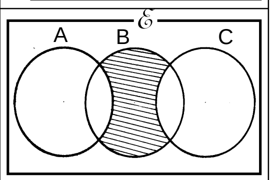
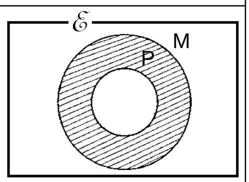
Consider the Table 4.2 below and try to **complete** the gaps left at the bottom. (Do not write in the book!)

Table 4.2 Set concepts

Verbal Description of the set	**Set Notation to represent the set**	**Pictorial (Diagrammatic) Representation of set by Shading**
Universal set	$\mathcal{E}$	
Complement of set A	A^{I}	
The intersection of set A and set B	$A \cap B$	
The union of set A and set B	$A \cup B$	
Set A complement intersection B	$A^{I} \cap B$	
The complement of $(A \cup B)$	$(A \cup B)^{I}$	
The intersection of sets A, B and C	$A \cap B \cap C$	





C is a subset of both set B and A	$C \subset (B \subset A)$	
–	$A \cup B'$	–
–	$(A \cap B) \cup (B \cap C)$	
–	$(A' \cap B') \cap B$	
–	–	

Hint

For problems involving symbols the building notations are: A', $A \cap B$ and $A \cup B$. All other problems are built by adding the complement symbol to these sets e.g.
$A' \cap B'$
$(A' \cup B)'$
$(A \cap B') \cup C'$ extra

Consider the following example

1.

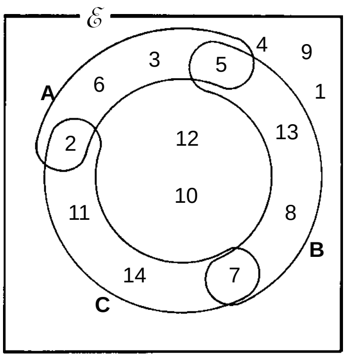
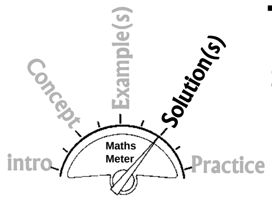


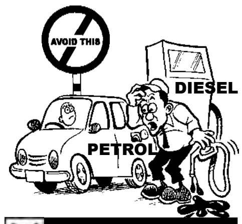
Fig. 4.11

In the Venn diagram illustrated in figure 4.11, three sets A, B and C intersect and are found within the universal set.

a) Describe the universal set using set notation.
b) List members of $(A \cup B \cup C)'$.
c) Find $n(A \cap B)$.
d) Find $n(A \cap C)'$.
e) Describe the set given by $(A \cap B) \cap C$.



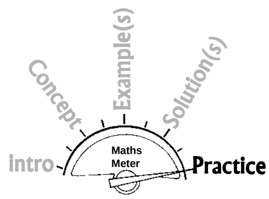
Solution



Common Error

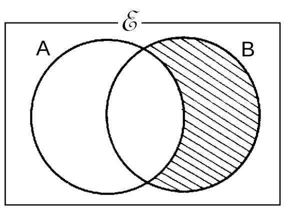
A common error is to list elements when required to give the number of elements in a set e.g n(A) is not the same as listing elements in Set A.

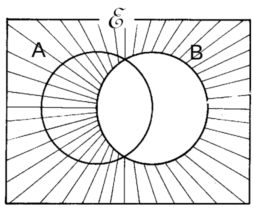
1. a) $\mathscr{E}$ = {x: x is an integer 1⩽ x⩽ 14}.
b) $(A \cup B \cup C)^{I}$ = {12; 10; 4; 9; 1}.
c) $n(A \cap B)$ = 1.
d) $n(A \cap B)^{I}$ = 13.
e) null set = Ø.

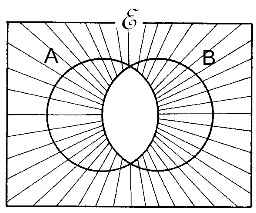


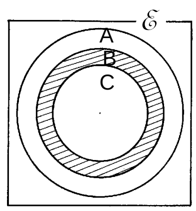
1. $\mathscr{E}$ = {x: x is an integer 1 ⩽x <10}.
A = {2; 4; 6; 8}.
B = {1; 2; 3; 4; 5; 6}.
C = {4; 5; 6; 9}.
List the following sets:
a) $A \cap B$
b) $(A \cap B)^{I}$
c) $(A \cap C) \cup B$
d) $(A \cup B) \cap C$
e) $(A \cap C) \cup (B \cap C)$
f) $(A \cup B)^{I} \cap C$
g) $(A \cap B)^{I} \cap C$

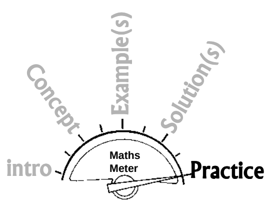
2. Write the set notations for the shaded part of the Venn diagrams in Fig. 4.12 (a – g).

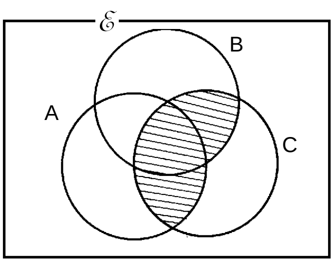
a)


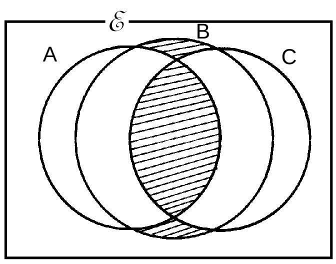
b)


c)


d)




e)


f)


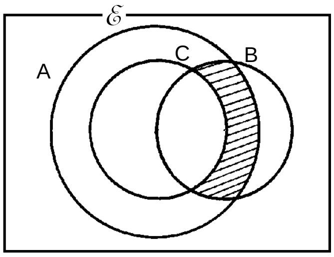
g) 

Fig. 4.12

3. Given that for the universal set A:
A = {x: $6 \leqslant x \leqslant 26$, x is an integer}.
B = {x: x is a multiple of 4}.
C = {x: x is an odd number}.

List the elements of the following sets:
a) B
b) C
c) $(B \cup C)'$
d) $(B \cap C)'$
e) $B \cup C$
f) $B \cap C$
g) $B' \cap C'$
h) $B' \cup C'$

4. $\mathscr{E}$ = {x: $1 \leqslant x \leqslant 15$, x is an integer}.
A = {x: x is a multiple of 4}.
B = {x: $x \leqslant 10$}.

a) Draw a Venn diagram to illustrate the above sets.
b) Write down the set resulting from $(A \cap B)'$.
c) Write $n(A \cap B)$.
d) Write $n(A)'$.

5. a) Define the term universal set.
b) Define the term complement set.
c) Explain what is meant by disjoint sets.
d) A and B are two disjoint sets. Show this using set notation.

6. $\mathscr{E}$ = { all the whole numbers between 5 and 20 inclusive}
A = {multiples of 5}.
B = {even numbers}.

List the following sets:
a) $A \cup B$
b) $A \cap B$
c) A'
d) $(A \cup B)'$

7.

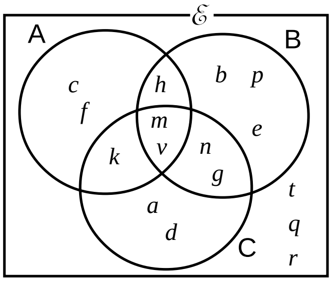


Fig. 4.13

Using the Venn diagram (Fig 4.13, list the elements of:

a) C
b) $C \cap B$
c) $A^I \cap B$
d) $A^I \cap B^I \cap C^I$
e) $A^I \cup B^I \cup C^I$
f) $(A \cap B)^I \cup (A \cup B)^I$
g) Deduce (i) n(C)
(ii) $n(B \cap A)$
(iii) $A^I\, n(B \cup C)$

D. WORD PROBLEMS IN SETS

Study the following example carefully:

1. In a class of 40 boys, 25 play soccer and 15 play basketball whilst 8 play neither.
Find the number of boys who play both soccer and basketball.

Hint

It is good practice to first list the sets using set notation. Also let the number you are trying to find be x and place x in the correct region on the Venn diagram.

Solution

ℰ = {Boys in the class}.
S = {Boys who play soccer}.
B = {Boys who play basketball}.

x = Number of boys who play both soccer and basketball.

Tip

Represent the information on a Venn diagram, as in Fig 4.14, placing the number of boys in each portion of the universal set correctly.

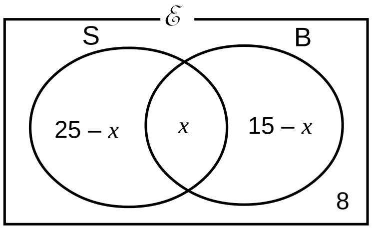


Fig. 4.14

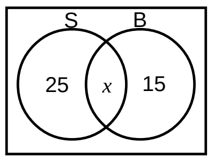
n(ℰ) = 40
$\therefore\ (25 - x) + (x) + (15 - x) + 8 = 40$

$(25 - x) + (x) + (15 - x) + 8 = 40$ remove brackets at this stage
$48 - x = 40$
$x = 48 - 40$
$x = 8$

$\therefore$ 8 boys play both soccer and basketball.

Common Error

Students tend to leave out the number of those who are not participating, in this case 8. e.g.



The diagram shows a common mistake.

1.

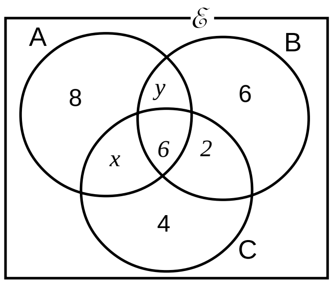


Fig. 4.15

The numbers in the Venn diagram (Fig 4.15) represent the number of elements in the sets A, B, and C.
The total number of elements in all the three sets is 36.
The number of elements in Set A is double the number in Set C.

a) Use the given information to produce two simultaneous equations using x and y.
b) Solve these equations.

Write down the number of elements in the following sets:
c) $A \cup B$ d) $B \cap C$ e) $C \cap (A \cap B)$

2. The results from a particular centre for Unit 1, Unit 2 and Unit 3 of the University examinations were as follows:
82 students attempted all the exams.
39 passed all three Units.
52 passed BOTH Unit 1 and Unit 2.
45 passed BOTH Unit 2 and Unit 3.
47 passed BOTH Unit 1 and Unit 3.
Nobody passed Unit 2 alone, but 3 failed all three Units.

a) By using x to denote the number of students who passed Unit 1 only, and y to denote the number who passed Unit 3 only, draw a Venn Diagram to show the numbers in each set.
b) If 8 more passed Unit 1 than passed Unit 3, derive a pair of simultaneous equations in x and y.
c) Solve these equations.

3. The results obtained from a particular 'A' level Centre for Physics, Chemistry and Mathematics in the Cambridge examinations were as follows:
151 students attempted all the exams.
37 passed all three subjects.
52 passed BOTH Physics and Chemistry.
49 passed BOTH Chemistry and Mathematics.

62 passed BOTH Physics and Maths.
17 passed Maths alone and 5 failed all three subjects.

a) By using x to denote the number of students who passed Physics only and y to denote the number of students who passed Chemistry only, draw a Venn diagram to show the numbers in each set.

b) If 15 more passed Physics than passed Chemistry, derive a pair of simultaneous equations in x and y.

c) Solve these equations.

d) How many students passed either Physics or Chemistry, but failed Maths?

4. $\mathcal{E}$ = $\{x$: $10 \leqslant x \leqslant 45$, x is an integer$\}$.
X = $\{x$: x is a prime number$\}$.
Y = $\{x$: x is even$\}$.
M = $\{x$: x is a multiple of 5$\}$.

a) List all the elements of set X.

b) Find (i) n(M),
(ii) n(Y $\cup$ M).

c) Express in set notation, using X,Y and/or M, the set {15; 25; 35}.

5. 37 tourists ordered a drink with at least one of the following flavours:
lemon
orange
pineapple
In the Venn diagram (Fig 4.16) the expressions represent the number of tourists in each subset.

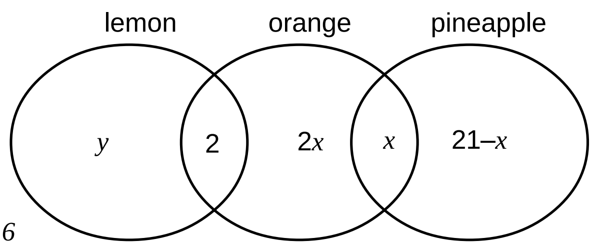


Fig. 4.16

a) Write down the total number of tourists who chose pineapple.

b) Express y in terms of x in its simplest form.

c) The number of tourists who chose pineapple only was three more than the number who chose orange only.

(i) Write down an equation in x.

(ii) Solve this equation.

(iii) Determine the number of pupils who had orange only.

6. A survey conducted amongst 80 students to determine which of three subjects, A, B or C they were studying showed that:
18 were studying none of the subjects.
5 were studying all three subjects.
14 were studying only B.
6 were studying only C.
17 were studying A and C.
12 were studying A and B.

a) Draw a Venn diagram with all sets enumerated as far as possible. Label in any order, the two subsets which cannot be enumerated as x and y.

b) If the number of students studying A is the same as the number studying B, write down a pair of simultaneous equations in x and y.

c) Solve these equations to find x and y.

SUMMARY

1. A set is a collection of objects with some common defining characteristics.

2. The following symbols are used in set notation.

Table 4.3

Symbol	Meaning
$A \cap B$	The intersection of the sets A and B.
$A \cup B$	The union of the sets A and B.
$A \subseteq B$	A is a super subset of B.
$A \subset B$	A is a proper subset of B.
$A \not\subseteq B$	A is not a subset of B.
$A \not\subset B$	A is not a proper subset of B.
$\mathcal{E}$	Universal set.
A^1	Complement of the set A.
A = {1;2;3;4;8}	A is the set whose elements are 1, 2, 3, 4 and 8.
$6 \in A$	6 is an element of the set A.
$7 \notin A$	7 is not an element of the set A.
A = B	A and B are equal sets.
n(A) = n(B)	The sets A and B have an equal number of elements. They are equivalent sets in terms of the number of elements. A may not be equal to B
n(A)	The number of elements in the set A.
Ø or { }	The empty set or null set.

3. Every set is a subset of itself and an empty set is a subset of every set.

4. Sets may be defined in many ways, for example, by verbal description, by listing, using Venn diagrams or by using symbols.

5. There are many types of sets, for example;

a) universal set
b) empty (null) set
c) complement set
d) joint sets (Venn diagrams)
e) disjoint sets (Venn diagrams)
f) finite set
g) infinite set
h) subset

6. Word problems can be solved using set concepts.

EXAM PRACTICE 4

The following example will help highlight key stages to take note of.

1. Of a group of tourists, 36 speak English,
30 speak French and 20 speak Zulu,
16 speak English only,
15 speak French only and
8 speak English and Zulu.
If 12 speak English and French and 6 speak French and Zulu,

a) how many speak: (i) Zulu only?
(ii) all three languages?

b) How many tourists were in the group?

Solutions

1. Let E = {Those who speak English}.
F = {Those who speak French}.
Z = {Those who speak Zulu}.

Also, let x be the number of tourists who speak all three languages.

Below is a Venn diagram (Fig 4.17) representing the information given above.

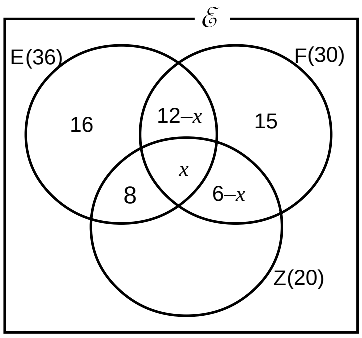


Fig. 4.17

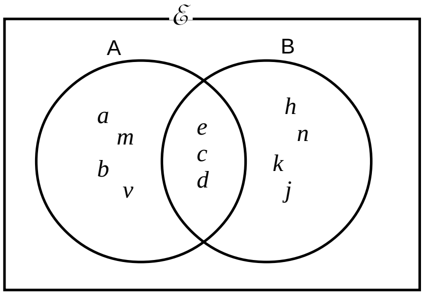
a) (i) Zulu only $= 20 - (8 + x + 6 - x)$
$= 20 - 8 - x - 6 + x$
$= 20 - 14$
$= 6$

(ii) n(F) = 30
$\therefore\ 12 - x + x + 6 - x + 15 = 30$
$33 - x = 30$
$x = 3$
∴ 3 speak all three languages.

b) Total in the group
$= 36 + 15 + 3 + 6$
$= 60$

Now do the following:

1. The Venn diagram below represents sets A and B.



Find a) $n(A' \cap B)$

b) $A \cup B$

c) B'

d) $A' \cap B'$

e) $n(A \cup B)'$

f) $n(A' \cap B')$

2. Say whether the following statements are true or false. You may use Venn diagrams, where necessary.

a) $A \subset B$ implies $A' \cup B' = 0$

b) $(A \cup B)' = A' \cap B'$

c) The number of elements in a set of letters appearing in the word "POLISHING", are 4.

d) The number of elements in a set of letters in the word "EXAMINATION", are 10.

e) $A \subset B$ implies $B' \subset A'$.

3. a) Given that:

$\mathscr{E}$ = {the set of workers working in a watermelon field}.

A = {the set of workers who picked more than 49 watermelons}.

B = {the set of workers who picked fewer than 51 watermelons}.

x workers picked more than 49 watermelons, $3x$ workers picked fewer than 51 watermelons, while 5 workers picked exactly 50 watermelons.

(i) Illustrate the above information clearly in a Venn diagram.

(ii) Given that the total number of workers who picked watermelons was 47, find x.

b) Two sets A and B are such that $n(A) = 9$, $n(B) = 12$ and $A \cap B = \emptyset$. Find:

(i) the smallest possible value of $n(A \cup B)$.

(ii) the largest possible value of $n(A \cup B)$.

4. $\mathscr{E} = \{x : 1 \leqslant x \leqslant 100,\ x \text{ is an integer}\}$
C $= \{x: x$ is a perfect square$\}$ and
T $= \{x: x$ is a multiple of 3$\}$

a) List the elements of C

b) Find $n(C \cap T)$.

c) Write down $n(C \cup T)$.

5. Among the 200 passengers on a plane, $(150 - x)$ are African, $(100 - x)$ are American and 20 passengers are neither African nor American.

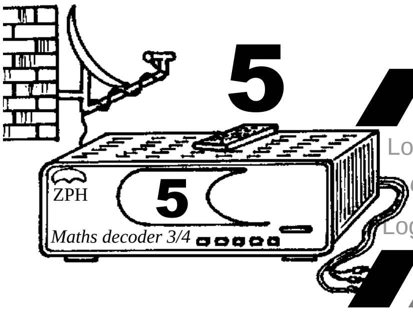
a) Represent the above scenario in a Venn diagram.
b) Deduce the value of x.
c) Find the number of passengers who are African.
d) How many passengers are America?

6. $\mathscr{E} = \{x : x$ is a positive integer less than 15$\}$.
A $= (3; 5; 7; 9; 11; 13\}$.
B $= \{2; 4; 6; 8; 14\}$.
C $= \{1; 2; 6; 9; 11; 13; 14\}$.

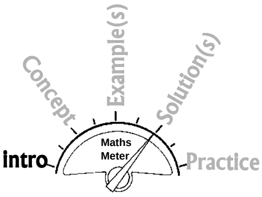
Find a) A' b) $n(B \cap C)$
c) $(A \cup C)'$ d) $(B \cap A)'$
e) B' f) $(A \cup B)' \cap C$

7. If A = {the letters in the word KUFAKOWADYA}
State whether the following are true or false, give reasons for your answers.

a) $M \in A$.
b) $K \subset A$.
c) $\{W\} \subset A$.
d) $n(A) = 11$.
e) The number of elements in a set of letters appearing in the word "KUFAKOWADYA" are 9.

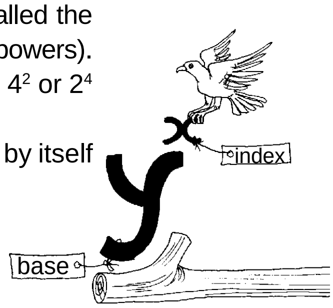


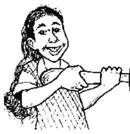
5 Indices and Logarithms



Numbers can be written in the form y^x. In this form y is called the base and x is called the index (singular of indices) or power (powers). Other examples of numbers in this form are: $8 = 2^3$, $16 = 4^2$ or 2^4 and $1000 = 10^3$.

2^3 means $2 \times 2 \times 2$ not 2×3. Thus y^x means y multiplied by itself x times.





Syllabus Expectations

By the end of this chapter, students should be able to:

1. write numbers from ordinary form to index form and vice-versa.
2. simplify expressions in index form by applying the basic laws of indices.
3. evaluate numbers given in index form.
4. solve equations in index form.
5. express statements like a = y^x in logarithm form and vice-versa.
6. use the three basic laws of logarithms to simplify expressions and/or evaluate given situations involving logarithms.

ASSUMED KNOWLEDGE

In order to tackle work in this chapter, it is assumed that students are able to:

- express basic arithmetic processes in letter symbols.
- manipulate problems involving the three basic laws of indices.
- solve simple linear equations.
- identify common factors in simple fractions.

A. LAWS OF INDICES

Law 1: $a^x \times a^y = a^{x+y}$

e.g. $a^2 \times a^4 = a^{2+4} = a^6$

i.e. $\underline{a \times a} \times \underline{a \times a \times a \times a}$

$= a^6$

i.e. When multiplying numbers in index form with the same base, add the powers and maintain the base.

Tip

The divisor is the denominator and dividend is the numerator.

Law 2: $a^x \div a^y = a^{x-y}$

e.g. $a^7 \div a^3 = a^{7-3} = a^4$

$$\frac{\cancel{a}^1 \times \cancel{a}^1 \times \cancel{a}^1 \times a \times a \times a \times a}{\cancel{a}_1 \times \cancel{a}_1 \times \cancel{a}_1}$$

$= a^4$

i.e. When dividing numbers in index form with the same base, subtract the power of the divisor from the power of the dividend and maintain the base.

Law 3: $(a^x)^y = a^{xy}$

e.g. $(a^2)^3 = a^{2 \times 3} = a^6$

$\underline{a \times a} \times \underline{a \times a} \times \underline{a \times a}$

$= a^6$

i.e. when raising a number in index form to another power, multiply the powers and maintain the base.

Zero and negative indices

The patterns below will help to clarify unity, zero and negative powers. Study them carefully.

Powers of 2				Powers of 3			
:		:		:		:	
2^4	=	16	Each answer is divided by 2 to get to the next.	3^4	=	81	Each answer is divided by 3 to get to the next.
2^3	=	8		3^3	=	27	
2^2	=	4		3^2	=	9	
2^1	=	2		3^1	=	3	
2^0	=	1		3^0	=	1	
2^{-1}	=	$\frac{1}{2}$		3^{-1}	=	$\frac{1}{3}$	
2^{-2}	=	$\frac{1}{4}$		3^{-2}	=	$\frac{1}{9}$	
2^{-3}	=	$\frac{1}{8}$		3^{-3}	=	$\frac{1}{27}$	
:		:		:		:	

These patterns suggest the following rules/observations

(i) Any number to the power of 1 is equal to the number itself.

e.g. $2^1 = 2$; $3^1 = 3$; $10^1 = 10$

This can be proved by this example

$2^3 \div 2^2 = 2^{3-2} = 2^1 = 2$

$8 \div 4 = 2$

$\therefore$ $\mathbf{a^1 = a}$

(ii) Any number raised to power 0 is equal to 1

e.g. $2^0 = 1$; $3^0 = 1$; $10^0 = 1$

This can be proved by this example

$3^4 \div 3^4 = 3^{4-4} = 3^0$

$81 \div 81 = 1$

$\therefore$ $\mathbf{a^0 = 1}$

However, be careful, $(-2)^0 \neq -2^0$

In $(-2)^0$, it is -2 which is being raised to power 0 giving 1.

But in -2^0 it is 2^0 which is negative. Thus $-2^0 = -(2^0) = -1$.

Also $2x^0 \neq 1$ but $= 2$ because $2x^0 = 2 \times x^0$ $(x^0 = 1)$

$$\therefore \quad 2 \times 1 = 2$$

$$\text{Thus } 2x^0 \neq (2x)^0$$

(iii) Any number raised to a negative power is equal to the inverse or reciprocal of the corresponding *positive* power.

e.g. $2^{-3} = \frac{1}{2^3} = \frac{1}{8}$, $\quad 3^{-2} = \frac{1}{3^2} = \frac{1}{9}$

Tip

$\frac{1}{2^3}$ is not the same as $2^{\frac{1}{3}}$ (2 to power $\frac{1}{3}$)

Simply put, the negative sign on a power is telling you to invert the number.

Note that 2^3 is now in the denominator.

Thus in general $\quad \boldsymbol{a^{-x} = \frac{1}{a^x}}$

Tip

From the above three illustrations, it follows that, when using the first three laws of indices (discussed earlier), an answer with unity, zero or a negative power can be simplified further.

e.g. $a^3 \div a^5 = a^{3-5}$ by *2nd law*

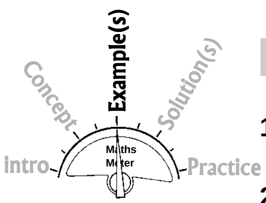
$= a^{-2}$

However by expansion

$$a^3 \div a^5 = \frac{a \times a \times a}{a \times a \times a \times a \times a}$$

$$= \frac{1}{a^2}$$

$$\therefore \quad a^{-2} = \frac{1}{a^2}$$

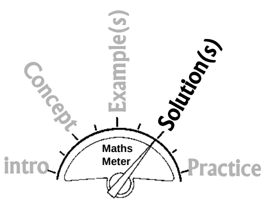


Consider the following examples

1. Simplify $a^5 \div a^6$

2. Evaluate a) $3^2 \times 3^{-2}$ b) $4^2 \div 4^5$ c) $(\frac{2}{5})^{-2}$

Solutions



1. $a^5 \div a^6$ Apply law 2

$= a^{5-6}$

$= a^{-1}$

$= \frac{1}{a^1}$ *meaning of negative power*

meaning of power 1

$= \frac{1}{a}$

2. a) $3^2 \times 3^{-2}$ Apply law 1

$= 3^{2+(-2)}$

$= 3^0$ *meaning of power 0*

$= 1$

b) $4^2 \div 4^5$ Applying law 2

$= 4^{2-5}$

$= 4^{-3}$ *meaning of negative power*

$= \frac{1}{4^3}$

$= \frac{1}{64}$

c) $(\frac{2}{5})^{-2}$

Means the reciprocal of $(\frac{2}{5})^2$. (Reciprocal means inverse of.)

$\therefore (\frac{2}{5})^{-2}$

$= \frac{25}{4}$

$= 6\frac{1}{4}$

Coefficients may be involved in the problems

Consider the following examples:

3. Simplify a) $3y^2 \times 4y^{-4}$ b) $15x^3y^2 \div 3x^{-2}y^3$

c) $(\frac{a}{b})^{-1}$

Solution

3. a) $3y^2 \times 4y^{-4}$

$= \underbrace{3 \times 4}_{} \times y^2 \times y^{-4}$

$= 12 \times y^{2+(-4)}$

Since $3y^2 = 3 \times y^2$ and $4y^{-4} = 4 \times y^{-4}$, multiplication can be done in any order. Applying law 1 on $y^2 \times y^{-4}$

$= 12y^{-2}$

y^{-2} means $\frac{1}{y^2}$, hence

$= \frac{12}{y^2}$

$12y^{-2}$ means $12 \times \frac{1}{y^2}$

b) $15x^3y^2 \div 3x^{-2}y^3$

This problem can be broken down to: $(15 \div 3) \times (x^3 \div x^{-2}) \times (y^2 \div y^3)$

$= 5x^5y^{-1}$

Apply meaning of y^{-1}.

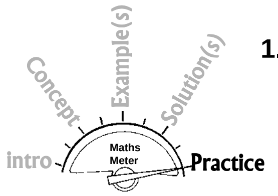
$= \frac{5x^5}{y}$

c) $(\frac{a}{b})^{-1}$ means the reciprocal of $(\frac{a}{b})$

$= \frac{b}{a}$

The negative power can appear in the denominator, e.g. $\frac{1}{a^{-2}}$.

The meaning is still the same, the inverse of $\frac{1}{a^{-2}} = 1 \times \frac{a^2}{1} = a^2$.



1. Find the value of:

a) 5^{-1} b) $10\ 000^{0}$ c) $(-12)^{0}$ d) -12^{0}

e) $(-1)^{4}$ f) $(-1)^{7}$ g) 2^{-3} h) -1^{0}

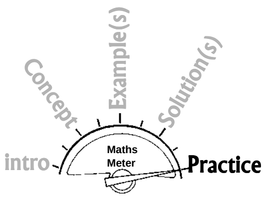
i) $(-1)^{0}$ j) $(-3)^{4}$ (k) $(-2)^{-2}$ l) $(-2)^{-3}$

2. Find the value of:

a) $(\frac{1}{4})^{-1}$ b) $(\frac{1}{4})^{-2}$ c) $(\frac{3}{4})^{-1}$ d) $(\frac{3}{4})^{-2}$

e) $(\frac{3}{4})^{0}$ f) $(\frac{3}{4})^{2}$ g) $(\frac{3}{2})^{-3}$ h) $(\frac{2}{3})^{-1}$

i) $(\frac{-1}{2})^{-1}$ j) $(\frac{-3}{5})^{0}$ k) $(\frac{-3}{5})^{-2}$ l) $-(\frac{3}{5})^{0}$



3. Simplify:

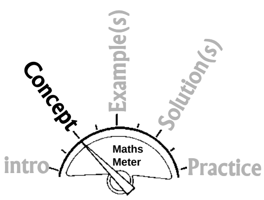
a) $2x^{-3} \times 5x^{5}$ b) $3x^{3} \div x^{5}$ c) $15x^{-2}y^{3} \times x^{-1}y^{2}$

d) $35xy^{3} \div 5x^{-2}y$ e) $7y^{2} \times xy^{-3}$ f) $11y^{-5} \times 5y^{2}$

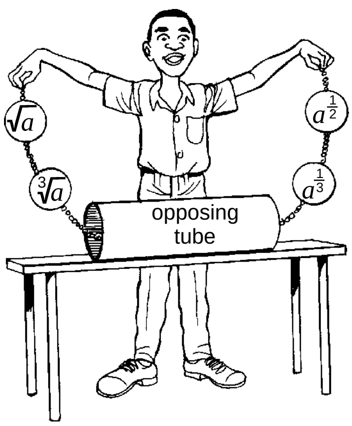
g) $6y^{-2}$ h) $\dfrac{1}{y^{-3}}$ i) $\dfrac{3}{y^{-2}}$

j) $(-5x^{2}y)^{3}$ k) $\dfrac{x^{5} \times (-x^{2})^{3}}{(-x^{3})^{3}}$ l) $\dfrac{(x^{3}y)^{-3} \times x^{3}y^{-2}}{(x^{-1}y^{4}) \times (x^{4}y^{-2})^{-5}}$

B. FRACTIONAL INDICES



So far the indices dealt with are integral (integers), so what is the meaning of a fractional index? Consider the following when answering this question:



Squaring implies power 2 e.g. a^{2}
The opposite of squaring is square root e.g. $\sqrt{a}$
The opposite of cube is cube root e.g. $\sqrt[3]{a}$
Now we want a power to stand for $\sqrt{a}$, $\sqrt[3]{a}$ etc. That power should be the opposite (reciprocal of 2 in the case of square root).

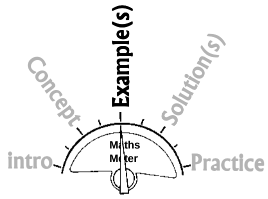
Thus $\therefore$ $\sqrt{a} = a^{\frac{1}{2}}$ $\sqrt[3]{a} = a^{\frac{1}{3}}$ $\sqrt[5]{a} = a^{\frac{1}{5}}$ etc

It can be seen that roots are represented by fractional powers where the root appears in the denominator of the power!
However, a number like $5^{\frac{3}{2}}$ has two powers joined together. (power 3 and power $\frac{1}{2}$)

Law 3 can help to clarify $(5^3)^{\frac{1}{2}} = 5^{3 \times \frac{1}{2}} = 5^{\frac{3}{2}}$

In general $a^{\frac{x}{y}} = \sqrt[y]{a^x}$ **or** $\left(\sqrt[y]{a}\right)^x$

Thus powers can be integral or fractional. In all cases, the basic laws of indices apply.



Consider the following examples:

Evaluate **1.** a) $7^{\frac{1}{2}} \times 7^{\frac{3}{2}}$ b) $125^{\frac{1}{3}}$ c) $16^{\frac{-3}{2}}$

Solution

1. a) $7^{\frac{1}{2}} \times 7^{\frac{3}{2}}$
$= 7^{\frac{1}{2} + \frac{3}{2}}$
$= 7^2$
$= 49$

Instead of worrying about what the powers mean, simply apply law 1.

Hint

Law 3 applies.

b) $125^{\frac{1}{3}}$
$= (5^3)^{\frac{1}{3}}$ law 3
$= 5^{3 \times \frac{1}{3}}$
$= 5$
or $125^{\frac{1}{3}} = \sqrt[3]{125} = 5$

First, write 125 in index form i.e. 5^3
If this is not obvious, find it i.e.
(By prime factorisation)

5	125
5	25
5	5
	1

Hint

From (b) and (c) always create a power which is the same as the denominator of the initial power, so the two will cancel each other off.

c) $16^{\frac{-3}{2}} = (4^2)^{\frac{-3}{2}}$
$= 4^{\cancel{2} \times \frac{-3}{\cancel{2}}}$
$= 4^{-3}$
$= \dfrac{1}{4^3}$
$= \dfrac{1}{64}$

or $\dfrac{1}{(16^3)^{\frac{1}{2}}}$
$= \dfrac{1}{(\sqrt{16})^3}$
$= \dfrac{1}{4^3} = \dfrac{1}{64}$

Hint

When root signs are appear in a problem, change them to powers and proceed with the relevant law.

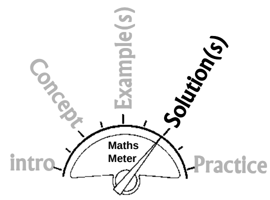
2. Simplify a) $\sqrt{a} \times \sqrt[3]{a}$ b) $\sqrt[3]{a^{10}b^{-5}} \div \sqrt{a^{-4}b^2}$
c) $\sqrt[4]{16a^{-4}}$

3. Evaluate a) $\sqrt[3]{8^{-2}}$ b) $\sqrt{0{,}0025}$

Solutions

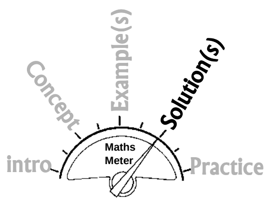
2. a) $\sqrt{a} \times \sqrt[3]{a} = a^{\frac{1}{2}} \times a^{\frac{1}{3}}$
$= a^{\frac{1}{2} \times \frac{1}{3}}$
$= a^{\frac{1}{6}}$

b) $\sqrt[5]{a^{10}b^{-5}} \div \sqrt{a^{-4}b^2}$
$= (a^{10}b^{-5})^{\frac{1}{5}} \div (a^{-4}b^2)^{\frac{1}{2}}$
$= a^2b^{-1} \div a^{-2}b$
$= a^4b^{-2}$
$= \dfrac{a^4}{b^2}$



c) $\sqrt[4]{16a^{-4}} = (16a^{-4})^{\frac{1}{4}}$
$= (2^4a^{-4})^{\frac{1}{4}}$
$= 2a^{-1}$
$= \dfrac{2}{a}$

3. a) $\sqrt[3]{8^{-2}} = 8^{-\frac{2}{3}}$
$= (2^3)^{-\frac{2}{3}}$
$= 2^{-2}$
$= \dfrac{1}{4}$

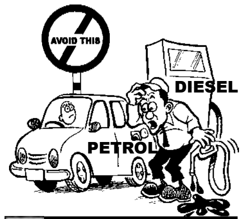


b) $\sqrt{0{,}0025} = \left(\frac{25}{10000}\right)^{\frac{1}{2}}$ or $\sqrt{\frac{25}{10000}}$

$= \left(\left(\frac{5}{100}\right)^2\right)^{\frac{1}{2}}$ $= \frac{\sqrt{25}}{\sqrt{10000}}$

$= \frac{5}{100}$ $= \frac{5}{100}$

$= \frac{1}{20}$ $= \frac{1}{20}$



Common Error

$a^3 \times b^2 = (ab)^5$

$3^3 \div 2^3 = 1{,}5^0$
$= 1$

This is wrong because laws are being used inappropriately.

Consider further examples below:

Hint

In examples 4 and 5, the bases are different and in example 6, an addition sign is between the numbers. No law of indices apply in such cases.

Watch out!

4. Simplify $a^3 \times b^2$

5. Evaluate $3^3 \div 2^3$

6. $8^2 + 8^1$

Solution

4. a) $a^3 \times b^2 = a^3b^2$
$= a^3b^2$

5. $3^3 \div 2^3 = 27 \div 8$
$= \frac{27}{8}$
$= 3\frac{3}{8}$

6. $8^2 + 8^1 = 64 + 8$
$= 72$

Find the value of:

1. $8^{\frac{2}{3}}$ **2.** $16^{-\frac{1}{2}}$ **3.** $3^{-\frac{2}{3}} \times 3^{-\frac{1}{3}}$

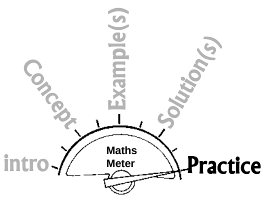
4. $4^{-\frac{1}{2}} \div 4^{-2}$ **5.** $7^{\frac{x}{3}} \times 7^{-\frac{x}{3}}$ **6.** $32^{-\frac{2}{5}}$

7. $\left(\frac{27}{8}\right)^{-\frac{2}{3}}$ **8.** $\sqrt[3]{125^2}$ **9.** $0{,}027^{-\frac{1}{3}}$

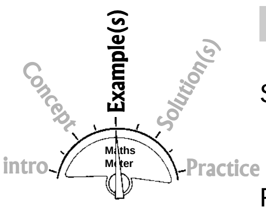
10. $49^{\frac{1}{2}} \times 5^{-2}$ **11.** $16^{-\frac{1}{2}} \times 7\left(\frac{4}{81}\right)^{\frac{1}{2}}$ **12.** $100^{-\frac{1}{2}} \times \left(3\frac{1}{3}\right)^0$

13. $a^6 \div y^2$ **14.** $a^2b^3 \times b^{-4}$ **15.** $4^2 + 4^0$

16. $9^2 - 9^0$



C. EQUATIONS WITH INDICES



Consider the following examples:

Solve the following equations:

1. $x^{\frac{1}{2}} = 3$ **2.** $3x^{-3} = 24$ **3.** $4^x = 32$

Remember when solving equations, the ultimate goal is to end up with the numerical value of the unknown letter.

Common Error

The 3 on the RHS is not raised by the same power as the LHS. The wrong answer will be $x = 3$!

Solutions

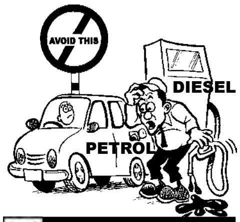
1. The unknown x is to the power of $\frac{1}{2}$ here. How can the power be transformed to a power of 1?

$x^{\frac{1}{2}} = 3$

$(x^{\frac{1}{2}})^? = 3^?$ — Think of *law 3* of indices. The number for the question mark should enable $(x^{\frac{1}{2}})^?$ to be simplified to x, with a power of 1.

Thus $(x^{\frac{1}{2}})^2 = 3^2$ — Do you ***notice*** that the inverse of the existing power will do exactly that i.e. it will change to power 1?

$x = 9$



Common Error

(i) $(3x^{-3})^{-\frac{1}{3}} = 3x$

(ii) $8^{-\frac{1}{3}}$ taken as $8 \times -\frac{1}{3} = -2\frac{2}{3}$.

(1) $32 = 4^8$
(2) Dividing both sides by 4 so that x = 8. Remember $4^x \div 4 = 4^{x-1}$. (law 2) Not x.

2. $3x^{-3} = 24$ — Both sides can be divided by 3.

$x^{-3} = 8$

$(x^{-3})^{-\frac{1}{3}} = 8^{-\frac{1}{3}}$ — $-\frac{1}{3}$ being the inverse of –3.

$x = (2^3)^{-\frac{1}{3}}$

$= 2^{-1}$

$= \frac{1}{2}$

3. $4^x = 32$ — What is needed here is to express each side in index form using the same base. ***Notice*** that 32 cannot be expressed in base 4 but both sides can be expressed in base 2.

i.e.. $2^{2x} = 2^5$ — since $4^x = (2^2)^x = 2^{2x}$

Since the bases are the same the powers have to be the same for the sides to be equal.

$\therefore 2x = 5$

$x = 2\frac{1}{2}$

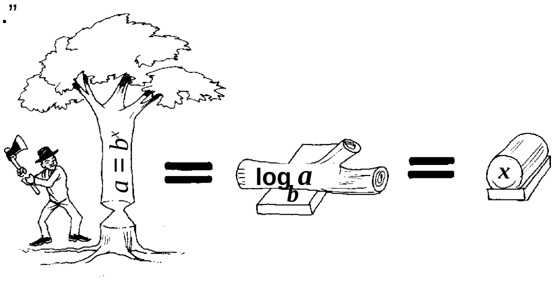
Solve the following equations

1.	$x^{\frac{1}{2}} = 4$	**2.**	$x^{-1} = 3$
3.	$x^{-\frac{1}{3}} = 2$	**4.**	$x^2 = 1$
5.	$3x^{-1} = -6$	**6.**	$3^x = 27$
7.	$4^{-x} = 16$	**8.**	$4^{-x} = 8$
9.	$5^{1-x} = 125$	**10.**	$8^{x+1} = 32$
11.	$7x^{\frac{1}{2}} = -21x$	**12.**	$5x = 40x^{-\frac{1}{2}}$
13.	$5^{3-2x} = 625$	**14.**	$3^{2(x-3)} \times 3^{5x} = 27$

D. THEORY OF LOGARITHMS

When a number is expressed in index form, such as $8 = 2^3$, this statement can be written differently as $\log_2 8 = 3$ which is read as "the logarithm (log) of 8 to base 2 is 3."

Conversely, given $\log_4 16 = 2$,
this can be expressed as $4^2 = 16$
Do you notice here that in $\log_2 8 = 3$?
This is the same 3 in 2^3.
So we can safely conclude that
a logarithm is a power.



In saying, "the logarithm of 16 to base 4 is 2" we mean that the power to which 4 must be raised to give us 16 is 2.
Thus in general, If $\mathbf{M} = x^a$, then $\log_x \mathrm{M} = a$ or $\log_y \mathrm{N} = a$ then $\mathrm{N} = y^a$.

The above equivalent statements can be used to find missing parts in given statements.

Consider the following examples:

Evaluate the following logarithms:
1. $\log_3 81$ **2.** $\log_{25} 0{,}2$ **3.** $\log_{0{,}2} 25$ **4.** $\log_5 5$

Solutions

1. Let $\log_3 81$ be x

So $\log_3 81 = x$ (similar to $\log_x \mathrm{M} = a$)

then $3^x = 81$ (similar to $x^a = \mathrm{M}$)
$3^x = 3^4$ Expressing both sides in *index form*.
$x = 4$
$\therefore \log_3 81 = 4$

2. Let $\log_{25} 0{,}2 = x$

$25^x = 0{,}2 = \frac{1}{5}$
$5^{2x} = 5^{-1}$
$2x = -1$
$x = \frac{-1}{2}$
$\therefore \log_{25} 0{,}2 = \frac{-1}{2}$

3. Let $\log_{0{,}2} 25 = x$

$0{,}2^x = 25$
$5^{-x} = 5^2$
$-x = 2$
$x = -2$
$\therefore \log_{0{,}2} 25 = -2$

4. Let $\log_5 5 = x$

$5^x = 5$
$x = 1$
$\therefore \log_5 5 = 1$

Question 4 is interesting. It is saying that the logarithm of a number with the same number as its base is 1.
In general **$\log_x x = 1$.** Also since $a^0 = 1$, it follows that **$\log_a 1 = 0$**.

Evaluate the following logarithms:

1. $\log_4 16$	**2.** $\log_2 16$	**3.** $\log_3 27$
4. $\log_{10} 10000$	**5.** $\log_6 216$	**6.** $\log_{0,25} 8$
7. $\log_{36} 6$	**8.** $\log_{16} 4$	**9.** $\log_{\frac{1}{7}} 7$
10. $\log_5 0{,}04$	**11.** $\log_4 \frac{1}{64}$	**12.** $\log_{64} 0{,}25$
13. $\log_{0,2} 125$	**14.** $\log_{0,001} 100$	**15.** $\log_{\frac{1}{8}} 0{,}0625$

E. LAWS OF LOGARITHMS

In the previous section we learnt that $\log_x M$ is a **single** expression (is a number).

Now suppose $M = x^a$ and $N = x^b$.

Then $MN = x^a \times x^b$
$= x^{a+b}$ (*law 1 of Indices*)

But from $M = x^a$ $\quad \log_x M = a$
and from $N = x^b$ $\quad \log_x N = b$

From $MN = x^{a+b}$, $\log_x (MN) = a + b$

$\therefore$ **$\log_x (MN) = \log_x M + \log_x N$**
(From the first law of indices).
Similarly the other two laws of logarithms can be derived using the other two laws of indices.

Laws of Indices	Laws of Logarithms
1. $x^a \times x^b = x^{a+b}$	$\log_x (MN) = \log_x M + \log_x N$
2. $x^a \div x^b = x^{a-b}$	$\log_x \left(\frac{M}{N}\right) = \log_x M - \log_x N$
3. $(x^a)^p = x^{ap}$	$\log_x (M)^p = p\log_x M$

Thus, just as there are three basic laws of indices, there also exist three corresponding basic laws of logarithms. They are simply the laws of indices written in logarithm form.

Go back to section D. The example $\log_4 16$ can now be evaluated using law 3 of logarithms,

i.e. $\log_4 16 = \log_4 4^2$

$= 2 \log_4 4$ but $\log_4 4 = 1$

$= 2 \times 1$

$= 2$

In fact, the three laws of logarithms can be used to simplify and/or evaluate given situations. In doing so, it must be remembered that our number system is in base 10. This means logarithms like $\log_{10} 10\ 000$ can be written without the base as $\log 10\ 000$.

The omission of the base implies base 10. Other bases **must** be stated.

Logarithms of numbers with 10 as the base are called **natural** logarithms or **common** logarithms.

Consider the following examples

Hint

Always create a statement similar to either law using the given situation.

Given that log 2 = 0,301, log 3 = 0,477 and log 7 = 0,845, evaluate:

1. log 6 **2.** $\log 3\frac{1}{2}$ **3.** log 0,8

Solutions

1. log 6
$= \log (2 \times 3)$ law 1
$= \log 2 + \log 3$ substitute the given values
$= 0{,}301 + 0{,}477$
$= 0{,}778$

2. $\log 3\frac{1}{2}$
$= \log (\frac{7}{2})$ law 2
$= \log 7 - \log 2$
$= 0{,}845 - 0{,}301$
$= 0{,}544$

3. log 0,8
$= \log (\frac{8}{10})$ law 2
$= \log 8 - \log 10$
$= \log 2^3 - \log 10$ law 3 on $\log 2^3$
$= 3\log 2 - \log 10$
$= 3 \times 0{,}301 - 1$ log 10 = 1 since its in base 10
$= 0{,}903 - 1$
$= -0{,}097$

Given log4 = a and log 11 = b, express the following in terms of a and/or b.

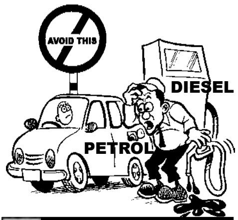
4. log 44 **5.** log 2 **6.** log 16

Solutions

4. $\log 44$
$= \log (4 \times 11)$
$= \log 4 + \log 11$
$= a + b$

5. $\log 2\frac{3}{4}$
$= \log \frac{11}{4}$
$= \log 11 - \log 4$
$= b - a$

6. $\log 16$
$= \log 4^2$
$= 2\log 4$
$= 2a$



Common Error

Log 4 is sometimes confused with $\log_4 4$ which is wrong.

Assume all logarithms (logs) are in base 10 in this section.

1. Using $\log 2 = 0{,}301$ and $\log 3 = 0{,}477$, find:

a) $\log 4$ b) $\log 9$ c) $\log 8$
d) $\log 27$ e) $\log 12$ f) $\log 18$
g) $\log 1\frac{1}{2}$ h) $\log \frac{2}{3}$ i) $\log 0{,}3$
j) $\log 0{,}16$ k) $\log 0{,}75$ l) $\log 3\frac{3}{8}$

2. Given that $\log 2 = 0{,}301$, $\log 3 = 0{,}477$, $\log 5 = 0{,}699$ and $\log 7 = 0{,}845$, find:

a) $\log 15$ b) $\log \frac{3}{7}$ c) $\log 21$
d) $\log 35$ e) $\log 10$ f) $\log 2\frac{1}{2}$
g) $\log 49$ h) $\log 24$ i) $\log 70$
j) $\log 625$ k) $\log 250$ l) $\log 0{,}7$

3. If $\log 5 = x$ and $\log 6 = y$, express the following in terms of x and/or y.

a) $\log 30$ b) $\log 36$ c) $\log 50$
d) $\log 0{,}6$ e) $\log 1\frac{1}{5}$ f) $\log \frac{5}{6}$

4. If $\log x = -3$ and $\log y = 2$, find:

a) $\log xy$ b) $\log \frac{x}{y}$ c) $\log \frac{1}{y}$
d) $\log x^2$ e) $\log \frac{1}{x}$ f) $\log y^{-3}$

5. Re-write the following equations without the logarithm (i.e. in index form).

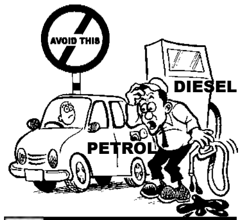
a) $\log x = y$ b) $\log m + \log n = 1$
c) $\log m - \log n = 0$ d) $\log x + 2\log y = 2$
e) $1 + \log x = 0$ f) $\log a - \log b = \log c$

F. SIMPLIFYING OR EVALUATING LOGARITHMS

Consider the following examples:

Evaluate

1. $\log_2 64$ **2.** $\dfrac{\log 16}{\log 8}$ **3.** $\frac{2}{3}\log_3 27$



Common Error

$\dfrac{\log 16}{\log 8} = \log 2$

by dividing 16 by 8. Remember log 16 is a number, hence 16 cannot be separated from log!

Solutions

1. $\log_2 64 = \log_2 2^6$

$= 6\log_2 2 \quad (\log_2 2 = 1)$

$= 6$

2. $\dfrac{\log 16}{\log 8} = \dfrac{4\log 2}{3\log 2}$ law 3

$= \dfrac{4}{3}$ Common factor log2

$= 1\frac{1}{3}$

3. $\frac{2}{3}\log_3 27 = \frac{2}{3}\log_3 3^3$

$= \cancel{3} \times \frac{2}{\cancel{3}}\log_3 3$

$= 2\log_3 3$

$= 2$

Thus when asked to simplify logarithms some answers will reduce to numerical values whilst others will remain in terms of a logarithm.

Simplify as far as possible or evaluate where possible.

1. $\log_3 81$ **2.** $\log_5 125$ **3.** $\log_2 32$

4. $\frac{3}{4}\log 16$ **5.** $\dfrac{\log 9}{\log 3}$ **6.** $\dfrac{\log \frac{1}{7}}{\log 7}$

7. $\dfrac{\log \sqrt{3}}{\log 9}$ **8.** $\dfrac{\log 0{,}2}{\log 25}$ **9.** $\dfrac{\log 27}{\log 243}$

10. $\dfrac{\log 125}{\log \sqrt{5}}$ **11.** $\dfrac{\log 36}{\log \frac{1}{6}}$ **12.** $\dfrac{\log 9 - \log 3}{\log 27 - \log 9}$

G. NATURAL OR COMMON LOGARITHMS (OPTIONAL)

This section is mainly for those doing the non-*calculator version* of their syllabi. The logarithms in base 10 are the natural logarithms since 10 is the base for our number system.

Consider Fig 5.1 below.

0,01	0,1	1	10	100	1000
10^{-2}	10^{-1}	10^{0}	10^{1}	10^{2}	10^{3}

Fig 5.1

It has previously been established that $100 = 10^2$, is the same as $\log_{10}100 = 2$.

From the number line above, the logarithm of a number say between 10 and 100, lies between 1 and 2.
This section of the chapter will be dealing with numbers between the obvious powers/logarithms of 10.

The Logarithm Tables

Suppose we are asked to find the logarithm of 6,3. The number line above helps us to locate the number 6,3 between appropriate powers of 10. i.e. between 1 and 10.
So if $6{,}3 = 10^x$, it follows that $\log 6{,}3 = x$. This means x lies between 0 and 1 and so x is 0, something (zero comma something). To find x we need logarithm tables (at the back) of this book.
Logarithm tables were established by a great mathematician, John Napier, and were refined to their present form by Henry Briggs, more than 300 years ago.

Fig 5.2 shows a typical line from 4-figure tables.

	3rd digit										difference 4th digit								
x	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
15	1761	1790	1818	1847	1875	1903	1931	1959	1987	2014	3	6	8	11	14	17	20	22	25

1st 2 digits

Fig. 5.2

The tables are called *4-figure* tables because that they deal with numbers up to 4 significant figures only.
The first two digits are under x, the third digit in row 0 to 9 row and the fourth digit in the difference column (1 to 9).

Consider the following examples on how to use these tables

Use Fig 5.2 above to find logarithms of

1. 1,5 **2.** 15 **3.** 159 **4.** 1 528

Solutions

Hint
Remember that the four digits of 1,5 are 1,500.

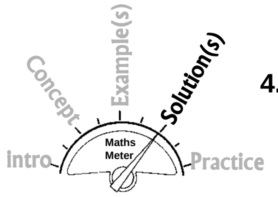
1. Since 1,5 lies between 1 and 10
log1,5 = 0, 'something'. (zero comma something)
The tables now give us the '*something*' i.e. the fraction part of the logarithm.
So go to line 15 of the tables under zero of third digits, i.e.

x	0
15 →	$\boxed{176}$

Thus, log 1,5 = 0,1761

Hint

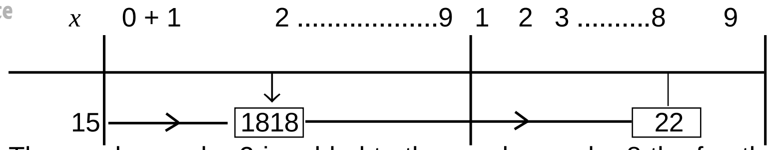
The 'difference' column does not have the digit zero. So only first 3 digits are applicable here.



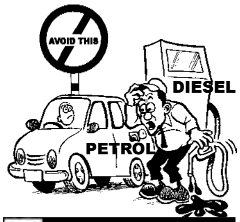
2. log15 = 1,1761
The whole number is now 1 since 15 lies between 10^1 and 10^2. So its logarithm is 1, 'something'. (one comma something)

3. log159 = 2,2014
Note that the whole number is 2 since 159 is more than 100 (10^2)
The fraction part is under 9 this time.

4. log1528 = 3,1840
Notice how the fraction is arrived at



The number under 2 is added to the number under 8 the fourth digit.

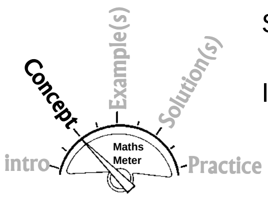


Common Error

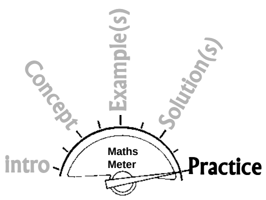
log 1,5 = 1761 This is wrong. We need the whole number part.

A logarithm has two parts, the *integer* and the *fraction*.

You are supposed to establish the integer before using the tables for the fraction.
Study the pattern for determining the integer of the logarithm.



In 1,5: 1 digit before the comma, so integer is 1 – 1 = 0
15: 2 digits before the comma, so integer is 2 – 1 = 1
159: 3 digits before the comma, so integer is 3 – 1 = 2
1528: 4 digits before the comma, so integer is 4 – 1 = 3 etc.



1. Use Fig 5.2 to find the logarithms of

a) 156	b) 1,56	c) 1 560
d) 1 506	e) 1 537	f) 1 57,9
g) 1 503	h) 153,5	i) 1 5,64

2. Use logarithm tables at the back of the book to find the logarithms of

a) 55	b) 2,42	c) 1,066
d) 47 500	e) 765 600	f) 989,9
g) 8,201	h) 301,9	i) 6 666 000

Multiplying and dividing with logarithms

The process depends on the laws of indices, since logarithms are powers.

Consider the following examples:

Evaluate

1. $61{,}63 \times 2{,}223$

2. $71{,}65 \div 7{,}034$

3. $(4{,}755)^2$

4. $\sqrt{56{,}71}$

You need to appreciate that the numbers in,

- question 1 can be written as $10^x \times 10^y$ where x and y are the logarithms of the numbers. (*First law of Indices*).

- question 2 can be written as $10^x \div 10^y$ where x and y are the logarithms of the numbers. (*Second law of Indices*).

- question 3 and 4 can be written as $(10^x)^y$ where x is the logarithm of the number and y the power in the number.

Hint

Powers are the logarithms of the numbers.

Thus 1 can be expressed as

$$61{,}63 \times 2{,}223 = 10^{1{,}7898} \times 10^{0{,}3470}$$
$$= 10^{1{,}7898 + 0{,}3470} \quad (\textit{First law of indices})$$
$$= 10^{2{,}1368}$$

Now, what is the ordinary form of $10^{2{,}1368}$? Before this question can be answered, let us introduce a much simpler layout of the above working.

Solutions

1. $61{,}63 \times 2{,}223$

Hint

Adding the logarithms is like adding the powers.

No	Log
61,63	1,7898
2,223	+ 0,3470
	2,1368

Now, what number has 2,1368 as its logarithm?

The answer to the question is found under **Antilogarithm Tables** at the back of this book which are the reverse of logarithm tables.

Studying the antilogarithm tables carefully, we notice that the number under x (the 4-figure number) is the fraction part of the logarithm.

Thus from 2,1368 above, it is the ,1368 which is used in the antilogarithm tables. Following the 4-figure procedure ,1368 gives 1371.

The integer of the logarithm tells us the number of digits before the comma of the ordinary form of the number. (*If you don't understand why, go back to data before the practice 5G1 above.*) The integer 2 in the logarithm indicates that there are 3 digits before the comma.

Thus, log 137,1 = 2,1368

Hence, 61,63 × 2,223 = 137,1

Snap check ☑: 62 × 2 = 124 (To the nearest whole number) which is close enough to 137,1.

2. 71,65 ÷ 7,034

No	Log
71,65	1,8552
7,034	– 0,8472
10,19	1,0080

Tip

Subtract since it is division. Use the second law of indices. Use ,0080 in Antilog Tables and use the integer to place the comma.

3. $4,755^2$

No	Log
4,755	0,6772
$(4,755)^2$	0,6772 × 2 *Third law*
22,61	1,3544

4. $\sqrt{56,71}$

No	Log	
56,71	1,7537	
$\sqrt{56,71}$	1,7537 ÷ 2	Since √ is power $\frac{1}{2}$.
	0,87685	
7,532	0,8769	to 4-figures

Tip

- *Log Tables are used when working from number to logarithm.*
- *Anti log tables are used when working from logarithm to number.*

Study the steps in these examples carefully before proceeding.

Use tables to evaluate the following to 3 significant figures:

1. 63,6 × 8,52 **2.** 63,6 ÷ 8,52

3. $\sqrt{63,6}$ **4.** $8,52^3$

5. 15,68 × 21,77 **6.** 476 ÷ 17,35

7. $\sqrt[3]{84,56}$ **8.** $13,6^2$

9. 359,9 × 58,8 **10.** 267,4 ÷ 31,8

11. $\sqrt[3]{128,6}$ **12.** $1,408^5$

13. $8{,}42 \times 12{,}3 \times 34{,}62$ **14.** $387{,}3 \div 47{,}09$

15. $\sqrt[4]{913{,}5}$ **16.** $2{,}765^4$

Negative integers of logarithms

Consider the pattern below:

$$
\begin{aligned}
100 &= 10^2 \\
10 &= 10^1 \\
1 &= 0^0 \\
\tfrac{1}{10} &= 10^{-1} \quad \Rightarrow \log 0{,}1 = -1 \\
\tfrac{1}{100} &= 10^{-2} \quad \Rightarrow \log 0{,}01 = -2 \\
\tfrac{1}{1000} &= 10^{-3} \quad \Rightarrow \log 0{,}001 = -3
\end{aligned}
$$

It follows that numbers between 0,01 and 0,001 have logarithm –1, something. (minus one comma something)
However, it should be appreciated that the negative sign here only applies to the integer of the logarithm (–1). The fraction is positive. This brings us to the **'bar' notation** of logarithms.

'**–1, something**' is correctly written as '**$\bar{1}$, something**' read as '**bar one comma something**'.
From the above procedure, 'the number of zeros (insignificant figures) in the number gives the integer part of the logarithm.

Consider the following examples:

1. Find the logarithm of the following
a) 0,15 b) 0,07462 c) 0,00347

Solution

1. a) $\log 0{,}15 = \bar{1}{,}1761$ b) $\log 0{,}07462 = \bar{2}{,}8728$

c) $\log 0{,}00347 = \bar{3}{,}5403$

Question 2 gives an excellent example of the manipulation of logarithms with negative integers.

2. Simplify a) $\bar{2}{,}3 + \bar{1}{,}8$

b) $\bar{3}{,}64 - \bar{1}{,}77$

c) $\bar{2}{,}65 \times 3$

d) $\bar{5}{,}8764 \div 4$

Solution

2. a) $\begin{array}{r} \bar{2},3 \\ +\bar{1},8 \\ \hline \bar{2},1 \\ \hline \end{array}$

Step 1: 3 + 8 = 11. Put down 1 and carry 1

Step 2: $1 + \bar{2} + \bar{1} = 1 - 2 - 1$
$= \bar{2}$

b) $\begin{array}{r} \bar{3},{}^{1}6\,{}^{1}4 \\ -{}^{0}\cancel{1},{}^{8}\cancel{7}\;7 \\ \hline \bar{3},8\;7 \\ \hline \end{array}$

c) $\bar{2},65 \times 3 = \bar{5},95$

d) $\bar{5},8764 \div 4$

For case d), create a 'bar' number that is divisible by the divisor (4) and maintain the original bar number (5) by adding a positive integer.
i.e. $\bar{5} = \bar{8} + 3$ or $\overline{12} + 7$ or $\overline{16} + 11$, (The first version is preferred).

Thus $\bar{5},8764 \div 4$
$= (\bar{8} + 3,8764) \div 4$ or $(\overline{12} + 7,8764) \div 4$ etc.
$= \bar{2} + 0,9691$ or $\bar{3} + 1,9691$
$= \bar{2},9691$ or $\bar{2},9691$

Whatever combination is used produces the required answer.
Study the examples carefully before proceeding.
The examples which follow give application to the techniques in example 2 above.

3. Evaluate
a) $4,73 \times 0,58$
b) $37,87 \div 43,8$
c) $(0,05721)^2$
d) $\sqrt[3]{0,0647}$

Solution

3. a) $4,73 \times 0,58$

No.	Log
4,73	0,6749
0,58	$+\bar{1},7634$
2,744	0,4383

Be clear about how the 0 has been obtained.

b) $37,89 \div 43,8$

No	Log
37,89	$1,{}^{1}5\;7\;8\,{}^{1}4$
43,8	$-1,\;6\;4\,{}^{2}\cancel{1}\;5$
0,8648	$\bar{1},\;9\;3\;6\;9$

Be clear about how the $\bar{1}$ has been obtained.

c) $(0,05721)^2$

No	Log
0,05721	$\bar{2}$,7575
$0,05721^2$	$\bar{2}$,7575 × 2
0,003273	$\bar{3}$,5150

Be clear about the $\bar{3}$.

d) $\sqrt[3]{0,0647}$

No	Log
0,0647	$\bar{2}$,8109
$\sqrt[3]{}$	$\bar{2}$,8109 ÷ 3
	($\bar{3}$ + 1,8109) ÷ 3
	$\bar{1}$ + 0,60363
0,4015	$\bar{1}$,6036

Evaluate the following using tables. Give answers to 1 significant figure.

1. 4,525 × 0,07575 **2.** 0,8106 × 0,5389

3. 0,00825 × 0,364 **4.** 0,6861 ÷ 0,0838

5. 7,466 ÷ 29,18 **6.** 3,165 ÷ 0,1278

7. $0,5919^2$ **8.** $0,0089^3$

9. $0,04816^4$ **10.** $\sqrt{0,9072}$

11. $\sqrt[3]{0,0589}$ **12.** $\sqrt[4]{0,04816}$

Combined Operations

Consider the following examples

Evaluate **1.** $\dfrac{6,83 \times 78,4}{798,6}$ **2.** $\sqrt[5]{\dfrac{37,5}{82,4}}$

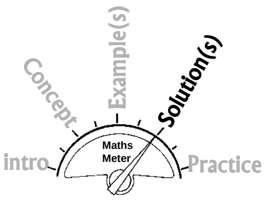
Solutions

1. $\dfrac{6,83 \times 78,4}{798,6}$

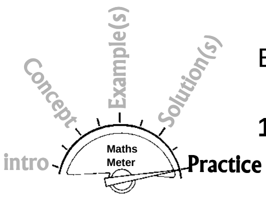
No	Log
6,83	0,8344
78,4	+1,8943
Numerator	2,7287
798,6	–2,9023
	$\bar{1}$,8264

No need to find the actual numerator.

2. $\sqrt[5]{\dfrac{37{,}5}{82{,}4}}$



No	Log
37,5	1,5740
82,4	$-1{,}9159$
	$\bar{1}{,}6581$
$\sqrt[5]{\ }$	$\bar{1}{,}6581 \div 5$
	$(\bar{5} + 4{,}6581) \div 5$
	$\bar{1} + 0{,}93162$
0,8543	$\bar{1}{,}9316$



Evaluate the following using tables to 2 significant figures.

1. $\dfrac{33{,}44 \times 8{,}84}{675{,}9}$

2. $\dfrac{0{,}19 \times 0{,}77}{0{,}044}$

3. $(0{,}598 \div 2{,}49)^3$

4. $\sqrt[4]{0{,}086 \div 0{,}48}$

5. $\sqrt[3]{0{,}748 \times 0{,}699}$

6. $(3{,}19 \times 0{,}087)^5$

7. $\sqrt[3]{0{,}578^2}$

8. $\sqrt{\dfrac{4{,}467 \times 12{,}74}{97{,}55}}$

SUMMARY

Laws of Indices	**Laws of Logarithms**
1. $x^a \times x^b = x^{a+b}$	$\log(MN) = \log M + \log N$
2. $x^a \div x^b = x^{a-b}$	$\log\left(\frac{M}{N}\right) = \log M - \log N$
3. $(x^a)^p = x^{ap}$	$\log_x (M)^p = p\log_x M$
Also ***note that*** $a^0 = 1$ $(-a)^0 = 1$ $-a^0 = -1$ $\sqrt[y]{a} = a^{\frac{1}{y}}$ $\sqrt[y]{a^x} = a^{\frac{x}{y}}$ $a^{-x} = \frac{1}{a^x}$	$\log_a a = 1$ $\log_x M = a$ then $x^a = M$ $\log_a 1 = 0$

When using Tables

- When multiplying – add the logs
- When dividing – subtract the logs
- When raising to a power – multiply the log by the power
- When finding roots – divide the log by the root.
- The whole number of a logarithm is 1 less than the number of digits the number has before the comma.
- The bar notation implies negative whole numbers.

EXAM PRACTICE 5

Hint

Where ever possible:
- *avoid introducing √ into your working. Keep working in index form.*
- *change ordinary numbers to index form where possible*
- *create the equivalent side using one of the laws of logarithms. When trying to solve a problem.*

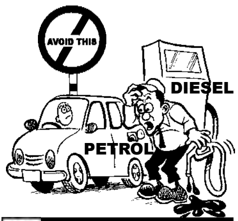
The following examples may help you master questions from this chapter.

1. Evaluate a) $5^{\frac{1}{3}} \times 5^{1\frac{2}{3}} \times 5$

b) $64^{-\frac{1}{6}}$

c) $16^{\frac{3}{4}} \times 25^{\frac{1}{2}}$

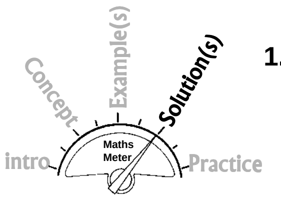
2. Given that $\log x = 5$ and $\log y = -7$
evaluate a) $\log \frac{x}{y}$ b) $\log \frac{1}{x}$ c) $\log xy^2$



Common Error

The last 5 is taken to be without a power. It has power 1.

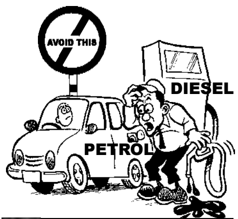
Solutions



1. a) $5^{\frac{1}{3}} \times 5^{1\frac{2}{3}} \times 5 = 5^{\frac{1}{3}+1\frac{2}{3}+1}$ *law 1 of indices*
$= 5^3$
$= 125$

b) $64^{-\frac{1}{6}} = (2^6)^{-\frac{1}{6}}$ *Change 64 to index form with power 6.*
$= 2^{-1}$ *Apply law 3.*

$= \frac{1}{2}$ Apply meaning of *negative index.*



Common Error

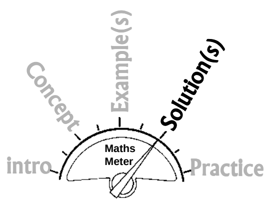
$(16 \times 25)^{\frac{3}{4}+\frac{3}{4}}$ or

$2^3 \times 5 = (2 \times 5)^{3+1}$
These numbers have different bases.

Hint

Note that the two bases i.e 2 and 5 are different so don't use the first law of indices.

c) $16^{\frac{3}{4}} \times 25^{\frac{1}{2}} = (2^4)^{\frac{3}{4}} \times (5^2)^{\frac{1}{2}}$ Change 16 and 25 to *index form.*
$= 2^3 \times 5$
$= 8 \times 5$
$= 40$



2. a) $\log \frac{x}{y} = \log x - \log y$ Create an equivalent side (*law 2 of logs*). Substitute with given values and simplify.
$= 5 - (-7)$
$= 5 + 7$
$= 12$

b) **Method 1**
$\log \frac{1}{x} = \log 1 - \log x$
$= 0 - 5$
$= -5$

Method 2
$\frac{1}{x} = x^{-1}$
So $\log \frac{1}{x} = \log x^{-1}$
$= -\log x$
$= -5$

Hint

Power 2 is for y only.

c) $\log xy^2 = \log x + \log y^2$
$= \log x + 2\log y$
$= 5 + 2(-7)$
$= 5 - 14$
$= -9$

3. Evaluate

a) $\dfrac{\log 5 - \log 7}{\log 25 - \log 49}$ b) $\log 30 + \log 20 - \log 6$

4. If $\log x - \log(x-3) = 1$ find x

5. If $5^{3x} \times 5^{2(2+x)} = 625$, find x

Solutions

Hint

Common factor
(log 5 – log 7)

3. a) $\dfrac{\log 5 - \log 7}{\log 25 - \log 49}$

Method 1

Simplify the denominator

i.e $\dfrac{\log 5 - \log 7}{2\log 5 - 2\log 7}$

$= \dfrac{(\log 5 - \log 7)}{2(\log 5 - 2\log 7)}$

$= \frac{1}{2}$

Method 2

Apply law 2 on numerator and denominator.

$\dfrac{\log(\frac{5}{7})}{\log(\frac{25}{49})}$ Apply law 3 on denominator.

$= \dfrac{\log(\frac{5}{7})}{2\log(\frac{5}{7})}$

Reduce fraction by $\log \frac{5}{7}$.

$= \frac{1}{2}$

b) $\log 30 + \log 20 - \log 6$

Method 1

$= \log \dfrac{(30 \times 20)}{6}$ *law 1 and 2*

$= \log 100$

$= 2\log 10$ but $\log 10 = 1$

$= 2 \times 1$

$= 2$

Hint

log10 = 1

Method 2

Expand the logs

$= \underbrace{\log 10 + \log 3}_{\text{(from log30}} + \underbrace{\log 10 + \log 2}_{\text{from log20)}} - \log 6$ Remember $\log 10 = 1$.

$= 1 + 1 + \log 3 + \log 2 - \log 6$ $\log 3 + \log 2 = \log 6$

$= 1 + 1 + \log 6 - \log 6$

$= 2$

Hint

Use Law 2 on LHS and use Log10 = 1 on RHS

4. $\log x - \log(x-3) = 1$

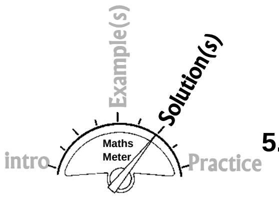
$\log_{10}\left(\frac{x}{x-3}\right) = \log_{10} 10$

$\frac{x}{x-3} = 10$

$x = 10x - 30$

$9x = 30$

$x = 3\frac{1}{3}$



5. $5^{3x} \times 5^{2(2+x)} = 625$ Re-write 625 in *index form*.

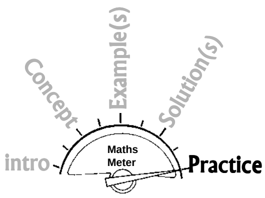
$5^{3x+4+2x} = 5^4$

$\rightarrow 3x+4+2x = 4$

$5x = 0$

$x = 0$

Now do the following:



1. Evaluate

a) $8^2 + 8^0$ b) $32^{-\frac{1}{5}}$

c) $2^{\frac{1}{4}} \times 8^{\frac{1}{4}}$ d) $\left(\frac{1}{9}\right)^{-\frac{3}{2}}$

e) $3^{2(x-3)} \times 3^{\frac{1}{2}(12-4x)}$

2. Simplify

a) $3\log_{10}3 + \log_{10}16 - \log_{10}36$

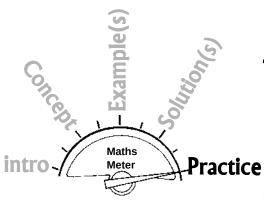
b) $\log 9 - \log 30 + 2\log 5$

c) $\log 1\frac{1}{2} - \log 3 + \log 2$

3. Evaluate

a) $\log_7 \frac{1}{7}$ b) $\log_5 \sqrt{5}$

c) $\log_4 64 + \log_4 16$ d) $\dfrac{\log 6 - \log 8}{\log 36 - \log 64}$



4. a) If $5^{3x} \div \frac{1}{125} = 5^{x(x+1)}$, find the values of x.

b) Given that $4^2 \times 2^{-3x} = 32$, find the value of x.

5. Given that log3 = 0,477 and log 5 = 0,699, evaluate:

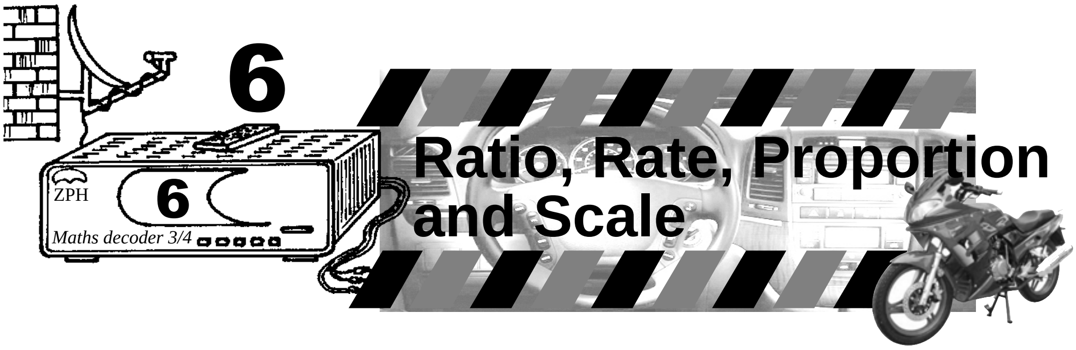
a) $\log_{10}15$ b) $\log_{10}1$ c) $\log_{10}\sqrt{5}$ d) $\log_{10}\frac{1}{9}$

6. If $\log m = 5$ and $\log n = 7$, evaluate:

a) $\log mn$ b) $\log \frac{m}{n}$ c) $\log \sqrt[3]{m}$ d) $\log \frac{1}{n}$

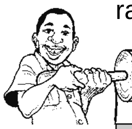
7. a) If $\log x + \log(3+x) = 1$, form an equation in x and show that it reduces to $x^2 + 3x - 10 = 0$.

b) Hence, or otherwise, find the value of x.



6 Ratio, Rate, Proportion and Scale

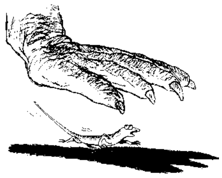
The concepts of ratio, rate and proportion are used in everyday life particularly in the business world. We will study the links between ratio and scale, ratio and proportion.



Syllabus Expectations

By the end of this chapter students should be able to:

1. make calculations based on ratio, rate and proportion.
2. find scale from given information.
3. express scale correctly in its different forms.
4. use given scales to calculate distances and areas.

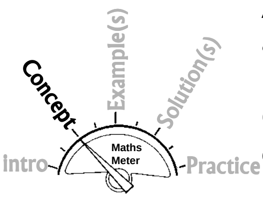


ASSUMED KNOWLEDGE

In order to tackle the work in this chapter it is assumed that students are able to:

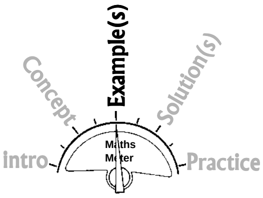
- express quantities in terms of units (unitary form) e.g. speed(m/s) and cost per unit area.
- carry out division and multiplication of numbers.
- reduce fractions to their lowest terms.

A. RATIO



A ratio compares quantities which are similar, for example, 60g and 150g can be expressed in the ratio 60:150 (said as '60 is to 150'). This ratio can be expressed in various ways, e.g. 60 : 150, 60 to 150 or $\frac{60}{150}$. The last form introduces us to the idea that ratio can be reduced to its simplest form,
i.e. 60 : 150 = 2 : 5

Consider the following examples



1. a) 115cm : 1,5m b) $\frac{5}{6} : \frac{7}{8}$

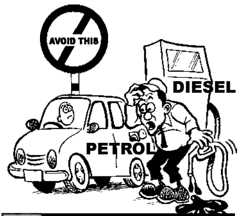
To simplify this ratio, one needs to express the parts in the same units.

a) 115cm : 150cm

$\frac{115\cancel{cm}}{150\cancel{cm}} = \frac{23}{30} = 23:30$

b) Multiply each part by the LCM of the denominator.

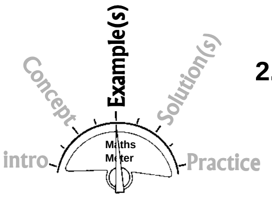
i.e. $\frac{5}{6} \times 24 : \frac{7}{8} \times 24 = 20 : 21$



Common Errors

There is a tendency to reduce before expressing the parts in the same unit!

i.e. 115: 1,5
1150 : 15
230 :3 Wrong!



2. If ratios can be reduced to simpler forms it follows that like fractions, they have equivalences. If given $x : 7 = 12 : 21$, we can find x by comparing these equivalences.

i.e. $\frac{x}{7} = \frac{12}{21}$

$x = \frac{12}{21} \times 7$

$x = 4$

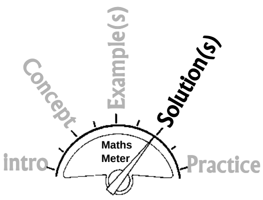
3. Just like fractions, ratios can be compared. For example, which ratio is greater 3:7 or 4:9? This question is the same as which fraction is greater $\frac{3}{7}$ or $\frac{4}{9}$?

Common Errors

(i) The Wrong decimalisation process is used.

(ii) 16 is used as the common denominator by addition of denominators, which is wrong.

Two methods are commonly used here.



Method 1

Change the fractions to decimals. Decimalise the fractions.

$\frac{3}{7} = 0,4285$ $\frac{4}{9} = 0,\dot{4}$

$\therefore \frac{4}{9}$ is greater

Method 2

Express the fractions to their equivalent fractions using the Lowest Common Multiple of their denominators.

Thus $\frac{3}{7} = \frac{27}{63}$

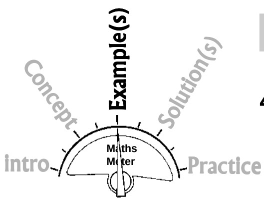
$\frac{4}{9} = \frac{28}{63}$

Now $\frac{28}{63} > \frac{27}{63}$

$\therefore \frac{4}{9}$ is greater

Quantities can be decreased or increased by given ratios.

Consider the following examples:



4. Decrease 726 kg in the ratio 7:11

By equivalences, x:726 = 7:11

Notice here the 726 has been linked to 11 since it is the bigger part. Decreasing will result in a smaller number, so the result must be linked with 7.

Thus, $\frac{x}{726} = \frac{7}{11}$

$x = \frac{7}{\cancel{11}} \times \overset{66}{\cancel{726}}$

$= 462$ kg

726kg is reduced to 462kg.

5. Increase 5,4cm in the ratio 5:8

By equivalences

5,4:x = 5:8

Be careful to note the linkage between the corresponding parts.

$$\frac{x}{5{,}4} = \frac{8}{5}$$

$$x = \frac{8 \times 5{,}4}{5}$$

$$= 8{,}64$$

$\therefore$ 5,4m increases to 8,64m

6. It is possible to express ratios in the form 1:n or n:1 where n is rational.

i.e. $5:8 = \frac{5}{5} : \frac{8}{5}$

$= 1 : 1{,}6$ or $1:1\frac{3}{5}$

or $5:8 = \frac{5}{8} : \frac{8}{8}$

$= \frac{5}{8} : 1$

7. Quantities can be shared in given ratios

e.g. Share 150 oranges in the ratio 11:19

		OR	
Thus: Total parts	= 11 + 19 = 30	Total parts	= 11 + 19 = 30
Value of each share	$= \frac{15\cancel{0}}{3\cancel{0}}$ = 5	Now, first share $\Rightarrow \frac{11}{30} \times 150 = 55$	
$\therefore$ 1st share	= 11×5 = 55	Second share $\Rightarrow \frac{19}{30} \times 150 = 95$	
2nd share	= 19×5 = 95		

Note that in both methods, after obtaining the first share, one can simply subtract it from the total quantity to get the second share.

PRACTICE 6A

> **Hint**
> *Use LCM to clear denominators and hence simplify.*

1. Write the following ratios in their simplest form.

a) 16:36 b) $\frac{1}{2} : \frac{2}{5}$

c) 2,6:39 d) 110c : \$2,20:\$5,50

e) $\frac{1}{3} : \frac{5}{8} : \frac{3}{4}$

2. Express each of the following ratios in the form:

(i) 1: n (ii) n:1

a) 8 : 12 b) 12 : 32 c) 28 : 7

d) 4,2 : 0,7 e) \$15 : 2400c f) $\frac{1}{2}$ km : $2\frac{1}{2}$ km

3. Find which of the ratios is greater in each case.

a) 2 : 7 or 3 : 8 b) 14 : 3 or 25 : 10

c) 40 seconds : 2 minutes or 9 hours : 1 day

d) 4,5m : 1,4m or 125 : 50

e) 6mm : 3cm or 300g : 1kg

f) $\frac{2}{9} : \frac{1}{3}$ or $\frac{2}{7} : \frac{1}{4}$

4. a) In each given quantity, find the larger part, if the quantity is shared in the given ratio.

(i) \$100 000 in the ratio 3 : 7

(ii) 25kg in the ratio 7 : 3

(iii) 405km in the ratio 10 : 17

b) In each given quantity, find the smaller part, if the quantity is shared in the given ratio.

(i) 14 hours in the ratio 2 : 5

(ii) 70m in the ratio 13 : 7

(iii) 91 shares in the ratio $\frac{1}{4} : \frac{1}{9}$

5. A trader increased the price of an item in the ratio 11 : 5. Find the new price of an article which was marked \$275.

6. Three business partners Kumbi, Vuma and Sipho shared the profit in the ratio 6 : 7 : 2. If Vuma received \$4 900

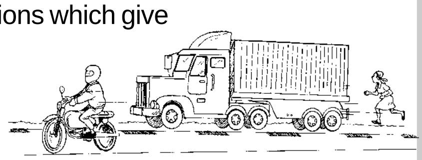
a) How much did Sipho get?

b) Find the total profit shared.

7. In the partnership in (6), Sipho is the executive director. The company policy is that the Executive Director is paid \$35 000 before sharing the profits. Find how much Kumbi got from a total profit of a \$345 000.

B. RATE

Rate connects quantities of different kinds. Examples of rate are speed (km/h) which compares distance with time and power e.g. Power = $\left(\frac{\text{Work}}{\text{Time}}\right)$ which compares work done with time. Rate is mostly associated with time. Can you think of other connections which give rate?



Consider the following examples

1. A car travelled 50km in 25 minutes. Find the speed in km/h.

Solution

1. Use simple proportion

50km	25min
? more	1h (60min)

$$\frac{60}{\cancel{25}} \times \frac{\overset{2}{\cancel{50}}\text{km/h}}{1} = 120\text{km/h}$$

2. A car used 20*l* of fuel to cover a distance of 250km. Find the rate of fuel consumption of the car per km.

Solution

2. The question here is, how much fuel is used per km?

$$\therefore \quad \frac{20}{250}\ \text{l/km} = \frac{20}{\cancel{250}} \times \frac{\overset{4}{\cancel{1000}}}{1}\ ml\text{/km}$$

$$= 80\ ml\text{/km}$$

3. Which of the two is a better deal, in terms of price, 5kg of pork for \$23,75 or 8kg of beef for \$50,00.
What is the price per kg?

Solution

3. Pork is \$ $\frac{23{,}75}{5}$ /kg = \$4,75/kg

Beef is \$ $\frac{50{,}00}{8}$ = \$4,25/kg

$\therefore$ Beef is the better deal.

1. A car travels 80km in 48 minutes. Find the speed of the car in km/h.

2. The density of aluminium is 2,7g/cm^3. Calculate the volume, in cm^3, of a piece of aluminium of mass 8,1kg.

3. A tank full of water lasts 6 days, if 30 litres are used daily. How long will the tank last if 18 litres are used daily.

4. A certain resettlement area in Zimbabwe has an adult population of 350, to the nearest ten. Find the density of the population if the settlement has an area of 1 750km^2.

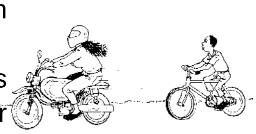
5. The distance between any two consecutive telephone poles along a road, is 15m. A car travels from the first to the ninth pole in 6 seconds. Calculate the speed of the car in km/h.

6. A certain District hospital recorded 260 deaths in one year. Calculate the average weekly death rate.

7. Sibongile bought 8 shirts for \$112. Tumai bought 13 similar shirts for \$195. Which of the two made the better buy?

8. Peter bought a dozen tyres from a certain shop and paid \$576. John went to another shop and bought 30 similar tyres for \$1 380. Who made the better deal?

C. PROPORTION



Proportion compares changes which affect one quantity when another quantity is changed.
For example, the time taken by a car to cover a given distance varies with the speed the car is travelling. The higher the speed, the shorter the time taken. Here speed and time are being compared on a constant distance. This brings us to the simple proportion table.

Consider the following examples

1. 15 packs of seed cost \$180. What will be the cost of 8 packs?

Solution

1.

15	180
8	? less

$\therefore$ Cost of 8 packs $= \frac{8}{\cancel{15}} \times \frac{\cancel{180}^{12}}{1}$

$= \$96$

Alternatively: 15 Packs cost \$180

1 Pack costs $\frac{\cancel{180}}{15} = \$12$

$\therefore$ Cost of 8 packs $= \$12 \times 8$

$= \$96$

2. 4 men can do a job in 9 days. How many days will 6 men take to do the same amount of work, assuming all are working at the same rate?

Solution

2.

4	9
6	less

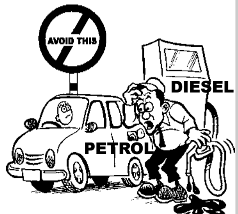
$\therefore \frac{4}{6} \times 9 = 6$ days

3. 50kg of Rock Salt, costing \$2,10 per kg, is mixed with 20kg of Fine Salt, costing \$2,55 per kg. Find the cost of the mixture per kg giving the answer to the nearest cent.

Solution

3.
Total cost of 50kg $= 2{,}10 \times 50$
$= \$105$
Total cost of 20kg $= 2{,}55 \times 20$
$= \$51$
Total cost of 70kg $= 105 \times 51$
$= \$156$

$\therefore$ Cost of mixture $= \frac{156}{70}$
$= \$2{,}27$/kg.



Common Errors

$\frac{2{,}10 + 255}{50 + 20}$

1. 5 adults can sow 150kg of seed per day.
How much seed per day will
a) 8 adults sow, if they are sowing at the same rate?
b) 3 adults sow, if they are sowing at the same rate?

2. Copper sulphate is made up of 32 parts of copper, 16 parts of sulphur, 32 parts of oxygen and 45 parts of water. Find the mass of water in 5,5kg of copper sulphate.

3. If there is 4,8kg of sulphur in a quantity of copper sulphate as in question 2, find the mass of the copper sulphate.

4. A motorist averages 50km/h for the first 60km of a journey and 75km/h for the next 80km. What is his average speed for the whole journey, to 2 significant figures?

5. The population density of a village is 8,5 people/km^2. If the village has an area of 15,2km^2, find its population to the nearest 10 people.

6. Convert 54km/h to a speed in m/s.

7. 5 men can complete a certain task in 9 days.
Find the time that:
a) 3 men will take to complete the same task if working at the same rate.
b) 8 men will take to complete the same task if working at the same rate.

8. A certain chemical costing \$137,50/50kg is mixed with another chemical costing \$112,50/50kg. If the mixture is sold at \$173,55/50kg calculate the percentage profit per kg to the nearest whole number.

D. SCALE

Map scales are linear. This means they give a relationship between distance on a map and the actual distance on the ground. Scale can be given in several ways, for example, 1cm to represent 20m can be given as 1cm:20m or 1:2000 or 1cm to 2000cm or $\frac{1}{2000}$
All this means is that 1cm on the map represents 20m on the ground. The scale with components reduced to the same basic unit is often called the **Representative Fraction (RF)** of the map.
Remember that

$$\text{RF} = \frac{\text{Distance between two points on the map}}{\text{Distance between the same points on the ground}}$$

Cecil John Rhodes would have used a scale to do calculations for his plans to extend British rule from Cape Town (South Africa) to Cairo (Egypt).
(Illustration by Edward Linley Sambourne)

Thus if a map gives a scale of 1cm to $\frac{1}{2}$ km,
RF = $\frac{1}{50000}$ or 1:50 000
i.e. $\frac{1}{2}$ km is changed to cm:

The scale for the area of the map can be derived from its linear scale. Remember ratios in similar figures? In actual fact, the linear scale of a map is the ratio of corresponding sides. It then follows that the area scale of a map is the square of the linear scale.
For example if a map has a scale of 1cm to $\frac{1}{5}$ km, the area scale is $(1 \times 1)\text{cm}^2$ to $(\frac{1}{5} \times \frac{1}{5})\text{km}^2 = 1\text{cm}^2$ to $\frac{1}{25}\text{km}^2$.

Consider the following examples

1. Calculate the Representative Fraction (RF) in the form $\frac{1}{n}$ of the following scales:
a) 1cm to represent 3km b) 2cm to represent 5,5m.

Solution

1. a) $1\text{cm} : 3\text{km} = \frac{1\text{cm}}{3\text{km}}$ change 3km to cm.

$$= \frac{1\text{cm}}{3 \times 100000\text{cm}}$$

$$= \frac{1}{300000}$$

b) $2\text{cm} : 5{,}5\text{m} = \dfrac{2\text{cm}}{5{,}5\text{cm}}$

$= \dfrac{2\text{cm}}{5{,}5 \times 100\text{cm}}$

$= \dfrac{5}{550}$

$= \dfrac{1}{275}$

2. The scale of a certain drawing is given as 15cm to represent 1,5km. Express this scale in the form 1:*n*.

Solution

2. $\dfrac{15\text{cm}}{1{,}5\text{km}} = \dfrac{15\text{cm}}{1{,}5 \times 100000}$

$= \dfrac{15}{1\,500\,000}$

$= \dfrac{1}{100\,000}$

$= 1{:}100\,000$

3. The scale of a map is given as 1:150 000.

a) Find the distance, in km, on the ground represented by 1cm on the map.

b) Find the map distance, in cm, of an actual distance of 7,5km.

c) Find the actual area, in km^2, represented by 6cm^2 on the map.

Hint

Scale in RF form uses the basic unit, cm, Thus change the RHS of the scale to the unit required in the question.
$1\text{km} = 100 \times 1000\text{cm}$

Solution

3. a) i.e. $150\,000\text{cm} = \dfrac{150\,000}{100\,000}$ km

$\therefore$ 1cm = 1,5km on the ground

Hint

Use simple proportion.

b)

1cm	1,5km
more ?	7,5km

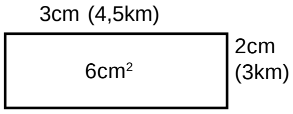
$7{,}5\text{km} = \frac{7{,}5}{1{,}5}$ cm

$= 5\text{cm}$

Hint

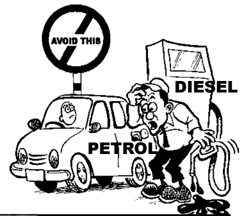
Break the area into convenient length and width e.g. 6cm² = 2cm by 3cm.

c) i.e.



Change the cm to actual distances on the ground (km).

Thus $6cm^2$ = $4{,}5km \times 3km$
= $13{,}5km^2$ on the ground



Common Errors

The number of zeros becomes confusing. One or two zeros are mistakenly left out in the process.

Notice that this approach (c) is very convenient and produces easy figures to work with. However, the Area Scale approach is also applicable.

i.e. Linear scale = 1:150 000
∴ Area scale = $1:150\ 000^2$
Hence Area for $6cm^2$ = $6 \times (150\ 000 \times 150\ 000)$
= 135 000 000 000

Hint

1km = 100 000cm.

Changing this to km^2 = $\dfrac{135\ 000\ 000\ 000}{100\ 000 \times 100\ 000}$

= $13{,}5km^2$

1. Calculate the Representative Fraction (RF), in the form $\frac{1}{n}$, of each of the following scales:

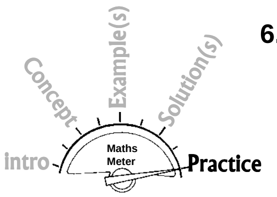
a) 1cm to 5km b) 4cm : 1km
c) 2cm : 3,4km d) 3cm to 6m
e) 5cm to 1,1m f) 5cm : 10km

2. The scale on a map is 25cm to 2km. Express this scale in the form 1: n.

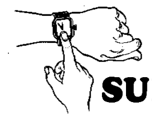
3. The RF of a map is given as 1 : 15 000.
Find: a) the map distance between two points which are 750m apart on the ground.
b) the actual distance, to the nearest km, between two points which are 5,7cm apart on the map.

4. One of a farmer's field has an area of 1ha. This is represented on a map by an area of $4cm^2$. Find the scale of the map in the form $\frac{1}{n}$. (1ha = 10 000m^2).

5. The scale on a map is 8cm to 10km.
a) Find the Representative Fraction of the map.
b) Find:
(i) the area, in km^2, represented by $3cm^2$ on the map. Give the answer to 3 significant figures.
(ii) the area on the map in cm^2, of a plot of area $28{,}125km^2$.



6. A town planner draws a plan of a shopping mall which covers an area of 3 872m^2. If the plan has an area of 2,2cm^2, calculate:
a) the area, in m^2, represented by 1cm^2 of the plan.
b) the map scale of the plan giving your answer in the form 1: n where n is to 1 significant figure.



SUMMARY

1. A ratio compares similar quantities, e.g. 60g and 150g is 60g : 150g which can simplify to 2 : 5.

2. A colon (:) is used between parts of a ratio.

3. Rate compares quantities of a different nature e.g. speed (km/h), price ($/kg).

4. Proportion compares the scale of changes between quantities. e.g. The higher you go, the cooler it becomes.

5. Scale compares the actuals with what is represented e.g. map scales compare the actual distance with distance on the map.

6. A representative fraction = $\dfrac{\text{Distance on the map}}{\text{Similar distance on the ground}}$

EXAM PRACTICE 6

Consider the following examples:

1. Tecla is paid a total of \$2 760 for some typing services. Calculate her rate of pay per hour if she takes 3 days to complete the job working 5 hours a day.

Common Errors

Total hours are often worked out.

$\frac{2760}{3}$ or $\frac{2760}{5}$

These are common wrong assumptions. Also 5 working hours is substituted as 12h a day i.e.

$\frac{2760}{36}$ is another common wrong working.

Solution

1. Total hours worked $= 3 \times 5$ h

$= 15$ h

Total amount paid $=$ \$2 760

$\therefore$ Rate of pay $= \dfrac{\overset{184}{\cancel{2\,760}}}{\cancel{15}}$

$=$ \$184/hour

2. A map of a town is drawn to a scale of 1cm : 8,5km.

a) A road on the map is 12cm long. Calculate the actual length of the road, giving the answer in kilometres.

b) The actual area of the town in 289km^2. Calculate, in cm^2, the area of the town on the map.

Common Errors

(1) 8,5km wrongly converted to cm.

(2) Wrong conversion from cm to km

(3) Scale 1cm : 8,5km is not converted. (Area scale is not found.)

(4) Area Scale as 1cm^2 to 17 km^2 is simplified as 17.

Solution

2. a) Scale = 1cm to 8,5km

= 1 : 850 000

$\therefore$ 12cm $= 8\,500\,000 \times 12$

$= 10\,200\,000$

$=$ 102km

b) Area scale = 1cm^2 to $(8,5)^2$ km^2

$289\text{km}^2 = \dfrac{289}{8,5 \times 8,5} = \dfrac{\overset{\overset{\overset{\overset{4}{\cancel{68}}}{\cancel{1156}}}{\cancel{5780}}}{\cancel{28900}}}{\underset{\cancel{17}_1}{\cancel{8,5}} \times \underset{\cancel{17}_1}{\cancel{8,5}}}$

$=$ 4cm^2

Now do the following:

1. A piece of alloy is a mixture of copper and zinc whose masses are in the ratio 11:16.

If there are 138g of zinc in the piece, find its total mass, in kg.

2.

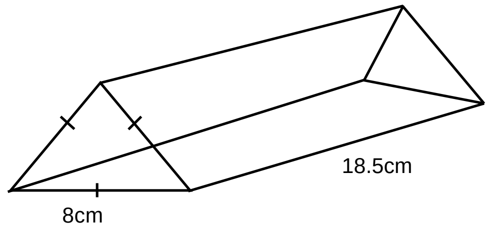


Fig. 6.1

Fig 6.1 shows a prism with a face of an equilateral triangle of sides 8cm.
If the prism is 18,5cm long and weighs 1,2kg, calculate the density of the prism, correct to 2 significant figures.

3. a) A Telecom company wants a 5km long trench dug for its service line.
If 10 men take 14 days to complete the job, how long will 7 men take, assuming the men work at the same rate?

b) Convert the density 1,3g/cm^3 to kg/m^3.

4. A contract worker, at a certain factory, is paid \$2 a day for working 8 hours a day, 5 days a per week.

a) Find the worker's hourly rate of pay.

b) Find the worker's total pay if he works at this factory for a month.

c) During a certain month, the man worked for only 5 hours per day. His pay, after working for 12 days, was \$24. Calculate the worker's new hourly rate of pay.

5. On a certain map, the distance from Murehwa Growth Point to Mutoko Growth Point is 50cm.
The actual distance is 40km.

a) Find the scale of the map, giving your answer in the form 1:n.

b) Calculate the actual distance, in km, between two schools which are 20cm apart on the map.

c) The actual area of Murehwa Growth Point is 7,5km^2. Find this area on the map, in cm^2, to 3 significant figures.

Table 6.1 shows the exchange rates on a certain day.
Use this table to answer questions 6 and 7 below.

Table 6.1 **Foreign exchange rates**

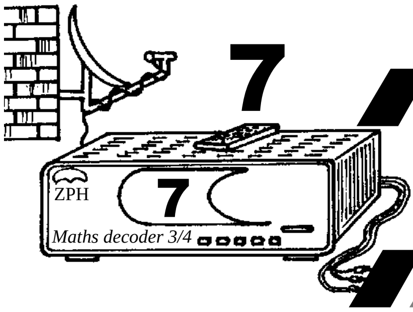
Country	Currency	Units to US$
Botswana	Pula (P)	6,1509
Kenya	Shilling (KES)	74,4260
South Africa	Rand (ZAR)	7,3439
Britain	Pound (£)	1,5967

6. A British tourist visits South Africa and declares £3 500 on arrival. She then changed half of this amount into Rands, using the US$ rate above.

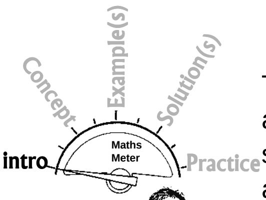
a) How many Rands did she get? Give your answer to the nearest whole number.

b) When she left South Africa for Botswana she was left with $\frac{1}{10}$ of the Rands she had obtained. She changed this amount in to Pulas. Calculate how many Pulas, to the nearest Pula, she received.

7. The tourist in number 6 flew to Kenya where she stayed for some time. On arrival in Kenya, she changed $\frac{2}{5}$ of her remaining pounds into Kenyan Shillings, using the exchange rate above. How many Shillings did she get?



7 Basic Geometric Concepts



This chapter will revise some of the geometric concepts you learned at lower levels of your study of mathematics and will introduce some rules of geometry you need to be familiar with. Polygons and their properties will also be discussed.

Syllabus Expectations

By the end of this chapter, students should be able to:

1. identify, interpret and apply concepts associated with points, line segments, parallels and perpendiculars.
2. identify types of angles like acute, right, obtuse, straight and reflex and deduce information associated with complementary, supplementary, vertically opposite angles formed by parallel lines and the transversal line and angles at a point.
3. use properties of polygons (from triangle to n-sided polygons) to calculate angles.
4. identify special names of n-sided polygons.
5. identify lines of symmetry and order of rotational symmetry of 2-dimensional figures.

ASSUMED KNOWLEDGE

In order to tackle work in this chapter, it is assumed that students are able to:

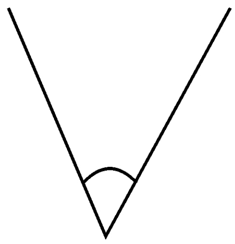
- use a protractor to draw and measure angles.
- use the terms: acute, obtuse, reflex, right and straight, with reference to angles.
- identify a diagonal in a polygon.
- use a ruler to draw straight lines.

A. THE PROTRACTOR

Study a protractor carefully. Do you remember its three main features? i.e. a) The two angle scales
b) The centre point
c) The base line

How comfortable are you with a protractor?
Let us revise this very important skill – **protractor use**.

1. Use the protractor to measure the marked angles.

a)


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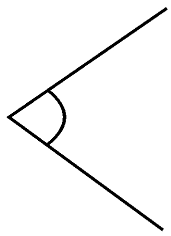

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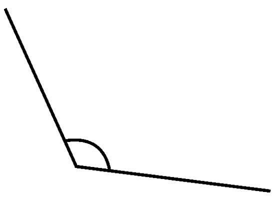

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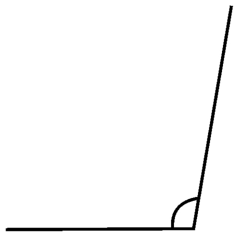

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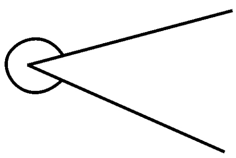

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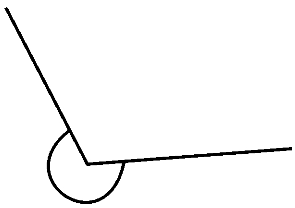


Fig 7.1

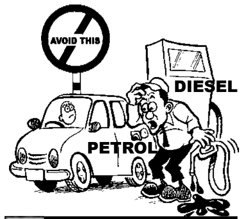
The art of estimation is very important in mathematics.

2. Using a ruler only, try to draw angles of the following sizes:

a) 30^0 b) 60^0 c) 150^0 e) 45^0

3. Now use a protractor on the angles drawn, to check how good your estimation is.

It is hoped that this exercise has helped you to visualise and internalise the angle sizes.



Common Error

Supplementary angles are two angles which add up to 180^0. Once '**two**' is missing, the definition becomes wrong.

Do you remember that:

(i) **Acute angles** are those between 0^0 and 90^0.
(ii) **Obtuse angles** are those between 90^0 and 180^0,
(iii) **Reflex angles** are those between 180^0 and 360^0?

The boundaries 90^0, 180^0 and 360^0 are special angles.
90^0 is called a right angle.
180^0 is called a straight angle.
360^0 is called a complete revolution or turn.

Also remember:
Complementary angles are two angles which add up to 90^0.
Supplementary angles are two angles which add up to 180^0.

Sometimes one is asked to draw rough diagrams usually called sketches. In the sketch, acute angles should look acute and obtuse angles also obtuse. This helps to avoid distortions in the diagram due to incorrect proportions.

Consider the following examples:

1. Sketch $\triangle$ ABC where $\hat{A} = 90^0$, AB = 5cm and AC = 8cm.

Solution

1.a)

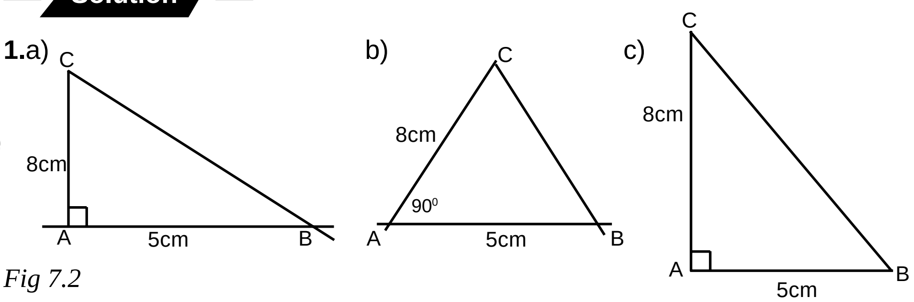


Fig 7.2

The sketches a) and b) are distorted. Can you say why?
Why is c) a better sketch?

Draw sketch diagrams of the triangles with the following measurements:

1. ABC in which $\hat{A} = 120^0$, $\hat{B} = 30^0$ and AB = 4cm.

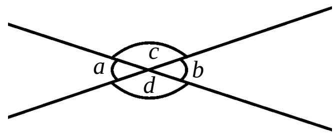
2. PQR in which PQ = 8cm, QR = 6cm and PR = 4cm.

3. $\triangle$ XYZ where $\hat{X} = 90^0$, XY = 9cm, XZ = 6cm.

4. $\triangle$CDE where CD = CE and $D\hat{C}E = 75^0$.

5. $\triangle$XYZ where all sides are equal.

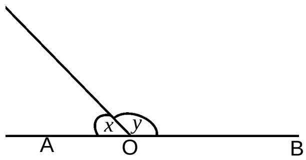
B. BASIC RULES OF ANGLES

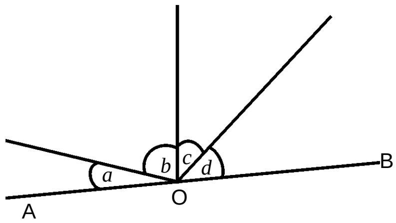


When two straight lines intersect (cross each other), **vertically opposite angles** are formed. a and b are vertically opposite angles. So are c and d.

By simple measurement with a protractor, one will find that:

* **Vertically opposite angles are equal.**

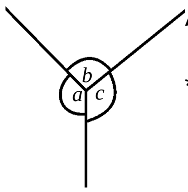




(ii) Angles x and y or a, b, c and d are adjacent angles on a straight line AOB.

* **Angles on a straight line add up to 180^0, ie $x + y$ =180^0, a + b + c + d = 180^0.**

(iii)

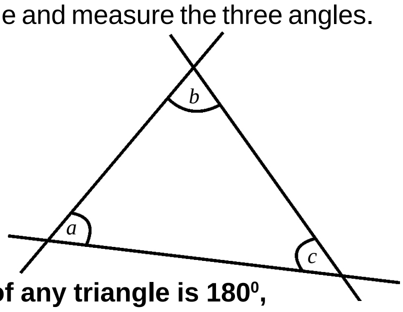


Angles a, b, and c are around a point.

* **Angles around a point add up to 360^0, a + b + c = 360^0.**

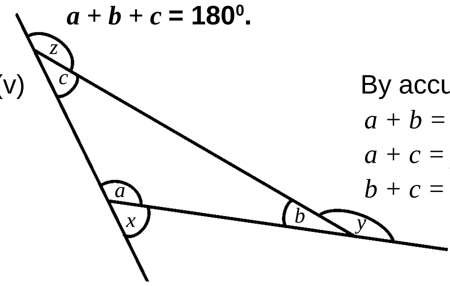
It is very important that you check the validity of each rule by drawing sketches and measuring the angles.

(iv) Draw any type of triangle and measure the three angles. What is their sum?



* **The sum of angles of any triangle is 180^0, $a + b + c$ = 180^0.**

(v)



By accurate measurement,

$a + b = z$

$a + c = y$

$b + c = x$

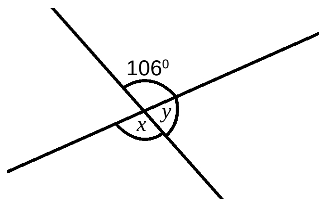
* **The sum of two interior angles of a triangle is equal to the exterior opposite angle.**

As you do the following exercise, identify the rule you have used to do the calculations.

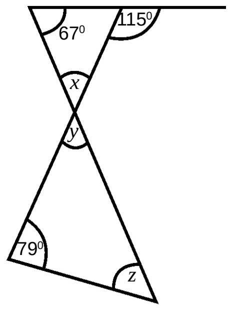
Consider the following examples:

Find the lettered angles in the diagrams.

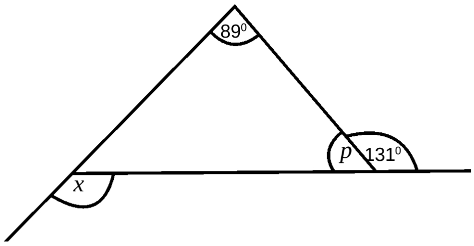
1.



2.



3.



Solutions

1. $x = 106^0$ vertically opposite angles.

$y + 106^0 = 180^0$ Adjacent angles on a straight line.

$y = 180^0 - 106^0$

$= 74^0$

2. $x + 67^0 = 115^0$ Sum of two interior angles equals the exterior opposite angle.

$\therefore\ x = 115^0 - 67^0$

$= 48^0$

$x = y$ Vertically opposite angles.

$\therefore\ y = 48^0$

$y + z + 79^0 = 180^0$ Sum of angles of a triangle.

$48^0 + z + 79^0 = 180^0$

$\therefore\ z = 180^0 - 127^0$

$= 53^0$

3. $p + 131^0 = 180^0$

$p = 180^0 - 131^0$

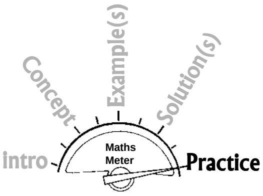
$= 49^0$

$\therefore\ x = 87^0 + 49^0$

$= 136^0$

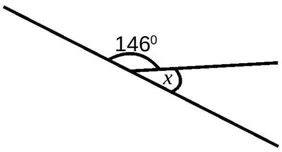
Snap check ☑ $87 + 49 + (180 - 136) = 180^0$ (Total of angles in a triangle)

$136 + 44 = 180^0$

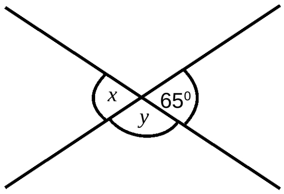


1. Calculate the value of x and the value of y in each case.

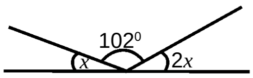
a)



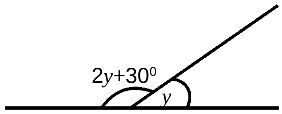
b)



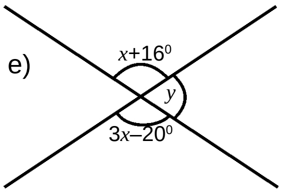
c)



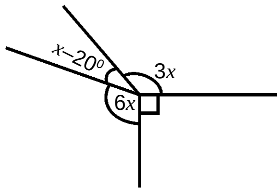
d)

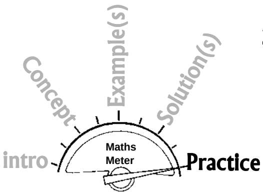


e)



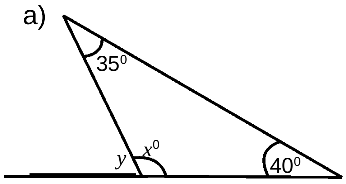
f)



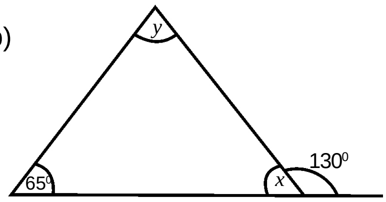


2. Calculate the value of x and the value of y in each case.

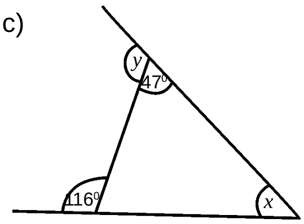
a)



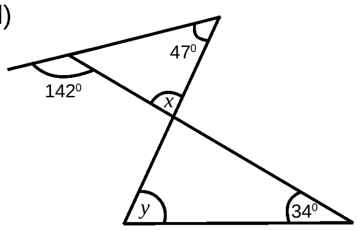
b)

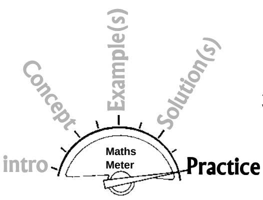


c)



d)





3. The three angles of a triangle are $x + 20^0$, $2x - 30^0$ and $x + 26^0$. Find x, then find the actual angles.

4. A triangle has the following angles, $2x$, $x + 60^0$ and $3x$. Find the smallest angle.

5. Define a) Supplementary angles
b) Complementary angles

C. ANGLES ON PARALLEL LINES

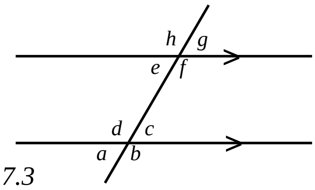


Fig 7.3

The diagram shows two parallel lines being cut by a third line called the **transversal line**.

Thus eight angles are formed (labelled a to h).
From the diagram, can you name:
(i) vertically opposite angles?
(ii) angles on a straight line?

h and d are called **corresponding angles** because they are on corresponding positions in the diagram i.e. on the left side of the transversal line and above the parallel lines. The diagram has three more pairs of such angles, name them. An accurately drawn diagram will show that:

*** Corresponding angles in parallel lines are equal.**

But f and d form a zig-zag pattern! Can you identify another pair which behaves in the same manner? These are called **Alternate Angles**. They are sometimes called Z angles, from the zig-zag pattern.

* **Alternate angles in parallel lines are equal.**

e and d are called **Co-interior or Allied angles**. Please identify another pair in the diagram.

* **Allied angles in parallel lines add up to 180⁰**.

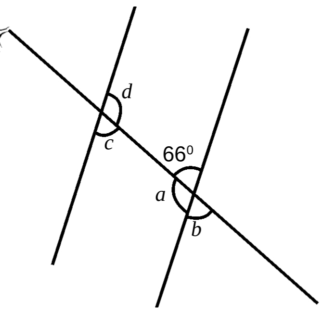
Note that these types of angles, alternate, corresponding and allied are discussed only in parallel lines, not anywhere else.

Consider the following examples:

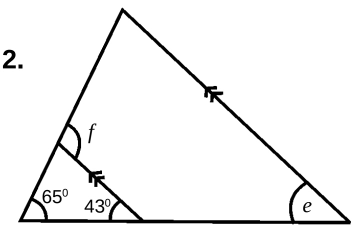
Hint
Compare each lettered angle to the given one.

Calculate all the labelled angles in the diagrams.

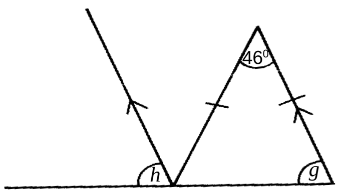
1.



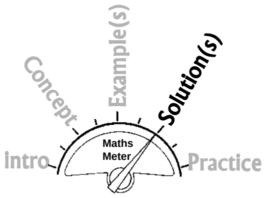
2.



3.



Solutions



1. $a + 66^0 = 180^0$ Adjacent angles on a straight line.

$a = 180^0 - 66^0$

$= 114^0$

$b = 66^0$ Vertically opposite angles.

$c = 66^0$ Z angles.

$d + 66^0 = 180^0$ Co-interior angles.

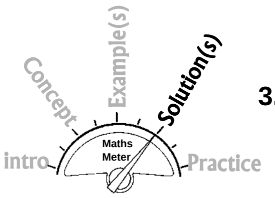
$d = 66^0 - 66^0$

$= 114^0$

2. $e = 43^0$ Corresponding angles.

$f = 65^0 + 43^0$ Sum of 2 interior angles.

$= 108^0$

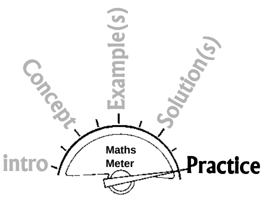


3.
$$g = \frac{180^0 - 46^0}{2}$$

$$= \frac{134^0}{2}$$

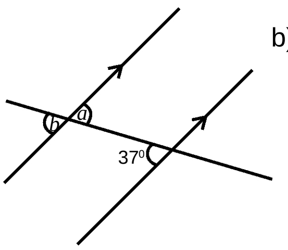
$$= 67^0$$

$h = 67^0$ corresponding angle with g.

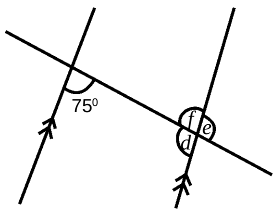


1. State the size of the lettered angles in each diagram. Make sure you always state the rule applied.

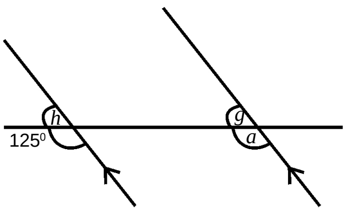
a)



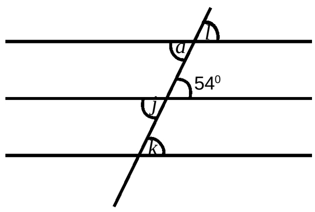
b)



c)

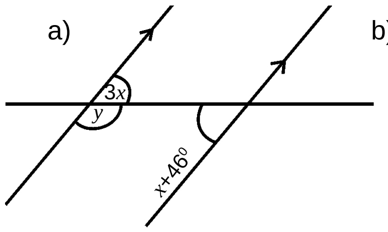


d)

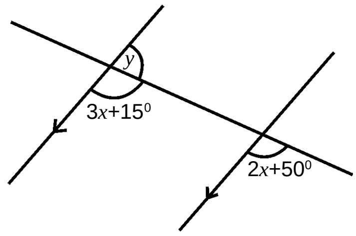


2. Calculate the value of x and the value of y in each case.

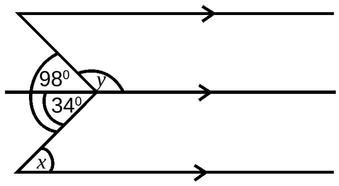
a)



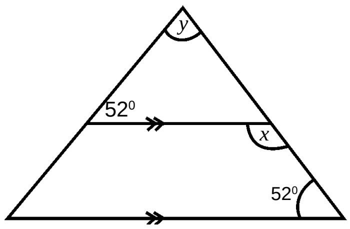
b)



c)



d)



D. ANGLES IN A POLYGON

A polygon is any closed shape whose sides are straight lines. The most basic one is a triangle (3 sides). From the triangle, we have polygons of any number of sides. Remember, a polygon has two kinds of angles as illustrated below.

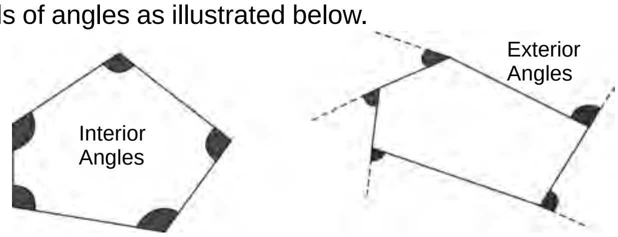


Fig 7.4

As can be seen from the two diagrams above,

INTERIOR ANGLES	are made by adjacent sides of the polygon and lie inside the polygon.
EXTERIOR ANGLES	are made by a side of the polygon and an extension of its adjacent side and lie outside the polygon.

It is very important to observe that the interior and its exterior angle form a straight angle, (180^0).

By accurate drawing and measuring you can prove that:

* **The exterior angles of any polygon always add up to 360^0.**

The sum of the interior angles depends on the type of polygon. We have already established that angles of a triangle add up to 180^0. This fact can be used to establish sums of other polygons. This is done by dividing the polygon into unintersecting triangles Below is an illustration of this:

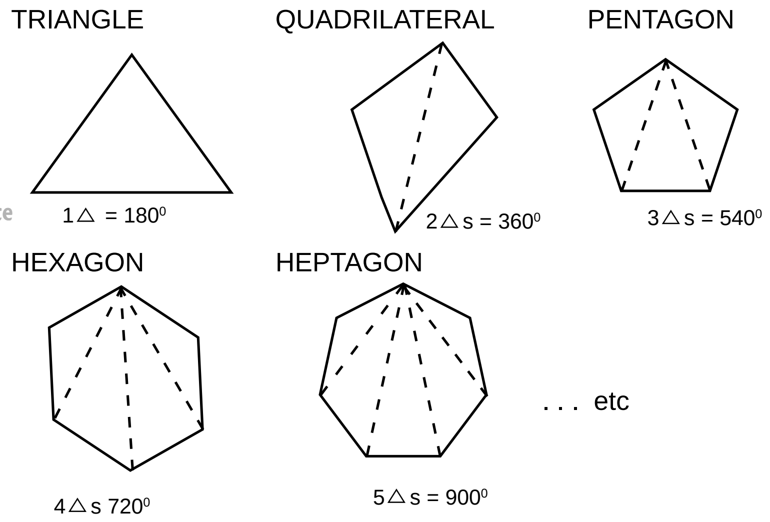


Fig 7.5

Study the table summarising the information regarding the above diagrams.

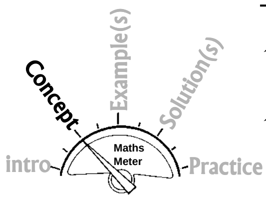
Table 7.1

Number of sides in SHAPE	NAME	Sum of INTERIOR ANGLES
3	Triangle	$1 \times 180^0 = 180^0$
4	Quadrilateral	$2 \times 180^0 = 360^0$
5	Pentagon	$3 \times 180^0 = 540^0$
6	Hexagon	$4 \times 180^0 =$
7	Heptagon	$5 \times 180^0 =$
8	Octagon	$6 \times 180^0 =$
9	Nonagon	$7 \times 180^0 =$
10	Decagon	$8 \times 180^0 =$
;	;	
;	;	
n	n-sided polygon	$(n - 2)180^0$

From the above, the number being multiplied by 180^0 is the number of triangles in the polygon.
This means, in an n-sided polygon there are $(n–2)$ triangles.

* Hence **the sum of the interior angles of any polygon is given by $(n–2)180^0$, where n is the number of sides of the polygon.** Other authorities give this as $(2n–4)90^0$ where $2n–4$ is the number of right angles in the polygon.

Polygons can be regular or irregular.
i.e.
Regular meaning all sides and angles are equal.
Irregular meaning not all sides or angles are equal.



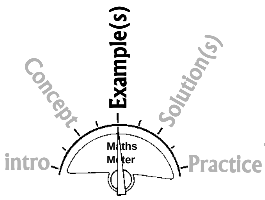
To summarise

- **Each exterior angle of any regular polygon is equal to $\frac{360^0}{n}$, where n is the number of sides or angles.**
- **Each interior angle of any regular polygon is equal to $180^0 - \frac{360^0}{n}$**
  i.e. by finding the exterior first then using the straight angle to find the interior angles

or $\frac{(n-2)180^0}{n}$ i.e. the sum total of the interior angles divided by the number of angles.

or $\frac{(2n-4)90^0}{n}$

Consider the following examples:



1. A polygon has 15 sides. Find the sum of its:
   a) exterior angles.
   b) interior angles.

2. Calculate the number of sides of a polygon whose interior angles add up to 1620^0.

3. Find the number of sides of a regular polygon whose exterior angle is 36^0.

4.

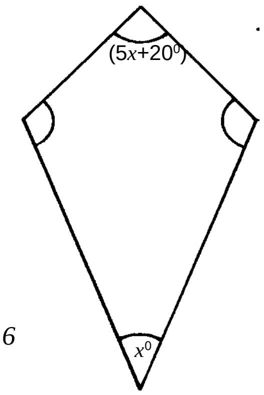


Fig 7.6

Find x.

Solutions

1. a) 360^0 always!

b) $(15-2)180^0 = 13 \times 180^0$
$= 2\ 340^0$

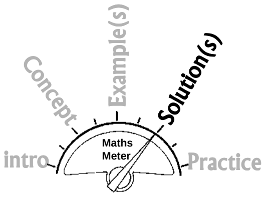
Hint

Connect the information that is given to the established facts.

2. Sum $= (n-2)180^0$ This is relevant since the sum is given.

$\therefore\ (n-2)180 = 1620$
$180n - 360 = 1620$
$180n = 1980$
$n = 11$

$\therefore$ The polygon has 11 sides.



3. Use $\frac{360^0}{n}$ This is relevant since the exterior is angle is given.

$\therefore \quad \frac{360^0}{n} = 36^0$

$\therefore \quad 36n = 360$

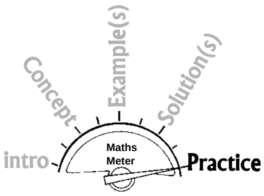
$n = 10$

$\therefore$ The polygon has 10 sides

4. $4x + x + 10^0 + 5x + 20^0 + x = 360^0$

$11x = 330^0$

$x = 30^0$



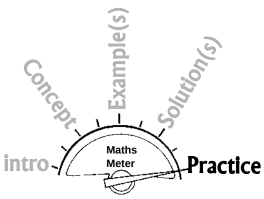
1. Calculate the sum of the interior angles of polygons with the following number of sides:
a) 7 b) 12 c) 21 d) 45 e) 112

2. Calculate the size of each interior angle of a regular polygon with the following sides:
a) 8 b) 15 c) 38 d) 74 e) 122

3. Find the number of sides of the polygon whose interior angles add up to:
a) 1800^0 b) 1260^0 c) 2520^0 d) 8640^0 e) 6120^0

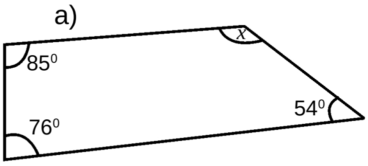
4. Find the number of sides of a regular polygon whose exterior angle is:
a) 24^0 b) 60^0 c) 90^0 d) 15^0 e) 10^0

5. Find the number of sides of a regular polygon whose interior angle is:
a) 150^0 b) 165^0 c) 171^0 d) 156^0 e) 140^0

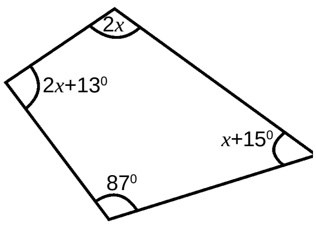


6. Calculate the value of x in each case.

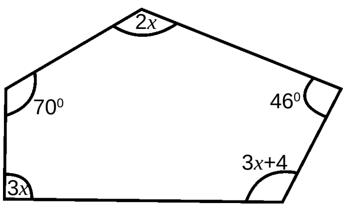
a)



b)



c)



7. The interior angle of a certain regular polygon is twice its exterior angle. Find out how many sides it has and give the special name of this polygon.

8. Wiseman, in Form 3B4, measured all the interior angles of a polygon and added them up.
Unfortunately, he skipped an angle when he was adding
Find out: a) the type of polygon Wiseman measured,
b) the size of the missing angle,
when he got a total of:
(i) 268^0 (ii) 452^0 (ii) 997^0

9. The exterior angles of a polygon are in the ratio 4: 5: 6: 7: 8.
a) What type of polygon is it?
b) Find the largest interior angle.

E. SPECIAL TRIANGLES

A triangle whose sides or angles are not equal is called a **scalene triangle**.
However, all the sides or angles can be equal. Such a triangle is an **Equilateral triangle.**
What is always the size of each angle in an equilateral triangle?

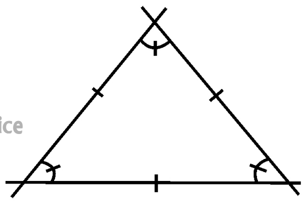


Fig 7.7

In some cases only two sides or angles are equal. Such a triangle is called an **Isosceles triangle**.

Notice that, the equal angles are always opposite the equal sides.
The equal angles are sometimes called base angles.

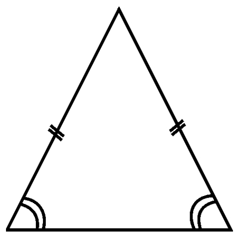
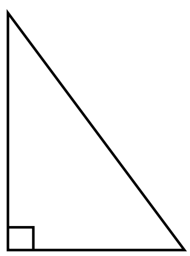
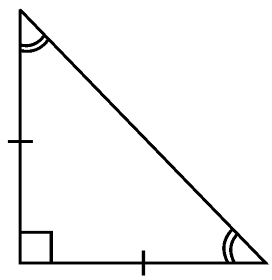


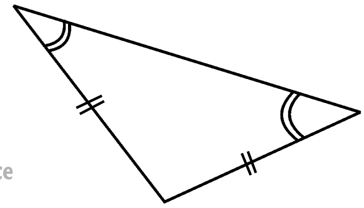
Fig 7.8

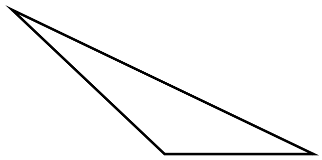
Other types of triangles we meet are:
(i) Right-angled (ii) Right-angled isosceles





(iii) Obtuse angled isosceles (iv) Obtuse angled scalene



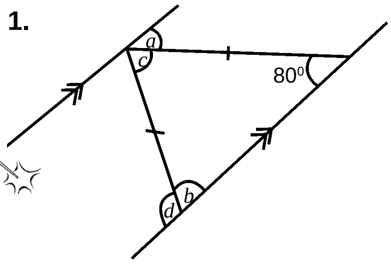


Consider the following examples:

Hint

Consider the diagram, Identify some important features in it. Then think of the angle properties of these.

1.



Find the lettered angles in the diagram

Solution

In this case there are parallel lines, and an isosceles triangle. The given angle is a base angle.

$a = 80^0$ alternate Ls

$b = 80^0$ base Ls

$c + b + 80^0 = 180^0$ Sum of Ls of a

$c = 180^0 - (80^0 + 80^0)$

$= 180^0 - 160^0$

$= 20^0$

$d + b = 180^0$ Ls on a straight line.

$d = 180^0 - 80^0$

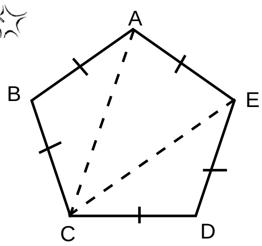
$= 100^0$

Hint

Draw a sketch first. A sketch should look in proportion i.e. equal features should look possible. oObtuse angles should look obtuse and acute angles should look acute.

2. Given that ABCDE is a regular pentagon, calculate:

a) $A\hat{B}C$ b) $C\hat{E}D$ c) $A\hat{C}D$



Solution

a) $A\hat{B}C$ is the interior angle

$\therefore \quad = \frac{(5-2)180^0}{5}$

$= 3 \times 36$

$= 108^0$

or

Exterior L at B $= \frac{360^0}{5}$

$= 72^0$

$\therefore \ A\hat{B}C = 180^0 - 72^0$

$= 108^0$

b) In $\triangle$CED

$\triangle$ is isosceles and $C\hat{D}E = A\hat{B}C = 108^0$

$\therefore\ C\hat{E}D = \frac{180^0 - 108^0}{2}$

$= \frac{72}{2}$

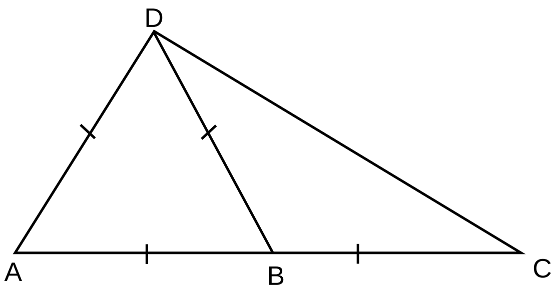
$= 36^0$

c) $A\hat{C}D = B\hat{C}D - B\hat{C}A$

$= 108^0 - 36^0$ $(B\hat{C}D = A\hat{B}C = 108^0;\ B\hat{C}A = C\hat{E}D = 36^0)$

$= 72^0$

3.



In the diagram,

$\triangle$ ABD is equilateral and

$\triangle$ BCD is isosceles

Find $A\hat{D}C$

Solution

$A\hat{D}C = A\hat{D}B + B\hat{D}C$

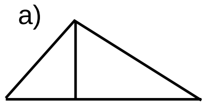
$B\hat{D}C = 60^0 \div 2$

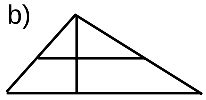
$= 30^0$

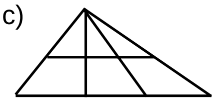
$\therefore\ A\hat{D}C = 60^0 + 30^0$

$= 90^0$

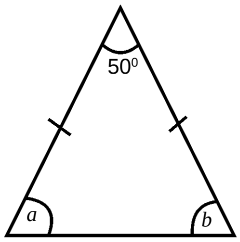
1. How many triangles are in the diagram?

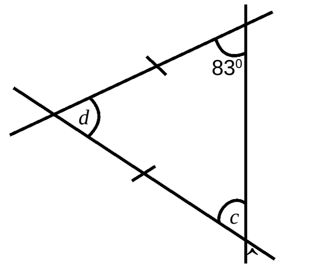
a) 

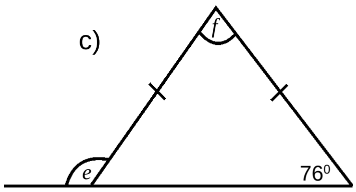
b) 

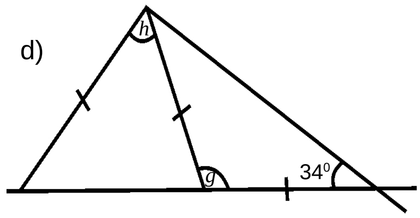
c) 

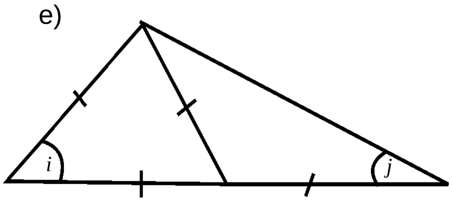
2. Calculate the lettered angles in each diagram.

a) 

b) 

c) 

d) 

e) 

3. The three angles of an isosceles triangle are x^0, $(2x - 5)$ and $(2x - 5)^0$. Find the actual sizes of the angles.

4. An isosceles triangle has an angle of 54^0. Draw sketches of two different possible triangles, which fit the given description, showing the sizes of the angles in each case.

5. Given that PQRST is a regular pentagon, calculate $P\hat{S}Q$.

6. Given that ABCDEF is a regular hexagon, calculate:
a) $B\hat{C}D$ b) $B\hat{C}A$ c) $F\hat{C}D$

7. Given that ABCDEFGH is a regular octagon, calculate:
a) $F\hat{G}H$ b) $H\hat{D}C$ c) $D\hat{F}H$

F. SPECIAL QUADRILATERALS

A quadrilateral is any four-sided figure.

The following special quadrilaterals and their angle properties are important to remember. Make sure you are able to show their properties in sketch diagrams.

a) **The Square**
All sides are equal.
All angles are 90^0.

Diagonals are equal in length.
Diagonals bisect each other at right angles at the centre of the square.
Diagonals bisect the angles they pass through.

b) **The Rectangle**
All angles are 90^0.
Opposite sides are equal.
Diagonals are equal in length.
Diagonals bisect each other at the centre.

c) **The Parallelogram**
Opposite sides are equal and parallel.
Opposite angles are equal.
Diagonals bisect each other at the centre.

d) **The Rhombus**
All sides are equal.
Opposite angles are equal.
Diagonals bisect each other at right angles.
Diagonals bisect the angles they pass through.

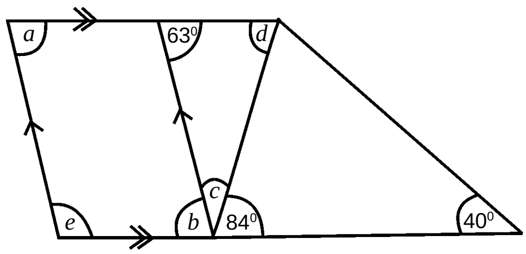
e) **The Trapezium**
Has a pair of parallel sides.

f) **The Kite**
Has two pairs of equal adjacent sides.
Has a pair of equal angles (Angles between the unequal sides).
The longer diagonal bisects the shorter diagonal at right angles.

The properties are used to calculate angles and sides of given figures. Make sure you recognise the properties where they exist.

Consider the following example:

1. Calculate the lettered angles.



Solution

1. $a = 63^0$ (Corresponding *L*s.)
$a = b$ (Opposite *L*s of a parallelogram.)
$\therefore\ b = 63^0$ (or Alternate *L*s.)

$b + c + 84^0 = 180^0$ ($\angle$s on a straight line.)

$c \quad = 180^0 - (84 + 63)$

$\quad = 180^0 - 147^0$

$\quad = 33^0$

$d \quad = 84^0$ Alternate $\angle$s.

$a + e = 180^0$ Allied $\angle$s.

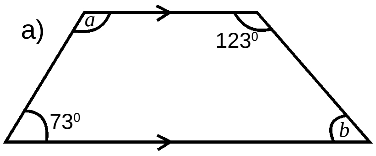
$e = 180^0 - 63^0$

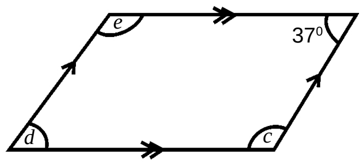
$= 117^0$

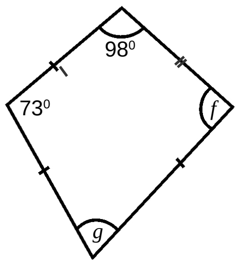
1. Are the following statements true or false? Explain your answer.

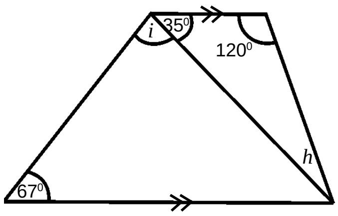
a) An equilateral triangle is an isosceles triangle.
b) An isosceles triangle is an equilateral triangle.
c) A rectangle is a parallelogram.
d) A square is a rhombus.
e) A rhombus is a square.
f) A rhombus is a kite.
g) A parallelogram is a trapezium.

2. Calculate the lettered angles in each case.

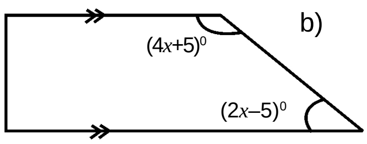
a)


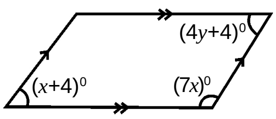
b)


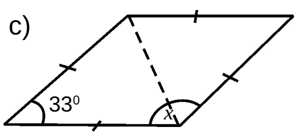
c)


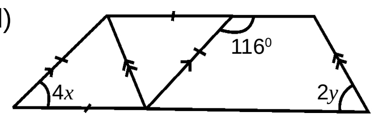
d)


3. Calculate the value of x and of y in each case.

a)


b)


c)


d)


6. Find the value of y in each of the quadrilaterals with the following angles. State the type of quadrilateral it is.

a) $y + 10^0$; $y + 30^0$; $y - 50^0$; $y - 30^0$

b) y; $y - 10^0$; $y - 10^0$; $y - 28^0$

c) $y + 15^0$; $5y + 15^0$; $2y - 10^0$; $6y - 10^0$

G. SHAPES AND SYMMETRY

Shapes have two types of symmetry.

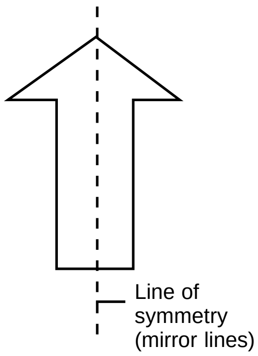
(i) **Bilateral** (lines of symmetry)

(ii) **Rotational** (Angles of turning)

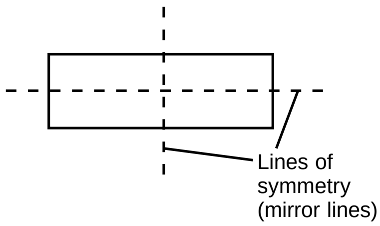
A line of symmetry is a line which can be drawn through a shape so that one side of the is a mirror image of the other side, or it is the line along which a shape can be folded to produce two identical parts.

Consider the following shapes

1.



2.



3.

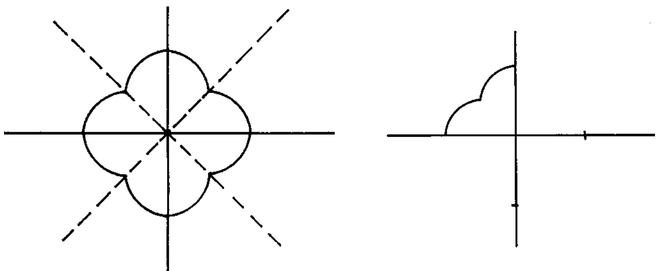


Fig 7.10

This diagram can be completed so that it has four lines of symmetry.

1. With the aid of sketches, copy and complete the table below.

Table 7.2

SHAPE	NUMBER OF LINES OF SYMMETRY
Isosceles △	
Equilateral △	
Square	

Table 7.3

SHAPE	NUMBER OF LINES OF SYMMETRY
Rectangle	
Rhombus	
e.g. Kite	1
Parallelogram	
Circle	
Regular Pentagon	

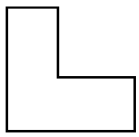
2. Consider all the capital letters of the alphabet. Write each one properly and see if it has any line or lines of symmetry. Copy and complete the table below.

Table 7.4

CAPITAL LETTER	NUMBER OF LINES OF SYMMETRY
e.g. A	1
⋮	
Z	

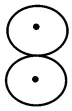
3. How many lines of symmetry does each of the following shapes have?

a)

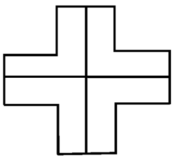


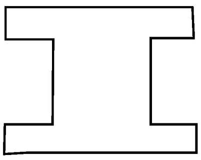
b)

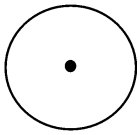
c)

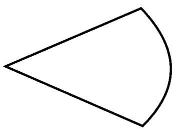


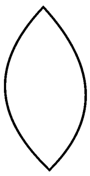
d)

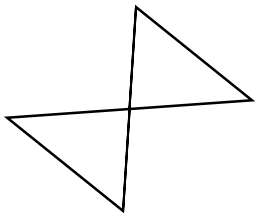






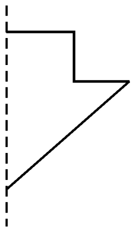




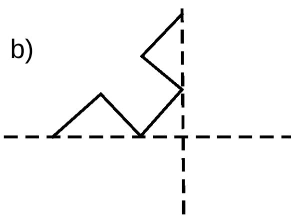


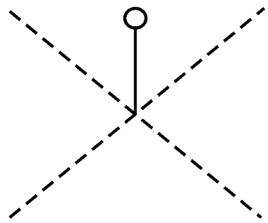
4. Copy and complete the following diagrams so that the dotted lines are the lines of symmetry of the completed shapes.

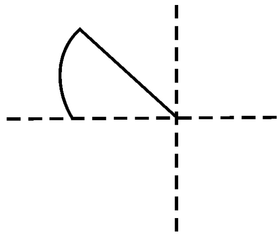
a)



b)



c) 

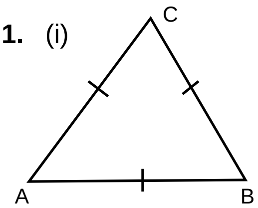
d) 

H. ROTATIONAL OR POINT SYMMETRY

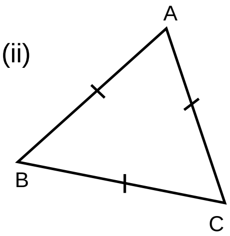
Unlike (normal) symmetry rotational symmetry involves turns about a point (usually the centre of the figure). The idea is to turn the figure until it mirrors itself. This can happen a number of times before the figure is back to its starting position. The number of times is the **order** of the rotational symmetry.

Consider an equilateral triangle:

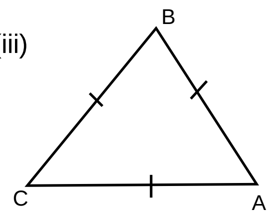
1. (i)



(ii)



(iii)



(iv)

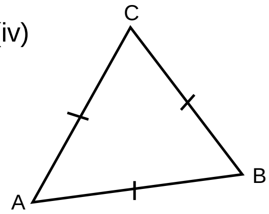


Fig 7.11

This illustrates an anticlockwise rotation about the centre of the triangle. We want to calculate the number of moves/stages say that A goes through before it comes back to its original position.

i.e. (i) Original position
(ii) A takes the position of B
(iii) A takes the position of C
(iv) A takes its original position

This shows that the figure fits exactly onto itself 3 times before it is back in its original position.

Thus an equilateral triangle has rotational symmetry of order 3.

Do you notice that at each rotation the figure has turned through 60^0?

N.B. A shape can be rotated ina clockwise or anticlockwise direction.

2. How about a rectangle

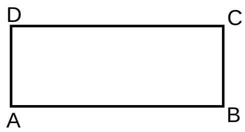


Fig 7.12

Notice that if rotated anti-clockwise about its centre. A cannot go to B to create a symmetry. It will not have moved enough for the rectangle to fit over itself.

A needs to move to C then back to itself. Hence a rectangle has rotational symmetry of order 2, as illustrated below.

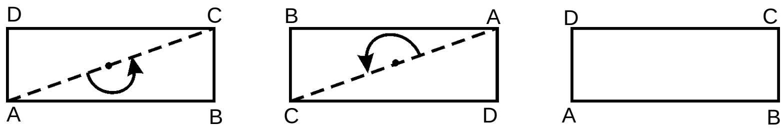


Fig 7.13

What is the angle of rotation in this case?

3. How about a regular octagon?

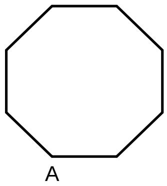


Fig 7.14

Point A stops at 7 other points before it comes back to its original position.
$\therefore$ order 8. Drawing the shape on tracing paper so you can physically see the moves helps understanding.

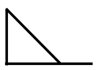
1. Investigate then copy and complete the given table below.

Table 7.5

SHAPE	ORDER OF ROTATIONAL SYMMETRY
Isosceles △	
Equilateral △	3
Square	
Rectangle	2
Rhombus	
Parallelogram	
Kite	
Trapezium	
Circle	
Regular Hexagon	

2. Compare the number of lines of symmetry and the order of rotational symmetry of regular shapes. What do you notice?

3. Consider capital letters of the alphabet
List those letters with rotational symmetry of order 2.

4. 

The diagram is incomplete. Complete the diagram given that the completed figure has rotational symmetry of order:
a) 2 b) 3 c) 4

5. How many lines of symmetry does each of the completed diagrams in question 4 have?

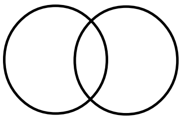
6. 

Fig 7.15

The diagram shows two equal circles intersecting at two points. Give
a) the number of lines of symmetry
b) the order of rotational symmetry of the diagram.

7. 

Fig 7.16

The diagram shows a rectangle between two semi-circles.
State: a) the number of lines of symmetry.
b) the order of rotational symmetry.

8. Draw a sketch of a square between two semi-circles, as in 7 above and state:
a) the number of lines of symmetry.
b) the order of rotational symmetry.

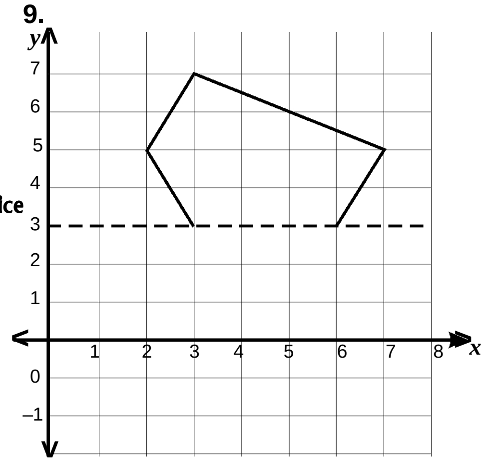
9. 

Fig 7.17

The shape shown is part of a shape which has the dotted line as its line of symmetry.
a) Copy the diagram to graph paper and complete the shape.
b) Give the order of rotational symmetry of the completed shape.

SUMMARY

1. Acute angles are those between 0^0 and 90^0.
Obtuse angles are those between 90^0 and 180^0.
Reflex angles are those between 180^0 and 360^0.

2. Vertically opposite angles are equal.

3. The angles around a point add up to 360^0.

4. Adjacent angles on a straight line add up to 180^0.

5. Of angles in parallel lines:
- Alternate angles are equal.
- Corresponding angles are equal.
- Co-interior or Allied angles are supplementary.

6. The internal angles of a triangle add up to 180^0.

7. The sum of two interior angles of a triangle add up to the exterior opposite angle.

8. The sum of exterior angles of any polygon is 360^0.

9. The sum of the interior angles of any polygon $(n - 2)1800$ where n is the number of sides or angles.

10. The table below summarises some properties of polygons:

Table 7.5

Shape	**Number of lines of symmetry**	**Order of Rotational symmetry**
Isosceles △	1	–
Equilateral △	3	3
Square	4	4
Rectangle	2	2
Rhombus	2	2
Kite	1	–
Parallelogram	0	2

EXAM PRACTICE 7

Please note that questions relating to this chapter are usually found in Paper 1 of 'O' Level examinations. One mark for most of the questions. This means candidates need to work with extreme caution to ensure their answers are correct and gain marks.

Consider the following examples:

Hint

A rhombus is often confused with a square or a parallelogram. Draw an accurate sketch or answers to relevant questions will depend on memory.

1. PQRS is a rhombus whose diagonals meet at O. State:

a) the number of lines of symmetry of the rhombus.

b) the order of rotational symmetry of the rhombus.

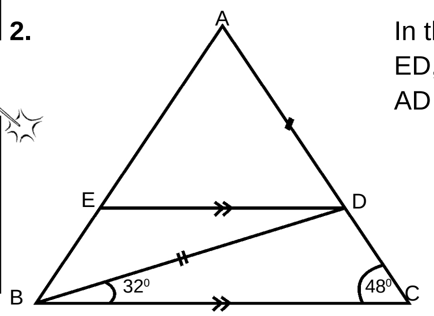
Solution

a) 2 b) 2

Hint

The diagram has an isosceles $\triangle$ BAD as well as parallel lines. Properties of these need to be used.

2.



In the diagram, BC is parallel to ED, $C\hat{B}D = 32^0$, $B\hat{C}D = 48^0$ and AD = BD. Calculate: a) $A\hat{D}B$
b) $A\hat{B}D$

Solution

$A\hat{D}E = 48^0$ Corresponding $\angle$s
$E\hat{D}B = 32^0$ Alternate $\angle$s
$\therefore A\hat{D}B = 48^0 + 32^0$
$= 80^0$

Parallel lines used

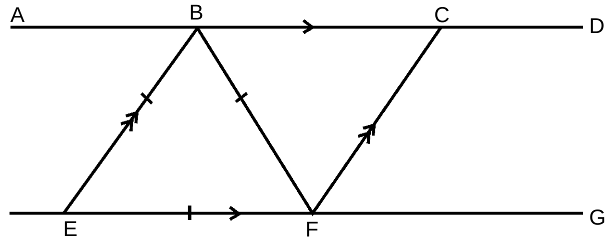
In $\triangle$BAD,
$B\hat{A}D = (180^0 - 80^0) \div 2$
$= 100 \div 2$
$= 50^0$

Isosceles $\triangle$ used

$D\hat{E}A = B\hat{A}D = A\hat{B}C$
$= 50^0$
$\therefore A\hat{B}D = 50^0 - 32^0$
$= 18^0$

Now do the following:

1.



In the diagram, the lines ABCD and EFG are parallel, BE is parallel to CF and $\triangle$ BEF is an equilateral triangle.

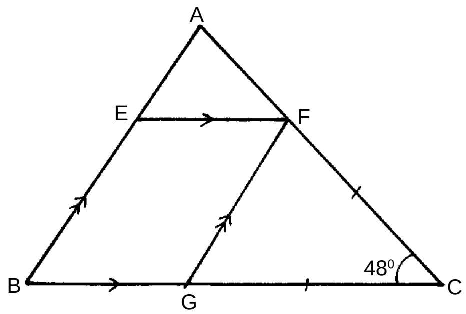
a) Calculate: (i) $C\hat{F}G$ (ii) $A\hat{B}E$

b) What type of triangle is BCF?

2. a) Find the size of each interior angle of a 25-sided regular polygon.

b) Find the number of sides of a regular polygon, whose interior angles are 108^0 each.

3.

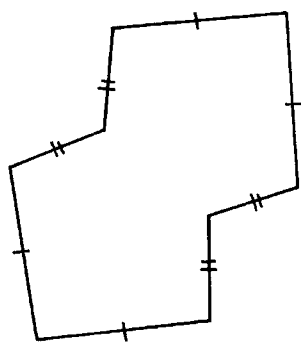


In the diagram, EF is parallel to BC and AB is parallel to FG; AEB and BGC are straight lines.

a) Given that CF = CG and $F\hat{C}G = 47^0$, find:
(i) $A\hat{F}E$ (ii) $E\hat{B}G$ (iii) $B\hat{E}F$.

b) Name the two triangles that are similar to triangle $A\hat{E}F$.

4.



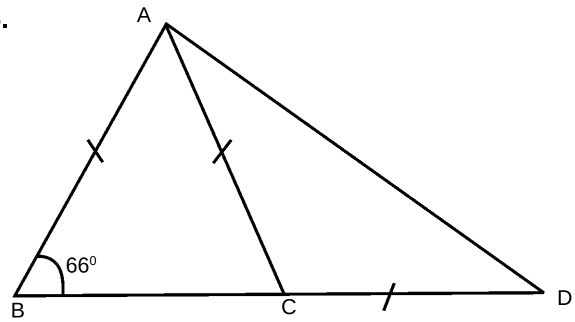
a) Give the number of lines of symmetry of the above shape.

b) State the order of rotational symmetry of the above shape.

5. a) 3 of the angles of a hexagon are $(2x)^0$, $(12 + x)^0$ and $(4 + 3x)^0$. Each of the other three is $(x + 10)^0$. Find the value of x.

b) Define: (i) complementary angles.
(ii) supplementary angles.

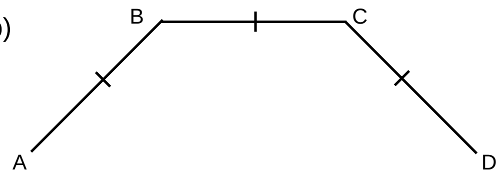
6.



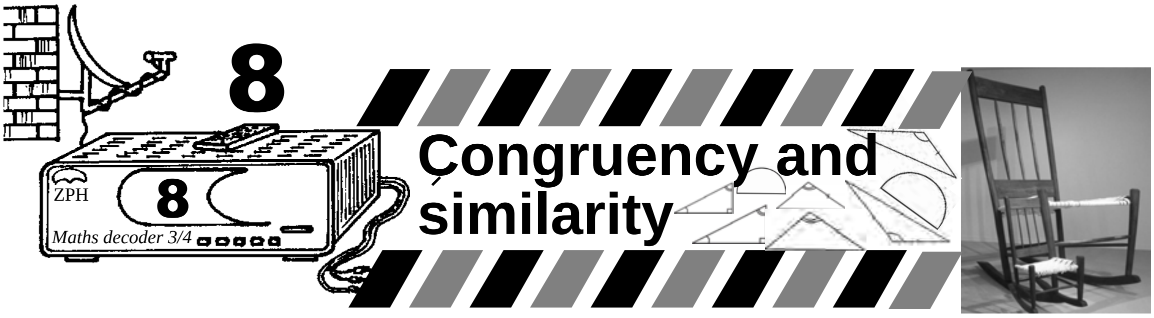
In the diagram, BCD is a straight line and triangles ABC and ACD are isosceles triangles. Given that $A\hat{B}C = 66^0$, find $B\hat{A}D$.

7. a) An isosceles triangle has an angle of 38^0. Draw sketches of two different possible triangles which fit the given description, showing the sizes of the angles in each case.

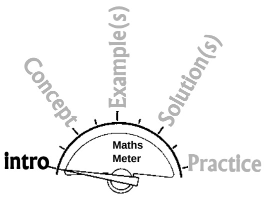
b)



The diagram shows three sides AB, BC and CD of a regular pentagon. Calculate: (i) $A\hat{B}C$
(ii) $B\hat{D}C$



8 Congruency and similarity



This chapter is going to teach you what is meant by congruent or similar figures. Two figures are said to be congruent when they exactly fit over each other. This means such figures are identical i.e. the same in all respects (shape and size). The following diagram shows four congruent triangles.

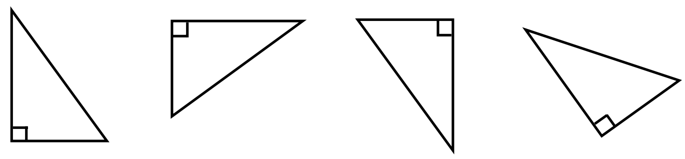
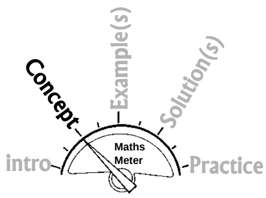


Fig 8.1

Notice that these triangles have the same shape and are all of the same size (i.e. are identical or congruent).

Figures are similar if they are **equiangular**. (i.e. corresponding angles are equal). In this case the figures have the same shape but differ in their sizes. A good example is you and your own photograph. Notice that the person on that card is really you but not exactly! This is because the person on the card is much smaller than you are. Below is an illustration of similar figures.



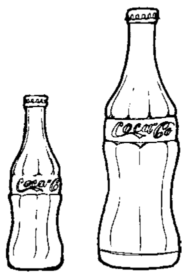


Fig 8.2a)

(i) 300ml coca cola bottles vs 1 litre coca cola bottles.

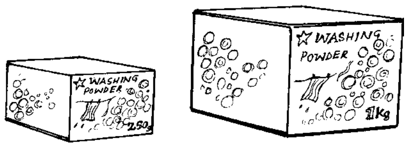


Fig 8.2b)

(ii) washing soap packs of different sizes.

(iii)

Fig 8.2c)

(iv)

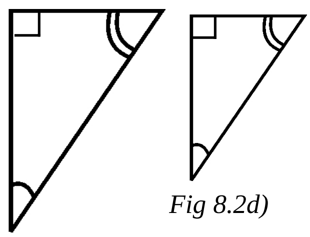
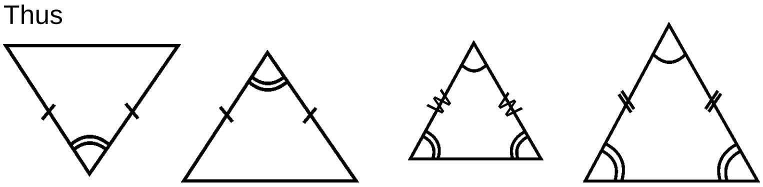


Fig 8.2d)

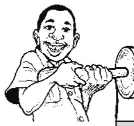
triangles with the same angles and shape but different side lengths.

Thus



Congruent　　　　　　　　　　　　Similar

Fig 8.3



Syllabus Expectations

By the end of this chapter, candidates should be able to:

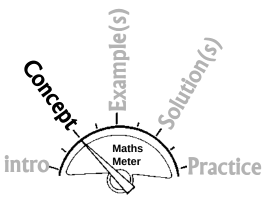
1. identify congruent as well as similar figures in given situations.
2. name congruent or similar figures correctly.
3. find sides or angles in congruent or similar figures.
4. solve problems using ratios of:
   (i) corresponding sides. (ii) areas. (iii) volumes.

ASSUMED KNOWLEDGE

In order to tackle work in this chapter it is assumed that students are able to:

- draw and construct triangles given relevant measurements.
- identify the type of triangle under discussion.
- find the missing angle in a triangle given any two of the angles.
- identify and apply angle properties in parallel lines.
- solve simple linear equations
- convert units of measurement or scales of maps in the form 1 :*n*.

A. CONGRUENT TRIANGLES



1. Cases of congruency

Triangles are congruent if: two sides and the angle between them are the same. This is identified as:

a) SAS (i.e. side-angle-side). ***Notice that*** the angle is between the given sides. This is **not** SSA nor ASS.

i.e.

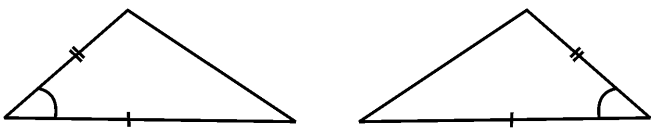


Fig 8.4

b) AAS (i.e. Angle-angle-side). This can also be identified as SAA or ASA. The angles are located on the given side.

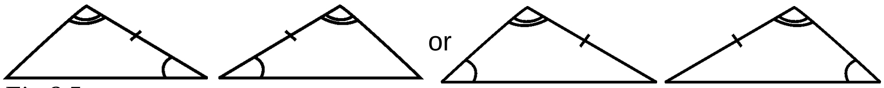


Fig 8.5

c) SSS (i.e. side-side-side). All the sides have the same measurement.

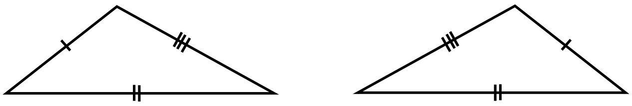


Fig 8.6

d) RHS (i.e. right angle-hypotenuse-side). The hypotenuse and one of the sides of a right angled triangle are the same.

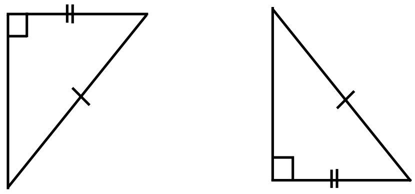


Fig 8.7

Notice that these conditions (cases) allow only one possible construction of the triangle.

These are the four cases or conditions of congruency, SAS, AAS, SSS and RHS. Next time you think the triangles you are looking at are congruent, make sure you see one of these cases in the triangles.

B. THE CONGRUENCY STATEMENT

Consider the following triangles

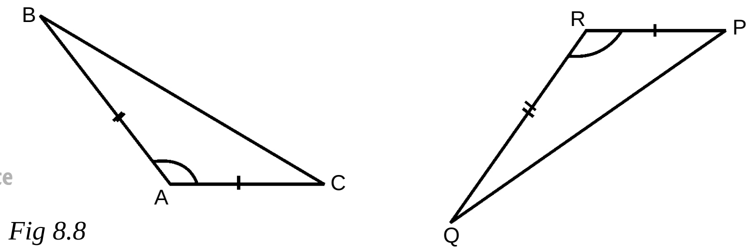


Fig 8.8

$\triangle ABC \equiv \triangle RQP$ (SAS)

(i.e. Triangle ABC is congruent to triangle RQP, the case being SAS)

$\equiv$ means is congruent to. Thus $\not\equiv$ means 'is not congruent to'.

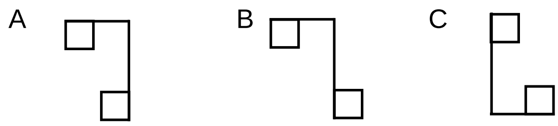
The statement above is correct since corresponding features of the triangles are in corresponding positions.

i.e. $\hat{A} = \hat{R}$, AB = RQ and AC = PR. These must be in corresponding positions in the statement.

This then means that $ABC \ncong PQR$
i.e. $\triangle$ ABC is not congruent to $\triangle$ PQR because the equal measurements do not correspond.
The order is important when naming congruent shapes.

Consider the following examples

1. Which shape is congruent to A?



2. Give the triangle that is congruent to $\triangle$ XYZ.

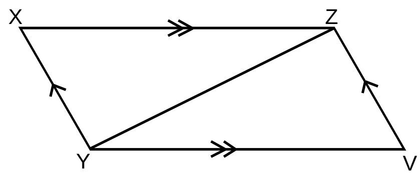


Fig 8.9

Solutions

2. C is congruent to A. **2.** $\triangle$VZY is congruent to $\triangle$XYZ.

1. State whether the pairs of shapes below are congruent or not congruent. Use your visual judgement.

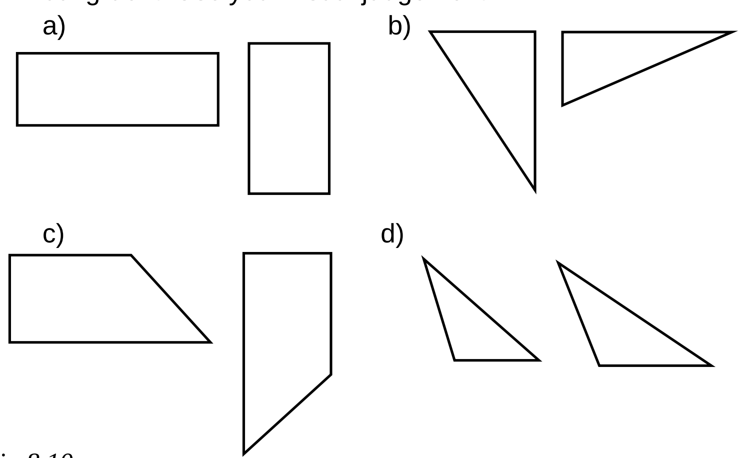
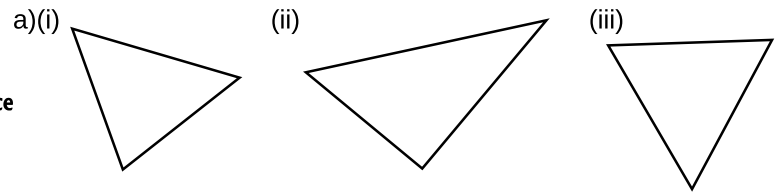
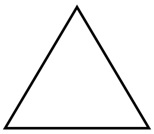


Fig 8.10

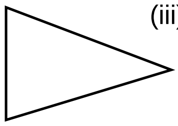
2. Which figure is not congruent to the other two?



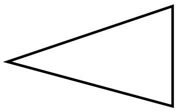
b)(i)



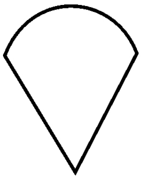
(ii)



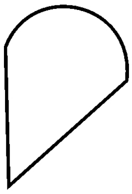
(iii)



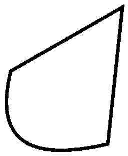
c)(i)



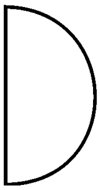
(ii)



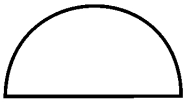
(iv)



d(i)



(ii)



(iii)

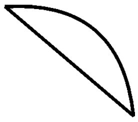
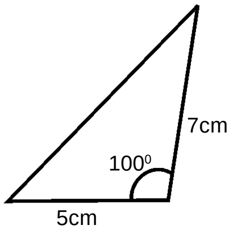
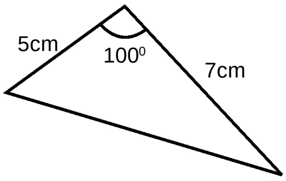


Fig 8.11

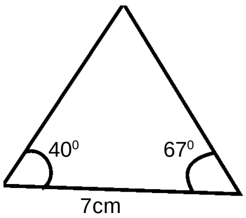
3. State whether the pairs of triangles are congruent or not congruent. If congruent, give the proof of congruency.

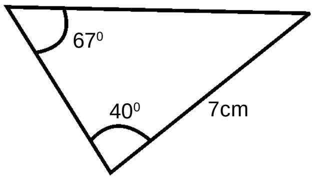
a)



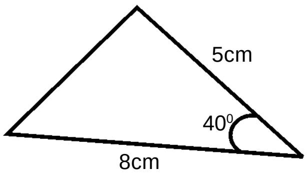


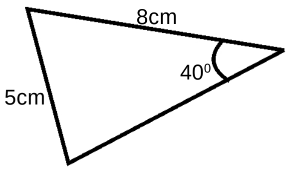
b)



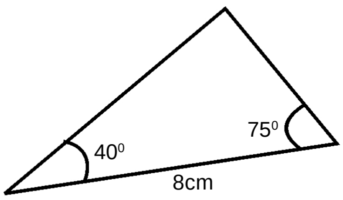


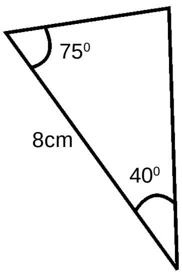
c)



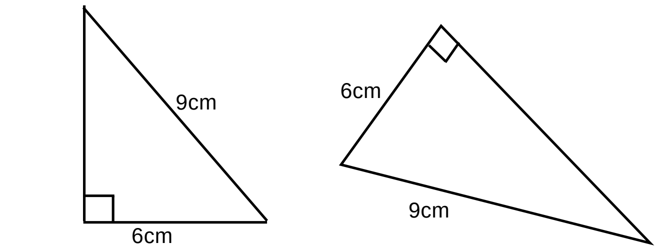


d)





e)



f)

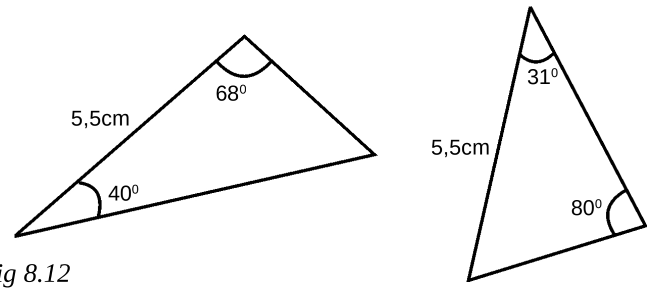


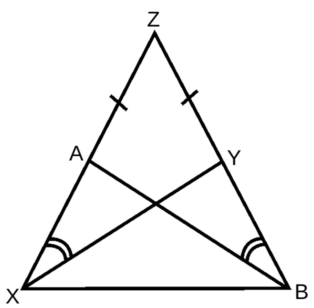
Fig 8.12

4. Draw a rectangle ABCD. Draw in the diagonals AC and BD. Name three triangles which are congruent to $\triangle$ABC.

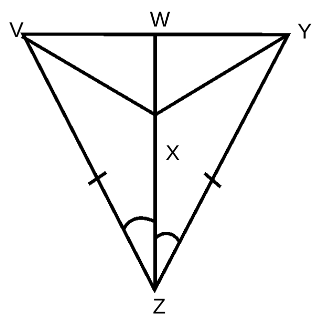
5. Draw a parallelogram PQRS. Draw in the diagonals PR and QS. Let the diagonals meet at O. Name the triangle congruent to: a) $\triangle$ OQR b) $\triangle$ POQ.

6. From each diagram, name the triangle that is congruent to $\triangle$XYZ. Use the correct order.

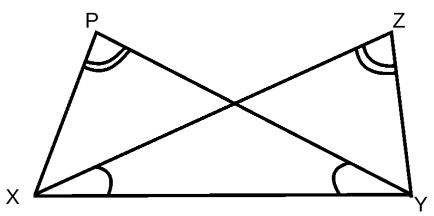
a)



b)



c)



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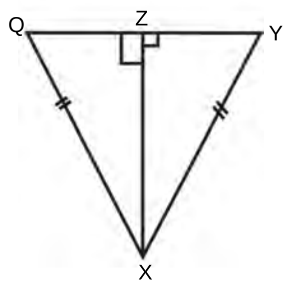


e)

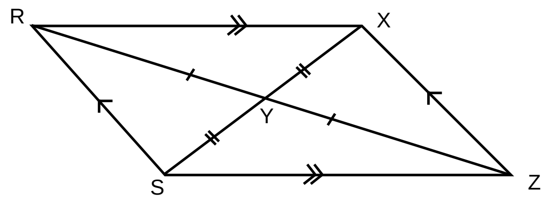


Fig 8.13

C. SIMILAR TRIANGLES

As previously stated, similar figures have the same shape but the size differs. This means angles are used to prove or show similar figures.

Consider the triangles below

1.

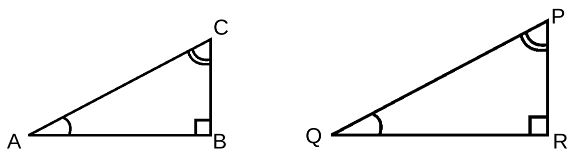


Fig 8.14

In this case △ABC III △QPR
(i.e. triangle ABC is similar to triangle QRP)
III means 'is similar to'
~~III~~ means 'is not similar to'
When triangles are similar and given in the correct order, **the ratio of their corresponding sides is the same**.

It then follows that the ratio of corresponding sides is the same i.e.
when △ ABC III △ QRP
then $\frac{AB}{QR} = \frac{BC}{RP} = \frac{AC}{QP}$ This ratio is often called **scale factor**.
It should be borne in mind that this ratio is very important, when figures are similar.

Consider the following examples:

1.

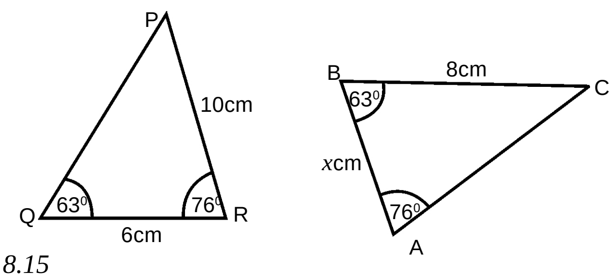


Fig 8.15

Find the value of x.

Solution

Hint

Write down the ratios right away.

1. Considering their angles, we find that the two triangles are similar.

$\triangle$ ABC ||| $\triangle$ RQP

$\therefore \quad \dfrac{AB}{RQ} = \dfrac{BC}{QP} = \dfrac{AC}{RP}$

Substitute the given lengths.

$\dfrac{x}{6} = \dfrac{8}{QP} = \dfrac{AC}{10}$

Solve the equation for x.

$\dfrac{x}{6} = \dfrac{8}{10}$

$x = 4{,}8$

Always ***make sure*** that:

(i) the order of the relevant measurements of triangles is correct.

(ii) numerators or denominators of the ratios are from the same triangle.

(iii) You work with two of the three ratios.
The third ratio with no values substituted is ignored.

2.

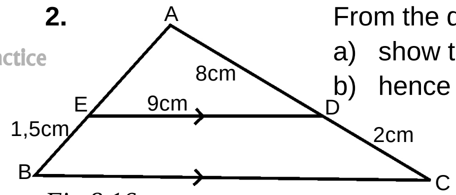


Fig 8.16

From the diagram:

a) show that $\triangle$ABC ||| $\triangle$AED

b) hence find (i) BC (ii) AE

Solution

Hint

Show means prove the angles are equal giving reasons why they are equal.

Hint

Sides are used in the ratio, so think of the ratio of corresponding sides.

2. a) $\hat{A}$ is common:

$\hat{B} = \hat{E}$ corresponding angles in parallel lines

$\hat{C} = \hat{D}$ corresponding angles in parallel lines or third angles are equal.

$\therefore \triangle$ ABC ||| $\triangle$ AED

b) i.e. $\dfrac{AB}{AE} = \dfrac{BC}{ED} = \dfrac{AC}{AD}$

(i) $\dfrac{BC}{9} = \dfrac{10}{8}$

$\therefore$ BC $= \dfrac{\overset{5}{\cancel{10}}}{\underset{4}{\cancel{8}}} \times 9$

$= \dfrac{45}{4}$

$= 11\frac{1}{4}$cm

(ii) $\dfrac{AB}{AE} = \dfrac{10}{8}$

(*but* AB = AE + 1,5)

$\dfrac{AE+1{,}5}{AE} = \dfrac{10}{8}$

8(AE+1,5) = 10AE

8AE + 12 = 10E

2AE = 12

AE = 6cm

> **Hint**
>
> *Always identify similar triangles, in the correct order. Then apply the ratio of corresponding sides to make further calculations.*

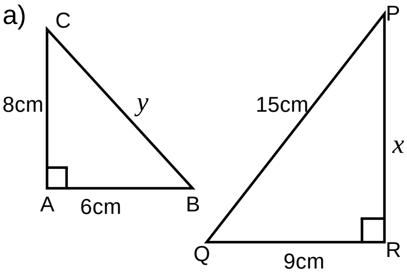
PRACTICE 8B

1. In each case:

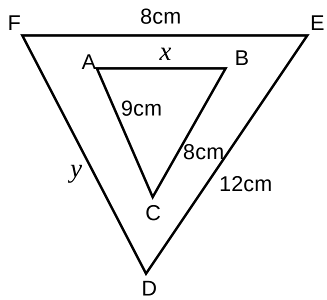
(i) state the pair of similar triangles.

(ii) find the length marked x or y.

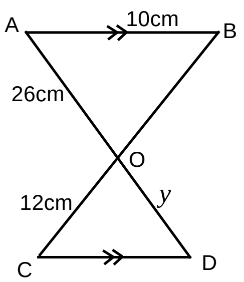
a)



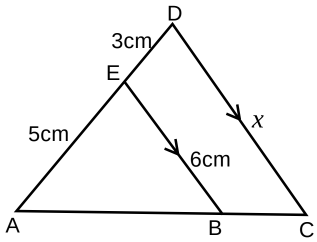
b)



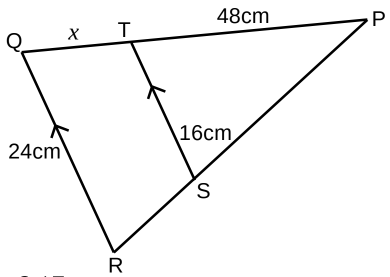
c)



d)



e)



f)

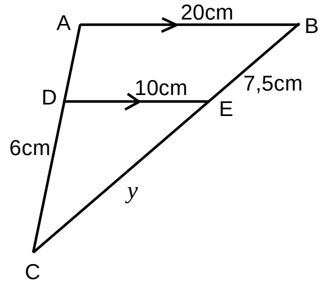


Fig 8.17

2.

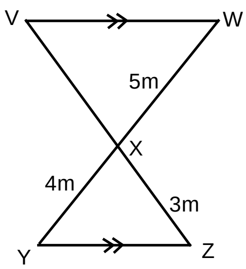


Fig 8.18

In the diagram XY = 4m, XZ = 3m and XW = 5m. VW*//*YZ.
WXY and VXZ are straight lines.

a) Show that $\triangle$VWX *///* $\triangle$ ZYX.

b) Find VX.

3. Find the height of a pole which casts a shadow of 1,2m, when at the same time in the same place, a girl of height 1,4m casts a shadow of 60cm.

D. RATIO OF AREAS AND THE RATIO OF VOLUMES OF SIMILAR FIGURES

Consider the diagram below.

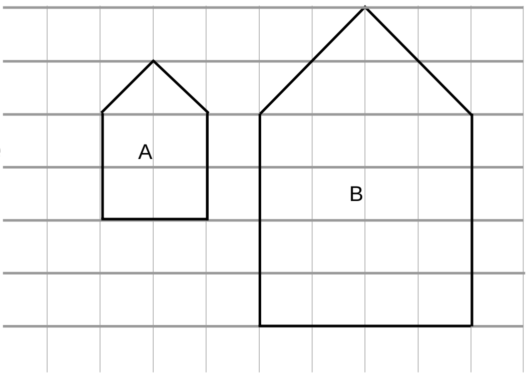


Fig 8.19

There is no doubt that the shapes are obviously similar.
Do you ***see that*** (1) that the ratio of corresponding sides = $\frac{A}{B} = \frac{2}{4} = \frac{1}{2}$?

Now consider the areas.
(Count the squares inside each shape).
Smaller area = 5 units^2
Bigger area = 20 units^2

$\therefore$ Ratio of areas $\frac{A}{B} = \frac{5}{20} = \frac{1}{4} = (\frac{1}{2})^2$

This indicates that:
(2) **The ratio of areas of similar shapes is the square of the ratio of their corresponding sides.**

Suppose the diagram is representing side views of two tins with square bases.
Then
Volume of the smaller tin A = $2 \times 2 \times 3$ units^3
= 12 units^3
Volume of the bigger tin B = $4 \times 4 \times 6$ units^3
= 96 units^3

$$\therefore \text{Ratio of their volumes} = \frac{12}{96} = \frac{1}{8} = \left(\frac{1}{2}\right)^3$$

This indicates that:
(3) **The ratio of the volumes of similar figures is the cube of the ratio of their corresponding sides.**

Make sure you understand these three ratios well.

Hint

Since the question is clearly based on similarity, deduce which ratio is relevant and make sure you bear in mind the other two.

Consider the following examples:

1. The ratio of the height of similar tins is 2:5.

a) Find the ratio of the surface areas of the tins.

b) If the smaller tin has a capacity of 160ml, find the capacity of the larger tin, in litres.

Solution

Hint

Capacity is volume!

Make sure components of corresponding volumes are on the same side e.g. bigger volumes are numerators.

1. a) Ratio of heights = 2:5

$\therefore$ Ratio of surface areas $= (2:5)^2$

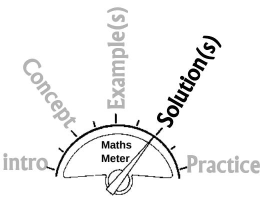
$= 4:25$

b) $\therefore$ Ratio of volumes $= (2:5)^3$

$= 8:125$

Let the volume of the larger tin be $x$ml

Hence $\frac{x}{160} = \frac{125}{8}$ i.e. Ratio of volumes

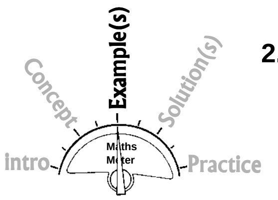


$$x = \frac{125}{\cancel{8}} \times \frac{\overset{20}{\cancel{160}}}{1}$$

$= 2\ 500$

Volume of larger tin $= 2500$ml

$= 2{,}5l$



2. The ratio of volumes of two cans is 8:27.

a) Find the ratio of the heights of the cans.

b) If the bigger can is 15cm high, calculate the height of the smaller can.

c) If the base area of the smaller tin is $12cm^2$, find the base area of the bigger tin.

Solution

Hint

Capacity is volume!

Make sure components of corresponding volumes are in the same position e.g. bigger volumes are numerators.

2. a) Ratio of volumes $= 8:27$

$\therefore$ Ratio of heights $= \sqrt[3]{\frac{8}{27}}$ (i.e. cube root of $\frac{8}{27}$)

$= \frac{2}{3}$

b) Let x be the height of the smaller can.

$$\frac{2}{3} = \frac{x}{15}$$

$$x = \frac{2}{\cancel{3}} \times \cancel{15}$$

$= 10$

$\therefore$ Height of smaller can = 10cm

c) Ratio of areas $= \left(\frac{2}{3}\right)^2$ not $\left(\frac{8}{27}\right)^2$

$= \frac{4}{9}$

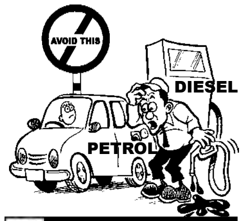
Let x be the base area of the bigger tin.

$\therefore \frac{x}{12} = \frac{9}{4}$

$x = \frac{9}{\cancel{4}_1} \times \cancel{12}^3$

$= 27$

$\therefore$ Base area of bigger tin = 27cm^2



Common Error

Ratio of areas $= \left(\frac{8}{27}\right)^2$

Using the given ratio incorrectly.

Hint

Scale is a ratio of corresponding sides.

3. A map is drawn to a scale of 1:25 000 and a forest on the map has an area of 20cm^2. Find the actual area of the forest in square kilometres.

Solution

3. Hence ratio of areas = (1:25 000)^2

$\frac{x}{20} = \frac{25\,000^2}{1}$

$x = (25\,000)^2 \times 20$

$= 1\,2500\,000\,000\text{cm}^2$

$\therefore$ Area of forest = 12 500 000 000cm^2

= 1,25km^2

Notice that this approach produces very large numbers like 25 000^2 = 625 000 000. One can easily miss one or two of the zeros in the working out. An alternative approach is:

Step 1. Change the 'actual' side of the scale to the required unit. i.e. 1:25 000 = 1cm : 0,25km. Note how.

Step 2. Split the given area into convenient sides e.g. 5cm by 4cm.
This then means that:
the actual length of the area $= 5 \times 0,25$
$= 1,25$km

the actual width of the area $= 4 \times 0,25$
$= 1$km

Hence the actual area $= 1,25 \times 1\text{km}^2$
$= 1,25\text{km}^2$

The figures used in this approach are relatively small.

1. Two similar containers have radii in the ratio 1:3.
Find the ratio of: a) their heights.
b) their surface areas.
c) their volumes.

2. Two circles have diameters $x$cm and $2x$cm.
What is the ratio of: a) their circumferences?
b) their areas?
Give both answers in their simplest form.

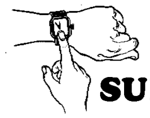
3. A solid has height 4cm and volume 50cm^3. A similar solid has volume 3 200cm^3. What is the height of this second solid?

4. A model tower is made to a scale of 1 to 200. The actual tower has a volume of 800m^3.
Calculate the volume of the model in cm^3.

5. Two similar solids have volumes V and v such that V = 27v. If their surface areas, A and a respectively, are such that A = ha, find the value of h.

6. A map of a growth point is drawn to a scale of 1:500 000. A road on the map is 8cm long.
a) Calculate the actual length of the road, giving the answer in kilometres.
The actual area of the growth point is 150km^2.
b) Calculate, in square centimetres, the area of the growth point on the map.

7. Two similar jugs have base diameters of 24cm and 18cm. Find the capacity of the larger jug if the smaller one has a capacity of 4 litres. Give your answer to the nearest litre.



SUMMARY

1. Congruent figures are identical.
   i.e. they have the same shape and size.

2. All corresponding features of congruent figures are equal, be it sides, angles, areas or volumes.

3. The sign '≡' is used to mean figures are congruent.

4. Similar figures have the same shape but differ in size.

5. The sign '**III**' means similar to.

6. When figures are similar:
   a) the ratio of their corresponding sides (scale factor) is the same.
   b) the ratio of their areas is the square of the scale factor.
   c) the ratio of their volumes is the cube of the scale factor.

7. Always place the corresponding measurement of congruent or similar figures in the correct order.

EXAM PRACTICE 8

Consider the following examples:

1. The diagram shows a parallelogram between parallel lines.

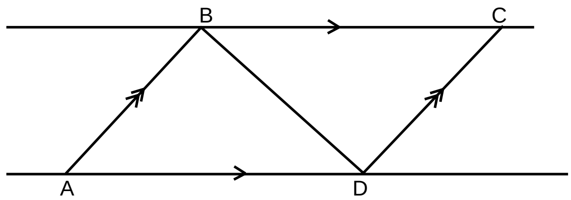


Fig 8.20

a) Name the congruent triangles, found in the diagram.
b) If $A\hat{D}B = 37^0$ and $A\hat{B}C = 105^0$, find $B\hat{C}D$.

Solution

Hint

Write down any one of the triangles then make sure the letters of the second follow the order of the first one.

1. a) $\triangle ABD \equiv \triangle CDB$

b) In $\triangle CDB$
$D\hat{B}C = A\hat{D}B = 37^0$
In $\triangle ABD$
$\hat{D} = 37^0$
$\hat{B} = 105^0 - 37^0$
$= 68^0$
$\therefore \hat{A} = 180^0 - (68^0 + 37^0)$
$= 180^0 - 105^0$
$= 75^0$
$\therefore B\hat{C}D = B\hat{A}D$
$= 75^0$

2. All the lengths in the diagram are in m.

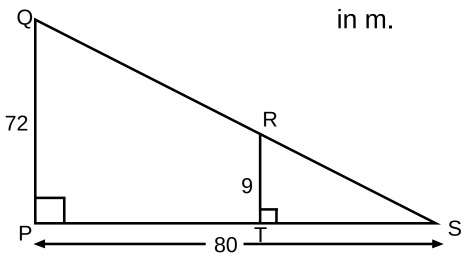


Fig 8.21

a) Which exist in the diagram, congruent or similar figures?
b) Name the congruent or similar figures.
c) Find PT.

Solution

2. a) The diagram has no congruent shapes.
There are similar triangles in the diagram.

b) $\triangle$ RST III $\triangle$PQS

c) $\frac{ST}{SP} = \frac{RT}{PQ} = \frac{RS}{QS}$

$\frac{ST}{80} = \frac{9}{72}$

$\therefore ST = \frac{\cancel{9}}{\cancel{72}_{\cancel{8}_{1}}} \times \cancel{80}^{10}$

$= 10m$

Hence PT $= (80 - 10)m$
$= 70m$

Now do the following:

1.

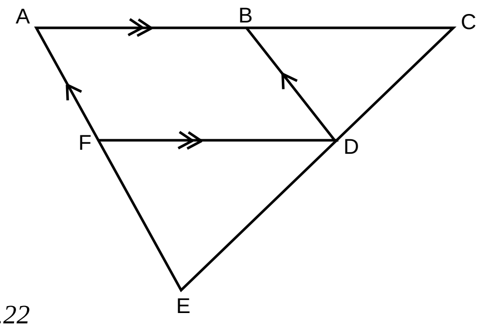


Fig 8.22

The diagram contains three pairs of similar triangles. Give the pairs.

2.

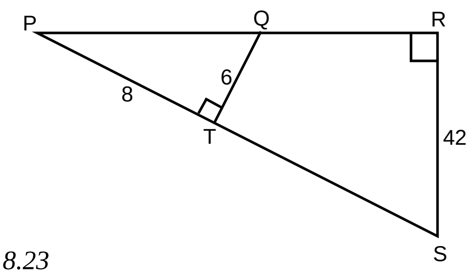


Fig 8.23

a) In the diagram, PT = 8cm, QT = 6cm and RS = 42cm.
Also $P\hat{T}Q = P\hat{R}S = 90^0$
Find QR.

3. The scale on a map is such that 3cm on the map represents 0,5km on the ground. Calculate:
   a) the length, in kilometres, of a stream which measure, 28,5cm on the map.
   b) the area on the map, in square centimetres that represents a farm of area 6km^2.

4. A school playground has an area of 726m^2. On the map, the playground is presented as an area of 24cm^2.
   a) Find the scale of the map.
   b) Find the actual length, in metres, of a classroom block which is represented by a line 3,2cm long on the map.

5.

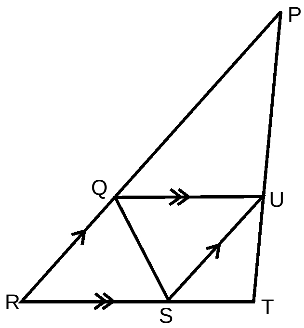


Fig 8.24

On the diagram PQR, RST and PUT are straight lines.
QU // RT AND RP // SU
   a) Name, in the correct order, the congruent triangles in the diagram.
   b) Name, in the correct order, two triangles which are similar to $\triangle$PQU.
   c) Given that PU:UT = 2:1, PQ = 7,5cm and the area of triangle PRT is 63cm^2.
      Calculate: (i) QR
      (ii) the area of triangle UST.

6.

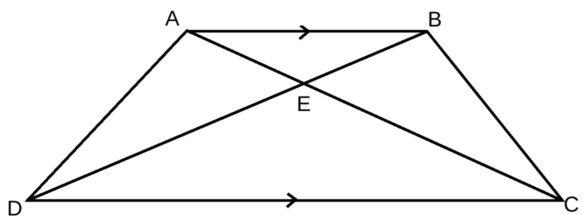


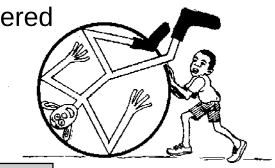
Fig 8.25

ABCD is an isosceles trapezium in which AB is parallel to DC. The diagonals AC and BD meet at E.

   a) Name two pairs of congruent triangles, other than $\triangle$ AED and $\triangle$BEC.

   b) If DE = 14cm, EB = 3,5and DC = 20cm,
      calculate: (i) AB
      (ii) the ratio of the area of $\triangle$AEB and the area of $\triangle$DEC.

9 Angles in a circle



This chapter seeks to explore mainly angles created inside and around a circle. Lengths of lines in a circle will also be considered in conjuction with the angles they create.

Syllabus Expectations

By the end of this chapter, students should be able to:

1. use properties of the radius, diameter, chord, tangent and cyclic quadrilateral to show relationships between angles.
2. use circle theorems in calculating angles and lengths of lines in a circle in the following situations:
   - angle at centre versus angle at the circumference.
   - angles in the same segment and semicircle.
   - angles in alternate segments.
   - angles in a cyclic quadrilateral.

ASSUMED KNOWLEDGE

In order to tackle work in this chapter, it is assumed that students are able to:

- correctly draw and identify circle parts created by special lines in a circle such as sector, segment, semi-circle.
- identify similar and congruent triangles.
- name correctly, lines drawn in and through a circle. Lines to include; radius, diameter, chord and arc.
- name and relate angles in parallel lines.
- carry out basic constructions using a ruler and compasses only.
- apply Pythagoras Theorem in right-angled triangles.
- solve simple simultaneous equations.
- use calculators for finding angles using appropriate trig ratios.

A. PARTS OF A CIRCLE

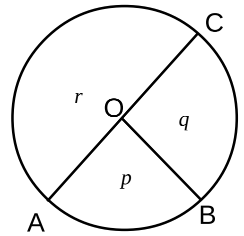


Fig 9.1

In Fig 9.1 A, B, C is a circle with centre O. The parts p, q and r are called **sectors**. r is a special sector, it is a **semi-circle**.
Line AC is a special chord called the **diameter**. When two diameters of the same circle meet perpendicularly, the sectors created are called **quadrants**.

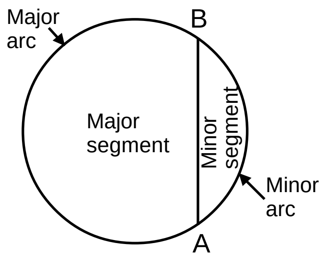


Fig 9.2

In Fig 9.2, line AB is called a chord. A chord divides the circle into **segments** (major and a minor). One is called a major according to its size. Along the circumference, AB is an arc (major and minor).

Hint

To subtend is to 'form' an angle. Lines AC and CB are from chord AB and meet at C. Thus chord AB subtends $A\hat{C}B = \theta$ at the circumference. In general – the outer letters of the angle form the chord or arc from where the arms of the angle are drawn.

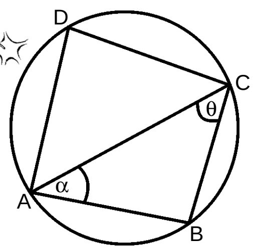


Fig. 9.3

In Fig 9.3, chords or arcs are subtending angles at the circumference of the circle.
e.g chord AB is subtending $A\hat{C}B = \theta$
chord BC is subtending $B\hat{A}C = \alpha$

1. Use Fig 9.3 to name the chord and the angle this chord is subtending.

Use Fig 9.4 to answer the questions 2 to 8 which follow.

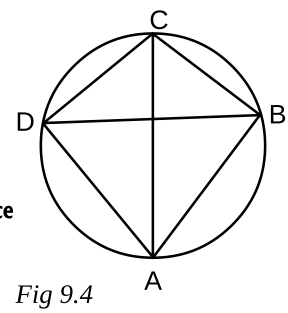


Fig 9.4

2. Name the chord which subtends $A\hat{D}C$.

3. Chord DB subtends $D\hat{A}B$ in the major segment. State the angle that is subtended by DB in the minor segment.

4. Chord AB subtends two angles in the major segment, name them.

5. Name all angles subtended by chord BC.

6. Chord DC subtends 2 angles in the major segment. Name them.

7. In which segment (major or minor) are these angles found?
a) $D\hat{B}A$ b) $D\hat{B}C$ c) $A\hat{D}B$
d) $B\hat{C}D$ e) $B\hat{D}C$

Hint
There are four pairs.

8. In Fig 9.4 list the pairs of angles being subtended by the same chord in the same segment.

B. CIRCLE THEOREMS

Put simply, a theorem is a proven fact. Draw a circle, centre O, with radius 5cm or more and copy the diagrams below. (Size of angles need not be copied).

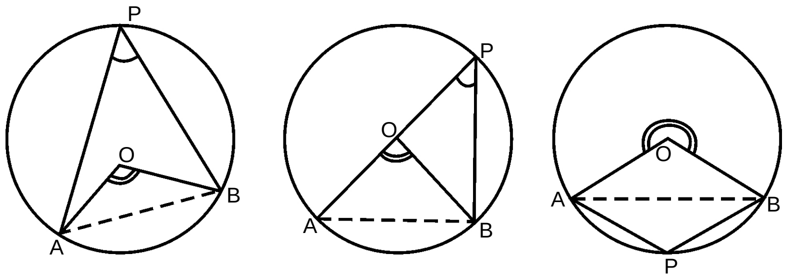


Fig 9.5

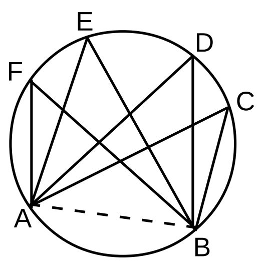
On your diagrams, measure angles $A\hat{O}B$ and $A\hat{P}B$ in each diagram. What do you notice?
The two angles $A\hat{O}B$ and $A\hat{P}B$ are both subtended by the same arc/chord AB, one at the centre and the other at the circumference. The relationship between these two angles gives the first theorem.

Theorem 1:

The angle subtended by an arc/chord at the centre is twice the angle subtended by the same arc/chord at the circumference

OR

The angle subtended at the circumference is half the angle subtended at the centre. (The two angles being subtended by the same arc/ chord.)



Copy this diagram onto a circle, centre O, of radius 5cm or more.
On your diagram measure the angles $A\hat{C}B$, $A\hat{D}B$, $A\hat{E}B$, $A\hat{F}B$. What do you notice?

Fig. 9.6

Theorem 2:

Angles in the same segment are equal if subtended by the same arc or chord.

OR

Angles subtended by the same arc/chord in the same segment are equal.

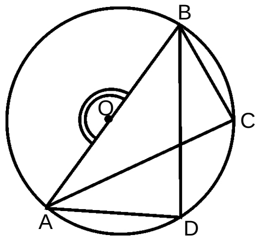


Fig. 9.7

There is now enough information learnt to establish the next THEOREM. In this diagram $A\hat{D}B$ and $A\hat{C}B$ are in the same segment, so they are equal

Also $A\hat{O}B = 180^0$ and is at O, the centre.

This then means $A\hat{C}B$ or $A\hat{D}B = \frac{1}{2}\ A\hat{O}B$

$= 90^0$

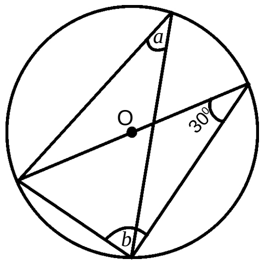
Theorem 3:

Diameters subtend right angles (90^0) at the circumference.

or The angle in a semi-circle is a right-angle

Consider the following examples

1.



Find the angles marked *a* and *b* in this circle, centre O.

Hint

Identify theorems in the diagram. Relate each to the question/ s.

Solution

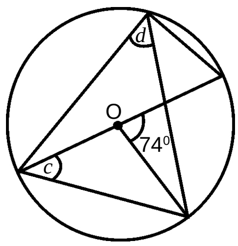
i.e. *a* and 30^0 are angles in the same segment,

$\therefore a = 30^0$

b is subtended by a diameter,

$\therefore b = 90^0$

2.



The circle has centre O.
Find the angles marked *c* and *d*.

Solution

74^0 is at the centre

$\therefore c = \frac{1}{2} \times 74^0$

$= 37^0$

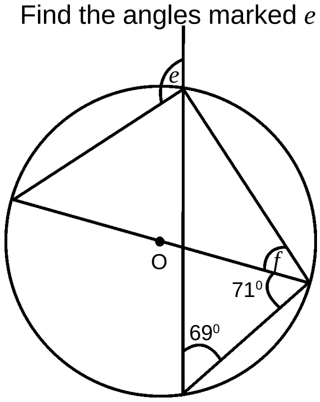
c and the angle next to d are in the same segment and so are equal.

d and the angle next to it make 90^0 .

$\therefore d = 90^0 - 37^0$

$= 53^0$

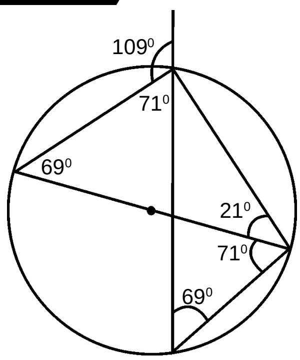
3. Find the angles marked e and f.



Solution

Hint

Mark all equal angles you can see in the diagram using the relevant concepts or theorems.

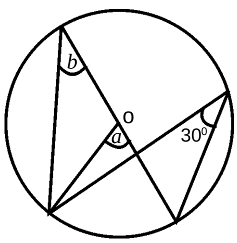


$e = 180^0 - 71^0 = 109^0$

$f = 90^0 - 69^0 = 21^0$

In these problems, O is the centre of the circle.
Find the lettered angles in each diagram.

1.



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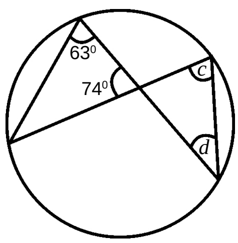

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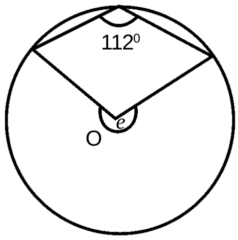

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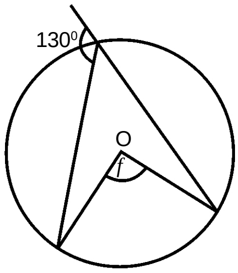

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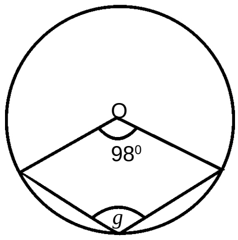

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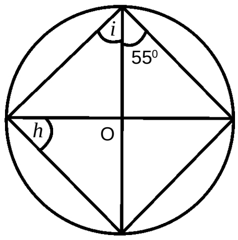

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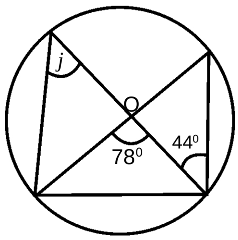

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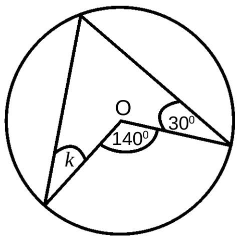


C. THE CYCLIC QUADRILATERAL

Remember, a quadrilateral is any four sided plane shape.

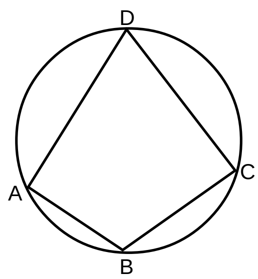
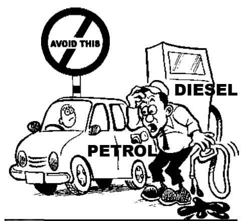


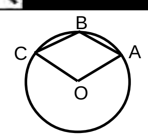
Fig. 9.8

In Fig 9.8 ABCD is a special quadrilateral, in the sense that all its vertices are on the circumference of a circle. Such a quadrilateral is called a ***cyclic quadrilateral***.

Cyclic comes from the word cycle/circle.



Common Error



ABCO is not a cyclic quadrilateral because O is not on the circumference.

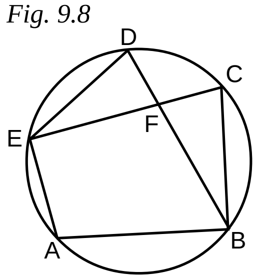


Fig. 9.9

In Fig. 9.9 ABCE, ABDE and ABFE are quadrilaterals but ABFE is **not** cyclic!

Angles of a cyclic quadrilateral are specially related.

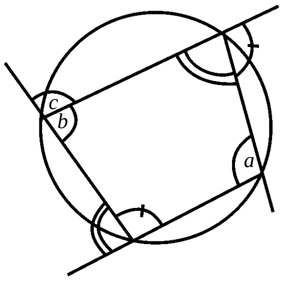
To establish the relationship, draw a circle of radius 5cm or more and insert any cyclic quadrilateral in the circle. Measure all the angles of the quadrilateral. What do you notice on the sum of the opposite angles?

Hint

Supplementary angles are two angles which add up to 180^0.

Theorem 4:

Opposite angles of a cyclic quadrilateral add up to 180^0 (they are supplementary).



If each side of the cyclic quadrilateral is produced or extended.

$a + b = 180^0$ Opposite angles of a cyclic quadrilateral.

$c + b = 180^0$ Adjacent angles on straight line.

Fig. 9.10

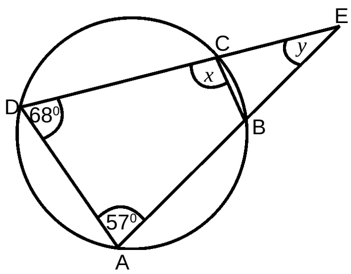
The two statements imply that $a = c$ a is an interior angle. c is the exterior opposite angle to a.

Theorem 5:

An interior angle of a cyclic quadrilateral is equal to its exterior opposite angle.

Consider the following examples

1. Find the angles marked x and y.



Solution

$x + 57^0 = 180^0$ Opposite angles of a cyclic quadrilateral

$x = 180^0 - 57^0$

$= 123^0$

$B\hat{C}E = 57^0$ and similarly

$C\hat{B}E = 68^0$

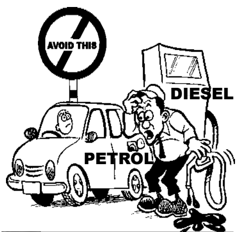
Thus to find y, either $\triangle$ AED or $\triangle$ BEC can be used.

In both cases

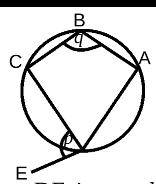
$y + 68^0 + 57^0 = 180^0$ Sum of angles of a $\triangle$

$y = 180^0 - 125^0$

$y = 55^0$



Common Error

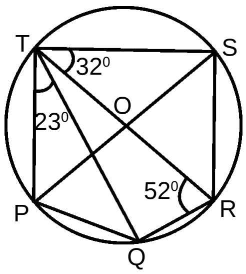


$p \neq q$ DE is not the extension of AD. ADE is supposed to be a straight line.

Hint

Identify theorems that apply to the diagram e.g. there are several cyclic quadrilaterals.

2.



In the diagram TR is the diameter of the circle. If $R\hat{T}S = 32^0$, $Q\hat{R}T = 52^0$ and $P\hat{T}Q = 27^0$, calculate:
a) $R\hat{T}Q$ b) $P\hat{Q}R$ (c) $P\hat{S}R$

Solution

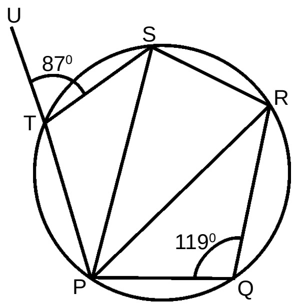
2. a) In $\triangle$ TRQ
$\hat{Q} = 90^0$
$\therefore R\hat{T}Q = 90^0 - 52^0$ or $180 - (90 + 52)$
$= 38^0$

b) $P\hat{Q}R$ is in a cyclic quadrilateral PQRT in which
$P\hat{T}R = 27^0 + 38^0$
$= 65^0$

$\therefore P\hat{Q}R = 180^0 - 65^0$
$= 115^0$

c) $P\hat{S}R = 180^0 - 115^0$
$= 65^0$
$P\hat{S}R$ is opposite $P\hat{Q}R$ in cyclic quad PQRS

3.



In the diagram, a circle passes through points P,Q,R,S and T. PT is produced to U. If $P\hat{Q}R = 119^0$ and $S\hat{T}U = 87^0$, calculate
a) $P\hat{S}R$ b) $S\hat{P}R$

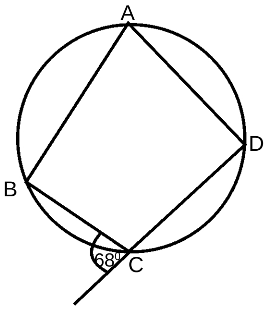
Solution

3. a) In cyclic quad. PQRS
$P\hat{S}R = 180^0 - 119^0$ opposite angles of a cyclic quad.
$= 61^0$

b) In cyclic quad. PRST,
$P\hat{R}S = 87^0$ Exterior vs. interior opposite angle

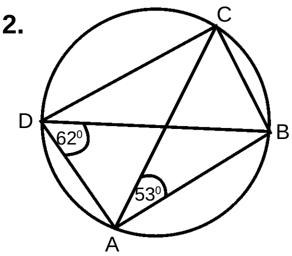
In $\triangle$ PSR
$S\hat{P}R = 180^0 - (61^0 + 87^0)$
$= 180^0 - 148^0$
$= 32^0$

1.



Given that $B\hat{C}E = 68^0$
Find $B\hat{A}D$.

2.

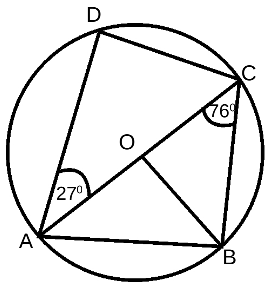


Given that $B\hat{A}C = 53^0$ and $A\hat{D}B = 62^0$. Find $A\hat{B}C$.

Hint

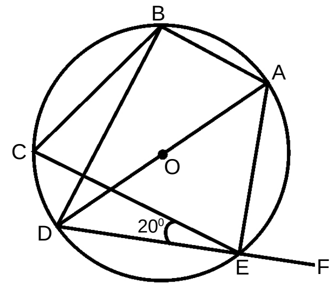
Some questions involve many unnecessary lines to confuse students e.g. Question 4.

3.



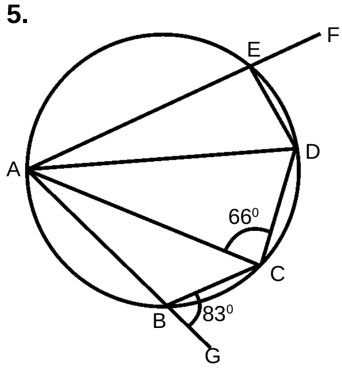
O is the centre of the circle. If $A\hat{C}B = 76^0$ and $D\hat{A}C = 27^0$,
Find a) $A\hat{O}B$ b) $O\hat{B}C$
c) $D\hat{A}B$

4.



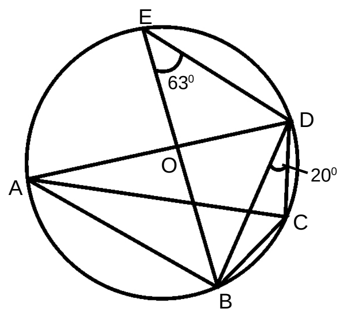
O is the centre of the circle. If $D\hat{E}C = 20^0$.
Find $C\hat{E}F$.

5.



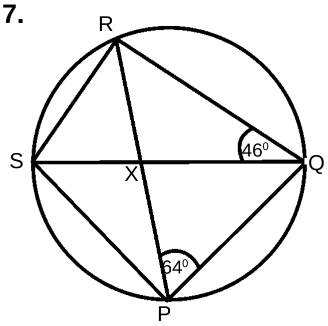
Given that $C\hat{B}G = 83^0$ and $A\hat{C}D = 66^0$, find
a) $A\hat{D}C$ b) $D\hat{E}F$

6.



O is the centre of the circle.
If $B\hat{E}D = 63^0$ and $B\hat{D}C = 20^0$, find
a) $B\hat{C}D$ b) $D\hat{A}C$

7.

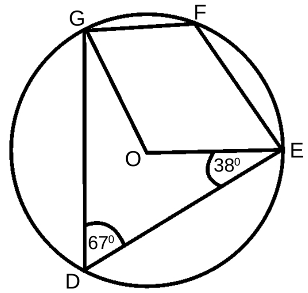


Chords PR and SQ intersect at X. It is given $Q\hat{P}R = 64^0$ and $RQS = 46^0$.

a) Find (i) $Q\hat{S}R$
(ii) $Q\hat{R}S$

b) Name in the correct order, a triangle which is similar to triangle RXS.

8.



O is the centre of the circle. Given that $E\hat{D}G = 67^0$ and $O\hat{E}D = 38^0$, Find

a) reflex $E\hat{O}G$

b) $D\hat{G}O$ (Hint: Join O to D)

D. THE TANGENT TO A CIRCLE

Hint

A tangent to a circle is a straight line which just touches the circle at one and only one point.

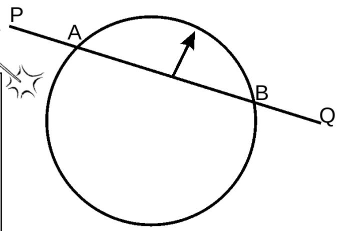


Fig. 9.11

In Fig 9.11 line PQ cuts the circumference at A and B.
Part AB of PQ is the chord not the whole PQ.
PQ is called a *secant*. A secant is a straight line cutting through a circle.

However if PQ is pushed in the direction shown by arrow, to sit on the circumference as illustrated in Fig 9.12 below, it becomes a **tangent** to the circle.

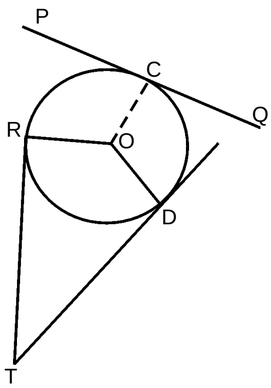


Fig. 9.12

PQ is a tangent to the circle at C. This PQ can be broken into two tangents PC and CQ.
Also, TR is a tangent to the circle at R and TD a tangent at D.
C, R and D are called **points of contact**. By measurement, it can be deduced that $T\hat{R}O$ and $O\hat{C}P$ are right angles (90^0).

Theorem 6:

The angle between a tangent and the radius of the circle at the point of contact is 90^0.

Also by symmetry

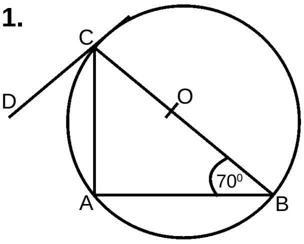
Theorem 7:

Tangents from the same external point to the same circle are equal in length.

e.g. TR = TD in the above diagram.

Consider the examples below:

1.



O is the centre of the circle.
Given $A\hat{B}C = 70^0$, find:
a) $A\hat{C}B$ b) $A\hat{C}D$

Solution

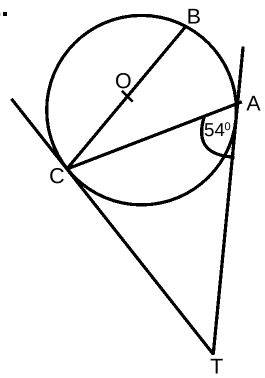
1. a) From $\triangle$ $A\hat{B}C$, $A = 90^0$

$\therefore A\hat{C}B = 90^0 - 70^0$
$= 20^0$

OR
$A\hat{C}B = 180^0 - (90^0+70^0)$
$= 180^0 - 160^0$
$= 20^0$

b) $D\hat{C}O = 90^0$ Tangent DC meet radius OC)
$\therefore A\hat{C}D = 90^0 - 20^0$
$= 70^0$

2.

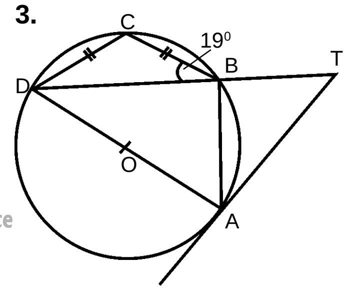


O is the centre of the circle
TC and TA are tangents to the circle from a point T. If $C\hat{A}T = 54^0$, find $A\hat{C}B$.

Solution

2. $\triangle$ TAC is isosceles since TC = TA
$A\hat{C}T = 54^0$
$\therefore A\hat{C}B = 90^0 - 54^0$ since $T\hat{C}B = 90^0$
$= 36^0$

3.



In the diagram, ABCD is a cyclic quadrilateral in which AD is a diameter and BC = CD.
DB is produced to meet the tangent at T. If $C\hat{B}D = 19^0$, find:

a) $B\hat{A}T$

B) $B\hat{T}A$

Solution

3. a) $B\hat{C}D = 180^0 - (19^0 + 19^0)$

$= 180^0 - 38^0$

$= 142^0$

$B\hat{A}D = 180^0 - 142^0$

$= 38^0$

b) $\therefore B\hat{A}T = 90^0 - 38^0$

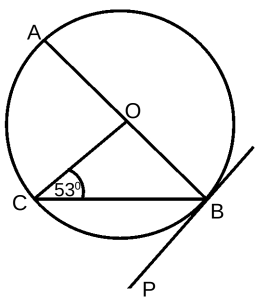
$= 52^0$

$\therefore B\hat{T}A = 90^0 - 52^0$

$= 38^0$

Take O to be the centre of the circle where it is given.

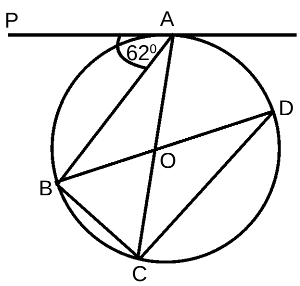
1.



Given that $B\hat{C}O = 53^0$, calculate:

a) $A\hat{O}C$ b) $P\hat{B}C$

2.

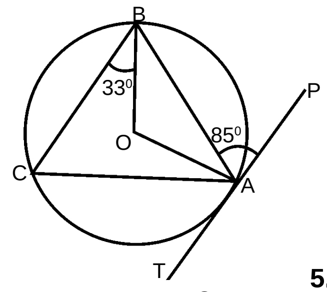


Given that $P\hat{A}B = 62^0$

Find: a) $B\hat{D}C$ b) $A\hat{B}D$

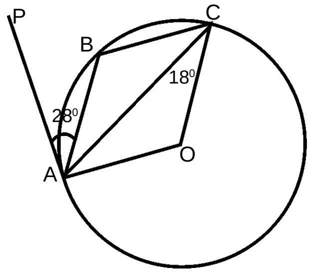
c) $B\hat{O}C$

3.



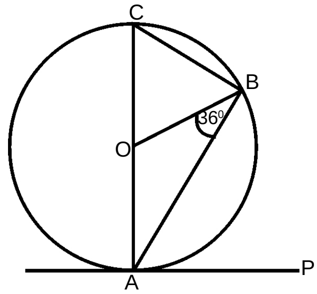
Given that $B\hat{A}P = 85^0$ and $C\hat{B}O = 33^0$, find:
a) $A\hat{C}B$ b) $B\hat{A}C$
c) $C\hat{A}T$

4.



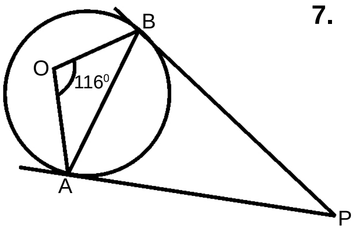
Given that $A\hat{C}O = 18^0$ and PAB = 28^0, calculate:
a) $A\hat{B}C$ b) $P\hat{A}C$

5.



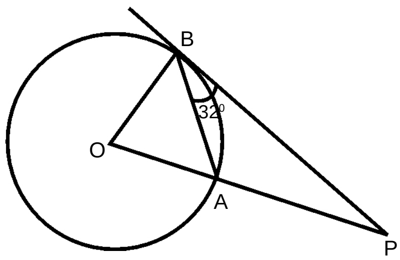
Given $O\hat{B}A = 36^0$, find:
a) $B\hat{A}P$ b) $A\hat{C}B$

6.



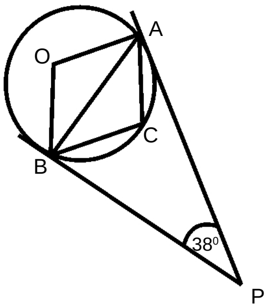
Given $B\hat{O}A = 116^0$, find:
a) $B\hat{A}P$ b) $A\hat{P}B$

7.



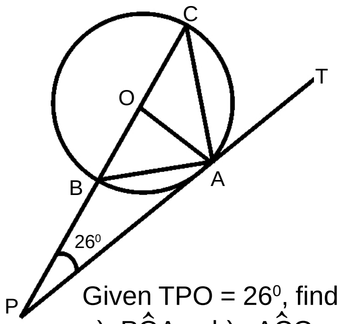
If $P\hat{B}A = 32^0$, find: a) $B\hat{A}O$
b) $A\hat{P}B$

8.



Given APB = 38^0, find:
a) $AO\hat{B}$ b) $A\hat{C}B$

9.



Given TPO = 26^0, find:
a) $P\hat{O}A$ b) $A\hat{C}O$
c) $C\hat{A}T$ d) $A\hat{B}P$

E. ANGLES IN ALTERNATE SEGMENTS

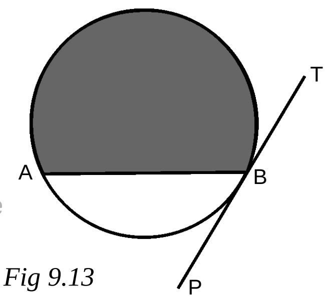
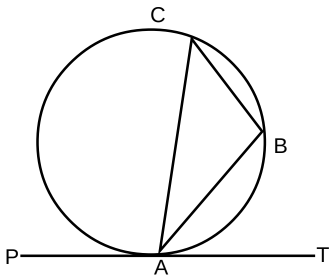


Fig 9.13

In Fig 9.13, $P\hat{B}A$ is between chord AB and tangent PB at the point of contact B. The shaded segment is called the alternate segment since this area is on the opposite or alternate side of the chord, making the angle with the tangent.

Similarly the unshaded segment is the alternate segment of $A\hat{B}T$.
Draw a circle of radius 5cm or more and copy the diagram below.



On your diagram, measure and compare the angles $T\hat{A}B$ and $A\hat{C}B$. Do you see that these are angles in alternate segments?

Fig 9.14

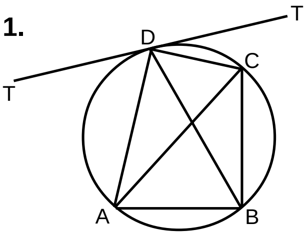
Measure and also compare $C\hat{A}P$ with $A\hat{B}C$. What do you notice?

Theorem 8:

Angles in alternate segments are equal.

Consider the example below

1.



In the diagram TDP is a tangent to the circle at D. Identify all the angles equal to the given angle.

a) $A\hat{D}T$ b) $B\hat{D}T$
c) $C\hat{D}P$ d) $B\hat{D}P$

Solution

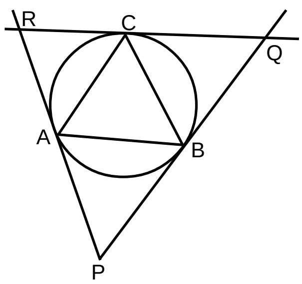
1. a) $A\hat{D}T = A\hat{B}D$ and $A\hat{C}D$ *Angles in alternate segments.*

b) $B\hat{D}T = B\hat{C}D$

c) $C\hat{D}P = C\hat{A}D = C\hat{B}D$

d) $B\hat{D}P = B\hat{A}D$

2.



In the diagram, three tangents RCQ, RAP and PBQ to the circle A, B, C meet at R, P and Q. If $\triangle$ ABC is equilateral, show that $\triangle$ PQR is equilateral as well.

Solution

2. $A\hat{B}C = R\hat{A}C = R\hat{C}A = 60^0$

$\therefore$ $A\hat{R}C = 60^0$

$B\hat{A}C = C\hat{B}Q = B\hat{C}Q = 60^0$

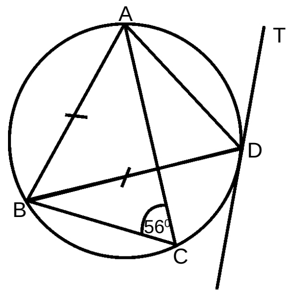
$\therefore$ $C\hat{Q}B = 60^0$

$B\hat{C}A = P\hat{A}B = P\hat{B}A = 60^0$

$\therefore \quad A\hat{P}B = 60^0$

$\therefore \quad \triangle PQR$ is equilateral

3.



In the diagram TD is the tangent of the circle at D. Given that AB = BD and $A\hat{C}B = 56^0$, find $A\hat{D}T$.

Solution

Hint

Study the diagram to find the relevant theorems.

3. Take note: (i) $A\hat{D}T = A\hat{B}D$ (Angles in alternate segments)

(ii) $\triangle ABD$ is isosceles

(iii) $B\hat{C}A = B\hat{D}A$ (Angles in the same segment)

How can one prove that $A\hat{B}D$ is equal to $A\hat{D}T$?

Use the isosceles triangle.

Since $B\hat{C}A = B\hat{D}A = 56^0$

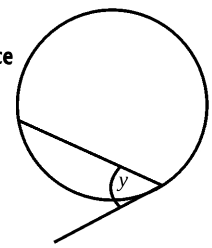
$A\hat{B}D = 180^0 - (56^0 + 56^0)$

$= 180 - 112$

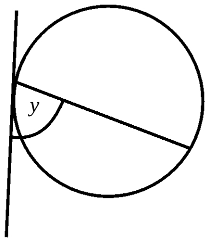
$= 68^0 \quad \therefore \; A\hat{D}T = 68^0$.

1. Copy the following diagrams (not to scale) and use the given chord to draw an angle in the alternate segment of the given angle.

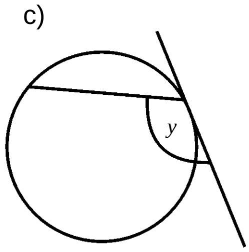
a)



b)



c)

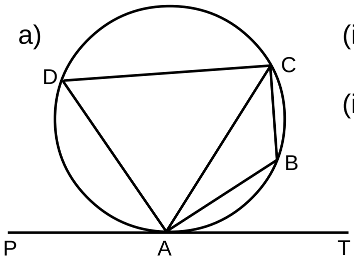


Hint

e.g $P\hat{A}D = A\hat{C}D$

2. The given diagrams have equal angles due to the "alternate segment" theorem. One angle is given. Give the angle, from the diagram, that is equal to this given angle.

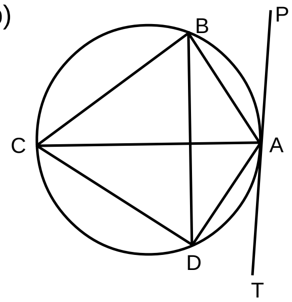
a)



(i) $P\hat{A}C$ (ii) $T\hat{A}B$

(iii) $T\hat{A}C$

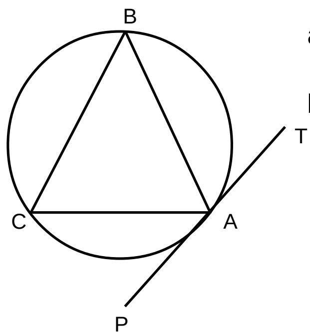
b)



(i) $P\hat{A}B$ (2 angles) (ii) $P\hat{A}C$

(iii) $A\hat{C}D$ (2 angles) (iv) $C\hat{A}T$

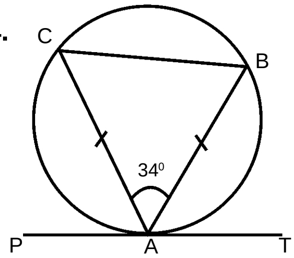
3.



a) Given $P\hat{A}C = 58^0$, find $A\hat{B}C$

b) On a separate diagram, given $A\hat{C}B = 49^0$, find $T\hat{A}B$

4.

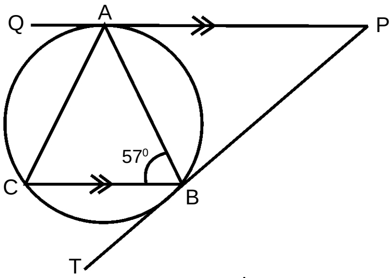


Given that AB = AC and $B\hat{A}C = 34^0$, find:

a) $A\hat{C}B$ b) $P\hat{A}C$

c) $B\hat{A}T$

5.

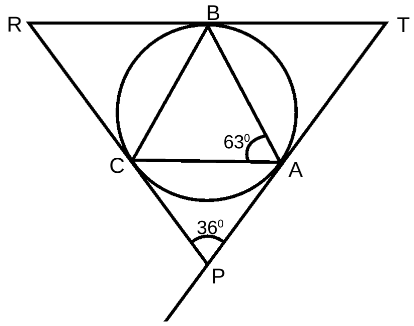


Given that $A\hat{B}C = 57^0$ find:

a) $P\hat{B}A$ b) $B\hat{C}A$

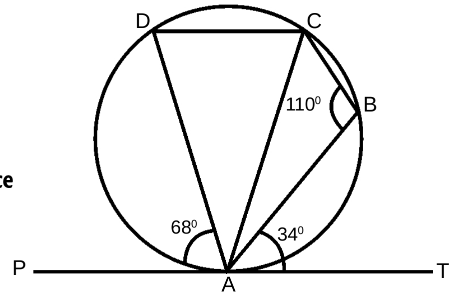
c) $C\hat{B}T$

6.



Given $B\hat{A}C = 63^0$, and $A\hat{P}C = 36^0$
find: a) $C\hat{B}R$ b) $B\hat{R}C$ c) $P\hat{A}C$
d) $A\hat{B}C$ e) $A\hat{T}B$

7.



Given $P\hat{A}D = 68^0$, $T\hat{A}B = 34^0$
and $A\hat{B}C = 110^0$, find:
a) $B\hat{A}C$ b) $A\hat{C}D$
c) $C\hat{A}T$

F. EQUAL CHORDS

Draw a circle of radius 5cm or more and copy the diagram below. Make sure AB = BC in the first diagram and AB = DC in the second diagram.

Hint

Open compasses to a reasonable radius. Use point A as centre to mark point B in both diagrams. Repeat with same radius to get BC and DC.

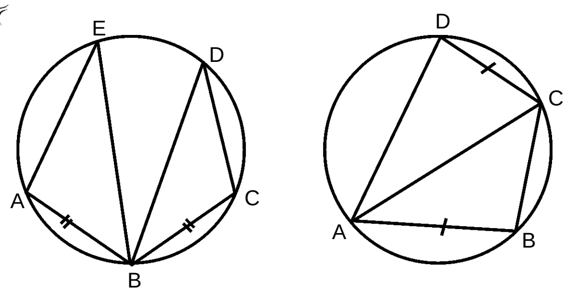


Fig 9.15

On your diagrams, measure
$A\hat{E}B$ and $B\hat{D}C$ from the circle ABCDE. What do you notice?
Now measure $A\hat{C}B$ and $C\hat{A}D$ from the circle ABCDE. What do you notice?
Accurate diagrams will produce equal angles!

Theorem 9:

Equal chords in the same circle subtend equal angles.

In this case use a ruler and compasses only.
Draw any circle and any chord on it as in Fig 9.16.

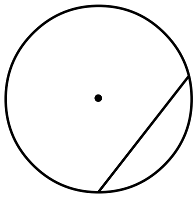


Fig 9.16

Now construct the perpendicular bisector of the chord. Draw another chord in the same circle and construct its perpendicular bisector. An accurate construction will reveal that:

Theorem 10:

The perpendicular bisector of any chord always passes through the centre of the circle.

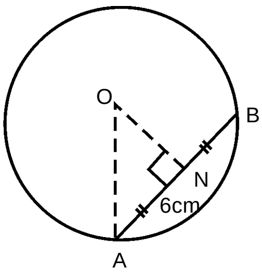
This conclusion helps us to find the distance of the chord from the centre of the circle.

Consider the example below:

1. The circle, centre O, has radius 4cm. A chord AB, 6cm long is drawn in the circle. How far is this chord from the centre O.

Solution

1. The diagram will resemble the one below.



Let the distance in question be ON.
This means N bisects AB and ON is perpendicular to AB. Triangle ONA or ONB is produced with OA = 4cm,
AN = 3cm

ON = $\sqrt{4^2 - 3^2}$ using Pythagoras' Theorem

$= \sqrt{7}$

= <u>2,65cm</u>

Draw 2 intersecting circles of different radii as below.

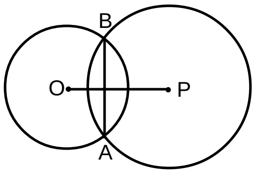


Fig. 9.17

You will ***notice that*** (By measurement of angles between the line joining the centres and the common chord).

Theorem 11:

The line joining the centres of two intersecting circles is perpendicular to the common chord.

G. CIRCLES TOUCHING INTERNALLY OR EXTERNALLY

Now draw circles touching internally and/or externally as illustrated below.

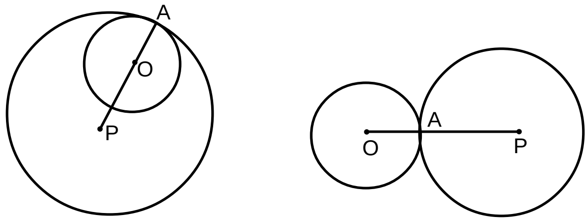


Fig. 9.18

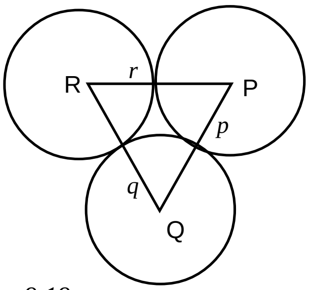
You will ***notice that***: point A, O and P form a straight line.

Theorem 12

When two circles touch each other internally or externally the two centres and the point of contact are always co-linear.

Consider the following example:

2.



Three circles with centres Q, P and R respectively touch each other externally.
If RP = 8cm, RQ = 9cm and PQ = 5cm, find the radii of the circles.

Fig. 9.19

Let the radii be r, p and q respectively.

This means

$r + p = 8$ —①

$r + q = 9$ —②

$p + q = 5$ —③

Now from ① $r = 8 - p$

substituting this into ② gives

$8 - p + q = 9$

$-p + q = 1$ solve simultaneously with ③

$p + q = 5$

$2q = 6$

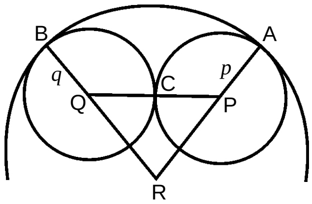
$q = 3$

$r = 6$

$p = 2$

∴ The radii are 6cm, 2cm and 3cm

Here is another version of touching circles.



Circles, centre Q and P, touch externally and they both touch the circle, centre R, on the circumference as shown in diagram.

Fig. 9.20

If the points A, B and C are the points of contact as shown, then QCP, BQR and APR are straight lines.
PA or PC (p) is the radius of circle centre P.
BQ or QC (q) is the radius of circle centre Q.
BR or AR (r) is the radius of circle centre R.
It then follows that:
$p + q$ = PQ
$r - p$ = PR
$r - q$ = QR
These three equations can be used to find the radii of the three circles.

H. INTERSECTING CHORDS (OPTIONAL)

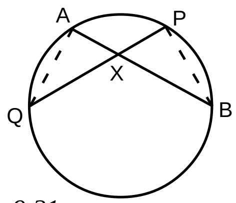


Fig. 9.21

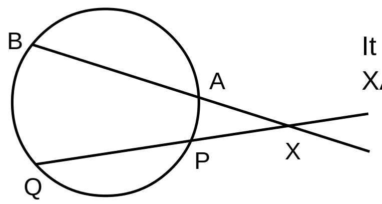
In the diagram chords AB and PQ do intersect at X.
A careful study of the diagram will reveal that $\triangle$XBP is similar to $\triangle$XQA.

Thus $\frac{XB}{XQ} = \frac{XP}{XA}$ Cross multiplying the ratios leads to

XA $\times$ XB = XP $\times$ XQ

Thus in general, **when chords of the same circle intersect, the product of the two parts of one chord is equal to the product of the two parts of the other chord.**

This relationship is always true of any intersecting chords even if they do meet outside the circle, as illustrated below.



It still follows that:
XA $\times$ XB = XP $\times$ XQ

Fig 9.22

Note that the measurements are from the point of intersection, in this case X.

Even when one chord becomes a tangent, this relationship will always stand.

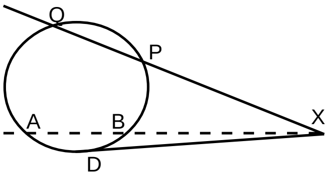


Fig. 9.23

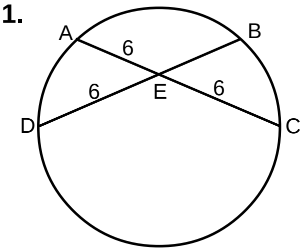
In this case

$XD^2 = XP \times XQ$

Note that the point of contact D is a merger of two points A and B, when the chord was pushed out of the circle to touch the circle at D. Thus $XD \times XD$ is taken as $XA \times XB$.

Consider the examples below

1.



In the diagram are two intersecting chords AC and DB intersecting at point E. If AE = 6cm, EC = 4cm, DE = 3cm, find BD.

Solution

1. Find BE first.

$$3 \times EB = 4 \times 6$$

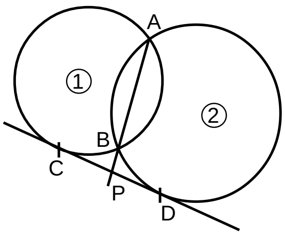
$$EB = \frac{4 \times 6}{3}$$

$$= 8cm$$

$$\therefore \quad BD = (8 + 3)cm$$

$$= 11cm$$

2.



In the diagram, CD is a common tangent and AB is a common chord to intersecting circles ①and ② .

a) If PC = 4cm and AB = 6cm, find BP.

b) Show that PC = PD.

Solution

Tip

When naming line BP is the same as PB.

2. a) ***Note that*** PC is a tangent to circle ① and is meeting chord AB produced at P.

$$\therefore PC^2 = PB \times PA$$

$$= PB (PB + BA)$$

$$4^2 = PB (PB + 6)$$

$$4^2 = PB^2 + 6PB$$

$PB^2 + 6PB - 16 = 0$ Let PB be x

$x^2 + 6x - 16 = 0$

$(x + 8)(x - 2) = 0$

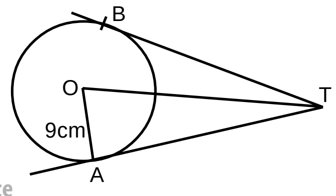
$x = -8$ or 2

PB = 2cm

Note that, –8 is not significant. Why?

b) PD is a tangent of circle ② meeting chord AB produced at P,
$\therefore$ PD = PB $\times$ PA and PC = PB $\times$ PA
Thus PC = PD

3.



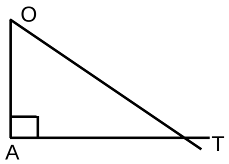
In the diagram, O is the centre of the circle of radius 9cm. TA and TB are tangents to the circle at A and B respectively.

a) Given that TA = 12cm
Calculate OT

b) Find $B\hat{T}O$.

Solution

3. a)



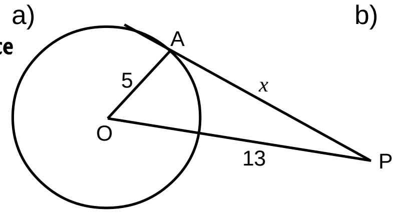
$OT = \sqrt{12^2 + 9^2}$ Pythagoras Theorem
$= \sqrt{144 + 81}$
$= \sqrt{225}$
$= 15\text{cm}$

b) By symmetry $B\hat{T}O = O\hat{T}A$

Thus $\tan O\hat{T}A = \frac{9}{12}$
$= 0,75$
$O\hat{T}A = 36,9^0$
$\Rightarrow B\hat{T}O = 36,9^0$

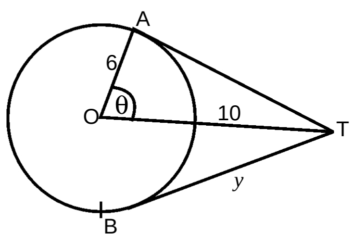
1. Find the lettered lengths and/or angles in the diagrams. O is the centre of the circle.

a)



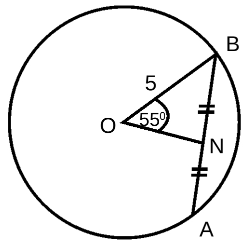
OA = 5cm, OP = 13cm

b)



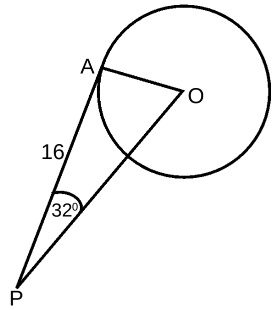
OA = 6cm, OP = 10cm

c)



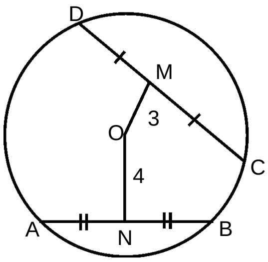
$O\hat{B}$ = 5cm, AN = NB
$N\hat{O}B = 55^0$
Find AB

d)



AP = 16cm, $A\hat{P}O = 32^0$
Find OP

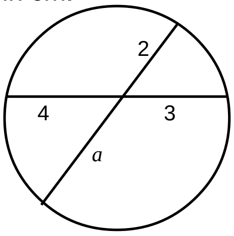
e)



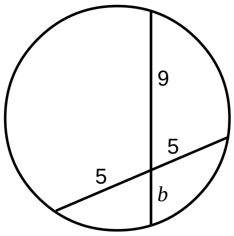
OM = 3cm, ON = 4cm
AB = 9cm
Find CD.

2. Find the lettered lengths in the following diagrams. All lengths are in cm.

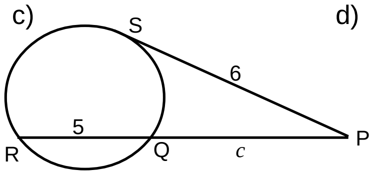
a)



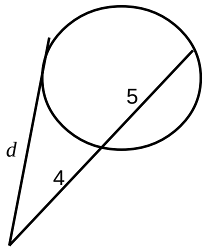
b)



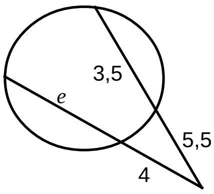
c)



d)



e)



3. Copy Fig 9.20 on page 200 and then answer the following questions.
Find the radii of the circles if
a) PR = 7cm, RQ = 9cm and PQ = 5cm.
b) PR = 6cm, RQ = 9cm and PQ = 5cm.
c) PR = 6,5cm, RQ = 7,5cm and PQ = 4,5cm.
d) PR = 8,5cm, RQ = 10cm and PQ = 5,5cm.

4. Use the same lengths given in number 3 (a and b) above to calculate the radii of the given circles in Fig 9.24 below.

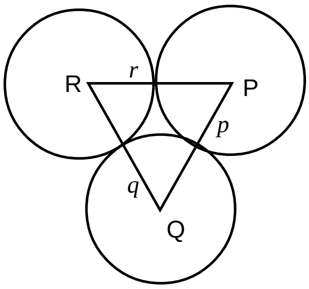
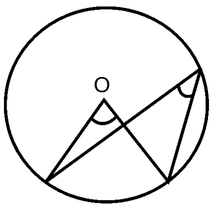


Fig. 9.24

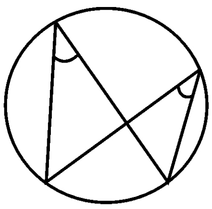
SUMMARY

THEOREMS

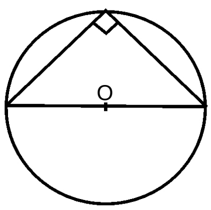
1. The angle at the centre is twice the angle at the circumference.



2. Angles in the same segment are equal.



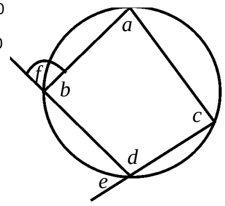
3. Angles in semicircles are right angles.



4. Opposite angles of a cyclic quadrilateral are supplementary.

$a + d = 180^0$

$b + c = 180^0$

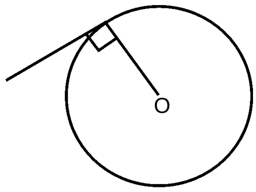


5. The interior angle of a cyclic quadrilateral is equal to the exterior opposite angle.

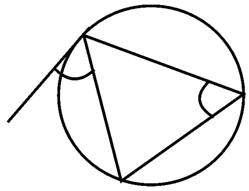
$a = e$.

$f = c$.

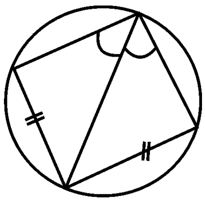
6. The angle between a radius and a tangent is 90^0.



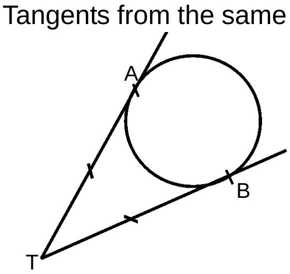
7. Angles in alternate segments are equal.



8. Equal chords subtend equal angles.

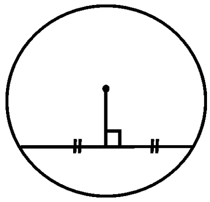


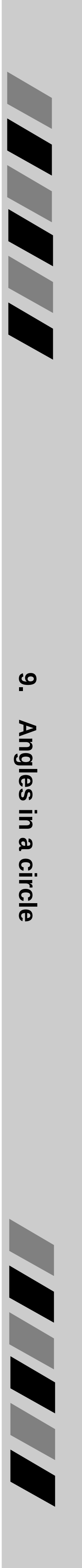
9. Tangents from the same point to the same circle are equal.



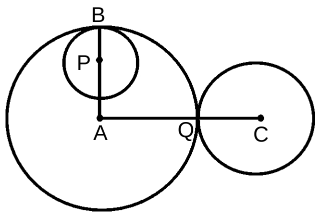
TA = TB

10. The perpendicular bisector of a chord passes through the centre of the circle.

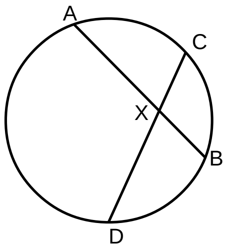




11. The centres of touching circles and the point of contact are co-linear. (form a straight line) i.e APB and AQC are both straight lines.



12. For intersecting chords the product of the two parts of one chord is equal to the product of the two parts of the other chord.

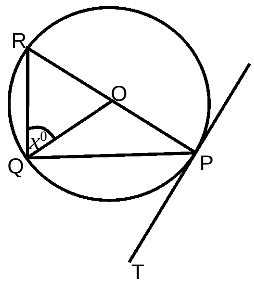


XB.XA = XC.XD.

EXAM PRACTICE 9

Consider the examples below:

1.

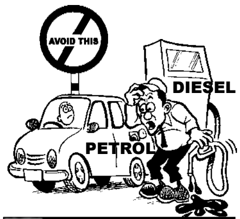


In the diagram O is the centre of the circle and PT is a tangent at P.
If $O\hat{Q}R = x^0$, express the following angles in terms of x.

a) $Q\hat{P}T$ b) $P\hat{O}Q$ c) $R\hat{P}Q$

Hint

Identify Theorems and key factors that apply to the diagram.



Common Error

Failure to deduce isosceles triangles in the diagram and solving for x instead of giving expressions as answers are common mistakes.

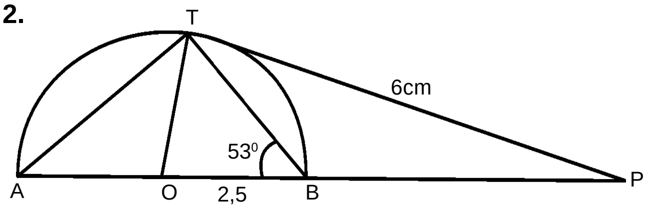
Solution

1. a) $Q\hat{P}T = x$ since $Q\hat{P}T = Q\hat{R}P = O\hat{Q}R$

b) $P\hat{O}Q = 2x$

c) $R\hat{P}Q = 90^0 - x$

2.



A semi-circle ATB, centre O, has PT as tangent at T and diameter AB is produced to meet TP at P.
Given that OB = 2,5cm, PT = 6cm and $A\hat{B}T = 53^0$, find
a) $B\hat{O}T$ b) $B\hat{A}T$ c) $B\hat{P}T$ d) BP

Solution

2. a) $B\hat{O}T = 180^0 - (53^0 + 53^0)$
$= 74^0$

b) $B\hat{A}T = \frac{1}{2}B\hat{O}T$ or $90^0 - 53^0$
$= \frac{1}{2} \times 74$
$= 37^0$

c) $B\hat{T}P = 37^0$

d) $PT^2 = PB.\ PA$

$6^2 = PB(5 + PB)$

$36 = 5PB + PB^2$ Let PB be x

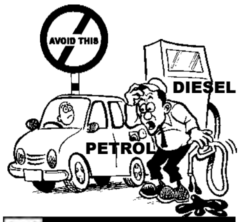
$PB^2 + 5PB - 36 = 0$

$x^2 + 5x - 36 = 0$

$(x + 9)(x - 4) = 0$

$\therefore x = -9$ or 4

BP = 4cm



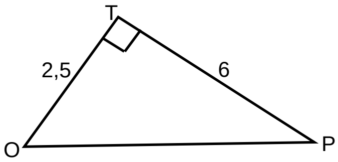
Common Error

$PT^2 = PB \times BA$
or
$PT = PB \times A$.

Alternatively

Triangle POT is right angled at T.

i.e



Using Pythagoras Theorem

$OP^2 = 6^2 + 2^5$

$= 36 + 6,25$

$= 42,25$

$\therefore OP = \sqrt{42,25}$

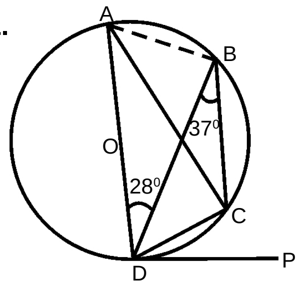
$= 6,25$

Hence BP = 6,5 –2,5

= 4cm

Now do the following:

1.

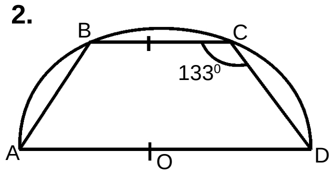


In the diagram, circle ABCD has centre O and tangent PD. If $A\hat{D}B = 28^0$ and $C\hat{B}D = 37^0$, calculate:

a) $B\hat{A}D$ b) $C\hat{A}B$ c) $B\hat{C}D$

d) $C\hat{D}P$

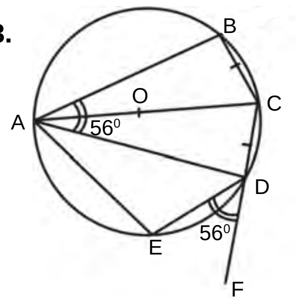
2.



The diagram shows a semi-circle ABCD centre O. Given that $B\hat{C}D = 133^0$ and that AB = BC, find:

a) $A\hat{B}C$ b) $D\hat{A}C$

3.



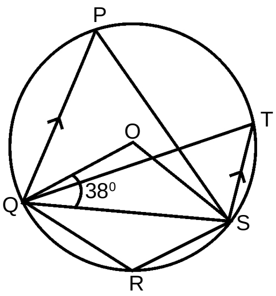
The diagram is a circle ABCDE centre O. Given that $B\hat{A}D = E\hat{D}F = 56^0$ and that BC = CD,

a) find: (i) $A\hat{C}D$ (ii) $A\hat{E}D$

(iii) $D\hat{A}E$

b) name the triangle that is congruent to triangle CDA.

4.

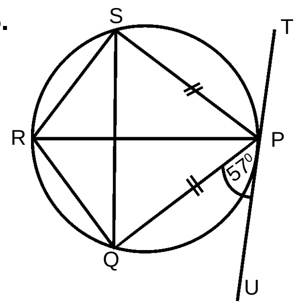


P, Q, R, S and T are points on the circumference of a circle centre O. Given that PQ is parallel to TS and $O\hat{Q}S = 38^0$.

a) calculate: (i) $Q\hat{P}S$ (ii) $Q\hat{R}S$

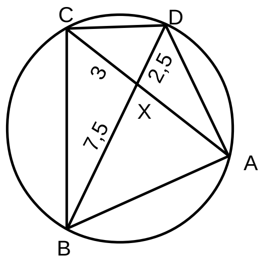
b) name three angles which are equal to $Q\hat{P}S$.

5.



The diagram is a circle PQRS and a tangent TPU to the circle at P. It is also given that PQ = PS. If $U\hat{P}Q = 57^0$, name five other angles in the diagram which are also 57^0.

6.

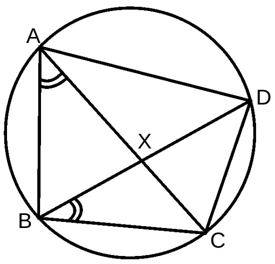


The diagram shows a circle ABCD with chords AC and BD intersecting at X.

a) Given that CX = 3cm, DX = 2,5cm and BX = 7,5cm, find AX.

b) Name, in correct order, two sets of similar triangles in the diagram.

7.



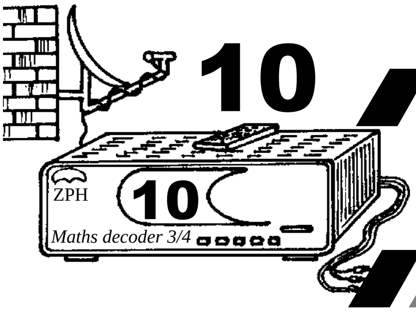
In the diagram, ABCD is a cyclic quadrilateral whose diagonals intersect at X.
Given that $B\hat{A}C = D\hat{B}C$.

a) name two arcs which are equal in length and state why.

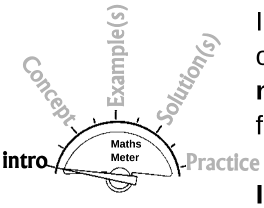
b) name two other angles which are equal to $B\hat{A}C$.

c) name, in correct order, the triangle which is similar to triangle AXB.

d) Given also that AX = XC and that BX = 2cm and XD = 8cm, calculate AX.



10 Functional Graphs



In mathematics, numbers and/or variables are connected using rules or relations. **The rule that changes one number into another number is called a function.** Below is a diagram illustrating some functions.

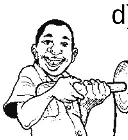
INPUT		FUNCTION		OUTPUT
x	→	1. Add 4 to x 2. Double x 3. Square x	→	1. $x + 4$ 2. $2x$ 3. x^2

The letter f is usually used to represent a function. The result (output) of applying a function to a number, represented by x, is denoted by $f(x)$, read as 'function of x' or, in short, 'f of x'.
From the above diagram:

1. $f(x) = x + 4$
2. $f(x) = 2x$
3. $f(x) = x^2$

Functions usually have a graphical representation. The aim of this chapter, is to teach you how to draw and interpret graphs of:

a) linear functions.
b) quadratic functions.
c) cubic functions.
d) inverse functions.



Syllabus Expectations

By the end of the chapter, students should be able to:

1. use and interpret the $f(x)$ notation.
2. construct a table of values from a given function and draw the respective graph in the given range.
3. use given scales correctly or find a suitable scale to apply to a given situation.
4. draw graphs of functions of the form $ax + by = 0$, $y = mx + c$, $y = ax^2 + bx + c$ and $y = x^n$ where $n = -2, -1, 0, 1, 2$ and 3.

5 find the area under a curve.

6 estimate gradients of curves by drawing tangents at given points.

7 solve (i) linear simultaneous equations graphically.
(ii) equations using points of intersection of graphs e.g. drawing $y = \frac{1}{x}$ and $y = 2x + 3$ to solve $2x^2 + 3x - 1 = 0$.

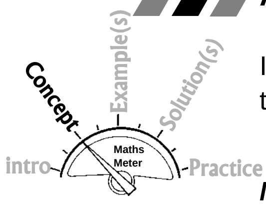
8 estimate area under a curve by:
(i) counting squares.
(ii) dividing the area into trapezia.

ASSUMED KNOWLEDGE

In order to tackle work in this chapter, it is assumed that students are able to:

- manipulate directed numbers.
- draw the cartesian plane and use scale correctly.
- plot points correctly.
- substitute given values into formulae and evaluate.
- solve quadratic equations.

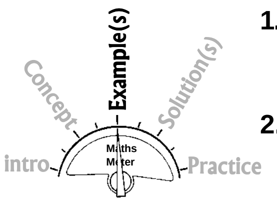
A. CALCULATIONS IN FUNCTIONAL NOTATION



If $f(x) = x + 4$
then $f(-5) = -5 + 4$
$= -1$

Notice $f(-5)$ simply means 'the function when x is -5'. On the RHS, we have simply substituted -5 for x and evaluated the result.

Consider the examples below:



1. Given that $f(x) = (x - 2)^2$
find a) $f(6)$ b) $f(0)$

2. If $f(x) = \dfrac{6x}{x + 2}$, find: a) $f(\frac{1}{2})$ b) $f(-3)$

3. If $f(x) = \dfrac{x + 2k}{1 - 3x}$ and $f(2) = 4$, find the value of k.

Solutions

1. a) $f(x) = (x - 2)^2$
$\therefore f(6) = (6 - 4)^2$
$= (2)^2$
$= 4$

b) $f(x) = (x - 2)^2$
$\therefore f(0) = (0 - 2)^2$
$= (-2)^2$
$= 4$

2. a) If $f(x) = \dfrac{6x}{x + 2}$

$\therefore f(\frac{1}{2}) = \dfrac{6\,(\frac{1}{2})}{\frac{1}{2} + 2}$

$= \dfrac{3}{2\frac{1}{2}}$

$= 3 \times \frac{2}{5}$

$= \dfrac{6}{5}$

$= 1\frac{1}{5}$

b) If $f(x) = \dfrac{6x}{x + 2}$

$\therefore f(-3) = \dfrac{6(-3)}{-3+2}$

$= \dfrac{-18}{-1}$

$= 18$

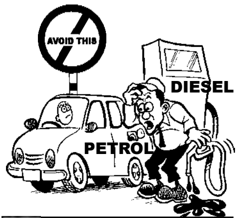
3. ***Note that***, unlike the first two functions, this one has 2 unknowns x and k. The second function $f(2) = 4$ is meant to help us form an equation so we can find k.

Thus in $\dfrac{x + 2k}{1 - 3x}$, $f(2) = \dfrac{2 + 2k}{1-3(2)}$

$f(2) = \dfrac{2 + 2k}{-5}$

Thus $\dfrac{2 + 2k}{-5} = 4$ since both sides are equal to $f(2)$.

$2 + 2k = -20$
$2k = -22$
$\therefore k = -11$



Common Error

1) $6(\frac{1}{2})$ taken as $6\frac{1}{2}$

$= \frac{13}{2}$

1. Given that $f(x) = 2x + 9$, find:
a) $f(6)$ b) $f(-6)$ c) $f(0)$ d) $f(-\frac{2}{3})$

2. If $f(x) = 8x - 5$, find:
a) $f(-1)$ b) $f(\frac{1}{2})$ c) $f(-0{,}75)$ d) $f(\frac{3}{8})$

3. Given that $f(x) = 2x^2 - x - 10$, find:
a) $f(-2)$ b) $f(0)$ c) $f(3)$ d) $f(\frac{1}{2})$

4. If $f(x) = (2x - 3)(x + 1)$, find:
a) $f(-3)$ b) $f(-2)$ c) $f(0)$ d) $f(5)$

5. If $f(x) = (x - 1)(3x + 1)(x + 2)$, find:
a) $f(-2)$ b) $f(0)$ c) $f(1)$ d) $f(3)$

6. If $f(x) = 3x^3 - 5x - 2$, find:
a) $f(-2)$ b) $f(-1)$ c) $f(2)$ d) $f(3)$

7. Given that $f(x) = \dfrac{x}{x-3}$, find:

a) $f(4)$

b) x if $f(x) = \frac{1}{7}$

8. If $f(x) = \dfrac{3x+1}{x-2}$, find: a) $f(-2)$
b) x if $f(x) = -1$

9. Given that $f(x) = kx + 4$, find the value of k if $f(\frac{1}{3}) = 5$.

10. If $f(x) = \dfrac{k+x}{3x-2}$ and $f(-\frac{1}{3}) = 7$, find the value of k.

B. GRAPHS OF LINEAR FUNCTIONS

Linear functions are those functions in which the highest power of the variable is one. Examples of such functions are $f(x) = 2x + 1$, $f(x) = 2 - \frac{3}{4}x$. These functions produce straight line graphs. Graphically x, being the input, its values are plotted on the horizontal axes (the x-axis) whilst $f(x)$ being the output, its values are plotted on the vertical axes (the y-axis).
Thus instead of $f(x) = 2x + 1$, it is more conventional to say $y = 2x + 1$. In general a linear function is in the form $mx + c$ where m is the gradient and c the y-intercept.
To draw a straight line graph, only two points are required. However a third point is recommended as a quality controller. It checks our accuracy in calculating the values of y.

Consider the following examples:

1. $y = 2x + 1$.

Solution

1. Using a Table of Values (T.O.V). Arrived at by substituting the values for x, –2, 0 and 3 in turn into the equation and solving it for y.

Table 10.1

x	–2	0	3
$2x$	–4	0	6
+1	+1	+1	+1
y	–3	1	7

$$\begin{aligned} \text{Notice } f(x) &= 2x + 1 \\ \therefore\ f(-2) &= 2(-2) + 1 \\ &= -4 + 1 \\ &= -3 \text{ which is the } y \text{ value, when } x = -2. \end{aligned}$$

Similarly when $x = 0$, $y = 2(0) + 1 = 1$. When $x = 3$, $y = 2(3) + 1$

$$\begin{aligned} y &= 6 + 1 \\ &= 7 \end{aligned}$$

Since the calculations are elementary a simpler Table of Values is usually drawn as illustrated in Table 10.2.

Table 10.2

x	–2	0	3
y	–3	1	7

The calculations for the y values can be done mentally.

From the T.O.V. points (–2;–3), (0;1) and (3;7) are plotted to produce the graph of $y = 2x + 1$. You should write the equation of the line as illustrated below.

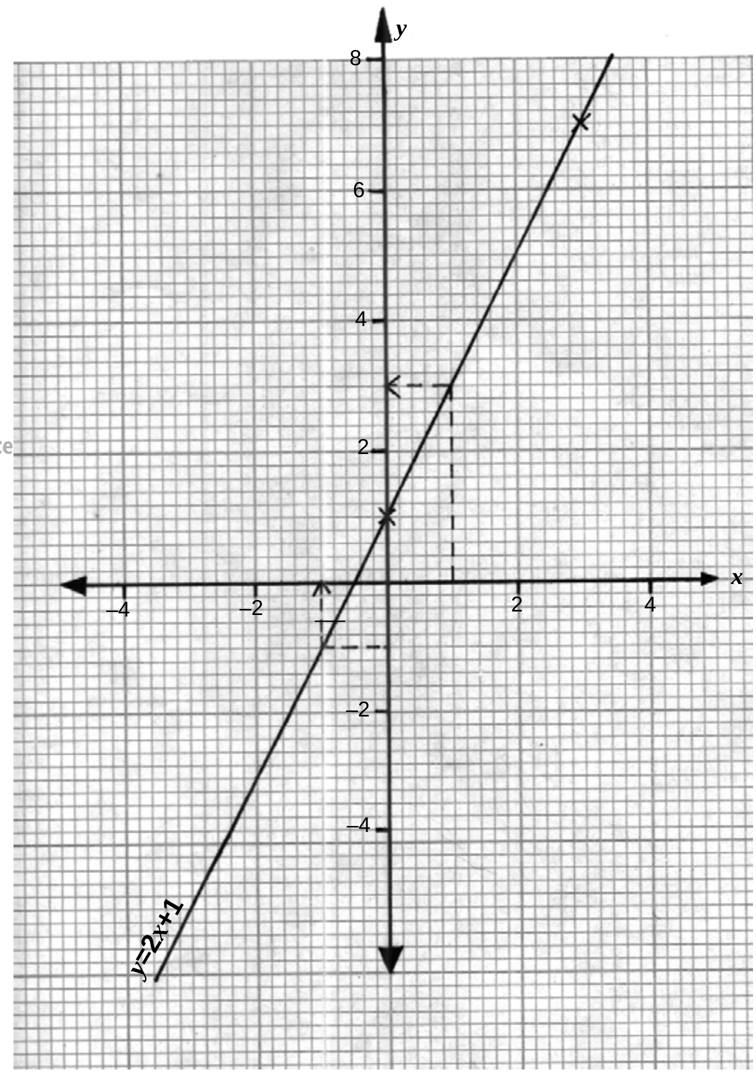


Fig 10.1

Intercepts can be used to find points instead of the table of values. The y-intercept is found by making x = 0, which gives you the result y = 1. The graph crosses the y-axis at (0;1).

The x-intercept is found by making y = 0, so that $x = \frac{1}{2}$. The graph crosses the x-axis at ($\frac{1}{2}$; 0). Draw the axes and plot these 2 points and draw the graph.

The graph can be used to find other points which may not be in the table of values e.g. when x = 1, y = 3; when y = –1, x = –1.

Graphs can be used to solve simultaneous equations.

2. Solve the following graphically: $y = 2x - 4$
$3x + y = 6$

Solution

2. Create a T.O.V. for each function.

$y = 2x - 4$

x	0	1	2
y	–4	–2	0

$3x + y = 6$

x	–1	0	1
y	9	6	3

The graphs are then drawn on the same axes as in Fig 10.2.

The two lines will meet at a point. That point gives the solutions of the equations
i.e. x = 2, y = 0

You can use other methods to check these.
Do you see the importance of labelling graphs?
It is necessary for clarity when there is more than one graph on the same axes.

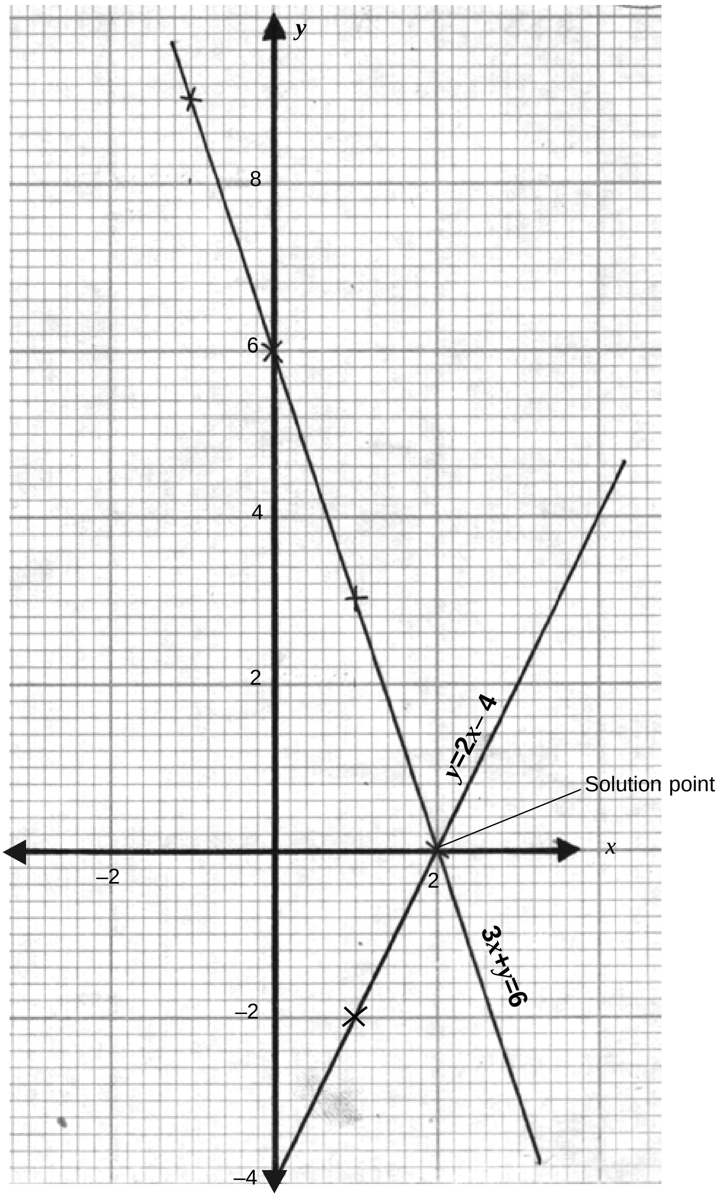


Fig. 10.2

1. On separate diagrams,

a) copy and complete the following table of values.

(i) $y = 3x - 6$

x	–1	0	3
y	–9	–6	

(ii) $y + 5x = 5$

x	–2	0	3
y		5	

b) Use suitable scales to draw the graphs of the respective functions.

c) From each graph:
   (i) give the coordinates of the x-intercept.
   (ii) find y when $x = -\frac{1}{2}$.
   (iii) find y when $x = 4$.

2. Given that $y + 2x = 6$, and $y = 4 - 2x$

a) Draw and complete T.O.Vs using –2, 0 and 2 for x in the first equation and –1, 0 and 3 for x in the second equation.

b) Using a scale of 2cm to 2 units on each axis, draw the two graphs on the same axes.

c) Use the elimination or substitution method to solve the two equations simultaneously. What do you notice?

3. a) Construct a convenient table of values for each of the following equations:

(i) $y - 2x = 0$, $x + y = 3$ (ii) $x - y = 1$, $y - 3x = 3$

(iii) $3x + 5y = 15$, $3y = 4x + 12$

b) Use suitable scales to draw the graphs of each pair of functions on the same axes.

c) Give the simultaneous solutions to the functions which intersect.

4. a) Draw the graphs of the following pairs of functions on the same axes, so that the two lines intersect.

(i) $2x - y = -2$, $2y + x = 2$ (ii) $y - 2x = 0$, $y = x + 2$

(iii) State the solutions to these equations.

C. THE GRAPH OF A QUADRATIC FUNCTION

You will recall that any expression having the variable of one of its terms to the power of 2, is called a **quadratic expression**. The general form of a quadratic function or expression is: $ax^2 + bx + c$, where a, b and c are rational numbers.

Below are examples of some quadratic functions

$f(x) = 3 + 2x - x^2$, $f(x) = 2x^2 - 3x$

$f(x) = 9 - 4x^2$ or $f(x) = (3 - 2x)(3 + 2x)$

1. Draw the graph of $f(x) = 3 + 2x - x^2$ for $-2 \leqslant x \leqslant 4$, using 2cm to represent 1 unit on the x-axis and 2cm to represent 2 units on the y-axis.

Solution

1. *Table 10.3:* $y = 3 + 2x - x^2$

x	–2	–1	0	1	2	3	4
3	3	3	3	3	3	3	3
$+2x$	–4	–2	0	2	4	6	8
$-x^2$	–4	–1	0	–1	–4	–9	–16
y	–5	0	3	4	3	0	–5

Step 1. Compile the table of values.
Step 2. Use the given scales to draw the axes.
Step 3. Plot the points from the table of values.
Step 4. Draw the curve smoothly through the plotted points.
Step 5. Label the graph.

With practice, the intermediary stages can be left out (done mentally) so the table of values will look like this:

Table 10.4

x	–2	–1	0	1	2	3	4
y	–5	0	3	4	3	0	–5

Graph of $f(x) = 3 + 2x - x^2$ is as illustrated in Fig 10.3.

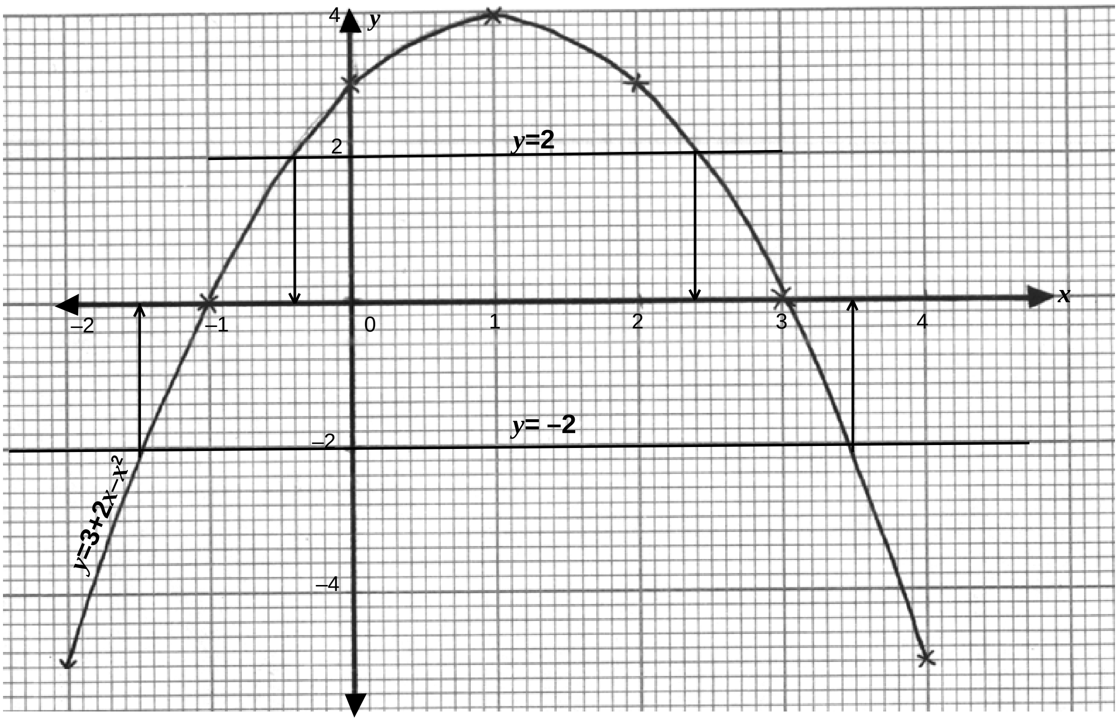


Fig. 10.3

2. Compare this general shape with that of $f(x) = 2x^2 - 3x$, Fig 10.4.

Solution

2. *Table 10.5:* $y = 2x^2 - 3x$

x	–2	–1	0	1	2	3	4
y	14	5	0	–1	2	9	20

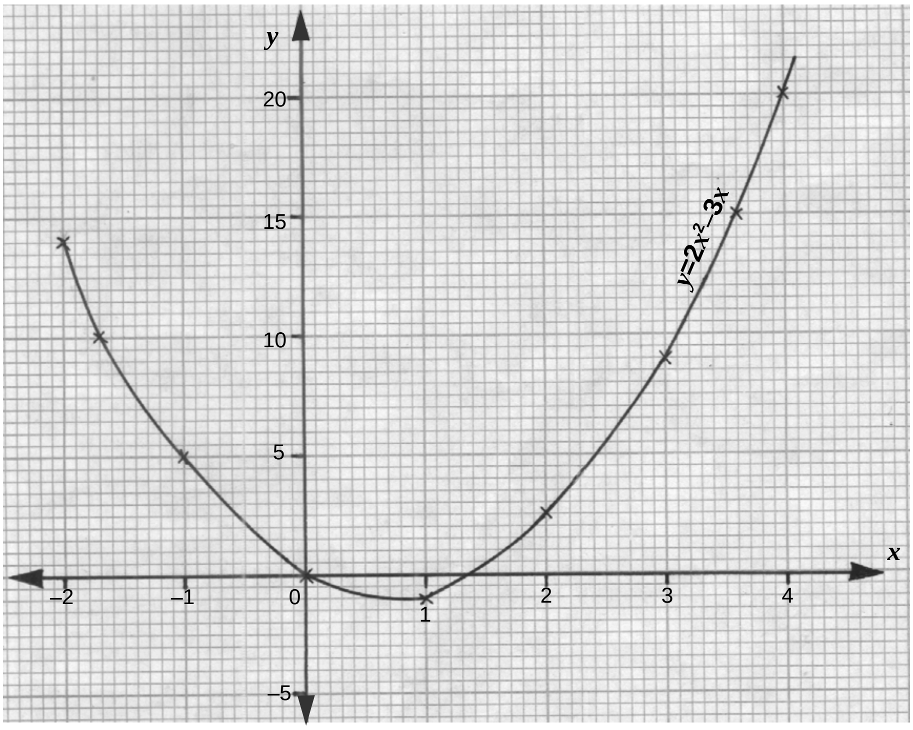
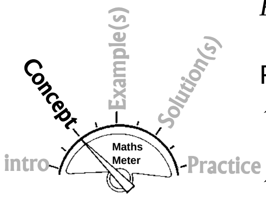


Fig. 10.4

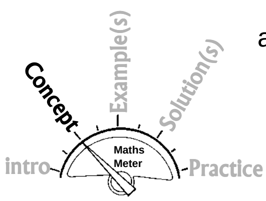


Features of **quadratic** graphs. A quadratic graph:

- is in the form of a cap ∩ when the coefficient of x^2 is negative.
- has a maximum turning point, if in the cap form.
- is in the form of a cup ∪ when the coefficient of x^2 is positive.
- has a minimum turning point, if in the cup form.
- gives two roots of the function (where the graph cuts the x-axis).

This curve is often called a **parabola**.

These shapes never change. In the diagrams below the non-dotted graphs are wrongly drawn because some points or a point is wrong. The dotted curves illustrate the cap and cup nature of the quadratic graphs.



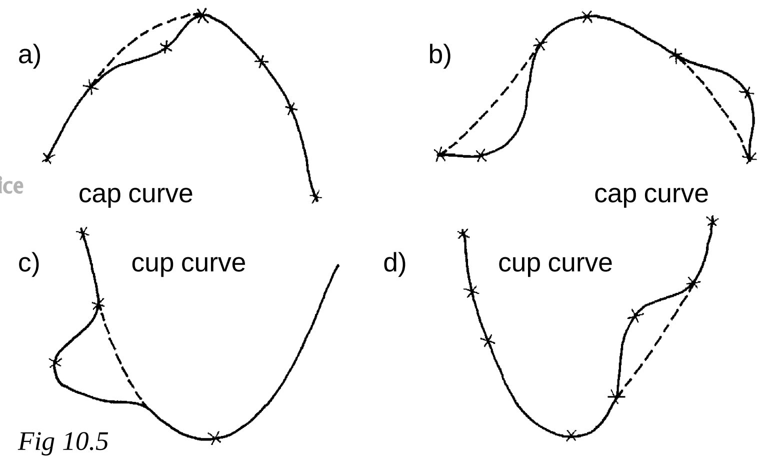
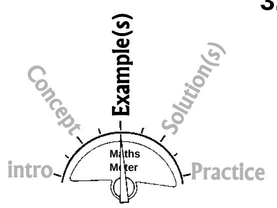


Fig 10.5

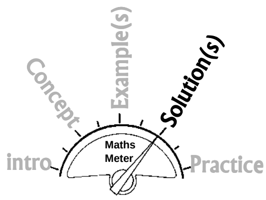
Make sure you don't produce incorrect graphs. (the non-dotted). The curves must be smooth caps or cups with no dents or bumps.



3. Referring to Fig 10.3 on page 218.
Use the graph to solve the equations.
a) $3 + 2x - x^2 = 0$
b) $5 + 2x - x^2 = 0$
c) $1 + 2x - x^2 = 0$

Solution

3. a) The function $3 + 2x - x^2$ is the graph function.
The roots of $3 + 2x - x^2 = 0$ are taken from where the graph cuts the x-axis (graph $y = 0$).
Thus when $3 + 2x - x^2 = 0$
$x = -1$ or 3



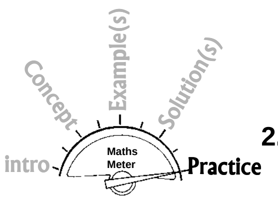
b) **Subtract the question function**, $(5 + 2x - x^2)$ **from the graph function**, $(3 + 2x - x^2)$.
i.e. $\begin{array}{r} 3 + 2x - x^2 \\ -(5 + 2x - x^2) \\ \hline -2 \end{array}$

This means the roots of $5 + 2x - x^2$ are taken from the points where the graph cuts the line $y = -2$.
Thus, when $5 + 2x - x^2 = 0$,
$x = -1{,}5$ or $3{,}5$

c) As above, $\begin{array}{r} 3 + 2x - x^2 \\ -(1 + 2x - x^2) \\ \hline 2 \end{array}$
i.e. draw the graph $y = 2$ and read x values when,
$1 + 2x - x^2 = 0$
$x = -0{,}45$ or $2{,}45$

Hint
Smoothness of graphs depends on the accuracy of the solutions.

1. Use the graph on Fig 10.4 to solve the equations:
a) $2x^2 - 3x = 0$
b) $2x^2 - 3x - 2 = 0$
c) $2x^2 - 3x - 7 = 0$



2. a) Copy and complete the table of values of the function $y = (x + 1)(x - 3)$.

Table 10.6

x	–3	–2	–1	0	1	2	3	4	5
y	12	5	0			–3	0		12

b) Using a scale of 2cm to 1 unit on the x-axis and 2cm to 2 units on the y-axis, draw the graph of $y = (x + 1)(x - 3)$.

c) Use the graph to solve the following equations:
(i) $x^2 - 2x - 3 = 0$
(ii) $x^2 - 2x - 2 = 0$
(iii) $x^2 - 2x - 5 = 0$
(iv) $x^2 - 2x - 10 = 10$

3. a) Copy and complete the table of values for the equation $y = 3 + 5x - 2x^2$

Table 10.7

x	–2	–1	0	1	2	3	4	5
y			3	6				

b) Using suitable scales, draw the graph of $y = 3 + 5x - 2x^2$.

c) Use the graph to solve the equations:
(i) $3 + 5x - 2x^2 = 0$
(ii) $1 + 5x - 2x^2 = 0$
(iii) $6 + 5x - 2x^2 = 0$

4. a) Draw the graph of $y = x^2 - 6x + 9$ using suitable scales for values of x from 0 to 6.

b) Use the graph to solve the equations:
(i) $x^2 - 6x + 9 = 0$
(ii) $x^2 - 6x + 3 = 0$
(iii) $x^2 - 6x + 5 = 0$

5. a) Draw the graph of $y = x^2 + 5x - 4$, using suitable scales for values of x from –7 to 2.
b) Use the graph to solve the equations:
(i) $x^2 + 5x - 4 = 0$
(ii) $x^2 + 5x = 0$
(iii) $x^2 + 5x - 9 = 0$
(iv) $x^2 + 5x + 2 = 0$

A variety of questions can be asked in association with graphs. Redraw the graph of $y = 3 + 2x - x^2$ Fig 10.3, on page 191.

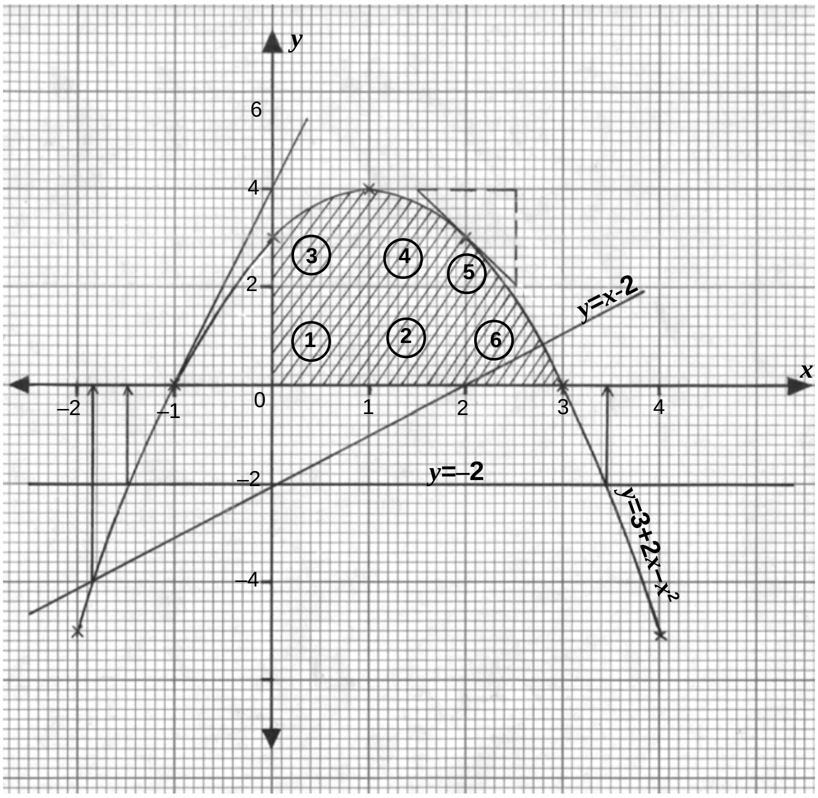
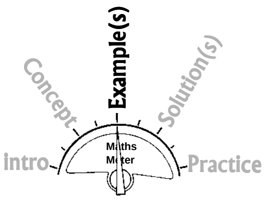


Fig 10.6



4. From this graph, find:

a) the equation of the line of symmetry of the graph.

b) the maximum value of $3 + 2x - x^2$.

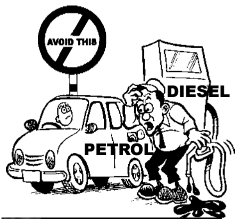
c) the gradient of the curve at (i) $x = 2$

(ii) $x = -1$

d) the range of values of x for which y is positive.

e) by drawing a suitable graph of the diagram the roots of the equation $3 + 2x - x = x - 2$.

f) the area bound by the curve, y-axis and the positive x-axis.



Common Error

a) Candidates simply draw the line and end there (they do not answer the question). $y = 1$ is common. Remember vertical lines are $x = a$ and horizontals $y = a$ where a is a rational number.

Solution

Hint

Note that this question is relevant to cap-type of curves. Minimum value is asked on cup-types. Again instead of $3 + 2x - x^2$, the question can use y since the two are equal. i.e. find the maximum value of y.

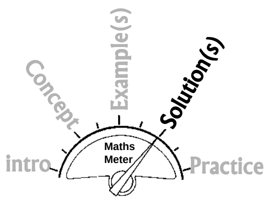
4. a) Remember line symmetry? It is the line of fold, i.e. $x = 1$

b) Maximum or Minimum value of y is at the **turning point** of the graph. For this question, maximum value for $y = 4$.

c) (i) Draw a tangent at $x = 2$ and a right-angled triangle on the tangent. Use the sides of this triangle to find the gradient.

$$\text{Gradient} = \frac{-2}{1}$$
$$= -2$$

or use points on the tangent e.g. (2;3) and $(2\frac{1}{2}\,;2)$ i.e. check for points lying on the tangent.



$$\text{Gradient} = \frac{2-3}{2{,}5-2}$$

$$= \frac{-1}{0{,}5}$$

$$= -2$$

(ii) Draw a tangent at $x = -1$
Using points (–1;0) and (–0,5;2) on the tangent.

$$\text{Gradient} = \frac{2-0}{-0{,}5-(-1)}$$

$$= \frac{2}{0{,}5}$$

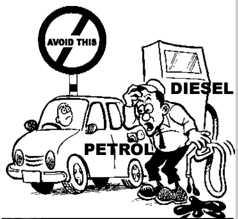
$$= 4$$

Hint

Use strict inequalities. Answers like from –1 to 3 includes the –1 and 3 which is wrong. So is $-1 \leqslant x \leqslant 3$. Between –1 and 3 is not clear whether –1 and 3 are excluded or not and so is unacceptable.

d) This question is concerned with that part of the graph which is on the positive side of the y-axis. In this case, values of x from –1 to 3 put the curve on the positive side of the y-axis. Thus, the range of values of x is $\mathbf{-1 < x < 3}$.

Notice here that inequalities illustrate the range better.



Common Error

Candidates draw the other straight line to cut the curve only once hence giving one value of x.

e) The given equation indicates that $y = x - 2$ and that this is the graph required. Use any means to draw the graph. e.g. use table of values.

When $3 + 2x - x^2 = x - 2$
$x = -1{,}85$ or $2{,}8$, from where the curve and the linear graph intersect.

Notice $3 + 2x - x^2 = x - 2$ could have been given as $5 + x - x^2 = 0$, i.e. if $x - 2$ is shifted to the LHS.
The procedure from example 3b and c on page 193, is then followed for the graph.

Hint

When ascertaining area by counting squares, a square whose bigger part is inside the region is counted as a full square. Those whose bigger part is outside the region is not considered.

f) Find the area of the shaded section Fig 10.6 on page 195.

Method 1
Counting squares to find area:
Number of small squares in the shaded area:

$$= \underset{①}{100} + \underset{②}{100} + \underset{③}{82} + \underset{④}{82} + \underset{⑤}{10} + \underset{⑥}{72}$$

$$= 446$$

Area of each small square $= 0{,}1 \times 0{,}2$ (using scale)
$= 0{,}02$

$\therefore$ Required area $= 446 \times 0{,}02$
$= 8{,}92$ square units

Hint

This method leaves out bits of area included in method 1. It gives an acceptable estimation though. However, more trapezia need to be created for better results.

Method 2
Dividing the area into Trapezia:
Do you see that ① + ③ – Trapezium
② + ④ – Trapezium
⑥ + ⑤ – Triangle

Total Area $= \frac{1}{2}(3+4)1 + \frac{1}{2}(4+3)1 + \frac{1}{2}1 \times 3$

$= \frac{1}{2} \times 7 + \frac{1}{2} \times 7 + \frac{1}{2} \times 3$

$= \frac{17}{2}$

$=$ 8,5 or $8\frac{1}{2}$ square units

5. a) Redraw Fig 10.4, on page 219 on graph paper.

b) Use the graph to estimate:

(i) the gradient at $x = 0$

(ii) the roots of a) $2x^2 - 3x - 3 = 0$ and (b) $2x^2 - 3x = x + 2$

(iii) the area bound by the curve, the x-axis and the lines $x = 2$ and $x = 3$

Solution

5a)

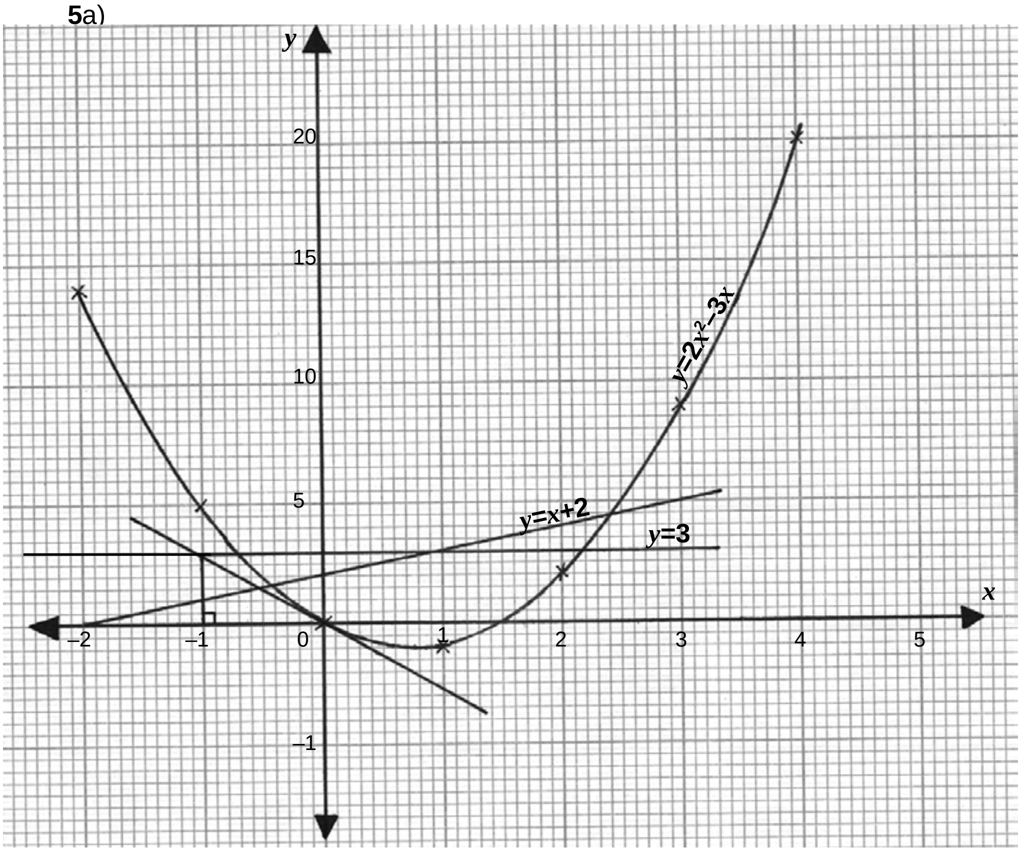


Fig 10.7.

b) (i) Gradient at $x = 0 = \frac{-3}{1}$ By drawing a tangent and taking measurements.

$= -3$

a) For $2x^2 - 3x - 3 = 0$

$2x^2 - 3x = 3$ Draw line $y = 3$ and solve.

$x = -0{,}6$ or $2{,}2$

b) For $2x^2 - 3x = x + 2$ Draw line $y = x + 2$ and solve.

$x = -0{,}4$ or $2{,}4$

(iii) Number of small squares $= 210$

Area of each square $= 0{,}1 \times 0{,}5$

$= 0{,}05$

$\therefore$ Total area of region $= 210 \times 0{,}05$

$= 10{,}50$ units^2

PRACTICE 10C2

1a) Copy and complete the following table of values for $y = (x - 2)(x + 5)$.

Table 10.8

x	–6	–5	–4	–3	–2	–1	0	1	2	3
y	8	0		–10		–12		–6	0	

b) Using a scale of 2cm to represent 2 units horizontally and 2cm to represent 5 units vertically draw the graph of $y = (x - 2)(x + 5)$ for $-6 \leqslant x \leqslant 3$.

Hint
Expand $(x-2)(x+5)$

c) Use the graph to find:

a) the minimum value of y.

b) the ranges of values of x for which $(x - 2)(x + 5)$ is negative.

c) the roots of (i) $x^2 + 3x - 12 = -2$.

(ii) $x^2 + 3x - 10 = x + 2$

(iii) $x^2 + 3x - 10 = 2x - 6$

2. a) Copy and complete the following table of values for $y = 4 - x^2$

Table 10.9

x	–3	–2	–1	0	1	2	3
y	–5	0				0	–5

b) Choose a suitable scale to draw the graph of $y = 4 - x^2$.

c) Use the graph to find:

(i) the equation of the line of symmetry.

(ii) the roots of (i) $4 - x^2 = 0$

(ii) $4 - x^2 = x + 1$

(iii) $4 - x^2 = 2 - \frac{1}{2}x$

d) Estimate the area bounded by the curve, the x-axis, $x = -\frac{1}{2}$ and $x = 1$.

e) Give the equation of the line of symmetry of the curve.

3. a) Draw and complete a table of values for $y = (x - 3)^2$, x in the range $0 \leq x \leq 6$.

b) Use a suitable scale to draw the graph of $y = (x - 3)^2$.

c) Use the graph to find:
the roots of (i) $(x - 3)^2 = 0$
(ii) $(x - 3)^2 - 4 = 0$

d) The minimum value of y.

e) Use the method of counting squares to estimate the area bound by the curve and the line $2y = x + 4$.

4. a) Copy and complete the following table of values for $y = 3x(4 - x)$.

Table 10.10

x	–1	0	1	2	2,5	3	4	5
y		0	9					–15

b) Using a horizontal scale of 2cm to 1 unit and a vertical scale of 2cm to 5 units, draw the graph of $y = 3x(4 - x)$.

c) By drawing the graph of $y = \frac{5}{2}x - 5$ on the same axis, find the roots of $3x(4 - x) = \frac{5}{2}x - 5$.

d) Find the area bound by the curve, the x-axis and the line $y = \frac{5}{2}x - 5$.

D. THE GRAPH OF THE CUBIC FUNCTION

The highest power of the variable in any term in a cubic function is 3. e.g. $y = x^3 - 2x^2 + 5x - 7$. Thus, a cubic function is of the form $ax^3 + bx^2 + cx + d$, where a, b, c and d are rational but $a \neq 0$.

What does the graph look like?

1. a) Consider the graph of $y = x^3$ with x ranging from –3 to 3 in Fig 10.8.

i.e. Table of values

Table 10.11

x	–3	–2	–1	0	1	2	3
y	–27	–8	–1	0	1	8	27

1a)

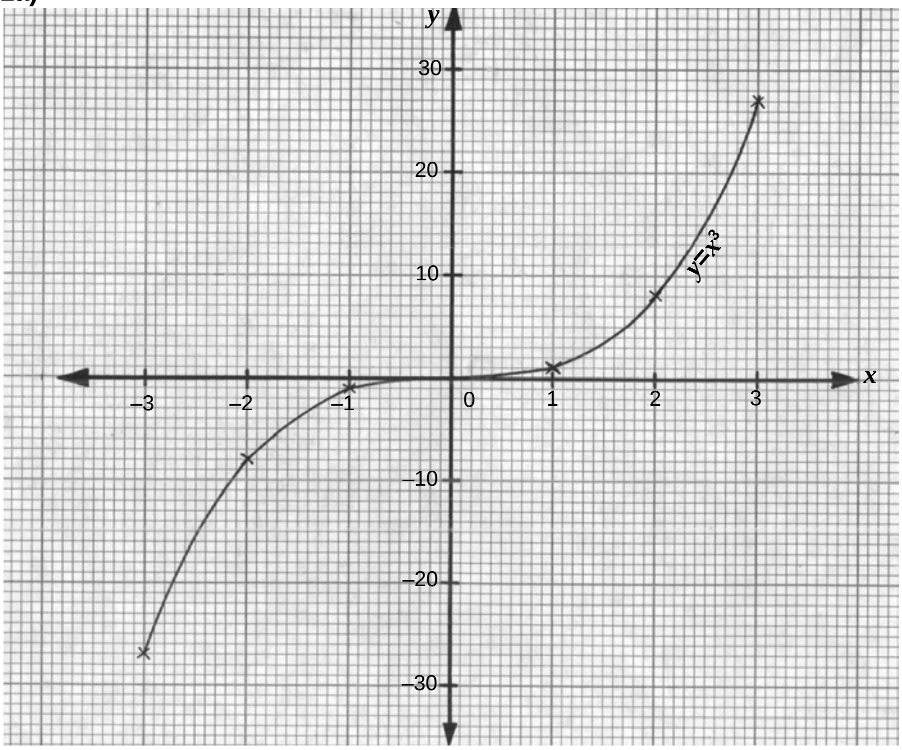


Fig. 10.8

Do you ***notice*** from –3 to 0, the graph tends to turn downwards but from 0 onwards it turns upwards?
We can safely conclude that cubic graphs turn twice!
Fig 10.8 illustrates the curves more fully.

b) The graph of $y = 12x + x^2 - x^3$ for $-4 \leqslant x \leqslant 4{,}5$.

Table 10.12

x	–4	–3	–2	–1	0	1	2	3	4	4,5
y	32	0	–12	–10	0	12	20	18	0	–16,9

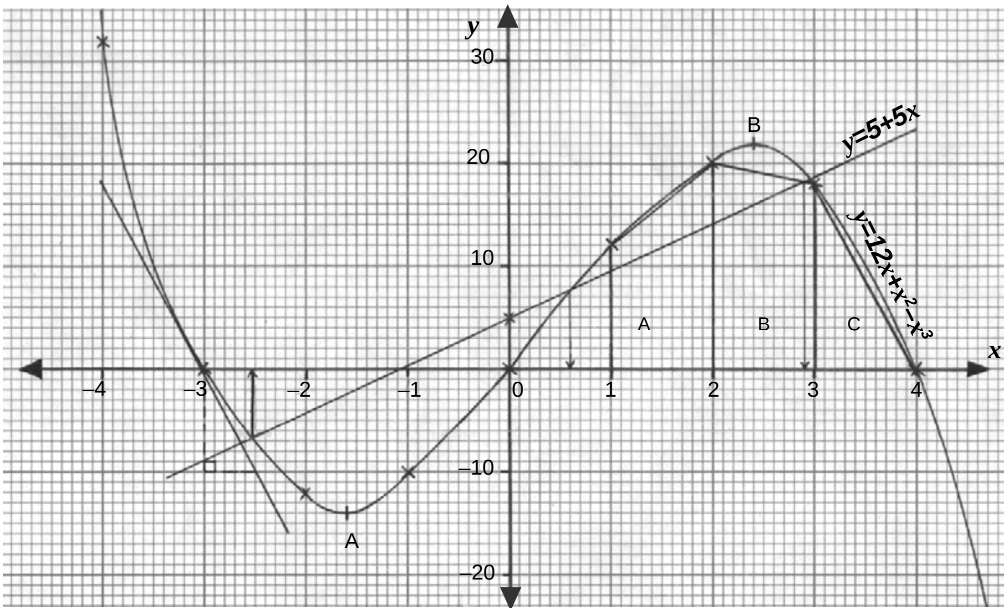
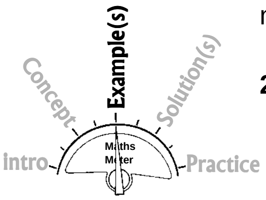


Fig.10.9

In Fig 10.9, point A is the minimum turning point and point B the maximum turning point of the graph.



2. Use the graph to find:

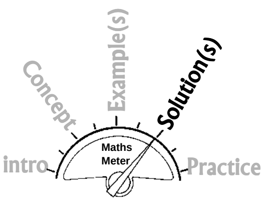
a) the minimum value of $12x + x^2 - x^3$.

b) the area bounded by the curve, the x-axis and lines $x = 1$ and $x = 3$.

c) by drawing a suitable graph on the diagram, the roots of $12x + x^2 - x^3 = 5 + 5x$.
(The equation can be given as $-5 + 7x + x^2 - x^3 = 0$).

d) the gradient of the curve at $x = -3$.

Solution



2. a) $y = -14$ From the lowest turning point.

b) Dividing area into trapezia:

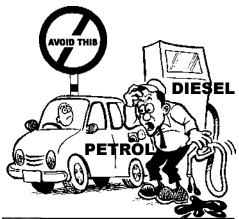
$= \quad x = 1$

$= \quad \frac{1}{2}(12 + 20)1 + \frac{1}{2}(20+18)1$

$= \quad 16 + 19$

$= \quad 35$ units ^{2}

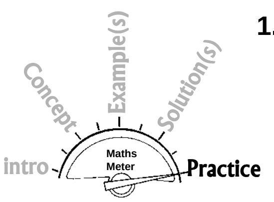
c) Graph of $y = 5 + 5x$ must be correctly drawn.
$\therefore$ If $12x + x^2 - x^3 = 5 + 5x$
then $x = -2{,}55$ or $0{,}55$ or $2{,}95$



Common Error

Candidates draw short line $y = 5 + 5x$ resulting in missing one or more values.

d) Tangent at $x = -3$ drawn on the curve and its gradient found as $\frac{-10}{0{,}5} = -20$



1. a) Copy and complete the table of values of $y = (x + 1)(x - 3)(x - 5)$ below.

Table 10.13

x	–2	–1	0	1	2	3	4	5	6
y		0	15	16			–5	0	

b) Use a suitable scale to draw the graph of $y = (x + 1)(x - 3)(x - 5)$, using the table above.

c) Use the graph to estimate:
(i) the roots of $(x + 1)(x - 3)(x - 5) = 5$.
(ii) the area bound by the curve, the y-axis and the positive x-axis, using the method of counting squares.

d) (i) the maximum value of y.

e) the gradient of the curve at (i) $x = 1$.
(ii) $x = 5$.

2. a) Draw the graph of $y = (x-2)^2(x-6)$ for values of x from 0 to 7.

b) Use the graph to:
(i) solve the equation a) $(x-2)^2(x-6) = 0$.
b) $(x-2)^2(x-6) = -4$.

(ii) give the value of y at the minimum turning point.
(iii) the gradient at $x = 2$.

c) By counting squares, find the area bound by the curve, the x-axis and the line $x = 3$.

3. a) Copy and complete the table of values for $y = 3 - x^3$, in table 10.14 below.

Table 10.14

x	–3	–2	–1	0	1	2	3
y	30			3			–24

b) Using a scale of 2cm to 1 unit on the x-axis and 2cm to 5 units on the y-axis, draw the graph of $y = 3 - x^3$, using the above table.

c) Use the graph to:
(i) solve the equation $3 - x^3 = 0$.
(ii) find the gradient of the curve at x = 2.

d) (i) Draw the graph of $y = 5x - 5$ on the same axes.
(ii) Estimate the roots of $8 - 5x - x^3 = 0$.

4. Below is a sketch of graph $y = x^3 + px^2 + qx + r$, where p,q and r are integers.

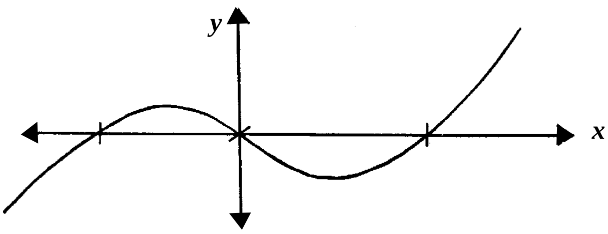


Fig. 10.10

The curve passes through (–3;0) the origin and (4;0)
Find the values of p, q and r in the given equation.

5.

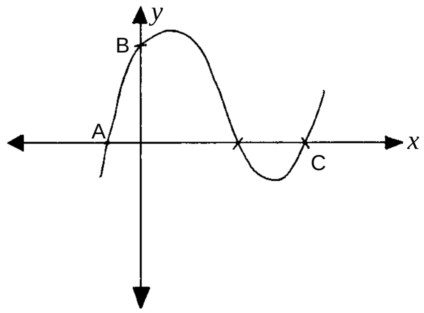


Fig. 10.11

Fig 10.11 shows a sketch graph of
$y = (x + 1)(x - 3)(x - 5)$.
Find the coordinates of points A, B and C.

E. THE GRAPH OF THE INVERSE FUNCTION.

An inverse or reciprocal function is one which involves division by the variable e.g. $f(x) = \frac{5}{x}$, $f(x) = 7 - \frac{2}{3}x$.
Graphically, these functions produce 'two part' curves which are distinctive (clearly separate).

1. Consider the graph of $y = \frac{2}{x}$ below.

Table 10.15

x	–4	–3	–2	–1	–0,5	0,5	1	2	3	4
y	–0,5	–0,7	–1	–2	–4	4	2	1	0,7	0,5

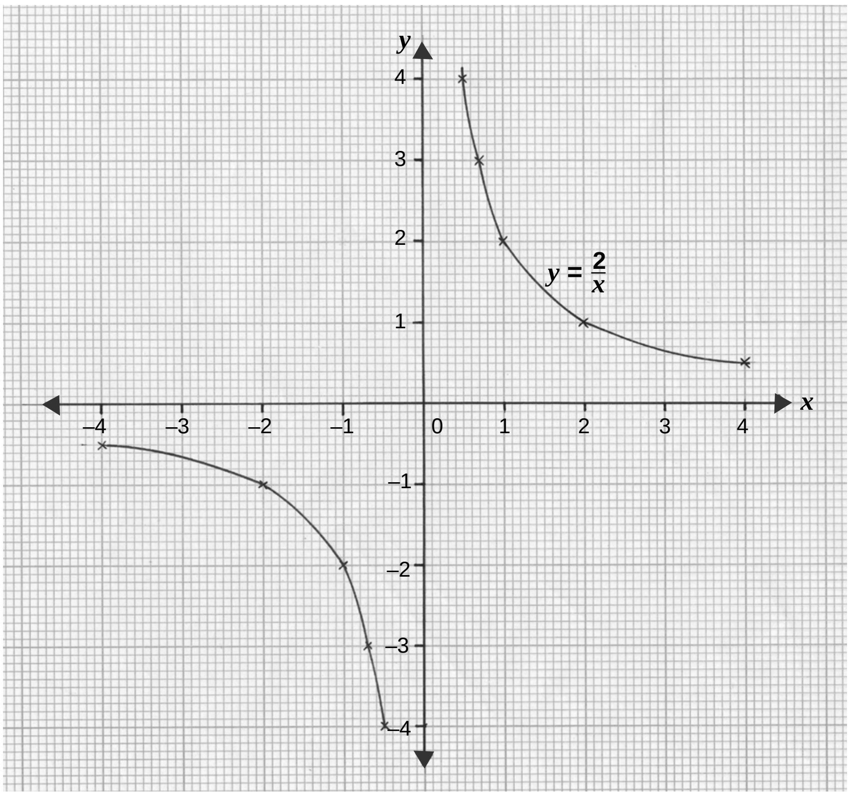


Fig 10.12

This type of graph is called a **hyperbola**.

Let us analyse the graph in relation to its equation

$y = \frac{2}{x}$

(i) The graph will never cut the y-axis and the x-axis i.e. x and y cannot be 0 in this equation

(ii) The equation $y = \frac{2}{x}$ can be rearranged as $x = \frac{2}{y}$ (making x the subject).
This is why the x-axis (y = 0) is also an asymptote of this graph.

Hint
An asymptote is a straight line that continually approaches a given curve but does not meet it at any finite distance

2. Let us consider another example:

$$y = \frac{3}{x-1}$$

Table 10.16

x	–4	–3	–2	–1	0	0,5	1,5	2	3	4	5	6
y	–0,6	–0,8	–1	–1,5	–3	–6	6	3	1,5	1	0,8	0,6

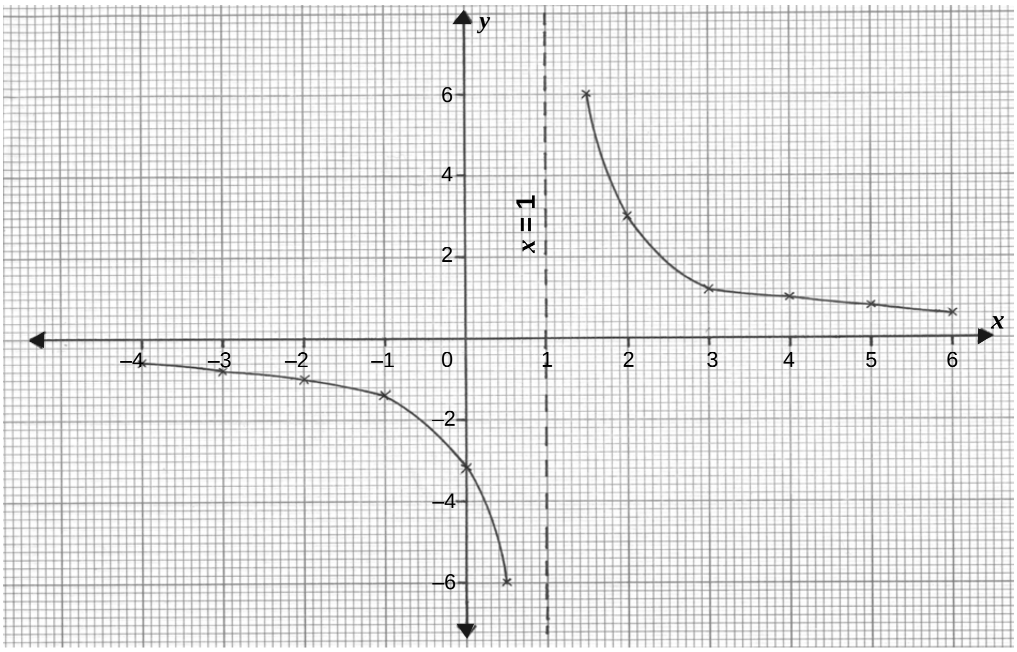


Fig 10.13

When $y = \frac{3}{x-1}$, x cannot be 1 because it is 1 which makes the denominator 0! Thus the vertical asymptote is x = 1.

When we change the subject to x, y goes into the denominator hence x-axis (y =0) is an asymptote. Do you see that asymptotes can be derived from the equation given? Hence it is possible to sketch or see the structure of these graphs before an actual one is drawn.

PRACTICE 10E

1. Fig 10.13, is a sketch of graph $y = \dfrac{12}{x}$.

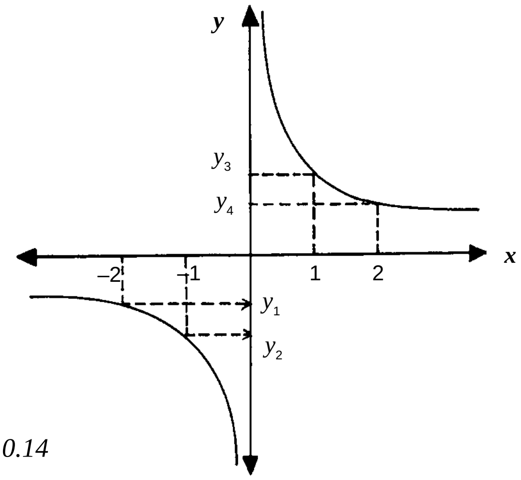


Fig. 10.14

Give the values of y_1, y_2, y_3 and y_4.

2. a) Copy and complete the table of values of $y = \frac{6}{x}$ below.

Table 10.17.

x	–6	–5	–4	–3	–2	–1	1	2	3	4	5	6
y		–1,2		–2	–3	–6	6			1,5		1

b) Using a scale of 2cm to represent 2 units on each axis, draw the graph of $y = \frac{6}{x}$ for $-6 \leqslant x \leqslant 6$.

c) Use the graph to find:
   (i) the area bound by the curve, $x = 2$, $x = 3$ and the x-axis using the method of counting squares.
   (ii) the gradient of the curve at $x = -3$.

d) (i) On the same axes, draw the graph of $y = x$.
   (ii) Hence, solve the equation $\frac{6}{x} = x$.

Sometimes only one part of the graph is required.

3. a) Draw the graph of $y = \dfrac{2}{x-2}$ for x from – 4 to 8, excluding 2.

b) Find the gradient of the curve at $x = 4$.

c) Find the area bound by the curve, the x-axis, the y-axis and line $x = -3$.

d) By drawing a suitable graph on the same axes give a root of the equation $\dfrac{2}{x-2} = x - 2$.

4. Below is an incomplete table of values of the function $y = \frac{1}{x^2}$.

Table 10.18

x	0,5	1	1,5	2	2,5	3
y	4				0,16	0,1

a) Copy and complete this table.

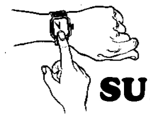
b) Using a scale of 2cm to 0,5 units on the x-axis and 2cm to 1 unit on the y-axis, draw the graph of $y = \frac{1}{x^2}$ for $0,5 \leqslant x \leqslant 3$.

c) On the same axis, draw the graph of $y = \frac{1}{x^2}$.

5. a) Using the same range of values of x as in Fig 10.12, draw and complete the table of values for $y = \frac{-1}{2x^2}$.

b) Use the same scale as in Fig 10.12, draw the graph of $y = \frac{-1}{2x^2}$.

c) What do you observe between the graphs of $y = \frac{2}{x}$ and $\frac{-1}{2x^2}$?



SUMMARY

1. $f(x)$ is read f of x and is a function in terms of x. e.g. $x + 2$, $x^2 + x - 3$, x^3. Similarly, $f(2)$ is read as "f of 2" or "f at 2".

2. Given $f(x)$, $f(a)$ means, a now takes the place of x in the function.

3. In summary:

name of independent variable

$$y = f(x) = \underbrace{3x - 6}_{\textbf{defining expression}}$$

value of the function

4. The four functions discussed here are:

(i) linears producing **linear/straight line** graphs.

(ii) quadratics producing **parabolas** which are in cap or cup form.

(iii) cubics producing **cubic** graphs.

(iv) inverses producing **hyperbolas.**

5. Use the table of values in most, if not all, cases where graphs are to be drawn.

6. Range uses strict inequalities > or < **not** $\geqslant$ or $\leqslant$.

7. Counting squares is the most accurate way to find area under a curve.
Remember some squares are counted, even if they are not whole.

8. Draw a tangent to the curve at the given point to find the gradient of the curve at that point.

EXAM PRACTICE 10

Consider the example below:

Hint

When an incomplete table of values is given and you are to find the missing values, make sure your calculation is accurate so that the structure of the curve is not spoilt.

Answer the whole of this question on a single sheet of graph paper.

1. Below is an incomplete table of values for the equation $y = x^2 + x - 4$.

Table 10.19

x	–3	–2	–1	0	1	2
y	2	p	–4	–4	q	2

a) Calculate the value of p and the value of q.

b) Using a scale of 2cm to represent $\frac{1}{2}$ unit on the x-axis and 2cm to represent 2 units on the y-axis, draw the graph of $y = x^2 + x - 4$ for $-3 \leqslant x \leqslant 2$.

c) Use the graph to:
(i) give the equation of the line of symmetry.
(ii) solve the equation $x^2 + x - 4 = 0$.
(iii) find the gradient of the curve at $x = \frac{1}{2}$.

d) By drawing a suitable graph on the same axes solve the equation $x^2 + x - 4 = x - 2$.

Solution

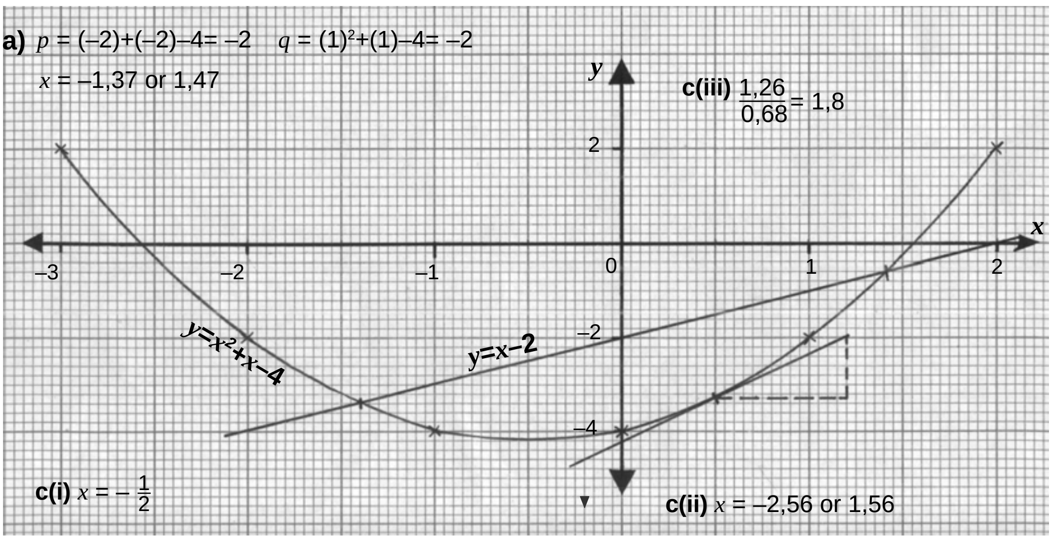


Fig. 10.15

Note that:

All answers are on the graph paper, in response to the instructions at the beginning of the question.

Make sure the following are observed in this work.

a) Make sure calculations for p and q are accurate. Check your answers.

b) Ensure you have used the correct scale.
Candidates are often confused by the range of x given i.e. $-3 \leqslant x \leqslant 2$. This simply means the values for x in the table range from –3 to 2 inclusive.

c) Does the curve look correct (i.e. no points off the curve). If not, correct that.
   (i) Give the equation, do not only draw the graph and end there. Remember $x = -\frac{1}{2}$ is not $y = -\frac{1}{2}$.

   (ii) Draw the tangent as accurately as possible. Students have a tendency to draw the tangent away from the curve (i.e. a gap between the curve and tangent or they draw a secant (one which cuts the curve).

d) Identify the type of graph indicated. Usually it is y = ? Which is on the RHS of the curve function. Draw the graph accurately. Use a table of values if need be.
Make sure the graph is long enough to cut the curve more than once if applicable.

Marks obtained from graph work are very easy to score. With practice, students should score full marks in this type of question. Practise calculating accurately, drawing smooth curves. Answer the whole question on a single sheet of graph paper, in each case.

Now do the following:

1. When a stone is thrown into the air its height (h metres) at any time (t) is given by the formula $h = 60 + 30t - 5t^2$
Below is an incomplete table of values of $h = 60 + 30t - 5t^2$.

Table 10.20

Time (t seconds)	0	1	2	3	4	5	6	7
Height (h metres)	60	85	p	105	100	q	60	25

a) Calculate the value of p and the value of q.

b) Using 2cm to represent 1 second on the horizontal axis and 2cm to represent 20 metres on the vertical axis, draw the graph of $h = 60 + 30t - 5t^2$ for $0 \leqslant x \leqslant 7$.

c) Use your graph to find:
   (i) the maximum height reached by the stone.
   (ii) the velocity of the stone at t = 6.
   (iii) the number of times when the stone is at a height of 75 metres.

2. The following is an incomplete table of values for the function

$$y = \frac{10}{x} - 2.$$

Table 10.21

x	1	2	3	4	5	6	7	8
y	8	3	m	0,5	0	–0,3	n	–0,8

a) Find the value of m and the value of n.

b) Using a scale of 2cm to represent 1 unit on each axis, draw the graph of $y = \frac{10}{x} - 2$ for $1 \leqslant x \leqslant 8$.

c) Use the graph to estimate:
(i) the value of y when $x = 3,5$.
(ii) the area of the region between the curve, the x-axis and the lines $x = 2$ and $x = 5$.

3.

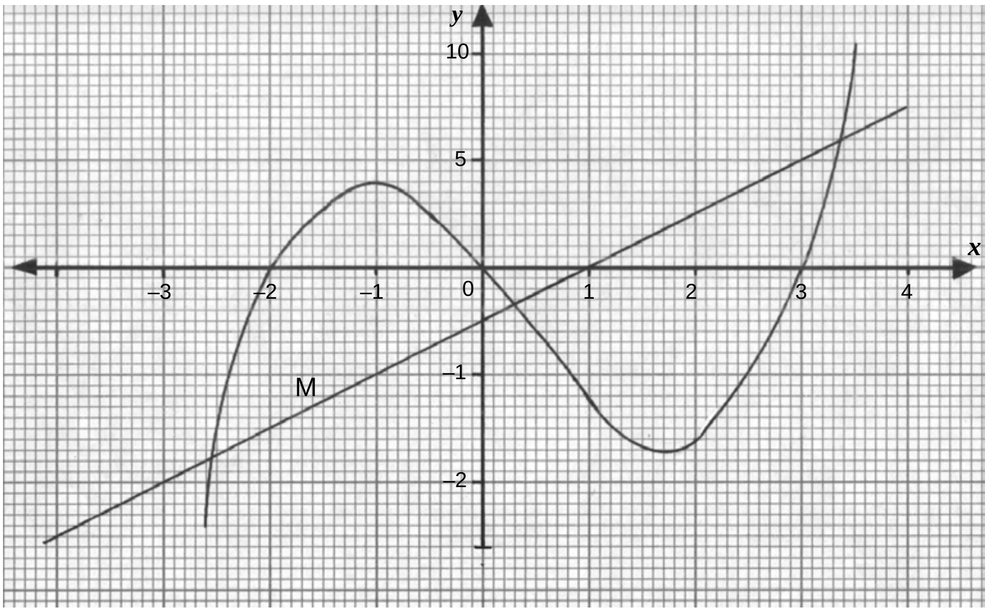


Fig. 10.16

Fig 10.16 shows the graph of a cubic function $y = ax^3 + bx^2 + cx + d$ where a, b, c and d are integers.

a) Find the values of a, b, c and d.

b) Find the range of values of x for which this function is positive.

c) Line M intersects the cubic function at three points.
(i) Find the equation of the line M in the form $y = mx + c$.
(ii) Find the solutions of $ax^3 + bx^2 + cx + d = mx + c$.

4. Given that $f(x) = (x - 3)^2 + 5$,
a) find the value of $f(1)$.

b) solve the equation $f(x) = f(-2)$.

5. a) Given that $f(x) = x(6 - x)$ and if $f(1) = k$, find the value of k.

b) Given that $f(x) = x^2 - 5x + 7$, find the values of x for which $f(x) = 7$.

6. Below is an incomplete table of values of the function $y = \frac{10}{x+1}$ for $-\frac{1}{2} \leqslant x \leqslant 9$.

Table 10.22

x	$-\frac{1}{2}$	0	1	2	3	4	5	6	7	8	9
y	20	p	5	3,3	2,5	2	q	r	1,3	1,1	1

a) Find the value of p, q and r.

b) Using a scale of 2cm to represent 5 units on the vertical axis, draw the graph of $y = \frac{10}{x+1}$ for $-\frac{1}{2} \leqslant x \leqslant 9$.

c) Use the graph to estimate:
(i) the gradient of the graph when $x = 1$.
(ii) the area bound by the curve, the y-axis , the x-axis and the line $x = 5$.

7. Fig 10.17 is a sketch showing graphs of straight line l_1 and curve l_2.

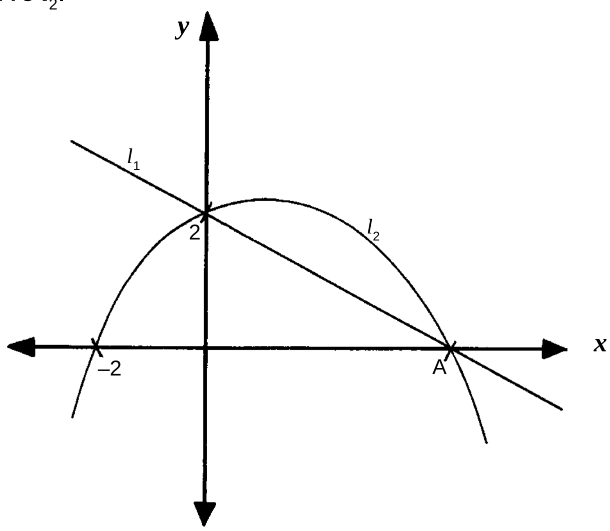
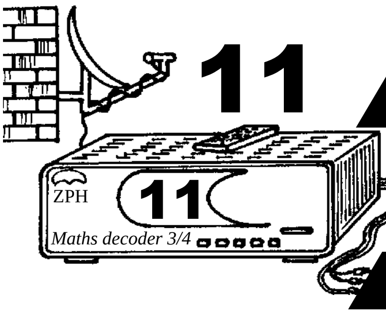


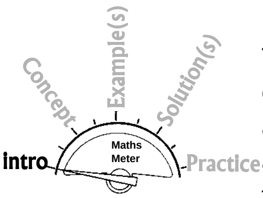
Fig 10.17

a) Given that the gradient of the straight line l_1 is $-\frac{1}{2}$, find the coordinates of the point labelled A.

b) Hence or otherwise find the equation of the:
(i) line l_1 in the form $y = mx + c$.
(iii) curve l_2 in the form $ax^2 + ax + c = 0$ where a, b and c are integers.



11 Surds



This chapter introduces you to some special irrational numbers called **surds**. A surd is a square root which cannot be reduced to a whole number e.g. $\sqrt{2}$, $\sqrt{7}$, $\sqrt{10}$. We will perform the four arithmetic operations with the surds and we will also express the trigonometrical ratios of 30^0, 45^0 and 60^0 in surd form.

Surds are irrational numbers.

Syllabus Expectations

By the end of this chapter, students should be able to:

1. simplify surds by making the number under the root sign as small as possible.
2. simplify surds by rationalising the irrational denominator.
3. carry out the four arithmetic processes with surds.
4. use rules of indices.
5. evaluate expressions involving surds.
6. give and use trigonometrical ratios of 30^0, 45^0 and 60^0 in surd form.

ASSUMED KNOWLEDGE

In order to tackle the work in this chapter, it is assumed that students are able to:

- identify square roots of perfect numbers.
- identify like and unlike terms.
- write numbers as products of their prime factors.

A. RULES OF OPERATIONS

(i) $\sqrt{a.b} = \sqrt{a} \times \sqrt{b}$

To prove this, let a = 4 and b = 9

L.H.S = $\sqrt{ab}$ RHS $= \sqrt{a} \times \sqrt{b}$

$= \sqrt{4 \times 9}$ $= \sqrt{4} \times \sqrt{9}$

$= \sqrt{36}$ $= 2 \times 3$

$= 6$ $= 6$

$\therefore$ LHS = RHS

This rule is very important when dealing with surds. It helps us to split a given surd into factors for convenience.

For example,

$\sqrt{24} = \sqrt{4} \times \sqrt{6}$ or $\sqrt{8} \times \sqrt{3}$ or $\sqrt{2} \times \sqrt{4} \times \sqrt{3}$ or $\sqrt{12} \times \sqrt{2}$ etc.

It also follows that, given $\sqrt{3} \times \sqrt{5}$, we can write the two as the square root of a single number i.e. $\sqrt{3 \times 5} = \sqrt{15}$

Similarly $\sqrt{2} \times \sqrt{5} \times \sqrt{7} = \sqrt{70}$.

(ii) $\mathbf{a\sqrt{b} = \sqrt{a^2 b}}$

when a = 4 and b = 9

LHS $= 4\sqrt{9}$ RHS $= \sqrt{4^2 \times 9}$

$= 4 \times 3$ $= \sqrt{16 \times 9}$

$= 12$ $= \sqrt{144}$

$= 12$

Hence LHS = RHS

Did you ***realise that*** $2\sqrt{3}$ means $2 \times \sqrt{3}$? Thus when the number outside the square root sign is put back into the root sign, it is squared.

i.e. $2\sqrt{3} = \sqrt{2^2 \times 3}$

$= \sqrt{12}$

(iii) $\sqrt{\dfrac{a}{b}} = \dfrac{\sqrt{a}}{\sqrt{b}}$ is also true.

You can check using $a = 9$ and $b = 4$

Simplifying surds

1. Looking back at $\sqrt{24}$ discussed earlier, we notice that some factor splits are more useful than others.

The split $\sqrt{4} \times \sqrt{6}$ helps to simplify $\sqrt{24}$, as there is a perfect root, $\sqrt{4}$, which simplifies to 2.

Thus $\sqrt{24} = \sqrt{4} \times \sqrt{6}$

$= 2\sqrt{6}$ This is a simplified surd as $\sqrt{6}$ does not contain a perfect root.

2. Given bigger numbers like, $\sqrt{245}$, it may be necessary to expose the perfect roots by writing the number 245 as a product of its prime factors, i.e. $245 = 5 \times 7 \times 7$

Thus $\sqrt{245} = \sqrt{5 \times 7 \times 7}$

$= \sqrt{5} \times \sqrt{7 \times 7}$

$= 7\sqrt{5}$

Note that $7\sqrt{5}$ is similar to an unknown e.g. $2y$. It cannot be written as $5\sqrt{7}$ just as we do not write $2y$ as y^2. The coefficient must come first.

3. We may be asked to simplify expressions like: $\sqrt{5}\times\sqrt{24}\times\sqrt{30}$. Although, we know that this is $\sqrt{3600}$, the surd $\sqrt{3600}$ is cumbersome to handle. It is more convenient to split the given numbers into useful parts.

i.e. $\sqrt{5}\times\sqrt{24}\times\sqrt{30} \quad = \sqrt{5}\times\sqrt{4}\times\sqrt{6}\times\sqrt{6}\times\sqrt{5}$
$= 5\times 2\times 6$
$= 60$

Hint
The 5 is from $\sqrt{5}\times\sqrt{5}$ or $(\sqrt{5})^2$
The 6 is from $\sqrt{6}\times\sqrt{6}$ or $(\sqrt{6})^2$
The 2 is from $\sqrt{4}$

4. Numbers like $(2\sqrt{5})^3$ simplify easily, by expanding.

$(2\sqrt{5})^3 = 2\sqrt{5}\times 2\sqrt{5}\times 2\sqrt{5}$
$= 2\times 2\times 2\times\sqrt{5}\times\sqrt{5}\times\sqrt{5}$
$= 2\times 2\times 2\times 5\times\sqrt{5}$
$= 40\sqrt{5}$

Simplify the following, leaving the answer in the simplest surd form.

1. $\sqrt{48}$ **2.** $\sqrt{63}$ **3.** $\sqrt{150}$

4. $\sqrt{800}$ **5.** $\sqrt{975}$ **6.** $\sqrt{1875}$

7. $\sqrt{50}\times\sqrt{18}$ **8.** $\sqrt{10}\times 3\sqrt{2}\times\sqrt{30}$

9. $\sqrt{5}\times\sqrt{3}\times\sqrt{12}\times\sqrt{15}\times\sqrt{10}$

10. $\sqrt{45}\times\sqrt{18}\times\sqrt{54}\times\sqrt{30}$

B. RATIONALISING THE DENOMINATOR

A fraction with a surd as denominator is not simplified e.g. $\dfrac{3}{\sqrt{5}}$.

Such surds are said to be simplified only if the denominator becomes a rational number. This is achieved getting rid of the square root in the denominator.

Consider the following examples:

1. Simplify by rationalising the denominators:

a) $\dfrac{3}{\sqrt{5}}$ b) $\dfrac{6}{\sqrt{3}}$ c) $\dfrac{5\sqrt{5}}{2\sqrt{10}}$

Hint

Multiply both the numerator and the denominator by the same surd.

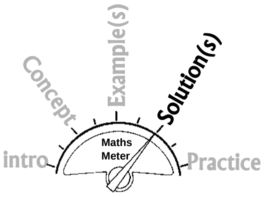
Solution

1. a) $\frac{3}{\sqrt5} = \frac{3\times\sqrt5}{\sqrt5\times\sqrt5}$

$= \frac{3\sqrt5}{5}$

b) $\frac{6}{\sqrt3} = \frac{6\sqrt3}{3}$

$= 2\sqrt3$

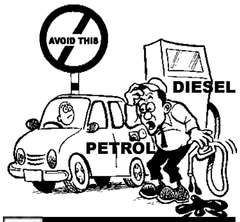


c) $\frac{5\sqrt2}{2\sqrt{10}} = \frac{\overset{1}{\cancel{5}}\sqrt2\times\sqrt{10}}{2\times\cancel{10}_2}$ multiply the numerator and denominator by $\sqrt{10}$ only

$= \frac{\sqrt2\times\sqrt2\times\sqrt5}{4}$

$= \frac{2\sqrt5}{4}$

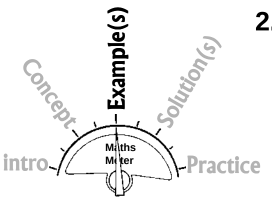
$= \frac{\sqrt5}{2}$



Common Error

At $\frac{\sqrt{2}.\sqrt{10}}{2}$, 2 is used to divide into the surds. e.g. $\frac{\sqrt{10}}{2} \neq \sqrt5$
This is incorrect.

Surds can also be added or subtracted.



2. Simplify

a) $\sqrt{32} - \sqrt{18} + \sqrt{50}$ giving the answer in its simplest surd form

b) $\frac{\sqrt{14}\times\sqrt{30}\times\sqrt2}{\sqrt{28}\times\sqrt{20}}$

3. Simplify

a) $\frac{\sqrt6\times\sqrt{27}\times\sqrt7}{\sqrt{108}}$

b) $\sqrt{98} + \sqrt2 - \sqrt{72}$

Solutions

2. a) $\sqrt{32} - \sqrt{18} + \sqrt{50} = 4\sqrt2 - 3\sqrt2 + 5\sqrt2$

$= 6\sqrt2$

This is like $4x - 3x + 3x$ where $x = \sqrt2$

b) $\frac{\sqrt{14}\times\sqrt{30}\times\sqrt2}{\sqrt{28}\times\sqrt{20}} = \frac{\overset{1}{\cancel{\sqrt{14}}}\times\sqrt3\times\overset{1}{\cancel{\sqrt{10}}}\times\overset{1}{\cancel{\sqrt2}}}{\sqrt2\times\underset{1}{\cancel{\sqrt{14}}}\times\underset{1}{\cancel{\sqrt2}}\times\cancel{\sqrt{10}}_1}$

$= \frac{\sqrt3}{\sqrt2}$ can be simplified to

$= \frac{\sqrt6}{2}$

Hint

Create like terms by considering the factors within surds.
e.g. 18 = 2×9 and 3×6
30 = 2×15; 3×10 and 5×6

Common Error

a) Wrong factors being used
e.g. 14 for $\sqrt{14}$ and $\sqrt{28}$.

$\frac{\sqrt{28}}{\sqrt{14}} = 2$. Wrong

$\frac{\sqrt{28}}{\sqrt{14}} = \sqrt2$. Correct

3. a) $\dfrac{\sqrt{6}\times\sqrt{27}\times\sqrt{7}}{\sqrt{108}} = \dfrac{\sqrt{21}}{\sqrt{2}}$

$= \dfrac{\sqrt{2}\times\sqrt{21}}{2}$

$= \dfrac{\sqrt{42}}{2}$

Snap Check ☑ : divide $\sqrt{6}$ into $\sqrt{6}$ and $\sqrt{108}$
$\sqrt{9}$ into $\sqrt{18}$ and $\sqrt{27}$

b) $\sqrt{98}+\sqrt{2}-\sqrt{72} = \sqrt{49\times 2}+\sqrt{2}-\sqrt{36\times 2}$

$= 7\sqrt{2}+\sqrt{2}-6\sqrt{2}$

$= 2\sqrt{2}$

1. Simplify by rationalising the denominators:

a) $\dfrac{1}{\sqrt{3}}$ b) $\dfrac{6}{\sqrt{8}}$ c) $\dfrac{21}{2\sqrt{3}}$

d) $\sqrt{\dfrac{9}{5}}$ e) $\sqrt{\dfrac{8}{50}}$ f) $\dfrac{2\sqrt{3}}{\sqrt{15}}$

2. Simplify:

a) $\sqrt{3}+\sqrt{75}$ b) $\sqrt{216}-4\sqrt{6}$

c) $2\sqrt{20}-\sqrt{180}+7\sqrt{5}$ d) $\dfrac{\sqrt{8}\times\sqrt{24}\times\sqrt{39}}{2\sqrt{3}\times\sqrt{26}}$

e) $\dfrac{\sqrt{12}\times\sqrt{40}\times\sqrt{32}}{\sqrt{6}\times\sqrt{24}\times\sqrt{50}}$ f) $\dfrac{\sqrt{13}\times\sqrt{15}\times\sqrt{75}}{\sqrt{5}\times\sqrt{65}\times\sqrt{10}}$

The addition and subtraction can also be done in fractions. Two approaches may be followed.

Consider the following example:

1. Simplify

$\sqrt{2}-\dfrac{3}{\sqrt{2}}+\dfrac{5}{\sqrt{8}}$

Solution

1. Method 1	Method 2
Step 1. Rationalise the denominators first.	**1.** Find Common denominator $= \sqrt{8}$.
Step 2. Bring all under a Common denominator.	**2.** Simplify $\sqrt{16}$ and $-3\sqrt{4}$.
Step 3. Simplify $\sqrt{8}$ in the numerator.	**3.** Rationalise the denominator.
Step 4. Reduce to lowest terms.	**4.** Simplify $\sqrt{8}$ in the numerator.
	5. Reduce to lowest terms.

Method 1:

Step 1 $= \sqrt{2} - \dfrac{3\sqrt{2}}{2} + \dfrac{5\sqrt{8}}{8}$

Step 2 $= \dfrac{8\sqrt{2} - 12\sqrt{2} + 5\sqrt{8}}{8}$

Step 3 $= \dfrac{8\sqrt{2} - 12\sqrt{2} + 10\sqrt{2}}{8}$

Step 4 $= \dfrac{6\sqrt{2}}{8}$

$= \dfrac{3\sqrt{2}}{4}$

Method 2:

Step 1 $= \dfrac{\sqrt{16} - 3\sqrt{4} + 5}{\sqrt{8}}$

Step 2 $= \dfrac{4 - 6 + 5}{\sqrt{8}}$

Step 3 $= \dfrac{3}{\sqrt{8}}$

Step 4 $= \dfrac{3\sqrt{8}}{8}$

Step 5 $= \dfrac{3 \times 2\sqrt{2}}{8}$

$= \dfrac{3\sqrt{2}}{4}$

2. Simplify:
$(\sqrt{9} - \sqrt{5})(\sqrt{9} + \sqrt{5})$

Solution

2. Do you remember that these are factors of "the difference of two squares"?

$\therefore \quad (\sqrt{9})^2 - (\sqrt{5})^2 = 9 - 5$

$= 4$

In some cases, you may be asked to evaluate given surds using root values of surds.

3. Given that $\sqrt{3} = 1{,}732$ and $\sqrt{5} = 2{,}236$ evaluate to 3s.f., without using tables or a calculator

a) $\dfrac{1}{\sqrt{3}}$ b) $\dfrac{10}{\sqrt{5}}$

Solution

Hint

Rationalise denominators first.

3. a) ***Notice*** $\dfrac{1}{\sqrt{3}} = \dfrac{1}{1{,}732}$ It is more difficult to calculate the problem with 1,732 as the denominator.

a) $\frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$

$= \frac{1{,}732}{3}$

$= 0{,}5773$

$= 0{,}577$ to 3s.f.

b) $\frac{10}{\sqrt{5}} = \frac{\overset{2}{\cancel{10}}\sqrt{5}}{\cancel{5}_1}$

$= 2\sqrt{5}$

$= 2 \times 2{,}236$

$= 4{,}472$

$= 4{,}47$ to 3 s.f

The working above demonstrates the importance of rationalising the denominator correctly.

PRACTICE 11B2

1. Simplify, leaving the answers in their simplest surd form:

a) $2\sqrt{3} - \frac{3}{\sqrt{3}} + \frac{2}{\sqrt{3}}$ b) $\frac{3}{\sqrt{2}} + \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{18}}$

c) $\frac{10}{\sqrt{45} + \sqrt{20}}$ d) $\frac{3}{\sqrt{8} + \sqrt{18}}$

e) $(\sqrt{11} + \sqrt{10})(\sqrt{11} - \sqrt{10})$ f) $(\sqrt{1{,}5} - \sqrt{6})^2$

2. Given that $\sqrt{3} = 1{,}732$ and $\sqrt{5} = 2{,}236$, evaluate the following, expressions without using tables or calculators. Give your answers to 3s.f.

a) $\frac{1}{2\sqrt{5}}$ b) $\frac{2}{\sqrt{5}}$ c) $\frac{5}{3\sqrt{3}}$

d) $\frac{\sqrt{8}}{\sqrt{6}}$ e) $\frac{12}{5\sqrt{3}}$ f) $\sqrt{2^3 \times 0{,}25 \times 10}$

C. SURDS IN TRIANGLES WITH 30^0, 45^0 OR 60^0 ANGLES

a)

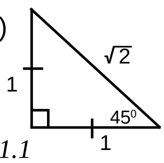
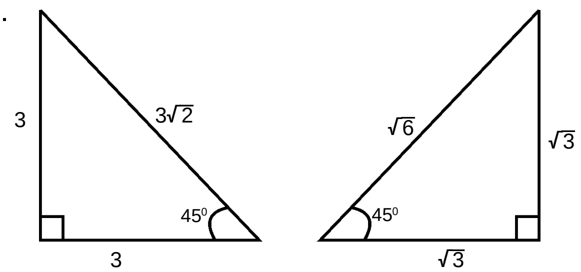


Fig. 11.1

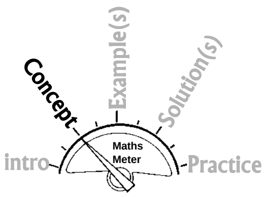
Consider a right-angled isosceles triangle whose equal sides are 1 unit long. The third side can be expressed as $\sqrt{2}$ using the Pythagoras' Theorem.

Notice that a lot of similar triangles can be derived from the above triangle.

e.g.



etc



b)

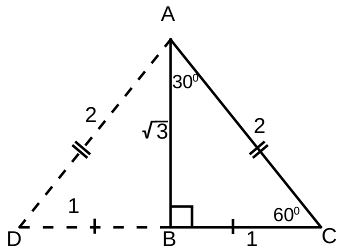
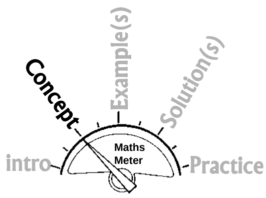
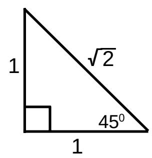


Fig 11.2

In Fig 11.2, triangle ADC is an equilateral triangle and the sides are 2 units long. $\triangle$ ABC is half the equilateral triangle.
Using Pythagoras Theorem in $\triangle$ABC we can deduce that, AB = $\sqrt{3}$.
Do the two triangles Fig 11.1 and Fig 11.2 reminded you of some instruments you often use?
Set-squares! Compare their shape with these triangles.





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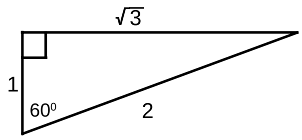
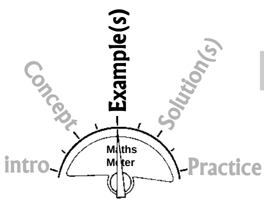


Fig. 11.3

These two triangles are the bases we need to work from. Memorise the measurements of their sides or you will not be able to correctly assign values to similar triangles.
Let us see how these are used in calculating sides of other similar triangles.



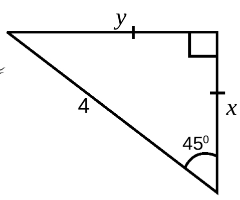
Consider the following examples:

Calculate the lengths marked x and y, giving the answers in simplified surd form.
All given lengths are in cm.

Hint

Compare corresponding sides with the similar triangle from the two in Fig18.3.

1.



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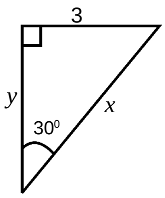

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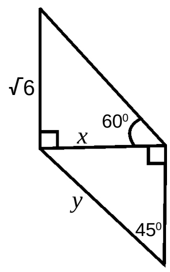


Remember:

Enlargement factor = $\dfrac{\text{Given side in question}}{\text{Corresponding side from original triangle}}$

Tip

Establish the enlargement factor first and use it to enlarge the other sides.

Solutions

1. Enlargement factor = $\frac{4}{\sqrt{2}} = 2\sqrt{2}$

$\therefore\ x = y$

$= 2\sqrt{2}$ cm

2. Enlargement factor = $\frac{3}{1} = 3$

$\therefore\ x = 3 \times 2$

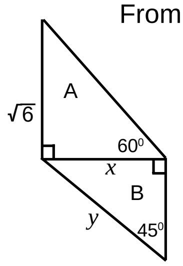
$= 6$cm

$y = 3 \times \sqrt{3}$

Hint

Work with the triangle, with a known side first.

3.



From $\triangle$A, enlargement factor $= \dfrac{\sqrt{6}}{\sqrt{3}}$

$= \sqrt{2}$

$\therefore x = \sqrt{2}$

$= \sqrt{2}$cm

Tip

The second triangle now has the known side $\sqrt{2}$.

From $\triangle$B enlargement factor $= \sqrt{2}$

$\therefore y = \sqrt{2} \times \sqrt{2}$

$= 2$cm

Trigonometry ratios of 45^0, 30^0 and 60^0 can also be used to make the same calculations.

For an example, to calculate x from example (a) above.

From 45^0 triangle in Fig 11.3, cos $45^0 = \dfrac{1}{\sqrt{2}}$

From the diagram in the question, cos $45^0 = \dfrac{x}{4}$

$$\frac{x}{4} = \frac{1}{\sqrt{2}}$$

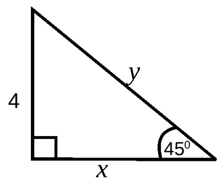
$$x = \frac{4}{\sqrt{2}}$$

$$= 2\sqrt{2}$$

PRACTICE 11C

In each case below, calculate the lengths marked x and y, giving the answers in their simplest surd form. All dimensions are in cm.

1.



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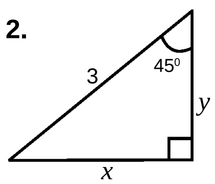

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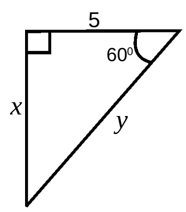

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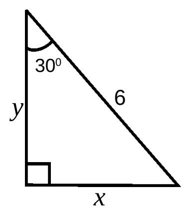

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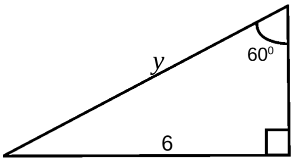

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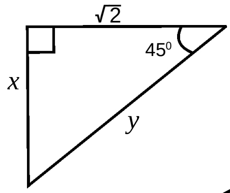

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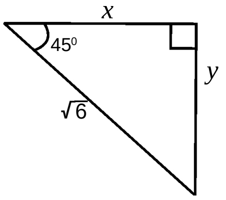

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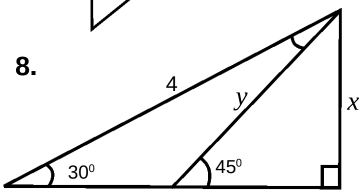

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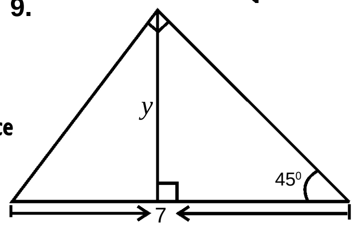

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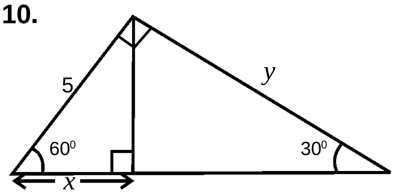
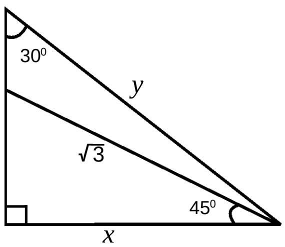
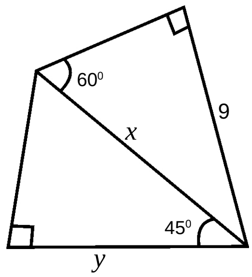


11.



12.



D. WORD PROBLEMS WITH SURDS

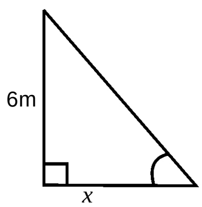
Some word problems can be solved using the skills you have learnt in the previous section.

Consider the following example

1. A tree is 6m high. The angle of elevation of the top of the tree from a point on the ground is 60^0. How far is this point from the foot of the tree? Give your answer to the nearest cm.

Solution

1. Sketch



Do you see that this is the 60^0 set square?

Thus enlargement factor $= \dfrac{6}{\sqrt{3}}$

$= 2\sqrt{3}$

$\therefore x = 2\sqrt{3} \times 1$

$= 2\sqrt{3}$

$= 2 \times 1{,}730$

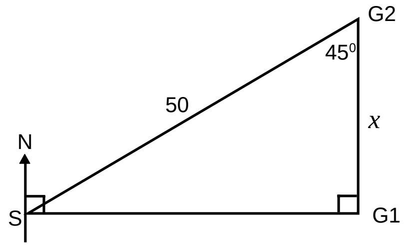
$= 3{,}46\text{cm}$

$x = 3\text{cm}$

2. Two girls were playing on the same spot. One of the girls moved due east of the playground. The other one moved on a bearing of 045^0 for 50m, until she was due north of the first girl. How far are they apart, now?

Solution

2.



This is a 45^0 triangle.

Scale factor $= \dfrac{50}{\sqrt{2}} = 25\sqrt{2}$

$\therefore$ Distance apart $x = 25\sqrt{2} \approx 35{,}4$m

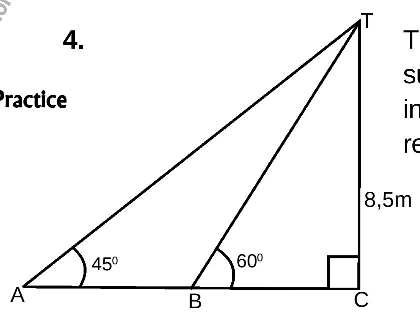
In these problems, give all dimensions to 3 significant figures, where necessary.

1. A girl walks 7km on a bearing 045^0. How far east has the girl walked?

2. The angle of depression, from the top of a building to a point on the ground, is 30^0. How high is the building if this point is 25m away from the foot of the building?

3. A rectangle is 5m long. If its diagonal makes an angle of 60^0 with its width, find the length of the diagonal.

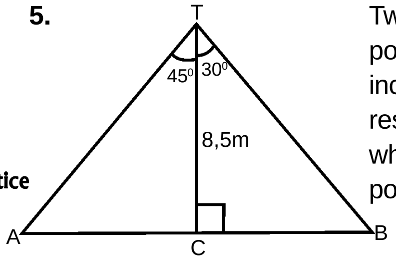
4.



The diagram shows a pole 8,5m high, supported by two wires AT and BT, inclined at 45^0 and 60^0 to the ground, respectively.

Calculate: a) AT b) BT

5. Two wires, TA and TB, support a pole TC = 8,5m high. If the wires are inclined 45^0 and 30^0 to the pole, respectively, find AB, the distance which is between the wires' anchor points.



SUMMARY

1. A surd is a square root which cannot be reduced to a whole number e.g. $\sqrt{2}$, $\sqrt{15}$.

2. Surds are irrational numbers.

3. Surds can be simplified using the following rules:

a) $\sqrt{ab} = \sqrt{a} \times \sqrt{b}$.   b) $\sqrt{a^2 b} = a\sqrt{b}$.

c) $\sqrt{\dfrac{a}{b}} = \dfrac{\sqrt{a}}{\sqrt{b}}$.

4. Denominators in surd form need to be rationalised by multiplying both numerator and denominator by the surd denominator. e.g. $\dfrac{3}{\sqrt{2}} = \dfrac{3\sqrt{2}}{\sqrt{2}.\sqrt{2}} = \dfrac{3\sqrt{2}}{2}$

5. These are the two basic triangles convenient for calculations of sides of similar triangles.

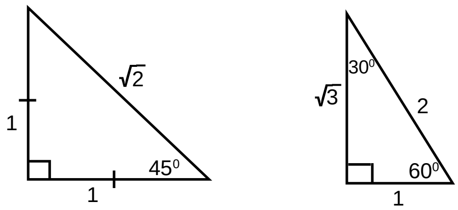


Fig 11.4

From the triangles

$\sin 30^0 = \dfrac{1}{2}$ $\cos 30^0 = \dfrac{\sqrt{3}}{2}$ $\tan 30^0 = \dfrac{1}{\sqrt{3}} = \dfrac{\sqrt{3}}{3}$

$\sin 45^0 = \dfrac{1}{\sqrt{2}}$ $\cos 45^0 = \dfrac{1}{\sqrt{2}}$ $\tan 45^0 = 1$

$\dfrac{\sqrt{2}}{2}$ $\dfrac{\sqrt{2}}{2}$

$\sin 60^0 = \dfrac{\sqrt{3}}{2}$ $\cos 60^0 = \dfrac{1}{2}$ $\tan 60^0 = \sqrt{3}$

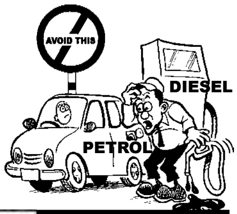
6. The enlargement factor can be used to make the same calculations.

EXAM PRACTICE 11

The following examples may help you master questions from this chapter. Study them carefully.

1. Simplify giving the answers in their simplest surd form.

a) $\sqrt{80}$ b) $\dfrac{2\sqrt{3} \times \sqrt{42}}{\sqrt{21} \times 5\sqrt{6}}$



Common Error

$\sqrt{80} = 8\sqrt{10}$ from $\sqrt{8} \times 10$

Solution

Hint

Rewrite 80 as a product of its prime factors.

1. a) **Method 1**

$$\sqrt{80} = \sqrt{2\times2\times2\times2\times5}$$
$$= 2 \times 2 \times \sqrt{5}$$
$$= 4\sqrt{5}$$

OR

Hint

a) Create a perfect factor of 80
b) Split $\sqrt{42}$, $\sqrt{21}$, and knock off common factors.

Method 2

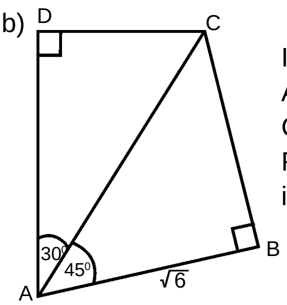
a) $\sqrt{80} = \sqrt{16} \times \sqrt{5}$
$= 4\sqrt{5}$

b) $$\frac{2\sqrt{3} \times \sqrt{42}}{\sqrt{21} \times 5\sqrt{6}} = \frac{2\times\cancel{\sqrt{3}}\times\cancel{\sqrt{7}}\times\cancel{\sqrt{6}}}{\cancel{\sqrt{3}}\times\cancel{\sqrt{7}}\times5\times\cancel{\sqrt{6}}}$$

$$= \frac{2}{5}$$

2. a) Given that $\tan 60^0 = 3\sqrt{3}$
Find $\cos 30^0$, leaving your answer in surd form.

b)



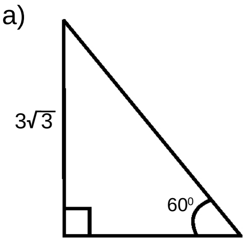
In the quadrilateral, ABCD.
$A\hat{D}C = A\hat{B}C = 90^0$, $B\hat{A}C = 45^0$,
$C\hat{A}D = 30^0$ and $AB = \sqrt{6}$cm
Find AD, leaving your answer in its simplest surd form.

Solution

Hint

Notice that the information given in the question is from a right angled triangle. (Its very important to link trigonometrical ratios with the right-angled triangle.)

2. a)



Given $\tan 60^0 = 3\sqrt{3}$
Scale factor $= 3$
Adjacent side $= 3$
Hypotenuse side $= 6$

$\therefore \quad \cos 30^0 = \dfrac{3}{6}$
$= \dfrac{1}{2}$

Common Error

In a) the djacent side of 60^0 is taken as 1 since $\tan 60^0 = \dfrac{3\sqrt{3}}{1}$

b) In $\triangle ABC$ the
enlargement factor = $\sqrt{6}$

$AC = \sqrt{6}.\sqrt{2}$
$= \sqrt{12}$
$= 2\sqrt{3}$

In $\triangle$ ACD the
Enlargement factor $= \sqrt{3}$. Do you see why?

$\therefore AD = \sqrt{3} \times \sqrt{3}$
$= 3$cm

Now do the following:

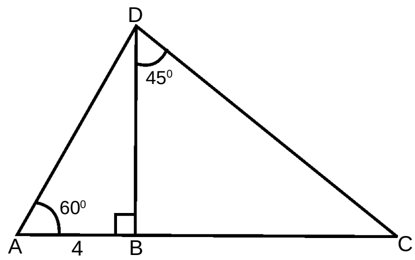
1. Without using tables, evaluate:

a) $(\sqrt{5} + \sqrt{7})(\sqrt{5} - \sqrt{7})$ b) $\dfrac{\sqrt{125}}{\sqrt{80} - \sqrt{5}}$

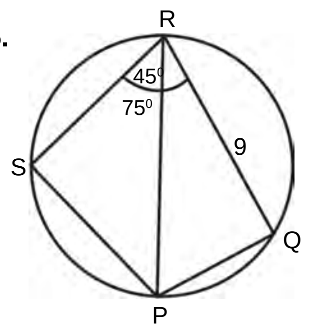
2. Simplify as far as possible:

a) $\dfrac{8\sqrt{2}}{\sqrt{24}}$ b) $(3\sqrt{2})^3$ c) $\dfrac{\sqrt{15} \times \sqrt{42} \times \sqrt{12}}{\sqrt{21} \times 5\sqrt{3} \times \sqrt{30}}$

3. In the diagram, ABC is a straight line, $\hat{B} = 90^0$. If $B\hat{A}D = B\hat{D}C = 60^0$ and AB = 4cm, find CD leaving the answer in surd form.



4. The diagonal of a rectangle is 55cm long and makes an angle of 30^0 with the length. Find, in cm, the length of the rectangle, giving the answer to 3 significant figures.

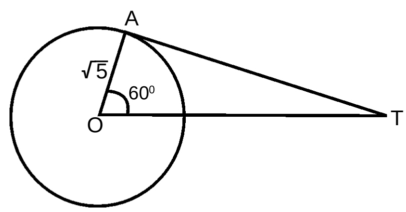
5. 

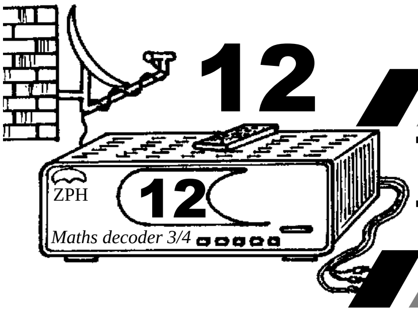
The diagram shows a circle, enclosing a quadrilateral PQRS, in which PR is a diameter.
If $P\hat{R}S = 45^0$. $Q\hat{R}S = 75^0$ and QR = 9cm, find SR in surd form.

6. Simplify as far as possible:

a) $\dfrac{2\sqrt{5}}{\sqrt{15}}$ b) $\dfrac{1}{\sqrt{2}} + \dfrac{3}{\sqrt{2}}$

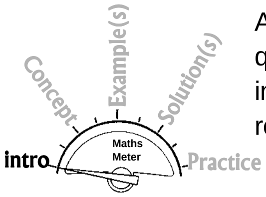
7. The diagram shows a circle centre O and the tangent TA at A from T. If the radius of the circle is $\sqrt{5}$ and $T\hat{O}A = 60^0$, find OT to 2 decimal places.



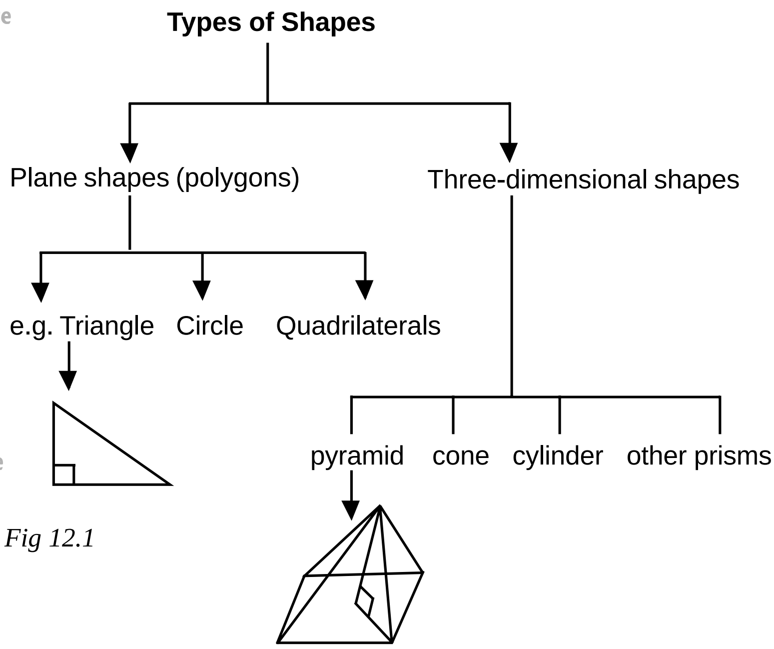


The Right-angled Triangle

A right-angled triangular building.



At 'O' Level, the mathematics associated with shapes constitutes quite an appreciable part of the syllabus. Fig 12.1, which is also found in the chapter on "Mensuration" – summary section, should help you recall aspects of shapes.



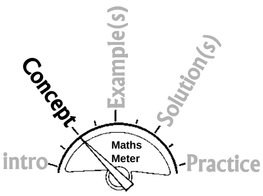
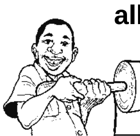


Fig 12.1

In this chapter we focus on the mathematics of the triangle. Notice that the right-angled triangle as a plane shape can exist alone or within other shapes, like a pyramid. "Tri" means three. This three-sided shape with three angles gives rise to a branch of mathematics called **trigonometry**.

Pyramids are triangular in shape.

Trigonometry is the branch of mathematics which deals with all the computations involving triangles.



Syllabus Expectations

By the end of this chapter, students should be able to:

1. apply Pythagoras Theorem to solve problems involving right-angled triangles.
2. define the trigonometrical ratios: tangent, sine and cosine.

3 use the trigonometrical ratios to find angles and sides of right-angled triangles.

4 use and interpret sine, cosine and tangent of obtuse angles.

ASSUMED KNOWLEDGE

In order to tackle work in this chapter, it is assumed that students are able to:

- name and define all types of triangles and identify their properties.
- construct angles and triangles using a ruler and compass only, and/or a protractor.
- construct parallel lines and perpendicular lines.
- use a protractor to measure the size of an angle.
- understand the concept of bearing.

A. THE RIGHT-ANGLED TRIANGLE

In the Junior Certificate course you learnt about equilateral, isosceles and scalene triangles. There is a fourth type of triangle called the **right-angled triangle**. This is a triangle with one internal angle equal to 90^0. The symbol $\perp$ is used in geometry to indicate a right angle (90^0).
A right-angled triangle can either be isosceles or scalene but never an equilateral triangle.

Every right-angled triangle has:

(i) one of the acute angles as the reference angle (θ).

(ii) a side opposite the reference angle known as the **opposite side**.

(iii) the side which defines the reference angle in relation to the hypotenuse known as the **adjacent side**.

(iv) the side opposite the right angle is known as the **hypotenuse.** It is always the longest side of the right-angled triangle. (See Fig 12.2)

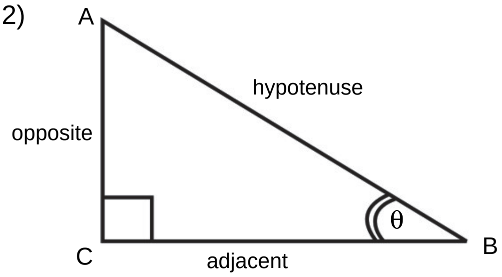
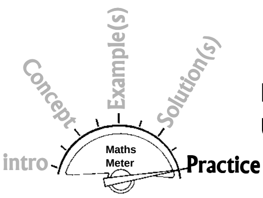


Fig. 12.2



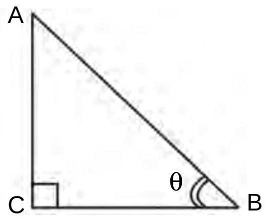
Label all the missing aspects of the right-angled triangles below.
Use the following for labelling the diagrams below:

a) θ (for reference angle) b) hypotenuse
c) opposite d) adjacent

Hint

Start by identifying the hypotenuse. Also note that the side directly opposite the reference angle is the opposite side and that directly opposite is the hypotenuse.

1.



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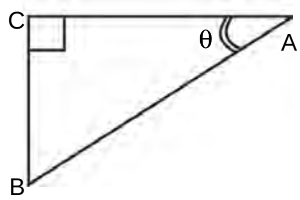

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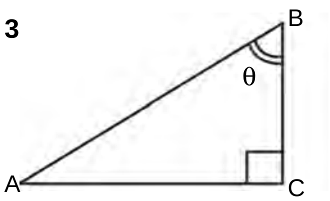
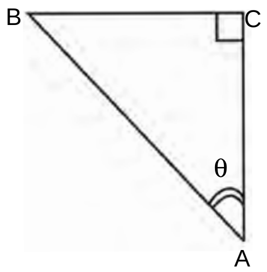
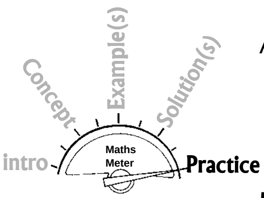


4.





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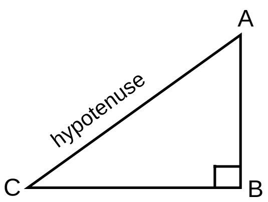

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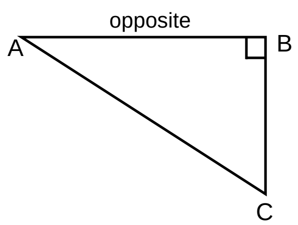
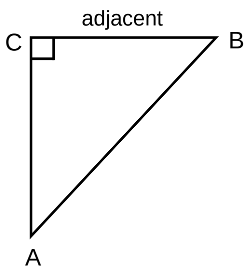


Hint

Draw a right-angled triangle and fully label it . Rotate it clockwise or anti-clockwise slowly observing the positions of the adjacent, hypotenuse and the opposite side.

7.



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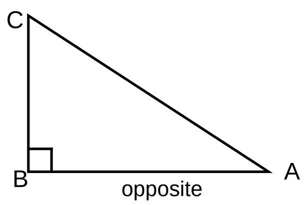


Fig 12.3

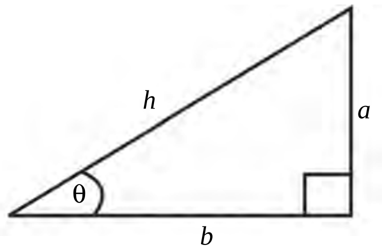
Hint

All calculations on the triangle at this level will involve one of the following:
(i) finding length of sides.
(ii) finding size of angle.
(iii) finding area of the triangle.

B. PYTHAGORAS THEOREM

Pythagoras was a mathematician who thoroughly studied the relationship between the length of the sides of a right-angled triangle. The Greek mathematician deduced a theorem referred to as Pythagoras Theorem, which is stated as:

For any given right-angled triangle the square of the hypotenuse is equal to the sum of the squares of the other two sides. (Fig 12.4)



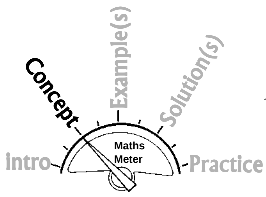
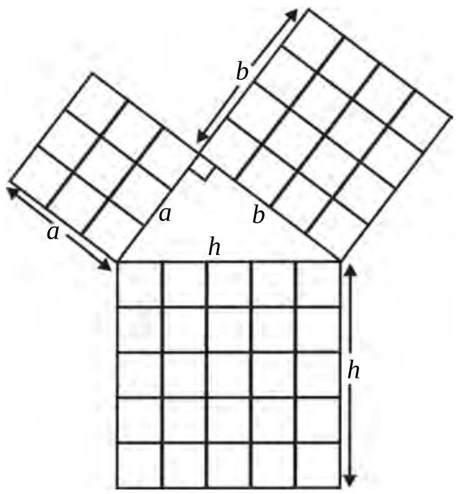


Fig. 12.4

$$\boxed{h^2 = a^2 + b^2}$$

This theorem may be verified experimentally for any right-angled triangle using squared or graph paper. (Fig 12.5)



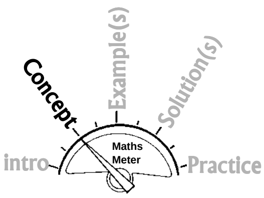


Fig. 12.5

The total number of squares in the $h \times h$ square is equal to the sum of those in the $b \times b$ square and those in the $a \times a$ square.

Hint

Pythagoras Theorem can be used to find one of the sides of a right-angled triangle, provided the other two sides are known. Always remember to square the lengths.

Consider the following examples:

1. Find a and b in figure 7 below.

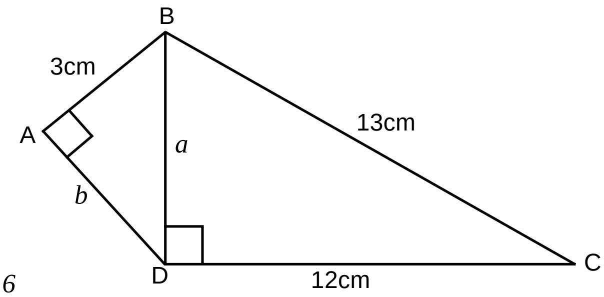


Fig. 12.6

Solution

1. To find a, use triangle BCD.
From Pythagoras:

$13^2 = 12^2 + a^2$

$13^2 - 12^2 = a^2$

$= 13^2 - 12^2$

$a^2 = 169 - 144$

$a^2 = 25$

$a = \sqrt{25}$

$a = 5\text{cm}$

To find b, use triangle ABD.
$a = 5$
From Pythagoras:
$5^2 = b^2 + 3^2$
$5^2 - 3^2 = b^2$
$= 25 - 9$
$b^2 = 16$
$b = \sqrt{16}$
$b = 4\text{cm}$

Common Error

A common error is to forget to square or to find the square root. e.g., giving your final answers as $a = 25$ and $b = 16$.
Watch out!

2. Use Fig 12.7 to find: a) AC
b) BC

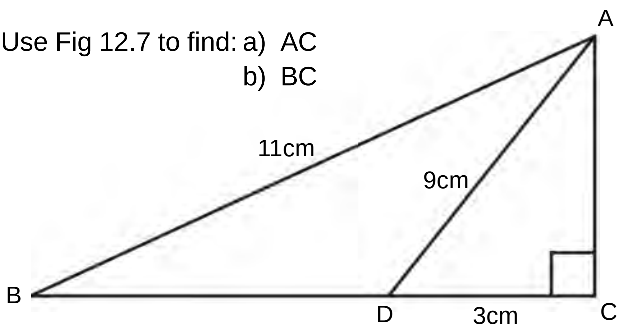
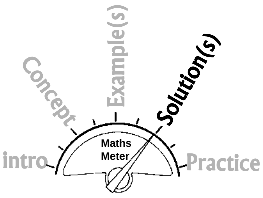


Fig 12.7

Solution



2. a) For triangle ADC, which is right-angled:

$AC^2 = 9^2 - 3^2$

$AC^2 = 81 - 9$

$AC = \sqrt{72}$

$AC = 8,485$

$= 8,5$cm (1 decimal place)

b) For triangle ABC:

$11^2 = BC^2 + AC^2$

$11^2 = BC^2 + (\sqrt{72})^2$

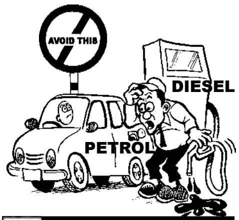
$BC^2 = 11^2 - (\sqrt{72})^2$

$BC^2 = 121 - 72$

$BC^2 = 49$

$BC = \sqrt{49}$

$BC = 7$cm



Common Error

A common error is to approximate value of AC, i.e. 8,485 or worse still round off 8,5 to 9 and then use it rather than the whole $\sqrt{72}$ note $(8.485)^2 \neq \sqrt{72}$.

Hint

Sketch triangle first.

Tip

AC is the hypotenuse and should be the longest side in all cases.

1. Given that triangle ABC is a right-angled triangle with $\hat{B} = 90^0$, calculate the missing side in each of the following:

a) AB = 20cm BC = 6cm.

b) BC = 10cm AC = 13cm.

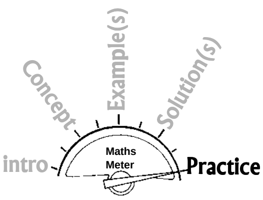
c) AB = 8cm AC = 10cm.

d) AC = 14cm BC = 12cm.

e) BC = 9cm AB = 15cm.

2. In Fig 12.8, triangle ABC is an isosceles triangle with AB = AC = 25cm, DA = 24 cm and CM = 2cm. Use pythagoras to find:

(i) DC. (ii) BC. (iii) DM.



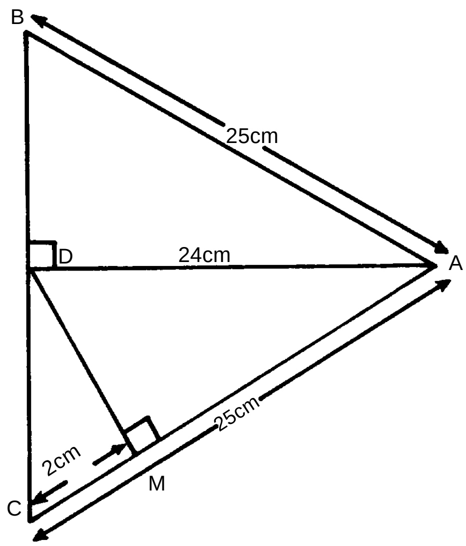


Fig. 12.8

3. A ladder is leaning against a vertical wall, such that its foot on the horizontal ground is 0,6m from the wall and it rests on the wall 3,5m from the ground. Calculate the length of the ladder.

4. Find the length of the diagonal of a square whose side length is 6cm.

5. Find the hypotenuse of an isosceles right-angled triangle which has one of its equal sides as 7cm.

C. PYTHAGOREAN TRIPLES

When all the sides of a right-angled triangle are integers then, these numbers are referred to as **Pythagorean triples**, for example, a 3, 4, 5 triangle. Other Pythagorean triples are (6, 8, 10) (5, 12, 13) (7, 24, 25) (8, 15, 17) and there are many more!

Hint

Multiples of these triples also form other Pythagorean triples e.g (6, 8 10) is a multiple of the (3,4,5) triangle. Another way to test or verify if a triangle is right-angled is by showing whether its sides are Pythagorean triples. The biggest number in the triple is the hypotenuse.

Consider examples below

Show whether the following are Pythagorean triples.

1. (15; 36; 39)
2. (32; 42; 50)

Solutions

1. $39^2 = 1521$
   $15^2 + 36^2 = 225 + 1296 = 1521$
   Hence $15^2 + 36^2 = 39^2$

   $\therefore$ (15; 36; 39) are Pythagorean triples

2. $50^2 = 2500$
   $42^2 + 32^2 = 1764 + 1024$
   $= 2788$

   $32^2 + 42^2 = 50^2$
   $\therefore$ (32; 42; 50) are not Pythagorean triples

1. Which of the following are Pythagorean triples?
   a) (18; 24; 30)
   b) (10; 24; 26)
   c) (14; 48; 26)
   d) (15; 22; 27)
   e) (16; 30; 34)

2.

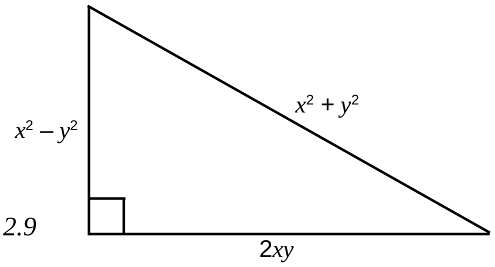


Fig 12.9

For the triangle in Fig 12.9, list the various integers which, when substituted, make the following *Pythagorean* triples. Part (b) of the question below has been done for you.

a) (3, 4, 5) b) (5, 12, 13)
c) (7, 24, 25) d) (9, 40, 41)

Hint

Note that the ratio of the triangle is $x^2 - y^2$; $2xy$; $x^2 + y^2$.

b) Suppose we choose the side ($x^2 - y^2$) as being the smallest length in the ratio of 5; 12; 13 Pythagorean triples, then $x^2 - y^2 = 5$,
Choosing 3 and 2 as x and y respectively we get,
$3^2 - 2^2 = 9 - 4 = 5$
$\therefore \quad x = 3 \quad y = 2$

Hint

The x and y value can be found using any of the terms i.e. instead of $x^2 - y^2$ you may also use $2xy$ or $x^2 + y^2$ to find the values of x and y.

The ratios of the sides of the triangle are:

$$\begin{array}{ccccc} \boxed{x^2 + y^2} & : & \boxed{2xy} & : & \boxed{x^2 - y^2} \\ \downarrow & & \downarrow & & \downarrow \\ \boxed{3^2 + 2^2} & : & \boxed{2 \times 3 \times 2} & : & \boxed{x^2 - y^2} \\ \downarrow & & \downarrow & & \downarrow \\ 13 & : & 12 & : & 5 \end{array}$$

3. Use Pythagoras theorem to find x, in Fig 12.10.

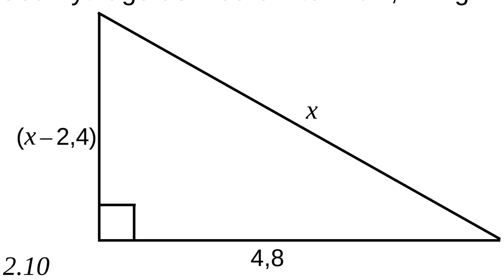


Fig. 12.10

D. RATIO OF SIDES: USING TANGENT, SINE OR COSINE

Once again the reference angle and the sides of a right-angled triangle have a special relationship. Consider the right-angled triangle in Fig 12.11.

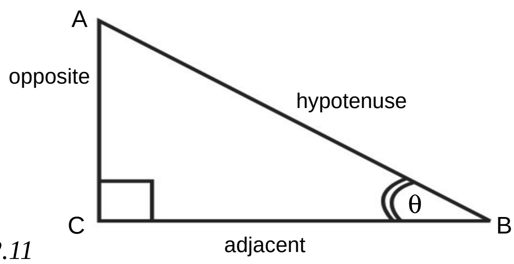


Fig 12.11

The sides may be divided to form ratios with specific trigonometric names. These are:

$$\frac{\textbf{opposite}}{\textbf{adjacent}} = \textbf{tangent } \theta$$

$$\frac{\textbf{opposite}}{\textbf{hypotenuse}} = \textbf{sine } \theta$$

$$\frac{\textbf{adjacent}}{\textbf{hypotenuse}} = \textbf{cosine } \theta$$

The above formulas should be memorised and the following mneumonic may be used.

SOH**C**AH**T**OA or **C**HA**S**HO**T**AO where S = sine
C = cosine
T = tan
O = opposite
H = hypotenuse
A = adjacent

Hint

If the value of the reference angle (θ) is changed then the values of the opposite, adjacent and hypotenuse sides also changes. The hypotenuse is always the side opposite the right-angle and always the longest side.

It is interesting to note that if any one side is increased or decreased while the reference angle θ is kept constant, but the lengths of the other two sides also change. See (Fig 12.12).

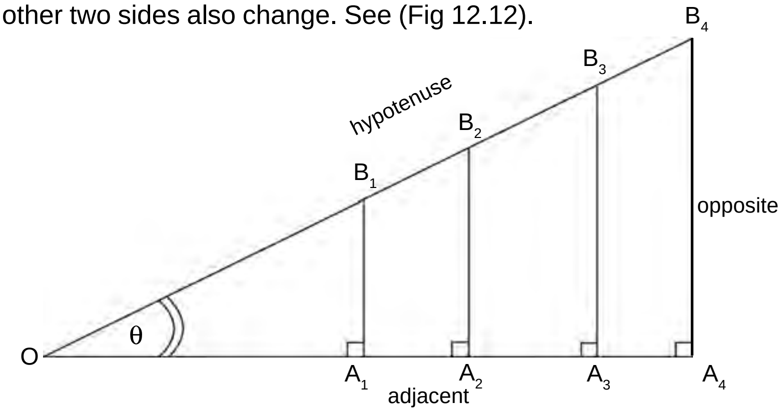
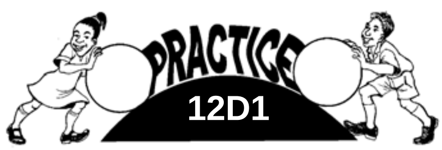


Fig 12.12



Draw and label triangular shapes, as illustrated in Fig 12.12. Use a set-square to ensure the triangles are right angles.

Task 1

To find tangent θ measure the lengths of opposite and that of the adjacent sides using a ruler then calculate the ratio. Tangent θ is usually written in short as tan θ. Write your answers in decimals.

a) $\text{tangent } \theta = \frac{\text{opp}}{\text{adj}} = \frac{A_1B_1}{OA_1} = \boxed{?}$

b) $\tan\theta = \dfrac{opp}{adj} = \dfrac{A_2B_2}{OA_2} = \boxed{?}$

c) $\tan\theta = \dfrac{opp}{adj} = \dfrac{A_3B_3}{OA_3} = \boxed{?}$

d) $\tan\theta = \dfrac{opp}{adj} = \dfrac{A_4B_4}{OA_4} = \boxed{?}$

Note that the value of θ is your choice but make sure it is less than 90^0 and bigger than 0^0.

Task 2

To find Sine θ, measure the lengths of the opposite sides and the hypotenuse side, work out the corresponding ratios. Sine θ is usually written in short as sin θ. Write your answers in decimal form.

a) $\sin\theta = \dfrac{opp}{hyp} = \dfrac{A_1B_1}{OB_1} = \boxed{?}$

b) $\sin\theta = \dfrac{opp}{hyp} = \dfrac{OA_2}{OB_2} = \boxed{?}$

c) $\sin\theta = \dfrac{opp}{hyp} = \dfrac{OA_3}{OB_3} = \boxed{?}$

d) $\sin\theta = \dfrac{opp}{hyp} = \dfrac{OA_4}{OA_4} = \boxed{?}$

Task 3

For cosine θ the same should be done, that is, to measure the adjacent and hypotenuse sides and calculate the ratio. Cosine is usually written as cos θ. Write your answers in decimal form.

a) $\cos\theta = \dfrac{adj}{hyp} = \dfrac{OA_1}{OB^1} = \boxed{?}$

b) $\cos\theta = \dfrac{adj}{hyp} = \dfrac{OA_2}{OB_2} = \boxed{?}$

c) $\cos\theta = \dfrac{adj}{hyp} = \dfrac{OA_3}{OB_3} = \boxed{?}$

d) $\cos\theta = \dfrac{adj}{hyp} = \dfrac{OA_4}{OB_4} = \boxed{?}$

Task 4

Comment on the answers you find in all tasks 1 to 3.

Do you notice that the ratio of a given angle does not change even if you change the lengths of the corresponding sides?

Consider the examples below:

Hint

Draw any right-angled triangle with one of its angles as 60^0.

1. Find by drawing and measurement, the values of:

a) $\tan 60^0$.

b) $\sin 60^0$.

c) $\cos 60^0$.

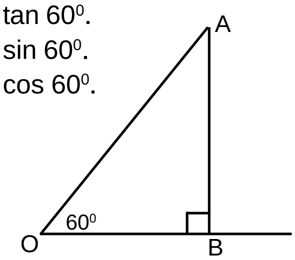


Fig 12.13

Solution

1. a) $\tan 60^0 = \frac{opp}{adj} = \frac{AB}{OB} = \boxed{-} = 1,7321$

b) $\sin \theta = \frac{opp}{hyp} = \frac{AB}{OA} = \boxed{-} = 0,8660$

c) $\cos \theta = \frac{adj}{hyp} = \frac{OB}{OA} = \boxed{-} = 0,5$

N.B. In the above example, the fractional values to be inserted in the empty boxes have been deliberately left out because they are infinite. In the “Practice” which follows do not leave empty boxes. Write the values you have used.

By drawing and measurement, find values for:

a) tangent,

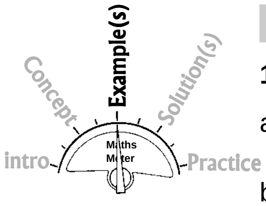
b) sine and

c) cosine, of the following angles:

1. 25^0	**2.** 68^0	**3.** 39^0	**4.** 80^0
5. 45^0	**6.** 80^0	**7.** 30^0	**8.** 42^0
9. 49^0	**10.** 66^0		

E. ANGLES FROM TRIGONOMETRIC RATIOS

It is also possible to find angles given the ratios of the sides.



Consider the following example:

1. Find the angles with the following trigonometric ratios.

a) $\tan\theta = \frac{5}{9}$

b) $\sin\theta = \frac{5}{9}$

c) $\cos\theta = \frac{5}{9}$

Hint

The lengths are in the ratio 5:9 and any multiples or submultiples would give the same answer. That is the lengths could be 10:18, 15:27 etc. Naturally the bigger the diagram the easier it is to measure accurately.

Solution

Step 1

1. Draw the triangles to scale on a separate sheet of paper. Fig 12.14 (a, b and c) give the information required on the diagrams.

N.B. *The diagrams are not drawn to scale.*

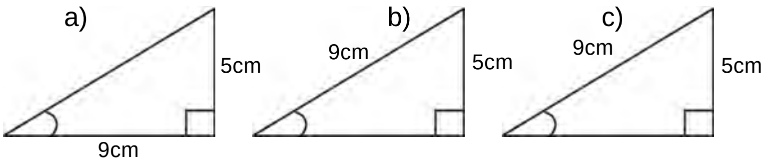
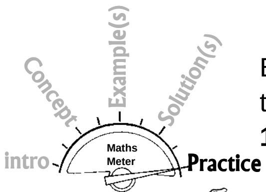


Fig 12.14

Step 2

Using a protractor, the following angles are found from the diagrams

(a) $\theta = 29^0$ (b) $\theta = 34^0$ (c) $\theta = 56^0$



By drawing and measurement, find angle θ, using the following trigonometric ratios:

1. tangent θ is:

a) $\frac{4}{5}$ b) $\frac{4}{3}$ c) $\frac{11}{3}$

d) $\frac{8}{10}$ e) $\frac{10}{8}$ f) $\frac{10}{10}$

2. sine θ is:

a) $\frac{6}{8}$ b) $\frac{3}{5}$ c) $\frac{9}{11}$

d) $\frac{7}{9}$ e) $\frac{5}{5}$ f) $\frac{7}{11}$

3. cosine θ is:

a) $\frac{4}{9}$ b) $\frac{3}{7}$ c) $\frac{7}{9}$

d) $\frac{6}{12}$ e) $\frac{3}{8}$ f) $\frac{6}{6}$

Hint

If the numerator and the denominator of the sine and cosine ratio are equal, then, for sine ratio, $\theta = 90^0$ and for cosine ratio $\theta = 0^0$ e.g. $\sin\theta = \frac{5}{5}$ Thus $\theta = 90^0$ and $\cos\theta = \frac{6}{6}$ thus $\theta = 0^0$

In fact the diagrams would be overlapping lines. Can you figure it out?

F. APPLICATION OF TANGENT, SINE AND COSINE OF AN ANGLE; – The unknown x as a numerator.

The value of the trigonometric ratios of tangent, sine and cosine for all angles $0^0 – 90^0$ have been calculated and they are found in tables. The use of the tables, or calculators, will be explained later in this chapter. For now, Table 12.1 gives the values of some chosen angles. All values in table 12.1 are given to 4 decimal places.

Table 12.1

Angle θ	tan θ	sin θ	cos θ
20^0	0,3640	0,3420	0,9397
35^0	0,7002	0,5736	0,8192
40^0	0,8391	0,6428	0,7660
47^0	1,0724	0,7314	0,6820
55^0	1,4281	0,8197	0,5736
60^0	1,7321	0,8660	0,5000
67^0	2,3559	0,9205	0,3907
75^0	3,7321	0,9659	0,2588

Consider the example below:

Hint

Remember, when solving trigonometry problems you are either:
a) finding one of the sides, given the angle and one side
b) finding the angles using the trigonometry ratios, given at least two sides
c) finding the area of triangle, given the perpendicular height. For now, shall we focus on (a) and later on (b).

1. In Figure 12.15 (a, b and c) find the value of x.

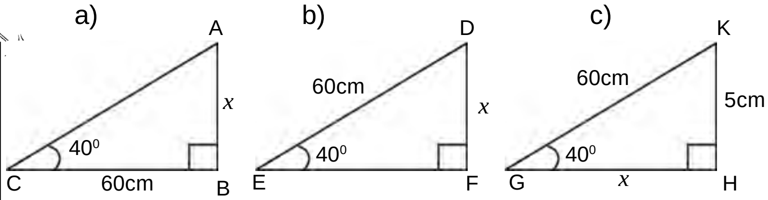


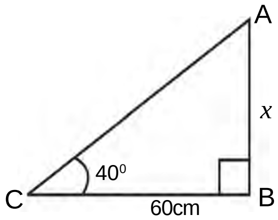
Fig 12.15

Solution

1. a) For fig 12.15 (a)

$$\frac{AB}{BC} = \tan 40^0$$

$$\frac{x}{60} = \tan 40^0$$



$$x = 60 \tan 40^0$$

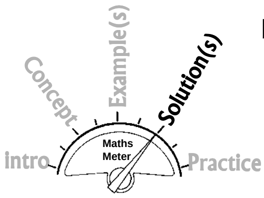
From table 12.1, $\tan 40^0 = 0,8391$

$\therefore\ x = 0,8391 \times 60$

$x = 50,3460\text{cm}$

The answer can be rounded off to 3 significant figures.

$x = 50,3\text{cm}$



b) For fig 12.15(b)

$\frac{DF}{DE} = \sin 40^0$

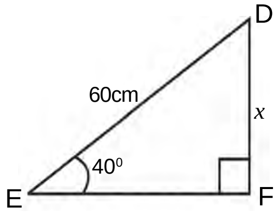
$\frac{x}{60} = \sin 40^0$

$x = 60\sin 40^0$

$x = 0,6428 \times 60$

$\quad 38,5680cm$

$x = 38,6cm$



c) For fig 12.15(c)

$\frac{GH}{KG} = \cos 40^0$

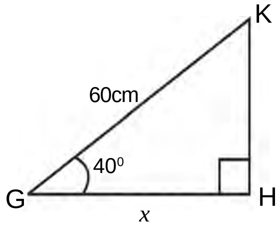
$\frac{x}{60} = \cos 40^0$

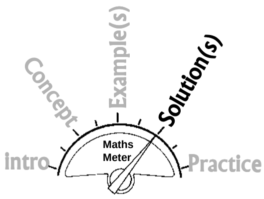
$x = 60\cos 40^0$

$x = 0,7660 \times 60$

$\quad 45,9600cm$

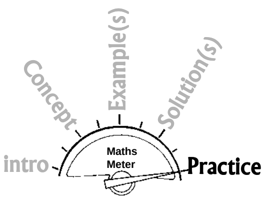
$x = 46,0cm$





Use table 12.1, on page 265 for this exercise. Give your answers to 3 significant figures.

1. Find the values of x in each of the triangles in Fig 12.16



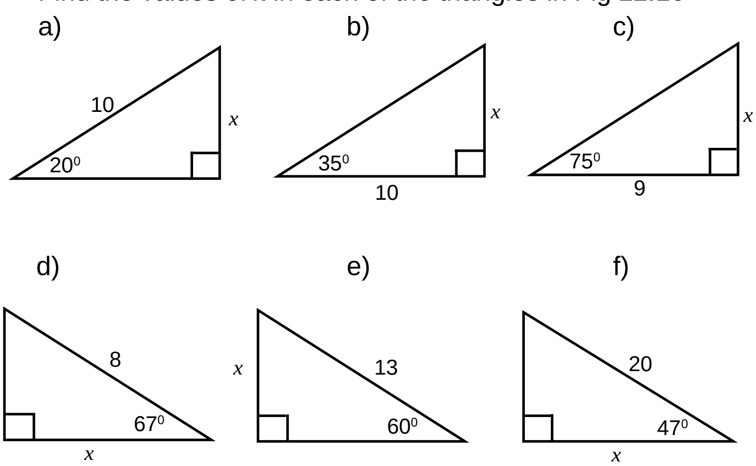


Fig 12.16

2. Find the values of x in each of the triangles in Fig 12.17 a to f.

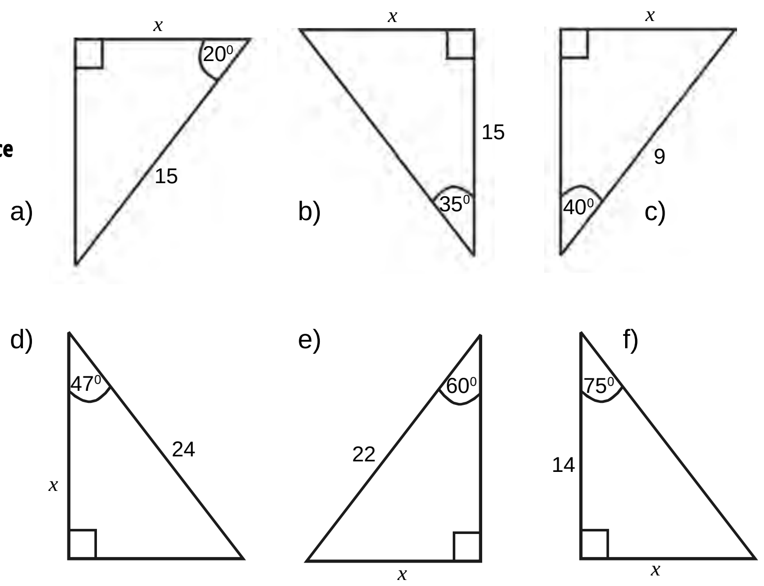


Fig 12.17

3. Find the values of x in each of the triangles a to f.

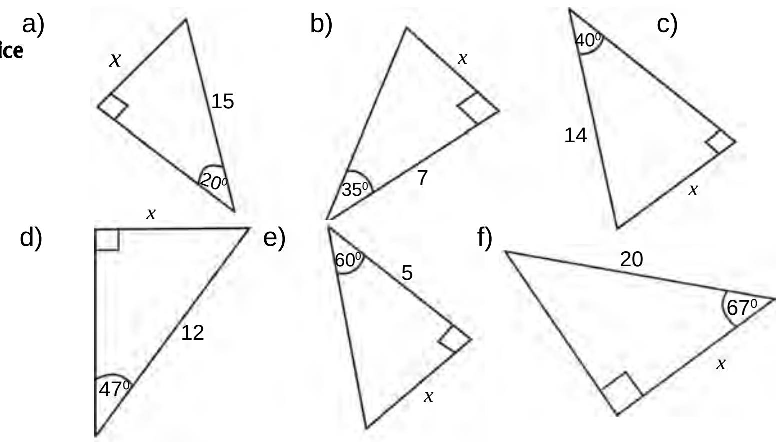


Fig 12.18

4. The National flag is flying at half-mast. If the angle of elevation of the flag from a point 4 metres away on level ground is 47^0, calculate the height of the flag. (Fig 12.19).

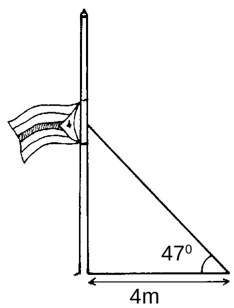


Fig 12.19

5. The angle of elevation to the top of the church roof, is 35^0, from a point 65 metres away on level ground. Calculate the height of the church. (Fig 12.20)

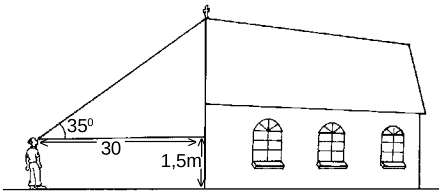


Fig 12.20

G. APPLICATION OF TRIGONOMETRIC RATIOS WITH UNKNOWN, x, AS THE DENOMINATOR

Consider the examples below:

1. In figure 12.21 (a) to (c) find the value of x. (All answers to be expressed correct to 1 decimal place). All measurements are in cm. Use the tables or a calculator.

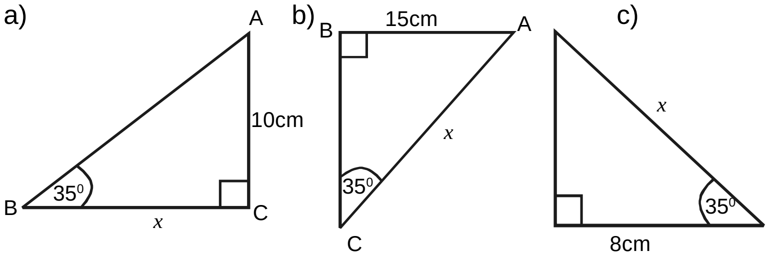


Fig. 12.21

Solution

1. a)

$$\tan 35^0 = \frac{AC}{BC}$$

$$\tan 35^0 = \frac{10}{x}$$

$$x = \frac{10}{\tan 35^0}$$

$$x = 14{,}28\text{cm}$$

$$x = 14{,}3\text{cm}$$

b)

$$\sin 35^0 = \frac{AB}{AC}$$

$$\sin 35^0 = \frac{15}{x}$$

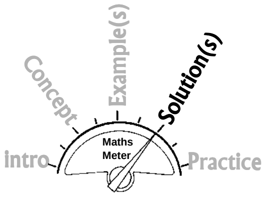
$$x = \frac{15}{\sin 35^0}$$

$$x = 26{,}1506$$

$$x = 26{,}2\text{cm}$$

Hint

The final answers for x are calculated by use of logs or calculators.



c) $\cos 35^0 = \frac{BC}{AC}$

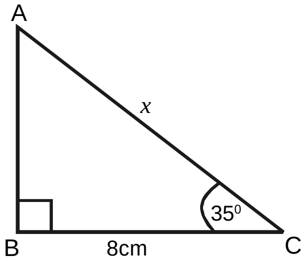
$\cos 35^0 = \frac{8}{x}$

$x = \frac{8}{\cos 35^0}$

$x = \frac{8}{0{,}8192}$

$x = 9{,}766\text{cm}$

$x = 9{,}8\text{cm}$



There is another method which can be used to find x. Consider the following figure, 12.21a).

2. Find x.

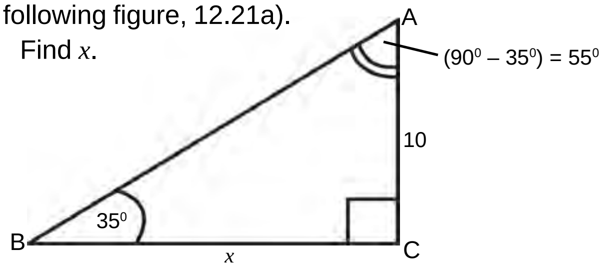


Fig. 12.22

Hint

***Special case** – using the complement of an angle – when x is the denominator, a simpler method, for the tangent ratio **only**, can be applied. The method involves finding the complement of the given angle. The above example fig 12.21(a) may be re-drawn as in fig 12.22, with $B\hat{A}C$ as the reference angle.*

Solution

2. $\hat{A} = 90^0 - 35^0 = 55^0$

$\therefore \frac{x}{10} = \tan 55^0$

$x = 10 \times \tan 55^0$

$x = 10 \times 1{,}4281$

$x = 14{,}28$

$x = 14{,}3$ cm which is the same answer as found before.

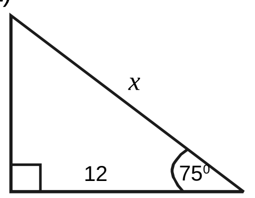
Hint

Use the shorter method where tangent is involved, in this exercise.

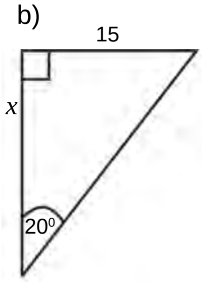
Find the value of x in each of the triangles in Fig 12.23.

Measurements in this exercise are in cm. (Answers to be written correct to 1 decimal place).

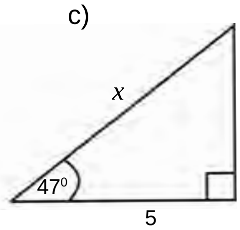
a)



b)



c)



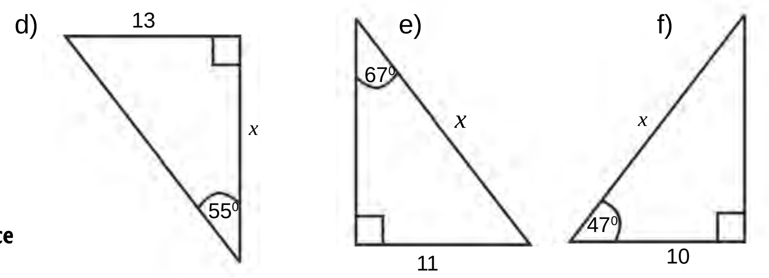


Fig. 12.23

H. DEGREES AND MINUTES IN ANGLES

Angles are measured in degrees or radians. At this stage we are concerned about the degrees. A degree (one degree) can further be divided into smaller units called *minutes*. In fact there are 60 minutes (60′) in one degree (1^0).

Tip

These minutes should not be confused with those used to measure time.

$$\boxed{1^0 = 60'}$$

Changing minutes to degrees and vice-versa.

$(28,5)^0 = 28,5^0$ is the same as $28^0 30'$.
The 30' are found as follows $\frac{5}{10} \times 60 = 30'$.

Similarly $(28,25)^0$ or $25,25^0 = 28^0\ 15'$
The 15′ are found as follows:
$\frac{25}{100} \times 60' = 15'$

Can you verify this using a scientific calculator?

Consider the examples below:

(Scientific calculators may be used.)

1. Change $(42,51)^0$ to degrees and minutes.

2. Change $42^0 27'$ into a decimal number of degrees.

Hint

When changing minutes to degrees you divide by 60′ when changing degrees to minutes, you multiply by 60′. (see examples a and b).

Solutions

1. $42,51^0 = 42^0 + \left(\frac{51}{100}\right)'$

$42,51^0 = 42^0 + 0,51'$

$42,51^0 = 42^0 + 0,51 \times 60'$
$= 42^0 + 30,6'$
$42,51^0 = 42^0 + 31'$

$42,51^0 = 42^0\ 31'$

2. $42^0 27' = 42^0 + \left(\frac{27}{60}\right)^0$

$42^0 27' = 42^0 + 0,45^0$

$42^0 27' = 42,45^0$

Note: Use calculators or tables in this chapter.

1. Change the following to degrees with a decimal.

a) $150'$ b) $28^0 40'$ c) $66^0 66'$

d) $29^0 21'$ e) $149'$

2. Change the following to degrees and minutes.

a) $(80\frac{5}{10})^0$ b) $0{,}6^0$ c) $40{,}45^0$

d) $0{,}8^0$ e) $23{,}23^0$

I. ADDITION AND SUBTRACTION OF DEGREES AND MINUTES

When you add minutes and the answer is equal to, or exceeds 60 then you convert them to degrees and minutes
e.g. $79' = 1^0\ 19'$
Similarly, if you are subtracting bigger minutes from smaller minutes you borrow one degree and convert it to 60 minutes and add it to the smaller minutes. The examples that follow illustrate this.

Consider the examples below:

Find the value of:

1. $20^0\ 46' + 31^0\ 32'$

2. $73^0\ 16' - 42^0\ 50'$

Hint

In (a) 46′ + 32’ = 78′ = 1⁰18′ hence the 1⁰ is added to 20⁰.
In (b) Since 16’ is less than 50′ we take 1⁰ from 73⁰ and convert it to minutes = 60′ and then add to 16′ to give 76′. The 73⁰ becomes 72⁰.

Solutions

1.
$$\begin{array}{r} 20^0\ 46' \\ +\ 31^0\ 32' \\ \hline 52^0\ 18' \\ \hline \end{array}$$

2.
$$\begin{array}{r} 73^0\ 16' \\ -\ 42^0\ 50' \\ \hline 30^0\ 26' \\ \hline \end{array} \quad \Rightarrow \quad \begin{array}{r} 72^0\ 76' \\ -\ 42^0\ 50' \\ \hline 30^0\ 26' \\ \hline \end{array}$$

1. Compute the following, using tables or calculators.

a) $29^0 32' + 46^0 41'$ b) $27{,}9^0 + 25^0\ 13'$

c) $33{,}7^0 + 42^0\ 54'$ d) $49^0\ 37' - 26^0\ 56'$

e) $56^0\ 13' - 42^0\ 28'$ f) $51{,}8^0 - 42{,}3^0$

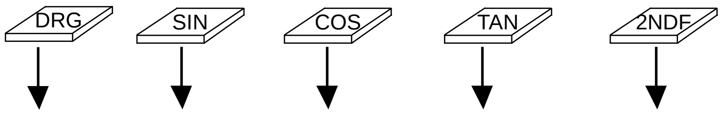
g) $59{,}5^0 - 39{,}8^0$ h) $49{,}6^0 + 54{,}9^0$

i) $47^0\ 39' + 52^0\ 54'$

J. USE OF A SCIENTIFIC CALCULATOR

The values of trigonometric ratios can be found using the scientific calculator

The following keys and their functions should be easily identifiable on your calculator they mean:



Degree or Radians or Gradients | Sine | Cosine | Tangent | 2nd function

Hint

Note that: sin^{-1}, tan^{-1} and cos^{-1} are read as arc sin, arc tan or arc cos respectively.

In addition, below the three trigonometrical ratio keys we have:

sin^{-1} .... below the sin key

cos^{-1}........ below the cos key

tan^{-1}......... below the tan key

The following steps may be followed when using the calculator.

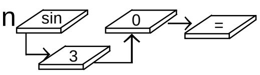
1. Switch it on.

Hint

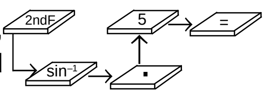
Some calculators require you to enter the angle first then ratio. Others expect you to enter the ratio then the angle. Watch Out!

2. Use the $\boxed{\text{DRG}}$ key to ensure the calculator is in the degree mode.

3. a) To find the value of an angle press the required ratio key and enter your degrees then press the equal sign e.g. Sin $30^0 = 0{,}5$



b) To change a given ratio to degrees, press the $\boxed{\text{2ndF}}$ key, then the trigonometrical ratio key, then enter the ratio and finally press the equal sign key e.g. $Sin^{-1}\ 0{,}5 = 30^0$.



This is just a guide. Students should read their calculator **manual** or experiment as they do with mobile phones.
It is recommended that you visit chapter 17 in this book, "**How to use the scientific calculator**".

PRACTICE 12J

Use a calculator to calculate the following:

1. cos $122^0 16'$ **2.** sin $76{,}9^0$ **3.** tan 149^0

4. $\cos^{-1} 0{,}7611$ **5.** $\cos^{-1} 0{,}3949$ **6.** $\sin^{-1} 0{,}6666$

7. $\tan^{-1} 6{,}4792$ **8.** tan 1,9792

9. Evaluate $\dfrac{(\cos 30)^2 - (\sin 49)^3}{\sin^2 30^0}$ **10.** Evaluate $\dfrac{(\sin 40^0 - \cos 152^0)^2}{2 \times \cos 144^0}$

K. TABLES OF TANGENT, SINES AND COSINES

We will now look at the manual method using tables, to find the trigonometrical ratios of given angles and vice versa.

The tables at the back of this text book can be used to find the values of trigonometrical ratios, i.e. tangent, sine and cosine. We are focussing on natural tangents, sines and cosines.

Hint

Always ensure that you are using the correct table.

Table in Fig 12.24 has been **extracted** from the tables at the end of this book.

TANGENTS tan $x°$

$x°$	0′ 0°.0	6′ 0°.1	12′ 0°.2	18′ 0°.3	24′ 0°.4	30′ 0°.5	36′ 0°.6	42′ 0°.7	48′ 0°.8	54′ 0°.9	1′ ADD	2′	3′	4′	5′
45	1.0000	0035	0070	0105	0141	0176	0212	0247	0283	0319	6	12	18	24	30
46	1.0355	0392	0428	0464	0501	0538	0575	0612	0649	0686	6	12	18	25	31
47	1.0724	0761	0799	0837	0875	0913	0951	0990	1028	1067	6	13	19	25	32
48	1.1106	1145	1184	1224	1263	1303	1343	1383	1423	1463	7	13	20	27	33
49	1.1504	1544	1585	1626	1667	1708	1750	1792	1833	1875	7	14	21	27	34

SINES sin $x°$

$x°$	0′ 0°.0	6′ 0°.1	12′ 0°.2	18′ 0°.3	24′ 0°.4	30′ 0°.5	36′ 0°.6	42′ 0°.7	48′ 0°.8	54′ 0°.9	1′ ADD	2′	3′	4′	5′
45	0.7071	7083	7096	7108	7120	7133	7145	7157	7169	7181	2	4	6	8	10
46	0.7193	7206	7218	7230	7242	7254	7266	7278	7290	7302	2	4	6	8	10
47	0.7314	7325	7337	7349	7361	7373	7385	7396	7408	7420	2	4	6	8	10
48	0.7431	7443	7455	7466	7478	7490	7501	7513	7524	7536	2	4	6	8	10
49	0.7547	7559	7570	7581	7593	7604	7615	7627	7638	7649	2	4	6	7	9

COSINES cos $x°$

$x°$	0′ 0°.0	6′ 0°.1	12 0°.2	18′ 0°.3	24′ 0°.4	30′ 0°.5	36′ 0°.6	42′ 0°.7	48′ 0°.8	54′ 0°.9	1′ SUBTRACT	2′	3′	4′	5′
45	0.7071	7059	7046	7034	7022	7009	6997	6984	6972	6959	2	4	6	8	10
46	0.6947	6934	6921	6909	6896	6884	6871	6858	6845	6833	2	4	6	9	11
47	0.6820	6807	6794	6782	6769	6756	6743	6730	6717	6704	2	4	6	9	11
48	0.6691	6678	6665	6652	6639	6626	6613	6600	6587	6574	2	4	7	9	11
49	0.6561	6547	6534	6521	6508	6494	6481	6468	6455	6441	2	4	7	9	11

Fig 12.24

Points to Note

(i) In relation to tangent and sines tables

- They range from 0^0 to 90^0 .
- 0^0 gives the smallest value while 90^0 gives the maximum values. The values of tangents increase rapidly towards 90^0.
- Each value is given correct to 4 decimal places.
- The difference column ranges from $1'$ to $5'$ and numbers in the difference column must be **added** when dealing with these two trigonometrical ratios.

(ii) In relation to cosine tables

- They also range from 0^0 to 90^0 .
- 0^0 gives the greatest value while 90^0 gives the **smallest** value.
- Each value is given correct to 4 decimal places.

Hint

Memorize
sin 0^0 = 0
sin 90^0 = 1
cos 0^0 = 1
cos 90^0 = 0

The difference column ranges from $1'$ to $5'$ and numbers in the difference column should be **subtracted**.

The tables are used to find values of angles e.g. tangent $39^0 19'$ or to find angles, whose ratio values are given, e.g. finding the angle whose value of sine is 0,4966.

Finding values of tangents and sines, from given values of angles.

Consider the examples below

Use tables to find tangent & sine values of the following:

1. $24'$ **2.** $38'$

3. $68^0 42'$ **4.** $29^0 16'$

Solutions

Using tangents tables:

1. $\tan 24' = \tan 0^0 24' = 0{,}0070$

2. $\tan 38' = 0^0 38' = 0{,}0070$

Since $38'$ is between $36'$ and $42'$ its difference with the nearest column, i.e. $36'$, is $2'$. $\therefore$ Under $2'$ difference column is 6 so we add 0,0006 to the value under $36'$ which is 0,0105.

tan 38′ = tan $0^0 38'$
= 0,0105 + 0,006
= 0,0111

3. 68^0 42′ – Read from 68^0 and move horizontally to the value directly under 42′ = 0,5649.

4. 29^0 16′ – Read value of 29^0 under 18′ = 0,5612 but we need 16′ hence difference is 2′ under the difference column 2′ gives a value = 8 which we subtract from 0,5612 since 16′ is smaller than 18′.

tan 29^0 18′	=	0,5612
–2′		– 0,0008
tan 29^0 16′	=	0,5612

Using the sine tables – The method is the same as above.

1. sin 24′ = 0,0070

2. sin 38′ = 0,0105
+ 0,0006
0,0111

3. sin 68^0 42′ = 0,9317

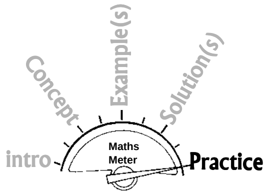
4. sin 29^0 16′ = 0,4894
– 0,0005
sin 29^0 16′ = 0,4889

1. Using the appropriate tables at the end of this textbook, or a calculator, find the tangents of the following:

a) 29′	b) 33′	c) 55^0
d) 63^0	e) 44^0 16′	f) 27^0 53′
g) 67^0 49′	h) 68^0 54′	i) 76^0 13′
j) 88^0 9′	k) 79^0 19′	l) 44^0 44′

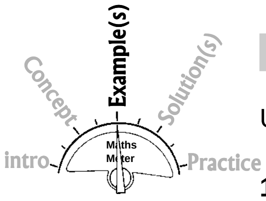
2. Using the appropriate tables at the end of this textbook, or a calculator, find the sine of the following:

a) 29^0	b) 37′	c) 47′
d) 47^0 28′	e) 79^0 55′	f) 82^0 37′



g) $71^0\ 41'$	h) $40^0\ 40'$	i) $39^0\ 49'$
j) $36^0\ 36'$	k) $49^0\ 59'$	l) $68^0\ 17'$
m) $78^0\ 43'$	n) $87^0\ 57'$	

Finding the cosine values of angles



Consider the following examples:

Using the tables at the end of this textbook, find cosine values for:

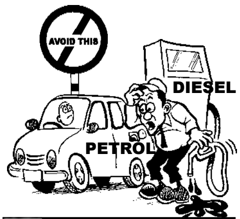
1. $47'$ **2.** $39^0\ 38'$ **3.** $29^0\ 53'$

Solutions

1. $47' = 0^0\ 47'$

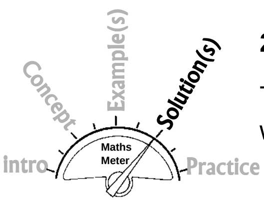
Under $48'$ = 0,9999 and the difference is $1'$ under the difference column = 0 $\therefore$ we add 0.

$$\begin{aligned} \cos 47' &= 0{,}9999 \\ & \underline{-0{,}0000} \\ &= \underline{0{,}9999} \end{aligned}$$



Common Error

The cosine trigonometrical ratio can be tricky – pause and think. Do not add the difference 0,0004 to 0,7705. Remember in cosine we **subtract**.



2. $39^0\ 38'$ – The cosine value of 39^0 under cos $36'$ = 0,7705

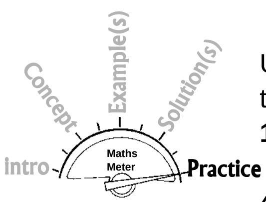
The difference with $38'$ = 2 hence the difference column give us 4 which should be subtracted since it is cosine.

$$\begin{array}{r} 0{,}7705 \\ \underline{-0{,}0004} \\ \underline{0{,}7701} \end{array} \quad \text{Cos } 39^0\ 38' = 0{,}7701$$

3. $29^0\ 53'$ Under $29^0\ 54'$ we get 0,8669. But the difference with $53'$ = 1 and under $1'$ the difference = 1.

We then **add** the difference.

$$\text{i.e.}\ \begin{array}{r} 0{,}8669 \\ \underline{+\ 0{,}0001} \\ \underline{0{,}8670} \end{array} \qquad \therefore\ \text{Cos } 29^0\ 53' = 0{,}8670$$



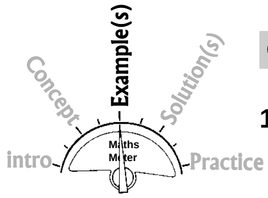
Using the tables at the end of this textbook, or a calculator, find the cosines of the following angles:

1. $15^0\ 15'$	**2.** 39^0	**3.** 47^0
4. $81^0\ 29'$	**5.** $36^0\ 47'$	**6.** $77^0\ 47'$

7. $52^0\ 17'$ **8.** $41^0 56'$ **9.** $33^0\ 33'$

10. $40^0\ 40'$

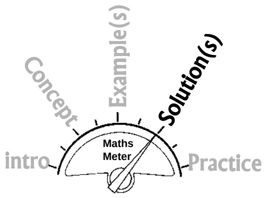
Finding angles when the values of the tangent, sine or cosine are given



Consider the following examples:

1. Use the tables to find the angles whose a) tan b) sin and c) cos respectively are 0,6935.

Solution



1. a) For tan, let angle be = θ.

$\therefore \tan\theta = 0{,}6935$

Step 1

– Search for 0,6935 or the closest two numbers which it falls between. Choose the nearer one. In this case it is = 0,6924 which is under $34^0\ 42$’. Record the $34^0\ 42$’.

Step 2

– The difference between the two numbers 0,6934 and 0,6924 is 9. Go to the difference column, to read the value for 9 or for the number nearest to it, in this case, it is 9.

Step 3

Move up the column to find the value of the minutes under which 9 is found in this case $2'$. Add $2'$ to $42' = 44'$

Hint

The bigger the tangent, the bigger the angle.

The final answer is $34^0\ 44'$

b) For sine, the same steps are followed, to give

$$\begin{array}{r} 43^0\ 54' \\ +\ 1' \\ \hline 43^0\ 55' \\ \hline \end{array}$$

Hint

The bigger the sine the bigger the angle.

c) For cosine, the same steps are followed.

$$\begin{array}{r} 46^0\ 6' \\ -\ \ 1' \\ \hline 46^0\ 5' \\ \hline \end{array}$$

Hint

*The bigger the cosine the **smaller** the angle.*

PRACTICE 12K3

1. Use the tables, or a calculator, to find the angles having the following tangents:

a) 0,6647
b) 1,7736
c) 0,4436
d) 0,9136
e) 2,5647

2. Use the tables, or a calculator, to find the angles with the sine values of:

a) 0,9612
b) 0,5672
c) 0,3648
d) 0,8714
e) 0,4444

3. Use the tables or a calculator, to find the angles with cosine values of:

a) 0,3747
b) 0,8123
c) 0,2478
d) 0,3214
e) 0,0461

SUMMARY

1. Pythagoras Theorem

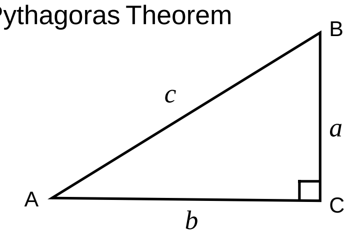


Fig 12.25

For any right-angled triangle
$\overline{AB}^2 = \overline{AC}^2 + \overline{BC}^2$ or

$$\boxed{c^2 = a^2 + b^2}$$

2. Simple trigonometrical ratios for an acute angle.

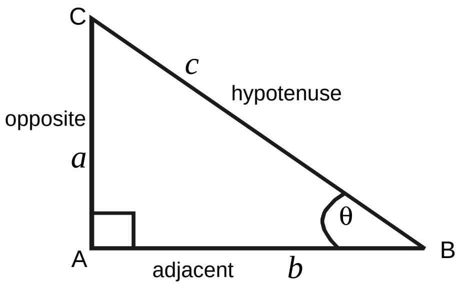


Fig 12.26

$$\text{Sin } \theta = \frac{\text{opp}}{\text{hyp}} = \frac{a}{c}$$

$$\text{Cos } \theta = \frac{\text{adj}}{\text{hyp}} = \frac{b}{c}$$

$$\text{Tan } \theta = \frac{\text{opp}}{\text{adj}} = \frac{a}{b}$$

3. Use '<u>**C**</u>HA<u>**S**</u>HO<u>**T**</u>AO' or '<u>**S**</u>OH<u>**C**</u>AH<u>**T**</u>OA' to remember the above ratios.

4. The above three formulae are used to calculate the ratios of the trigonometric functions. The ratios may also be used to find the value of the angle and vice versa.

5. Read the manual of your calculator and produce a summary of its application in trigonometry.

EXAM PRACTICE 12

Consider the following example:

1. Find the values of a) x and b) θ in Fig 12.23.
All measurements are in cm.

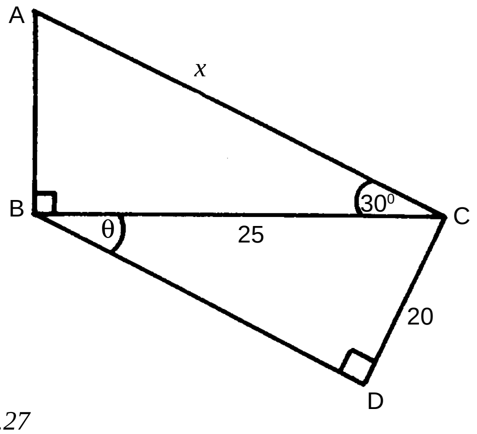


Fig 12.27

Solution

1. a) Using trigonometry, x is found as follows:

$$\frac{25}{x} = \text{Sin } 30^0$$

$$x = \frac{25}{\text{Sin } 30^0}$$

But $\text{Sin } 30^0 = 0,5$

$$x = \frac{25}{0,5}$$

$$x = 50\text{cm}$$

b) Using trigonometrical ratios, θ is calculated as follows:

$$\frac{20}{25} = \tan \theta$$

$$0,8 = \tan \theta$$

$$\theta = \tan^{-1}(0,8)$$

$$\theta = 38,66^0 \text{ or } 38^0 4'$$

Now do the following:

1.

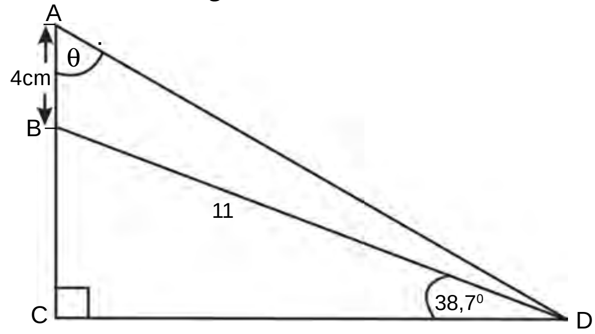


Fig 12.28

In Fig. 12.28, Calculate:

a) the length of DC.

b) the length of AC.

c) the size of the angle θ.

d) the size of $A\hat{D}C$.

e) the area of triangle ABD.

2. The diagonal of a rectangle is 12cm and the longer side is 8cm.

a) Find the angle between the diagonal and the shorter side.

b) Find the width of the rectangle.

3. Find the length of the sides of a square which fits exactly in a circle, of diameter 6cm, with all the corners of the square touching the circumference of the circle.

4. A ladder 5 metres long leans against a vertical wall with its foot on a horizontal floor 4 metres away from the wall. Find the angle between the ladder and the horizontal floor.

5.

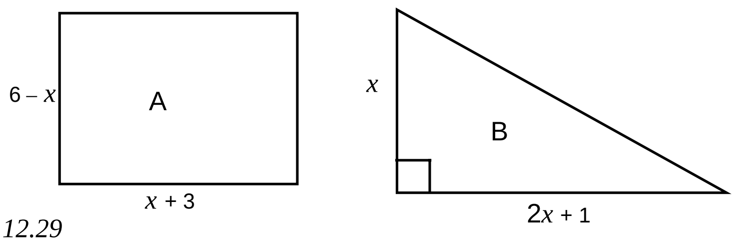


Fig 12.29

The diagrams show a rectangle and a right-angled triangle. The lengths of the sides of the rectangle A are $(x + 3)$cm and $(6 – x)$cm. The lengths of two of the sides of the triangle B are x cm and $(2x + 1)$cm, as shown.

a) Write down an expression, in terms of x, for the area of:
   (i) rectangle A.
   (ii) triangle B.

b) Given that the area of rectangle A is twice the area of triangle B, form an equation in x and show that it reduces to $3x^2 - 2x - 18 = 0$.

c) Solve the equation in (b), giving your answers correct to 3 significant figures.

d) Write down the dimensions of rectangle A correct to the nearest millimetre.

6.

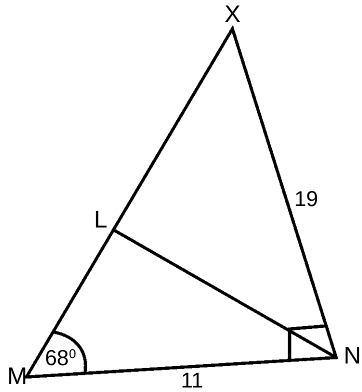


Fig 21.30

In the diagram, MNX is an acute angled triangle in which MN = 11cm, XN = 19cm and $X\hat{M}N = 68^0$.

a) Calculate:
   (i) $M\hat{X}N$
   (ii) the perpendicular distance from X to MN.
   (iii) MX.

b) Given that point L lies on MX and that ML = 6cm, calculate NL.

7.

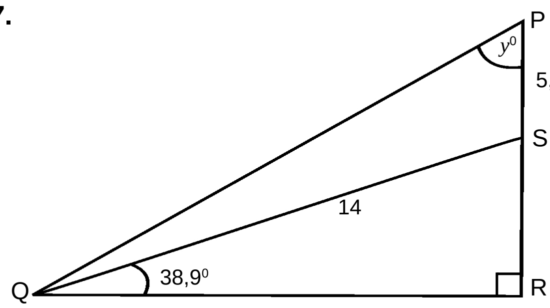


Fig 21.31

In the diagram, PS = 5,2cm, SQ = 14cm, $S\hat{Q}R = 38{,}9^0$ and $Q\hat{R}S = 90^0$.

Calculate:

a) the length of QR,

b) the length of PR,

c) the size of the angle marked y^0,

d) the area of the triangle PSQ.

8. In the digram, $D\hat{A}B = D\hat{C}B = 90^0$, AB = AD, DC = 9cm and CD = 2cm. Calculate the length of line AD.

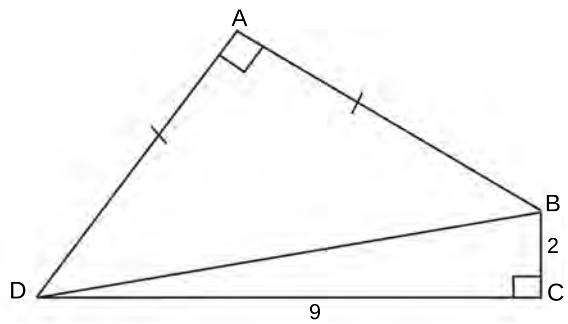
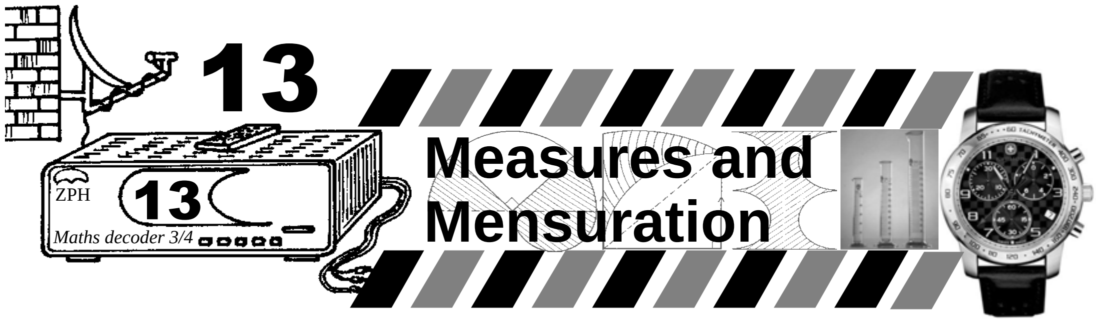
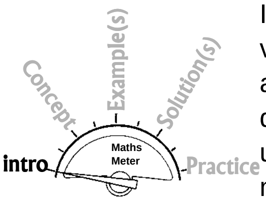


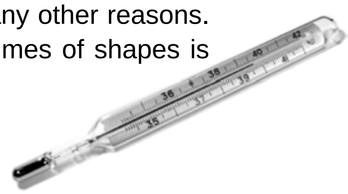
Fig 12.32

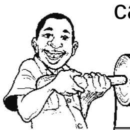


13 Measures and Mensuration



In everyday life, we encounter useful quantities which we use for various purposes. Time, mass, temperature, distance, perimeter, area, volume and density are examples. Measurements of these quantities may be found by using an instrument or by calculation, using a defined formula. The purpose of taking these measurements or doing these calculations may be for recording, pricing, packaging, loading research or for many other reasons. The calculation of perimeters, areas and volumes of shapes is called **mensuration**.





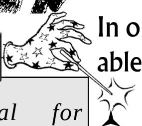
Syllabus Expectations

By the end of this chapter, students should be able to:

1. read time on both the 12 and 24 hour clocks.
2. use System International (SI) units of mass, temperature, distance, area, volume and density.
3. express quantities in terms of larger or smaller units.
4. define other SI units, including the hectare.
5. carry out calculations involving the perimeter and area of a rectangle, triangle, parallelogram and trapezium.
6. find the circumference of a circle, length of a circular arc and the length of a sector.
7. find the area of a circle and the area of a sector of a circle and the area of a segment.
8. calculate the surface area and volume of a cylinder, cuboid, prism of uniform cross-section, pyramid, cone and sphere.

ASSUMED KNOWLEDGE

In order to tackle work in this chapter it is assumed that students are able to:



Hint

The plural for rhombus is rhombi and for trapezium is trapezia.

- draw polygons such as triangles and quadrilaterals (including kites, rhombi and trapezia).

- draw regular n-sided polygons.
- change the subject of a formula and substitute in to formulae, including those from other subjects e.g Science.
- draw three dimensional-shapes such as pyramids, cubes, cuboids and prisms of uniform cross section.
- solve quadratic equations.
- manipulate algebraic functions.
- find the squares and square roots of numbers.

A. MEASURES

A quantity is any scientific measure which may be used to define a concept, for example, time, mass and density. A quantity may have all or some of the following characteristics: symbol, units, formula to calculate it, instrument to measure it and a definition. Most quantities are related to other quantities in one way or the another. Table 13.1 gives the characteristics of basic measurements you are required to know.

Table 13.1

Quantity	Unit	Relationship to other quantities (formula)	Instrument used to measure quantity	Definition
Time (T)	second (s)	$T = \frac{\text{Distance}}{\text{Speed}}$	12 or 24 hour clock	Not necessary at this stage but may be measured in seconds, hours, days, weeks, months or years.
Mass (M)	kilogram (kg)	M = Density x Volume	Scale	Amount of material which makes up matter.
Density (D)	kg/m^3	$D = \frac{\text{Mass}}{\text{Volume}}$	Calculated	Density is mass per unit volume.
Temperature (T)	Kelvin ^0C	–	Thermometer	A measure of the degree of hotness or coldness of something.
Distance /length (l)	m	Varies	Metre rule	Not necessary at this stage.
Area (A)	m^2	Varies for different shapes	Calculated	A measure of space in a plane which is enclosed by boundaries
Volume or Capacity (V)	m^3	Varies for different shapes	Calculated	Space occupied by an object

Time

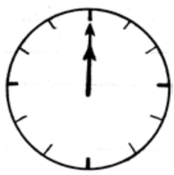
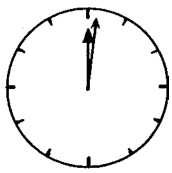
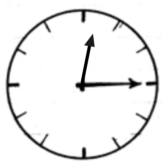
Time is a very valuable quantity which we use daily in order to manage our activities. The basic unit of time is the second. These accumulate into minutes, hours, days, weeks, months and years.

60 seconds	=	1 minute
60 minutes	=	1 hour
24 hours	=	1 day
7 days	=	1 week
4 weeks	=	1 month
12 months	=	1 year
52 weeks	=	1 year
365 days	=	1 year

In this section we concentrate on the **time of the day**.
Time may be measured using the 12 hour clock or the 24-hour clock. A day starts at midnight and ends at midnight. A mechanical (analogue) 12 hour clock must have its hour hand rotating two revolutions in order to complete a day (24 hours). A digital clock will show the exact time in terms of hours, minutes and seconds for either the 12-hour or 24-hour clock.

Consider the following times in table 13.2.

Table 13.2

Time in words	12-hr clock	12-hour Notation	24 hour Notation
Midnight (Beginning of new day)		12 midnight	0000
a minute past 12 midnight		12.01am	0001
quarter past 12 mid-day		12.15pm	1215
eight o'clock in the evening		8.00pm	2000

Consider the following examples:

1. A motorist starts a journey at five minutes past twelve during the day and travels for 6,25 hours in order to arrive at her destination. Express the arrival time in 12 hour and 24-hour clock.

Solution

	12 hour clock	24 hour clock
1. Starting time: →	12.05pm	1205

6,25 hours $= 6 \text{ hours} + \frac{\cancel{25}^{1}}{\cancel{100}_{4}} \times 60 \text{ mins}$

$= 6 \text{ hours} + 15 \text{ mins}$

For 12 hour clock

00.05 + 6hr 15min

$$\begin{array}{r} 00.05 \\ +\ 6.15 \\ \hline 6.20 \end{array} \quad = 6.20\text{pm}$$

For 24 hour clock

1205 + 6hr 15 min

$$\begin{array}{r} 1205 \\ 0615 \\ \hline 1820 \end{array}$$

Common Error

Note that 6,25 hours is not 6 hours and 25 minutes similarly 2,10 minutes is equivalent to 2 minutes + $\left(\frac{10}{100} \times 60\right)$ seconds which reduces to 2 mins and 6 sec.

Also note the following:

18 20hrs → wrong
18:20 → wrong
18 20 → correct
06:20pm → wrong
6.20pm → correct

2. A plane leaves Harare airport on a Sunday at 2300hrs heading for London. Given that Harare time is ahead of London time by 2 hours and the journey takes 10,5hrs

Calculate

a) The time in Harare when the plane lands in London (on the 12 hour clock).

b) The London time on the 24 hour clock, when the plane lands in London.

Solution

	12-hour clock	24-hour clock
2. Departure time (Harare time) →	11.00 pm	2300

Two hours ahead means when it was 2300 in Harare, it was (2300 – 2)hours in London = 2100

	12-hour clock	24-hour clock
Departure time (London Time) →	9.00 pm	2100

$$10{,}5 \text{ hours} = 10\text{hrs} + \left(\frac{5}{10} \times 60 \text{ mins}\right)$$

a) Arrival time (Harare time) = 10hrs + 30mins

$$= \begin{array}{r} 2300 \\ +1030 \\ \hline 3330\text{hr} \end{array} \qquad \begin{array}{r} 3330 \\ -2400 \\ \hline 0930 \end{array}$$

Harare time = 9.30 am on Monday

b) London is 2 hours behind so
Arrival time (London time)

$$= \begin{array}{r} 0930\text{hrs} \\ -0200\text{hrs} \\ \hline 0730\text{hrs} \end{array} = 7.30\text{am on Monday}$$

Using 24hr clock, London arrival time = 07 30

$$\text{OR} \quad \begin{array}{r} 21\ 00\text{hrs} \\ +\ 10\ 30\text{hrs} \\ \hline 31\ 30\text{hrs} \end{array} \Rightarrow \begin{array}{r} 31\ 30\text{hrs} \\ -\ 24\ 00\text{hrs} \\ \hline 07\ 30 \end{array}$$

1. Convert the following times to 24-hour notation:
a) 1.06pm. b) 11.59pm.
c) Half past twelve midnight.
d) Twenty minutes to one daytime.

2. Convert the following to 12 hour clock given that its 24 hr notation:
a) 0002 b) 1313 c) 2400
d) 1210 e) 0000

3. A cyclist starts his journey at 0600 and arrives at 3.40pm, the same day. Calculate the total time taken to complete the journey giving your answer in minutes.

4. An omnibus travelled 600km from one city, A to another city, B, at an average speed of $v$km/hr. The omnibus returned, non-stop, by the same route, at an average speed of $2v$km/hr.
a) Write down an expression for the time in hours, taken for the return journey.

b) Given that the difference between these two times is 2 hours, form an equation in v and solve it.

5. A Sunday service starts at 1400. John arrived 24,25 minutes late. Calculate the time John arrived and express it in 12 hr clock notation.

6. A plane leaves London at 6.00am, Zimbabwean time heading for Bulawayo. Given that, Bulawayo is ahead of London by 2,20 hours and the journey takes 11 hours, calculate the arrival time in Bulawayo. Express your answer in 24-hour notation.

7. A train left a station at 1326 and arrived at its destination after travelling for $2\frac{3}{4}$ hours. Find the time, on the 12-hour clock, at which the train arrived at its destination.

B. THE METRIC SYSTEM AND (SI) UNITS

The System International (SI) units of quantities are the metric system which uses the following:

Length $\longrightarrow$ metre.
Volume $\longrightarrow$ m^3.
Mass $\longrightarrow$ kg or gram.
Capacity $\longrightarrow$ litre.

Table 13.3 illustates how these prefixes may be used for the gram.

A basic unit like gram, is converted into multiples or sub-multiples by using prefixes. These prefixes are: kilo-, hecto-, deka-, deci-, centi-, milli-.

Table 13.3

Unit	Abbreviation	Basic Units
kilo – 1 kilogram	1kg	1 000g
hecto – 1 hectogram	1hg	100g
deka – 1 dekagram	1dag	10g
1 gram	1g	1g
deci – 1 decigram	1dg	0,1g
centi – 1 centigram	1cg	0,01g
milli – 1 milligram	1mg	0,001g

The beauty of the System International (SI) units is that they are universal and they use multiples and sub-multiples of 10. The expression of quantities in terms of larger or smaller units can be easily done.

Consider the example below:

1. Convert the following:
a) $10m^2$ to cm^2.
b) $500cm^2$ to m^2.
c) 0,2kg to g.

d) $3000cm^3$ to m^3.
e) $20kg/m^3$ to g/cm^3.
f) 100m/s to km/hr.

Solution

1. a) Always start with the bigger unit, i.e.

$$1m = 100cm.$$

Square both sides.

$$(1m)^2 = (100cm)^2$$
$$1m^2 = 10\ 000cm^2$$

Use ratio.

$$1m^2 = 10\ 000cm^2$$
$$10m^2 = \boxed{?} \text{ ...... more}$$

$$\therefore \frac{10\cancel{m^2}}{1\cancel{m^2}} \times 10\ 000cm^2$$

$$= 100\ 000cm^2$$

Hint

Units multiply i.e
$cm \times cm \times cm = cm^3$
$(1m)^2 = 1m \times 1m$
$= 1m^2$
Units divide i.e.
$\frac{m^{\cancel{2}}}{\cancel{m}} = m$
$\frac{1\cancel{m^3}}{1\cancel{m^3}} = 1$

b) $$500cm^2 = \boxed{?}\ m^2$$
$$1m = 100cm$$

$$\therefore (1m)^2 = (10\ 000cm)^2 \text{ (to find square units)}$$

$$10\ 000\cancel{cm^2} = 1m^2$$
$$\Rightarrow 500\cancel{cm^2} = \boxed{?} \text{ .... less}$$

$$\therefore 500cm^2 = \frac{500\cancel{cm^2}}{10000\cancel{cm^2}} \times 1m^2$$

$$= 0{,}05m^2$$

c) $$0{,}2kg = \boxed{}\ g$$

$$1kg = 1000g$$
$$0{,}2\ kg = \boxed{?} \text{ .... less}$$

$$\therefore 0{,}2kg = \frac{0{,}2\cancel{kg}}{1\cancel{kg}} \times 1000g$$

$$= 200g$$

d) $$3000cm^3 = \boxed{?}\ m^3$$

$$1m = 100cm$$
$$(1m)^3 = (100cm)^3$$

$$1m^3 = 1\ 000\ 000cm^3$$

$$1\,000\,000\text{cm}^3 = 1\text{m}^3$$
$$3000\text{cm}^3 = \boxed{?} \text{ .... less}$$

$$\therefore 3\,000\text{cm}^2 = \frac{3\,000\cancel{\text{cm}^3}}{1000000\cancel{\text{cm}^3}} \times 1\text{m}^3$$

$$= 0{,}003\text{m}^3$$

e) $\dfrac{20\text{kg}}{\text{m}^3} = \dfrac{\boxed{?}\ \text{g}}{\boxed{?}\ \text{cm}^3}$

$\Rightarrow \dfrac{20\text{kg}}{\text{m}^3} = \dfrac{\boxed{?}\ \text{g}}{\boxed{?}\ \text{cm}^3}$

In this case, change 20kg to g and 1m^3 to cm^3.

for kg, $1\text{kg} = 1000\text{g}$
$20\text{kg} = \boxed{?}$

$$\frac{20\cancel{\text{kg}}}{1\cancel{\text{kg}}} \times 1000\text{g}$$

$$= 20\,000\text{g}$$

for 1m^3, $1\text{m} = 100\text{cm}$
$(1\text{m})^3 = (100 \times 100 \times 100)\ \text{cm}^3$

$$1\text{m}^3 = 1\,000\,000\text{cm}^3$$

Substitute respectively:

$$\frac{20\text{kg}}{1\text{m}^3} = \frac{2\cancel{0}\ \cancel{000}\text{g}}{100\cancel{0}\ \cancel{000}\text{cm}^3}$$

$$\therefore \quad 20\text{kg/m}^3 = 0{,}02\text{g/cm}^3$$

f) $100\text{m/s} = \boxed{?}\ \text{km/hr}$
$1\text{hr} = 3\,600\text{s}$

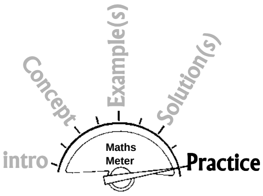
Use simple proportion.

$$\frac{100\text{m}}{1\text{s}} = \frac{\left(\frac{100\text{m}}{1000}\right)\text{km}}{\left(\frac{1}{3600}\right)\text{h}}$$

$\boxed{?}$ km covered in 3600 seconds

$$\therefore\ 100\text{m/s} = \frac{10\cancel{0}}{100\cancel{0}} \times 3600\text{km/h}$$

$$= 360\text{km/h}$$



1. Convert the following:

a) 400m^2 to cm^2.

b) 2000mm^3 to cm^3.

c) 0,004kg to grams.

d) 4 litres to millilitres.

e) 20g/cm^3 to kg/m^3.

2. One winter morning the temperature at the bottom of a tall mountain is 9^{0}C and the temperature at the top is –6^{0}C. Calculate the difference between these temperatures.

3. a) The speed of an object is 10km/hr. Convert its speed to m/s.

b) Convert 100m/s to km/hr.

c) Express 48 minutes as a percentage of 2 hours.

4. A solid object has a mass of 200kg and a volume of 40m^3. Express its density in terms of g/cm^3.

Hint

Density = $\dfrac{Mass}{Volume}$

5. From question 4, calculate the volume of the same object, given that its mass is 0,004grams.

6. Calculate the mass of 400ml of water, given that the density of water is 1g/cm^3.

Hint

1ml = 1cm^3

7. a) The area of a triangle is 250cm^2. Express this area in m^2.

b) Sound travels at 330m/s. Express this speed in km/hr.

8. A playground has a length of 120m and a width of 100m. Calculate its area and express it in hectares.

Hint

100m = 1 hectmetre
10 000m^2 = 1 hectare
students confuse hectare and acre
1 Acre = 0,40468 hectares.

9. A cyclist starts a 40km journey at 0800. She maintains an average speed of 30km/hr for the first three quarters of an hour and then rests. Thereafter, she continues her journey at an average speed of 35km/hr, arriving at her destination at 1100.

a) Calculate the distance covered in the first three quarters of an hour.

b) Calculate, in hours, the time taken to cover the last part of the journey.

c) Calculate, in minutes, the duration of her rest.

Hint

100m = 1 hectmetre

10. Say which of the units: millimetres, centimetres, meters or kilometers would be the best to use in these cases:

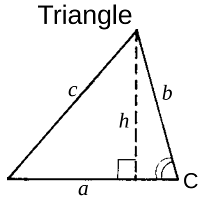
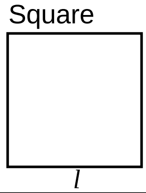
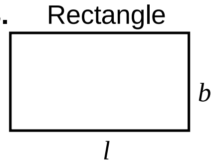
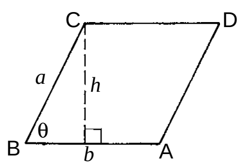
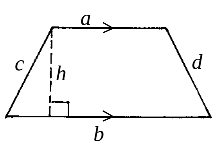
a) the height of a grown up person.

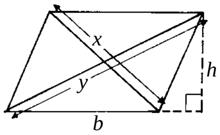
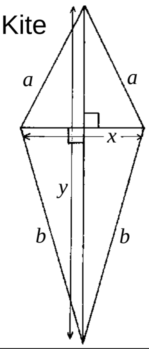
b) the height of a building.
c) the distance from Bulawayo to Masvingo.
d) the length of a rugby pitch.
e) the thickness of a plough-share.

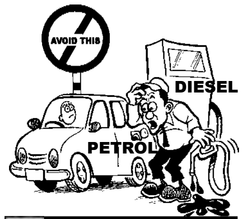
C. AREA AND PERIMETER OF PLANE SHAPES

The perimeter of a plane shape is the distance around that shape. Area is area is a measure of the space enclosed by the perimeter quoted in squares of the unit involved. Table 13.4 below shows a summary of formulae used to calculate these two quantities, for each shape.

Table 13.4

Shape	**Formula for Area (A) (units²)**	**Formula for perimeter (P) (units)**
1. Triangle 	Area = $\frac{1}{2}$ base × height a) A = $\frac{1}{2}$ b × h OR b) A = $\frac{1}{2}ab$ Sin C	P = $a + b + c$
2. Square 	Area = side × side = l^2	P = $4l$
3. Rectangle 	Area = length × breadth = $l \times b$	P = $2(l + b)$
4. Parallelogram 	Area = base × height Area = $b \times h$ OR Area = $ab\text{Sin}\theta$	P = $(2a + 2b)$ = $2(a + b)$
5. Trapezium 	Area = $\frac{1}{2}$(sum of parallel lines) × height Area = $\frac{1}{2}(a + b)h$ (in units²)	P = $(a + b + c + d)$

6.	Rhombus 	Area = base $\times$ height $= b \times h$ OR $A = \frac{1}{2}$ product of diagonals $A = \frac{1}{2}xy$	$P = 4b$
7.	Kite 	Area = longer diagonals $\times$ half length of short diagonal $A = \frac{1}{2}xy$	$P = (2a + 2b)$ $= 2(a + b)$



Common Error

Ensure x is half of the shorter diagonal.

Consider the examples below:

1. An artist has a rectangular piece of paper which is 54cm long and 33cm wide. (Fig 13.1)

a) Calculate the perimeter of the piece of paper giving your answer in centimetres and in metres.

b) The artist wishes to cut the paper into squares each with a length of 6cm. Calculate the largest number of whole squares the artist can cut.

54cm

33cm

Fig 13:1

Solution

a) Perimeter of rectangle $= 2l + 2b$

$= 2 \times 54 + 2 \times 33$

$= 108 + 66$

$= 174\text{cm} = 1{,}74\text{m}$

b) First find the area of the rectangle

$= 54\text{cm} \times 33\text{cm}$

$= 1815\text{cm}^2$

The area of the squares to be cut $= l^2$

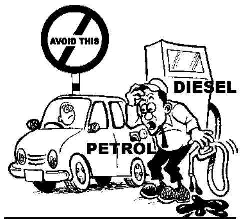
$(6\text{cm})^2 = 36\text{cm}^2$

$\therefore$ Number of squares to be cut = lengthwise $= \frac{54}{6}$

$= 9$

Number of squares to be cut widthwise $= \frac{33}{6} = 5$

Together $= 9 \times 5 = 45$ whole squares



Common Error

Number of squares $= \frac{1815}{36}$ That is wrong!

2. Shape ABCD is a trapezium where AD is parallel to BC and $A\hat{B}C = 90^0$, AD = $x + 4$, AB = $(x - 4)$ and BC = $(4x - 1)$, as illustrated in Fig 13.2.

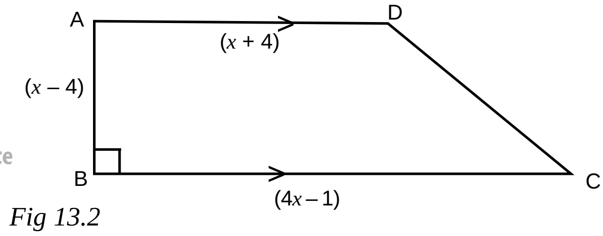


Fig 13.2

a) Write down, in terms of x, an expression of the area.

b) Given that the area of the trapezium is 20cm², form an equation in terms of x and show that it reduces to:
$5x^2 - 17x - 52 = 0$.

c) Solve the equation and hence find the length AB.

Solution

2. a) Area of a trapezium = $\frac{1}{2}(BC + AD) \times AB$

Area of trapezium $= \frac{1}{2}[(x + 4) + (4x - 1)] \times (x - 4)$

Area of trapezium $= \frac{1}{2}[x + 4 + 4x - 1] \times (x - 4)$

Area $= \frac{1}{2}[(5x + 3)(x - 4)]$

$= \frac{1}{2}(5x^2 - 17x - 12)$ units²

b) From (a) Area of trapezium = $\frac{1}{2}(5x^2 - 17x - 12)$

But given Area = 20cm²

$\therefore$ 20cm² $= \frac{1}{2}(5x^2 - 17x - 12)$

$\therefore 5x^2 - 17x - 12 = 40$

$5x^2 - 17x - 52 = 0$ (shown)

c) $5x^2 - 17x - 52 = 0$
Since it is a quadratic equation we may use the quadratic formula

i.e. $$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$\therefore \ x = \frac{17 \pm \sqrt{17^2 - 4 \times 5 \times 5^2}}{10}$$

$$x = \frac{17 \pm \sqrt{289 + 1040}}{10}$$

$$x = \frac{17 \pm 36.45545227}{10}$$

$x = 5{,}346$ or $x = -1{,}946$

Hint

Use the positive value of x since length is a positive value.

$\therefore$ Length of AB $= x - 4$

$= 5{,}346 - 4$

$= 1{,}346$cm

3. The quadrilateral ABCD has dimensions as shown, with a rectangle, EBFD, enclosed in it. All lengths are in cm. (Fig 13.3)

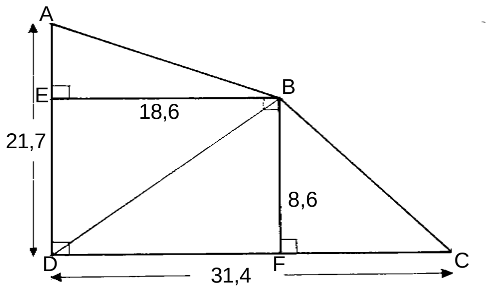


Fig. 13.3

a) Calculate the area of $\triangle$BCD.

b) Calculate the area of quadrilateral ABCD.

Solution

3. a) Area of $\triangle$BDC $= \frac{1}{2} b \times h$

$= \frac{1}{2} \times 31{,}4\text{cm} \times 8{,}6\text{cm}$

$= 135{,}02\text{cm}^2$

Hint

The area of a quadrilateral ABCD can also be found by adding the area of the trapezium EBCD and the triangle AED or the trapezium ABFD and the triangle BFC.

b) Area of the quadrilateral ABCD.

= (Area of $\triangle$BAD + Area of $\triangle$BDC)

Area of $\triangle$ ABD $= \frac{1}{2} b \times h$

$= \frac{1}{2} \times 21{,}7\text{cm} \times 18{,}6\text{cm}$

$= 241{,}81\text{cm}^2$

$\therefore$ Area of quadrilateral ABCD $= 135{,}02\text{cm}^2 + 241{,}81\text{cm}^2$

$= 376{,}83\text{cm}^2$

4. The diagram in Fig 13.4 shows a tile ABCD in the form of a kite. The 'shaded part' is painted blue while the rest is white.

a) Calculate the blue (shaded) area, given that the length of the longer diagonal for the external kite = 42cm and also that the length of the longer diagonal for the internal kite = 21cm. Give your answer in cm^2.

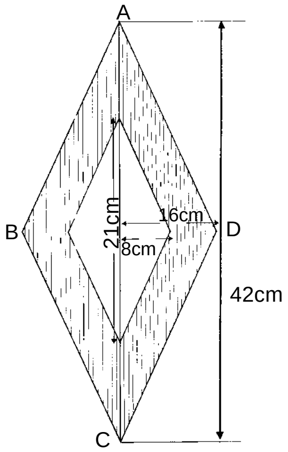


Fig 13.4

Solution

4. Shaded area $= \begin{pmatrix}\text{Area of} \\ \text{external kite}\end{pmatrix} - \begin{pmatrix}\text{Area of} \\ \text{internal kite}\end{pmatrix}$

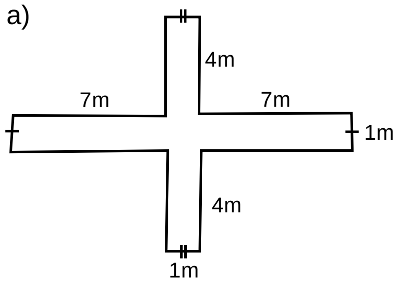
$= (42 \times 16)\text{cm}^2 - (21 \times 8)\text{cm}^2$

$= (672 - 168)\text{cm}^2$

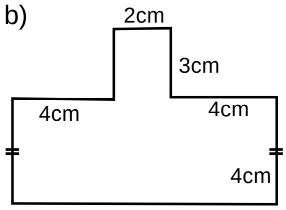
$= 504\text{cm}^2$

1. Find the perimeters and areas of the shapes in Fig 13.5(a to d).

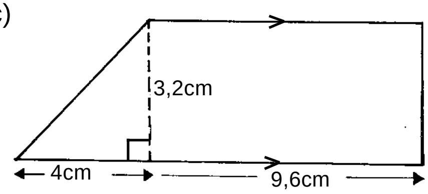
a)



b)



c)



d)

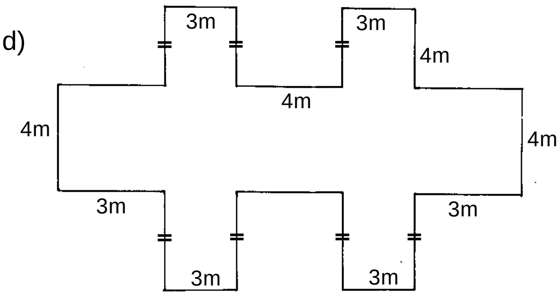


Fig 13.5

2. A rectangular lawn measuring 3 metres by 1 metre, is surrounded by a path of width 60cm. Calculate the area of the path, giving your answer in square meters.

Hint

Regular means all sides are equal.

3. The area of a regular hexagon, of side 15cm is required. This may be achieved by dividing the hexagon into a rectangle and four right-angled triangles, as shown in Fig 13.6.

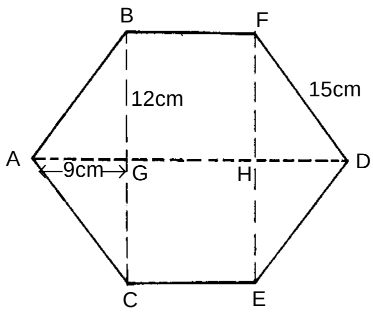


Fig 13.6

Given that BG = 12cm:

a) calculate the area of triangle ABG.

b) calculate the area of rectangle BGHF.

c) calculate the area of the whole hexagon.

4. A pattern is made up of large and small squares. The sides of the larger squares are x long. The sides of the smaller squares are $\frac{1}{4}x$ long. See (Fig 13.7)

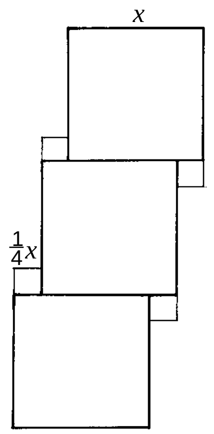


Fig 13.7

Find the following expressions, in terms of x, simplifying as far as possible:

a) the perimeter of the diagram.
b) the area of the diagram.

5. A rectangular floor space measures 15m by 9,8m. How many **whole** rectangular tiles, measuring 25cm by 20cm, will be needed to cover this floor?

6.

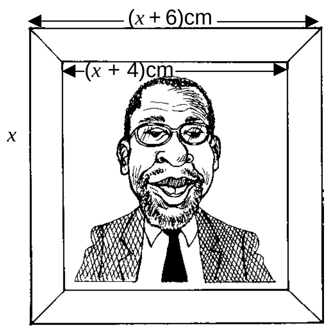


Fig 13.8

a) Find the area of the picture frame in Fig 13.8.
b) Show that the area of the frame can be reduced to $(x^2 - 2x - 16)\text{cm}^2$ square units. Assume the frame is flat and of uniform cross-sectional area.

7. If the sides of a kite are measured to the **nearest** centimetre. They are 16cm and 13cm respectively. Find the **smallest** possible perimeter of the kite.

8. ABCD is a parallelogram (Fig 13.9) with AE perpendicular to EC. Given that the area of triangle ABD is $40cm^2$, AB = 15cm and ED = 12cm.

a) Calculate the length AE.

b) Calculate the area of parallelogram ABCD.

c) Find the area of quadrilateral EDBA.

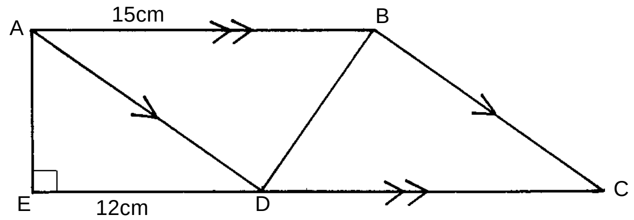


Fig. 13.9

D. AREA AND PERIMETER OF THE CIRCLE, SECTOR AND SEGMENT

A circle is unique because it can enclose a plane shape with all the shape's vertices touching the circumference. The following parts of a circle should be noted. (Fig 13.10)

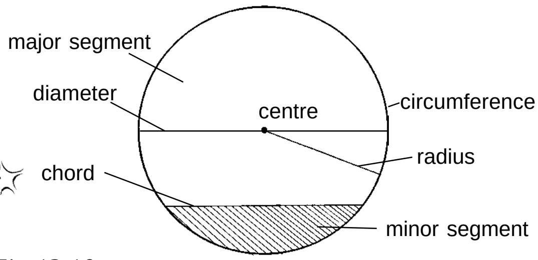


Fig 13.10

Hint

While a circle's diameter is a chord, its radius is not.

Chord – Any straight line drawn within the circle with both ends touching the circumference of the circle. The chord divides the circle into two segments – the minor segment and the major segment.

The **diameter** is a special chord which divides the circle into two equal segments called semi-circles. It passes through the centre

Arc of a circle

Any part of the circumference between two points is called an arc. It should be noted that a chord also divides the circumference of a circle into minor and major arc. (Fig 13.11)

The diameter divides the circumference into two equal arcs.

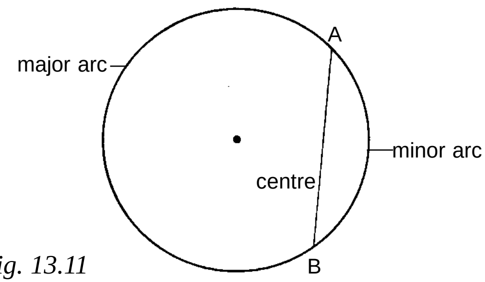
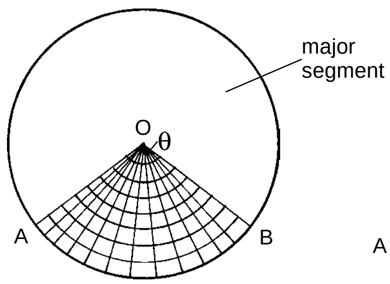


Fig. 13.11

Moving along the circumference
Length AB clockwise = minor arc
Length AB anticlockwise = major arc

Sector of circle

If two radii are drawn within the same circle with an angle θ in between them, less than 180^0, then the formed shape is a minor sector of the circle. (Fig 13.12).



Shaded AOB = minor sector

Fig. 13.12(a)

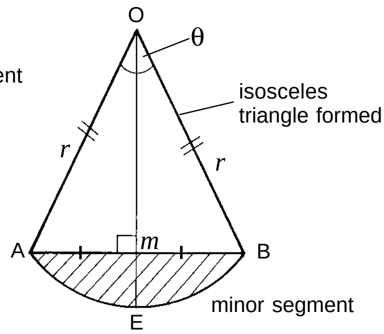
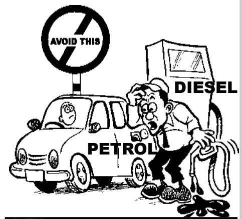


Fig. 13.12(b)

Hint

The sector can further be divided into two shapes: a triangle and a minor segment (Fig 13.12(b). The triangle formed is always an isosceles. Throughout this chapter the word "sector" will refer to the minor sector.

As a matter of fact the triangle in any minor sector is always an isosceles triangle and its perpendicular bisects the base Fig 13.12(b).

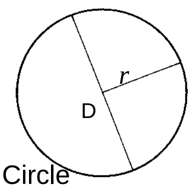
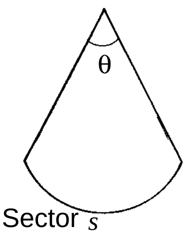
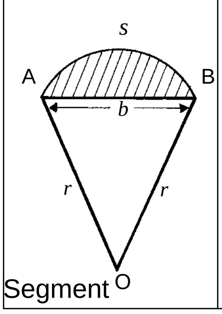


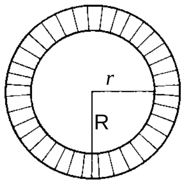
Common Error

The arc in a sector is not part of the triangle of the sector. Most importantly, note that in Fig. 13.12 (b) OE =OA=OB since they are all radii of the same circle. However, OM is not a radius of the circle but the perpendicular bisector of the triangle base AB.

Table 13.5 gives a summary of formulas used to make calculations related to the circle, sectors and segments.

Table 13.5

Shape	Formula for Area (A) (units^2)	Formula for perimeter (P) (units)
 Circle	Area = πr^2	Circumference (Perimeter) = $2\pi r$ or πd
 Sector	Area = $\frac{\theta}{360^0} \times \pi r^2$ OR If θ is in radians Area = $\frac{1}{2}r^2\theta$ or $\frac{1}{2}rs$	P = $\left(\frac{\theta}{360^0} \times 2\pi r^2 + 2r\right)$ where the bracketed formula is that for finding length of the arc of sector.
 Segment	Area of shaded segment = Area of sector − Area of triangle	Perimeter = length of arc + length of chord

 Annulus/Ring	Shaded Area = $\pi R^2 - \pi r^2$ $= \pi(R^2 - r^2)$	Not applicable

Consider the examples below:

1. In a pattern design process, a designer cut, from a rectangular cloth (four equal semi circles), (Fig 13.13). Taking π to be 3,0, work out the following:

a) the area of the original rectangle.

b) the area of the four semi-circles.

c) the area of the shaded shape.

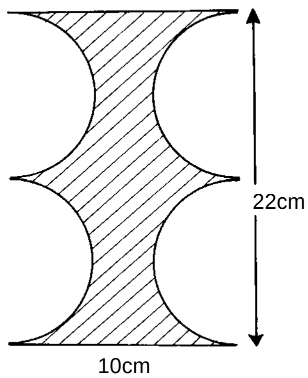


Fig 13.13

Solution

1. a) Area of the original rectangle

$= 22 \times 10$

$= 220\text{cm}^2$

b) To find the area of the semicircle:

Find, the diameter $= \dfrac{22}{2}$

$= 11\text{cm}$

$\therefore$ Radius for each semicircle $= \dfrac{11}{2}$

$= 5{,}5\text{cm}$

Area of the semicircle $= \dfrac{\pi r^2}{2}$

Area $= \left(\dfrac{3{,}0 \times 5{,}5 \times 5{,}5}{2}\right) \text{cm}^2$

Tip

Students forget to divide by two to get area of a semicircle.

$\therefore$ For the semi-circles $= \dfrac{3{,}0 \times 5{,}5 \times 5{,}5 \times \cancel{4}^{2}}{\cancel{2}}$

$= 181{,}50\text{cm}^2$

Hint

The value of π maybe given as 3,142, $\frac{22}{7}$ or simply 3. Always follow the examiner's question and use the value given in that question.

c) The area of the shaded part

$= 220{,}00$
$\underline{\;181{,}50\;}$
$\underline{\;\;38{,}50\text{cm}^2}$

2. In this question take π to be $\frac{22}{7}$

A semi-circle, ADB, has a diameter equivalent to the length of a rectangle, measuring 8cm $\times$ 6cm, is placed on the length of the rectangle as ADB, (Fig 13,14). A sector OACB, is formed inside the semi-circle. O is the centre of the rectangle.

Calculate:

a) length OB.
b) angle AOB.
c) area of sector OACB.
d) the shaded area.

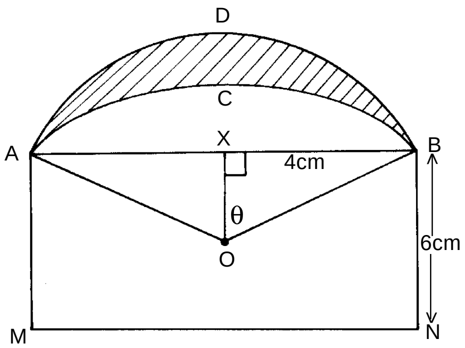


Fig. 13.14

Solution

2. a) Using Pythagoras:

$OB^2 = OX^2 + XB^2$
$OB^2 = 3^2 + 4^2$
$OB^2 = 25$
$OB = \sqrt{25}$
$OB = 5$

b) Angle $A\hat{O}B$ $\quad \tan\theta = \dfrac{4}{3}$

$\tan\theta = 53{,}13^0$

$\therefore \; A\hat{O}B = 53{,}13^0 \times 2$

$A\hat{O}B = 106{,}26^0$

c) Area of sector OACB $= \dfrac{106{,}26^0}{360^0} \times \pi r^2$

$= \dfrac{106{,}26^0}{360^0} \times \dfrac{22}{7} \times 5^2$

$= 23{,}19\text{cm}^2$

d) First find the areas of the semi-circle, $\triangle$OAB and segment XACB.

(i) Area of semi-circle $= \dfrac{\pi r^2}{2} = \dfrac{1}{2} \times \dfrac{22}{7} \times 4^2$

$= 25{,}14\text{cm}^2$

(ii) Area of $\triangle$OAB $= \frac{1}{2}\, b \times h$

$= \dfrac{1}{2} \times 8 \times 3$

$= 12\text{cm}^2$

(iiI) Area of segment XACB = Area of sector OACB – area of $\triangle$ OAB.

$= 23{,}19 - 12$

$= 11{,}19\text{cm}^2$

(iv) Shaded area = area of semi circle – area of segment

$= (25{,}14 - 11{,}19)\text{cm}^2$

$= 13{,}95\text{cm}^2$

3. The sides of a kite form the radii of two circles which intersect at points O and B as shown in Fig 13.15. If the shorter and longer sides of the kite are 12cm and 15cm respectively, calculate the shaded area of the intersection. (Take $\pi = 3{,}14$)

Hint

In such complicated problems it is important to state the steps to be done before you start working, so that you have a clearly defined path to follow.

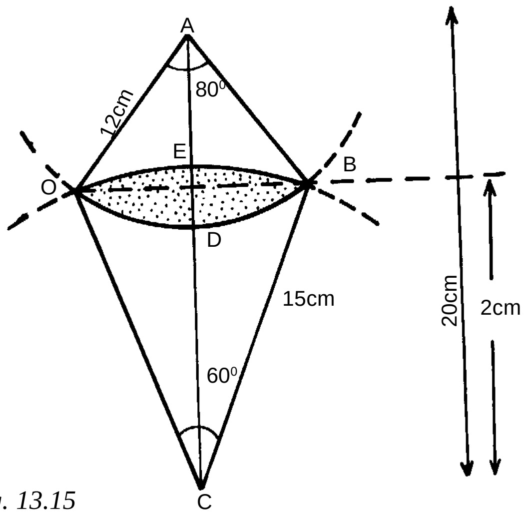


Fig. 13.15

Solution

3. **Step 1:** Find the area of the larger sector COEB.
Step 2: Find the area of $\triangle$COB.
Step 3: Find the area of minor segment OEB.
Step 4: Find the area of smaller sector AODB.

Step 5: Find the area of $\triangle$ OAB.
Step 6: Find the area of minor segment ODB.
Step 7: Find the shaded area (i.e step 3 + Step 6).

Step 1: Area of sector COEB

$$= \frac{\theta}{360} \times \pi r^2 = \frac{60^0}{360^0} \times 3{,}14 \times 15^2$$

$$\text{Area of COEB} = \frac{1}{6} \times 3{,}14 \times 15 \times 15$$

$$= 117{,}75\text{cm}^2$$

Step 2: Area of $\triangle$COB $= \frac{1}{2}$ base $\times$ height

$$= \frac{1}{\cancel{2}_1} \times 18 \times \cancel{12}^{6}$$

$$= 108\text{cm}^2$$

Step 3: Area of minor segment
= area of sector COEB – area of $\triangle$COB
= (117,75 – 108)cm^2
= 9,75cm^2

Step 4: Area of the smaller sector OABD $= \frac{\theta}{360} \times \pi r^2$

$$= \frac{80^0}{360^0} \times 3{,}14 \times 12^2$$

$$= \frac{2}{9} \times 3{,}14 \times 144$$

$$= 100{,}48\text{cm}^2$$

Step 5: Find the area of $\triangle$OAB $= \frac{1}{2}$ base $\times$ height

$$= \frac{1}{\cancel{2}} \times 18 \times \cancel{8}^{4}$$

$$= 18 \times 4$$

$$= 72\text{cm}^2.$$

Step 6: Area of minor segment ODB = Area of sector OABD – Area of $\triangle$OAB

$$= (100{,}48 - 72)\text{cm}^2$$

$$= 28{,}48\text{cm}^2.$$

Step 7: The shaded area is the sum of the areas of the two minor segments.

= Area of segment ODB + Area of segment OEB

= $(9,75 + 28,48)\text{cm}^2$

= $38,23\text{cm}^2$

PRACTICE 13D

Hint

5 minutes movement = 30^0 at centre.

1. The minute hand of a watch is 1,3cm long. Using $\pi = \frac{22}{7}$, calculate:
a) the distance moved by the minute hand in 25 minutes.
b) the area swept by the minute hand in this time.

2. The machine wheel, of radius 0,9m, rolls a distance of 2m along the ground. Calculate, to the nearest degree the angle the wheel turns through. ($\pi = \frac{22}{7}$).

3. The area covered by lawn, in Fig 13.16, is the shaded part.

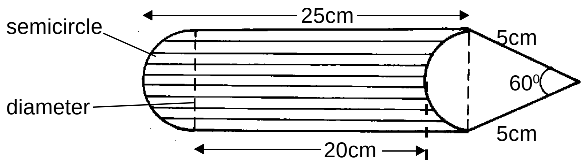


Fig 13.16

Calculate:
a) the shaded area.
b) the perimeter of the shaded shape.

4. In Fig 13.17, OA is parallel to CB.
OB is an arc of a circle with centre C.

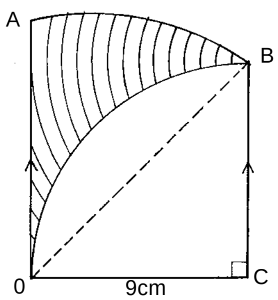


Fig. 13.17

AB is an arc of a circle with centre O. Taking $\pi = \frac{22}{7}$, find the area of the shaded region.

5. ABC and OEC are semi-circles in Fig 13.18. Given that semi-circle ABC has its centre as O and a diameter of 22cm, calculate the shaded area. Take π to be 3,142cm.

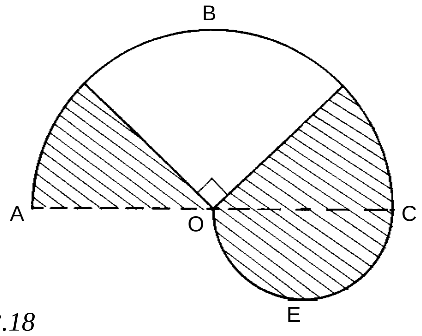


Fig 13.18

6. Part of the tiling pattern of a floor is shown in Fig 13.19. The pattern consists of equal circles and semi-circles, of the same radii as the circles. Calculate the area of the shaded region, given that the circle's diameter is equal to 14cm. Take $\pi = \frac{22}{7}$.

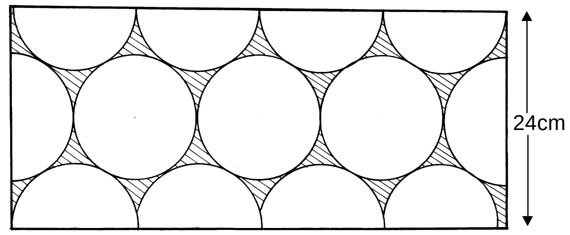


Fig. 13.19

7. The smaller radius(r) of the annulus is 3,5cm (Fig 13.20). Using $\pi = \frac{22}{7}$, calculate:
a) the area of the shaded part on the diagram.
b) the area of all the unshaded part of the diagram.
c) the perimeter of the shape labelled Fig 13.20.

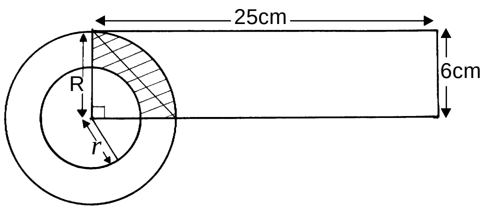
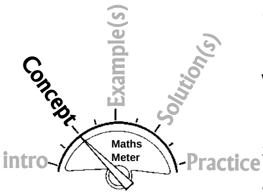


Fig 13.20

E. SURFACE AREA AND VOLUME OF SOLIDS/ (THREE DIMENSIONAL SHAPES)

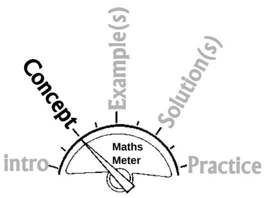
All plane shapes can easily be represented on a plane (flat) surface e.g the cartesian plane. They are always "glued" to a particular surface. They cannot be lifted alone without lifting the plane where they are "glued to". No wonder they are called plane shapes.

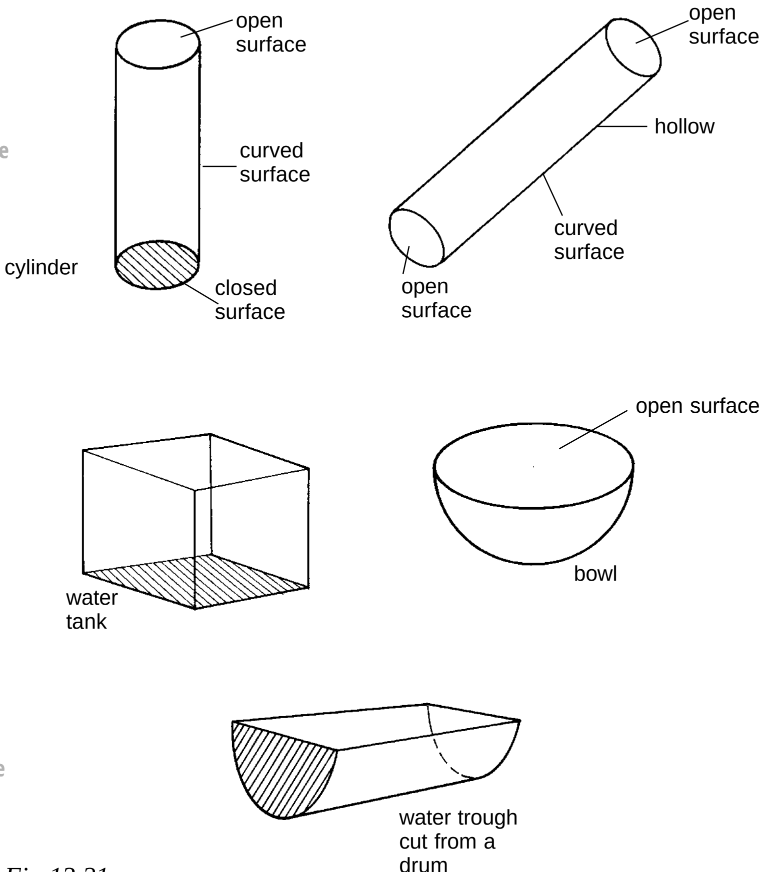


Solid (three dimensional) shapes have a third dimension called breadth (thickness). This height is measured from the surface where the object rests. The most defining characteristic of solids is that they occupy space and hence have volume (capacity). If small, they may be lifted and moved about in space without being attached to any resting surface. Examples of such objects are: football (sphere), cube, cuboid, pyramid, cylinder and many others (refer to Fig 13.22).

The following characteristics should be noted concerning three dimensional objects.

1. **Hollow object:** This refers to a three dimensional object which has free space (capacity) inside it. An opening may be made by removing one or more surfaces. Hollow objects may have the capacity to carry objects or allow objects to pass through them.(Fig 13.21)





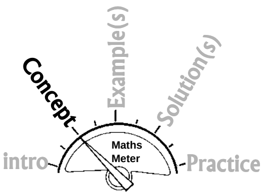
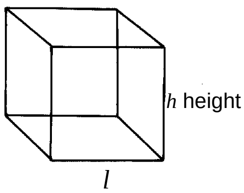


Fig 13.21

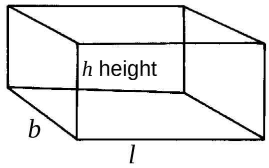
2. Solid Objects: Have no open surface.

a)



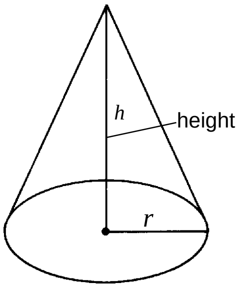
cube

b)



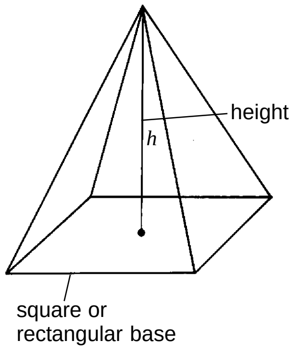
cuboid

c)



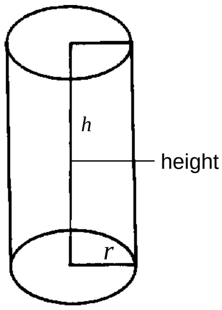
cone

d)



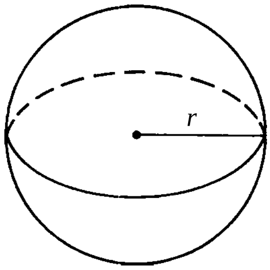
pyramid

e)



closed cylinder

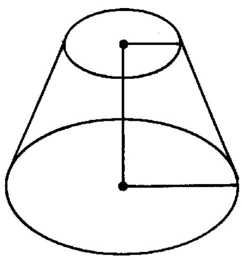
f)



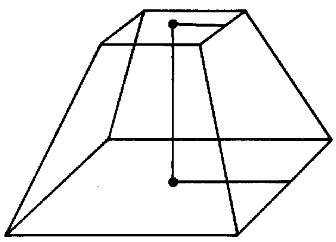
sphere

Fig 13.22

Frustum of a cone or pyramid – This is the solid figure remaining when a smaller, similar cone or pyramid is removed by a cut which is parallel to the base of the original shape.



frustrum of a cone



frustrum of a pyramid

The conical tower at Great Zimbabwe ruins.

Fig 13.23

Hint

Most solid shapes are called prisms. The shapes which are not prisms are: cone, pyramid, sphere frustum and semi-sphere.

Prisms: A prism is a solid which has a uniform cross-section. The cross-section is achieved by cutting the solid by a plane perpendicular to the horizontal. Usually the shape defined by the cross-section is used to define the prism. Fig 13.24.

These buildings are some of the recent marvels man has built. Can you identify which shapes were employed in the structures?

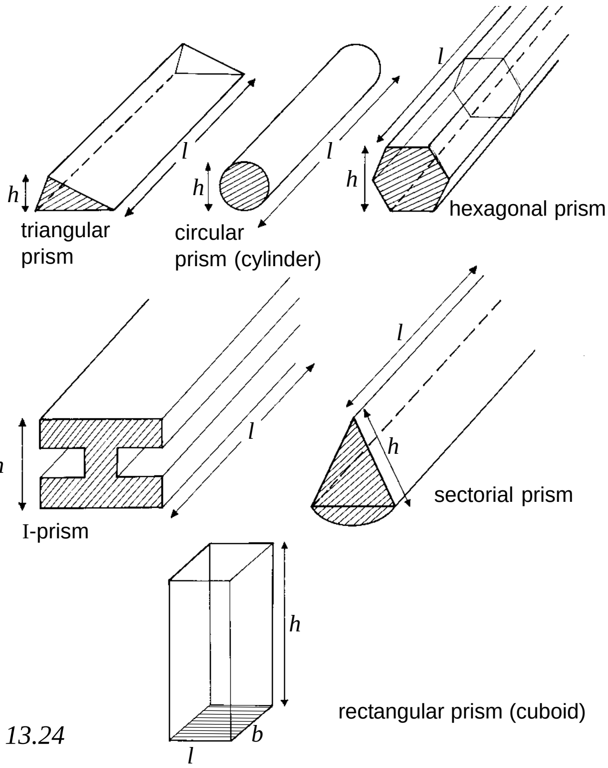


Fig 13.24

Hint

In a prism, the cross section (shaded portion) is the same throughout.
– when the prism is vertical, its length becomes its height.
Problems concerning solid objects require us to find:
(i) surface area of the solid.
(ii) volume of the solid.
(iii) mass of the solid.
(iv) density of the solid.

1. State whether the following objects are hollow or solid.
- a) a cup.
- b) a farm brick.
- c) a loaf of bread.
- d) a closed textbook.
- e) a swimming pool.
- f) a dip-tank.
- g) a water glass.

2. Draw the following prisms and shade the cross section in each case:
- a) a chalk duster.
- b) a round gum pole.
- c) a your ruler.
- d) a rectangular plank.
- e) a new box of chalk.

3. Draw the following shapes labelling the height in each case.
   a) cone.
   b) pyramid.
   c) solid hemisphere.
   d) frustrum of a cone or pyramid.
   e) cylinder.
   f) cube.
   g) cuboid.

F. CALCULATIONS INVOLVING SOLID AND HOLLOW OBJECTS

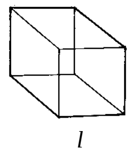
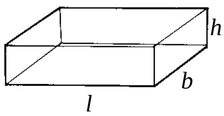
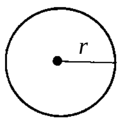
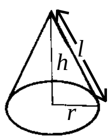
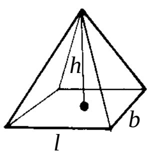
Hint

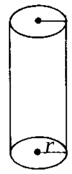
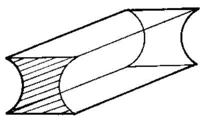
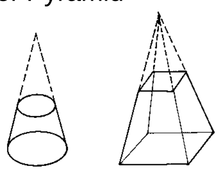
Hollow shapes have both external and internal surface areas whereas solid shapes only have external surface area. The formulas used to calculate the volume of solids are the same as those used to calculate the volume of the hollow shapes.

Solid objects and hollow objects have two things in common i.e. they both have **surface area** and **volume**.

Table 13.6 below gives a summary of formulae of areas and volumes of solid objects.

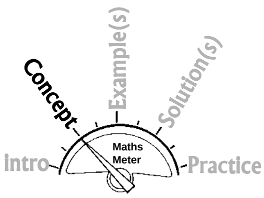
Table 13.6

Solid Object	**Surface Area**	**Volume**
Cube 	Sum of areas of faces = $6l^2$	l^3
Cuboid 	Sum of areas of faces $2lb + 2lh + 2hb$	$l \times b \times h$
Sphere 	$4\pi r^2$	$\frac{4}{3}\pi r^3$
Cone 	Curved surface area $= \pi rl$ Total surface area $= \pi rl + \pi r^2$	$\frac{1}{3}\pi r^2 h$
Pyramid 	Area of base + area of the four triangular faces	$\frac{1}{3}$ base area $\times$ height $\frac{1}{3}lbh$

Cylinder 	Curved surface area + Circular surface area A = $2\pi rh + 2\pi r^2$ A = $2\pi r(h + r)$	V = area of base × height Volume = $\pi r^2 h$
Prism 	Sum of areas of all faces	Area of cross section × length
Frustum of a Cone or Pyramid 	Area of large cone or pyramid – Area of smaller cone or pyramid	Volume of larger cone/pyramid – Volume of smaller cone/pyramid

Hint

Curved surface area = $2\pi rh$.
Area of cylinder with both ends open = $2\pi rh$.
Area of cylinder with one end closed = $2\pi rh + \pi r^2$.
Area of cylinder with both ends closed = $2\pi rh + 2\pi r^2$.



Consider the following examples:

1. A cylindrical drum, of height 1,5m, is half filled with water. Fig 13.25. Given that, its base radius is 35cm and $\pi = \frac{22}{7}$, calculate:

a) the volume of the water in m^3.

b) the internal surface area not occupied by water in m^2.

Hint

Most problems involving solids are easier to visualise if you sketch a diagram before you proceed. Then state the steps. Also remember that dimensions must be in the same unit (mm must not be multiplied with cm or m).

Solution

1. a)

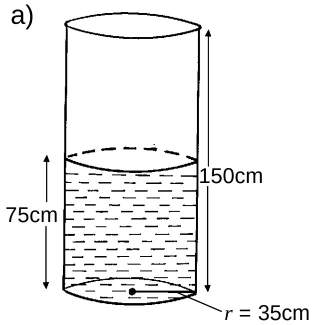
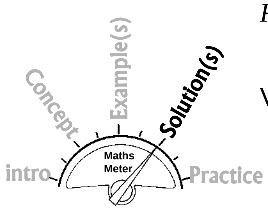


Fig. 13.25



Volume of water = base area × height

$= \pi r^2 \times 75\text{cm}$

$= \frac{22 \times \cancel{35}^{5} \times 35 \times 75}{\cancel{7}}$

= 288 750cm^3.

To change volume to m^3.

1m = 100cm

$(1m)^3 = (100cm)^3 \Rightarrow 1m^3 = 1\ 000\ 000cm^3$

$$\therefore 288\ 750cm^3 = \frac{288750cm^3}{1000\ 000} = 0{,}289m^3.$$

b) Internal surface area = curved surface area of half drum

$$\text{Internal surface area} = 2\pi r \times h$$
$$= 2 \times \frac{22}{\cancel{7}} \times \overset{5}{\cancel{35}} \times 75$$
$$= 16\ 500cm^2$$

To convert to m^2 = 100cm

$(1m)^2 = (100cm)^2$

$1m^2 = 10000cm^3$

$$\therefore 16\ 500cm^2 = \frac{16\ 500}{10\ 000} = 1{,}65m^2$$

2.

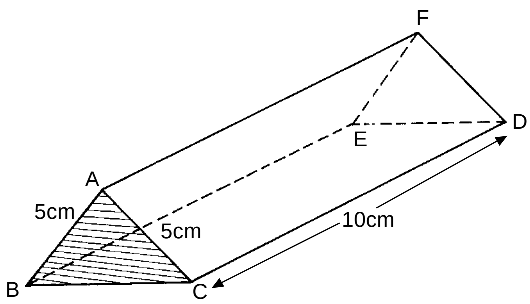


Fig 13.26

Fig 13.26 represents a solid wooden triangular prism. Calculate:

a) the volume of the prism.

b) the total surface area.

c) calculate the density of the prism, given that its mass = 2409g. $\left(d = \frac{m}{v}\right)$.

Solution

2. a) **Step 1:** Find the area of $\triangle$ABC.
Step 2: Find the volume of prism.

Step 1: $\triangle$ABC is an isosceles triangle so its perpendicular height bisects its base.

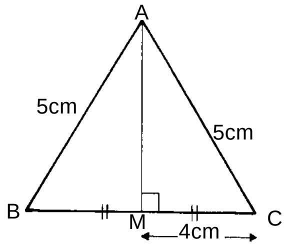


Fig 13.27

AMC is a right-angled triangle.
Using Pythagoras $AM^2 = 5^2 - 4^2$

$AM^2 = 9$

$AM = 3$

Area of $\triangle ABC = \frac{1}{2}$ base $\times$ height

$= \frac{1}{\cancel{2}_1} \times \cancel{8}^{4} \times 3$

$= 12cm^2$

Volume of prism = cross sectional area $\times$ length

$= 12cm^2 \times 10cm$

$= 120cm^3$

b) **Step 1:** Find the area of the two triangular faces.
Step 2: Find the area of the base.
Step 3: Find the area of the other two surfaces.
Step 4: Find the total surface area.

Step 1: Area of the two triangular faces

$= 2 \times 12cm^2$

$= 24cm^2$

Step 2: Area of rectangular base

$= l \times w$

$= 10 \times 8$

$= 80cm^2$

Hint
AFDC is a rectangle

Step 3: Area of AFDC $= l \times w$

$= 10cm \times 5$

$= 50cm^2$

Similarly ABEF $= 50cm^2$ (Congruent triangle –ASS)

$\therefore$ Sum area of the other two sides

$= (50 + 50)cm^2$

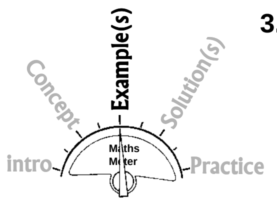
$= 100cm^2$

Total surface area of the prism $= (24 + 80 + 100)cm^2$

$= 204cm^2$

c) $\text{Density} = \dfrac{\text{Mass}}{\text{Volume}} = \dfrac{240\text{g}}{120\text{cm}^3}$

$\text{Density} = 2\text{g/cm}^3$

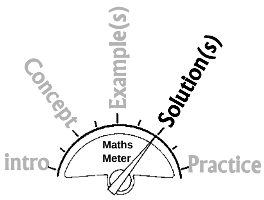


3. A factory receives its metal in cuboidal sheets of length 2m and width 40cm and thickness 10cm. If the metal is used to make solid pyramids each with a square base of length 10cm and height 15cm, calculate the following:

a) the volume of each pyramid.

b) the number of pyramids obtained from each metal sheet.

Solution



3. a) Make a sketch of the shape (Fig 13.28).

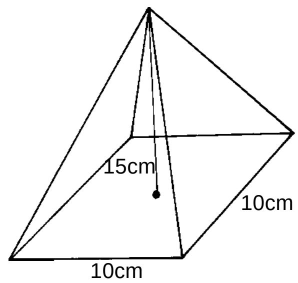


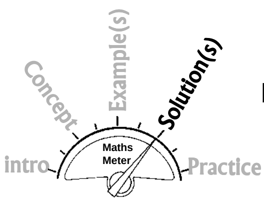
Fig 13.28

Volume of pyramid = $\frac{1}{3}$ base area $\times$ height

$= \dfrac{1}{3} \times 10^2 \times 15$

$= \dfrac{1}{\cancel{3}} \times 100 \times \overset{5}{\cancel{15}}$

$= 500\text{cm}^3$



b) Volume of the cuboidal sheets (Fig 13.29)

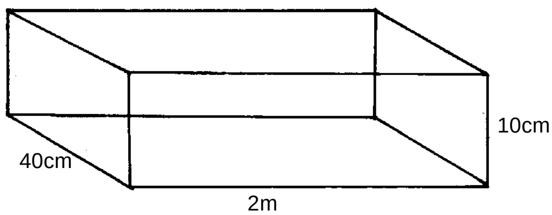


Fig 13.29

Volume of cuboid $= (40 \times 200 \times 10)\text{cm}^3$
$= 80\ 000\text{cm}^3$

$\therefore$ Number of pyramids obtained from each metal sheet $= \dfrac{80\ 00\cancel{0}}{50\cancel{0}}$
= 160 pyramids.

Common Error

Multiplying mixed units is a common mistake e.g. 40×10×2 without changing the 2m to cm.

4. A shape consists of a solid hemi-sphere attached to a solid cylinder as shown in the diagram (Fig 13. 30). The height of the cylinder is 25cm and the area of its base is 440m².

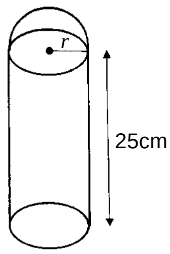


Fig 13.30

a) calculate the volume of the model.
(Take $\pi = \frac{22}{7}$)

Solution

4. **Step 1:** Find the radius of the base of the cylinder.
Step 2: Find the volume of the cylinder.
Step 3: Find the volume of the hemisphere.
Step 4: Find the volume of the model.

Step 1: Radius of base

Area of base = $44cm^2$

$\pi r^2 = 44cm^2$

$$\frac{22r^2}{7} = 44$$

$$r^2 = \overset{2}{\cancel{44}} \times \frac{7}{\cancel{22}}$$

$$r^2 = 14$$

$$r = \sqrt{14}$$

$$r = 3{,}742cm$$

Step 2: Volume of cylinder = cross-sectional area × height
= 44 × 25
= $1100cm^3$

Step 3: Volume of hemisphere = $\frac{1}{2}$ that of sphere.

Volume of hemisphere = $\frac{1}{2} \times \frac{4\pi r^3}{3}$

But Radius of sphere = Radius of base surface

$$\text{Volume} = \frac{1}{\cancel{2}} \times \frac{\cancel{4}}{3} \times \frac{22}{\cancel{7}} \times \overset{2}{\cancel{14}} \times 3{,}742$$

$= 109{,}76533cm^3$.

$$\text{Volume of the model} = \left(\begin{array}{l}\text{Volume}\\\text{of cylinder}\end{array}\right) + \left(\begin{array}{l}\text{Volume of}\\\text{hemisphere}\end{array}\right)$$

$$V = 1100\text{cm}^3 + 109{,}76533\text{cm}^3$$

$$V = 1209{,}7653\text{cm}^3$$

$$V = 1210\text{cm}^3$$

1. Use your ruler to measure the dimensions of this Mathematics textbook when it is closed. Find its approximate volume in cm^3.

2. Fig 13.31 represents a solid metal prism,
Using the dimensions given, find:
a) the volume of the object.
b) the total surface area.
c) its density, given that its mass is 40kg.

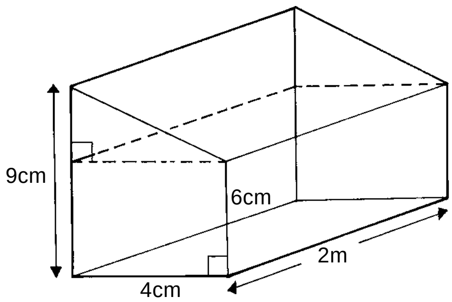


Fig 13.31

3. A pyramid with a height of 15cm has a square base of length 8cm. It stands with its square face in contact with the top face of a square prism vessel, of height 20cm. If the square base of the pyramid fits exactly on the square prism, calculate:
a) the volume of the solid prism.
b) the total surface area of the solid figure.

4.

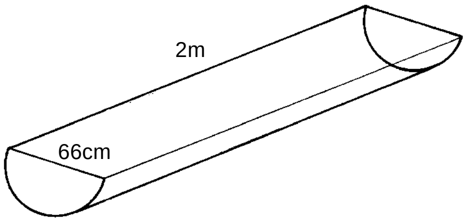


Fig 13.32

The above diagram (Fig 13.32), shows a water trough whose cross-section is a semi-circle. Using $\pi = \frac{22}{7}$, calculate:

a) the volume of the trough.

b) the total internal surface area of the trough.

5. From Fig 13.33 work out:

a) the total surface area.

b) the volume.

(Take $\pi = 3{,}14$)

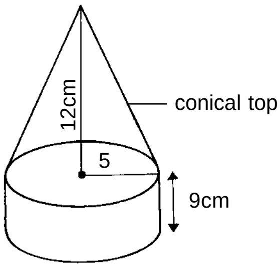


Fig 13.33

6. A thick hollow sewage pipe, with a uniform cross-sectional area of an annulus, is shown in Fig 13.34, take $\pi = 3{,}142$.

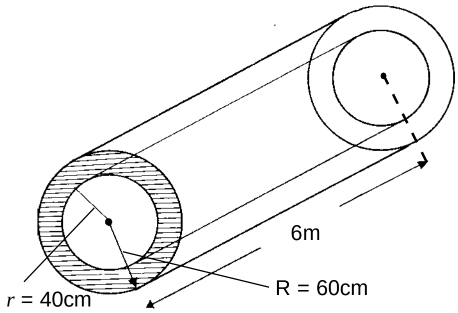


Fig. 13.34

a) Work out the volume of the material which was used to construct the pipe.

b) Given that its mass = 50kg, calculate the density of the pipe.

G. MORE ABOUT SOLID AND HOLLOW SHAPES

Study the examples below carefully:

1. A cuboid trough has a cylindrical metallic pole inside it, (Fig 13.35). If water is poured in, to fill the trough to the brim, calculate the volume of water in the trough ($\pi = \frac{22}{7}$).

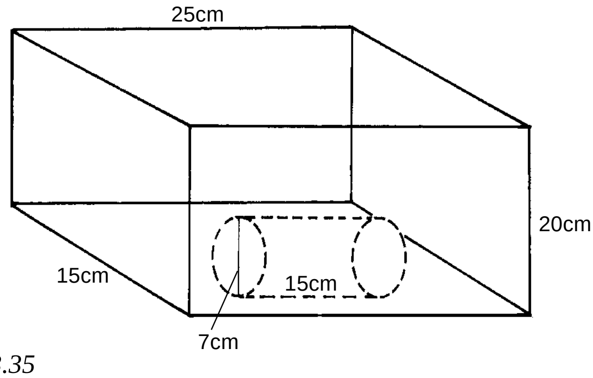


Fig 13.35

Step 1: Find the volume of the solid pole.
Step 2: Find the volume of the trough.
Step 3: Find the volume of water.

Solution

1. **Step 1:** Volume of the solid pole $= \pi r^2 h$

$$= \frac{\overset{11}{\cancel{22}}}{\cancel{7}} \times \cancel{3,5} \times 3,5 \times 15$$

$$= 577,5\text{cm}^3$$

Step 2: Volume of the cuboid $= l \times b \times h$

$$= 15 \times 25 \times 20$$
$$= 7500\text{cm}^3$$

Step 3: Volume of water $= \begin{pmatrix}\text{Volume of} \\ \text{trough}\end{pmatrix} - \begin{pmatrix}\text{Volume of} \\ \text{solid pole}\end{pmatrix}$

Volume of water $= 7500 - 577,5$
$= 6922,5\text{cm}^3$

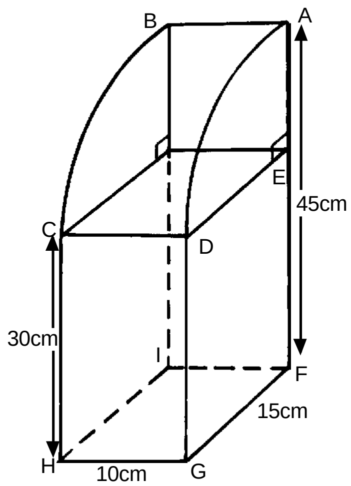


Fig. 13.36

2. Fig 13.36 is a diagram of a hollow plastic sweet container, CB and DA are arcs of quadrants of a circle,
   a) calculate the area of sector DAE.
   b) calculate the area of face GDAF.
   c) find the volume of the sheet container.
   ($\pi = \frac{22}{7}$)

Solution

2. a) Sketch the sector

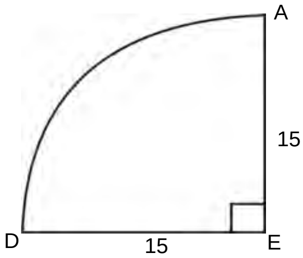


Fig 13.36a)

$$\text{Area of sector ADE} = \frac{90^0}{360^0} \times \pi r^2$$
$$= \frac{1}{4} \times \frac{22}{7} \times 15 \times 15$$
$$= 176{,}79$$

b) Area of GDAF = Area of rectangle + Area of sector

$= (30 \times 15)\text{cm}^2 + 176{,}79\text{cm}^2$

$= 626{,}79\text{cm}^2$

c) Volume of the plastic container = face area $\times$ height

$\therefore$ Volume $= 626{,}79 \times 10$
$= 6267{,}9 \Rightarrow 6\ 268\text{cm}^3$

Hint
The "height" is the length GH.

3. Fig 13.37 shows a frustum of a pyramid. Top and bottom square bases being of lengths 6cm and 10cm, respectively. Find the volume of the frustum, given that its depth is 5cm.

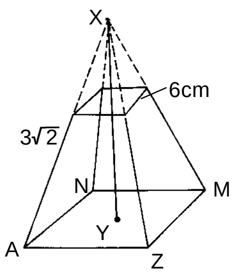


Fig. 13.37

Solution

3. Extract △ NZM from the square base of the sketch. Also extract triangle XYZ (Fig 13.38).

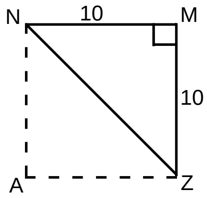


Fig. 13.38(a)

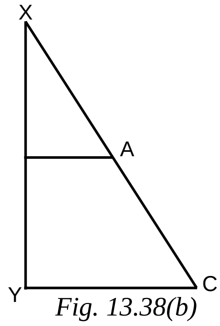


Fig. 13.38(b)

$$NZ^2 = 10^2 + 10^2$$
$$NZ^2 = 200$$
$$NZ^2 = \sqrt{200}$$
$$\therefore YZ = \frac{\sqrt{200}}{2} = 10\sqrt{2} = 5\sqrt{2}$$

For the smaller square, $\frac{1}{2}$ of the diagonal $= 3\sqrt{2}$

△ XBA and △ XYZ are similar

∴ Ratio of corresponding sides are equal

$$\frac{5 + x}{x} = \frac{5\cancel{\sqrt{2}}}{3\cancel{\sqrt{2}}}$$

$$15 + 3x = 5x$$
$$15 = 5x - 3x$$
$$15 = 2x$$
$$7{,}5 = x$$

∴ Height XY $= 7{,}5 + 5$
$= 12{,}5$cm

Volume of frustum $= \begin{pmatrix}\text{Volume of larger pyramid} - \text{Volume of smaller pyramid}\end{pmatrix}$

$$= \tfrac{1}{3}\, l^2 \times h - \tfrac{1}{3}\, l^2 h$$

$$= \frac{1}{3} \times 10^2 \times 12{,}5 - \frac{1}{3} \times 6^2 \times 7{,}5$$

$$\text{Volume} = \left(\frac{1}{3} \times 100 \times 12{,}5\right) - \left(\frac{1}{3} \times 36 \times 7{,}5\right)$$

$$\text{Volume} = 416{,}\ 67 - 90$$

$$= 326{,}67\text{cm}^3$$

The above example was done without writing steps – was it easy to follow?

Now re-do it, using the following steps:

Step 1: Find the diagonal of the bigger square using Pythagoras' Theorem.

Step 2: Find the diagonal of the smaller square using Pythagoras' Theorem.

Step 3: Extract triangles XBA and XYZ and find YZ and BA as half the diagonals.

Step 4: Find the height of triangle XYZ.

Step 5: Use the formula to find the volume of the frustum.

1. A swimming pool, viewed from one side, has the shape of a prism whose cross-section is a trapezium (Fig 13.39).

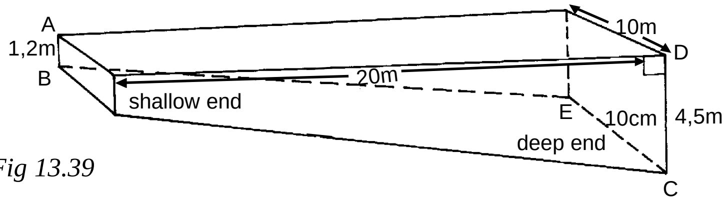


Fig 13.39

Calculate the following:

a) the cross-section area ABCD, (which is a trapezium).

b) the volume of the pool.

c) the length of BC.

d) the area of the pool's surface which is in contact with the water.

2. A **solid** cone of radius 12cm and height 25cm, stands inside a cylinder. The cone's circular base is in contact with the base of the cylindrical vessel, which has a radius which has a radius of 12cm (Fig 13.40). Water is poured into the vessel until the vessel is full.

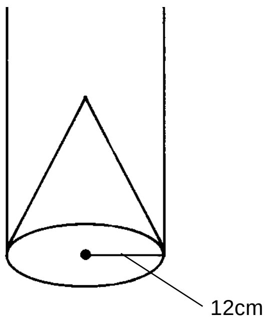


Fig. 13.40

Find:

a) the volume of the cone which is submerged.

b) the surface area of the cone which is in contact with the water.

c) the new height reached by the water in the cylinder, if the cone was to be removed.

3. A spherical clay pot, with an internal radius of 14cm, is filled with water to about $\frac{3}{4}$ of its capacity (Fig 13.41.)

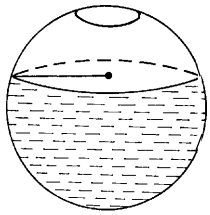


Fig 13.41

The horizontal surface of water has a radius of 10cm.

a) Work out the height of the water surface above the centre of the clay pot.

b) The depth of the water in the clay pot. Take $\pi = \frac{22}{7}$

4. The volume of a cylindrical jar, of base radius 6cm and height 10cm, is twice that of a hermispherical bowl. Calculate the internal radius of the bowl. Take $\pi = 3{,}14$.

5. A frustum of a cone has the top radius = 3cm and a bottom radius = 7cm. Given that the depth of the frustum is 5cm, find the volume of the frustrum.

SUMMARY

1. Quantities are scientific terms used to express concepts.

2. Most quantities are associated with a symbol, unit, formula to calculate it and/or an instrument to measure it.

3. The System International (SI) units use multiples and sub-multiples of 10 so they can easily be converted from larger to smaller units or vice versa.

4. In mensuration of shapes the three quantities most often involved are:
   a) perimeter.
   b) area.
   c) volume.

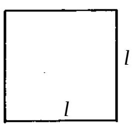
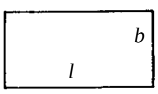
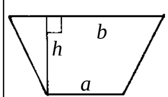
5. Table 13.7 below gives a summary of the formulae used to calculate these three quantities.

6. Shapes may be grouped into plane shapes and (two dimensional shapes) three dimensional shapes.

7. The three dimensional shapes are further divided into hollow and solid objects.

8. A prism is a solid object with a uniform cross-sectional area.

Table 13.7 **Plane Shapes**

PLANE SHAPES	**Formula for perimeter**	**Formula for area**	**Formula for volume**
 square	P = $4l$	A = l^2	not applicable
 rectangle	P = $2(l + b)$	A = lb	not applicable
 trapezium	–	A = $\frac{1}{2}(a+b)h$	not applicable

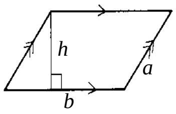
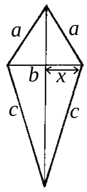
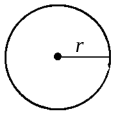
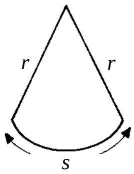
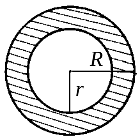
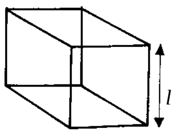
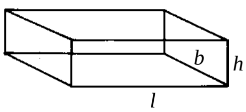
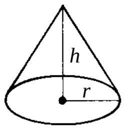
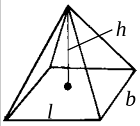
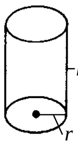
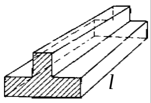
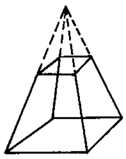
 parallelogram	$2(a+b)$	$A = bh$	not applicable
 kite	$2(a+c)$	$A = \frac{1}{2}xb$	not applicable
 circle	$C = 2\pi r$ or πd	$A = \pi r^2$	not applicable
 cone	$P = 2r+s)$	$\frac{\theta}{360} \times \pi r^2$	not applicable
 ring	$C = 2\pi r$	$A = \pi(R^2 - r^2)$	not applicable

Table 13.8 **Solid Shapes**

3-DIMENSIONAL SHAPES	**Formula for for perimeter**	**Surface area**	**Volume**
 cube	not applicable	$A = 6l^2$	$V = l^3$
 cuboid	not applicable	$2(lb + lh + bh)$	$V = lbh$
 sphere	not applicable	$C = 4\pi r^2$	$V = \frac{4}{3}\pi r^3$

 cone	not applicable	$A = \pi r h$	$V = \frac{1}{3}\pi r^2 h$
 pyramid	not applicable	A = Area of base + area of the triangular faces	V = base area × perpendicular height $V = \frac{1}{3} lbh$ (for a rectangular pyramid)
 cylinder	not applicable	Curved area $= 2\pi r h$ Total surface area $2\pi r h + 2\pi r^2$ $A = 2\pi r(h+r)$	$V = \pi r^2 h$
 prism	not applicable	Sum of areas of the faces	V=area of cross section × length
 frustum of a pyramid	not applicable	Area of larger cone or pyramid – Area of smaller upper cone or pyramid.	Volume of large cone or pyramid – Volume of smaller upper cone or pyramid

Concepts from this chapter may be summarised using the flowchart below.

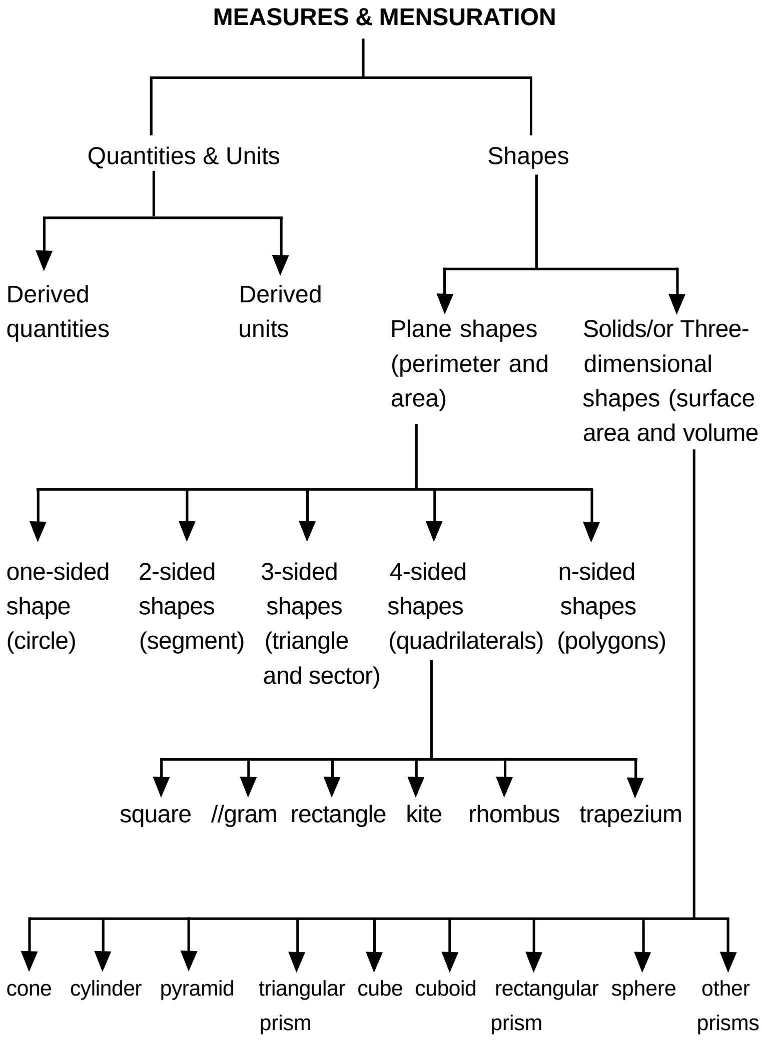


Fig. 13.42 Flowchart

EXAM PRACTICE 13

Intro Concept Example(s) Solution(s) Practice

The following questions may help you master skills taught in this chapter.

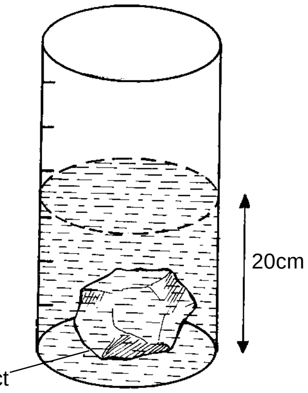


Fig 13.43

Intro Concept Example(s) Solution(s) Practice

1. A solid metal object is completely immersed in water contained in a vertical measuring cylinder as shown in Fig 13.43. The volume of water measured after the object was immersed was $880cm^3$. The solid object was then removed from the water. Presuming all the water remained in the cylinder and the new volume of water in thecylinder was found to be $660cm^3$. Taking $\pi = \frac{22}{7}$

a) Calculate the height of the water after the solid object was removed.

b) Given that the density of the solid object is $2g/cm^3$, calculate its mass.

Solution

Hint

Write down the steps to follow.

1. a) **Step 1:** Find the radius of the cylinder.

Step 2: Find the new height.

Volume of water before solid is removed $= 880cm^3$

$$\therefore \quad \pi r^2 h = 880cm^3$$

$$h = 20cm$$

$$\therefore \quad \pi r^2 = \frac{880}{h}cm^3$$

$$\frac{22r^2}{7} = \frac{88}{24}$$

$$r^2 = \frac{880}{20} \times \frac{7}{22}$$

$$r^2 = 14$$

$$r = \sqrt{14}$$

Intro Concept Example(s) Solution(s) Practice

Step 2: $\pi r^2 h = 660$

$$\frac{22}{7} \times 14h = 660$$

$$h = \frac{\cancel{660} \times \cancel{7}}{\cancel{22} \times \underset{2}{\cancel{14}}}$$

$$h = 15\text{cm}$$

b) Volume of the solid object

$= (880 - 660)\text{cm}^3$

$= 200\text{cm}^3$

$$\text{Density} = \frac{\text{Mass}}{\text{Volume}}$$

Mass $=$ Density $\times$ Volume

$= 2\text{g/cm}^3 \times 200\text{cm}^3$

Mass $= 400\text{g}$

Now do the following:

1.

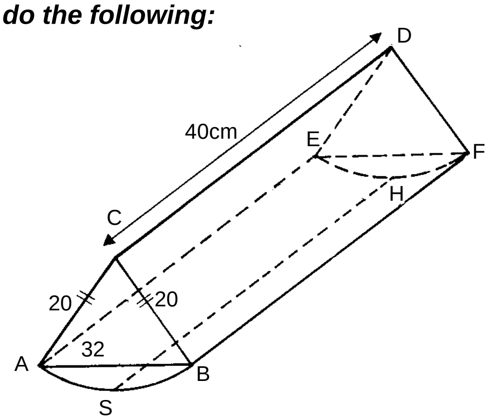


Fig 13.44

Take π to be 3,142.

Fig 13.44 represents a metallic block in the form of a prism. AB and EF are diameters of the semi-circles. Using the dimensions given on the diagram, calculate:

a) the area of the semi-circle, AGB.
b) the volume of the metallic block.
c) the mass of the block, in kilograms, given that its density is $1{,}6\text{g/cm}^3$.

2.

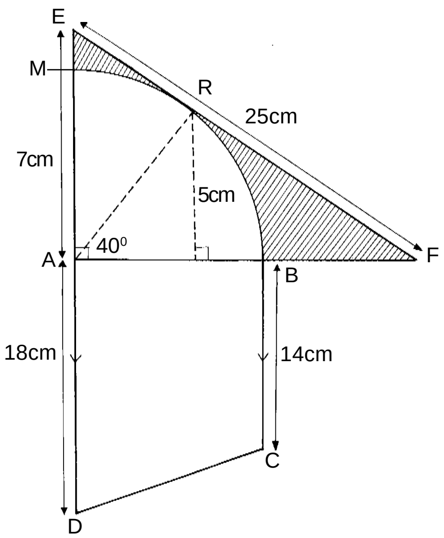


Fig. 13.45

Fig 13.45 consists of a trapezium ABCD, sector AMRB and a right-angled triangle, EAF. Using the given dimensions on the diagram. (Taking $\pi = \frac{22}{7}$),
work out:

a) the area of a triangle AEF.
b) the area of sector AMRB.
c) the area of shape DMRBC.
d) area of the shaded space BRF.

3.

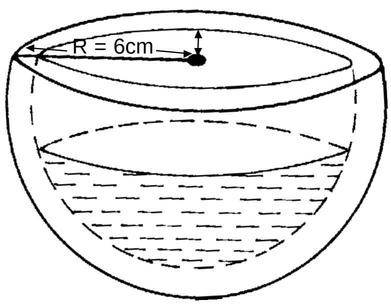


Fig 13.46

A hemispherical shaped vessel has a thickness of $x$cm. (Fig 13.46).
It is given that the external radius, R = 6cm and the internal curved surface area is 121cm^2. The water in the vessel is half the capacity of the internal hemisphere.
Calculate:

a) the internal radius of the vessel.
b) the volume of the material which makes up the vessel.
c) the volume of the water contained in the vessel.
d) the density of the material which makes up the vessel, given that its mass, without water, is 100 grams.

(Taking $\pi = \frac{22}{7}$) Volume sphere = $\frac{4}{3}\pi r^3$, Area of sphere = $4\pi r^2$.

4. Three holes of radius 2cm each are drilled through a cylindrical prism. (Fig 13.47). The values of the cross section is 7cm.

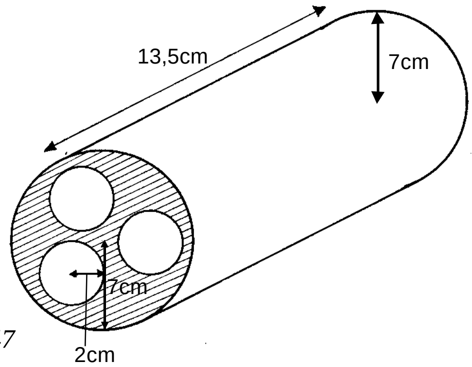


Fig.13.47

The cylindrical prism is 13,5cm long. The material drilled from the prism is used to make solid conical objects, all of base radius 2cm and height 5cm. Taking $\pi = \frac{22}{7}$,
work out:

a) the volume of the solid cylinder after drilling through all the holes have been drilled.

b) the number of cones made from the material from all three drilled holes.

5.

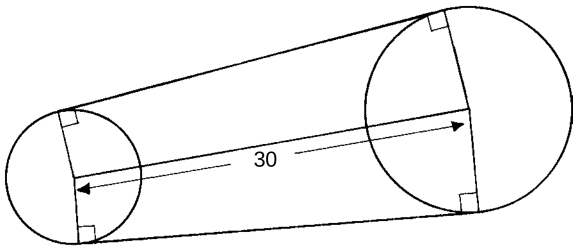


Fig 13.48

Fig 13.48 shows a belt passing round two pulley wheels of radii 14cm and 21cm whose centres are 30cm apart. Calculate the length of the belt. Take $\pi = \frac{22}{7}$

6.

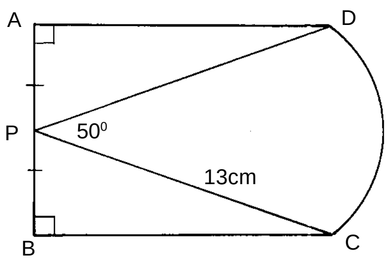


Fig. 13.49

In Fig 13.49, AD = BC and DC is an arc of a circle, radius 13cm, which subtends an angle of 50^0 at the mid-point of AB, at point P. Find:

a) the perimeter of the figure.

b) the area of the figure. Take $\pi = \frac{22}{7}$.

7. A hollow pipe has a cross sectional area that resembles a kite (Fig 13.50). The internal kite has the following dimensions:
length of shorter diagonal = 16cm.
length of longer diagonal = 21cm.
length of shorter side = 10cm.
length of longer side = 17cm.

The external kite has the following dimensions:
length of shorter diagonal = 32cm.
length of longer diagonal = 42cm.
length of shorter side = 20cm.
length of longer side = 34cm.

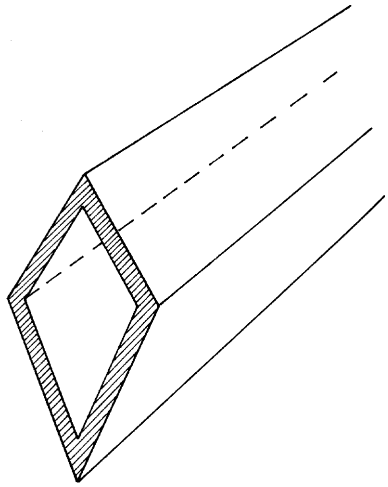
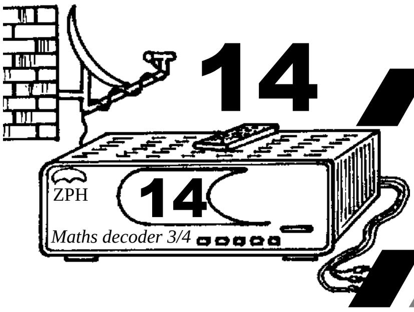


Fig 13.50

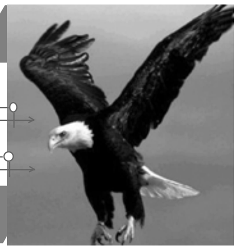
Given that the hollow pipe is 2,5cm long, calculate the volume of material used to make the pipe, giving your answer in m^3.

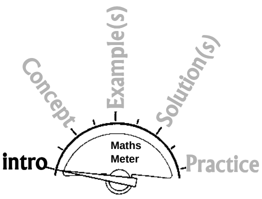


14 Inequalities

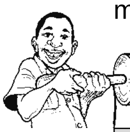
a, x, b, y, c, 1, 2, 3, 4, 5, 6 $\quad ax + b > c$

$> \ \ngtr \ < \ \nless \ \geqslant \ \ngeqslant \ \leqslant \ \nleqslant \ = \ \neq$





The term inequalities implies things are not equal. When a equals b, we say $a = b$.
$a \neq b$ means a is not equal to b. This gives us two possibilities.
If a is not equal to b, then either a is greater than b ($a > b$) or a is less than b ($a < b$). This chapter also introduces other signs like: a $\ngtr b$ meaning a is not greater than b implying $a \leqslant b$, or $a = b$. Also $a \nless b$ meaning a is not less than b, implying $a \geqslant b$ or $a = b$. $a \geqslant b$ means that a takes values that are equal to b and greater. $a \leqslant b$ means that a takes values that are equal to b and less. The meanings of $\nleqslant$ and $\ngeqslant$ will be discussed later.



Syllabus Expectations

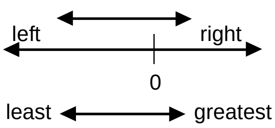
By the end of this chapter, students should be able to:

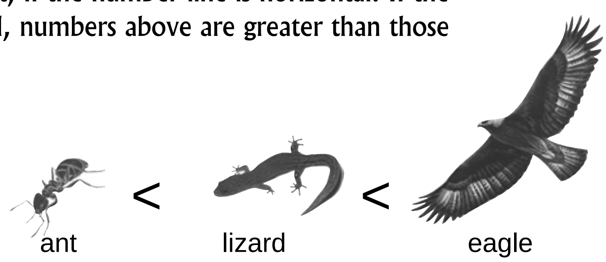
1. use the following signs in appropriate situations $=, >, \ngtr, <, \nless, \geqslant, \ngeqslant, \leqslant, \nleqslant, \neq$.
2. solve linear inequalities in the form $ax + b > c$ and/or $c < ax + b < d$ where a, b, c, d are rational numbers.
3. represent inequalities and their solution sets on a number line and/or a cartesian plane.
4. use simple linear programming methods to solve problems.

ASSUMED KNOWLEDGE

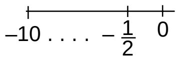
In order to tackle work in this chapter it is assumed that students are able to:

- appreciate that all numbers are on a number line and are arranged in order of size i.e. numbers to the left are smaller than those to the right, if the number line is horizontal. If the number line is vertical, numbers above are greater than those below them.





know that, on a horizontal line, negatives are to the left of zero whilst on a vertical line the negatives are below zero.
This knowledge helps to compare numbers and correctly place them on a diagram. Consider $-\frac{1}{2}$ and -10. People tend to see the higher number and think -10 is greater than $-\frac{1}{2}$. By putting these on a number line we can see this is incorrect.



-10 is to the left of $-\frac{1}{2}$ which makes $-\frac{1}{2}$ much greater.

A. SOLUTION SETS AND THE NUMBER LINE

Consider the following examples:

1. Give the solution sets of the following, given that x is an integer.

a) $x > 2\frac{1}{2}$ b) $x \leqslant -2$

c) $x \nless 3$ d) $-3 < x \leqslant 3$

Solution

NB. Unlike an equation, x will have several values.

1. a) $x > 2\frac{1}{2}$, values of x should be greater than $2\frac{1}{2}$.
$\therefore\ x = \{3, 4, 5, 6, \ldots\}$ ***Notice that*** the set starts from 3 since it is the first integer after $2\frac{1}{2}$.

b) $x \leqslant -2$
$x = \{-2; -3; -4; -5; ...\}$
or $x = \{...-5; -4; -3; -2\}$

The inequality sign here reads 'less than or equal to; hence the values start from the –2.

c) $x \nless 3$
$x \geqslant 3$
$x = \{3; 4; 5; 6; 7;...\}$

This reads x is not less than 3, meaning either x is equal to 3 or it is bigger than 3. ($x \geqslant 3$)

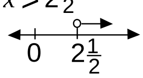
d) $-3 < x \leqslant 3$
$x = \{-2; -1; 0; 1; 2; 3\}$

Do you appreciate why –3 is not a member of the set whilst 3 is a member? What type of sets are a, b and c? What about d?

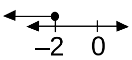
2. Illustrate the solution sets of (1) on a number line.

Solution

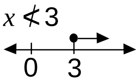
a) $x > 2\frac{1}{2}$



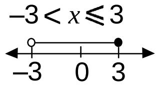
b) $x \leqslant -2$



c) $x \nless 3$

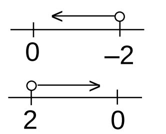


d) $-3 < x \leqslant 3$

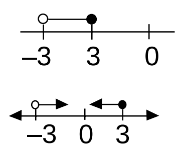


Common Errors

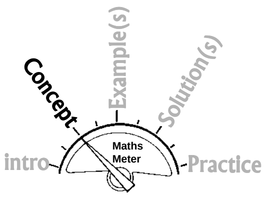
b)



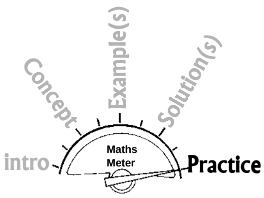
d)



Make sure numbers are on the correct side of zero.

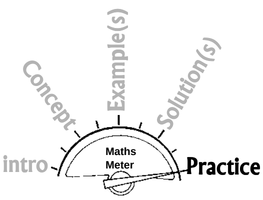


Notice that:
- an open circle is drawn above 2 and –3 to mean these numbers are not members of the solution set.
- a shaded circle means the number being referred to is a member of the solution set.
- the arrow in a, b and c shows that the solution set is an infinite set going in the direction the arrow is pointing.
- the zero is a reference point for positioning the rest of the numbers.

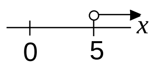


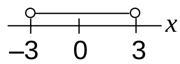
1. (i) List the members of the solution set of each of the following, taking x to be an integer.
(ii) Represent each solution set on a number line.

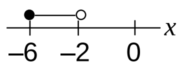
a) $x < -1$
b) $x \geqslant -1$
c) $x < 3$
d) $x \ngeqslant 20$
e) $-5 < x < 10$
f) $-10 < x \leqslant -5$
g) $-7 \leqslant x \leqslant 0$
h) $x \nless -21$
i) $x \ngtr -5$
j) $-3 \leqslant x < 4\frac{1}{2}$
k) $-3,8 < x < 6,5$
l) $-10\frac{1}{3} < x < -3\frac{1}{4}$

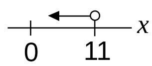


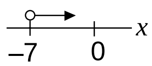
2. The following are solution sets given on a number line. Give the inequalities in x.

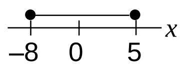
a) 

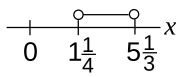
b) 

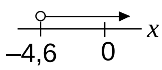
c) 

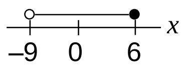
d) 

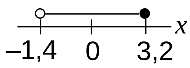
e) 

f) 

g) 

h) 

i) 

j) 

B. SOLVING INEQUALITIES

1. Solve the following and illustrate the solution sets on number lines.

a) $2x - 3 < 5$ b) $-11 \leqslant 3x - 2 < 4$

Solution

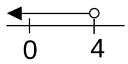
The rules for solving equations, apply in solving inequalities.

1. a) $2x - 3 < 5$

$2x < 5 + 3$ change side and sign on –3

$2x < 8$ divide both sides by 2

x 4

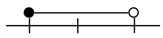


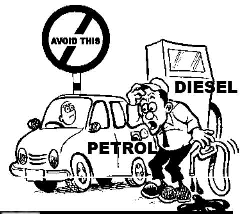
b) These are two inequalities joined at $3x - 2$

i.e. $-11 \leqslant 3x - 2$	and	$3x - 2 < 4$
$-11 + 2 \leqslant 3x$		$3x < 4 + 2$
$-9 \leqslant 3x$		$3x < 6$
$-3 \leqslant x$		$x < 2$

Now join the two solutions into one

$-3 \leqslant x < 2$

i.e.  .



Common Errors

* It is a common mistake to change the inequality sign to an equal sign thereby changing the question to an equation.

Hint

List the values in the problem.
i.e. x = 4, –3, –2, –1, 0, 1, 2 and y = 6, 7, 8, 9, 10, 11, 12.

2. Given that $-4 \leqslant x \leqslant 2$ and $6 \leqslant y \leqslant 12$, find

a) the greatest value of $y - x$

b) the least value of xy

c) the least value of $x^2 + y^2$

Solutions

2. a) Greatest $y - x$
$12 - (-4)$
$= 12 + 4$
$= 16$

For the value to be greatest, y must be the biggest whilst x is the smallest. Take these from the list of values given in the hint box.

b) Least $x\,y$
i.e. $-4(12)$
$= -48$

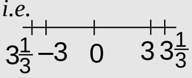
Since x has negatives, the biggest negative is the least.

c) Least $x^2 + y^2$
$= 0^2 + 6^2$
$= 36$

Both x^2 and y^2 must be smallest. The smallest x^2 is 0 not –16. Remember x^2 when $x = -4$ is $(-4)^2 = 16$ not $-4^2 = -16$.

Hint

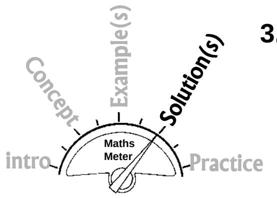
Draw a rough number line and place the given boundaries on it.
i.e.



3. List the integer values of x which satisfy the following sets:

a) $\{x: -3\frac{1}{3} < x < 3\frac{1}{3}\}$ b) $\{x: x \text{ is even and } 10 < x < 21\}$

Solution



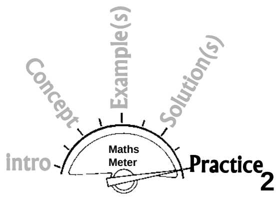
3. a) The sketch shows that $x = \{-3; -2; -1; 0; 1; 2; 3\}$ with or without the set brackets.

b) Here two conditions are given, $\{x{:}x$ is even and $10 < x < 21\}$

(i) x is even

(ii) x is between 10 and 21.

Thus $x = 12, 14, 16, 18, 20$. Note that 10 and 21 cannot be members of this set. Check the inequality sign being used.



1. Solve the following inequalities:

a) $x + 3 \geqslant 5$ b) $7 - x < 9$

c) $5x - 20 \leqslant 0$ d) $-4x < 24$

e) $6x - 8 \geqslant x + 7$ f) $8x - (2x - 7) \geqslant 37$

g) $5(x - 4) < 2(2x + 11)$ h) $\frac{x+8}{7} - 1 < \frac{2x-4}{3}$

2 Represent each of the solution sets in question 1 on a number line.

3. x is such that $2x + 5 < 11$ and $5x \geqslant 2x - 9$
Find the range of values of x which satisfy both inequalities.

4. List the integer values of x which satisfy the following sets.

a) $\{x: -4\frac{1}{2} < x < 2\frac{2}{3}\}$

b) $\{x: -5 \leqslant x < -1\frac{3}{4}\}$

c) $\{x: x$ is a multiple of 7 and $40 < x < 69\}$

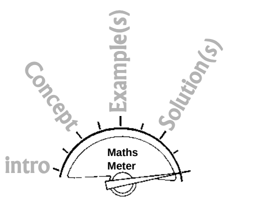
5. If $-6 \leqslant x \leqslant 4$ and x is an integer, for the following expressions find:

a) $7 - x$ b) $x - 7$ c) $9x^2$

d) $2 + 3x$

(i) the smallest and,

(ii) the largest possible values.



Practice

6. If $-2 \leqslant x \leqslant 5$ and $-4 \leqslant y \leqslant 3$, x and y being integers, find:

a) the minimum value of $x - y$.

b) the maximum value of $x^2 + y^2$.

c) the least value of xy.

d) the greatest value of $(xy)^2$.

7. x is a perfect square which satisfies both $2 - x < 3x - 10$ and $x - 17 \leqslant 32$. Find all the possible values of x.

8. x is such that $\frac{x}{2} - \frac{3}{4} < \frac{5x}{6} + \frac{7}{12}$ and $\frac{2x}{5} \leqslant 7 - x$.

Find the values of x given that x is an odd whole number.

C. INEQUALITIES IN THE CARTESIAN PLANE

Remember the cartesian plane is made up of an x and a y axis. In a cartesian plane vertical lines are $x = a$ and the horizontal lines are $y = a$ where a is rational i.e. the y-axis is $x = 0$ and the x-axis is $y = 0$. We can represent inequalities on the cartesian plane, but in this case, the solution is not numerical but an area. The area can be shown by shading the unwanted region so that the unshaded area is the wanted region.

Shading on vertical or horizontal lines

Consider the following examples

1. Show, by shading the **UNWANTED** region, the region being represented by the following inequalities.

a) $x \leqslant 4$ b) $x < 4$

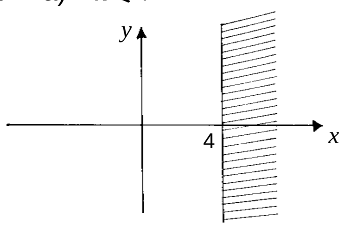
c) $y > -3$ d) $-5 < y \leqslant -3$

Hint

Draw the graph of x=4 (i.e. a vertical line passing through 4 of the x-axis)

Solution

1. a) $x \leqslant 4$



The line $x = 4$ is drawn as a solid or continuous line because of the 'equal to' in the inequality. The unshaded region is the one being described by the inequality.

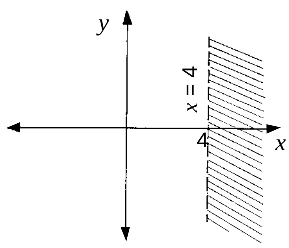
The difference between the diagram for (a) and the diagram for (b) is on the line $x = 4$. In b), the line is broken since the inequality does not include 4.

Hint

Draw a broken boundary for >, < and a solid boundary for ⩽ ⩾ .

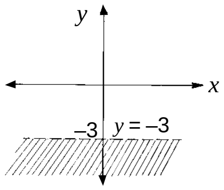
i.e.

b) $x < 4$

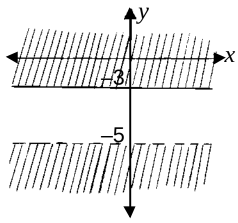


c) $y > -3$

This time the boundary is horizontal (i.e. $y = -3$) and broken



d) $-5 < y \leqslant -3$



Notice that the shadings in a), b) and c) indicate infinite sets. The (d) shading shows definite values for y but the x coordinates are infinite.

2. Show by shading the unwanted regions the area represented by the following sets:

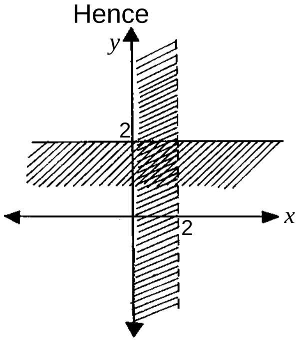
a) $\{(x,y): x > 2; y \leqslant 2\}$

b) $\{(x,y): x \leqslant -2; y < -1\}$

Solutions

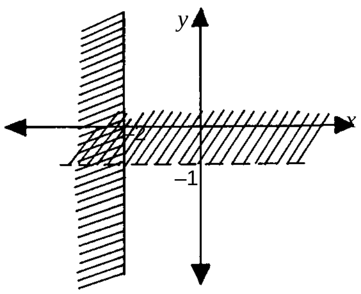
2. a) Set (a) reads the set of points (x,y) such that x is less than 2 and y is greater than or equal to 2.

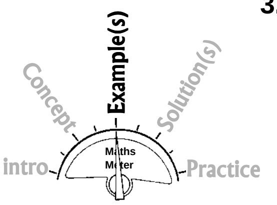
Hence



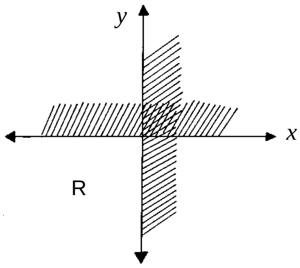
Notice that the boundaries $x = 2$ and $y = 2$ come from the respective inequalities. *Do you see why one line is solid and the other one broken?*

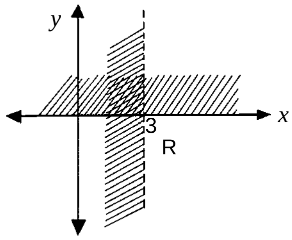
b) $\{(x,y): x \geqslant -2; y < -1\}$

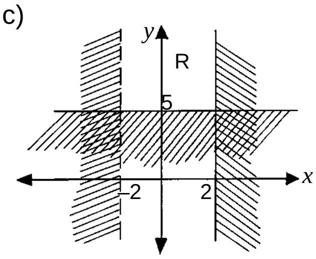


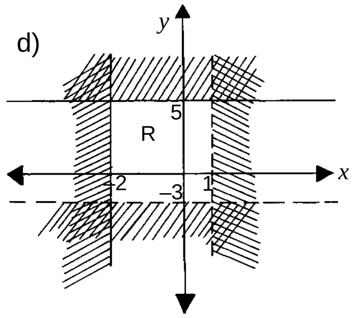


3. Describe, in set notation, the unshaded regions labelled R. Give the set of points represented by the following diagrams:

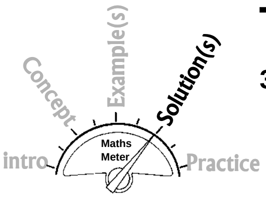
a)


b)


c)


d)


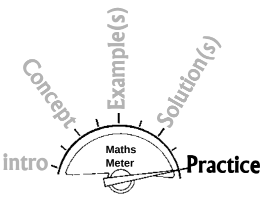
Solutions



3. a) $\{(x, y): x \leqslant 0;\ y \leqslant 0\}$ b) $\{(x,y): x > 3,\ y \leqslant 0\}$

c) $\{(x,y): -1 < x \leqslant 2;\ y \geqslant 5\}$ d) $\{(x,y): -2 \leqslant x < 7,\ -3 < y \leqslant 5\}$

Study these above answers carefully and see how simple the diagrams are.



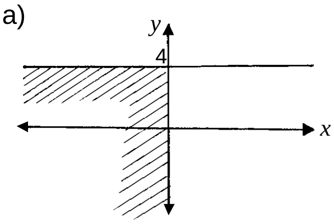
1. Show by shading the unwanted regions, the region represented by the following sets:

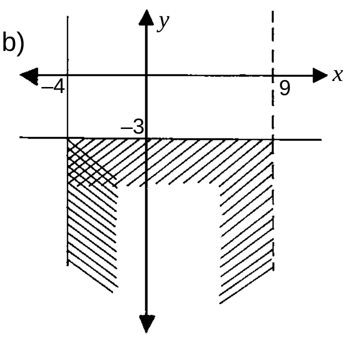
a) $\{(x,y): x \geqslant 0,\ y \leqslant 4\}$ b) $\{(x,y): x < 5,\ y \geqslant -5\}$

c) $\{(x,y): 6 \leqslant x < 11,\ y < -3\}$ d) $\{(x,y): -3 \leqslant x \leqslant 8,\ -4 \leqslant y < 2\}$

e) $\{(x,y): -2 < x,\ 7 \leqslant y \leqslant 13\}$ f) $\{(x,y): x \leqslant 2,\ -1 < y < 5\}$

2. Give the sets describe the shaded regions.

a)


b)


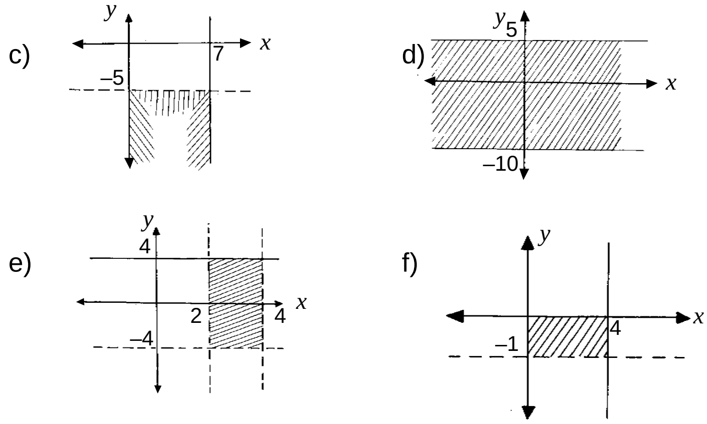


Fig. 14.1

Shading with oblique lines

The shadings can also be made on oblique lines (slanting lines).

Consider the following examples:

1. Show, by shading the unwanted regions, the region which satisfies the following inequalities.
a) $x + 2y < 6$ b) $x + y \geqslant 5$ and $x - y < 5$

Solutions

1. a) The inequality describes a wanted side (an area) with the boundary $x + 2y = 6$. This means we need to establish (draw) this boundary first so we can identify which side to shade. Using knowledge of graph drawing, $x + 2y = 6$ is a straight line graph passing through (0,3) and (6,0) i.e. when $x = 0$, $y = 3$ when $y = 0$, $x = 6$.

i.e.

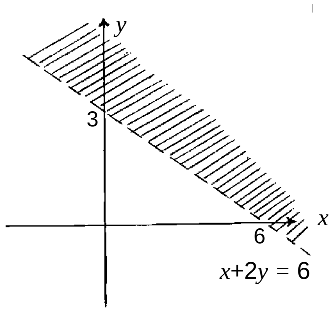


Fig 14.2

Hint

Check the y-term in the inequality.
It is 'less than', it implies that the wanted region is below the line.
If it is on the 'bigger than' side, it implies that the wanted region is above the line.

Do you see why the line is broken or dotted?
How shall we establish which side to shade?
Make the y term positive first (already so in this case).

In this case the wanted region, the unshaded side, is below the line, as illustrated.

b) Here we need to draw the two boundaries on the same axes.

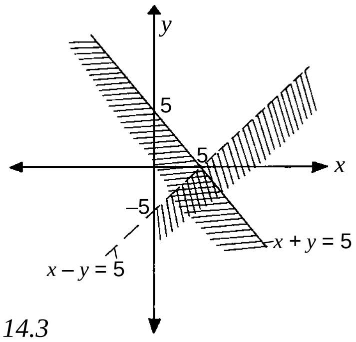


Fig 14.3

For $x + y \geqslant 5$, the wanted region is above the line.

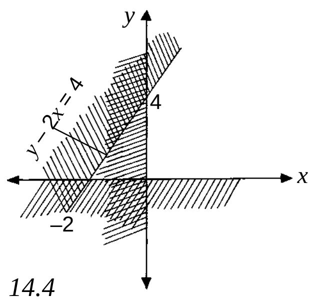
For $x - y < 5$, the wanted region is above the line.

Now, the single shading should respect the two inequalities.

There can be one, two or even more inequalities on the same axes. All should define a common area.
That is the wanted region.
Sometimes a diagram is given where you are asked to give the inequality or inequalities which define the unshaded region.

Consider the example below

2.



In this case the unshaded area is bound by three boundaries.

Fig. 14.4

Find the inequalities which describe the unwanted region.

Solution

2. The equations of the boundaries are
$y = 0$, $x = 0$ and $y - 2x = 4$
(x-axis) (y-axis)

Since the region is above the x-axis and is continuous, which means $y \geqslant 0$.
The region is to the right of the y-axis and is continuous, $x \geqslant 0$.

For $y - 2x = 4$, the region is below the line, so $y - 2x \leqslant 4$

because y is already positive the boundary is solid.

The inequalities are $y \geqslant 0$, $x \geqslant 0$ and $y - 2x \leqslant 4$.

PRACTICE 14C2

intro Concept Example(s) Solution(s) Practice

1. Show by shading the unwanted regions, using the following inequality or inequalities:

a) $y \geqslant \frac{1}{2}$ b) $x < -13$

c) $x + y > 4$ d) $x - y \leqslant 7$

e) $x - 2y > 6$ f) $3x + y < -3$

g) $6 \leqslant 3x + y$ and $x + 2y < 4$ h) $x + 2y < -4$ and $y - x > -2$

2. Give the inequality or inequalities which define the unshaded regions.

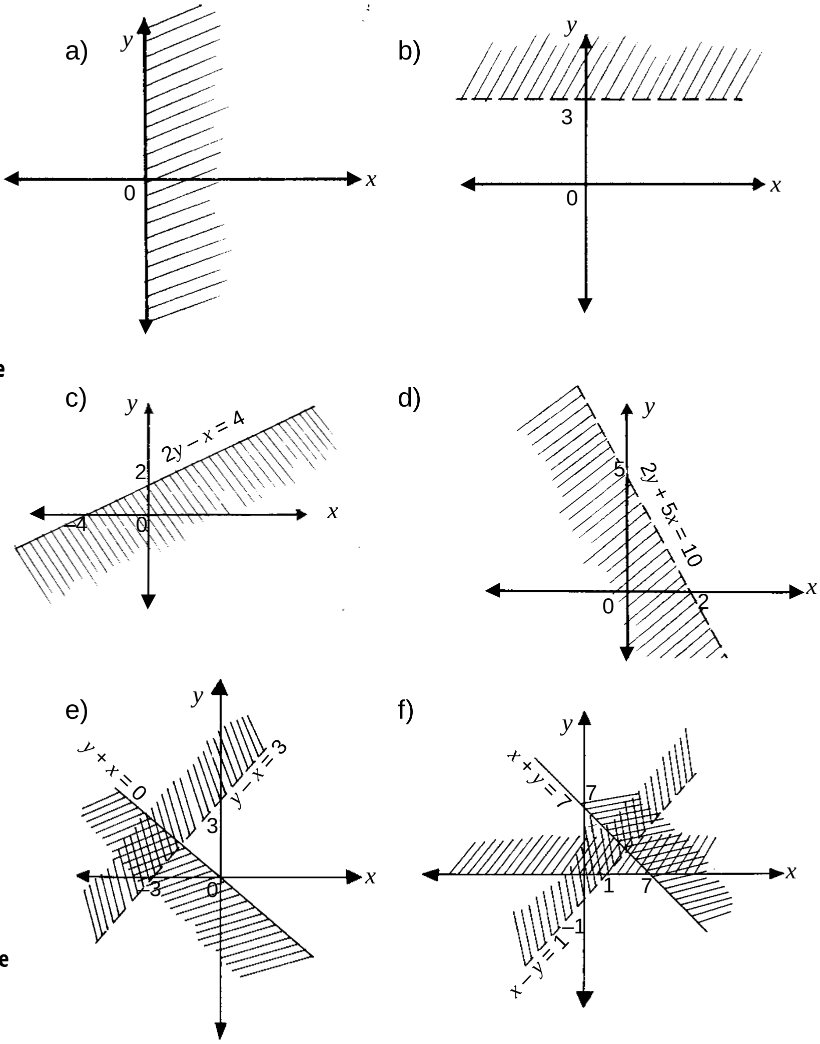


Fig. 14.5

D. MORE BOUNDARIES ON THE SAME AXES

Consider the following illustration

1. On graph paper, show by shading the ***unwanted*** regions, the areas indicated by $x \geqslant -2$, $y \leqslant 3$ and $y > x - 2$

Use the arrow system to indicate the wanted side of each line. These arrows will point to a common area. That is the wanted region which must not be shaded.

It is not necessary to shade all the space occupied by the unwanted region on the whole graph paper. A fairly wide shading around the wanted region indicates the wanted region more clearly.

inequality:	$x \geqslant -2$	$y \leqslant 3$	$y > x - 2$
equation:	$x = -2$	$y = 3$	$y = x - 2$
			(0, –2); (2, 0)

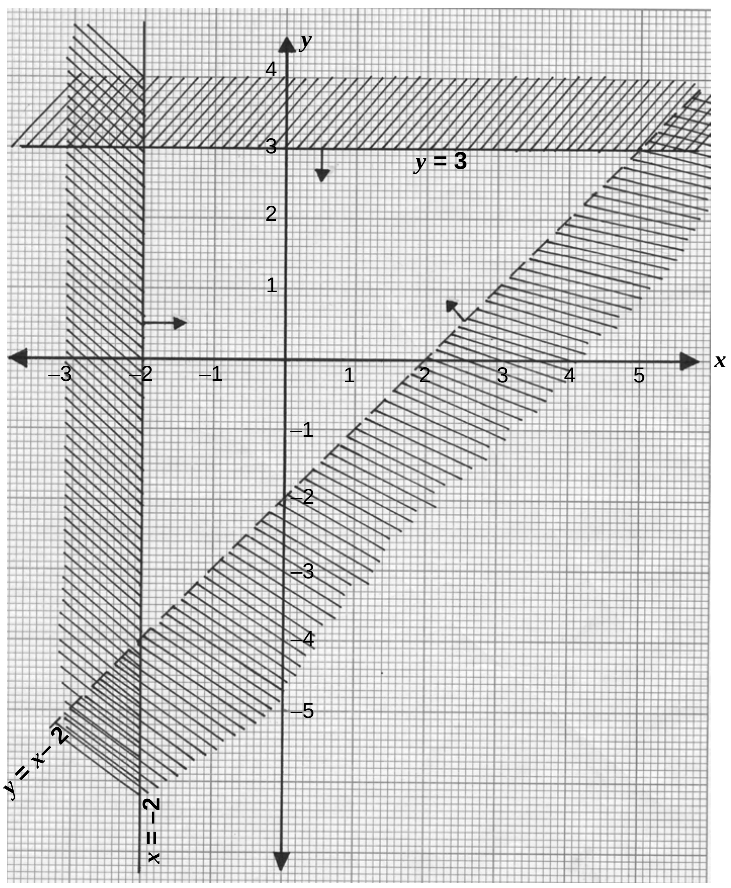


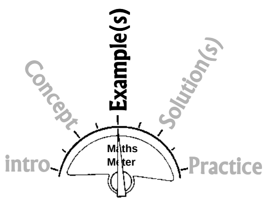
Fig. 14.6

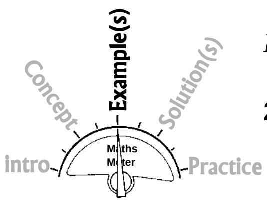
2. Consider the graph in Fig 14.7.
Write down the three inequalities which define the unshaded region.

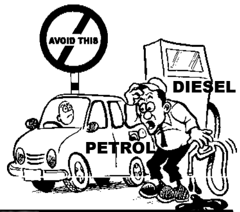
Hint

Refer to Chapter 10 for ways of drawing the boundaries.

1. *Place the inequalities well spaced out across the top of the graph paper.*
2. *Re-write the inequalities as equations below the respective inequalities.*
3. *Where necessary plot suitable points to plot and draw the boundaries. Remember, we said two points are enough for linear graphs.*
4. *Watch out for conditions which dictate solid and broken lines.*
5. *Make sure the boundaries (lines drawn) cross each other.*







Common Errors

* Giving (0,0) and (0,0) as two points to locate lines which pass through the origin e.g $y = 2x$. These are the same point, so one still needs a second point to draw the line accurately.
* All boundaries are drawn as solid lines, irrespective of the inequalities. Check the inequality sign before drawing each boundary.
* Wrong graphs for vertical and horizontal boundaries e.g. $y = 3$ drawn vertically and $x = -2$ is drawn horizontally.
* Drawing short boundaries which do not intersect. Always avoid making these possible errors.

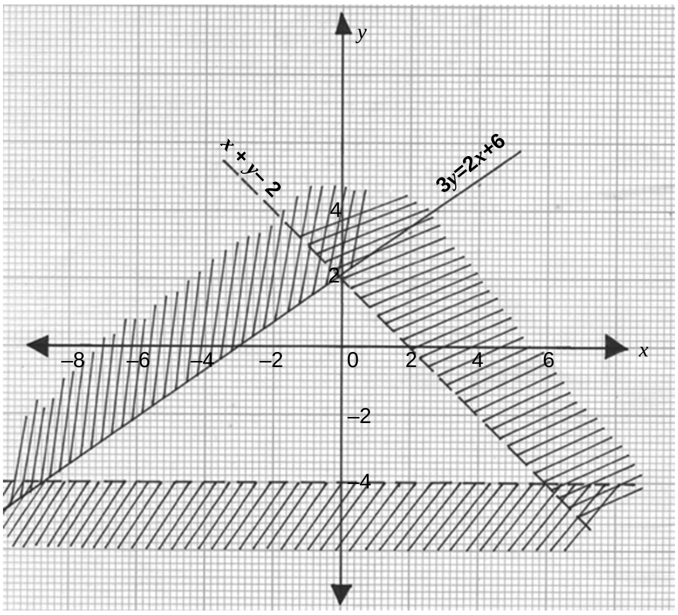


Fig. 14.7

Let us start with the boundary $y = -4$

Here the wanted region is above the line.

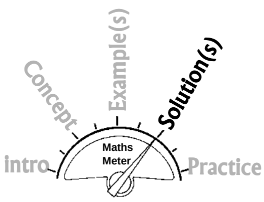
$y > -4$ will describe the side. The line is broken so there is no line under the '$>$' because -4 is not included.

Next, $x + y = 2$. Since the wanted region is below the boundary.

So $x + y < 2$. ***Note*** the line is broken.

Finally $3y = 2x + 6$. As above.

$3y < 2x + 6$ is the correct inequality



$\therefore$ The inequalities are $y > -4$, $x + y < 2$ and $3y \leqslant 2x + 6$

Going back to the diagram, the unshaded region contains many points (x,y). Examples of such points are (0;0), (1;2); (2;1) and many others.

However it is very important to ***note that:***

- All points on solid boundaries are in the wanted region. e.g. (–2;0); (2;3); (–2;–3)
- All points on broken boundaries are ***not*** in the wanted region. e.g. (1;–1); (0;–2); (5;–3); (–2;–4).
- A point like (5;–3) is not in the wanted region.

Hint

Use previous knowledge of finding the equation of a straight line.

1. Give all the inequalities which define each of the following wanted regions

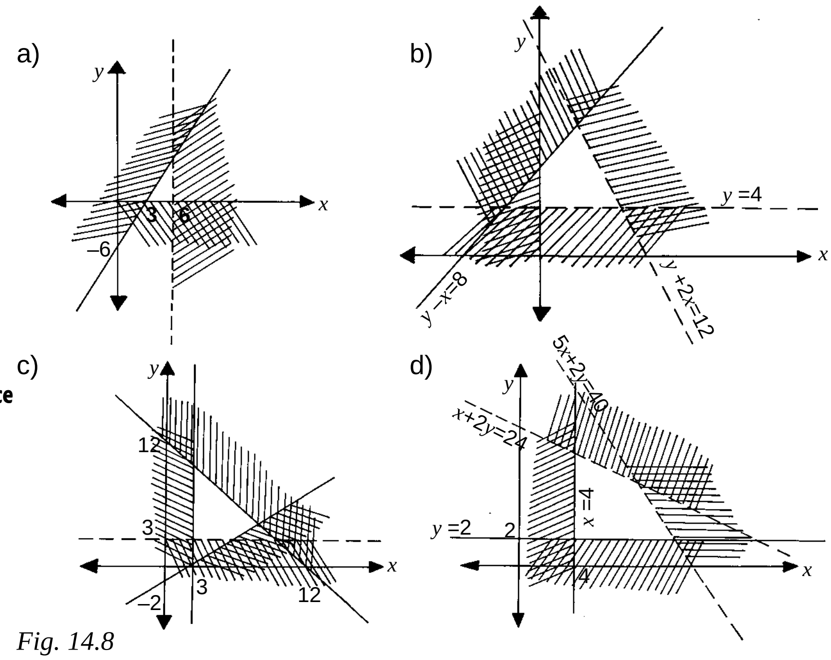


Fig. 14.8

Note that in (a) and (c) you have to start by finding the equations of the boundaries, then the inequalities.

2. Solve the following graphically (that is, draw and identify the wanted region by shading the unwanted regions and give all the points in it)$(x;y)$ where x and y are integers.

a) $y \geqslant 0$, $x - y > 2$, $3x + 4y \leqslant 12$

b) $y > 2$, $y + x < 7$, $y - 2x \leqslant 0$

c) $x \geqslant 0$, $y > -2$, $x + y \leqslant 2$

d) $x + y \leqslant 4$, $x - y \leqslant -2$, $2x + y \geqslant 2$

e) $y < 4$, $x < 3$, $3x + 2y + 4 \geqslant 6$, $x - y - 2 \leqslant 0$

f) $y \leqslant 4$, $x - y \leqslant 1$, $2x + 3y > 12$

g) $y \geqslant 0$, $x \geqslant 0$, $x - 2y \leqslant 2$, $3x + 2y < 12$

3. In each of the wanted regions in question 2 above, find which point gives:

a) the greatest value of $x + 3y$.

b) the minimum value of $2x - y$.

E. LINEAR PROGRAMMING

This is a method of meeting all the requirements in a given situation. The method involves straight lines (Linear). Using the lines as boundaries, a region is established which will then be used to answer further questions. It is important to be able to derive the correct inequalities from given graphs.

Consider the following example

1. Mr Murimi visits ZIMA Farmers Inn to buy some fertilizer and some seed. He decides to buy at least 4 bags of fertilizers and, at least, 2 packs of seed.
If x is the number of fertilizer bags and y the number of seed packs bought,

a) (i) write down two inequalities which satisfy the above conditions.

(ii) Mr Murimi has only \$1 200 to spend. A bag of fertilizer costs \$150 and a seed pack \$60. Write down an inequality in x and y and show that it reduces to $5x + 2y \leqslant 40$.

(iii) Mr Murimi intends to buy less seed packs than fertilizer bags.
Write down another inequality in x and y which satisfies this condition.

b) The point (x,y) represents x bags of fertilizers and y packs of seed.
Using a scale of 2cm to represent 1 bag of fertilizer horizontally and 2cm to represent 2 seed pack vertically, draw and indicate by shading the **UNWANTED** regions, the region in which (x,y) must lie.

Hint

Please read the whole question more than once. Take note of guides which lead to formation of inequalities.

c) From the diagram, find:

(i) the number of possible combinations of bags of fertilizer and seed packs Mr. Murimi could buy.

(ii) the combination, (x,y), which equals all the money available to Mr Murimi.

d) Mr Murimi, however, needs some change to cover transport charges. How much change would he get if he buys as many bags of fertilizer as possible?

Solution

1. In a the inequalities required are found in the second sentence of the problem. $x \geqslant 4$ and $y \geqslant 2$

a) (ii) is in two parts. First form an inequality using the given prices i.e. $150x + 60y \leqslant 1\ 200$. Here we are saying, the number of items (x), times the unit price (150). The cost of fertilizer combined with the cost of seed should not exceed the amount available! Hence the $\leqslant$ sign.
The second part of the question is to reduce the inequality found to the given one.

It is important to note the word, ‘show’.
This means you need to show how you reduce the inequality. See Fig 14.9 for the process.

a) (iii) The first sentence in the question gives the inequality $y < x$.

b) We now have the required four inequalities, four of them. Refer to work on number 1 (Practice 14C2) and follow that method to establish the wanted region.

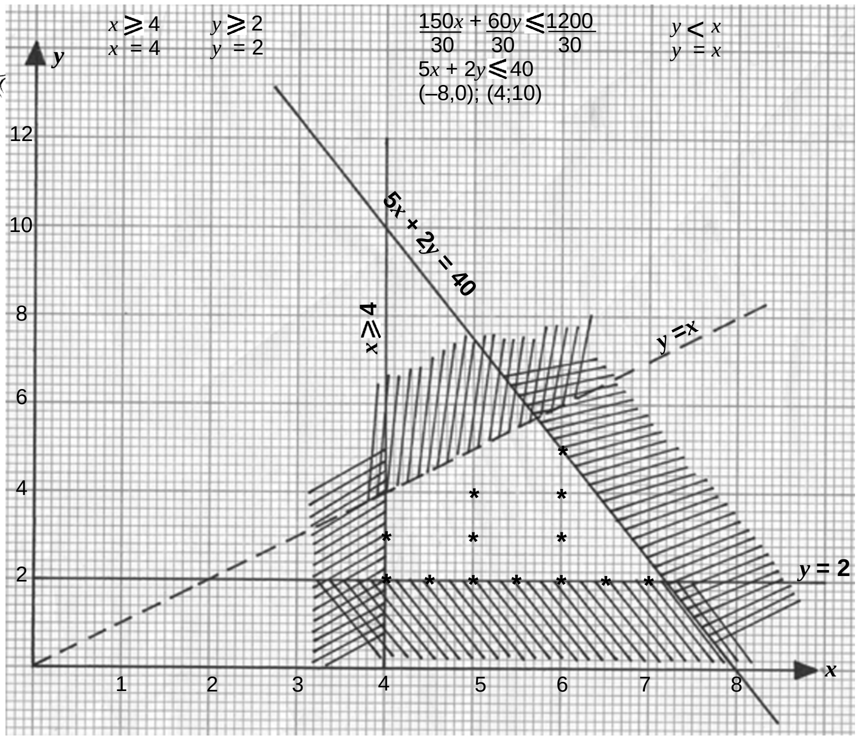


Fig. 14.9

The question c(i) requires you to note the points which fall in the wanted region. Remember the points on solid lines are in whilst those on broken lines are out.

(ci) 10 possible combinations, i.e (4;2), (4;3), (5;2), (5;3), (5;4), (6;2), (6;3), (6;4), (6;5), (7;2).

c(ii). Of the combinations in c(i) there are one or more which lie on the line $5x + 2y = 40$. This is the equation which deals with costs. A point on this line gives the combination which equals the whole amount of money. (i.e. 6;5)

d) The objective here is for you to find the largest number of fertiliser bags Mr Murimi can buy with his money. This calls for the maximum number of bags allowed by the restrictions. Use this point to find how much is used so you can find the change.

i.e. (7;2) gives $150 \times 7 + 60 \times 2 = \$1170$.

$\therefore$ change = \$1200 – \$1170 = \$30

Hint

1. When solving inequalities:
* *avoid dividing by a negative number.*
* *do not replace inequalities with equal signs*
* *all rules for solving equations apply to solving inequalities.*

2. When identifying required regions:
* *use easy points to work with e.g. (0, 0), if it is not on the line.*

3. In Linear programming:
* *make sure you understand the correct meanings of terms like minimum, at least, at most etc.*
* *the question 'show' requires you to show your working.*
* *mark or note all points which lie with in the wanted region and use these to find the one with the most suitable answer. Candidates have a tendency to pick a point without checking other possibilities.*
* *be careful about the type of line you are dealing with, (solid or broken).*

PRACTICE 14E

1. Fig 14.10 shows the region R.

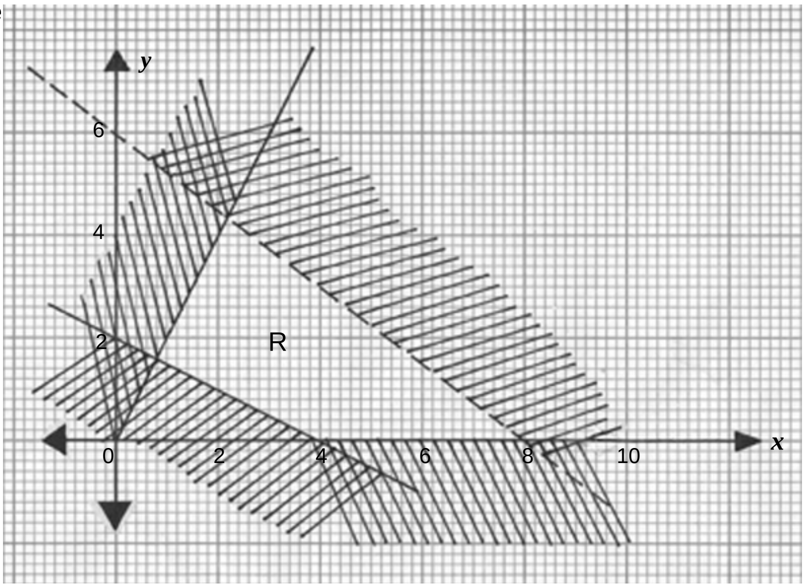


Fig. 14.10

a) Give the number of points found in this region given that x and y are integers.

b) Give the
   (i) maximum value of $y - 2x$ from the region R.
   (ii) minimum value of $y - 2x$ from the region R.

Use graph paper for the following questions.

2. A shopkeeper sells two brands of soft drink, Cooler and Juicy. She ordered x crates of Cooler and y crates of Juicy. She realises that there is space for only 80 crates. She intends to order more crates of Juicy than Cooler since Juicy is more popular The shopkeeper finally decides that she will order more than 10 crates of Cooler and, at most, 50 crates of Juicy.

a) Write down the four inequalities in x and/or y which will satisfy the restrictions given.

b) Using a suitable scale, draw and indicate, by shading the unwanted regions, the region in which (x,y) must lie.

c) Use your diagram to find the number of combinations which will use the whole space available.

d) A crate of Cooler makes a profit of \$3 whilst that of Juicy makes a profit of \$2,50. Use your diagram to estimate the combination which gives the maximum profit.

e) Give the maximum profit.

3. A contractor has more than \$40 000 and up to 8 hectares of land. He decides to build medium density and high density houses on this land. Each medium density house will cost \$4 000 to build and requires 0,5 ha of land whilst each High density house will cost \$1 000 to build and requires 0,1ha of land.
The responsible authority said permission to build would only be granted if the contractor agreed to build at least 5 medium density houses and at least 10 high density houses.
If (x,y) means x number of medium density houses and y number of high density houses,

a) write down the four inequalities which satisfy the above conditions.

b) using a scale of 2cm to represent 2 medium density houses horizontally and 2cm to represent 10 high density houses vertically, draw and indicate, by shading clearly, the unwanted regions, and the region in which (x,y) must lie.

c) From the diagram, find:
(i) the greatest number of medium density houses the contractor can build.
(ii) the combination that gives the largest number of houses that can be built.

4. A small manufacturing company plans to buy new machines. The table below shows the cost, the needed floor space required and the output of each machine.

Machine Type	Cost	Floor	Output
X	\$3 500	4m^2	150 per day
Y	\$4 500	2,5m^2	250 per day

The company is prepared to spend \$31 500 on the project. The company wants to rent a factory shell which has a floor space of more than 20m^2.
To save on foreign currency, customs officials recommend that the company buys at least 2 of machine X and at least 3 of Machine Y.

a) Find the maximum number of machines the company can buy.
b) Which arrangement gives the greatest output?

5. *Answer the whole of this question on a single sheet of graph paper.*

There were x rows of ordinary seats and y rows of superior seats for a school play.

a) Each row of ordinary seats had 25 seats and each row of Superior seats had 20 seats.
Fire regulations limit the total number of seats to 400.
Write down an inequality which expresses this restriction and show that it reduces to $5x + 4y \leqslant 80$.

b) There is space in the hall for no more than 18 rows of seats. The Head insisted that there be at least 5 rows of each type of seat.
Write down three further inequalities, other than $x \geqslant 0$ and $y \geqslant 0$, which express these restrictions.

c) The point (x,y) represents x rows of ordinary seats and y rows of superior seats. Using a scale of 2cm to represent 2 rows on each axis, draw the x and y-axes taking values of x and y from 0 to 20.
Construct, and indicate clearly by shading the **UNWANTED** region, the region within which (x,y) must lie.

d) All of the seats were occupied. Write down the coordinates of the points which represent the largest permitted audience.

e) Each ordinary seat costs 50c and each superior seat costs $1,50.
Write down, in terms of x and y, an expression for the total income, in dollars.

f) Find the greatest possible income, which satisfies all the conditions.

6. A newly resettled farmer wishes to grow maize and soya beans. He realises he has, at most, 72 of labour days available and is prepared to spent at least $3 000 on the two crops.
The table below shows the requirements of each of the two crops.

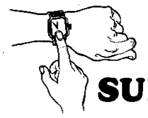
	Maize	Soyabeans
Minimum area to be sown (ha)	5	4
Labour-days per hectare	3	6
Cost of production per hectare ($)	300	200

a) If the farmer used x full hectares for maize and y hectares for soyabeans, write down four inequalities other than $x \geqslant 0$ and $y \geqslant 0$, which satisfy the conditions given above.

b) Using a scale of 2cm to represent 5 hectares horizontally and vertically, draw a suitable diagram to show the region within which (x,y) must lie. Show this region by shading the **UNWANTED** regions.

c) Find the number of combinations which will allow the farmer to use the whole of the man-days available.

d) Which combination gives (i) the maximum area?
(ii) the minimum area which can be used?

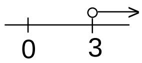


SUMMARY

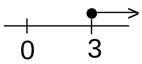
1. $x \neq 3$ means $x > 3$ or $x < 3$; $x < 3$ means $x \geqslant 3$ and $x \ngtr 3$ means $x \leqslant 3$.

2. An inequality describes a boundary whether on a number line or on a cartesian plane.

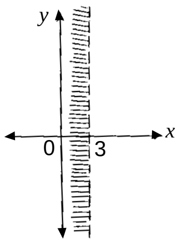
i.e.

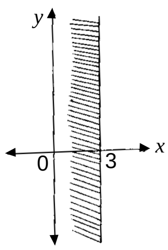
a) 

$x > 3$

b) 

$x \geqslant 3$

c) 

d) 

In (a) and (c) 3 is not a member of the solution set.

In (b) and (d), 3 is a member of the solution set.

3. An inequality can be solved just like an equation. Watch out when dividing by a negative number.

$2x - 7 < 15 - 9x$
$2x + 9x < 15 + 7$
$11x < 22$
$x < 2$

If $-11x > -22$ or $\frac{22}{11} < \frac{11x}{11}$

$\frac{-11x}{-11} < \frac{-22}{-11}$

$x < 2$

Remember to reverse the sign when dividing or multiplying by a negative number. If possible avoid dividing by a negative number by using the "change side change sign approach $2 > x$".

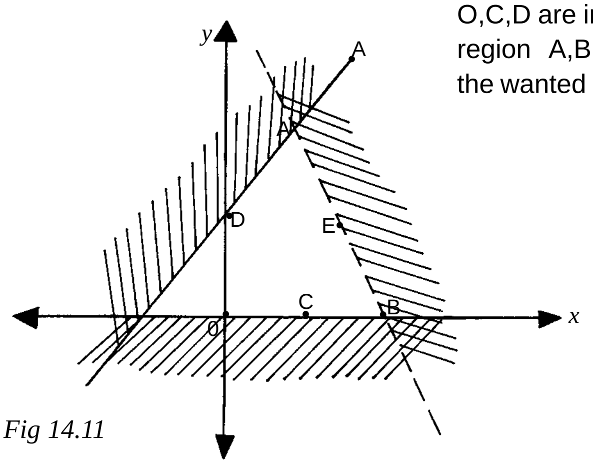
4. An unshaded circle on a number line and a broken line on a cartesian plane indicate that the integer is not included i.e. $<$ or $>$.

5. A shaded circle on a number line and a solid line on a cartesian plane indicate that the integer is included i.e. $\leqslant$ or $\geqslant$.

6. Certain words or statements in linear programming problem give the inequalities required to solve the problem e.g. more than, at most, at least, etc.

7. Points on a solid line are in the wanted region whilst those on a broken line are out of the wanted region.

8. The intersection point of a solid and a broken line is not in the wanted region either.



O,C,D are in the wanted region A,B,E are outside the wanted region.

Fig 14.11

EXAM PRACTICE 14

The following problems may help you master the concepts taught in this chapter.

1. a) List the integer values of x that satisfy the inequality $17 - 2x < 3x < 20$.

2. Fig 14.12 shows the unshaded region R.

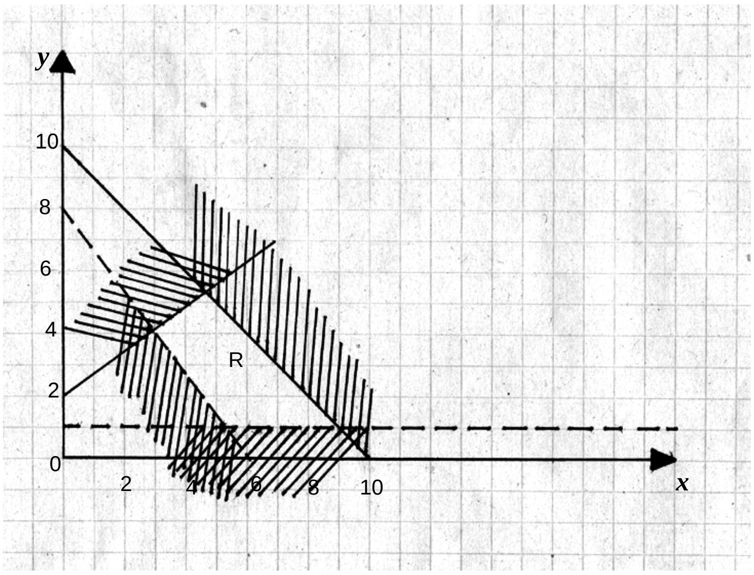


Fig 14.12

Hint

Create two inequalities and solve them separately. Then join the solution sets to form one.

a) Find the other three inequalities which define the unshaded region, R, given that one of the inequalities is $3y \leqslant 2x + 6$.

Given that x and y are integral values in the region R,

b) (i) find: the minimum value of $y + 2x$.

(ii) find: the maximum value of $3y - 2x$.

Solutions

1.

$17 - 2x < 3x$ and $3x < 20$

$17 < 5x$ $\qquad x < 6\frac{2}{3}$

$3\frac{2}{5} < x$

$\therefore 3\frac{2}{5} < x < 6$

$x = 4, 5$

(Use the number line to find the final answer.)

Hint

Identify the boundary for the given inequality. Ascertain if (i) the boundary is continuous (ii) the gradient is positive.

2. a) *The boundary is the one slanting from bottom left to top right then the inequality is $3y \leqslant 2x + 6$. The other three are $y > 1$ (from $y = 1$ which is broken)*

Common Errors

*Common mistakes are interchanging of $>$ with $\geqslant$

*Inability to derive a correct equation from a given line.

Hint
Both x and y must be as small as possible i.e. (4,3)

y must be the biggest

$8x + 6y > 48$ (from $8x + 6y = 48$ which is broken)
$x + y \leqslant 10$ (from $x + y = 10$ which is continuous)

b(i) Minimum of $y + 2x = 3 + 2(4)$
$= 3 + 8$
$= 11$

b(ii) Maximum of $3y - 2x = 3(5) - 2(5)$
$= 15 - 10$
$= 5$

Now do the following:

1.

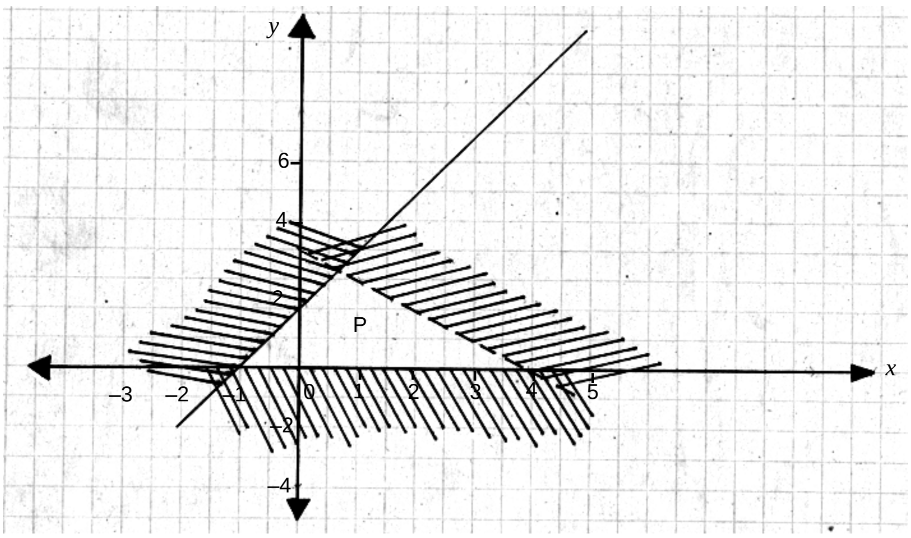


Fig. 14.13

Fig 14.13 shows a region marked P.

a) Write down the three inequalities which define region P.

b) How many points (x,y) are in region P, given that x and y are integers?

c) Use the region to find the greatest value of $x + 3y$ given that x and y are integral values.

2. The points (x, y) satisfy the following inequalities:
$y \geqslant 0$; $x + y \leqslant 6$; $4x + 3y > 12$; $2y < x$

a) Using a scale of 2cm to represent 2 units on each axis, draw on graph paper and clearly indicate, by shading the unwanted regions, the region within which the points (x,y) must lie.

b) Use your diagram to find the minimum value of $2x + y$, given that x and y are intergers.

3. A farmer plans to establish an orchard of orange and mango trees. He decides to have at least 5 orange saplings plants and at least 10 young mango plants.

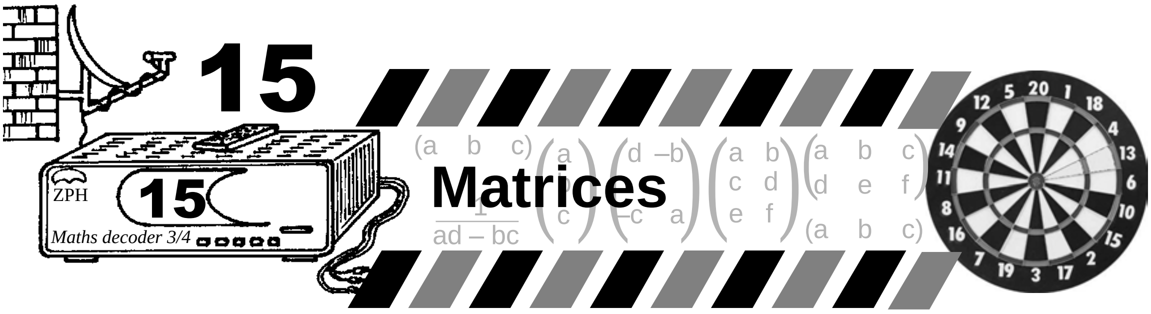
a) If the farmer plants x orange trees and y mango tree, write down two inequalities, other than $x \geqslant 0$ and $y \geqslant 0$ which satisfy the above conditions.

b) Each orange tree requires an area of $2m^2$ whilst a mango tree requires $3m^2$. The farmer has $120m^3$ of land available for the orchard.
Write down an inequality for this condition.

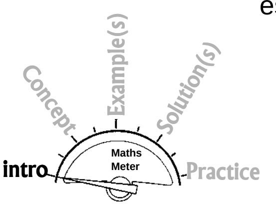
c) The farmer does not wish to have more than 50 plants in the orchard.
Give an inequality for this condition.

d) Using a scale of 2cm to represent 10 plants horizontally and vertically, draw and clearly indicate, by shading the unwanted regions, the region which satisfies all the above inequalities.

e) (x, y) is a point in the wanted region. Use the region to find the highest possible number of mango trees which could be planted in the orchard.



15 Matrices



A matrix (plural – matrices) is a smart way of storing information, especially that involving numbers. Consider the table below.

SOCCER LEAGUE LOG

Team	Won	Draw	Lost
A	3	1	1
B	1	2	2
C	0	2	3

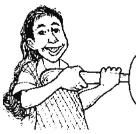
The numbers show a team performance and can be written in matrix form as

$$\underbrace{\begin{pmatrix}3 & 1 & 1\\ 1 & 2 & 2\\ 0 & 2 & 3\end{pmatrix}}_{\text{Columns}}\begin{array}{l}\rightarrow\\ \rightarrow\\ \rightarrow\end{array}\Bigg\}\ \text{Rows}$$

Features of a matrix are:

1. the members (elements) are written in rows and columns.

2. the members (elements) are always enclosed in brackets.



Syllabus Expectations

By the end of this chapter, students should be able to:

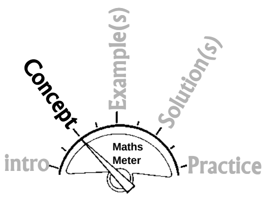
1. use and interpret a matrix as a store of information.
2. use the idea of order of a matrix in operations.
3. multiply matrices with a scalar and/or with another matrix where possible.
4. find the determinant of any 2×2 matrix.
5. find and/or use the inverse of a 2×2 non-singular matrix.

ASSUMED KNOWLEDGE

In order to tackle the work in this chapter, it is assumed that students are able to:

- manipulate directed numbers.
- find solutions to simultaneous linear equations.
- solve simple quadratic equations.

A. BASIC FACTS ABOUT MATRICES



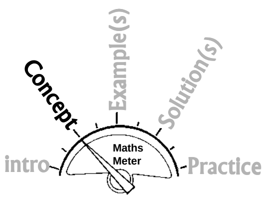
1. A matrix has an order. This is given as 'the number of rows by the number of columns'.

e.g. $\begin{pmatrix} 3 & 1 & 1 \\ 1 & 2 & 2 \\ 0 & 2 & 3 \end{pmatrix}$ the matrix has order 3 by 3 or (3×3) i.e. 3 rows by 3 columns.

$\begin{pmatrix} 3 \\ 1 \end{pmatrix}$ the matrix has order 2×1 meaning 2 rows by 1 column

$\begin{pmatrix} 2 & 2 & 2 \end{pmatrix}$ has order 1×3

Do you ***see that*** the product of the order gives the total number of elements in the matrix?



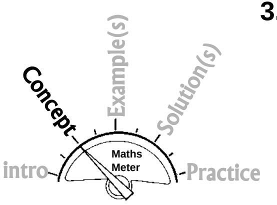
2. Matrices are grouped into 4 main types

a) **Row matrices** (made up of single rows) e.g. $(-1 \quad 5)$

b) **Column matrices** (single columns)
e.g. $\begin{pmatrix} 3 \\ 1 \end{pmatrix}$; $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$.

c) **Square matrices** (equal number of rows and columns)
e.g. $\begin{pmatrix} 3 & 1 & 1 \\ 1 & 2 & 2 \\ 0 & 2 & 3 \end{pmatrix}$; (5)

d) **Rectangular matrices** (number of rows is different from the number of columns)
e.g. $\begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix}$; $\begin{pmatrix} a & b \\ c & d \\ e & f \\ g & h \end{pmatrix}$



3. Matrices can be equal.
This means the matrices have the same order and all corresponding elements are the same.
e.g. Given that $\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$

It follows that a = 1, b = 2, c = 3 and d = 4

$\begin{pmatrix} 3 \\ 1 \end{pmatrix} \neq (3 \quad 1)$ why?

1. Write down the order of each of the following matrices

a) $\begin{pmatrix} 4 \\ 6 \end{pmatrix}$ b) $\begin{pmatrix} 3 & 1 \end{pmatrix}$ c) $\begin{pmatrix} 0 & -2 \\ 3 & 1 \end{pmatrix}$

d) $\begin{pmatrix} 4 & 0 & -4 \\ -3 & 1 & 0 \end{pmatrix}$ e) (8) f) $\begin{pmatrix} -2 & 0 \\ 1 & 3 \\ 5 & 7 \end{pmatrix}$

2. Write down an example of a matrix of the following order

a) 2×3 b) 3×2 c) 1×4

d) 4×4 e) 4×1 f) 1×1

3. Given the matrix $\begin{pmatrix} -2 & 0 & 6 \\ 1 & 3 & -1 \\ 5 & 7 & 4 \end{pmatrix}$,

write down the element on the corresponding position:

a) first row, second column.
b) second row, first column.
c) third row, third column.
d) second row, third column.
e) first row, third column.
f) third row, second column.

4. Write down the type of each matrix.

a) $\begin{pmatrix} 0 \\ -4 \end{pmatrix}$ b) $\begin{pmatrix} 2 & -2 \\ 1 & 5 \end{pmatrix}$ c) $\begin{pmatrix} 0 & 6 \\ 3 & 1 \\ 7 & 4 \end{pmatrix}$

d) $\begin{pmatrix} 3 & 0 & -3 \end{pmatrix}$ e) $\begin{pmatrix} -2 & 0 & 6 \\ 1 & 3 & -1 \\ 5 & 7 & 4 \end{pmatrix}$

5. If $\begin{pmatrix} x & -3 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 5 & -3 \\ 1 & y \end{pmatrix}$, find the values of x and y?

B. ADDITION AND SUBTRACTION OF MATRICES

It is sometimes possible to add and subtract matrices. The process is carried out by combining all corresponding elements.

Consider the following examples

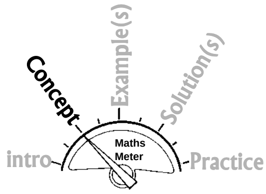
Hint

Be aware of directed numbers when carrying out these operations.

1. $\begin{pmatrix} 4 & 0 \\ 6 & -4 \end{pmatrix} + \begin{pmatrix} 0 & -2 \\ 3 & 1 \end{pmatrix} = \begin{pmatrix} 4+0 & 0+(-2) \\ 6+3 & -4+1 \end{pmatrix}$

$= \begin{pmatrix} 4 & -2 \\ 9 & -3 \end{pmatrix}$

2. $\begin{pmatrix} 2 & -3 \\ 4 & 0 \\ 6 & -4 \end{pmatrix} - \begin{pmatrix} 3 & 5 \\ 0 & -2 \\ 3 & 1 \end{pmatrix} = \begin{pmatrix} 2-3 & -3-5 \\ 4-0 & 0-(-2) \\ 6-3 & -4-1 \end{pmatrix}$

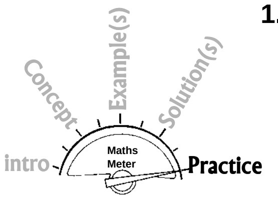


$$= \begin{pmatrix} -1 & -8 \\ 4 & 2 \\ 3 & -5 \end{pmatrix}$$

Consider this one

3. $\begin{pmatrix} 0 & -2 \\ 3 & 1 \end{pmatrix} + \begin{pmatrix} 4 & 0 & -4 \\ -3 & 1 & 0 \end{pmatrix}$

Here some elements of the second matrix have no corresponding elements in the first matrix. So addition is not possible. Thus put simply **addition or subtraction is possible only when the matrices are of the same order.**



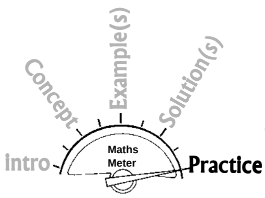
1. Given, $A = \begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 5 & 2 \\ 3 & -1 \end{pmatrix}$ and $C = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}$

Find: a) A + A b) B + B c) C + C
d) A + B e) B – A f) B – C
g) C – B

2. Simplify where possible
a) $\begin{pmatrix} 4 & 1 & 7 \\ -2 & 4 & 1 \\ 3 & 2 & 6 \end{pmatrix} - \begin{pmatrix} 0 & 0 & -1 \\ 0 & -2 & 1 \\ 3 & 1 & -11 \end{pmatrix}$

b) $\begin{pmatrix} 3 \\ 0 \\ -3 \end{pmatrix} + (3 \quad 0 \quad -3)$

3. Using the matrices in question 1, find:
a) A – B + C b) B – C – A c) C – A + C
d) A + A – B e) C + C f) B + B



4. If $\begin{pmatrix} x & 2 \\ -2 & 4 \end{pmatrix} - \begin{pmatrix} -4 & 5 \\ 0 & 6 \end{pmatrix} = \begin{pmatrix} 6 & -3 \\ -2 & y \end{pmatrix}$, find the value of x and the value of y.

5. If $\begin{pmatrix} -6 & 9 & 5 \\ 1 & 0 & 7 \end{pmatrix} = \begin{pmatrix} 1 & -1 & -5 \\ -8 & b & -6 \end{pmatrix} + \begin{pmatrix} -7 & -4 & c \\ 9 & 4 & 13 \end{pmatrix}$

find the values of a, b and c.

C. MATRIX MULTIPLICATION

Matrices can be multiplied by a number (scalar). Consider the question number 3f) in Practice 15B. B + B is the same as 2B.

1. $\therefore$ B + B = 2B $= 2\begin{pmatrix} 5 & 2 \\ 3 & -1 \end{pmatrix}$ $= \begin{pmatrix} 10 & 4 \\ 6 & -2 \end{pmatrix}$

2. $\frac{1}{3}\begin{pmatrix} -12 & 9 \\ 0 & -3 \end{pmatrix}$ $= \begin{pmatrix} -4 & 3 \\ 0 & -1 \end{pmatrix}$

Simply put: **the scalar multiplies each and every element in the matrix.**
The scalar itself can be either a positive or a negative integer or fraction.
A matrix can be multiplied with another matrix. Sometimes this is not possible.

Consider the following examples:

2. Given $A = \begin{pmatrix} 4 & -1 \\ -2 & 3 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 3 & 2 \\ -1 & 4 & 0 \end{pmatrix}$

$$AB = \begin{pmatrix} 4 & -1 \\ -2 & 3 \end{pmatrix} \begin{pmatrix} 1 & 3 & 2 \\ -1 & 4 & 0 \end{pmatrix}$$

Firstly, is it possible? The orders of the matrices will answer this. Write down the orders of the matrices side by side, in the order the matrices are given.

i.e. $\overset{A}{2 \times \underline{2}} \quad \overset{B}{\underline{2} \times 3}$

The inner numbers are the same, meaning it is possible to multiply the two matrices.

This tells us that **the number of columns of the first matrix should be the same as the number of rows of the second matrix for multiplication to be possible.**
If not, it is not possible.

Let us now multiply. Each row of the first matrix is used on each and every column of the second matrix.

Hint

The first row is used to create the first row of the answer, the second row is used to create the second row of the answer and so on.

Thus $\begin{pmatrix} 4 & -1 \\ -2 & 3 \end{pmatrix}\begin{pmatrix} 1 & 3 & 2 \\ -1 & 4 & 0 \end{pmatrix} = \begin{pmatrix} 4\times1 + -1\times-1 & 4\times3 + -1\times4 & 4\times2 + -1\times0 \\ -2\times1 + 3\times-1 & -2\times3 + 3\times4 & -2\times2 + 3\times0 \end{pmatrix}$

$$= \begin{pmatrix} 5 & 8 & 8 \\ -5 & 6 & -4 \end{pmatrix}$$

What is the order of the answer? Look at where we checked the possibility of the process. Do you see that when it is possible the outer numbers give the order of the answer? In the case above, the answer is a 2×3 matrix from $2 \times \underline{2 \quad 2} \times 3$

Is BA possible? Let us check with orders.

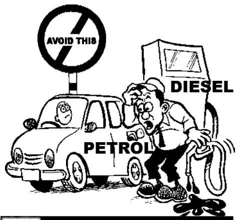
$$\begin{array}{ccc} B & & A \\ 2 \times \underline{3} & & \underline{2} \times 2 \end{array}$$

The inner numbers are different, meaning it is not possible.

3. How about A^2? Since A^2 can be expressed as $A \times A$, it means

$$A^2 = \begin{pmatrix} 4 & -1 \\ -2 & 3 \end{pmatrix} \begin{pmatrix} 4 & -1 \\ -2 & 3 \end{pmatrix}$$

$$= \begin{pmatrix} 18 & -7 \\ -14 & 11 \end{pmatrix}$$



Common Error

On A^2, the elements are simply squared to get $\begin{pmatrix} 16 & 1 \\ 4 & 9 \end{pmatrix}$ No!

Hint

– Multiplication of matrices is either possible or not possible. Check this first using orders. If it is not possible just say so.
– Copy the matrices as given in the question e.g. CA means copying C first then A. AC is the other way round. CA and AC are different questions.

1. Given $P = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$, $Q = (1 \quad 2)$, $R = \begin{pmatrix} -1 & 1 \\ 1 & -2 \end{pmatrix}$

$S = \begin{pmatrix} 1 & 0 & 1 \\ 2 & 1 & 0 \end{pmatrix}$ and $T = \begin{pmatrix} 1 & 2 \\ 0 & 3 \\ -4 & -5 \end{pmatrix}$

a) Say which of the following matrices can be multiplied and give the order of the answer to be.

(i) PQ,	(ii) QP,	(iii) PR,	(iv) RP,
(v) SP,	(vi) ST,	(vii) TS,	(ix) P^2,
(x) R^2,	(xi) T^2	(xii) SR,	(xiii) RT,
(xiv) TP.			

b) Find the products of all the possibles in (a).

2. If $F = \begin{pmatrix} 1 & -1 \\ -1 & 0 \end{pmatrix}$ and $G = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$, find

a) FG b) GF

What do you notice about these two matrices?

3. If $\begin{pmatrix} a & 1 \\ 2 & -3 \end{pmatrix} \begin{pmatrix} 3 & -5 \\ -1 & b \end{pmatrix} = \begin{pmatrix} 11 & c \\ d & -16 \end{pmatrix}$, find a, b, c and d.

4. If $3M + \begin{pmatrix} -3 & 0 \\ -2 & 5 \end{pmatrix} = \begin{pmatrix} 6 & 3 \\ 4 & 2 \end{pmatrix}$, find the matrix M.

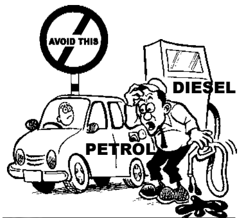
5. Given that $A = \begin{pmatrix} -3 & 0 \\ -2 & 5 \end{pmatrix}$, find A^2.

D. THE INVERSE OF A MATRIX

One important component of the inverse of a matrix is its determinant (det). The symbol for the determinant is $\triangle$.

Given a matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ the determinant of all such 2×2 matrices is defined as **ad – bc**. Take note that the arrangement '**product of diagonal (ad)** ***minus*** **the product of diagonal (bc)**' is strictly followed.

This means bc – ad is wrong!



Common Error

In (b) –18–1 = –19 Remember it is 'minus the product –1'. In (c) 8 – 2 = 6. This happens because the first diagonal has a smaller product. (2–8 is less favoured.)

The following examples will clarify this concept

1. Find the determinant of

a) $\begin{pmatrix} 4 & 1 \\ 2 & 3 \end{pmatrix}$ b) $\begin{pmatrix} -6 & 1 \\ -1 & 3 \end{pmatrix}$ c) $\begin{pmatrix} 1 & 4 \\ 2 & 2 \end{pmatrix}$

Solution

1. a) det $= 4\times3 - 2\times1$
$= 12 - 2$
$= 10$

b) det $= -18-(-1)$
$= -18 + 1$
$= -17$

c) det $= 2 - 8$
$= -6$

The inverse of matrix A (symbol A^{-1}) is one which multiplies A to get the identity matrix (I). This is similar to inverses of fractions.

e.g. $\frac{2}{3} \times \frac{3}{2}$ i.e. a fraction $(\frac{2}{3})\times$ its inverse $(\frac{3}{2}) = 1$, the identity element for multiplication of numbers. The identity matrix for all 2×2 matrices is $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$,

That of all 3×3 matrices is $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ etc

Suppose we want the inverse of $A = \begin{pmatrix} 4 & 1 \\ 2 & 3 \end{pmatrix}$

Let $A^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

This means A^{-1} = $\begin{pmatrix} 4 & 1 \\ 2 & 3 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ or

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 4 & 1 \\ 2 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Now, multiplying out from the first arrangement and equating gives:

$4a + c = 1$ $4b + d = 0$
$2a + 3c = 0$ $2b + 3d = 1$

Solving these simultaneously gives:

$a = \frac{2}{3}$, $b = \frac{-1}{10}$, $c = \frac{-2}{10}$ and $d = \frac{4}{10}$.

Verify these answers using methods for solving simultaneous equations previously learnt.

$$\text{So } A^{-1} = \begin{pmatrix} \frac{3}{10} & \frac{-1}{10} \\ \frac{-2}{10} & \frac{4}{10} \end{pmatrix}$$

$$= \frac{1}{10}\begin{pmatrix} 3 & -1 \\ -2 & 4 \end{pmatrix}$$

Hint

The inverse is made of 2 parts

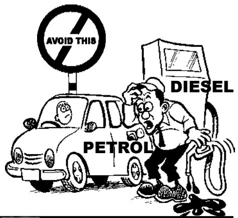
*• **Inverse** of the determinant*

*• **Bracket** where a and d interchange positions and b and c change signs.*

Let us analyse this final version of A^{-1}.

- det A =10 and appears as inverse in A^{-1}
- elements of diagonal ad have been interchanged whilst those of diagonal bc have changed signs!

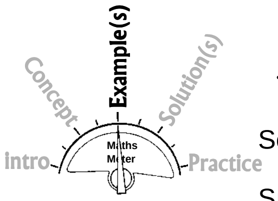
These are the only two processes needed to find the inverse of a matrix.



Common Error

• Where the determinant is negative, e.g. –16, $\frac{-1}{16}$ is usually left as $\frac{1}{-16}$. As in fractions, this is not acceptable at the final stage.

• Process of one diagonal is done on the wrong diagonal.



2. Find the inverse of $\begin{pmatrix} -6 & 2 \\ -1 & 3 \end{pmatrix}$

det $= -18-(-2)$

$= -16$

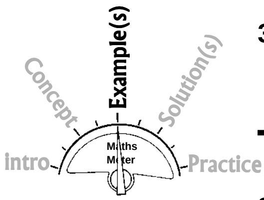
$\therefore$ The inverse is $\frac{-1}{16}\begin{pmatrix} 3 & -2 \\ 1 & -6 \end{pmatrix}$ Simple!

Some matrices like, $\begin{pmatrix} 2 & 4 \\ 1 & 2 \end{pmatrix}$, $\begin{pmatrix} 3 & 0 \\ -2 & 0 \end{pmatrix}$ have a determinant zero.

Such matrices have no inverse, since the inverse of zero is undefined.

Matrices with the determinant zero or which have no inverse are called ***singular matrices*.**

Consider the example below



3. Given that $\begin{pmatrix} x & 3 \\ 1 & x+2 \end{pmatrix}$ has no inverse, find the possible values of x.

Solution

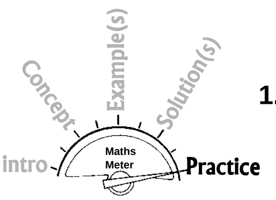
3. No inverse means the determinant is zero!

Thus $x(x + 2) - 3 = 0$

$x^2 + 2x - 3 = 0$

$(x + 3)(x - 1) = 0$

$x = -3$ or 1



1. Find the determinants of the following matrices

a) $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ b) $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ c) $\begin{pmatrix} 0 & 2 \\ -3 & 4 \end{pmatrix}$

d) $\begin{pmatrix} 2 & -4 \\ -1 & 3 \end{pmatrix}$ e) $\begin{pmatrix} -4 & 1 \\ 3 & 3 \end{pmatrix}$ f) $\begin{pmatrix} 1 & -2 \\ 3 & -1 \end{pmatrix}$

g) $\begin{pmatrix} 2 & 6 \\ 1 & -3 \end{pmatrix}$ h) $\begin{pmatrix} 2 & 6 \\ 1 & 3 \end{pmatrix}$ i) $\begin{pmatrix} 7 & 0 \\ 4 & 0 \end{pmatrix}$

2. Find the inverses of the matrices in (1), above. If there is no inverse, say so.

3. Given that $\begin{pmatrix} x & 3 \\ -3 & 4 \end{pmatrix}$ has a determinant of zero, find x.

4. If $\begin{pmatrix} y & 2 \\ 1 & y-1 \end{pmatrix}$ is a singular matrix, find the possible values of y.

E. SOLVING SIMULTANEOUS EQUATIONS: MATRIX METHOD

The method we want to practise here involves matrices, hence its known as the **matrix method.**

Consider the examples below:

1. Solve simultaneously a) $2x - y = 7$
$x + 2y = 1$

Using the elimination or the substitution method
$x = 3$, $y = -1$. Verify this.

The matrix method is as follows:

$2x - y = 7$
$x + 2y = 1$

Rearrange the coefficients:

$$\begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 7 \\ 1 \end{pmatrix}$$

From the coefficients of x and y in the two equations. Do you ***notice* the coefficients of one letter form one column**? This should always be so.

To proceed, we need the inverse of this matrix of the coefficients.

Inverse of $\begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix} = \frac{1}{5}\begin{pmatrix} 2 & 1 \\ -1 & 2 \end{pmatrix}$

Then **pre**-multiply the matrix equation with the inverse.

$$\underbrace{\frac{1}{5}\begin{pmatrix} 2 & 1 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix}} = \underbrace{\frac{1}{5}\begin{pmatrix} 2 & 1 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} 7 \\ 1 \end{pmatrix}}$$

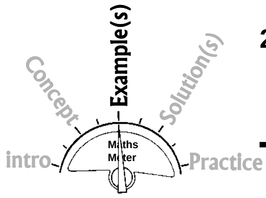
Gives $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{5}\begin{pmatrix} 15 \\ -5 \end{pmatrix}$ continue multiplying

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 \\ -1 \end{pmatrix}$$

$\therefore\ x = 3\ ,\ y = -1$

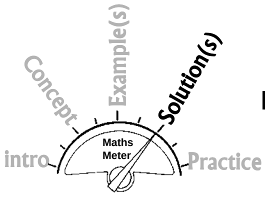
As you do more and more practice you will notice that the left hand side (LHS) always reduces to $\begin{pmatrix} x \\ y \end{pmatrix}$.

With this in mind, the amount of working can be reduced.



2. $3x = y - 5$
$4x - y + 8 = 0$ ***Watch out!***

Solution



2. Re-arrange the equations first.
The proper arrangement will give

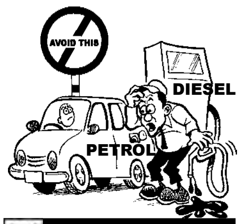
$$\begin{pmatrix} 3 & -1 \\ 4 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -5 \\ -8 \end{pmatrix}$$

Inverse of $\begin{pmatrix} 3 & -1 \\ 4 & -1 \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ -4 & 3 \end{pmatrix}$

So $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ -4 & 3 \end{pmatrix}\begin{pmatrix} -5 \\ -8 \end{pmatrix}$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -3 \\ -4 \end{pmatrix}$$

$x = -3, \quad y = -4$

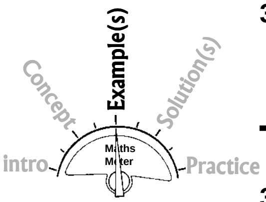


Common Error

$\begin{pmatrix} 3 & 1 \\ 4 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -5 \\ 8 \end{pmatrix}$

Wrong Wrong

The equations are not properly arranged!



3. $3x - 2y = 14$
$y - 5x = 0$ ***Watch out!***

Solution

3. The proper arrangement leads to

$$\begin{pmatrix} 3 & -2 \\ -5 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 14 \\ 0 \end{pmatrix}$$

Inverse $\begin{pmatrix} x \\ y \end{pmatrix} \quad \frac{-1}{7}\begin{pmatrix} 1 & 2 \\ 5 & 3 \end{pmatrix} = \begin{pmatrix} 14 \\ 0 \end{pmatrix}$

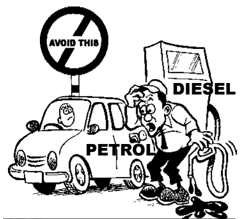
$$= \frac{-1}{7}\begin{pmatrix} 14 \\ 70 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -2 \\ -10 \end{pmatrix}$$

$\therefore \quad x = -2, \; y = -10$

Tip

Mind! The inverse of -7 is not $\frac{1}{-7}$ but $\frac{-1}{7}$



Common Error

$\begin{pmatrix} 3 & -2 \\ 1 & -5 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 14 \\ 0 \end{pmatrix}$

Wrong! Coefficients of different letters are forming columns.

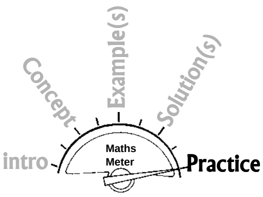
1. Solve for x and y.

a) $\begin{pmatrix} 4 & 1 \\ 3 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 9 \\ 5 \end{pmatrix}$ b) $\begin{pmatrix} 3 & 2 \\ 3 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 7 \\ 1 \end{pmatrix}$

d) $\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 4,7 \\ 3 \end{pmatrix}$ d) $\begin{pmatrix} 3 & -1 \\ 2 & -4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 \\ 5 \end{pmatrix}$

Hint

Do not leave your answer at $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}$ stage as $\begin{pmatrix} a \\ b \end{pmatrix}$ is a vector.



2. Use the matrix method to solve these simultaneous equations.

a) $x - 4y = 4$
$x + 4y = -4$

b) $3x - 2y = -1$
$3x + 2y = 7$

c) $2x + y = 2$
$2x + 2y = 0$

d) $2x = 23 - 3y$
$2x - y = 3$

e) $y = 2 - 5x$
$x = 4 - 0{,}5y$

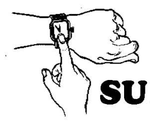
f) $6x - y = 7$
$2y + 5x = 3$

g) $2x = 5 - 3y$
$y = 7 + 10x$

h) $5x - 3y = -8$
$y = \frac{1}{2} - \frac{1}{2}x$

i) $x - y + 7 = x - 2y = 8$

j) $x + \frac{1}{2}y = \frac{1}{2}$
$\frac{5}{3}x + y = 1$



SUMMARY

The following are the basic facts to remember

1. The order is the number of rows followed by the number of columns e.g. 2 by 5 or 2×5
Product gives the total elements.

2. Addition and subtraction is possible when orders are the same and combine corresponding elements. e.g.
Orders 2×3 and 3×2 Impossible.
Orders 2×3 and 2×3 Possible.

3. Multiplication is possible when the number of columns of the first matrix is the same as the number of rows of the second matrix.
Order of the answer e.g. 2×3 and 3×1 are possible while 5×3 and 5×3 are not possible.

4. Use the rows of first matrix against the columns of the second matrix to create corresponding elements of the answer.

5. The determinant of a matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is (ad – bc). Watch out for directed numbers.

6. Inverse of matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ has two parts.

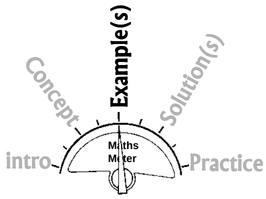
(i) inverse of determinant $\dfrac{1}{ad - bc}$

(ii) bracket $\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$ i.e. $\dfrac{1}{ad - bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$

7. Matrix Method: to solve simultaneous equations, arrange equation in the form $ax + by = c$ before restructuring to matrix form. Be sure of the inverse. Do not leave the answer at the stage $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}$ but write the answer as $x = a$, $y = b$.

EXAM PRACTICE 15

Consider the examples below



1. Given that $A = \begin{pmatrix} 3 & -2 \\ 7 & 4 \end{pmatrix}$, $B = \begin{pmatrix} 3 & -2 \\ 1 & 3 \end{pmatrix}$ and $C = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$

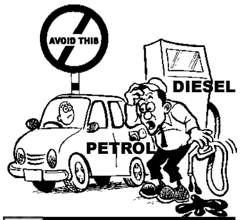
a) Write down the order of C.

b) Find (i) B – 2A

(ii) A^{-1}

c) Find the value of y which makes the matrix $\begin{pmatrix} y & 2 \\ -3 & 4 \end{pmatrix}$ singular.

Solution

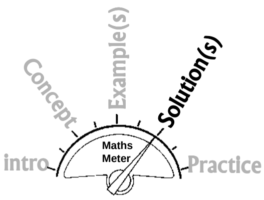


1. a) Order of C is 2×1

b) (i) $B - 2A = \begin{pmatrix} 3 & -2 \\ 1 & 3 \end{pmatrix} - 2\begin{pmatrix} 3 & -2 \\ 7 & 4 \end{pmatrix}$

$= \begin{pmatrix} 3 & -2 \\ 1 & 3 \end{pmatrix} - \begin{pmatrix} 6 & -4 \\ 14 & 8 \end{pmatrix}$ or $\begin{pmatrix} 3 & -2 \\ 1 & 3 \end{pmatrix} + \begin{pmatrix} -6 & 4 \\ -14 & 8 \end{pmatrix}$

$= \begin{pmatrix} -3 & 2 \\ -13 & -5 \end{pmatrix}$



Common Error

$\begin{pmatrix} 3 & -2 \\ 1 & -5 \end{pmatrix} - \underbrace{\begin{pmatrix} -6 & 4 \\ -14 & -8 \end{pmatrix}}_{}$

Wrong!

Compare with the 'or' part of the working.

*****

–2 is taken for the determinant of matrix A.

(ii) Det. of A = $3 \times 4 - 7 \times (-2)$

$= 12 + 14$

$= 26$

$A^{-1} = \frac{1}{26}\begin{pmatrix} 4 & 2 \\ -7 & 3 \end{pmatrix}$

c) A singular matrix has no inverse (its determinant is zero)

Det. of $\begin{pmatrix} y & 2 \\ -3 & 4 \end{pmatrix} = 4y - (2x - 3)$

$= 4y + 6$

Now, $4y + 6 = 0$ (singular matrix)

$4y = -6$

$y = -1\frac{1}{2}$

Hint

The examination question does not usually specify the method to use. The choice is yours.

2. Solve these simultaneously

a) $5x - 6y = 10$
$6x - 5y = 1$

b) $\frac{1}{2}x + \frac{1}{2}y = 2$
$2x + 3y = 13$

Solution

2. a) In this case the coefficients are fairly large. This makes the elimination and/or substitution method more difficult to use.

Using the Matrix Method.

$$\begin{pmatrix} 5 & -6 \\ 6 & -5 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 10 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{11}\begin{pmatrix} -5 & 6 \\ -6 & 5 \end{pmatrix}\begin{pmatrix} 10 \\ 1 \end{pmatrix}$$

$$= \frac{1}{11}\begin{pmatrix} -44 \\ -55 \end{pmatrix}$$

$$= \begin{pmatrix} -4 \\ -5 \end{pmatrix}$$

$\therefore = x = -4, y = -5$

Hint

Clear fractions in the first equation by multiplying through by LCM 6.

b) $\frac{1}{2}x + \frac{1}{3}y = 2$ (i)

$2x + 3y = 13$ (ii)

$3x + 2y = 12$ ((i)×6)

$$\begin{pmatrix} 2 & 3 \\ 3 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 13 \\ 12 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = -\frac{1}{5}\begin{pmatrix} -10 \\ -15 \end{pmatrix}$$

$$= \begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

$\therefore\ x = 2, y = 3$

Now do the following:

1. a) Given that $M\begin{pmatrix} -4 & -3 \\ 5 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, find the matrix M.

b) If $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} N = \begin{pmatrix} 7 & 0 \\ 2 & -1 \end{pmatrix}$ find the matrix N.

2. a) If $P = \begin{pmatrix} 3 & -2 \\ -1 & -1 \end{pmatrix}$ and $P\begin{pmatrix} 1-x \\ -1-2 \end{pmatrix} = \begin{pmatrix} 8 \\ 2y \end{pmatrix}$, find the values of x and y.

b) Given that $\begin{pmatrix} 5 & -2x \\ 0 & 7 \end{pmatrix} - \begin{pmatrix} -1 & 4 \\ 9 & -4 \end{pmatrix} = \begin{pmatrix} 6 & -7 \\ 3y & 11 \end{pmatrix}$

Find the value of x and the value of y.

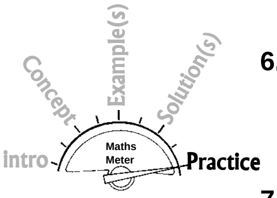
3. Solve the simultaneous equations.

$7x + 2y = 14$

$x - \frac{1}{4}y = 2$

4. Given that $A = \begin{pmatrix} 3 & 2 \\ 2 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & -2 \\ 3 & 4 \end{pmatrix}$, find

a) $A - 2B$ b) B^2 c) AB d) A^{-1}



5. Solve for x and y.

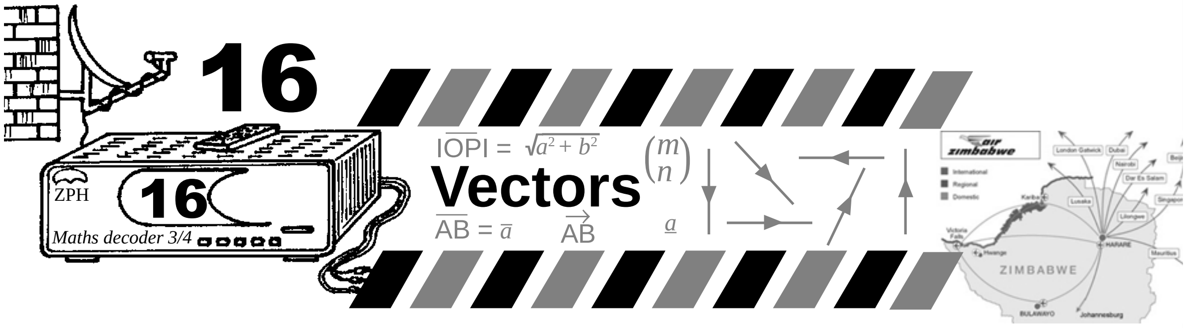
$$\begin{pmatrix} -4 & -3 \\ 5 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

6. Given that P = $\begin{pmatrix} -4 & -3 \\ 5 & 3 \end{pmatrix}$ and Q = $\begin{pmatrix} 1 & -3 \\ 1 & 7 \end{pmatrix}$

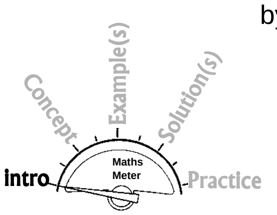
find a) P^2 b) QP c) Q^{-1}

7. a) If the determinant of the matrix $\begin{pmatrix} 2y+3 & -3 \\ y-3 & 2 \end{pmatrix}$ is –10, find the value of y.

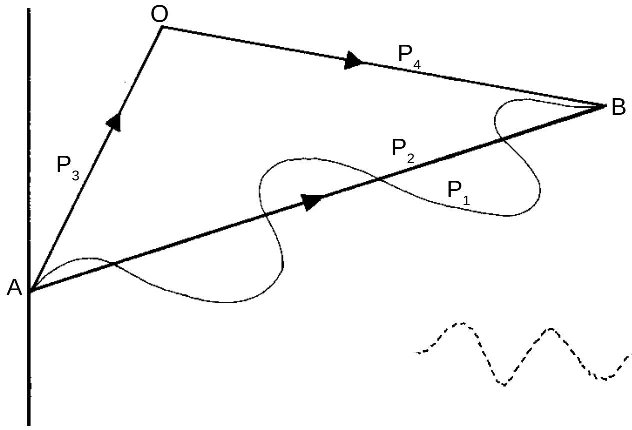
b) Given that the matrix $\begin{pmatrix} x+2 & 4 \\ 2 & x \end{pmatrix}$ has no inverse, find two possible values of x.



16 Vectors



In Fig 16.1, an insect moves from A and follows the path described by line P_1 until it reaches point B.



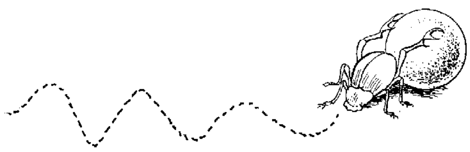


Fig 16.1.

Hint

Bear in mind that when you are dealing with vectors, you are actually dealing with quantities although we may not mention or refer to these quantities. Vector quantities are special because they are associated with direction. Their manipulation requires special attention. Consider two scalar quantities, mass and temperature, in terms of mass 20kg of water and 30kg of water would give us 50kg of water. However water at 20⁰C when added to water at 30⁰C does not result in the water being 50⁰C. Temperature is a non-additative quantity!

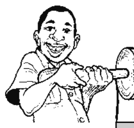
It can be seen that the path, P_1, followed by the insect has no specific direction. The insect could have travelled in a specific direction from A to B using path P_2. The insect could also have taken the P_3 and P_4 route. In this case, it could have moved in a specific direction. In mathematics the length covered by path P_1 is directionless and is called a **scalar quantity**.

The lengths covered by P_2, P_3 and P_4 are described as **vectors** since they have specific directions.

1. **A vector is a quantity with both magnitude and direction, e.g. force.**

2. **A scalar is a quantity described by magnitude only, e.g. mass.**

In the above example, length P_1 is defined as **distance** and is directionless while P_2, P_3 and P_4 are described as **displacements** since these are lengths covered in a specified direction. Other examples of vectors are force, velocity and acceleration while some examples of scalars are time, mass etc. What is the relationship between P_2, P_3 and P_4? Can vectors be added and subtracted just like scalars? Do vectors take different forms? All these and many others, are questions that will be answered in this chapter.



Syllabus Expectations

By the end of this chapter, students should be able to:

1. define and represent vectors in the form of directed line segments as well as bold letters and other notations.
2. represent a translation by a column vector.
3. add and subtract vectors.
4. apply the concept of position vector.
5. multiply vectors by a scalar.
6. identify equal vectors.
7. calculate the magnitude of a vector $\begin{pmatrix} x \\ y \end{pmatrix}$ as $\sqrt{x^2 + y^2}$
8. use the notation $|\overrightarrow{AB}|$ or $|\overrightarrow{a}|$
9. represent position vectors, equal vectors and parallel vectors.
10. use link vectors and matrices using the cartesian plane.
11. use parallel vectors in problem solving.

ASSUMED KNOWLEDGE

In order to tackle work in this chapter, it is assumed that students are able to:

- represent or draw a point or line accurately on a cartesian plane.
- use the Pythagoras Theorem
- appreciate the meaning of basic quantities like force, mass and time.
- draw to scale.
- understand basic trigonometry.
- add and subtract 2×1 matrices

A. REPRESENTING VECTORS

Four ways of representing vectors are as follows:

(i) A vector may be represented by a directed line segment.

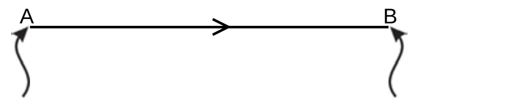


Fig. 16.2

ii) A vector may also be represented by letters with a line above or below them. The letters need not be bold if a line is drawn below or above them. Thus for figure 16.2 the vector is represented as

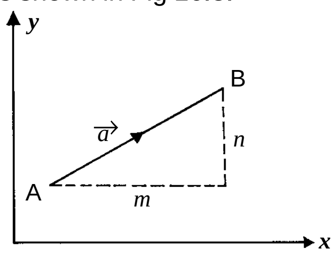
$\overrightarrow{AB}$ or $\underline{AB}$ or or **AB**, $\bar{a}$ or $\underline{a}$

Note the the order of the letters indicates the direction of the vector

(iii) By making bold letters e.g.

$\overrightarrow{AB} = \boldsymbol{p} = \mathbf{AB}$

(iv) A vector may also be represented using a column matrix as shown in Fig 16.3.



magnitude in x direction

$$\overline{AB} = \bar{a} = \begin{pmatrix} m \\ n \end{pmatrix}$$

magnitude in y direction

Fig. 16.3

B. MAGNITUDE OF A VECTOR

The magnitude of vector $\overrightarrow{AB}$ is represented as $|\overline{AB}|$. The magnitude of a vector is also called its **modulus** or **size**.
Pythagoras Theorem may be used to calculate the magnitude. This is covered in detail under section E of this chapter.

C. TRANSLATION AND VECTORS ON CARTESIAN PLANES

In mathematics translation is movement in a straight line, i.e. in a specified direction without turning. In chapter 21 “Transformation” we shall look at this concept in detail. For now let us revisit Fig. 16.1 once more, P_2, P_3, P_4 are examples of translation as there is movement **without turning**. Obviously P_1 is not a translation!

If you have a cartesian plane, a point $(x;y)$ may be used to describe a translation. The path it describes obviously defines a **vector** since the point would be moving (without turning) in a specified direction.

Fig 16.4 illustrates examples of translations. In all cases the movement starts from the *tail of the vector* and moves to the *head of the vector*. Consider, for example, vector $\overline{OB}$, the point O (0;0) translates to a point B(6;2). Similarly, for vector $\overline{PQ}$, begins at P(1;6) and translates to Q(6;6). There are two types of vectors which can be represented on a cartesian plane.

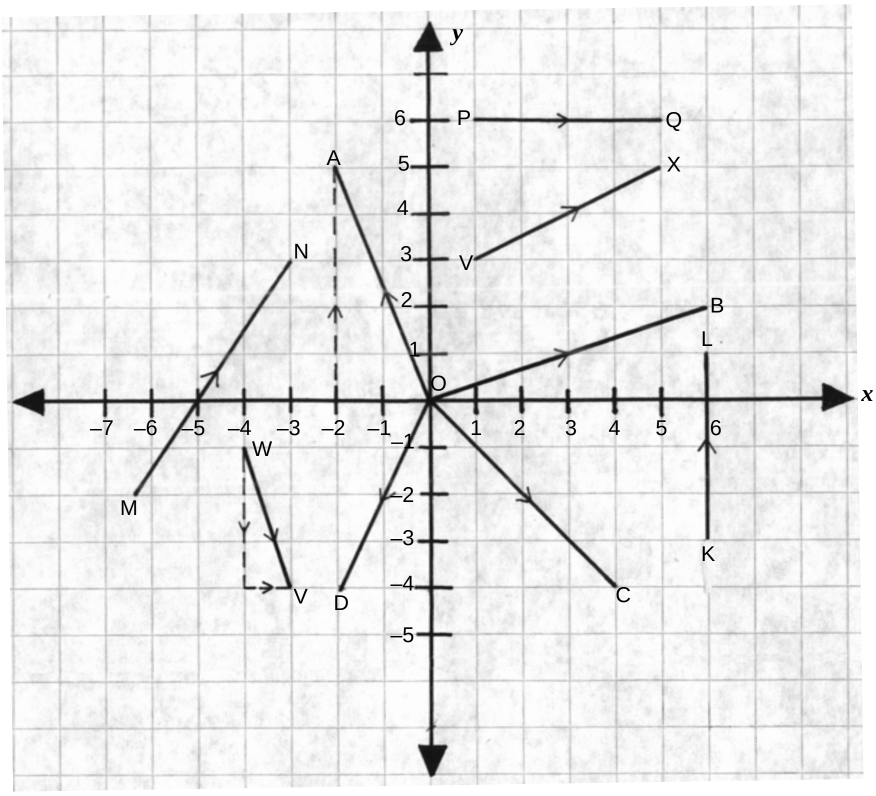


Fig 16.4

D. TYPES OF VECTORS

A translation, forming a vector on a cartesian plane, can either have its starting point at the origin (0,0) point or at any other point on the cartesian plane.

Position vector

A vector whose initial point is the origin 0 is called a position vector. Position comes from the fact that its point of origin is known, or the position of its tail is known. It is **always** (0,0).

Displacement (free) Vector – Any other vector on the cartesian plane which does not start at the origin (0,0).

In Fig 16.4;

- Position vectors are:
  $\overline{OA}$ $\overline{OB}$ $\overline{OC}$ $\overline{OD}$.

- Free (displacement) vectors are:
  $\overline{PQ}$ $\overline{VX}$ $\overline{MN}$ $\overline{WV}$ $\overline{KL}$

Both position vectors and free vectors can be represented in i.e. **column form** i.e. $\binom{x}{y}$ where x represents the x horizontal units and y represents the y vertical units.

For Fig 16.4

$$\overrightarrow{OA} = \begin{pmatrix} -2 \\ 5 \end{pmatrix}$$

$$\overrightarrow{WV} = \begin{pmatrix} 1 \\ -3 \end{pmatrix}$$

Find the column vectors of all other vectors in Fig 16.4, except for $\overline{OA}$ and $\overline{WV}$.

E. POSITION VECTORS

Since the tail (origin) of any position vector is known, this means that any point (P) on the cartesian plane has a position vector which can be drawn from the origin to that point, hence we tend to talk of a position vector of a point P. The magnitude and direction of all position vectors can be found by use of formulas. As shown in Fig 16.5, the position vector of a point P(a,b), denoted as $\overrightarrow{OP}$ is written as $\binom{a}{b}$.

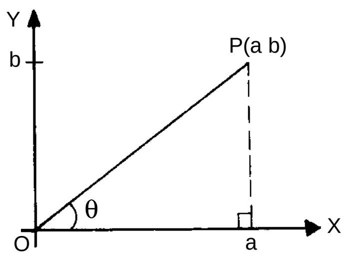


Fig. 16.5

The magnitude of the vector $\overrightarrow{OP} = \binom{a}{b}$ denoted as $|\overline{OP}|$ is the distance of P from O.

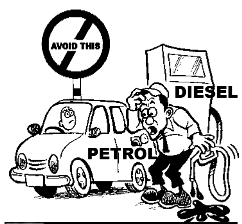
$$\boxed{\therefore\ |\overrightarrow{OP}| = \sqrt{a^2 + b^2}}$$ **(Using Pythagoras Theorem)**

The direction of the vector $\overrightarrow{OP}$ is found by first finding

$$\boxed{\tan\theta = \frac{b}{a}}$$

Hint

Once θ has been found, the direction of the vector of OP can be stated using the bearing.



Common Errors

After finding the value $\tan\theta$ most students end here thinking this the final answer . This is wrong.

F. UNIT VECTOR

A unit vector is a vector of length, one unit in a given direction, i.e. it is a vector whose magnitude is one.

$|\overrightarrow{OP}| = 1$ if $\overrightarrow{OP}$ is a unit vector.
$\overrightarrow{OP}$ is a unit vector if $|\overrightarrow{OP}| = 1$

Consider the following examples:

Hint
It is important to do a sketch of the diagram first.

Find the magnitude and direction of the position vectors of the following points **1.** M(3;4)
2. B(–6; –8)

Solution

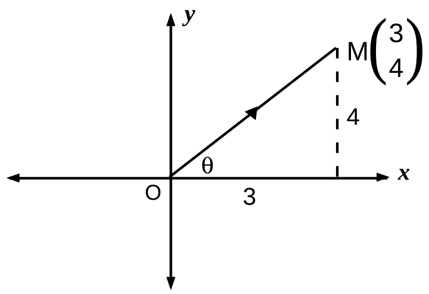
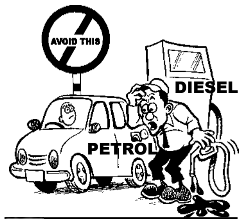


Fig. 16.6



Common Errors

Students confuse OM as length or magnitude of a vector. The correct thing is
$\overrightarrow{OM}$ = vector with magnitude.
$|\overrightarrow{OM}|$ = magnitude of vector only = OM.

1. Magnitude:- $OM^2 = 3^2 + 4^2$ (Pythagoras)

$OM^2 = 9 + 16$

$|\overrightarrow{OM}| = \sqrt{25}$

$|\overrightarrow{OM}| = 5$

Hint
Magnitude cannot be negative since it is distance, i.e. +5 will be considered wrong.

Direction $\tan\theta = \frac{4}{3}$

$\tan\theta = 1{,}3333$

$\theta = 53^0\,3' \text{ or } 53{,}1^0$

Direction from the North line $= 90^0 - 53^0\,3'$
$= 36^0\,53' \text{ or } 36{,}9^0$

$\therefore$ Vector $\overrightarrow{OM}$ is in the direction $036^0\,53'$ or $36{,}9^0$

Solution

2.

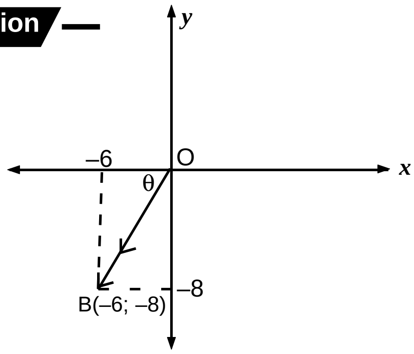
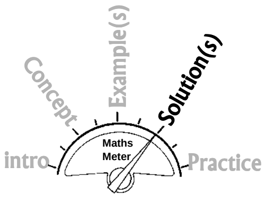


Fig. 16.7

Tip

The value of your answer using tables is $53^0 3'$ and $53{,}1^0$ using a calculator.



Magnitude

$$OB^2 = (-6)^2 + (-8)^2$$

$$OB^2 = 36 + 64$$

$$|\overrightarrow{OB}| = \sqrt{100}$$

$$|\overrightarrow{OB}| = 10$$

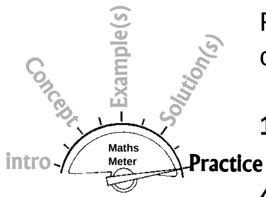
Direction

$$\tan\theta = \frac{-8}{-6}$$

$$\tan\theta = 1{,}3333$$

$$\theta = 53^0\,3' \text{ or } 53{,}1^0$$

$\therefore$ Direction of $\overrightarrow{OB}$ is $180^0 + 36^0 53' = 216^0 53'$



Find the magnitude (modulus) and direction of the position vectors of the following points:

1. (6; 7)	**2.** (12; 7)	**3.** (–9; 12)
4. (–6; –6)	**5.** (13; –9)	**6.** (7; 7)
7. (–8;–10)	**8.** (–4; –2)	**9.** (7; –6)

G. DISPLACEMENT OR FREE VECTORS

As already noted, a free vector is also a form of translation, hence, it can also be written in column form $\begin{pmatrix} x \\ y \end{pmatrix}$.

Tip

Remember you don't need the origin to depict this but you need to identify the tail of the vector.

Where x represents a translation *parallel* to the x-axis and y represents a translation *parallel* to the y-axis. Movements to the right and upwards are **positive** while movements to the left and downwards are **negative**.

Consider the example below:

1. Write down column vectors to represent the line segments in Fig 16.8.

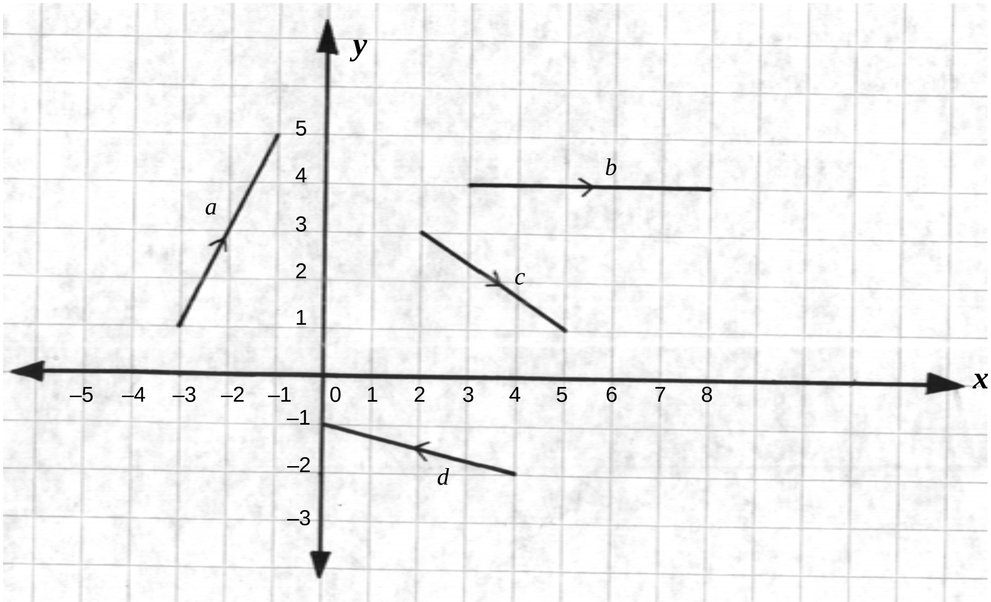
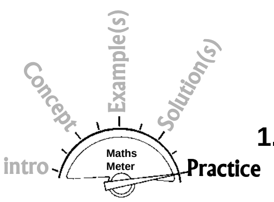


Fig. 16.8

Solution

1. $\underline{a} = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$ $\underline{b} = \begin{pmatrix} 5 \\ 0 \end{pmatrix}$

$\underline{c} = \begin{pmatrix} 3 \\ -2 \end{pmatrix}$ $\underline{d} = \begin{pmatrix} -4 \\ 1 \end{pmatrix}$



1. Write down the column vectors which represent the line segments in Fig 16.9. *(Each big square represents 1 unit).*

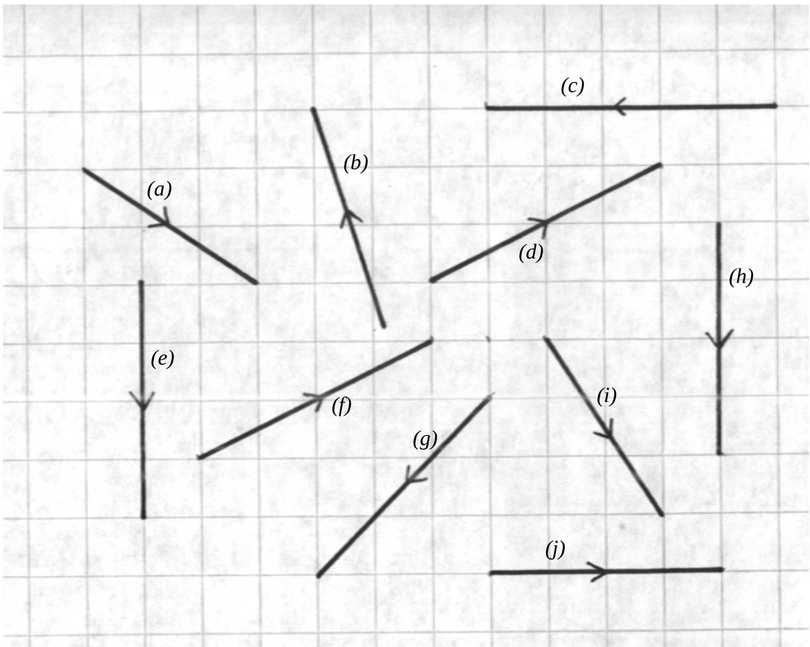


Fig. 16.9

2. In this question, use graph or squared paper. Starting at 0 in the middle of a square of sides with 11 units, draw the following vectors:

a) $\overline{OB} = \begin{pmatrix} 3 \\ -5 \end{pmatrix}$ b) $\overline{BC} = \begin{pmatrix} -3 \\ 2 \end{pmatrix}$

c) $\overline{CD} = \begin{pmatrix} -3 \\ -2 \end{pmatrix}$ d) $\overline{DE} = \begin{pmatrix} 5 \\ 9 \end{pmatrix}$

e) $\overline{EF} = \begin{pmatrix} -3 \\ -2 \end{pmatrix}$ f) $\overline{FG} = \begin{pmatrix} -3 \\ 2 \end{pmatrix}$

g) $\overline{GH} = \begin{pmatrix} 3 \\ -5 \end{pmatrix}$ h) $\overline{HJ} = \begin{pmatrix} -5 \\ -3 \end{pmatrix}$

i) $\overline{JK} = \begin{pmatrix} 9 \\ 5 \end{pmatrix}$

3. On graph paper, using a scale of 1cm: 1 unit on both axis:

a) Plot the point P(2;1).

b) Draw the three vectors:

$\overrightarrow{PQ} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$ $\overrightarrow{QR} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$

$\overrightarrow{RS} = \begin{pmatrix} -2 \\ 0 \end{pmatrix}$.

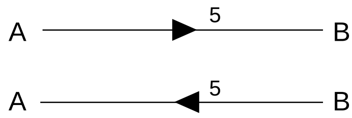
c) Draw and write the sum of PQ + QR + RS as a column vector.

H. NEGATIVITY OF A VECTOR

Hint

*Note that when you reverse the direction of a vector, don't reverse the **order** of the vectors or they cease to be negative.*

A vector becomes negative once its **direction is reversed**.



AB and BA have the same magnitude but move in opposite directions.

Hence $\overrightarrow{AB} = -\overrightarrow{BA}$

Similarly $\overrightarrow{BA} = -\overrightarrow{AB}$

But $\overrightarrow{AB} \neq \overrightarrow{BA}$ (because they have different directions)

A vector whose magnitude is zero is called a **zero vector** denoted by $\begin{pmatrix} \mathbf{0} \\ \mathbf{0} \end{pmatrix}$.

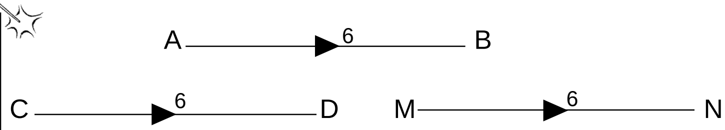
$\overline{AB} + (-\overline{AB})$ or $\overrightarrow{AB} + \overrightarrow{BA} = 0$

I. EQUALITY OF VECTORS

Hint

*All equal vectors are **parallel** and have the same magnitude. This includes vectors which are in line. Remember the angle between parallel lines is zero.*

Two or more vectors are equal if and only if, they are the same in magnitude and direction.
Consider the vectors below:



$\overrightarrow{AB} = \overrightarrow{CD} = \overrightarrow{MN}$ . . . (same magnitude and direction)

Hence $\overrightarrow{AB}$ // $\overrightarrow{CD}$ // $\overrightarrow{MN}$ . . . (where // implies is "parallel to")

$|\overrightarrow{AB}| = |\overrightarrow{CD}| = |\overrightarrow{MN}| = 6$

J. ADDITION AND SUBTRACTION OF VECTORS (Expressing one Vector in terms of the other two in a triangle)

Consider any triangle made of vectors (triangle of vectors) Fig 16.10.

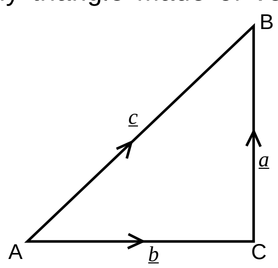


Fig 16.10

Any one of three vectors given in triangular form can be expressed in terms of the other two vectors. We call a vector, expressed in terms of the other two, a **resultant vector**. This is the triangle rule concept.

Hint

*The **most important** point to note in the triangle rule is that the resultant vector's tail must start from the tail of the first vector. The head of the resultant vector must meet the head of the second vector. The arrows of the vectors $\underline{a}$ and $\underline{b}$ must $\underline{\text{all}}$ be facing towards the head of the resultant vector. If the arrow is $\underline{\text{not}}$ in the correct direction (facing the tail of resultant) then, the vector is treated as $\underline{\text{negative}}$.*

In Figure 16.10, supposing we take:

a) $\overrightarrow{AB}$ to be the resultant vector, then we can express it in terms of the other vectors as follows:

$$\overline{\mathbf{AB}} = \underline{\boldsymbol{b}} + \underline{\boldsymbol{a}}.$$

b) $\overline{AC}$ to be the resultant vector, then we can express it in terms of the other vectors as:

$$\overline{\mathbf{AC}} = \underline{\boldsymbol{c}} - \underline{\boldsymbol{a}}.$$

c) $\overline{CB}$ to be the resultant vector, then we can express it in terms of the other vectors as:

$$\overline{\mathbf{CB}} = -\underline{\boldsymbol{b}} + \underline{\boldsymbol{c}}.$$

Consider the following examples:

1. Write the resultant vector $\underline{r}$ in terms of the other vectors $\underline{a}$ and $\underline{b}$ in the diagrams that follow.

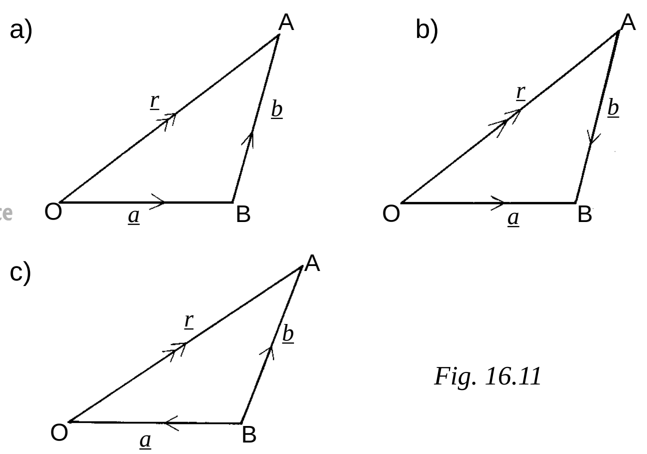


Fig. 16.11

Solution

1. a) $\bar{r} = \bar{a} + \bar{b}$ (the arrows of $\underline{a}$ and $\underline{b}$ are facing the right direction i.e. towards the head of the resultant)

b) For this diagram the direction of $\underline{b}$ is not facing the favoured direction, so we deliberately **reverse it** in our minds but then reversing the direction of a vector will result in it being negative.
$\therefore$ $\underline{b}$ becomes $-\underline{b}$.

Thus $\underline{r} = \underline{a} + (-\underline{b})$
Hence $\underline{r} = \underline{a} - \underline{b}$

Hint

Strictly speaking, subtracting a vector is the same as adding a negative vector to another.

c) $\underline{a}$ is not facing the favoured direction so we reverse it to follow our favoured direction and thus $\underline{a}$ becomes negative.

$\therefore \; r = (-\underline{a}) + \underline{b}$

$r = \underline{b} - \underline{a}$

2. Using the triangle in Fig 16.12, write each of the following as a single vector, in its simplest form.

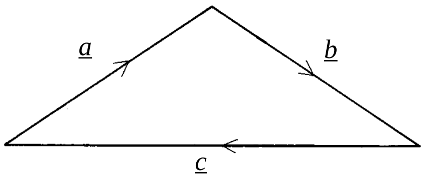


Fig. 16.12

2. a) $\underline{a} + \underline{b}$

b) $\underline{c} + \underline{a}$

c) $\underline{a} + \underline{b} + \underline{c}$

Solution

2. a) $\underline{a} + \underline{b} = -\underline{c}$

b) $\underline{c} + \underline{a} = -\underline{b}$

c) $\underline{a} + \underline{b} + \underline{c}$

from above $\underline{a} + \underline{b} = -c$

$\therefore \; a + b + c = -\text{c} + \text{c}$
$= 0$

1. In Fig 16.13, EBCD and ABCE are both parallelograms.

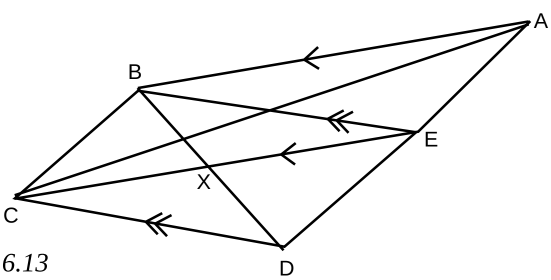


Fig. 16.13

Write the following as single vectors:

a) $\overline{AB} + \overline{BC}$

b) $\overline{EC} + \overline{CD}$

c) $\overline{EX} + \overline{XD}$

d) $\overline{AB} + \overline{BD} - \overline{AD}$

e) $\overline{ED} + \overline{EB}$

f) $\overline{AB} + \overline{BD} + \overline{DC}$

2. a) In Fig 16.14, express each vector $\underline{a}$, $\underline{b}$, $\underline{c}$ and $\underline{d}$ in column form.

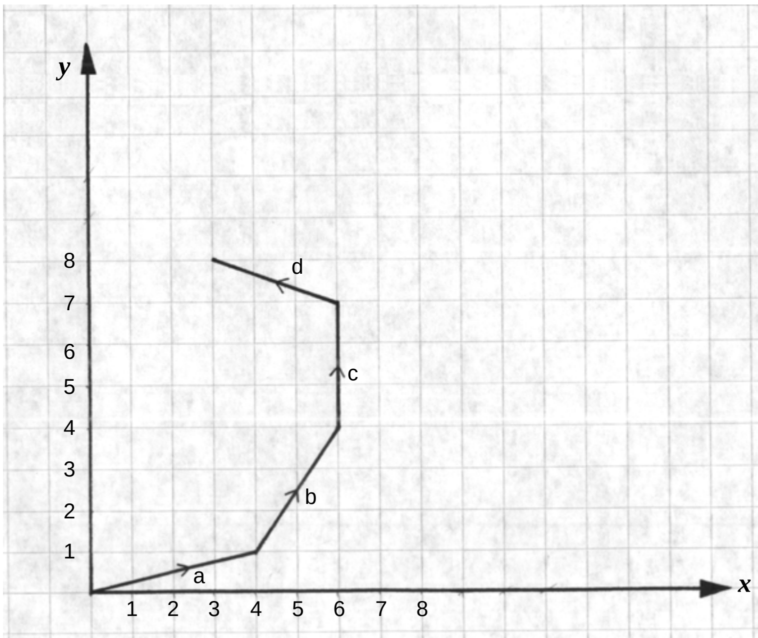


Fig. 16.14

Using the column vector form find the following:

(i) $\underline{a} + \underline{b}$

(ii) $(\underline{a} + \underline{b}) + \underline{c}$

(iii) $\underline{c} + \underline{d}$

(iv) $\underline{a} + (\underline{b} + \underline{c})$

(v) $\underline{a} + \underline{b} + \underline{c} + \underline{d}$

Hint

Sketch the diagram first!

3. Show diagrammatically, or otherwise, that any free vector. $\overline{PQ}$, can be expressed in terms of the position vectors, $\overline{OQ}$ and OP, as:

$\overline{PQ} = \overline{OQ} - \overline{OP}$

K. ADDITION AND SUBTRACTION OF VECTORS IN COLUMN (MATRIX) FORM

As already mentioned, a vector written in column form looks the same as a single column matrix. In fact column vectors are a special form of matrices.

Consider two vectors $\underline{p}$ and $\underline{q}$ Fig. 16.15 where:

$$\underline{p} = \begin{pmatrix} 3 \\ 3 \end{pmatrix} \qquad \underline{q} = \begin{pmatrix} 4 \\ 1 \end{pmatrix}$$

$\underline{p} + \underline{q} = \bar{r}$ (triangle law rule of vectors)

But the column matrix of $\bar{r} = \begin{pmatrix} 7 \\ 4 \end{pmatrix}$

Using column matrix, only it follows that:

$$\bar{p} + \bar{q} = \bar{r}$$

$$\begin{pmatrix} 3 \\ 3 \end{pmatrix} + \begin{pmatrix} 4 \\ 1 \end{pmatrix} = \begin{pmatrix} 7 \\ 4 \end{pmatrix}$$

NB. $3 + 4 = 7$
$3 + 1 = 4$

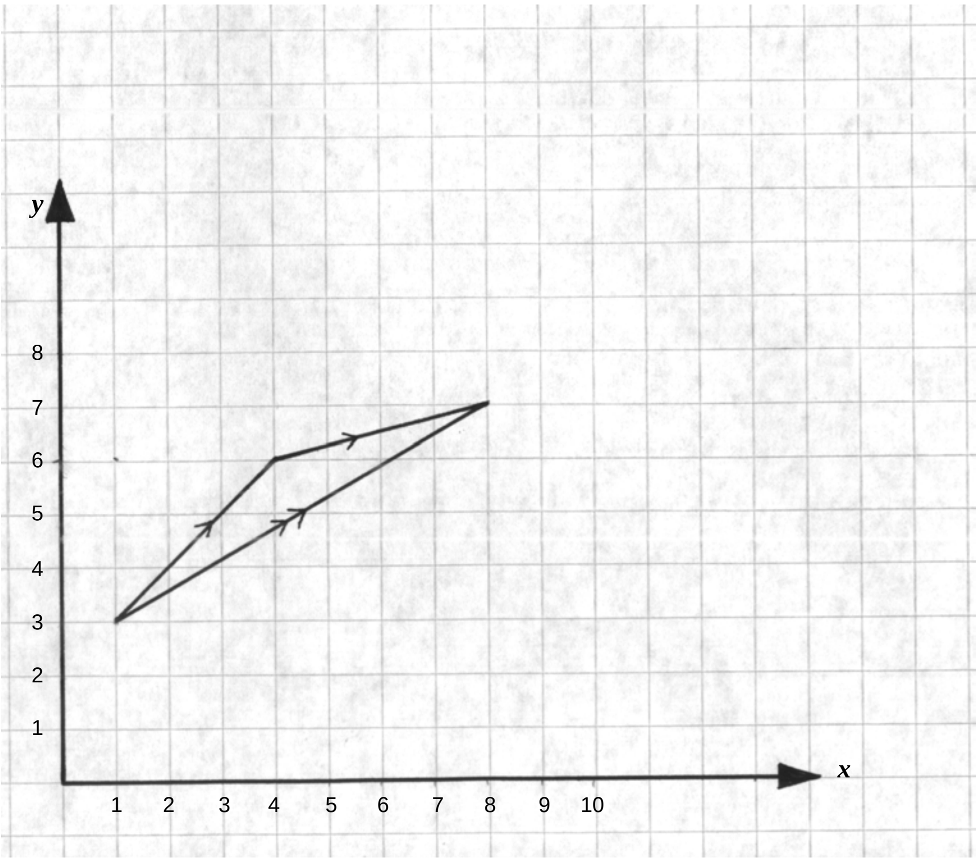


Fig. 16.15

Two given vectors expressed in column form

$$\underline{m} = \begin{pmatrix} a \\ b \end{pmatrix} \quad \text{and } \underline{n} = \begin{pmatrix} c \\ d \end{pmatrix}$$

$$\text{then } \underline{m} + \underline{n} = \begin{pmatrix} a \\ b \end{pmatrix} + \begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} a+c \\ b+d \end{pmatrix}$$

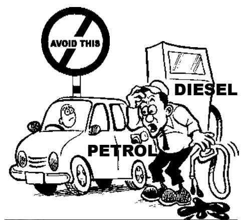
$$\text{Similarly } \underline{m} - \underline{n} = \begin{pmatrix} a \\ b \end{pmatrix} - \begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} a-c \\ b-d \end{pmatrix}$$

K. SCALAR MULTIPLICATION OF A VECTOR

A vector $\underline{a}$ is multiplied by a scalar, k.

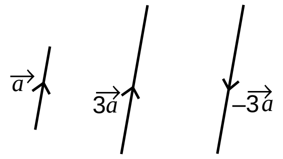
(i) For $k > 0$, vector $k\underline{a}$ has the **same** direction as $\underline{a}$ and has a magnitude equal to k times the magnitude $\underline{a}$.

(ii) For $k < 0$, vector $k\underline{a}$ has the direction **opposite** to that of $\underline{a}$ and a magnitude equal to k times the magnitude of $\underline{a}$.



Common Errors

Students fail to realise that $k < 0$ means a negative number.



where $k = 3$ and -3, respectively

Study the examples that follow:

Solve the following:

1. $2\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} -2 \\ 4 \end{pmatrix} = \begin{pmatrix} 12 \\ 16 \end{pmatrix}$

2. Given that,

$$\underline{a} = \begin{pmatrix} -3 \\ 4 \end{pmatrix} \qquad \underline{b} = \begin{pmatrix} 4 \\ -2 \end{pmatrix}$$

$$\underline{c} = \begin{pmatrix} 0 \\ -5 \end{pmatrix}$$

find $\underline{x}$ such that $3\underline{a} + \underline{x} = \underline{b}$ and $\underline{y}$ such that $\underline{a} - 3\underline{y} = \underline{c}$.

Solutions

1. $2\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} -2 \\ 4 \end{pmatrix} = \begin{pmatrix} 12 \\ 16 \end{pmatrix}$

$\begin{pmatrix} 2x \\ 2y \end{pmatrix} + \begin{pmatrix} -2 \\ 4 \end{pmatrix} = \begin{pmatrix} 12 \\ 16 \end{pmatrix}$

$2\underline{x} - 2 = 12$	$2\underline{y} + 4 = 16$
$2\underline{x} = 14$	$2\underline{y} = 16 - 4$
$\underline{x} = 7$	$2\underline{y} = 12$
	$\underline{y} = 6$

2. $3\underline{a} + \underline{x} = \underline{b}$

$$3\begin{pmatrix} -3 \\ 4 \end{pmatrix} + \underline{x} = \begin{pmatrix} 4 \\ -2 \end{pmatrix}$$

$$\begin{pmatrix} -9 \\ 12 \end{pmatrix} + \underline{x} = \begin{pmatrix} 4 \\ -2 \end{pmatrix}$$

$$\underline{x} = \begin{pmatrix} 4 \\ -2 \end{pmatrix} - \begin{pmatrix} -9 \\ 12 \end{pmatrix}$$

$$\underline{x} = \begin{pmatrix} 13 \\ -14 \end{pmatrix}$$

$\underline{a} - 3\underline{y} = \underline{c}$

$$\begin{pmatrix} -3 \\ 4 \end{pmatrix} - 3\underline{y} = \begin{pmatrix} 0 \\ -5 \end{pmatrix}$$

$$-3\underline{y} = \begin{pmatrix} 0 \\ -4 \end{pmatrix} - \begin{pmatrix} -3 \\ 5 \end{pmatrix}$$

$$-3\underline{y} = \begin{pmatrix} 3 \\ -9 \end{pmatrix}$$

$$\underline{y} = \begin{pmatrix} -1 \\ 3 \end{pmatrix}$$

1. Simplify:

a) $\begin{pmatrix} 3 \\ -2 \end{pmatrix} + \begin{pmatrix} -5 \\ -3 \end{pmatrix}$

b) $\begin{pmatrix} -6 \\ 4 \end{pmatrix} - \begin{pmatrix} -2 \\ -5 \end{pmatrix}$

2. Given that:

$\underline{m} = \begin{pmatrix} -3 \\ 2 \end{pmatrix}$ $\quad\underline{n} = \begin{pmatrix} -2 \\ 5 \end{pmatrix}$

$\underline{p} = \begin{pmatrix} -6 \\ 4 \end{pmatrix}$ $\quad\underline{r} = \begin{pmatrix} 4 \\ 7 \end{pmatrix}$

evaluate:

(i) $\underline{m} + \underline{r}$ (ii) $2\underline{m} + 3\underline{n}$

(iii) $\underline{m} + \underline{n} + \underline{r}$ (iv) $\underline{r} - \underline{p}$

(v) $2(\underline{p} - \underline{n})$

3. Solve the following vector equations:

a) $\begin{pmatrix}2\\3\end{pmatrix} + 3\underline{x} = \begin{pmatrix}7\\-4\end{pmatrix}$

b) $\begin{pmatrix}-2\\-4\end{pmatrix} - \underline{x} = \begin{pmatrix}8\\9\end{pmatrix}$

c) $4\underline{x} - \begin{pmatrix}3\\2\end{pmatrix} = \begin{pmatrix}-6\\4\end{pmatrix}$

d) $3\underline{x} - \begin{pmatrix}2\\-3\end{pmatrix} = \begin{pmatrix}-4\\2\end{pmatrix} - \underline{x}$

e) Find the two vectors $\underline{m}$ and $\underline{n}$ such that

$\underline{m} + \underline{n} = \begin{pmatrix}-2\\3\end{pmatrix}$ and $\underline{m} - \underline{n} = \begin{pmatrix}6\\-6\end{pmatrix}$

L. PARALLEL VECTORS

Hint

(i) Vectors which are parallel are equal to each other (provided their magnitudes are equal).
(ii) If two vectors are parallel then either they are:
a) equal or
b) one is a fraction of the other or
c) one is a multiple of the other.
(iii) Vectors in a line (one behind the other) are said to be parallel to each other. They are referred to as ***collinear****.*

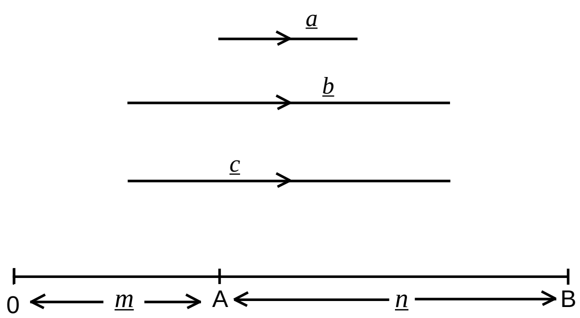


Fig. 16.16

For the vectors in Fig 16.16.

$\underline{a}$ // $\underline{b}$ ($\underline{a}$ is a fraction of $\underline{b}$)

$\underline{b} = \underline{c} \Rightarrow \underline{b}$ // $\underline{c}$ ($\underline{b} = \underline{c}$ as well as being parallel to each other)

$\underline{m}$ // $\underline{n}$ // $\overline{OB}$ ($\underline{m}$ is a fraction of $\overline{OB}$
$\overline{OB} = k\underline{m}$ means that $\overline{OB}$ is a multiple of $\underline{m}$ in terms of magnitude).

In summary,

1. $\underline{a} = k\underline{b} \Rightarrow |\underline{a}| = k|\underline{b}|$ and $\therefore \underline{a}$ // $\underline{b}$.

2. If $\underline{a}$ // $\underline{b}$, then $\underline{a} = k\underline{b}$ (where k can be 1 or a fraction or an integer (+ve or –ve).

3. If $p\underline{a} + q\underline{b} = h\underline{a} + k\underline{b}$ and $\underline{a}$ is not parallel to $\underline{b}$, then $p = h$ and $q = k$.

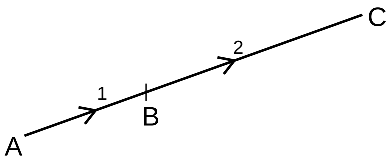
Hint

Collinear means on the same straight line.

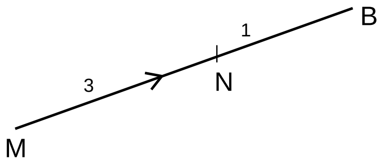
4. If $\overline{AB} = k\overline{BC}$, $\overline{AB}$ // $\overline{BC}$ and since B is a common point, A, B and C are **collinear**.

M. RATIO CONCEPTS IN VECTORS

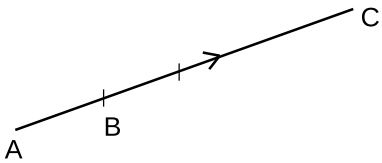
1. If AB = $\frac{1}{3}$ AC, then the ratio of AB : BC : 1 : 2



2. If $\overline{MN}$ = $\frac{3}{4}$ $\overline{MB}$, then, $\overline{MN}$: $\overline{NB}$ = 3 : 1



3. Consider a straight line ABC with AB = $\frac{1}{4}$ AC then:



$\overline{AB}$ = $\frac{1}{4}$ $\overline{AC}$

or $\overline{BC}$ = $\frac{3}{4}$ $\overline{AC}$

or $\overline{BC}$ = $3\overline{AB}$

or $\overline{AC}$ = $4\overline{AB}$

4. The Ratio Theorem of Vectors (Fig 16.17) states that if P lies **between** (not mid-point) A and B and

$\overline{AP}$: $\overline{PB}$ = m : n then;

$$\overline{OP} = \frac{n\underline{a}}{m+n} + \frac{m\underline{b}}{m+n}$$

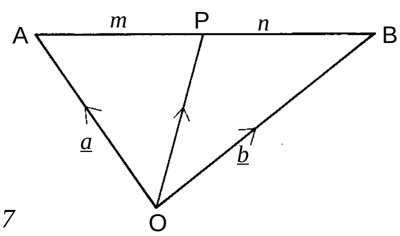


Fig. 16.17

If P is the **mid-point** of $\overline{AB}$, then $m = n$ hence $m + n = 2m$ or $2n$ and

$$\overline{OP} = \frac{m\underline{a}}{m+m} + \frac{m\underline{b}}{m+m}$$

$$\overline{OP} = \frac{m\underline{a}}{2m} + \frac{m\underline{b}}{2m}.$$

Common Errors

"Lies between A and B" to mean the mid point of A and B. This is wrong.

Hint

Since P is the midpoint
$m = n$

$\therefore \quad \overline{OP} = \dfrac{n\underline{a} + m\underline{b}}{2m}$

$n = m$

$\therefore \quad \overline{OP} = \dfrac{\cancel{m}\underline{a} + \cancel{m}\underline{b}}{2\cancel{m}}$

$\overline{OP} = \frac{1}{2}(\underline{a}+\underline{b})$

$$\overline{OP} = \frac{m(\underline{a} + \underline{b})}{2m}$$

$$\overline{OP} = \tfrac{1}{2}(\underline{a} + \underline{b})$$

Consider the following example:

1. Given that

$\overline{OA} = \begin{pmatrix} -2 \\ 4 \end{pmatrix}$, $\overline{OB} = \begin{pmatrix} 3 \\ 6 \end{pmatrix}$ and Q is a point on $\overline{AB}$, such that

$\overrightarrow{AQ} = \frac{1}{4}\overrightarrow{AB}$,

express as column vectors:

a) $\overline{AQ}$ b) $\overline{OQ}$

Solution

a) First sketch a diagram.

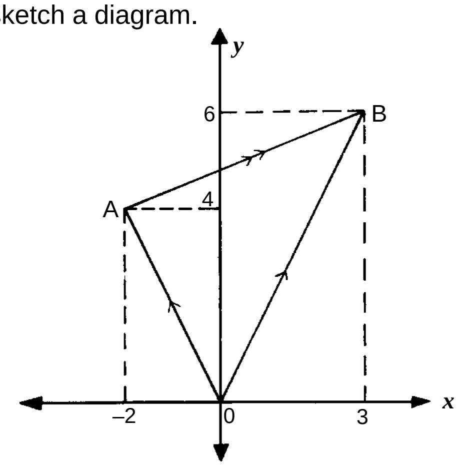


Fig 16.18

$\overline{AB} = -\overline{OA} + \overline{OB}$

$\overline{AB} = \overline{OB} - \overline{OA}$

$\overline{AB} = \begin{pmatrix} 3 \\ 6 \end{pmatrix} - \begin{pmatrix} -2 \\ 4 \end{pmatrix}$

$\overline{AB} = \begin{pmatrix} 5 \\ 2 \end{pmatrix}$

$\overline{AQ} = \frac{1}{4}\overline{AB}$

$= \frac{1}{4}\begin{pmatrix} 5 \\ 2 \end{pmatrix}$

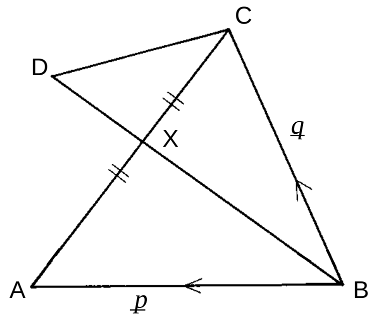
$\overline{AQ} = \begin{pmatrix} \frac{5}{4} \\ \frac{1}{2} \end{pmatrix}$

b) $\overline{OA} = \overline{OQ} + \overline{AQ}$

$\overline{OQ} = \begin{pmatrix} -2 \\ 4 \end{pmatrix} + \begin{pmatrix} \frac{5}{4} \\ \frac{1}{2} \end{pmatrix}$

$\overline{OQ} = \begin{pmatrix} -\frac{5}{4} \\ 3\frac{1}{2} \end{pmatrix}$

1.



In Fig 16.19

In Fig 16.19, $\overline{BA} = \underset{\sim}{p}$ and $\overline{BC} = \underset{\sim}{q}$ and $\overline{AX} = \overline{XC}$.

a) Express in terms of $\underset{\sim}{p}$ and $\underset{\sim}{q}$

a) $\overline{AX}$

b) $\overline{BX}$

b) Given that $\overline{BX} = \frac{3}{4}(\vec{p} + \vec{q})$, write down the numerical value of the ratio $\dfrac{\overline{BX}}{\overline{XB}}$.

2. a) The co-ordinates of Q are (5;10) and the coordinates of M are (–3;2). It is also given that:

$\overrightarrow{MN} = \begin{pmatrix} -4 \\ -7 \end{pmatrix}$

Find: (i) $\overline{QM}$

(ii) $|\overline{QM}|$

(iii) the co-ordinates of the point N.

b) X and Y are two points such that

$\overline{XY} = \begin{pmatrix} -10 \\ 6 \end{pmatrix}$.

Find $|\overline{XY}|$

3. Given that $\overrightarrow{OA} = \begin{pmatrix} x \\ 5 \end{pmatrix}$.

and $\overrightarrow{OB} = \begin{pmatrix} 6 \\ 0 \end{pmatrix}$, find:

a) the value of $|\overline{OB}|$

b) a value for x if $\overline{OA}$ and $\overline{OB}$ are two sides of a rhombus.

4. a) The vector $\underline{p}$ and $\underline{q}$ are defined as

$$\underline{p} = \begin{pmatrix} 6 \\ 5 \end{pmatrix} \text{ and } \underline{q} = \begin{pmatrix} -2 \\ 5 \end{pmatrix}$$

If $\underline{p} + 3\underline{t} = 5\underline{q}$ express $\underline{t}$ as a column vector.

b)

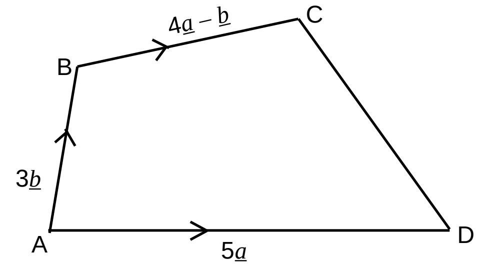


Fig 16.20

From Fig 16.20, express as simply as possible:

a) $\overline{AC}$.

b) $\overline{DC}$.

5. It is given that $\underline{a} = \begin{pmatrix} 5 \\ 7 \end{pmatrix}$ $\underline{b} = \begin{pmatrix} -5 \\ 2 \end{pmatrix}$ and $\underline{r} = \begin{pmatrix} x \\ y \end{pmatrix}$.

a) Find $|\underline{a}|$.

b) Express $3\underline{a} + 4\underline{b}$ as a column vector.

c) Given $\underline{a} - \underline{b} = 2\underline{r}$ find the value of x and the value of y.

6. The lines AB, PQ meet at M, and M is the midpoint of AB and PQ. Let $\overline{MA} = \overrightarrow{a}$ and $\overline{MP} = \overrightarrow{p}$.

a) Express $\overline{MB}$ and $\overline{MQ}$ in terms of $\overrightarrow{a}$ and $\overrightarrow{p}$.

b) Use the triangle law to prove that $\overline{AP} = \overline{QB}$. What kind of quadrilateral is APBQ?

7.

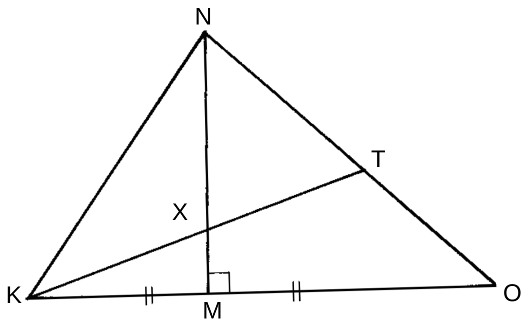


Fig. 16.21

Fig. 16.21 represents an equilateral triangle, KNO, with sides 14cm. M is the midpoint of KO and OT = 5cm.

Given that $\overrightarrow{ON} = 3\overrightarrow{p}$ and $\overrightarrow{MO} = \overrightarrow{q}$,

a) express as simply as possible, in terms of $\underline{p}$ and/or $\underline{q}$

(i) $\overline{KO}$ (ii) $\overline{OT}$ (iii) $\overline{MT}$ (iv) $\overline{MN}$

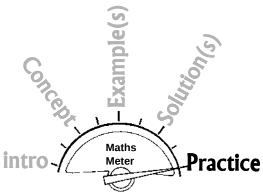
b) X is the point where NM intersects KT.

(i) Given that $\overline{MX} = h\overline{MN}$, find $\overline{KX}$

(ii) Given also that $\overrightarrow{KX} = k\overrightarrow{KT}$, form an equation involving p, q, h and k.

(iii) Use this equation to find the value of h.

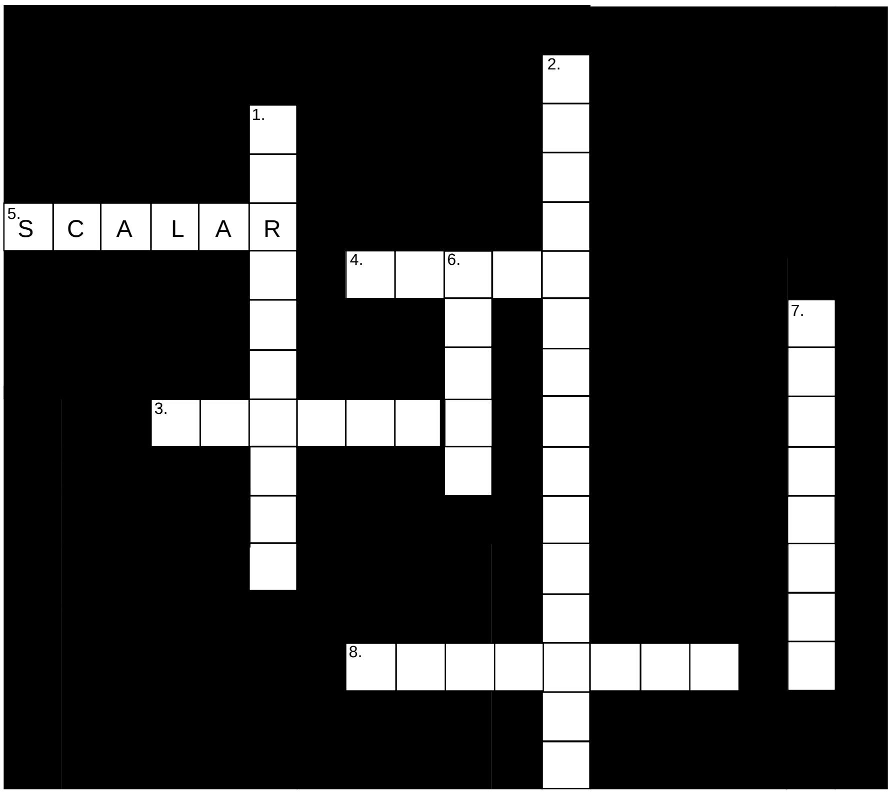
(iv) By using Pythagoras Theorem in triangle MNO. calculate $\overline{NM}$.
Find the length of $\overline{XM}$.



PUZZLER!

Complete the puzzle.

1. A vector whose magnitude is zero.
2. The result of adding or subtracting two vectors.
3. A quantity with both magnitude and direction.
4. Vectors that have the same magnitude and direction are said to be ____ .
5. A quantity with magnitude only. ____
6. A unit vector is a vector whose magnitude is ____ .
7. $\overrightarrow{c}$ $\overrightarrow{d}$ vectors c and d are ______ .
8. A vector whose tail is the origin 'O' on the cartesian plane.



SUMMARY

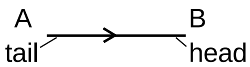
1. A scalar is a quantity with magnitude (size) only.

2. A vector is a quantity with both magnitude and direction.

3. Vectors may be denoted by:

$\overline{AB}$ or $\underline{a}$ or $\underline{AB}$ or **a**

They can also be represented by a directed line segment:



4. A vector whose tail is the origin O on the cartesian plane, is called a *position vector*.

5. The position vector of a point P(a:b), OP, is written as $\begin{pmatrix} a \\ b \end{pmatrix}$.

6. The magnitude of a vector, OP = $\begin{pmatrix} a \\ b \end{pmatrix}$ is denoted by OP.

It is the distance of P from O as given by:

$|OP| = \sqrt{a^2 + b^2}$.

7. The value of θ – vector direction of $\overrightarrow{OP} = \begin{pmatrix} a \\ b \end{pmatrix}$ given by $\tan\theta = \frac{b}{a}$.

8.

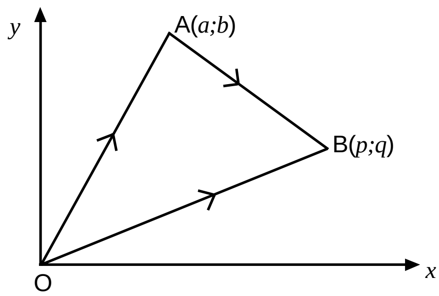


Fig 16.22

Refer to Fig. 16.22

If A(a,b) and B (p:q) then OA = $\begin{pmatrix} a \\ b \end{pmatrix}$ OB = $\begin{pmatrix} p \\ q \end{pmatrix}$

$\Rightarrow \overline{AB} = -\overline{OA} + \overline{OB}$

$\Rightarrow \overline{AB} = \overline{OB} - \overline{OA}$

$\overline{AB} = \begin{pmatrix} p \\ q \end{pmatrix} - \begin{pmatrix} a \\ b \end{pmatrix}$

a) $\overline{AB} = \begin{pmatrix} p-a \\ q-b \end{pmatrix}$

9. Vectors are equal if they are equal in both magnitude and direction.

10. When the direction of a given vector is reversed the vector becomes negative.

11. A zero vector is one whose magnitude is zero i.e. $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$
$\overline{AB} + \overline{BA} = 0$

12. A unit vector is a vector whose magnitude is unity, i.e.
$|\overline{OP}| = 1$ if $\overrightarrow{OP}$ is a unit vector.

13. A resultant vector is found when vectors are added or subtracted. The triangle rule and/or the parallelogram rule of vectors may be used to add or subtract vectors.

14. Any vector $\underline{a}$ may be multiplied by a scalar, k, to be vector $k\underline{a}$ which has the same direction as $\underline{a}$, and has a magnitude equal to k times the magnitude of $\underline{a}$.
NB. If k is negative the vector $k\underline{a}$ has the opposite direction to that of $\underline{a}$, but still has a magnitude equal to k times the magnitude of $\underline{a}$.

15. Parallel vectors can be co-linear, equal or a fraction of the other.

16. The Ratio Theorem states that if P lies between A and B (Fig 16.23), and AP: PB = m:n, then
$$\overline{OP} = \frac{n\underline{a}}{m+n} + \frac{m\underline{b}}{m+n}.$$

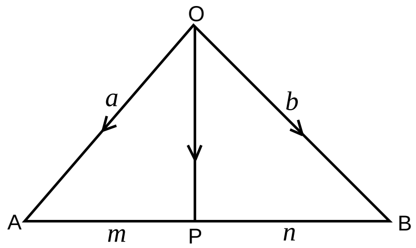


Fig. 16.23

And if P is the midpoint of the AB, then

$$\overrightarrow{OM} = \tfrac{1}{2}(\vec{a} + \vec{b}).$$

EXAM PRACTICE 16

Consider the example below:

1. It is given that $\overline{OM} = 4\underline{a} + \underline{b}$,
$\overline{ON} = 3\underline{a} + 4\underline{b}$ and OMPN is a parallelogram (Fig 16.24).

a) Express, as simply as possible, in terms of $\underline{a}$ and $\underline{b}$,
(i) $\overline{MP}$ (ii) $\overline{MN}$.

b) Find $\overline{OP}$ in terms of $\underline{a}$ and $\underline{b}$.

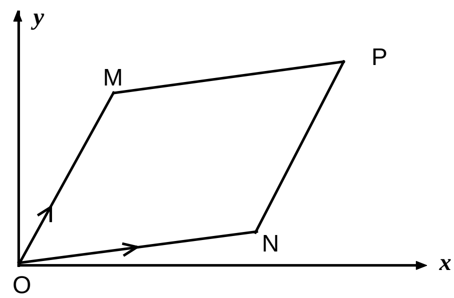


Fig. 16.24

Solution

1. a) (i) $\overline{MP} = \overline{ON}$
$= 3\underline{a} + 4\underline{b}$

(ii) $\overline{MN} = -\overline{OM} + \overline{ON}$

$\overline{MN} = \overline{ON} - \overline{OM}$

$\overline{MP} = (3\underline{a} + 4\underline{b}) - (3\underline{a} + 4\underline{b})$

$\overline{MN} = 3\underline{a} + 2\underline{b} - 3\underline{a} - 4\underline{b}$

$\overline{MN} = -2\underline{b}$

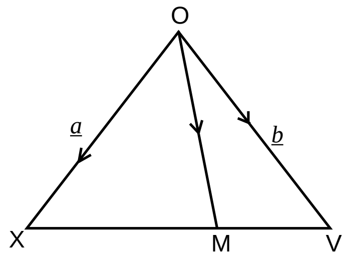
b) $\overline{OP} = \overline{OM} + \overline{MP}$

$\overline{OP} = 4\underline{a} + \underline{b} + 3\underline{a} + 4\underline{b}$

$\overline{OP} = 7\underline{a} + 5\underline{b}$

Now do the following:

1.



In the diagram, the point M is such that, $\overline{XM} = 2\overline{MV}$ and the point Z is such that, $\overline{XV} = \overline{VZ}$. Given that $\overline{OX} = \underline{a}$ and $\overline{OV} = \underline{b}$, express, as simply as possible, in terms of $\underline{a}$ and/or $\underline{b}$:

a) $\overline{XV}$
b) $\overline{XM}$
c) $\overline{OM}$
d) $\overline{XZ}$

2.

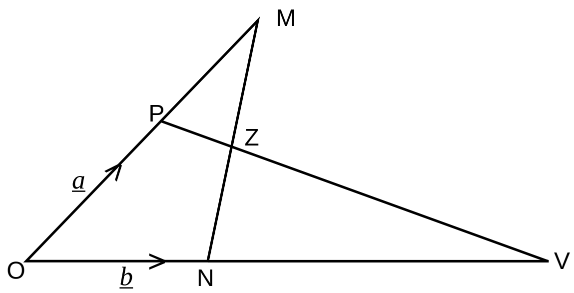


Fig. 16.25

In the diagram OPM and ONV are straight lines. PV intersects MN at Z, $\overline{OM} = \underline{a}$, $\overline{ON} = \underline{b}$, $\overline{OP} = 4$ and $\overline{PM}$ and $\overline{OV} = 3\overline{ON}$.

a) Express in terms of a and/or b:
(i) $\overline{MN}$.

(ii) $\overline{PV}$.

b) Given that $\overline{PZ} = h\overline{PV}$, express $\overline{PZ}$ in terms of h, $\underline{a}$ and $\underline{b}$.

3. P is the point (7; 4) and Q is the point (3; 7).
Calculate:

a) $\overline{PQ}$

b) $\overline{QP}$

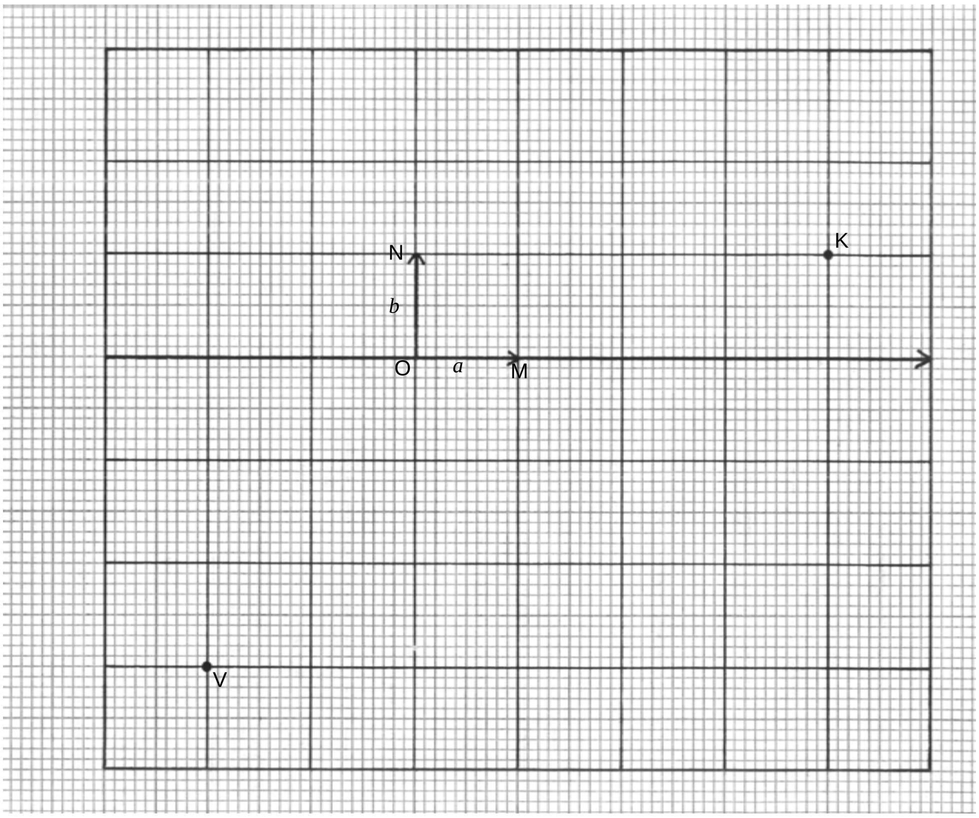


Fig 16.26

In the graph, $\overline{OM} = \underline{a}$ and $\overline{ON} = \underline{b}$.

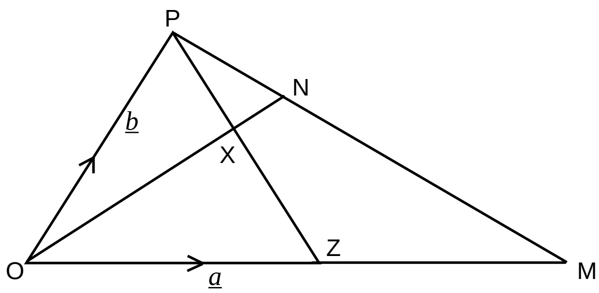
The points K and V are also shown on the graph.
Copy the above diagram on a graph paper and perform the following:

a) Mark and label clearly on the graph, the point X, such that $\overline{OX} = 2\underline{a} + 3\underline{b}$.

b) Write down $\overline{OK}$ in terms of $\underline{a}$ and $\underline{b}$.

c) Draw $\overline{VE}$ such that $\overline{VE} = \overline{OK}$.

5.



In the diagram $\overline{OM} = \underline{a}$, $\overline{OP} = \underline{b}$.
The point N is such that MN = 3NP and the point Z is, such that, OZ = ZM.

a) Express, in terms of $\underline{a}$ and/or $\underline{b}$, the vectors:

(i) $\overline{MP}$

(ii) $\overline{OZ}$

(iii) $\overline{MN}$

(iv) $\overline{ON}$

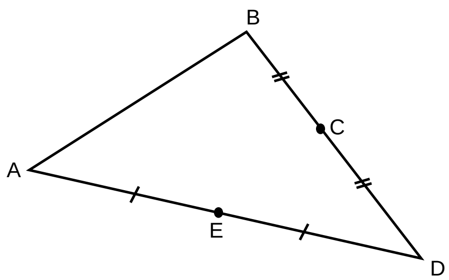
b) $\overline{ON}$ and $\overline{PZ}$ meet at X. Given that $\overline{PX} = k\overline{PZ}$, express $\overline{PX}$ in terms of a, b and k. Hence, show that
OX = $\frac{1}{2}k\underline{a} + (1 - k)\underline{b}$.

c) Given that $\overline{OX} = h\overline{ON}$, express $\overline{OX}$ in terms a, b and h.

d) Using these two expressions for $\overline{OX}$, find the values of h and k.

e) Find the numerical value of the ratio PX : XZ.

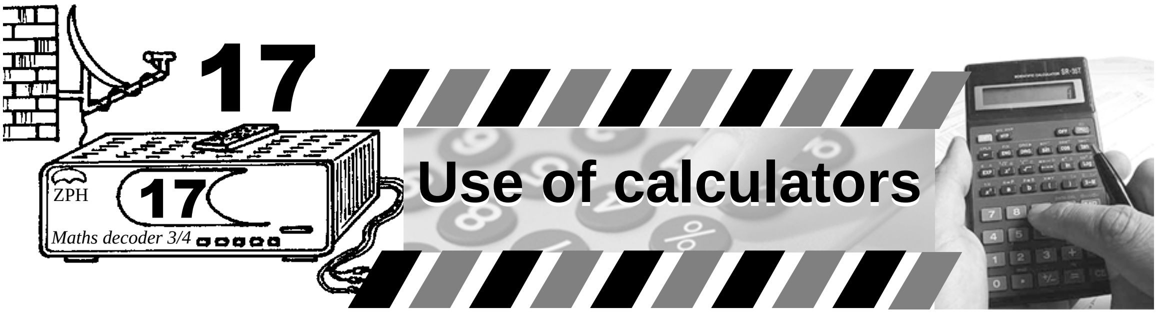
6.



In triangle ABD, E is the midpoint of AD and C is the mid-point of BD. Also given is that $\overline{AD} = \underline{p}$ $\overline{AB} = \underline{q}$. Express in terms of $\underline{p}$ and $\underline{q}$:

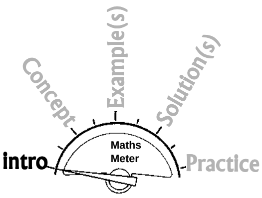
a) $\overline{BD}$
b) $\overline{DC}$
c) $\overline{AC}$
d) $\overline{BC}$
e) X lies on AC such that,
AX = $\frac{2}{3}$ AC. Express $\overline{BX}$ in terms of $\underline{p}$ and $\underline{q}$.

7. ABCD is a rectangle whose diagonals intersect at O.
If $\overline{AB} = \underline{a}$ and $\overline{AD} = \underline{b}$, express in terms of $\underline{a}$ and /or $\underline{b}$:
a) $\overline{CD}$
b) $\overline{BO}$
c) If $|\underline{a}| = 36$ and $|\underline{b}| = 10$,
calculate $C\hat{A}B$.



17 Use of calculators

An introduction to Scientific Calculators

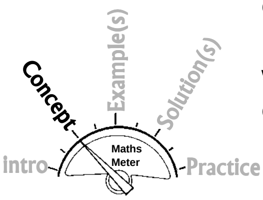


The past four decades have witnessed a revelation in the use and dependence on computers in general and the calculator in particular. Prior to the 70's the computations in mathematics were enhanced by the use of tables of logarithms and slide rules. With the widespread availability of calculators, the use of logarithm tables has become less and less popular. Slide rules are no longer used. Calculators come in a large array of different types, sizes and prices.

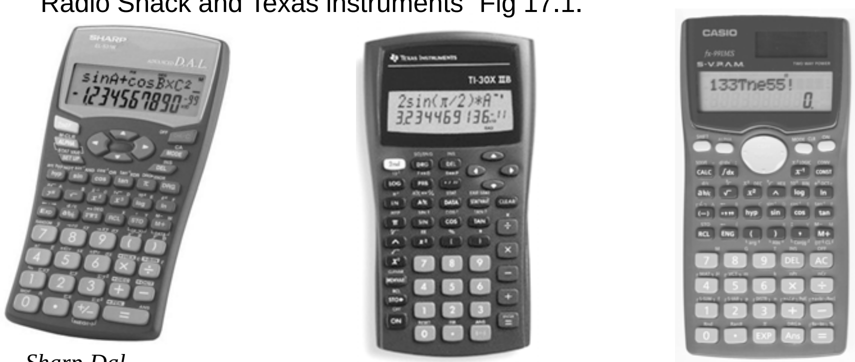
Table 1

Calculators

- Non-scientific Calculators
- Scientific calculators
- Graphing calculators



In the 'O' Level course for which this book is intended, the most appropriate type is the **scientific calculator**. Most of these scientific calculators use **algebraic logic** (AL) and not **Reverse Polish Notation** (RPN). In this chapter we explain the use of calculators with algebraic logic which are most popular and required by the course. Although calculators vary among manufacturers and models we recommend the following for this course: 'Sharp, Casio, Radio Shack and Texas instruments" Fig 17.1.



Sharp Dal *Texas instruments* *Casio*

Fig 17.1

A typical scientific calculator is illustrated in Fig 17.2. Also note that this introduction is a guide and not intended to take the place of **your own user's manual**.

TYPICAL SCIENTIFIC CALCULATOR OUTLOOK

Keys and their uses. In the diagram below, only the necessary 2nd Function notations for this level have been shown above the main keys.

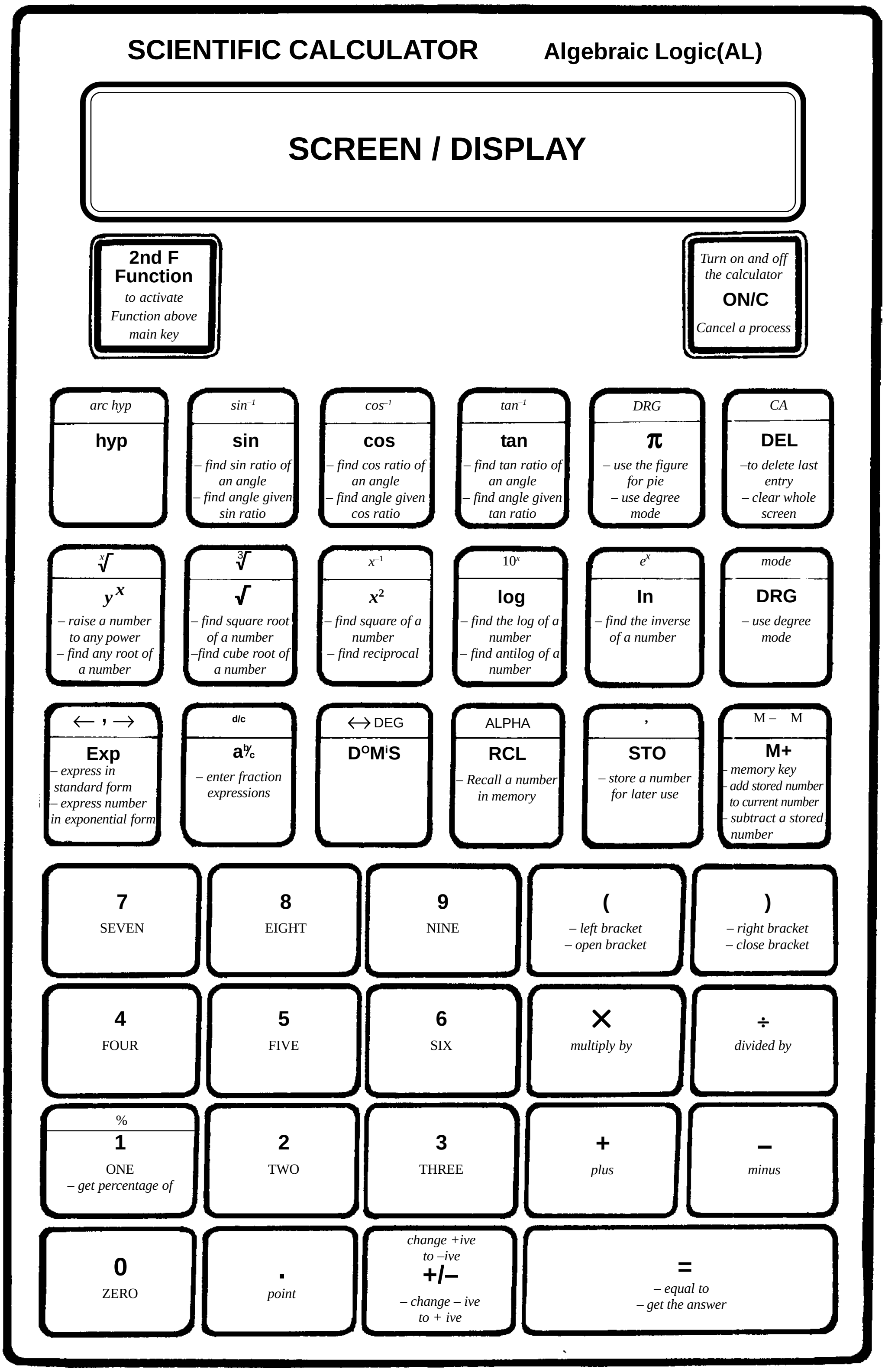


Fig 17.2

Syllabus Expectations

By the end o f this chapter the student should be able to use a scientific calculator to:

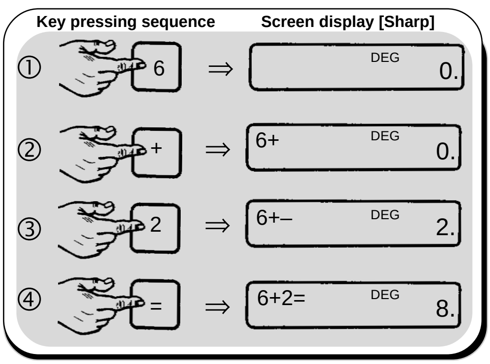
1. Perform arithmetic operations (+, -, x, ÷).
2. Find the square root or cube root of any number.
3. Find the square or cube of any number.
4. Raise a number to any power.
5. Find any root of a number.
6. Find the natural logarithm and the logarithm of a number.
7. Find the reciprocal of a number.
8. Store and recall a number using the memory key.
9. Enter a negative number on the display.
10. Use the 2nd function key in general.
11. Write a number in exponential form using the exponential key.
12. Find the Tan, Cos and Sin of an angle.
13. Find arc Tan, arc Cos and arc Sin.

ASSUMED KNOWLEDGE

In order to tackle work in this chapter, it is assumed that pupils are able to:

- Perform arithmetic operations of simple numbers manually.
- Understand the following definitions square, cube, power, square root, cube root, exponential, reciprocal, pie, tan, Cosine, Sine, inverse and 2nd function.

CONCEPT 1: To perform an arithmetic operation [6 + 2 =]
The important operation keys to note are the +, -, x, ÷ and =. You first enter the numbers or signs the way you write or read the arithmetic operation. Proceed as shown in the box:



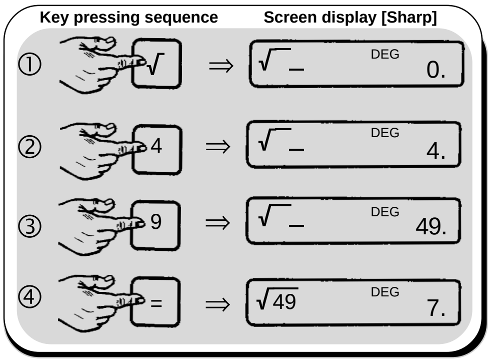
Tip

The key marked +/– allows you to change the sign of a number on display. Hence to enter –2 use the key strokes 2+/–2.

CONCEPT 2: To find the square root of any number. [$\sqrt{49}$ =]

To find the square root of any number:

Identify the square root key marked $\sqrt{\ }$ or $\sqrt{x}$. Proceed as shown in the box:

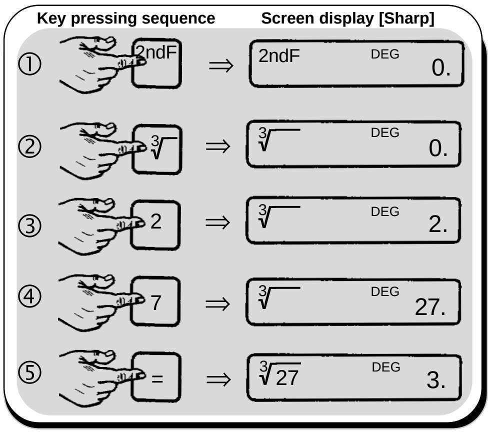


Tip

Experiment with your calculator to find out which method it uses.

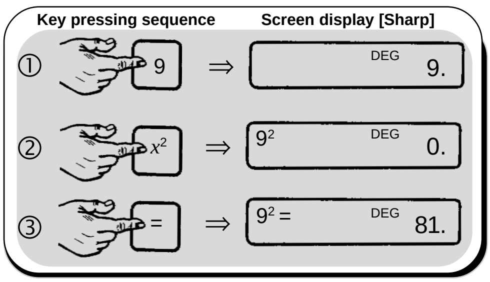
Concept 3: To find cube root of any number. [$\sqrt[3]{27}$ =]

Identify the cube root key marked $\sqrt[3]{\ }$. Proceed as shown in the box:



Concept 4: To find the square of any number. [9^2 =]

Identify the squaring key marked (x^2) and proceed as shown in the box:

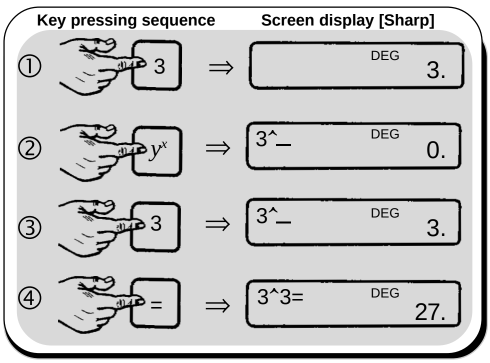


Hint

Note that the square root key and the squaring key maybe found on the same key with one of them being a second function.

Concept 5: To find the cube of any number. [3^3 =]

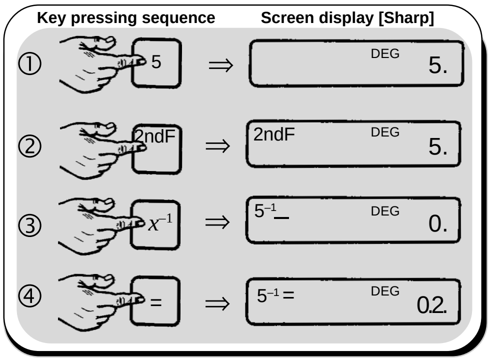
Identify the cubing key marked. Proceed as shown in the box:



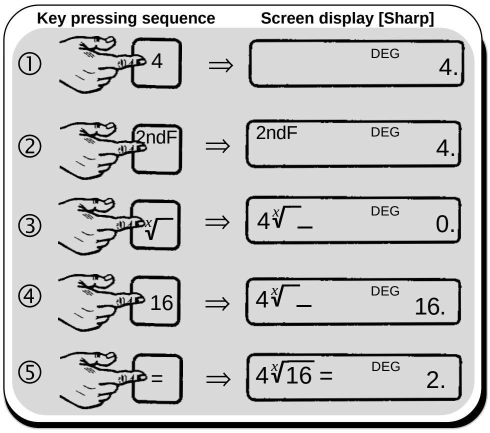
Tip

Again experiment with your calculator to find if you need to use the 2nd function key.

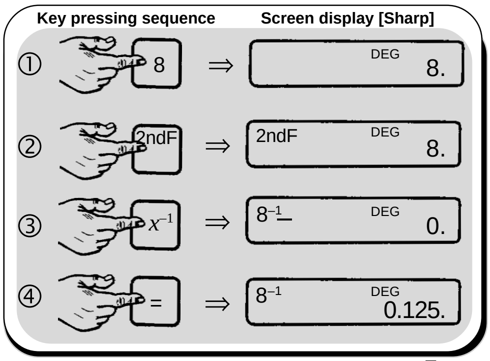
Concept 6: Raising a number to any power. [2^4 =]
Identify the exponential key marked x^y or y^x. This keystroke allows you to raise a number to any power. Proceed as shown in the box:



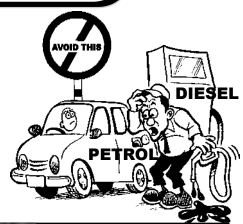
Concept 7: Finding any root of a number. [$\sqrt[4]{16}$ =]
Identify the root key marked or $\sqrt[y]{x}$ or $\sqrt[x]{y}$ and experiment with it. Some calculators do not have this sign. In this case you may use the inverse key in conjuction with the exponential key e.g. to find the fifth root of 32 using the following keystrokes. 32 inv x^y 5 = 2
Proceed as shown in the box:



Concept 8: To find the reciprocal of a number. [8^{-1} =]
To find the reciprocal use the key marked $\frac{1}{x}$ or x^{-1}.Proceed as shown in the box:



NOTE that the when two number have a product = 1, they are called reciprocals, in this case, 0,125 is a reciprocal of 8 and vice-versa.
Inverse operations "undo" each other e.g. squaring and taking the square root are inverse operations. Experiment this with your calculator.

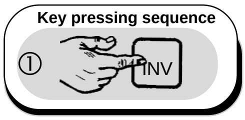


Common Error

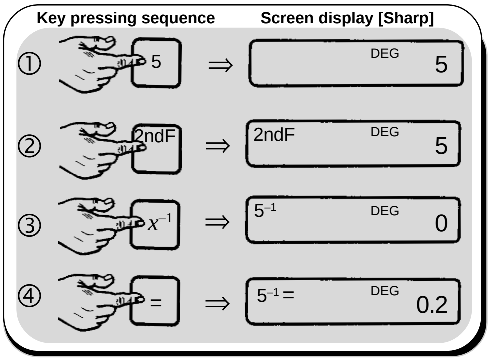
Student confuse the reciprocal key $\frac{1}{x}$ and the inverse key inv.

Concept 9: Use of the inverse key. Inv.

The use varies among different calculators. Please refer to your Manual. Proceed as shown in the box:



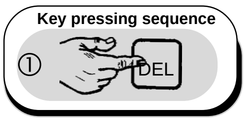
Concept 10: Use of the second function key 2ndF [5^{-1} =]



This key is usually marked as 2^{nd}F or Shift and is always used in conjuction with another key. The 2^{nd}F key is used to activate a function which is printed above an operation key and not the key itself. Most calculators use different colours for the 2^{nd}F key and corresponding functions above the keys.
For example if you want to find the reciprocal function, ($\frac{1}{x}$) is printed above another key. Proceed as shown in the box:

Concept 11: Use of the clearing or delete key

This key is usually marked CE or C or DEL. This key allows you to delete the last entry entered into the display. Continuous pressing the delete key will clear the entire operation on display. Experiment this with your calculator.

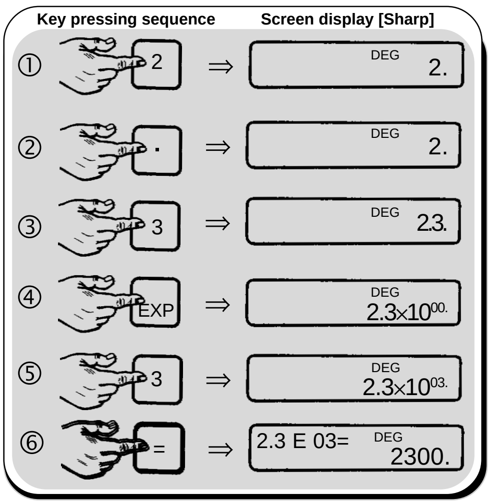


Tip

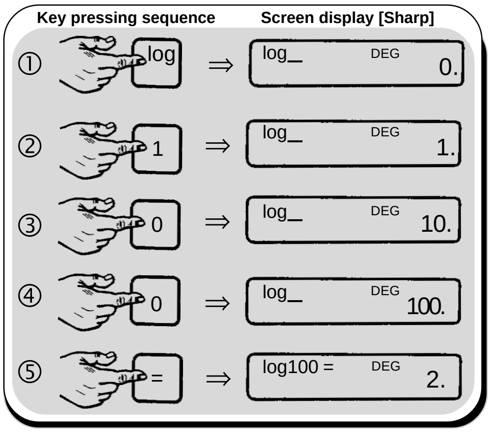
Note that when very large or small numbers are obtained as answers most scientific calculators express such answers in scientific notation (standard form)

Concept 12: Entering numbers in standard form into the calculator.

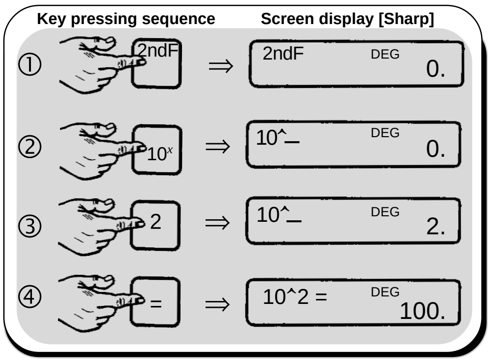
Any number can be expressed in standard form i.e. in the form a x 10^b e.g. 2300 = 2,3 x10^3 proceed as follows. 2.3 exp 3
Again this varies among different calculators models so experiment with your calculator.



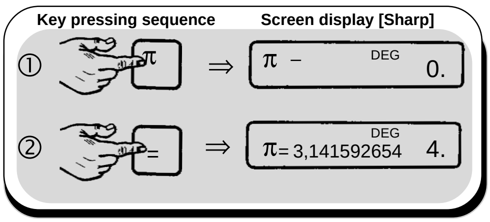
Concept 13: To find the natural logarithm of a number i.e. the power to which 10 must be raise to get the number.[log 100 =]
Proceed as shown in the box:



Concept 14: To find the anti-logarithm of a number.
[Antilog of 2]
Identify the 10^x key the use the 2nd function to manipulate the problem.Proceed as shown in the box:



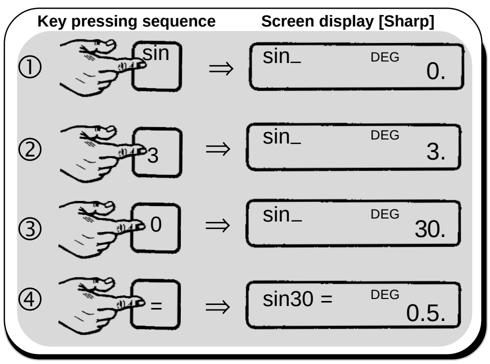
Concepts 15: The pie key
This is an irrational number and is required mostly in the mathematics of a circle. Its value on most calculators = 3,1415927. It is found by pressing the π key. Experiment with your calculator to see if you can get its value.



Concept 16: The Trigonometric keys [sin30⁰ =]

Sin^{-1} Cos^{-1} Tan^{-1}
Sin Cos Tan

These are used to find ratios of angles and vice versa. Above each of the three trigonometric keys there is also the $\sin^{-1}$, $\cos^{-1}$ and $\tan^{-1}$ function with the 2nd function key. Two concepts must be understood:



a) To find the ratio given an angle e.g. find Sin 30^0, Cos30^0, Tan 30^0 .
b) To find the angle give the ratio

c) E.g. Sinx^0 = 0.5, Cos x^0 = 0.5, Tan x^0 = 0.5

Use your calculators and experiment to see if you get the following.

Sin 30^0 = 0.5

Cos 30^0 = 0.8660

Tan 30^0 = 0.5774

Sin x = 0.5 $\Rightarrow$ $x = 30^0$

Cos x = 0.5 $\Rightarrow$ $x = 60^0$

Tan x = 0.5 $\Rightarrow$ $x = 26.57^0$

Hint

The chapters on solving the right-angled triangle will help you understand more about trigonometric ratios.

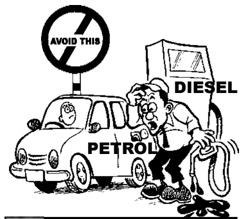
Concept 17: Use of the brackets

Opening and closing brackets may also be used with the scientific calculators. The round bracket are often on the relevant keys i.e. (and) Consider the following 2x(2+3 – 1) -4 you proceed as shown in the box:

	Key pressing sequence		Screen display [Sharp]
①	2	$\Rightarrow$	DEG 2.
②	×	$\Rightarrow$	2×_ DEG 0.
③	(	$\Rightarrow$	2×(_ DEG 0.
④	2	$\Rightarrow$	2×(_ DEG 2.
⑤	+	$\Rightarrow$	2×(2+_ DEG 0.
⑥	3	$\Rightarrow$	2×(2+_ DEG 3.
⑦	–	$\Rightarrow$	2×(2+3–_ DEG 0.
⑧	1	$\Rightarrow$	2×(2+3–_ DEG 1.
⑨	)	$\Rightarrow$	2×(2+3–1)_ DEG 0.
⑩	–	$\Rightarrow$	2×(2+3–1)–_ DEG 0.
⑪	4	$\Rightarrow$	2×(2+3–1)–_ DEG 4.
⑫	=	$\Rightarrow$	2×(2+3–1)–4= DEG 4.

Concept 18: Use of the $\boxed{\text{DRG}}$ key

D on the key stands for degree mode.
R on the key stands for radian mode.
G on the key stands for gradient mode
For almost all our computations in this course ensure that the screen is in degree (D) mode. Rarely do we need to operate the calculator when its in radian (R) or gradient G form.



Common Error

Always ensure the screen is in degree mode. If its another mode you don't get the correct answer (Experiment this with your calculator).

Concept 19: Memory key, $\boxed{\text{M}}$

This key allows you to store a number for later use. Three aspects are associated with memory.

a) Storing a number in the memory by using the key labelled M or Sto.
b) To add to or subtract from the value currently in memory by using the key labelled M+ and M- keys respectively,
c) Recalling or retrieve the value in memory by using the key labelled MR RM or RCL.
   Supposing you want to store the number 8 in memory
   Step 1 enter the number 8.
   Step 2 press the memory key.

You may then perform other calculators and when you need to use 8 you may retrieve it by pressing the memory recall key.
You may experiment with your calculator on how your can add to or subtract from the value currently in memory.

The word exponential is used to describe various situations in Maths e.g., exponential number, Exponential form , exponential graph. Discuss this with your teacher.

Concept 20: The $\boxed{e^x}$ key

e is a special number in maths = 2.7183. Although its use is beyond the scope of this book, this key should not be confused with the EXP key.

1. Using your scientific calculator, write down the relevant keys for the following operations:
   a) To switch on your calculator.
   b) To switch off your calculator.
   c) To delete the last digit appearing on your screen.
   d) To change your calculator from degree mode to radian mode and vice-versa.
   e) To find the square of a number.

2. Evaluate the following using the calculator
   a) $\left(\frac{7}{3}\right)^2 + 1.92 \times 10^2$

b) $\dfrac{4\pi \times 10^3}{\text{Log } 20 + 36} + \dfrac{\sqrt{52}}{\sqrt{36}}$

c) $(8400020 \text{ x } 4^{-3}) + (9\pi)^2 - 6.002$

d) $\dfrac{48 \times 102 + 3^5}{5^3 + 3 \times 10^4}$ e) $\dfrac{0.00047 \times 270\ 000}{420 \times 10{-}3 \times \sqrt[5]{50}}$

3. Find the following using the calculator:

a)	Sin 46^0	b)	Sin 120^0	c)	Sin 36^0
d)	Cos 56^0	e)	Cos 150^0	f)	Cos 0^0
g)	Tan 67^0	h)	Tan 162^0	i)	Tan 45^0

4. Evaluate the following using the calculator:

(i) Find the sine of the following

a)	0.6679	b)	0.1112	c)	1.000
d)	0.8660	e)	0.4679		

(ii) Find the cosine of the following:

a)	0.8946	b)	0.7121	c)	0.4136
d)	1.000	e)	0.0000		

(iii) Find the tangent of the following:

a)	0.3789	b)	2.7113	c)	1.4978
d)	3.369	e)	6.117		

5. Manipulate the following using the scientific calculator:

a) $\dfrac{[(-3) - 2\pi \times 4^5]^2 - 436.00 \times 1.022 \times 10^2}{[4\pi + (-3)(5)4] + (3.6)^2}$

b) $\dfrac{6213678 - 47778}{(4 \times 10^2)^2 - 3.6 - 4 \times (-6)3}$

c) $\dfrac{420 - [6.2 + 1.6^2]^3}{\text{Sin } 140^0 - \text{Cos}^2 30^0}$

d) $\dfrac{\text{Tan}^2\ 40^2 + \text{Sin}^2\ 100}{\text{Cos } 160^0 \times \sqrt[4]{42}}$

e) $\dfrac{(4 \text{ x } 10^{-3})^2 + (-3)^5 + (3.3)^2}{4\pi \text{ x } 10^{-6} - 4921 + 3 - \frac{2}{5}}$

f) $\dfrac{4\frac{1}{2} + 8 - \frac{2}{3} - (0.1115)^3}{\sqrt{(25.6 + 37) - (0.22)5}}$

SUMMARY

1. This chapter is only a guide and is not intended to take the place of your owner's manual. The best summary for this chapter is to read your manual thoroughly to understand it.

2. Give special attention to the following operation keys as you read your user's manual or experiment with your calculator.

(i) Arithmetic operation keys **+ – x ÷ =**
(ii) Clearing key **C** or **Ce** .
(iii) 2nd function key.
(iv) Memory key and its associates **M** or **Sto**.
(v) Square root key $\sqrt{\ }$ or $\sqrt{x}$
(vi) Squaring key x^2
(vii) Reciprocal key $\frac{1}{x}$
(viii) Exponential key x^y or y^x
(ix) Root key $\sqrt[y]{x}$ or $\sqrt[x]{y}$
(x) Change sign key **+/–**
(xi) Log key **Log**.
(xii) Anti-log key $\mathbf{10}^x$.
(xiii) The trigonometric ratio keys **Sin** **Cos** **Tan** as well as $\mathbf{Sin}^{-1}$ $\mathbf{Cos}^{-1}$ $\mathbf{Tan}^{-1}$.
(xiv) The pie key π or **pie**.
(xv) The **DRG** keystroke.
(xvi) The $\mathbf{e}^x$ key.