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Mathematics Today

A-LEVEL MATHEMATICS

EXAMINATION-TYPE QUESTIONS WITH SUGGESTED SOLUTIONS

Balah ARITHOPPAH
Ishwara DUVA-PENTIAH

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BOOKCENTRE Co Ltd

FOREWORD

This new title in the **MATHEMATICS TODAY** series comes at an opportune time to help students in the learning and understanding of mathematical concepts at GCE Advanced Level. Being the key focus of this book, its main objective is to help students, whether working on their own or receiving private coaching, to prepare for their A-Level Maths examination by developing their problem-solving skills.

Written by two experienced Mathematical educators, Balah ARITHOPPAH and Ishwara DUVA-PENTIAH, **A-LEVEL MATHEMATICS – Examination-type Questions with Suggested Solutions** covers in one single volume all the topics outlined in the pure mathematics section of single subject A-Level Mathematics 9709 Syllabus (P1, P2 and P3).

The suggested solutions given by the authors to the A-Level examination-type questions strip down the questions to their essential components, and show students how to set about answering them. What do the keywords mean? How does the information in one part help with the answer to a different part? How do you cut through the words to find out exactly what is being asked? The material has been class-tested by the authors through interaction with their A-Level Mathematics students and hence confirms the accessibility of the material of this book at this level.

The authors' experiences with students at various levels have shown that most students have difficulties when topics that are entirely new to them first appear. To help students with the transition from the Mathematics Syllabus D and/or Additional Mathematics class to a more rigorous A-Level course, the authors have included motivation for concepts most students have not seen before and provided more details when they introduce new methods. In addition, they have tried to give students ample opportunity to see these new tools in action.

This book has been written in a simple style mainly to allow an opportunity to provide motivational discussions in a number of places. Hence, a key feature of this textbook is that it follows, chapter-wise, the same sequence as the 9709 A-Level Maths Syllabus and in this format, it should prove an ideal book for revision work. The chapters are arranged in the following order but can be studied independently:

CONTENTS

UNIT 1

REMAINDER AND FACTOR THEOREMS

1. (a) The polynomial $2x^3 - 3ax^2 + ax + b$ has a factor $x - 1$ and, when divided by $x + 2$, a remainder of -54 is obtained.

Find the values of a and b.

(b) With these values, factorise the polynomial completely.

(c) Hence, or otherwise, find all the real factors of

(i) $2x^6 - 9x^4 + 3x^2 + 4$,

(ii) $4x^3 + 3x^2 - 9x + 2$.

2. Given that $(x - 2)$ and $(x + 2)$ are factors of $x^3 + ax^2 + bx + 4$, find the value of a and the value of b.

3. The polynomial $x^5 - 3x^4 + 2x^3 - 2x^2 + 3x + 1$ is denoted by $f(x)$.

(a) Show that neither $(x - 1)$ nor $(x + 1)$ is a factor of $f(x)$.

(b) Use the fact that $f(x) = (x^2 - 1)\, q(x) + ax + b$ to find the remainder when $f(x)$ is divided by $(x^2 - 1)$.

(c) Show that when $f(x)$ is divided by $(x^2 + 1)$, the remainder is $2x$.

(d) Hence find all the real roots of the equation $f(x) = 2x$.

4. Given that $f(x) = x^4 + ax^3 + bx^2 - 2x - 4$ has factors $(x + 2)$ and $(x - 1)$.

(a) Find the value of a and the value of b.

(b) What is the third real factors of $f(x)$?

(c) Prove that this third real factor is positive for all real values of x.

(d) If $f(x)$ is positive, find the set of values of x.

5. Factorise $(x^4 - 16)$ completely into real factors.

6. $f(x) = 6x^3 + cx^2 + 3$

(a) If $(2x + 1)$ is a factor of $f(x)$, find the value of c.

(b) Hence factorise $f(x)$ completely.

7. Find the values of x, given that

$$(x-2)(x^2-2x) = (x-2).$$

8. (a) If (x^2-1) is a factor of f (x), where

f $(x) = 3x^4 + ax^3 + bx^2 - 7x - 4$,

find the value of a and the value of b.

(b) Factorise f (x) completely, using the factor theorem.

(c) Hence, sketch the graph of $y = \mathrm{f}(x)$.

(d) If f $(x) < 0$, write down the set of values of x.

(e) Now sketch the graph of $y = |\mathrm{f}(x)|$.

9. Given that g $(x) = x^4 - 3x^3 + ax^2 + 15x + 50$, a being a constant.

It is also given that $(x+2)$ is a factor of g (x),

(a) Find the value of the constant a.

(b) Calculate the value of g (5).

(c) Factorise g (x) completely.

(d) If g $(x) > 0$, find the set of values of x.

(e) Given th[redacted] the set of values of x.

10. Given that f $(x) = 3x^3 + ax^2 + x - 2$ and that $x - 2$ is a factor of f (x).

(a) Find the value of a.

(b) Using this value of a, show that f $(x) = 0$ has only one real root.

11. Using the factor theorem, or otherwise, find the three roots of the following equation:

$$4x^3 + 8x^2 = 3 - x$$

12. If $(x-2)$ is a factor of f $(x) = 2x^3 + cx^2 - 5x + 6$, find the value of the constant c.

13. (a) f $(x) = 6x^3 + 11x^2 - 5x - 12$. Show that $(x-1)$ is a factor of f (x).

(b) Factorise f (x) completely.

14. Given that

$$x^2 + x + 1 = \frac{2x^4 + 2x^3 + 5x^2 + 3x + 3}{ax^2 + bx + c} \text{ for all values of } x.$$

Find the value of (a) a,

(b) b,

and (c) c.

15. $f(x) = 2x^3 + ax^2 + 16x + 6$

(a) If $(2x + 1)$ is a factor of $f(x)$, find the value of a.

(b) Factorise $f(x)$ completely, using this value of a.

(c) Using completing the square method, or any other method, show that the quadratic factor obtained is positive for all real values of x.

16. $(x + 1)$ and $(x + 2)$ are factors of the cubic polynomial $x^3 + ax^2 + bx - 8$.

a and b are constants.

Find the value of a and the value of b.

17. Use the factor theorem to factorise $f(x) = x^4 + x^3 - x^2 - 3x - 6$.

Show that there are 2 quadratic factors.

UNIT 2

PARTIAL FRACTIONS

1. Express $\frac{5x+4}{(x-1)(x+2)}$ into partial fractions.

2. Change into partial fractions: $\frac{4}{(x-3)(x+1)}$.

3. Express $\frac{1}{(x+2)(x+1)}$ into partial fractions.

4. Change into partial fractions: $\frac{4x-11}{(2x+1)(x^2+3)}$.

5. Express $\frac{x^2-11}{(3x-1)(x+2)^2}$ into partial fractions.

6. Given that f (x) = $\frac{3x+8}{(2x+1)(x^2+3)}$,
   change f (x) into partial fractions.

7. Change the following fraction into partial fractions: $\frac{1+3x+x^2}{(x+2)(x+1)}$.

8. Express the following fraction into partial fractions: $\frac{5}{x(x-1)(x+2)}$.

9. If f $(x) = \dfrac{x^2+2x}{(x-1)(x+1)^2}$, express f (x) into partial fractions.

10. Express $\dfrac{2x^2-3}{x(x+5)^2}$ in the form $\dfrac{A}{x}+\dfrac{B}{(x+5)}+\dfrac{C}{(x+5)^2}$.

11. Express $\dfrac{1}{x^2(x-3)}$ in the form $\dfrac{A}{x}+\dfrac{B}{x^2}+\dfrac{C}{x-3}$.

12. Change $\dfrac{16}{x^2-9}$ into partial fractions.

13. Express $\dfrac{3x}{(x^2-1)(x+2)}$ in the form $\dfrac{A}{x+1}+\dfrac{B}{x-1}+\dfrac{C}{x+2}$.

14. Change $\dfrac{5x}{(x^2+1)(x+2)}$ into partial fractions.

15. Express $\dfrac{3x}{(2x-1)(x^2-3)}$ into the form $\dfrac{A}{2x-1}+\dfrac{Bx+C}{x^2-3}$.

16. Change into partial fractions: $\dfrac{x-5}{(3x-1)(x-2)^2}$.

UNIT 3

INDICES, LOGARITHMIC AND EXPONENTIAL EQUATIONS

1. Without using a calculator or mathematical tables, find the exact value of

$$\left(1-\sqrt{3}\right)^4 - 4\left(1-\sqrt{3}\right)^2 - 8\left(1-\sqrt{3}\right) - 4.$$

Hence write down one linear factor of

$$x^4 - 4x^2 - 8x - 4$$

in the form $x + a + \sqrt{b}$, a and b being integers.

2. In the following equations, $a > 0$ and $a \neq e^2$.

Solve each equation and find x in terms of a.

(a) $a^x = e^{2x+1}$,

(b) $2\ln(2x) = 1 + \ln a$.

3. Find the value of x in the following equation:

$$4^x = 8^{2x+1}.$$

4. Given that $2(4^x) + 4^{-x} = 3$.

Using the substitution $y = 4^x$, find the two values of x.

5. Find the value(s) of x given that

$$x^{\frac{1}{2}} + x^{-\frac{1}{2}} = 2\left(x^{\frac{1}{2}} - x^{-\frac{1}{2}}\right).$$

6. Use the substitution $y = x^{-\frac{1}{2}}$ and solve the equation $x^{-\frac{1}{2}} + 2x^{-1} = 10$.

7. Find the exact value of x, given that

$$4x^{\frac{1}{2}} = 3 - 7x.$$

8. Solve the following equation:

$$3^{2x} = 4^{2-x}.$$

Give your answer to three significant figures.

9. Solve the following equation:

$$e^{2-2x} = 2e^{-x}.$$

10. Express $e^{\frac{x}{2}}$ in the form K^x.

Give the value of K to 3 S.F.

11. Rewrite $\left(3+\sqrt{2}\right)^4$ in the form $a + b\sqrt{c}$.

12. Express $\ln\left(2\sqrt{e}\right) - \frac{1}{3}\ln\left(\frac{8}{e}\right) - \ln\left(\frac{e}{3}\right)$ in the form $a + \ln b$.

13. Given that $x^{\frac{1}{3}} - 2x^{-\frac{1}{3}} = 1$.

Use the substitution $y = x^{\frac{1}{3}}$, to find the values of x.

14. By using the substitution $u = e^x$, or otherwise, solve the equation below:

$$e^x - 2e^{-x} = 1.$$

15. Solve the simultaneous equations:

$x + y = 1$

and $2^x = 3^y$.

UNIT 4

QUADRATIC EQUATIONS AND EXPRESSIONS

1. Draw the graph represented by each of the following. Then draw the line(s) of symmetry.

   (a) $y = x^2 - 3$,

   (b) $y^4 = (x + 3)^2 + 1$.

2. If $3x^2 - 5x + 1 = a(x + b)^2 + c$, find all the possible values of x, given that a, b and c are constants.

3. Write down the coordinates of the minimum point on the graph of

   $$y = 4x^2 + 12x + 1.$$

4. Write the expression $9x^2 - 36x + 52$ in the form $(ax - b)^2 + c$, where a, b and c are integers. Hence find the range of the expression $9x^2 - 36x + 52$ for all real values of x.

5. Express $2x^2 - 4x - 5$ in the form $A(x + B)^2 + C$, writing down the values of A, B and C. Hence find the range of $2x^2 - 4x - 5$ for all real values of x.

6. Sketch the graph of the curve

   $$y = x(1 - x).$$

   Hence find the maximum and minimum values of y for $-2 \leq x \leq 2$.

7. Express $25x^2 - 20x + 11$ in the form $(px + q)^2 + r$.

   p, q and r are integers.

   Hence find the set of values of x for which

   $$25x^2 - 20x + 11 > 23.$$

8. Given that $f(x) = 3x^2 - 7x + 1$, find the maximum or minimum value of $f(x)$.

9. Express $2x^2 - 7x + 5$ in the form $a(x + b)^2 + c$.

   a, b and c are constants.

   Hence solve the inequality

   $$2x^2 - 7x + 5 > 0.$$

✓**10.** In the equation

$$4x^2 + 16xy + y^2 = -16x - 13 - 14y,\ x \text{ and } y \text{ have distinct values.}$$

Using the correct substitution, or otherwise, find the values of x and y.

11. It is given that $f(x) = x^2 - 4x + 9$.

If x is real, prove that $f(x)$ is always positive.

Hence solve the inequality

$$x^3 + 2x^2 + x + 14 > (x + 1)(x^2 + 5).$$

12. Express the expression

$$f(x) = 2x^2 - 3x + 7 \text{ in the form } A(x + B)^2 + C.$$

Hence, write down the minimum value of $f(x)$ and the value of x for which it occurs.

13. If $(2x + 1)$ is a factor of $f(x)$, where

$$f(x) = 2x^3 + ax^2 + 16x + 6, \text{ find the value of } a.$$

(a) Also find the quadratic factor of $f(x)$.

(b) Using completing the square, or otherwise, show that the quadratic factor is positive for any real values of x.

14. Prove that the equation

$$(2a - 1)x^2 + (3a + 1)x + a + 1 = 0$$

has always 2 real and distinct roots for all real values of a.

15. Given that the roots of

$$x^2 - 3x + 1 = 2x + a$$

are real and distinct, find the least value of a.

a is an integer.

16. If $f(x) = (k + 1)x^2 + (k + 2)x - (k - 1)$, has two real roots, find the range of values of k.

UNIT 5

ARITHMETIC AND GEOMETRIC PROGRESSIONS

1. Given that the sum of the first twenty terms of an A.P. is 1 220, and the sum of the next twenty terms is 3 620.

(a) Find the first term.

(b) What is the common difference?

(c) Hence find the sum of the first eighteen terms of the progression.

2. The first term of an A.P. is 3, and the twenty-first term is 8.

(a) Find the common difference.

(b) Find also the least value of n such the sum of the first n terms exceeds 20.

3. Given that a geometric series has first term 10 and the common ratio r is greater than 0. The sum of the first 7 terms is three times the sum of the 8th and 9th terms. Prove that $r^7(4 - 3r^2) = 1$.

4. Look at the following sequence:

$$7.23, 7.28, 7.33, 7.38, \ldots, 9.68.$$

Then find:

(a) the common difference,

(b) the number of terms and,

(c) the sum of all the terms.

5. The following series is an arithmetic progression

$$x, a_1, a_2, a_3, a_4, \ldots, y.$$

(a) If there are 11 terms in all, find the common difference.

(b) Show that $a_3 = \dfrac{7x + 3y}{10}$.

6. (a) If a, b and c are in arithmetic progression, find b in terms of a and c.

(b) If p, q and r are in geometric progression, find q in terms of p and r.

7. Given that $|r| < 1$ and that r is the common ratio in a geometric progression.

S_n is the sum of the first n terms and $S_\propto$ is the sum to infinity.

(a) Express r in terms of S_n, $S_\propto$ and n.

(b) Find the sum of the first $2n$ terms.

8. In an $A. P.$, the sum of the first 5 terms is 48 and the sum of the first 6 terms is 64.5. Find the ninetieth term.

9. In a $G. P.$, the first term is 15 and the fifth term is $\frac{5}{27}$.

Find the sum of the first 20 terms.

Find also the sum to infinity and calculate the least value of n, the number of terms, for which the sum to n terms is greater than 22.

10. The sum of the first 70 terms of an arithmetic progression is 6 387.5.

The first, third and seventh terms are the three consecutive terms of a geometric progression.

Find:

(a) the first term,

(b) the common difference of the $A. P.$,

(c) the common ratio of the $G. P.$

11. A geometric progression is given by

$$1+\frac{1}{e^x}+\frac{1}{e^{2x}}+\ldots$$

Show that this $G. P.$ has a sum to infinity for any positive real value of x.

12. Given that $x = 5$, in the $G. P.$ $1 + e^{-x} + e^{-2x} + \ldots$, work out an expression for S_n which is the sum to n terms.

Also write down an expression for the sum to infinity.

Hence write down an expression for

$$S_\propto - S_n.$$

13. Given that the sum to infinity of a geometric progression, whose first term is x and common ratio is $\frac{1}{\sqrt{3}}$, is 1.

Find the value of x.

14. The sum to infinity of a $G.P.$ is $3(2 + \sqrt{2})$. Find the common ratio if the first term is 3.

15. A geometric progression has first term a and common ratio r, where $0 < r < 1$.
If the sum of the first three terms is one quarter the sum to infinity, find the value of r, correct to 3 S.F.
Find also the 10th term, given that $a = 2$. (*Answer to 2 d.p.*)

16. An $A.P.$ has first term a and common difference 5.
The sum of the first n terms, S_n, is 5 000.
Express a in terms of n.
Find also the n^{th} term of the progression.

UNIT 6

THE BINOMIAL THEOREM

1. (a) Find the cœfficient of x^8 in the expansion of $(1 + x^2)^{-1}$.

(b) Find the term independent of x in the expansion of $\left(x + \frac{3}{x}\right)^{10}$.

2. (a) Find the first three terms in the expansion of $(1 + x)^{\frac{1}{2}}$.

(b) Hence, find the value of $(1.02)^{\frac{1}{2}}$.

3. If $y = \dfrac{1}{\sqrt{(1 + 2x)} + \sqrt{(1 + x)}}$, show that

$$y = \frac{1}{x}\left\{\sqrt{(1 + 2x)} - \sqrt{(1 + x)}\right\}$$

provided $x > -\frac{1}{2}$ and $x \neq 0$.

Hence expand this expression up to x^2.

Also if $x = 0.01$, prove that

$$\frac{10}{\sqrt{102} + \sqrt{101}} \approx \frac{79\,407}{160\,000}$$

4. Expand $(1 + x)^{\frac{1}{3}}$ up to x^3, where $|x| < 1$.

5. Find the cœfficient of x^2 in the expansion of $(3 - x)^{\frac{1}{3}}$.

Hence write down the set of values of x for which the expansion is valid.

6. Find the first five terms in the expansion of $(2 + x)^{-\frac{1}{2}}$.

Write down the set of values of x for which the expansion is valid.

7. Expand $(1 + x)^{\frac{1}{2}}$ up to x^2.

If $x = \frac{1}{36}$, find the value of $\sqrt{37}$ to 3 d.p.

8. (a) Expand $(x + 1)^7$.

(b) Hence find the expansion of $(x - 1)^7$.

(c) Calculate the exact value of

$$\left(\sqrt{3} + 1\right)^7 - \left(\sqrt{3} - 1\right)^7.$$

9. Expand $(1 + 3x)^{\frac{1}{3}}$ in ascending powers of x, given that $|x| < \frac{1}{3}$, up to x^4.

10. Expand $(1 + 3x^2)^{-2}$ as far as x^6.

11. Find the first three terms in the expansion of $\left(1 + \frac{1}{x}\right)^{\frac{1}{2}}$.

Hence write down the term in $\left(\frac{1}{x}\right)^3$.

12. Write down the expansion of $(1 - 3x)^{-\frac{1}{2}}$ up to the term in x^3. Also state the range of values of x for which this expansion is valid.

13. Find the expansion of $\frac{1}{\sqrt{1-3x}}$ as a series of ascending powers of x up to x^3, given that $|x| < \frac{1}{3}$.

14. Write down the expansion of $(1-x)^{-3}$ up to the term in x^3, where $|x| < 1$.

Hence show that $\{(1-x)^{-3} + (1+x)^{-3}\}$ is approximately equal to $2(1+6x^2)$.

15. Expand $(7-2x)^{\frac{1}{2}}$ as far as x^4.

Then state the values of x for which the expansion is valid.

16. Find the first four terms in the expansion of $(1-4x)^{-2}$, where $|x| < \frac{1}{4}$.

UNIT 7

INEQUALITIES

1. Use an appropriate diagram to solve the following simultaneous inequalities

$$8 \leqslant 3x + 2y \leqslant 16$$
$$-5 \leqslant x - y \leqslant 0$$

2. Solv[redacted] give the conditions which may make the inequality

3. If $x(x - 2)(x + 3) > 0$, find the value(s) of x.

4. Find the solution set of x given that

$$x - \frac{5}{x} < \frac{7}{3}$$

5. First solve the simultaneous equations:

[redacted]ntegers (x, y) which satisfy the inequalities

6. Given that $x \in \mathbb{R}$, solve each of the following inequalities:

(a) $\frac{x}{x + 3} < 8$, [redacted]

(b) $x(x + 3) < 8$,

(c) $|x| < 7|x + 3|$.

7. Find the value(s) of x by solving each of the following inequalities.

(a) $\dfrac{x+2}{x-2} < 5,$

(b) $\dfrac{|x|+2}{|x|-2} < 5,$

(c) $\left|\dfrac{x+2}{x-2}\right| < 5.$

8. (a) Sketch the graph $y = 3|x|$

and $y = |x-2|$.

(b) Hence solve the equation $3|x| = |x-2|$.

(c) Find the range of values of x for which

$3|x| > |x-2|$

9. Given that $x \in \mathbb{R}$, sketch the graphs of

$y = |x+3|$

and $y = 3x + 1$

Hence solve the inequality

$|x+3| > 3x + 1.$

10. Solve the inequality

$x^3 > 5x^2 - 6x.$

11. Draw the graphs $y = x$

and $y = x^3$

Hence solve the inequality $x^3 - x > 0$.

12. Solve the following inequality

$\dfrac{x+2}{5-x} < 4.$

13. Find the range of values of x for which

$$|x - 3| > 2 - 3x.$$

14. Sketch the graphs of $y = |3x + 2|$

and $y = |x|$.

Hence, or otherwise, solve the inequality

$$|x| > |3x + 2|.$$

15. Draw, on graph paper, the lines

$y = x - 3$

and $y = 2x$.

Then on the same diagram draw the graphs of

$$y = |x - 3| \text{ and } y = |2x|.$$

Hence find the values of x for which

$$|2x| < |x - 3|.$$

16. Solve the inequality

$$|(x + 3)| > 2|(5x - 2)|.$$

UNIT 8

CURVE SKETCHING

1. Sketch the following curve

$$4x^2 + 9y^2 = 144.$$

2. Sketch each of the following curves and give the equation(s) of the line(s) of symmetry.

(a) $y = x^2 - 7x + 10,$

(b) $y = \dfrac{1}{x^2}.$

3. On the same graph, draw the graphs of

$$y = x^2$$

and $y = \sqrt{x}.$

Write down the coordinates of all points of intersection.

4. Sketch the curves represented by

$$y = \sin \theta$$
$$x = \cos \theta$$

Hence find the cartesian equation of the curve.

5. Sketch the curve represented by the equation

$$(3x)^2 + (4y)^2 = 5^2.$$

Also write down the coordinates of the points of intersection with the axes.

6. Sketch the following curves:

(a) $y = \dfrac{1}{x},$

(b) $y = \dfrac{1}{x^2},$

(c) $y = x^3,$

on separate diagrams.

7. Draw the curve represented by the equation

$$y = 6 - x - x^2.$$

Write down the coordinates of its intersection with the axes.

Also find the equation of its line of symmetry.

8. Sketch the curve $y = x^2 - 3$.

Hence, draw the curve $y = |x^2 - 3|$.

Write down the coordinates of the points where the curve cuts the axes.

9. Sketch the following graphs on separate diagrams

(a) $y = -x$,

(b) $y = \sqrt{x}$,

(c) $y = -x^3$.

10. Draw the curves represented by the equations

(a) $y = x + 2$,

(b) $y = \dfrac{1}{x + 2}$.

11. Sketch the curves represented by

$$y = \sin 2\theta$$

and $x = \cos 2\theta$

Find the Cartesian equation represented by these curves.

Hence find the coordinates of the point of intersection of these curves with the x-axis and the y-axis.

12. Sketch the curve $y = \tan x$, where $0 < x < \pi$.

Show the value of x when $\tan x = x$, by sketching a straight line.

13. Draw the graphs of $y = e^x$ and $y = \ln x$ on the same diagram, stating the coordinates of the points where the curves intersect the axes.

Find the relation between the two graphs.

14. Sketch the graph of $y = x^3 - 9x^2 + 23x - 15$ by first factorizing the expression. State the range of values of x for which the curve is positive.

15. Sketch two graphs to solve the following equation

$$x^3 + x - 2 = 0.$$

UNIT 9

PERMUTATIONS AND COMBINATIONS

1. Each card of a number of cards has a single capital letter printed on it. An arrangement of the eight cards forms a code. A player has eight cards lettered A, B, C, C, D, F, F, F.

(a) Find the number of different ways in which these letters can be arranged.

(b) How many of these codes will begin and end with the letter C?

2. Ten cards are lettered A, B, C, D, ..., J respectively. Sonia chooses 3 cards from the lot.

(a) In how many ways can she do this?

(b) How many arrangements of three will she get where there are no vowels at all?

3. A group of 12 persons set off on a journey in 3 cars; each car will contain 4 persons. If each car is driven by its owner, find the number of ways in which the remaining 9 persons will travel.

4. A board of juries has to select 5 winners out of a total of 12.

Calculate the number of ways in which this can be done.

5. A group of judges has to select 7 finalists out of a total of 25, without placing them in order.

Find the number of ways in which this can be done.

6. Four objects, p, q, r, and s are to be placed in four containers P, Q, R and S.

An object is correctly placed if it is put in its right container (i.e. p in P, etc.)

Find the number of ways in which the objects can be placed in such a way that there are 1 correct and 3 incorrect.

7. 10 boys take part in a race.

If there have been no dead-heats, how many ways can the results be given for the boy who comes first, second or third.

8. A group of representatives of 5 pupils is to be chosen from a class of 20 boys and 13 girls.

Find the number of ways in which this can be done if at least one boy and at least one girl are to be chosen.

9. 12 juries sit all round a circular table with twelve chairs equally spaced.

In how many ways can they be seated?

10. Twenty guests are seated all round a rectangular table where there are 20 seats well placed and equally spaced.

In how many ways can the twenty guests be seated?

11. How many even numbers, greater than 600 000, can be made from the digits 1, 2, 3, 4, 5, 6, without repetitions?

12. Simplify:

$$\frac{n!}{(n-r)!(r!)} - \frac{n!}{(n-r+2)!(n-r-1)!}$$

. How many arrangements can be made with the letters of the word "MAURITIUS"?

. In how many ways can eleven players be chosen to play a football match if they are to be selected from 6 players in Group A, 3 in Group B and 2 in Group C? At least one of each group should be in the football team, and no. of players in A = 8, in B = 5 and in C = 4.

. Simplify:

(a) ${}^{n+1}C_{n-1}$,

(b) ${}^{n+1}P_{n-1}$.

UNIT 10

FUNCTIONS

1. Functions f and g are defined as follows:

$$f : x \mapsto x^2$$

$$g : x \mapsto 3 - 2x \qquad \text{for } x \in \mathbb{R}.$$

Find the value of each of the following:

(a) $fg(x)$,

(b) $gf(x)$,

(c) $f^{-1}(x)$,

(d) $g^{-1}(x)$.

2. A certain function f maps x onto e^x, where $x \in \mathbb{R}$.

Find:

(a) $f^2(x)$,

(b) $f^{-1}(x)$,

(c) $f(-2)$.

3. Functions f and g are defined by

$$f : x \mapsto e^{3x}, \qquad x \in \mathbb{R}$$

$$g : x \mapsto \sqrt{x}\ , \qquad x \geqslant 0$$

Simplify each of the following:

(a) $fg(x)$,

(b) $(fg)^{-1}(x)$.

4. The function g maps x onto $1 + e^x$, for all real values of x.

Find $g^{-1}(x)$ in terms of x.

5. Functions f and g are defined by

$$f : x \mapsto \ln x, \qquad x \in \mathbb{R},\ x > 0$$

$$g : x \mapsto (x + 1), \quad x \in \mathbb{R}.$$

(a) Find expressions for (i) $f^{-1}(x)$,

(ii) $g^{-1}(x)$.

(b) Simplify $(fg)^{-1}(x)$.

6. The function h maps x onto $x^2 - 9x$, $x \in \mathbb{R}$

and $|x| < 3$.

Find an expression for $h^{-1}(x)$.

7. Given that $f : x \mapsto x + 4$, $x \in \mathbb{R}$

and $g : x \mapsto 3x$, $x \in \mathbb{R}$

(a) Find the value of x for which $f(x) = g(x)$.

(b) Also find the value of x if $f^{-1}(x) = g^{-1}(x)$.

8. The functions h and k are defined as follows:

$$h : x \mapsto 2x^2 + 3, \quad x \in \mathbb{R}$$

and $$k : x \mapsto \sqrt{\frac{x-3}{2}}, \quad x \geqslant 3.$$

Find the expression for $hk(x)$.

9. Given that functions f and g are defined as follows:

$f : x \mapsto x - 1$,

$g : x \mapsto x^2$, $x \in \mathbb{R}$.

(a) Write down an expression for

(i) $f^{-1}(x)$,

and (ii) $g^{-1}(x)$.

(b) If $fg(x) = gf(x)$, find the set of values of x.

10. The functions g and h are defined as follows:

$$g : x \mapsto |x|, \quad x \in \mathbb{R}$$

$$h : x \mapsto \sqrt{x}, \quad x > 0$$

Find the value(s) of x for which

$$g^{-1} h^{-1} (x) = h^{-1} g^{-1} (x).$$

11. A function f is defined by

$$f : x \mapsto \ln x, \quad x > 0$$

Find

(a) $f(2)$,

(b) $f^{-1}(x)$,

(c) $f^2(x)$.

12. Given that $x \in \mathbb{R}$ and that the functions f and g are defined as follows:

$$f : x \mapsto x^2$$

$$g : x \mapsto 3 - 2x$$

Find

(a) $fg(x)$,

(b) $f^{-1}(x)$,

(c) $g^{-1}(x)$,

(d) $(fg)^{-1}(x)$.

13. If $x \in \mathbb{R}$ and $f(x) = 3x^{\frac{1}{2}} + 7$,

find the value of $f^{-2}(x)$.

14. Given that $f(\theta) = \sin \theta$

and $g(\theta) = \cos \theta$ for $0 \leqslant \theta \leqslant \pi$

Find θ when (a) $f(\theta) = 0.5$,

Find θ when (b) $g(\theta) = \dfrac{1}{\sqrt{2}}$,

(c) $fg\left(\dfrac{\pi}{2}\right)$.

15. Functions g and h are defined as follows:

$$g : x \mapsto (x-1)(x-2), \quad x \in \mathbb{R}$$
$$h : x \mapsto (x^2-1), \quad x \in \mathbb{R}$$

(a) Find gh (x).

(b) What are the exact values of x, when gh $(x) = 0$?

16. Given that the function of f is defined by

$$f : x \mapsto 16 - x^2, x \in \mathbb{R}$$

Find the exact value of

(a) $f^{-1}(x)$, and

(b) $f^2(x)$.

17. Functions f and g are defined by

$$f(x) = \frac{1}{x}$$

$$g(x) = \frac{1}{x^2}$$

(a) Find the range of f (x).

(b) Calculate the value of $g^{-1}(2)$.

18. Given that $f(x) = 3x - 1, \ x \in \mathbb{R}$

and $g(x) = 2x + 5, \ x \in \mathbb{R}$.

Find :

(a) fg (x),

(b) gf (x),

(c) the value of x for which fg (x) = 2gf (x).

UNIT 11

TRIGONOMETRY

1. Rewrite the equation $7\sin^2\theta = \frac{1}{2}(8 + 5\cos\theta)$ as a quadratic equation in $\cos\theta$.

Hence solve the equation, for $0° \leqslant \theta \leqslant 360°$, giving your answers to the nearest 0.1°.

2. (a) Prove that $\tan\theta - \cot\theta = -2\cot 2\theta$.

(b) Hence solve the equation $\tan\theta - \cot\theta = 5$, where $0° < \theta < 360°$.

3. ✓Express $(5\sin\theta - 12\cos\theta)$ in the form $R\sin(\theta - \alpha)$, where $R > 0$ and $0 < \alpha < \frac{\pi}{2}$.

Hence find the solution set of θ, given that $5\sin\theta - 12\cos\theta = 5$, where $0° \leqslant \theta \leqslant 180°$.

4. ✓ (a) Show that cos x eot x may be written as $\frac{1 - \sin^2 x}{\sin x}$.

(b) Hence solve the equation $2\cos x \cot x = 3$, where $0° \leqslant x \leqslant 360°$.

5. ✓ (a) Change $2\cos\theta - 3\sin\theta$ to the form $R\cos(\theta + \alpha)$, where $R > 0$ and $0° < \alpha < 90°$.
Write down the value of R and of α.

(b) Hence solve the equation $2\cos\theta - 3\sin\theta = 3$, giving all solutions in the interval $0° \leqslant \theta \leqslant 360°$.

(c) Find the coordinates of the stationary point on the curve
$y = 2\cos\theta - 3\sin\theta$.

6. Prove the identity $\cot\theta + \tan\theta = 2\operatorname{cosec} 2\theta$.

7. ✗ (a) Show that the equation $3\cot\theta = 2\sin\theta$ can be expressed as $2\cos^2\theta + 3\cos\theta - 2 = 0$.

(b) Hence solve the equation $3\cot\theta = 2\sin\theta$, for $0° \leqslant \theta \leqslant 360°$.

8. (a) Show that the equation $\sin(x + 30^\circ) = 2\sin(x + 60^\circ)$ may be simplified to

$\tan x = -\left(4 + 3\sqrt{3}\right)$

(b) Find the values of x between 0° and 360°.

(c) Write down the exact value of $\sin 2x$.

9. ✓(a) Express $3\sin\theta + 4\cos\theta$ in the form $R\sin(\theta + \alpha)$, where $R > 0$ and $0^\circ < \theta < 90^\circ$. Give the value of α to 1 d.p.

(b) Hence solve the equation $3\sin\theta + 4\cos\theta = 3$, giving values of θ such that $0^\circ < \theta < 360^\circ$.

(c) Also write down the maximum value of $\dfrac{1}{3\sin\theta + 4\cos\theta - 3}$

10. Given that θ has values in the interval $0^\circ \leqslant \theta \leqslant 180^\circ$.
Find the values of θ given that $\cos 3\theta + 2\sin 3\theta = 0$

11. (a) Show that the equation $\sin(30^\circ - x) = 3(\cos 30^\circ - x)$ can be written as

$\tan x = 5 - 2\sqrt{3}$

(b) Hence solve the equation $\sin(30^\circ - x) = 3(\cos 30^\circ - x)$, where $0^\circ < x < 360^\circ$.

12. (a) Show that the equation $\cos(x + 30^\circ) + \sin(x + 30^\circ) = 0$ can be written in the form $\tan x = k$, k being a constant.

(b) Solve the equation $\cos(x + 30^\circ) + \sin(x + 30^\circ) = 0$, given that $0^\circ < x < 180^\circ$.

Prove the identity $\cot x - \operatorname{cosec} 2x \equiv \cot 2x$.

Given that $x = \cos^2\theta$ and that $2x^2 + 3x - 2 = 0$, find all the possible values of θ between 0° and 360°.

✓. (a) Express $\sqrt{3}\cos\theta + \sin\theta$ in the form $R\cos(\theta - \alpha)$,

where $R > 0$ and $0 < \alpha < \frac{1}{2}\pi$, and find the exact value of α.

(b) Hence find the set of values of θ given that $\sqrt{3}\cos\theta + \sin\theta = \sqrt{3}$ and that $0 < \theta < 2\pi$.

16. Solve the equation $\sin 2x - \cos x = 0$, given that $0^\circ \leqslant x \leqslant 360^\circ$.

17. (a) Change the equation $4\cos^2\theta - \sin^2\theta - 3\sin\theta\cos\theta = 0$ into a quadratic equation in $\tan\theta$.

(b) Hence solve the equation for $0^\circ \leqslant \theta \leqslant 180^\circ$.

UNIT 12

DIFFERENTIATION

1. (a) Differentiate $\sqrt{(x^3 - 1)}$ with respect to x.

   (b) The equation of a curve is given by

   $$y = 2x^3 - 3x^2.$$

   (i) Find the coordinates of two turning points on the curve.

   (ii) Determine the nature of each of these two points.

   (iii) What is the set of values of x for which $(2x^3 - 3x^2)$ is an increasing function of x?

2. A curve has equation $y = x^2 - 8x + 7$. M is a point on the curve, with x-coordinate 1.

   (a) Find the gradient at any point on the curve.

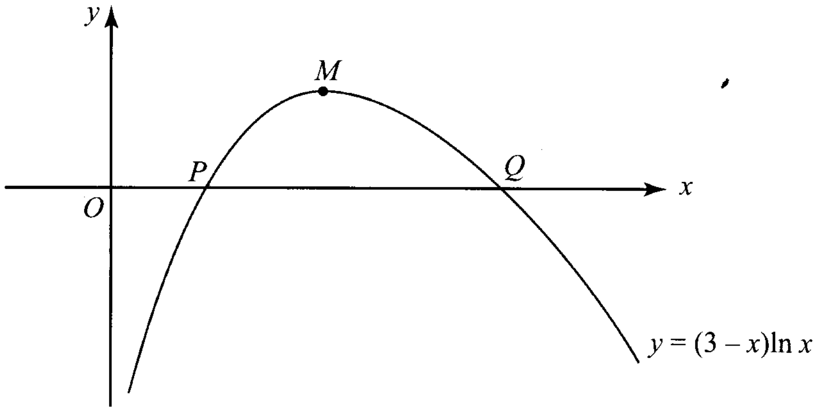
   (b) Hence find the equation of the tangent to the curve at M.

The equation of a curve is

$$y = 3x^2 - 2x - 1.$$

(a) Find the coordinates of the turning point on the curve and determine its nature.

(b) If X with coordinates $(0, -1)$ is a point on the curve, find the equation of the normal to the curve at X.



The drawing shows the curve $y = (3 - x)\ln x$. Its maximum point is the point M. The curve cuts the x-axis at the points P and Q.

(a) Find the coordinates of P and Q.

(b) Show that the x-coordinate of M satisfies the equation

$$x = \frac{3}{(1 + \ln x)}$$

5. A cylinder, open at one end, is made of a thin metal sheet. The total surface area, the curved surface and the bottom base, is 96π cm^2.
The radius of the cylinder is r cm and its height is h cm.

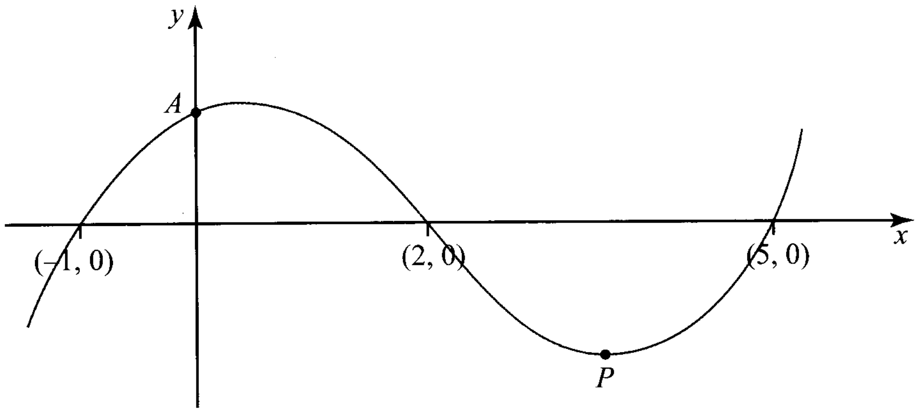
(a) Express h in terms of r.

(b) Show that the volume of the cylinder is $\frac{1}{2}\pi r\left(96 - r^2\right)\text{cm}^3$

(c) If V has a stationary value, find the length of r.

(d) Find this stationary value and determine its nature.

6.



The curve on the diagram is represented by the equation $y = ax^3 + bx^2 + cx + d$.

(a) Find the values of a, b, c and d.

(b) Find the coordinates of the point A.

(c) Write down the exact x-coordinate of its minimum point P.

7. A curve on the diagram is represented by the equation $y = xe^x$.

(a) Find $\frac{dy}{dx}$.

(b) Write down the coordinates of its stationary point.

8. The equation of a curve is

$$y = \cos 2x + 2\sin x.$$

Find the x-coordinates of its stationary points for which $0 < x < \pi$.
Also determine whether these points have maximum or minimum values.

9. A curve has equation $y = x^3 - x^2 - 8x + k$, k being the constant

(a) Write down an expression for $\frac{dy}{dx}$.

(b) Find the x-coordinates of its stationary points.

(c) Hence find the value(s) of k for which the curve has a stationary point on the x-axis.

10. A curve has equation $y = x^3 - 3x + 2 + k$, where k is a constant.

(a) Find an expression for $\frac{dy}{dx}$.

(b) Write down the coordinates of the two turning points.

(c) Hence find the value(s) of k for which the curve has a stationary value on the x-axis.

11. The equation of a curve is

$$3x^2 - 2y^2 + xy = 5$$

(a) Find the value of $\frac{dy}{dx}$.

(b) Hence write down the coordinates of the points on the curve where the tangent is parallel to the x-axis.

12. The curve $y = 3e^{-x} + 2e^{x}$ has one stationary point.

(a) Find the x-coordinate of this point.

(b) Also determine whether this point is a maximum or minimum.

13. A curve has equation $y = x\ln x$.

(a) Show that this curve have one stationary point.

(b) Find the x-coordinate of this point.

(c) Say whether this point has a maximum or minimum value.

14. The equation of a curve is

$$y = \frac{1}{1 - \tan x}$$

(a) Find an expression for $\frac{dy}{dx}$.

(b) Find whether the gradient of the curve is positive or negative.

15. A solid cuboid has a base of length $3x$ cm and breadth $2x$ cm.

The height of the cuboid is y cm and its volume is 144 cm^3.

(a) Find y in terms of x.

(b) Show that the total surface area of the cuboid is $A = 12\left(x^2 + \frac{20}{x}\right)$ cm^2.

If x can vary,

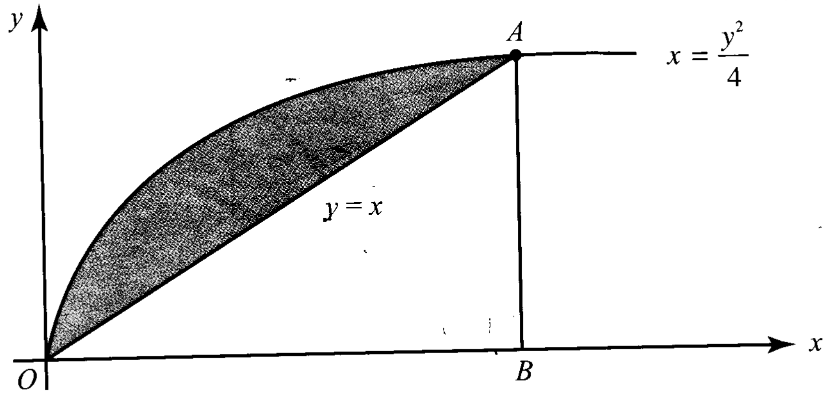
(c) Find the value of x for which A has a stationary value.

(d) Find this stationary value and find out whether it is a maximum or a minimum.

UNIT 13

INTEGRATION

1.



The drawing above shows the curve $x = \frac{y^2}{4}$ which intersects the line $y = x$ at O and A.

Find (a) the coordinates of O and A,

(b) the area of the sector OAB,

(c) the shaded area, by solving $\int_0^4 \sqrt{4x}\, dx$.

2. (a) Find the exact value of $\int_0^{\frac{1}{3}\pi} \sin 2x\, \mathrm{d}x$.

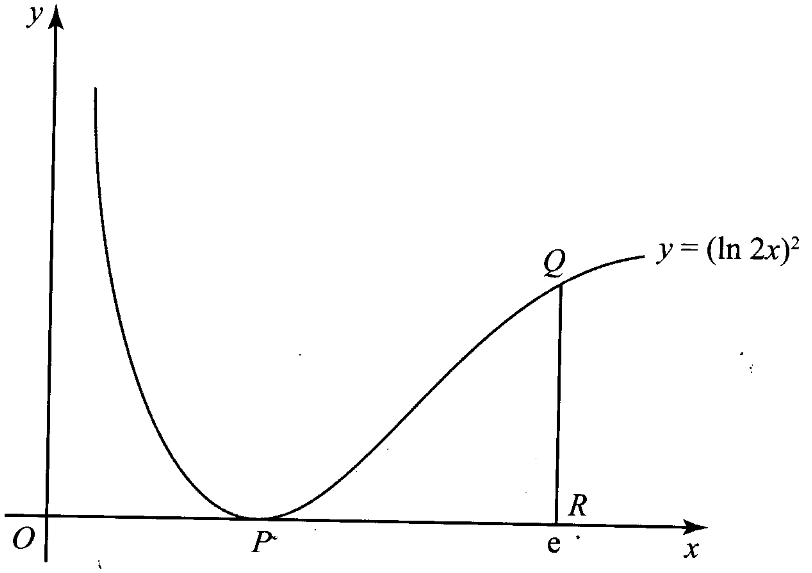
(b) Use an appropriate trigonometrical identity to find the exact value of $\int_0^{\frac{1}{3}\pi} \cos^2 x\, \mathrm{d}x$.

(c) By using the trapezium rule with 2 intervals, estimate the value of $\int_0^{\frac{1}{3}\pi} \cos^2 x\, \mathrm{d}x$, giving your answer to 3 significant figures.

(d) Sketch an appropriate part of the graph of $y = \cos^2 x\, \mathrm{d}x$. Determine if the trapezium rule gives an over-estimate or under-estimate of the exact value.

3. Prove that $\int_1^2 \mathrm{f}(x)\, dx = 0.06$, given that $\mathrm{f}(x) = \frac{2x}{(x-1)(2x+1)^2}$.

4.



Given that the function is $f(x) = (\ln 2x)^2$ for $x > 0$.

The above diagram shows a sketch of $y = f(x)$. The minimum point on the curve is P and the x-coordinate of Q is e.

(a) Find the x-coordinate of P.

(b) Find the value of (i) $\frac{dy}{dx}$,

(ii) $\frac{d^2y}{dx^2}$.

(c) Using an appropriate substitution $x = e^u$, show that the area bounded by the x-axis, the vertical line QR and the part of the curve between P and Q, is $\int_{\ln(0.5)}^{1} 2\ln 2(e^u) + 2ue^u \, du$.

5. The gradient of a curve is given by

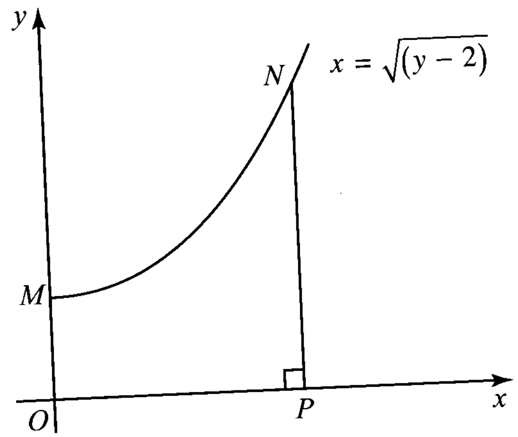
$$\frac{dy}{dx} = \sqrt{1 + 3x}$$

The curve passes through the point $(1, -1)$.

Find (a) the equation of the curve,

(b) the points at which the curve intersects the 2 axes.

6.



The graph shows the curve $x = \sqrt{(y - 2)}$. M and N are points on the curve.

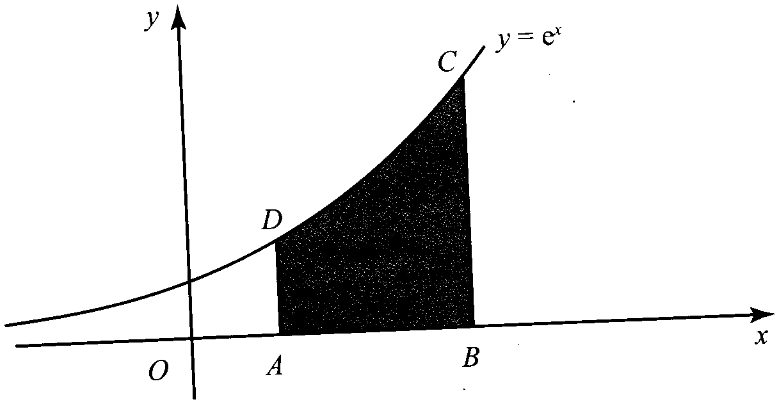
The coordinates of P are $(3, 0)$.

Find (a) the coordinates of M and N.

- (b) the area of the curve bounded by the two axes, the curve and the vertical line NP.
- (c) the volume of the solid formed when the region $MOPN$ is rotated about the x-axis through an angle of 360°.

7. (a) Find the value of $\int_0^{\frac{\pi}{6}} (\sin x + \cos 2x)\, dx$.

(b)



Given the x-coordinates of A and B in the above diagram are 1 and 3 respectively.

(i) Find the coordinates of the points C and D on the curve.

of the shaded part.

(iii) Also find the volume of the solid formed when the shaded region is rotated through 360° about the x-axis.

8. Find the exact value of $\int_1^3 x^2 \ln x \, dx$

9. (a) Find the value of $\int 3x - \dfrac{2}{x^2} \, dx$.

(b) The equation of a curve is $y = \sqrt{2x + 3}$

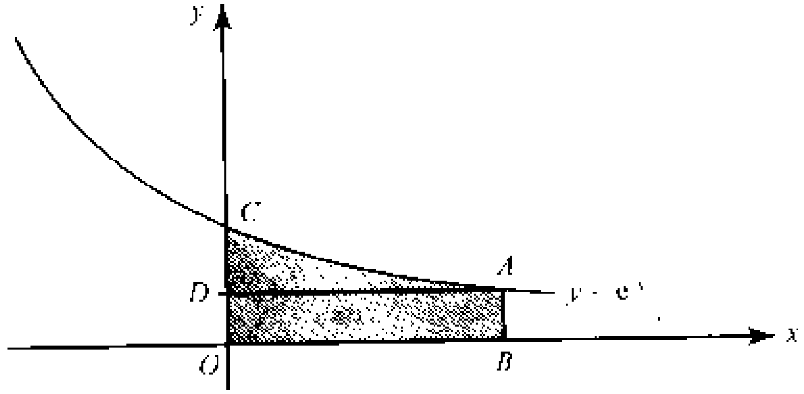
(i) Calculate the gradient of the curve at the point where $x = 2$.

(ii) A point with coordinates (x, y) moves along the curve so that its rate of increase has the constant value of 0.02 units per second.
Find the rate of increase of y when $x = 2$.

(iii) Find the area bounded by the curve, the x-axis and the y-axis and the line $x = 2$.

(iv) Find the volume generated by this region when it is rotated about the y-axis through 360°.

10.



The diagram shows the curve $y = e^{-x}$. The shaded region is bounded by the curve, the x-axis, the y-axis and the line $x = 3$.

(a) Write down the coordinates of A, B and C.

(b) Find the shaded area.

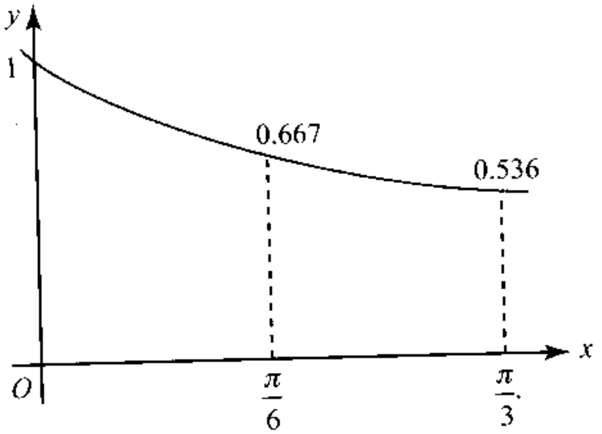
(c) The shaded area is rotated about the x-axis through 360°
Find the volume generated.

11. (a) Use the trapezium rule with 2 intervals to estimate the value of

$$\int_0^{\frac{\pi}{3}} \frac{1}{1 + \sin x}\, dx$$

Give your answer correct to 3 significant figures.

(b)



The diagram shows a sketch of curve $y = \dfrac{1}{1 + \sin x}$ for $0 \leqslant x \leqslant \dfrac{\pi}{3}$.

State, giving your reason, whether the trapezium rule gives an over-estimate or an under-estimate of the true value of the integral in part (a) above.

12. Find the exact value of $\int_0^3 2xe^{3x^2} dx$

13. A curve is such that $\dfrac{dy}{dx} = 2x^2 - 3x + 1$

The curve passes through the point with coordinates (0, 3)

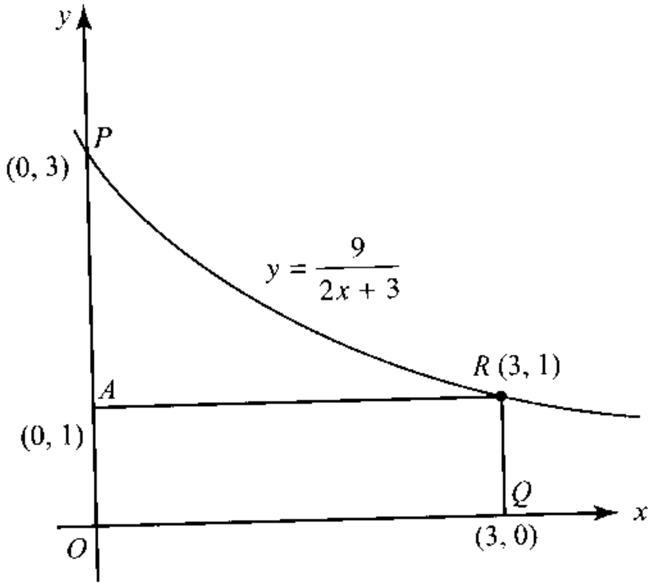
(a) Find the equation of the curve.

(b) Also find the set of values of x for which the gradient of the curve is positive.

14. The diagram shows the curve $y = \dfrac{9}{2x + 3}$

The point $P(0, 3)$ and $R(3, 1)$ are found on the curve.

The point Q has coordinates $(3, 0)$.



(a) Find the equation of the tangent to the curve at R.

(b) Also find the volume of the solid formed when the region $OPRQ$ is rotated about the y-axis through an angle of 360°.

15. (a) By differentiating $\frac{\sin x}{\cos x}$, show that $\frac{dy}{dx} = \sec^2 x$, given that $y = \tan x$.

(b) Hence show that the integral of $\sec^2 x$ for $\frac{\pi}{4} \leqslant x \leqslant \frac{\pi}{3}$ is $\left(\sqrt{3} - 1\right)$.

(c) If $y = \cot x$, find the value of $\frac{dy}{dx}$.

UNIT 14

VECTORS

1. Two lines **a** and **b** have vector equations

$$\mathbf{r} = \mathbf{i} - 2\mathbf{k} + \lambda(2\mathbf{i} + \mathbf{j} + 3\mathbf{k})$$

and $\mathbf{r} = 6\mathbf{i} - 5\mathbf{j} + 4\mathbf{k} + \varphi(\mathbf{i} - 2\mathbf{j} + \mathbf{k})$ respectively.

(a) Prove that the two lines intersect.

(b) Write down the position vector of their point of intersection.

(c) Find the cartesian equation of the plane which contains both **a** and **b**.

2. The position vectors of the points P, Q, R and S relative to an origin O are given by

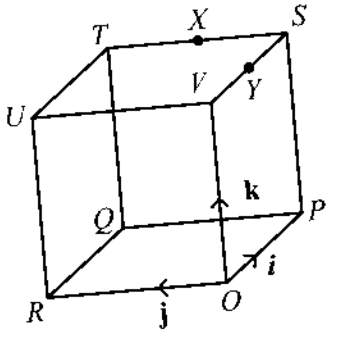
$$\mathbf{OP} = \begin{pmatrix}1\\2\\1\end{pmatrix},\ \mathbf{OQ} = \begin{pmatrix}2\\-1\\2\end{pmatrix},\ \mathbf{OR} = \begin{pmatrix}3\\1\\p\end{pmatrix} \text{ and } \mathbf{OS} = \begin{pmatrix}1\\0\\q\end{pmatrix},$$ p and q being constants.

Find (a) the unit vector in the direction of **PQ**,

(b) the value of P for which $P\hat{O}R = 90^\circ$,

(c) the value(s) of Q such that $|\mathbf{OS}| = \sqrt{5}$ units.

3.



$OPQRSTUV$ is a cube of edge 8 units. The unit vectors **i**, **j** and **k** are along **OP**, **OR**, **OV**. X and Y are the mid-points of TS and VS respectively.

Find (a) the vector **OX**, in terms of **i**, **j** and **k**.

(b) the vector **OY**, in terms of **i**, **j** and **k**.

(c) the scalar product of **OX** and **OY** and find angle XOY.

4. P and Q have position vectors $10\mathbf{i} + 3\mathbf{k}$ and $7\mathbf{i} - 3\mathbf{j} + 6\mathbf{k}$ respectively.

A plane p has vector equation

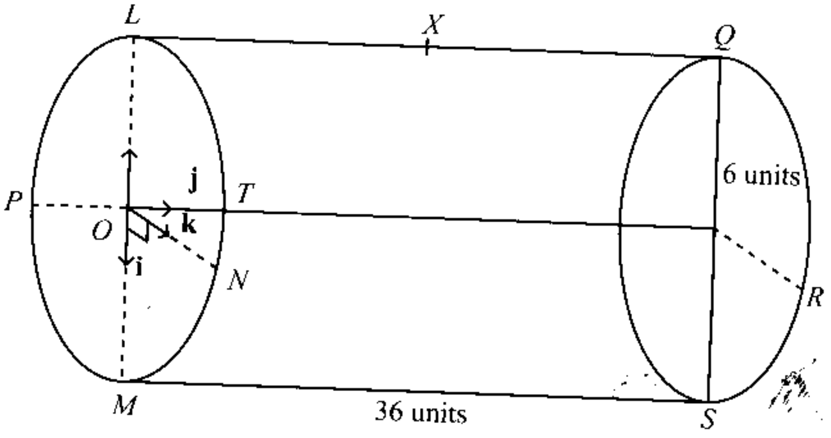
$$\mathbf{r} \cdot \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix} = 8$$

(a) Show that **PQ** is parallel to the plane r.

(b) A point S is the foot of the perpendicular from P to the plane p.

Find a vector equation of the line which passes through S and which is parallel to the line PQ.

5.



This drawing shows the water tank of a solar water heater system.

The length of the tank is 36 units and the radius of its circular base is 6 units.

LM is the diameter of one base. The radius ON is perpendicular to LM. Unit vectors **i**, **j** and **k** are parallel to **OM**, **OL** and **ON** respectively.

X is the mid-point of LQ.

(a) Express **XO** and **XS** in terms of **i**, **j** and **k**.

(b) Hence calculate $X\hat{O}S$.

6. A plane p has cartesian equation

$$x + 3y - 2z = 3.$$

X and Y are two points with position vectors $4\mathbf{i} - \mathbf{j} + 3\mathbf{k}$ and $\mathbf{i} + \mathbf{k}$ respectively.

(a) The line XY intersects the plane p at M.
Write down the position vector of M.

(b) Another plane q contains XY and is perpendicular to p.
Find the cartesian equation of q.

7. **p**, **q** and **r** are three vectors such that

$$\mathbf{p} = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix},$$

$$\mathbf{q} = \begin{pmatrix} 3 \\ -2 \\ 5 \end{pmatrix},$$

and $$\mathbf{r} = \begin{pmatrix} x \\ -x \\ x-1 \end{pmatrix}.$$

Find (a) the angle between the directions of **p** and **q**,

(b) the angle between the directions of **q** and **r**,

(c) the value of x for which **q** and **r** are perpendicular.

8. Relative to the origin O, the points A, B, C and D have position vectors such that

$\mathbf{OA} = 3\mathbf{i} - 2\mathbf{k}$,

$\mathbf{OB} = 2\mathbf{i} + 3\mathbf{j} - 4\mathbf{k}$,

$\mathbf{OC} = 3\mathbf{i} - \mathbf{j} + \mathbf{k}$,

and $\mathbf{OD} = -\mathbf{i} - \mathbf{j} - \mathbf{k}$.

(a) Find the angle between **AC** and **BD**.

(b) Find the position vector of the point of intersection of **AB** and **CD**, if any.

(c) The point M has position vector $3\mathbf{i} + 2\mathbf{j} + \mathbf{k}$.

Find the perpendicular distance from M to **BC**.

9. The points P, Q, R and S have position vectors such that

$\mathbf{OP} = 2\mathbf{i} - 2\mathbf{j} + 5\mathbf{k}$

$\mathbf{OQ} = 3\mathbf{i} + 2\mathbf{k}$

$\mathbf{OR} = -2\mathbf{i} + 10\mathbf{j} + 7\mathbf{k}$

and $\mathbf{OS} = 2\mathbf{j} + 7\mathbf{k}$

(a) Using the scalar product, show that **PQ** and **PS** are perpendicular.

(b) Show also that **PS** and **QR** are parallel, and find the ratio of the length of **PS** to the length of **QR**.

10. p and q are two planes with equations

$2x - y + 3z = 4$

and $x + 2y - 4z = 6$ respectively.

The two planes intersect in the straight line XY.

(a) Find the vector equation of the line XY.

(b) Also find the acute angle between the two planes.

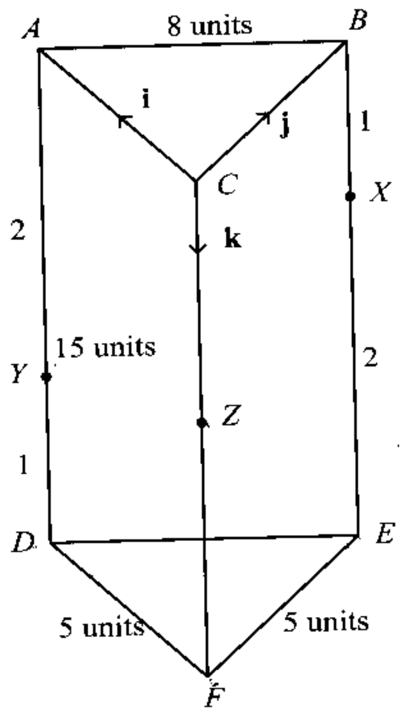
11. The given drawing is a triangular prism. $AD = 15$ units and Y divides AD in the ratio 2 : 1. $AB = DE = 8$ units.

$BE = 15$ units and X divides BE in the ratio 1 : 2. Z is the midpoint of CF.

Unit vectors **i**, **j** and **k** are parallel to **CA**, **CB** and **CF** respectively.

(a) Express each of the vectors YZ and XZ in terms of **i**, **j** and **k**.

(b) Find the value of $YZ.\ XZ$ and hence find angle XZY. Answer to the nearest degree.



12. The points P and Q have position vectors $2\mathbf{i} - 3\mathbf{j} + \mathbf{k}$ and $3\mathbf{i} + 4\mathbf{j} - 2\mathbf{k}$ respectively, relative to an origin O.

(a) Using a scalar product, calculate angle POQ to the nearest $0.1°$.

(b) Another point R is such that $PQ = 3QR$.

Find the vector in the direction of $\mathbf{OR}$.

13. Two lines l and m have cartesian equations

$$l: \ \frac{x-1}{2} = \frac{y-2}{3} = \frac{z-5}{1}$$

and $$m: \ \frac{x-2}{1} = \frac{y-3}{5} = \frac{z-1}{2}$$

(a) Show that the two lines do not intersect.

(b) A point X lies on the line l and another point Y has position vector $\mathbf{i} - 2\mathbf{j} + 3\mathbf{k}$.

Given that XY is perpendicular to l, find the position vector of X.

(c) Show that Y does not lie on m and that XY is not perpendicular to m.

14. The lines *a* and *b* have cartesian equations

$$\frac{x-2}{1}=\frac{y+1}{1}=\frac{z-4}{-1}$$

and $$\frac{x+2}{-2}=\frac{y-2}{1}=\frac{z-1}{1}$$ respectively.

(a) Show that *a* and *b* do not intersect.

(b) Find the acute angle between the two lines.

15. Relative to an origin *O*, the position vectors of the point *S* and *T* are given by

$$\mathbf{OS} = 4\mathbf{i} - 3\mathbf{j} + 2\mathbf{k}$$

and $\mathbf{OT} = 2\mathbf{i} + 3\mathbf{j} - \mathbf{k}$ respectively.

(a) Using a scalar product, calculate acute $\angle SOT$ to the nearest degree.

(b) Find the vector **ST** in terms of **i**, **j** and **k**.

(c) Hence find the unit vector in the direction of **ST**.

UNIT 15

PARAMETERS

1. The parametric equations of a curve are given by

$$x = \frac{1 + \sin \theta}{\cos \theta}$$

and $y = \dfrac{1}{\cos \theta}$, for $0^\circ \leqslant \theta \leqslant 180^\circ$

(a) Find the value of $\dfrac{dy}{dx}$.

(b) Hence find the coordinates of the point on the curve where the gradient is $\dfrac{1}{2}$.

2. The parametric equations of a curve are

$$x = \theta + \sin \theta$$

and $y = 1 - \cos \theta$

Find the value of $\dfrac{dy}{dx}$.

3. Two parametric equations of a curve are given by

$$x = t^2$$

$$y = 1 - t$$

(a) Find the cartesian equation of the curve.

(b) Hence, or otherwise, find $\dfrac{dy}{dx}$.

4. The parametric equations of a curve are

$$x = 2\theta + \cos 2\theta$$

and $y = 1 - \sin 2\theta$

Find the value of $\dfrac{dy}{dx}$.

5. The parametric equations of a curve are given by

$x = t^2 - 1$

and $y = 2t + \ln(t + 1), t > -1$

(a) Find $\frac{dy}{dx}$ in terms of t.

(b) Find the coordinates of point(s) on the curve at which the gradient is -4.

6. Two parametric equations are such that

$x = \cos 2\theta - 1$

and $y = 1 - 2\sin 2\theta$

(a) Show that $\frac{dy}{dx} = 2 \cot 2\theta$

(b) Give the cartesian equation of these parametric equations.

7. The parametric equations of a curve are given by

$y = 2\sin\theta$

and $x = 1 - 3\cos 2\theta$, for $-\frac{1}{2}\pi \leqslant \theta \leqslant \frac{1}{2}\pi$.

Find the value of $\frac{dy}{dx}$ in terms of θ, giving the answer as simply as possible.

8. The parametric equations of a curve are

$x = a\cos^2 t$

and $y = b\sin^2 t$, a being a positive constant and $0 < t < \frac{1}{2}\pi$

(a) Find (i) $\frac{dx}{dt}$,

(ii) $\frac{dy}{dt}$,

(iii) $\frac{dy}{dx}$.

(b) Express the parametric equations as a cartesian equation.

9. Two parametric equations are such that

$x = b\sin^3 \theta$

and $y = b\cos^3 \theta$; where b is a positive constant and θ lies between 0 and $\frac{1}{2}\pi$.

(a) Find $\frac{dy}{dx}$.

(b) Find the equation of the tangent to the curve at the point with parameter θ.

10. The parametric equations are given by

$x = \lambda\sin^3 \theta$

and $y = \lambda\cos^3 \theta$, λ being a positive constant and $0 \leqslant \theta \leqslant \frac{1}{2}\pi$.

(a) Find the equation of the tangent to the curve at the point with parameter θ.

(b) This tangent cuts the x-axis at A and the y-axis at B.
Show that the length of AB is equal to λ.

11. The parametric equations of a curve are

$x = t + e^t$

and $y = t - e^t$, where $t \in \mathbb{R}$

(a) Find $\frac{dy}{dx}$ in terms of t.

(b) Hence find the value of t for which the gradient of the curve is $\frac{1}{2}$.

12. A curve has parametric equations

$x = 4 - e^{2t}$

and $y = 5 + e^{2t}$, $t \in \mathbb{R}$

(a) Express the parametric equations as a cartesian equation.

(b) Find $\frac{dy}{dx}$, and hence show that there is no greatest value of x for points on the curve.

13. The parametric equations of a curve are given by

$x = a\sin t$

and $y = at\cos t$, a being a positive constant, and $0 < t < \frac{1}{2}\pi$

(a) Find $\frac{dy}{dx}$ in terms of t.

(b) Hence show that, when $t = \cot t$, the gradient of the curve is zero.

(c) Sketch a suitable pair of graphs to show that the equation $t = \cot t$ has only one value of t in the given range.

14. A curve has parametric equations

$x = \theta + 2\ln\theta$

and $y = 2\theta - \ln\theta$, where θ can have any positive value.

(a) Express $\frac{dy}{dx}$ in terms of θ.

(b) Write down the equation of the tangent to the curve at the point where $\theta = 1$.

(c) It is given that the curve has one stationary point.
Find the y-coordinate of this point, and say whether it is a maximum or a minimum.

15. A curve has parametric equations

$x = at^2$

and $y = 2at$, a being a positive constant. The tangent to the curve at a point X on the curve meets the x-axis at A.

(a) Find $\frac{dy}{dx}$ in terms of t.

(b) Calculate the equation of the tangent to the curve at A.

16. The parametric equations of a curve are

$$x = a\cos^3\theta$$

and $y = a\sin^3\theta$,

a is a positive constant and $0 \leqslant \theta \leqslant 2\pi$.

The tangent to the curve at any point M on the curve cuts the x-axis at A and y-axis at B.

Show that the length of AB is constant.

UNIT 16

NUMERICAL SOLUTION OF EQUATIONS

1. The set of values given by the iterative formula

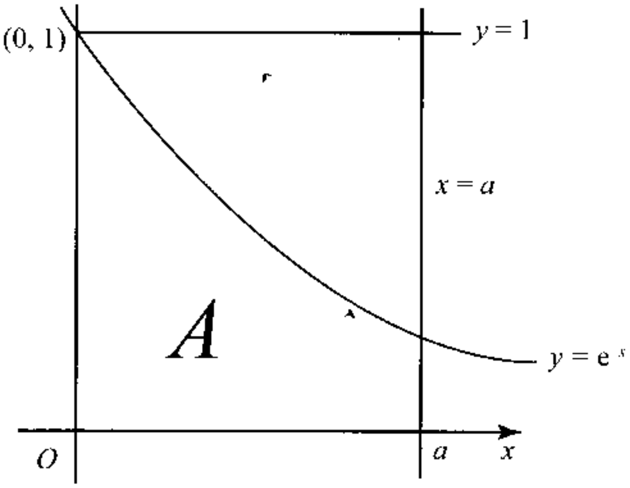
$$x_{n+1} = \frac{2x_n}{3} + \frac{7}{x_n^3},$$

taking $x_1 = 2$ as initial value, converges to α.

(a) Using this iterative formula, find α correct to 3 decimal places. Give the result of each iteration to 4 d.p.

(b) Write an equation satisfied by α and show that $x = \sqrt[4]{21}$.

2.



The region A in the above diagram is bounded by the curve $y = e^{-x}$, the line $x = a$ and the line $y = 1$, a being a constant.

(a) The area of A can be given in terms of the constant a.

Find this area.

(b) If this area is equal to $\frac{1}{2}a$, show that

$$a = \frac{2(e^a - 1)}{e^a}.$$

(c) Use the iterative formula

$$a_{n+1} = \frac{2(e^{a_n} - 1)}{e^{a_n}}$$

starting with $a_1 = 2.5$, to find the value of a correct to 2 decimal places. Give the result of each iteration to 4 decimal places.

3. The equation $x^3 - 2x^2 + x - 2 = 0$ has only one real root.

(a) By evaluating f (0), f (1), f (2), and f (3), find this root.

(b) Prove that, if a sequence of values given by the iterative formula

$$x_{n+1} = \sqrt[3]{2x_n^2 - x_n + 2}$$

converges, then it will converge to this root.

4. The equation $x^3 - 2x - 2$ has only one root.

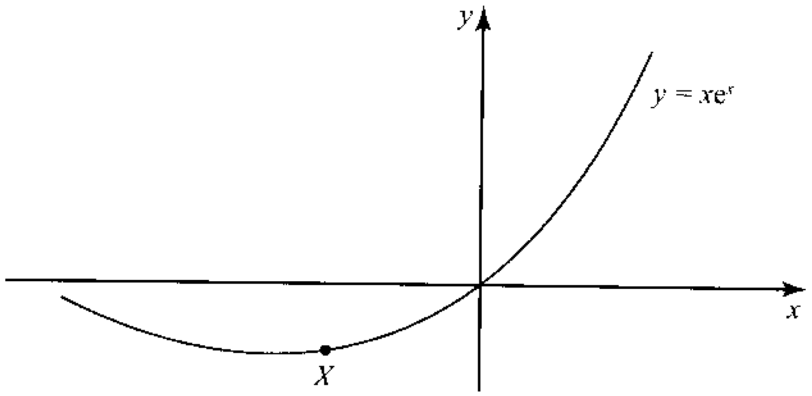
(a) Show that this root lies between $x = 1$ and $x = 2$.

(b) Show that, if a sequence of values given by the iterative formula

$x_{n+1} = \dfrac{2x_n^3 + 2}{3x_n^2 - 2}$ converges, then it will converge to this root.

(c) Hence calculate this root to 2 d.p., giving the result of each iteration to 4 d.p.

5.



In the above diagram the given curve is $y = xe^x$ and its minimum point is X

(a) Find the coordinates of X.

(b) Show that the x-coordinate of the point where the curve cuts the line $y = 7$ is the root of the equation

$$x = \ln\left(\frac{7}{x}\right).$$

(c) Use the iterative formula

$x_{n+1} = \ln\left(\frac{7}{x_n}\right)$, with initial value $x_1 = 1.5$, to calculate this root to 2 d.p. Also give the result of each iteration to 4 d.p.

6. (a) Given the equation $e^{\frac{1}{2}x}(x-2) = 1$.

Sketch two suitable graphs to show that this equation has only one root.

(b) Show that this root α lies between 2 and 2.5.

(c) Using an iterative formula, calculate the value of α correct to 2 d.p., giving the result of each iteration to 4 d.p.

7. Using the iteration formula

$$x_{n+1} = x_n^3 - 0.74\,x_n^2 + 0.74,$$

find whether the root is convergent or divergent. Let the first approximation be $x_1 = 0.7$.

8. Show that the equation $x^2 = \frac{1}{\sin x}$, has a root or more betwen 0 and $\frac{1}{2}\pi$.

Sketch the graph of $y = \sin x$ and the graph of $y = \frac{1}{x^2}$, for $0 < x < \frac{1}{2}\pi$.

Using the iteration $x_{n+1} = \sqrt{\dfrac{1}{\sin x_n}}$, find the root(s) correct to 3 significant figures.

9. The iterative formula

$x_{n+1} = \dfrac{2}{3}x_n + \dfrac{5}{3x_n}$ has a sequence of values which converges to α, with initial value $x_1 = 1.3$.

(a) Using this iterative formula, find α to 2 d. p., giving each iteration to 4 d.p.

(b) Write down an equation satisfied by α and hence find the exact value of α.

10. The equation $x^3 = 2x + 2$ has one real root.

(a) Show that this root of x lies between 1 and 2.

(b) The sequence of values given by the iterative formula

$x_{n+1} = \sqrt[3]{2x_n + 2}$ converges, show that it converges to the root between 1 and 2.

(c) Using the given iterative formula, calculate the root to 2 d.p.
Give the result of each iteration to 4 d.p.

11. (a) The equation given by $x^3 + 2x - 1 = 0$ has one real root.
Find f (0) and f (1).
Between what values does this root lie?

(b) If a sequence of values given by the iterative formula

$x_{n+1} = \dfrac{3x_n^2 - 1}{2x_n + 1}$ converges, show that it converges to a root between 1 and 2.

(c) Using an iterative formula from (a) above starting with $x = 0.5$, determine the root correct to 2 decimal places, giving the result of each iteration to 4 decimal places.

12. Determine the number of roots of each of the following equations.

(a) $e^x + x - 2 = 0$,

(b) $e^{-x} - 2\cos x = 0, \quad 0 \leqslant x \leqslant 2\pi$,

(c) $3\sin x + x - 1 = 0, \quad 0 \leqslant x \leqslant 2\pi$.

13. Show that the equation $e^{-x} - 2x + 3 = 0$ has only one root and locate the root. Hence use an iteration formula of the form $x_{n+1} = g(x_n)$ to calculate the root to 1 d.p.

14. The sequence of values given by the formula $x_{n+1} = \frac{2}{x_n^2} + \frac{3}{5}x_n$ converges to the value p.
Find the value of p.

15. Write down all the iteration formula derived from each of the following equations:

(a) $x^3 - 3x^2 + 5 = 0$,

(b) $2x^3 - 3x + 7 = 0$,

(c) $x - 2\sin x + e^{-x} = 0$,

(d) $x^2 + 4 = 4x + \ln x$.

16. The sequence of values given by the formula

$$x_{n+1} = \frac{2}{x_n^3} + \frac{3}{4}x_n$$

converges to the value p.

Find the value of p.

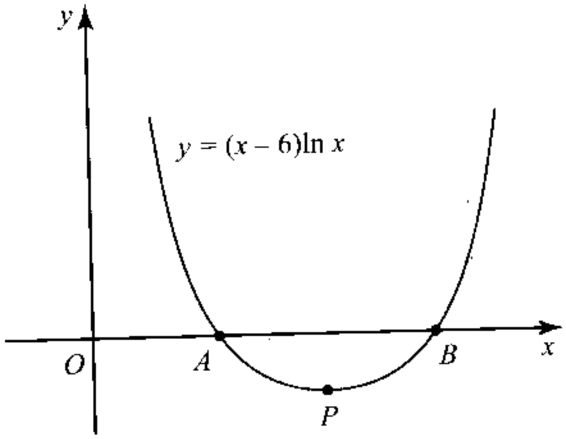
17. The diagram shows the curve
$y = (x - 6)\ln x$ and its minimum point P.
The curve cuts the x-axis at the points A and B.

(a) Write down the coordinates of A and B.

(b) Show that the x-coordinate of P satisfies the equation $x = \dfrac{6}{1 + \ln x}$

(c) Use the iteration formula $x_{n+1} = \dfrac{6}{1 + \ln x_n}$

to find the x-coordinate of P correct to 1 decimal place, showing the result of each iteration to 4 decimal places.



18. Show that the equation $x^4 - 2x - 3 = 0$ has only 2 real roots and hence find the greater root, correct to 2 decimal places.

19. Starting with $x_1 = 1$, use the iterative formula $x = \ln(4\sin x_n)$, to find x_2, x_3, x_4 and x_5.

Give each answer to 4 decimal places.

Then find the value to which this sequence converges, correct to 2 decimal places.

20. Find the positive roots of $x^3 - 4x + 1 = 0$, correct to 4 decimal places, starting with $x_1 = 1$ and using the iteration formula

$$x_{n+1} = \frac{1}{4}\left(x_n^3 + 1\right) \cdot$$

UNIT 17

COMPLEX NUMBERS

1. If $z_1 = 3 - 2i$ and

$z_2 = -4 + 5i$, simplify each of the following:

(a) $3z_1 + iz_2$,

(b) $iz_1 - z_2$,

(c) $z_1 z_2$,

(d) z_2^2,

(e) $z_2 \div z_1$.

2. Find the square root of $-16 - 30i$.

3. If one root of the equation

$x^2 + ax + b = 0$ is $(2 - i)$,

find the (a) other root,

(b) value of a and

(c) value of b.

4. A, B and C represent the complex numbers

$z_1 = 2 - 3i$,

$z_2 = 3 + 2i$ and

z in the complex plane

Find z, given that

(a) $OABC$ is a parallelogram

(b) $\angle CAB = 90°$ and $AC = 2AB$

5. If $z_1 = 3 + 2i$

and $z_2 = 5 - 3i$,

find

(a) $|z_1 - z_2|$ and

(b) $\arg(z_1 - z_2)$.

6. If $z = 2 - 3i$

Find

(a) $|z^3|$ and

(b) $\arg z^3$

7. If $\arg(z + 6) = \frac{\pi}{6}$, find the least value of $|z|$..

8. If $\arg(z + 4) = \frac{\pi}{6}$ and

$\arg(z - 6) = \frac{5}{6}\pi$,

Find $|z|$.

•9. If $|z| = 4$, find the least and greatest value of:

(a) $|z + 8|$,

(b) $\arg(z + 8)$.

10. Express (a) $2 + 3i$

(b) $3 - 5i$, in polar form $k = e^x$

11. Sketch each of the following sets in an Argand diagram.

(a) $1 \leqslant |z - 2| < 2$,

(b) $-\frac{\pi}{4} \leqslant \arg(z + i) \leqslant \frac{\pi}{3}$,

(c) $|(z-2)| \leqslant 1$ and $-\frac{\pi}{4} \leqslant \arg(z-1) \leqslant \frac{\pi}{4}$.

12. In an Argand diagram, O is the origin, the point A represents $(2-i)$, the point B represents $(x+iy)$, x and y being positive, and triangle OAB is equilateral.

Find the values of x and y.

13. The complex number z is represented in an Argand diagram by the point X.

Given that $z = \frac{1}{2+ix}$, where x is a real variable, and z^* is the complex conjugate of z, show that $z + z^* = 4zz^*$.

Hence show that as x varies, X lies on a circle.

14. In an Argand diagram, the points X, Y and Z represent the complex numbers r, s and $3+4i$ respectively.

O is the origin.

$OXYZ$ is a square described in a clockwise sense.

(a) Find the complex numbers r and s.

(b) On separate diagrams show the points O, X, Y and Z.

Also draw the loci

(i) $|z-3-4i| = 5$,

(ii) $|z| = |z-6-8i|$,

(iii) $\arg(z-r) - \arg z = \frac{1}{2}\pi$.

15. Let u denote the complex number $(1-i)$.

(a) If u is a root of the equation

$x^3 - 5x^2 + 8x - K = 0$, where K is real, find the value of K.

(b) Write down the other complex root of this equation.

(c) Find the modulus and argument of u.

(d) On an Argand diagram, show the point representing u.

(e) Shade the region on the diagram which shows the points representing the complex numbers z satisfying both the inequalities:

$$|z| < |z - 1| \text{ and } 0 < \arg (z - u) < \frac{2}{3}\pi .$$

16. The equation $x^3 + 3x^2 + x + 3 = 0$ has one real root and two complex roots.

(a) Show that i is one of the complex roots.

(b) Write down the other complex root of this equation.

(c) On an Argand diagram, show the point representing the complex number i.

On the same diagram, show the set of points representing the complex numbers z which satisfy:

$$|z| < |z - i|$$

17. The complex number $(1 + 2i)$ is denoted by u. The complex number with modulus 3 and argument $\frac{1}{3}\pi$ is denoted by w.

(a) Find in the form $(a + ib)$, where a and b are real, the complex numbers

(i) w,

(ii) uw,

(iii) $\frac{u}{w}$.

(b) On an Argand diagram, sketch the points X, Y and Z representing respectively the complex numbers $(0 + \sqrt[3]{3}i)$, $(-3 + oi)$ and $(3 + oi)$.

(c) What kind of triangle is ΔXYZ?

18. Let u and v represent the complex numbers $(1 + 2i)$ and $(3 - i)$ respectively.

(a) Find, in the form $(a + ib)$, where a and b are real, the complex numbers $u + v$ and uv.

(b) Find the argument of uv.

(c) With reference to an origin O, the points P, Q and R represent the complex numbers, u, v and $u + v$ respectively.
Find angle POR and and QOR, in radians.

19. (a) Solve the equation $z^2 + 3z + 4i + 2 = 0$, giving your numbers in the form $x + iy$, where x and y are real.

(b) Find the modulus and argument of each root.

(c) Sketch an Argand diagram showing the points representing the roots.

20. The complex number z is given by

$$u = \frac{2 + i}{3 - i}.$$

(a) Express u in the form $a + ib$, where a and b are real.

(b) Find the modulus and argument of u.

(c) Sketch an Argand diagram showing the point representing the complex number u.

(d) On the same diagram draw the locus of the point representing the complex number z such that $|z - u| = 2$.

UNIT 18

FIRST ORDER DIFFERENTIAL EQUATIONS

1. Solve the following differential equations:

(a) $y\dfrac{dy}{dx} = \cos x$,

(b) $\dfrac{dy}{dx} = xy$.

2. ✓ Solve the following differential equations:

(a) $\dfrac{1}{x}\dfrac{dy}{dx} = \dfrac{y}{x^2+1}$, given that $y = 2$ when $x = 1$.

(b) $\dfrac{dy}{dx} = 2y$, given that $y = 3$ when $x = 0$.

3. The rate at which a certain bacteria multiply is proportional to the actual number present.
If the original amount doubles in 3 hours, in how many hours will it triple?

✓ **4.** Find the particular solution of the differential equation $\dfrac{dy}{dx} = 2x^2 - 4x$, given that $y = 3$ when $x = 1$.

5. Find the displacement, s cm, from O of a particle at time ts, given that its velocity, V cm/s, is given by the differential equation

$$V = \frac{ds}{dt} = t^2 - t + 4$$

and the displacement is 250 cm at time 5s.

Find also the initial displacement.

6. (a) Find the solution of the differential equation $\dfrac{dy}{dx} = 2xy^2$, expressing y in terms of x.

(b) Solve $\frac{dy}{dx} = \frac{e^{2x}}{y}$, given that $y = 3$ when $x = 0$.

7. A circular inkblot, with radius r cm, is enlarging such that the rate of increase of the radius at time t seconds is given by

$$\frac{dr}{dt} = \frac{0.1}{r}$$

Initially the radius of the inkblot is 0.4 cm.

Find the area of the inkblot after 2 seconds.

8. A spherical balloon, which is being inflated, has radius r cm at time t seconds.

It takes 3 seconds to inflate the balloon to a radius of 16 cm from its initial value of 1 cm. In a simple model, the rate of increase of r is proportional to $\frac{1}{r^2}$.

Express this statement as a differential equation and find the time it would take to inflate the balloon to a radius of 20 cm.

Answer in seconds, to 2 S.F.

9. A population of insects, y, is increasing at a rate proportional to the total population at time t, and is such that when $t = 0$, $y = y_0$.

(a) Show that $y = y_0 e^{kt}$, where k is a positive constant.

(b) If the population doubles in 5 years, find by what factor the initial population has been multiplied after a further 10 years.

10. A mathematical model for a number of seeds, x, in a culture, states that x is increasing at a rate proportional to the number present.

At noon, there are 1 000 seeds, at 13.00, there are 1 672 seeds. According to the model, at what time will the number of seeds have increased to 2 000?

11. A certain substance decays at a rate proportional to the amount, y, present at time t.

If it takes 2 000 years for half the original amount to decay, find the percentage of the original amount that remains after 100 years.

12. Given that the rate at which the temperature of a body falls is proportional to the amount by which the temperature exceeds that of its surroundings.

A closed area has a constant temperature of 17°C.

An object has a temperature of 60°C when it is placed in the closed area.

10 minutes later its temperture is 48°C. What will its temperature be in a further 10 minutes?

13. A tank is being filled with liquid. Initially the tank is empty. t hours after filling starts, the volume of the liquid in the tank is V m^3, and the depth of the liquid is hm.

Given that $V = \frac{4}{3} h^3$. The liquid is being poured at a rate of 10 m^3 per hour. But due to a leakage, the liquid is being lost at a rate proportional to h^2. When $h = 114^2$, $\frac{dh}{dt} = 10$.

(a) Form a differential equation and show that $\frac{dh}{dt} = h + \frac{79}{40}h^2$

(b) Express t in terms of h.

14. At a certain time t, the pressure of air in a container is P.

The outside atmospheric pressure of the air is constant and is equal to X.

Air is being escaped from the container at a rate which is proportional to the square root of the difference of the pressures, i.e. $(P - X)$.

Show that the differential equation connecting P and t and X is

$\frac{dp}{dt} = -k\left(\sqrt{(P - X)}\right)$, k being a positive constant.

(a) Find the general solution of this differential equation,

(b) It is given that $P = 3x$ when $t = 0$, and that $P = X$ when $t = 1$, find the value of k.

(c) Find the value of t when $P = 2x$.

(d) Hence find P in terms of X and t.

15. The rate at which the temperature (α) of an object cools down is inversely proportional to its temperature at time t.

(a) Write down a differential equation and find α in terms of t.

(b) Given that the temperature α decreases from 60°C to 50°C in 10 minutes, find its temperature after 12 minutes.

16. At a fish breeding station, the birth rate of fish at any time t is equal to $\frac{1}{8}$ the number N of fish present in the lake at that time.

The fish are harvested regularly at a constant rate of 600 per day.

When $t = 0$, the number of fish in the lake is 1 000.

If N is a continuous variable, write down a differential equation connecting t and N.

Solve the differential equation.

When will the population of fish in the lake be 15 000?

17. A balloon is being inflated. Its radius at time t seconds is r cm. The initial radius of the balloon is 1 cm. In 3 seconds the radius of the balloon reaches 16 cm.

The rate of increase of the radius is inversely proportional to the square of r.

(a) Using these data, form a differential equation.

(b) Find the time it will take the balloon to reach a radius of 20 cm.

Answer in seconds, to 2 s.f.

✓

18. The variables x and y are such that $\frac{dy}{dx} = \left(\frac{y}{x}\right)^2$

(a) Find the general solution of the curve, giving y in terms of x.

(b) Also find the particular solution of the curve, given that it passes through the point $(1, 2)$.

19. The rate of decay of a certain substance is proportional to the mass x kg of the substance at that time.

Initially the mass of the substance is M kg.

(a) Form a differential equation connecting M, x and t, where t is the time in hours after decay has started, and solve it.

(b) Solve the differential equation, given that when $t = 20$, $x = \frac{1}{2}M$.

(c) Find the value of t when $x = \frac{1}{3}M$. (*Answer to 2 d.p.*)

20. According to a cooling system, the rate at which a body cools is proportional to S, where S°C is the temperature of its surroundings.

A body at 70°C is put in a room where the temperature is 18°C.

After 10 minutes, it has cooled to 50°C.

Find its temperature after 10 more minutes.

UNIT 1

REMAINDER AND FACTOR THEOREMS

Suggested Solutions

1. (a) $f(x) = 2x^3 - 3ax^2 + ax + b$

When $x - 1 = 0, x = 1$

$f(1) = 2(1)^3 - 3a(1)^2 + a(1) + b = 0$

$2 - 3a + a + b = 0$

$-2a + b + 2 = 0$

$2a - b = 2$ -- (1)

When $x + 2 = 0, x = -2$

$f(-2) = 2(-2)^3 - 3a(-2)^2 + a(-2) + b = -54$

$(2 \times -8) - (3a \times 4) - 2a + b = -54$

$-16 - 12a - 2a + b = -54$

$-14a + b = -54 + 16$

$-14a + b = -38$

$14a - b = 38$ -- (2)

From (1) take (2), we obtain

$-12a = -36$

$a = 3$ and $b = 4$ (Answer)

(b) Replacing a by 3 and b by 4, we have

$f(x) = 2x^3 - 3ax^2 + ax + b$

$= 2x^3 - 9x^2 + 3x + 4$

$= (x - 1)(2x^2 - 7x - 4)$ (by division)

$= (x - 1)(x - 4)(2x + 1)$ (Answer)

(c) (i) $2x^6 - 9x^4 + 3x^2 + 4$

Replacing x by x^2 in equation (b), we obtain

$(x^2 - 1)(x^2 - 4)(2x^2 + 1)$

$= (x + 1)(x - 1)(x + 2)(x - 2)(2x^2 + 1)$ (Answer)

(ii) $4x^3 + 3x^2 - 9x + 2$

$= x^3\left(4 + \frac{3}{x} - \frac{9}{x^2} + \frac{2}{x^3}\right)$

Replacing x by $\frac{1}{x}$ in (b), we have

$= x^3\left(\frac{1}{x} - 1\right)\left(\frac{2}{x} + 1\right)\left(\frac{1}{x} - 4\right)$

$= (1 - x)(x + 2)(4x - 1)$ (Answer)

2. Let $f(x) = x^3 + ax^2 + bx + 4$

If $(x - 2)$ is a factor,

$\therefore \quad x - 2 = 0$

$x = 2$

i.e. $f(2) = 0$

$f(2) = 8 + 4a + 2b + 4 = 0$

$4a + 2b = -12$

$2a + b = -6$ ---------------------------- (1)

$f(-2) = -8 + 4a - 2b + 4 = 0$

$4a - 2b = 4$

$2a - b = 2$---------------------------- (2)

Solving (1) and (2), we obtain

$4a = -4$

$a = -1$ (Answer)

and $b = -4$ (Answer)

3. Given that $f(x) = x^5 - 3x^4 + 2x^3 - 2x^2 + 3x + 1$

(a) $f(1) = 1 - 3 + 2 - 2 + 3 + 1$

$= 2$, which is not 0.

Also $f(-1) = -1 - 3 - 2 - 2 - 3 + 1$

$= -10$, which is not 0, again.

$\therefore$ neither $(x - 1)$ nor $(x + 1)$ is a factor of $f(x)$. (Shown)

(b) $f(x) = (x^2 - 1)\,q(x) + ax + b$

$f(1) = a + b = 2$ ---------- (1)

$f(-1) = -a + b = -10$ ---------- (2)

Solving (1) and (2), we obtain $a = 6$ and $b = -4$

$\therefore$ $f(x) = (x^2 - 1)\,q(x) + 6x - 4$

i.e. the remainder is $(6x - 4)$ (Answer)

(c) When $f(x)$ is divided by $(x^2 + 1)$, the remainder is $2x$ and the quotient is $x^3 - 3x^2 + x + 1$. (Shown)

(d) $f(x) = (x^2 + 1)(x^3 - 3x^2 + x + 1) + 2x$

Since $f(x) = 2x$

$f(x) = (x^2 + 1)(x^3 - 3x^2 + x + 1) + 2x = 2x$

$\therefore$ $(x^2 + 1)(x^3 - 3x^2 + x + 1) = 0$

Now because $(x^2 + 1) \neq 0$ and x is real

$\therefore$ $x^3 - 3x^2 + x + 1 = 0$

Applying factor theorem

if $f(x) = x^3 - 3x^2 + x + 1$

then $f(1) = 0$

$\therefore$ $(x - 1)$ is a factor of $f(x)$.

Thus the real roots will be

$x = 1$ (Answer)

$$\begin{array}{r|l}
 & x^2 - 2x - 1 \\
\hline
x - 1 & x^3 - 3x^2 + x + 1 \\
 & x^3 - x^2 \\
\hline
 & -2x^2 + x + 1 \\
 & -2x^2 + 2x \\
\hline
 & -x + 1 \\
 & -x + 1 \\
\hline
 & 0
\end{array}$$

or $x = \dfrac{2 \pm \sqrt{8}}{2} = 1 \pm \sqrt{2}$ (Answer)

4. (a) Since $(x + 2)$ and $(x - 1)$ are factors of f (x), f $(-2) = 0$

and f $(x) = x^4 + ax^3 + bx^2 - 2x - 4$

f (-2) $16 - 8a + 4b = 0$

$8a - 4b = 16\ (\div 4)$

$2a - b = 4$ -- (1)

Also f $(1) = 0$

$1 + a + b - 2 - 4 = 0$

$a + b = 5$ -- (2)

Solving (1) and (2), we obtain $a = 3$ and $b = 2$ (Answer)

(b) $(x - 1)(x + 2) = x^2 + x - 2$

Dividing f (x) by $x^2 + x - 2$,

the 3rd-factor will be $x^2 + 2x + 2$. (Answer)

(c) $x^2 + 2x + 2 = x^2 + 2x + 1 + 1$

$= (x + 1)^2 + 1$

$(x + 1)^2$ being a square number is positive

$\therefore$ $(x + 1)^2 + 1$ is positive for all values of x. (Proved)

(d) Given f $(x) > 0$, and since $x^2 + 2x + 2$ is positive

$\therefore$ $(x + 2)(x - 1) > 0$

$x > 1$ or $x < -2$

$\therefore$ The set of values of x is

$\{x: x < -2 \text{ or } x > 1, x \in \mathbb{R}\}$ (Answer)

5. $x^4 - 16 = (x^2)^2 - (4)^2$

$= (x^2 + 4)(x^2 - 4)$

$= (x^2 + 4)(x + 2)(x - 2)$ (Answer)

6. f $(x) = 6x^3 + cx^2 + 3$

(a) Since $(2x + 1)$ is a factor of f (x)

When $(2x + 1) = 0$, f $(x) = 0$

i.e. when $x = \frac{-1}{2}$, f $(x) = 0$.

$$\therefore \quad f\left(\frac{-1}{2}\right) = 0$$

$$\text{i.e.} \quad 6\left(\frac{-1}{2}\right)^3 + c\left(\frac{-1}{2}\right)^2 + 3 = 0$$

$$\frac{-6}{8} + \frac{1}{4}c + 3 = 0$$

$$\frac{1}{4}c = -3 + \frac{3}{4}$$

$$\frac{1}{4}c = \frac{-9}{4}$$

$\therefore \quad c = -9$ (Answer)

(b) $f(x) = 6x^3 - 9x^2 + 3$

$= 3(2x^3 - 3x^2 + 1)$

Using factor theorem, $(x - 1)$ is also a factor of $f(x)$.

Hence $f(x) = 3(2x + 1)(x - 1)(x - 1)$

$= 3(2x + 1)(x - 1)^2$ (Answer)

7.

$$(x - 2)(x^2 - 2x) = (x - 2)$$

$$(x - 2)(x^2 - 2x) - (x - 2) = 0$$

$$(x - 2)\{(x^2 - 2x) - 1\} = 0$$

$$(x - 2)(x^2 - 2x - 1) = 0$$

$\therefore \; x - 2 = 0$ or $x^2 - 2x - 1 = 0$

$x = 2$ $\quad x = 1 \pm \sqrt{2}$

(Answer)

8. (a) $(x^2 - 1) = (x + 1)(x - 1)$

if $x^2 - 1 = 0$

$x = -1$ or 1

$f(x) = 3x^4 + ax^3 + bx^2 - 7x - 4$

$\therefore \quad f(1) = 3 + a + b - 7 - 4 = 0$

$a + b = 8$ ---------------------------- (1)

$$f(-1) = 3 - a + b + 7 - 4 = 0$$

$$-a + b = -6 \text{ ------------------------- } (2)$$

Solving (1), and (2) gives

$a = 7,\ b = 1$ (Answer)

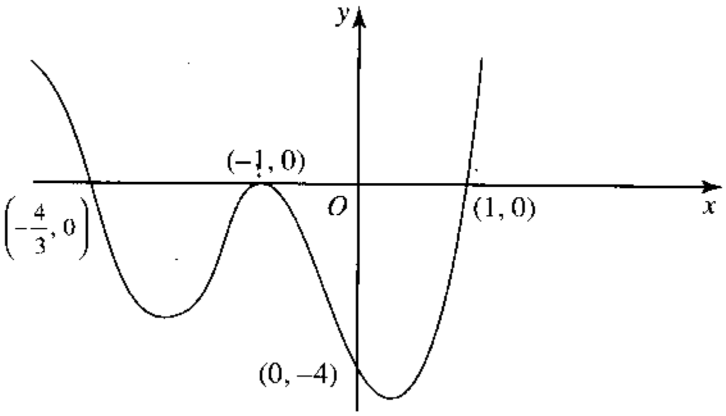
$\therefore\quad f(x) = 3x^4 + 7x^3 + x^2 - 7x - 4$

(b) Divide f (x) by $(x^2 - 1)$ gives $3x^2 + 7x + 4$

$$\begin{aligned} \therefore\quad f(x) &= (x^2 - 1)(3x^2 + 7x + 4) \\ &= (x + 1)(x - 1)(3x + 4)(x + 1) \\ &= (x - 1)(x + 1)^2(3x + 4) \end{aligned}$$

(Answer)

(c)



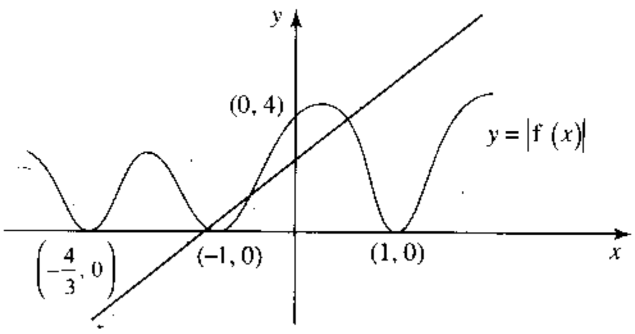
(d) When f $(x) < 0$,

$$\left\{\frac{-4}{3} < x < -1\right\} \text{ or } \{-1 < x < 1\}$$

(Answer)

Taking the parts of the graph below the x-axis.

(c)



(Answer)

9. $g(x) = x^4 - 3x^3 + ax^2 + 15x + 50$

(a)

$$\begin{aligned} x + 2 &= 0 \\ x &= -2 \\ g(x) &= x^4 - 3x^3 + ax^2 + 15x + 50 \\ \therefore \quad g(-2) &= (-2)^4 - 3(-2)^3 + a(-2)^2 + 15(-2) + 50 = 0 \\ &= 16 + 24 + 4a - 30 + 50 = 0 \\ 4a &= -60 \\ a &= -15 \end{aligned}$$

(Answer)

(b)

$$\begin{aligned} g(5) &= (5)^4 - 3(5)^3 + -15(5)^2 + 15(5) + 50 \\ &= 625 - 375 - 375 + 75 + 50 \\ &= 0 \end{aligned}$$

(Answer)

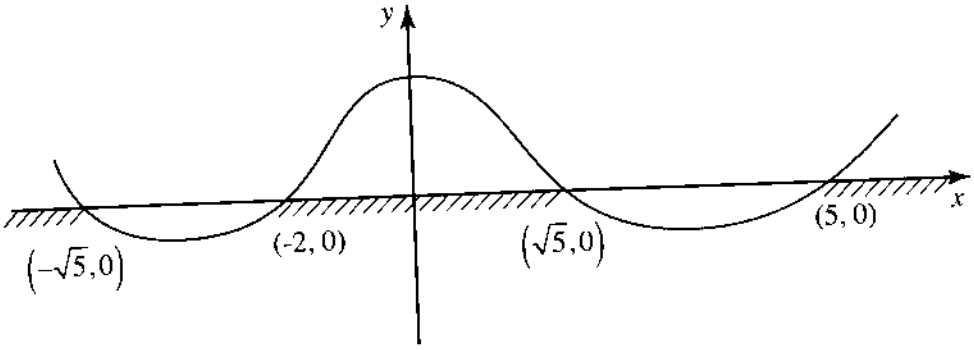
(c)

$$\begin{aligned} g(x) &= x^4 - 3x^3 - 15x^2 + 15x + 50 \\ &= (x + 2)(x - 5)(x^2 - 5) \end{aligned}$$

(Referring to (a) and (b). (Answer)
or refer to the following division)

$$\begin{array}{r|l} & x^2 - 5 \\ \hline x^2 - 3x - 10 & x^4 - 3x^3 - 15x^2 + 15x + 50 \\ & x^4 - 3x^3 - 10x^2 \\ \hline & \qquad -5x^2 + 15x + 50 \\ & \qquad -5x^2 + 15x + 50 \\ \hline & \qquad\qquad\qquad 0 \end{array}$$

(d) Sketching the graph of $y = g(x)$, we have



When $g(x) = 0$, $x = -\sqrt{5}$, -2, $\sqrt{5}$ and 5.

Set of values of x for which $g(x) > 0$ is

$\{x : x \in \mathbb{R},\ -2 < x < \sqrt{5}\ \text{ or }\ x > 5\ \text{ or }\ x < -\sqrt{5}\}$ (Answer)

(e) For $g(|x|) > 0$,

$0 \leq |x| < \sqrt{5}$ or $|x| > 5$

$\therefore$ Solution set $= x \in \mathbb{R}$ (Answer)

10. (a) Since $(x-2)$ is a factor of $f(x)$,

$f(2) = 3(2)^3 + a(2)^2 + (2) - 2 = 0$

$4a = 2 - 2 - 24$

$a = -6$ (Answer)

(b) By division, or otherwise

$3x^3 + ax^2 + x - 2$

$= 3x^3 - 6x^2 + x - 2$

$= (x-2)(3x^2 + 1)$

$3x^2 + 1$ has no real root, since the discriminant $= -12$ i.e. < 0

$\therefore$ $f(x)$ has only one real root. (Shown)

11. Given $4x^3 + 8x^2 = 3 - x$

$4x^3 + 8x^2 + x - 3 = 0$

Using factor theorem, $f(-1) = 0$

$4(-1)^3 + 8(-1)^2 + (-1) - 3$

$= -4 + 8 - 1 - 3 = 0$

$\therefore$ $(x+1)$ is a factor.

By division,

$4x^3 + 8x^2 + x - 3$

$= (x+1)(4x^2 + 4x - 3) = 0$

$= (x+1)(2x+3)(2x-1) = 0$

$\therefore$ $x = -1, \dfrac{-3}{2}$ or $\dfrac{1}{2}$ (Answer)

12. Since $(x-2)$ is a factor of $f(x)$,

$f(2) = 2(2)^3 + c(2)^2 - 5(2) + 6 = 0$

$16 + 4c - 10 + 6 = 0$

$4c = -12$

$c = -3$

(Answer)

13. (a) $f(1) = 6(1)^3 + 11(1)^2 - 5(1) - 12$

$= 6 + 11 - 5 - 12 = 0$

$\therefore$ $(x - 1)$ is a factor of $f(x)$.

(b) By division, $6x^3 + 11x^2 - 5x - 12$

$= (x - 1)(6x^2 + 17x + 12)$

$= (x - 1)(3x + 4)(2x + 3)$

(Answer)

14.

$$x^2 + x + 1 = \frac{2x^4 + 2x^3 + 5x^2 + 3x + 3}{ax^2 + bx + c}$$

$\therefore$ $(x^2 + x + 1)(ax^2 + bx + c) = 2x^4 + 2x^3 + 5x^2 + 3x + 3$

$ax^4 + bx^3 + cx^2 + ax^3 + bx^2 + cx + ax^2 + bx + c$

$= 2x^4 + 2x^3 + 5x^2 + 3x + 3$

$ax^4 + x^3(b + a) + x^2(a + b + c) + x(c + b) + c$

$= 2x^4 + 2x^3 + 5x^2 + 3x + 3$

Equating cœfficients of x^4, we have $a = 2$

x^3, we have $a + b = 2$

$b = 0$

x^2, we obtain $a + b + c = 5$

$2 + 0 + c = 5$

$c = 3$

(Answer)

15. (a) $2x + 1 = 0$

$$x = \frac{-1}{2}$$

$f(x) = 2x^3 + ax^2 + 16x + 6$

$$f\left(\frac{-1}{2}\right) = 2\left(\frac{-1}{2}\right)^3 + a\left(\frac{-1}{2}\right)^2 + 16\left(\frac{-1}{2}\right) + 6 = 0$$

$$\frac{-2}{8} + \frac{a}{4} - 8 + 6 = 0$$

$-2 + 2a - 64 + 48 = 0$

$2a = 18$

$a = 9$ (Answer)

(b) By division $2x^3 + 9x^2 + 16x + 6$

$= (2x + 1)(x^2 + 4x + 6)$ (Answer)

(c) $x^2 + 4x + 6$

$= x^2 + 4x + 4 + 2$

$= (x + 2)^2 + 2$ which is positive for all values of x (Answer)

16. Since $(x + 1)$ and $(x + 2)$ are factors of f (x),

$\therefore$ f $(-1) = (-1)^3 + a(-1)^2 + b(-1) - 8 = 0$ ---------- (1)

and f $(-2) = (-2)^3 + a(-2)^2 + b(-2) - 8 = 0$ ---------- (2)

from (1) $-1 + a - b - 8 = 0$ ---------- (1)

$a - b = 9$ ---------- (2)

from (2) $-8 + 4a - 2b - 8 = 0$ ---------- (3)

$4a - 2b = 16$

$2a - b = 8$ ---------- (4)

Solving for (2) and (4), we have

$a = -1$

and $b = -10$ (Answer)

17. f $(1) = 1 + 1 - 1 - 3 - 6 \neq 0$

f $(-1) = 1 - 1 - 1 + 3 - 6 \neq 0$

f $(2) = 16 + 8 - 4 - 6 - 6 \neq 0$

f $(-2) = 16 - 8 - 4 + 6 - 6 \neq 0$

f $(3) = 81 + 27 - 9 - 9 - 6 \neq 0$

f $(-3) = 81 - 27 - 9 + 9 - 6 \neq 0$

f $\left(\sqrt{3}\right) = 9 + 3\sqrt{3} - 3 - 3\sqrt{3} - 6 = 0$

$\therefore$ $\left(x - \sqrt{3}\right)$ is a factor

So is $\left(x + \sqrt{3}\right)$

$\therefore$ One quadratic factor is $(x^2 - 3)$ which is the same as $\left(x + \sqrt{3}\right)\left(x - \sqrt{3}\right)$.

By division, the other quadratic factor is $(x^2 + x + 2)$ (Answer)

UNIT 2

PARTIAL FRACTIONS

Suggested Solutions

1. Let $\frac{5x+4}{(x-1)(x+2)} = \frac{A}{(x-1)} + \frac{B}{(x+2)}$

$$\frac{5x+4}{(x-1)(x+2)} = \frac{A(x+2)+B(x-1)}{(x-1)(x+2)}$$

$\therefore$ $5x + 4 = A(x + 2) + B(x - 1)$

$5x + 4 = Ax + 2A + Bx - B$

Equating cœfficients of x: $A + B = 5$ ---------------------- (1)

Equating constants: $2A - B = 4$ ---------------------- (2)

solve (1) and (2), we get

$A = 3$ and $B = 2$

Or let $x = -2$,

then $-10 + 4 = 0 + B(-3)$

$-6 = -3B$

$B = 2$

When $x = 1$,

$5 + 4 = 3A + 0$

$A = 3$

Hence $\frac{5x+4}{(x-1)(x+2)} = \frac{3}{(x-1)} + \frac{2}{(x+2)}$ (Answer)

2. Let $\frac{4}{(x-3)(x+1)} = \frac{A}{(x-3)} + \frac{B}{(x+1)}$

$$\frac{4}{(x-3)(x+1)} = \frac{A(x+1) + B(x-3)}{(x-3)(x+1)}$$

$\therefore \quad 4 = A(x+1) + B(x-3)$

When $x = -1$

$4 = B(-4)$

$B = -1$

When $x = 3$

$4 = 4A + 0$

$A = 1$

$\therefore \quad \frac{4}{(x-3)(x+1)} = \frac{1}{(x-3)} - \frac{1}{(x+1)}$ (Answer)

3. Let $\frac{1}{(x+2)(x+1)} = \frac{A}{(x+2)} + \frac{B}{(x+1)}$

$$\frac{1}{(x+2)(x+1)} = \frac{A(x+1) + B(x+2)}{(x+2)(x+1)}$$

$\therefore \; A(x+1) + B(x+2) = 1$

When $x + 1 = 0$

$x = -1$

then $0 + B = 1$

$B = 1$

When $x + 2 = 0$

$x = -2$

$A(-1) + 0 = 1$

$A = -1$

$$\therefore \quad \frac{1}{(x+2)(x+1)} = \frac{-1}{(x+2)} + \frac{1}{(x+1)}$$ (Answer)

4. Let $\dfrac{4x-11}{(2x+1)(x^2+3)} = \dfrac{A}{(2x+1)} + \dfrac{Bx+C}{(x^2+3)}$

$$\frac{4x-11}{(2x+1)(x^2+3)} = \frac{A(x^2+3)+(Bx+C)(2x+1)}{(2x+1)(x^2+3)}$$

$4x - 11 = A(x^2 + 3) + (Bx + C)(2x + 1)$

When $2x + 1 = 0$

$$2x = -1$$

$$x = \frac{-1}{2}$$

By substitution,

$$-2-11 = A\left(\frac{1}{4}+3\right)+0$$

$$3\frac{1}{4}A = -13$$

$$\frac{13}{4}A = -13$$

$$A = -13 \times \frac{4}{13} = -4$$

$4x - 11 = -4(x^2 + 3) + 2Bx^2 + Bx + 2Cx + C$

$4x - 11 = -4x^2 - 12 + 2Bx^2 + Bx + 2Cx + C$

Cœfficient of x^2 on the right = Cœfficient of x^2 on the left.

We obtain $-4 + 2B = 0$

$$2B = 4$$

$$B = \frac{4}{2} = 2$$

Comparing cœfficients of x, we have

$$4 = 2 + 2C$$

$$2C = 2$$

$$C = \frac{2}{2} = 1$$

$$\therefore \quad \frac{4x-11}{(2x+1)(x^2+3)} = \frac{-4}{(2x+1)} + \frac{2x+1}{(x^2+3)}$$ (Answer)

5. Let $\frac{x^2-11}{(3x-1)(x+2)^2} = \frac{A}{(3x-1)} + \frac{B}{(x+2)} + \frac{C}{(x+2)^2}$

Then $\frac{x^2-11}{(3x-1)(x+2)^2} = \frac{A(x+2)^2 + B(3x-1)(x+2) + C(3x-1)}{(3x-1)(x+2)^2}$

$$x^2 - 11 = A(x + 2)^2 + B(3x - 1)(x + 2) + C(3x - 1)$$

When $x = -2$

$$(-2)^2 - 11 = 0 + 0 + C\{3(-2) - 1\}$$

$$4 - 11 = C(-6 - 1)$$

$$-7 = -7C$$

$$C = 1$$

When $x = \frac{1}{3}$,

$$\left(\frac{1}{3}\right)^2 - 11 = A\left(\frac{1}{3}+2\right)^2 + 0 + 0$$

$$A\left(\frac{7}{3}\right)^2 = \frac{1}{9} - 11$$

$$\frac{49}{9}A = \frac{-98}{9}$$

$$49A = -98$$

$$A = -2$$

When $x = 0,\ 0 - 11 = 4A + -2B - C$

$$4A - 2B - C = -11$$

$$4(-2) - 2(B) - 1 = -11$$

$$-8 - 2B - 1 = -11$$

$$-2B = -2$$

$$B = 1$$

$$\therefore \quad \frac{x^2 - 11}{(3x-1)(x+2)^2} = \frac{-2}{(3x-1)} + \frac{1}{(x+2)} + \frac{1}{(x+2)^2}$$ (Answer)

6. Let $\dfrac{3x+8}{(2x+1)(x^2+3)} = \dfrac{A}{(2x+1)} + \dfrac{Bx+C}{(x^2+3)}$

$$\frac{3x+8}{(2x+1)(x^2+3)} = \frac{A(x^2+3) + (Bx+C)(2x+1)}{(2x+1)(x^2+3)}$$

$$\therefore \quad 3x + 8 = A(x^2 + 3) + (Bx + C)(2x + 1)$$

When $2x + 1 = 0$

$$2x = -1$$

$$x = \frac{-1}{2}$$

$$3\left(\frac{-1}{2}\right) + 8 = A\left\{\left(\frac{-1}{2}\right)^2 + 3\right\} + 0$$

$$A\left(\frac{1}{4}+3\right) = \frac{-3}{2}+8$$

$$A \times \frac{13}{4} = \frac{-3+16}{2} = \frac{13}{2}$$

$$A = \frac{13}{2} \times \frac{4}{13} = 2$$

Comparing constant terms, we have

$$3A + C = 8$$
$$3(2) + C = 8$$
$$6 + C = 8$$
$$C = 2$$

Comparing cœfficients of x^2, we obtain

$$Ax^2 + 2Bx^2 = 0$$
$$A + 2B = 0$$
$$2 + 2B = 0$$
$$2B = -2$$
$$B = -1$$

$$\therefore \quad \frac{3x+8}{(2x+1)(x^2+3)} = \frac{2}{(2x+1)} + \frac{-x+2}{(x^2+3)}$$ (Answer)

7. Let $\dfrac{1+3x+x^2}{(x+2)(x+1)}$

$$= \frac{1+3x+x^2}{x^2+3x+2}$$

$$= \frac{x^2+3x+1}{x^2+3x+2}$$

By division,

$$\begin{array}{r l} & \qquad 1 \\ \underline{x^2 + 3x + 2} \,\big) & \overline{\cancel{x^2} + 3\cancel{x} + 1} \\ & \cancel{x^2} + \cancel{3}x + 2 \\ \cline{2-2} & \qquad\quad -1 \end{array}$$

$$\therefore \quad \frac{x^2+3x+1}{(x+2)(x+1)} = +1 + \frac{-1}{(x+2)(x+1)}$$

Let $$\frac{-1}{(x+2)(x+1)} = \frac{A}{(x+2)} + \frac{B}{(x+1)}$$

$$\frac{-1}{(x+2)(x+1)} = \frac{A(x+1)+B(x+2)}{(x+2)(x+1)}$$

$$-1 = A(x+1) + B(x+2)$$

When $x + 1 = 0$

$$\begin{aligned} x &= -1 \\ -1 &= 0 + B(-1+2) \\ -1 &= B(1) \\ B &= -1 \end{aligned}$$

When $x + 2 = 0$

$$\begin{aligned} x &= -2 \\ -1 &= A(-2+1) + 0 \\ -A &= -1 \\ A &= 1 \end{aligned}$$

$$\therefore \quad \frac{1+3x+x^2}{(x+2)(x+1)} = 1 + \frac{1}{(x+2)} - \frac{1}{(x+1)}$$ (Answer)

8. Let $\dfrac{5}{x(x-1)(x+2)} = \dfrac{A}{x} + \dfrac{B}{(x-1)} + \dfrac{C}{(x+2)}$

$$\frac{5}{x(x-1)(x+2)} = \frac{A(x-1)(x+2) + Bx(x+2) + Cx(x-1)}{x(x-1)(x+2)}$$

$\therefore \quad 5 = A(x-1)(x+2) + Bx(x+2) + Cx(x-1)$

$$\begin{aligned} \text{When } x-1 &= 0 \\ x &= 1 \\ 5 &= 0 + B(1+2) + 0 \\ 3B &= 5 \\ B &= \frac{5}{3} \end{aligned}$$

$$\begin{aligned} \text{When } x+2 &= 0 \\ x &= -2 \\ 5 &= 0 + 0 + C(-2)(-2-1) \\ 5 &= 6C \\ C &= \frac{5}{6} \end{aligned}$$

$$\begin{aligned} \text{When } x &= 0 \\ 5 &= A(0-1)(0+2) + 0 + 0 \\ -2A &= 5 \\ A &= \frac{-5}{2} \end{aligned}$$

$$\therefore \quad \frac{5}{x(x-1)(x+2)} = \frac{-5}{2x} + \frac{5}{3(x-1)} + \frac{C}{6(x+2)}$$ (Answer)

9. Let $$\frac{x^2+2x}{(x-1)(x+1)^2} = \frac{A}{(x-1)} + \frac{B}{(x+1)} + \frac{C}{(x+1)^2}$$

then $$\frac{x^2+2x}{(x-1)(x+1)^2} = \frac{A(x+1)^2 + B(x-1)(x+1) + C(x-1)}{(x-1)(x+1)^2}$$

$$x^2 + 2x = A(x+1)^2 + B(x-1)(x+1) + C(x-1)$$

$$\begin{aligned} \text{When } x+1 &= 0 \\ x &= -1 \\ (-1)^2 + 2(-1) &= 0 + 0 + C(-1-1) \\ -2C &= 1-2 \\ -2C &= -1 \\ C &= \frac{1}{2} \end{aligned}$$

$$\begin{aligned} \text{When } x-1 &= 0 \\ x &= 1 \\ (1)^2 + 2(1) &= A(1+1)^2 + 0 + 0 \\ 4A &= 1+2 \\ A &= \frac{3}{4} \end{aligned}$$

$$\begin{aligned} \text{When } x &= 0, \\ 0+0 &= A(0+1)^2 + B(0-1)(0+1) + C(0-1) \\ 0 &= A - B - C \\ 0 &= \frac{3}{4} - \frac{1}{2} - B \\ B &= \frac{1}{4} \end{aligned}$$

$$\therefore \quad \frac{x^2+2x}{(x-1)(x+1)^2} = \frac{3}{4(x-1)} + \frac{1}{4(x+1)} + \frac{1}{2(x+1)^2} \qquad \text{(Answer)}$$

10. Let = $\frac{2x^2-3}{x(x+5)^2} = \frac{A}{x} + \frac{B}{x+5} + \frac{C}{(x+5)^2}$

$$\frac{2x^2-3}{x(x+5)^2} = \frac{A(x+5)^2 + Bx(x+5) + Cx}{x(x+5)^2}$$

$$2x^2 - 3 = A(x+5)^2 + Bx(x+5) + Cx$$

When $x = 0$

$$2(0) - 3 = A(0+5)^2 + 0 + 0$$

$$25A = -3$$

$$A = \frac{-3}{25}$$

When $x = -5$,

$$2(-5)^2 - 3 = 0 + 0 + C(-5)$$

$$-5C = 50 - 3$$

$$-5C = 47$$

$$C = \frac{-47}{5}$$

Comparing cœfficient of x^2, we have

$$Ax^2 + Bx^2 = 2x^2$$

$$A + B = 2$$

$$\frac{-3}{25} + B = 2$$

$$B = 2 + \frac{3}{25}$$

$$B = \frac{53}{25}$$

$$\therefore \quad \frac{2x^2-3}{x(x+5)^2} = \frac{-3}{25x} + \frac{53}{25(x+5)} - \frac{47}{5(x+5)^2}$$ (Answer)

11. Let = $\dfrac{1}{x^2(x+3)} = \dfrac{A}{x} + \dfrac{B}{x^2} + \dfrac{C}{(x-3)}$

$$\frac{1}{x^2(x-3)} = \frac{Ax(x-3) + B(x-3) + Cx^2}{x^2(x-3)}$$

$$1 = Ax(x-3) + B(x-3) + Cx^2$$

$$\begin{aligned}
\text{When } x &= 0 \\
1 &= 0 + B(0-3) + 0 \\
-3B &= 1 \\
B &= -\frac{1}{3}
\end{aligned}$$

$$\begin{aligned}
\text{When } x - 3 &= 0 \\
x &= 3 \\
1 &= 0 + 0 + 9C \\
C &= \frac{1}{9}
\end{aligned}$$

Comparing cœfficients of x^2, we have

$$\begin{aligned}
A + C &= 0 \\
A + \frac{1}{9} &= 0 \\
A &= -\frac{1}{9}
\end{aligned}$$

$$\therefore \quad \frac{1}{x^2(x-3)} = -\frac{1}{9x} - \frac{1}{3x^2} + \frac{1}{9(x-3)}$$ (Answer)

12. Let = $\dfrac{16}{x^2-9} = \dfrac{16}{(x+3)\,(x-3)} = \dfrac{A}{(x+3)} + \dfrac{B}{(x-3)}$

$$\frac{16}{(x+3)(x-3)} = \frac{A(x-3) + B(x+3)}{(x+3)(x-3)}$$

$$16 = A(x-3) + B(x+3)$$

$$\begin{aligned}
\text{When } x-3 &= 0\\
x &= 3\\
16 &= 0+6B\\
B &= \frac{16}{6}=\frac{8}{3}
\end{aligned}$$

$$\begin{aligned}
\text{When } x+3 &= 0\\
x &= -3\\
16 &= A(-3-3)+0\\
-6A &= 16\\
A &= \frac{-16}{6}=\frac{-8}{3}
\end{aligned}$$

$$\therefore \quad \frac{16}{x^2-9}=\frac{-8}{3(x+3)}+\frac{8}{3(x-3)} \qquad \text{(Answer)}$$

13. Let $\dfrac{3x}{(x^2-1)(x+2)}=\dfrac{A}{x+1}+\dfrac{B}{x-1}+\dfrac{C}{x+2}$

$$\frac{3x}{(x^2-1)(x+2)}=\frac{A(x-1)(x+2)+B(x+1)(x+2)+C(x+1)(x-1)}{(x^2-1)(x+2)}$$

$$\therefore \quad 3x=A(x-1)(x+2)+B(x+1)(x+2)+C(x+1)(x-1)$$

$$\begin{aligned}
\text{When } x &= 1\\
3 &= 0+B(2)(3)+0\\
6B &= 3\\
B &= \frac{1}{2}
\end{aligned}$$

$$\begin{aligned} \text{When } x &= -2 \\ -6 &= 0+0+C(-1)(-3) \\ 3C &= -6 \\ C &= -2 \end{aligned}$$

$$\begin{aligned} \text{When } x &= -1 \\ -3 &= A(-2)(1)+0+0 \\ -2A &= -3 \\ A &= \frac{3}{2} \end{aligned}$$

$$\therefore \quad \frac{3x}{(x^2-1)(x+2)} = \frac{3}{2(x+1)} + \frac{1}{2(x-1)} + \frac{-2}{(x+2)}$$ (Answer)

14. Let $\dfrac{5x}{(x^2+1)(x+2)} = \dfrac{Ax+B}{(x^2+1)} + \dfrac{C}{(x+2)}$

$$\frac{5x}{(x^2+1)(x+2)} = \frac{(Ax+B)(x+2)+C(x^2+1)}{(x^2+1)(x+2)}$$

$$\therefore \quad 5x = (Ax+B)(x+2)+C(x^2+1)$$

$$\begin{aligned} \text{When } x &= -2 \\ -10 &= 0+C(4+1) \\ 5C &= -10 \\ C &= \frac{-10}{5} = -2 \end{aligned}$$

$$\begin{aligned} \text{When } x &= 0 \\ 0 &= B(2)+C(1) \\ 0 &= 2B-2 \\ 2B &= 2 \\ B &= 1 \end{aligned}$$

$$5x = (Ax + 1)(x + 2) + -2x^2 - 2$$

$$5x = Ax^2 + x + 2Ax - 2x^2 - 2$$

Equating cœfficients of x^2

$$0 = Ax^2 - 2x^2$$

$$A - 2 = 0$$

$$A = 2$$

$$\therefore \quad \frac{5x}{(x^2+1)(x+2)} = \frac{2x+1}{x^2+1} - \frac{2}{x+2}$$ (Answer)

15. Let $\dfrac{3x}{(2x-1)(x^2-3)} = \dfrac{A}{(2x-1)} + \dfrac{Bx+C}{(x^2-3)}$

$$\frac{3x}{(2x-1)(x^2-3)} = \frac{A(x^2-3) + (Bx+C)(2x-1)}{(2x-1)(x^2-3)}$$

$$\therefore \quad 3x = A(x^2 - 3) + (Bx + C)(2x - 1)$$

When $2x - 1 = 0$

$$2x = 1$$

$$x = \frac{1}{2}$$

$$3 \times \frac{1}{2} = A\left(\left(\frac{1}{2}\right)^2 - 3\right) + 0$$

$$A\left(\frac{1}{4} - 3\right) = \frac{3}{2}$$

$$A\left(\frac{-11}{4}\right) = \frac{3}{2}$$

$$A = -\frac{3}{\cancel{2}} \times \frac{\cancel{4}^{\,2}}{11} = \frac{-6}{11}$$

Comparing cœfficients of x^2, we have

$$\begin{aligned} Ax^2 + 2Bx^2 &= 0 \\ A + 2B &= 0 \\ \frac{-6}{11} + 2B &= 0 \\ 2B &= \frac{6}{11} \\ B &= \frac{3}{11} \end{aligned}$$

$$\begin{aligned} \text{When} &= 0 \\ 0 &= A(0-3) + (0+C)(0-1) \\ -3A + -C &= 0 \\ -3\left(-\frac{6}{11}\right) &= C \\ C &= \frac{18}{11} \end{aligned}$$

$$\therefore \quad \frac{3x}{(2x-1)(x^2-3)} = \frac{-6}{11(2x-1)} + \left(\frac{\frac{3}{11}x + \frac{18}{11}}{x^2-3}\right)$$

i.e. $\dfrac{3x}{(2x-1)(x^2-3)} = \dfrac{-6}{11(2x-1)} + \left(\dfrac{3x+18}{11(x^2-3)}\right)$ (Answer)

16. Let $\dfrac{x-5}{(3x-1)(x-2)^2} = \dfrac{A}{(3x-1)} + \dfrac{B}{(x-2)} + \dfrac{C}{(x-2)^2}$

$$\frac{x-5}{(3x-1)(x-2)^2} = \frac{A(x-2)^2 + B(x-2)(3x-1) + C(3x-1)}{(3x-1)(x-2)^2}$$

$$\therefore \quad x - 5 = A(x-2)^2 + B(x-2)(3x-1) + C(3x-1)$$

$$\begin{aligned} \text{When } x - 2 &= 0 \\ x &= 2 \\ 2 - 5 &= 0 + 0 + C(6 - 1) \\ 5C &= -3 \\ C &= \frac{-3}{5} \end{aligned}$$

$$\begin{aligned} \text{When } 3x - 1 &= 0 \\ 3x &= 1 \\ x &= \frac{1}{3} \end{aligned}$$

$$\frac{1}{3} - 5 = A\left(\frac{1}{3} - 2\right)^2 + 0 + 0$$

$$A \times \frac{25}{9} = \frac{-14}{3}$$

$$A = \frac{-14}{\cancel{3}} \times \frac{\cancel{9}^{\,3}}{25} = \frac{-42}{25}$$

Equating cœfficients of x^2, we get

$$\begin{aligned} 0 &= Ax^2 + 3Bx^2 \\ A + 3B &= 0 \\ 3B &= -A \\ 3B &= \frac{42}{25} \\ B &= \frac{42}{25} \times \frac{1}{3} = \frac{14}{25} \end{aligned}$$

$$\therefore \quad \frac{(x-5)}{(3x-1)(x-2)^2} = \frac{-42}{25(3x-1)} + \frac{14}{25(x-2)} + \frac{-3}{5(x-2)^2} \qquad \text{(Answer)}$$

UNIT 3

INDICES, LOGARITHMIC AND EXPONENTIAL EQUATIONS

Suggested Solutions

1. $$\left(1-\sqrt{3}\right)^4 = 1+4\left(-\sqrt{3}\right)+6\left(-\sqrt{3}\right)^2+4\left(-\sqrt{3}\right)^3+\left(-\sqrt{3}\right)^4$$

$$= 1-4\sqrt{3}+18-12\sqrt{3}+9$$

$$= 28-16\sqrt{3}.$$

$$-4\left(1-\sqrt{3}\right)^2 = -4\left(1-2\sqrt{3}+3\right) = -16+8\sqrt{3}$$

So that $\left(1-\sqrt{3}\right)^4 - 4\left(1-\sqrt{3}\right)^2 - 8\left(1-\sqrt{3}\right) - 4$

$$= 28-16\sqrt{3}-16+8\sqrt{3}-8+8\sqrt{3}-4$$

$$= 0$$

(Answer)

Let $\left(1-\sqrt{3}\right) = x$

The expression becomes

$$x^4 - 4x^2 - 8x - 4 = 0$$

$\therefore$ since $x = 1-\sqrt{3} = 0$

$$x - 1 + \sqrt{3} = 0$$

Hence $\left(x-1+\sqrt{3}\right)$ is a linear factor of the given expression. (Answer)

2. (a)

$$a^x = e^{2x+1}$$

$$\ln a^x = \ln e^{2x+1}$$

$$x\ln a = (2x+1)\ln e$$

$$x\ln a = (2x+1)\times 1$$

$$x\ln a = 2x+1$$

$$x\ln a - 2x = 1$$

$$x(\ln a - 2) = 1$$

$$x = \frac{1}{(\ln a - 2)}$$ (Answer)

(b)

$$2\ln(2x) = 1 + \ln a$$

$$2\ln(2x) = \ln e + \ln a$$

$$\ln(2x)^2 = \ln(ea)$$

$$(2x)^2 = ea$$

$$4x^2 = ea$$

$$x^2 = \frac{ea}{4}$$

$$x = \pm\sqrt{\frac{ae}{4}} = \pm\frac{1}{2}\sqrt{ae}$$ (Answer)

3.

$$4^x = 8^{2x+1}$$

$$(2^2)x = (2^3)^{2x+1}$$

$$2^{2x} = 2^{6x+3}$$

$$\therefore \quad 2x = 6x + 3$$

$$2x - 6x = 3$$

$$-4x = 3$$

$$4x = -3$$

$$\therefore \quad x = \frac{-3}{4}$$ (Answer)

4.

$$2(4^x) + 4^{-x} = 3$$

Substituting 4^x by y.

$$2(4^x) + \frac{1}{4^x} = 3$$

$$2y + \frac{1}{y} = 3$$

$$2y^2 + 1 = 3y$$
$$2y^2 - 3y + 1 = 0$$
$$(2y - 1)(y - 1) = 0$$

$$2y - 1 = 0 \quad \text{or} \quad y - 1 = 0$$
$$2y = 1 \qquad\qquad y = 1$$
$$y = \frac{1}{2}$$

$$\therefore \quad 4^x = \frac{1}{2} \quad \text{or} \quad 4^x = 1$$
$$4^x = 4^0$$
$$\therefore \quad x = 0$$

$$2^{2x} = 2^{-1}$$
$$2x = -1$$
$$x = -\frac{1}{2}$$

$$\therefore \quad x = -\frac{1}{2} \quad \text{or} \quad 0$$ (Answer)

5. $$x^{\frac{1}{2}} + x^{-\frac{1}{2}} = 2\left(x^{\frac{1}{2}} - x^{-\frac{1}{2}}\right)$$

Let $x^{\frac{1}{2}} = \text{y}$

$$x^{\frac{1}{2}} + \frac{1}{x^{\frac{1}{2}}} = 2\left(x^{\frac{1}{2}} - \frac{1}{x^{\frac{1}{2}}}\right)$$

$$y + \frac{1}{y} = 2\left(y - \frac{1}{y}\right)$$

$$y^2 + 1 = 2(y^2 - 1)$$
$$y^2 + 1 = 2y^2 - 2$$
$$y^2 = 3$$
$$y = \sqrt{3}$$

$$\therefore \quad x^{\frac{1}{2}} = \sqrt{3}$$

$$x^{\frac{1}{2}} = 3^{\frac{1}{2}}$$ (Answer)

$$\therefore \quad x = 3$$

6.

$$x^{\frac{-1}{2}} + 2x^{-1} = 10$$

$$y + 2y^2 = 10$$

$$2y^2 + y - 10 = 0$$

$$(2y + 5)(y - 2) = 0$$

$$2y + 5 = 0 \qquad \text{or} \qquad y - 2 = 0$$

$$2y = -5 \qquad\qquad y = 2$$

$$y = \frac{-5}{2}$$

$$x^{\frac{-1}{2}} = \frac{-5}{2} \qquad\qquad x^{\frac{-1}{2}} = 2$$

$$\frac{1}{x^{\frac{1}{2}}} = \frac{-5}{2} \qquad\qquad \frac{1}{x^{\frac{1}{2}}} = 2$$

$$\frac{1}{x} = \frac{25}{4} \qquad\qquad \frac{1}{x} = 4$$

$$x = \frac{4}{25} \qquad\qquad x = \frac{1}{4}$$ (Answer)

7.

$$4x^{\frac{1}{2}} = 3 - 7x$$

$$16x = (3 - 7x)^2$$

$$16x = 9 - 42x + 49x^2$$

$$49x^2 - 58x + 9 = 0$$

$$(49x - 9)(x - 1) = 0$$

$$49x - 9 = 0 \qquad \text{or} \qquad x - 1 = 0$$

$$49x = 9 \qquad\qquad x = 1$$

$$x = \frac{9}{49}$$ (Answer)

8.

$$\begin{aligned}
3^{2x} &= 4^{2-x} \\
\lg 3^{2x} &= \lg 4^{2-x} \\
2x\lg 3 &= (2-x)\lg 4 \\
2x \times 0.4771 &= (2-x) \times 0.6021 \\
0.9542x &= 1.2042 - 0.6021x \\
0.9542x + 0.6021x &= 1.2042 \\
1.5563x &= 1.2042 \\
x &= 0.774 \text{ (3 S.F.)}
\end{aligned}$$

(Answer)

9.

$$\begin{aligned}
e^{2-2x} &= 2e^{-x} \\
\frac{e^2}{e^{2x}} &= \frac{2}{e^x} \\
\frac{e^{2x}}{e^x} &= \frac{e^2}{2} \\
e^x &= \frac{1}{2}e^2 \\
x &= \ln\left(\frac{1}{2}e^2\right) \\
x &= \ln\frac{1}{2} + \ln e^2 \\
x &= \ln\frac{1}{2} + 2\ln e \\
x &= \ln\frac{1}{2} + 2 \times 1 \\
\therefore\ x &= \ln\frac{1}{2} + 2
\end{aligned}$$

(Answer)

10. $e^{\frac{x}{2}} = \sqrt{e^x}$

$$e^{\frac{x}{2}} = \left(e^{\frac{1}{2}}\right)^x = k^x$$

$$\therefore\quad k = e^{\frac{1}{2}} = 1.65 \text{ (3 S.F.)}$$

11. $$\left(3+\sqrt{2}\right)^4 = 3^4 + 4\left(3^3\right)\left(\sqrt{2}\right) + 6(3)^2\left(\sqrt{2}\right)^2 + 4(3)\left(\sqrt{2}\right)^3 + \left(\sqrt{2}\right)^4$$

$$= 81 + 108\sqrt{2} + 108 + 24\sqrt{2} + 4$$

$$= 193 + 132\sqrt{2}$$ (Answer)

12. $$\ln\left(2\sqrt{e}\right) - \frac{1}{3}\ln\left(\frac{8}{e}\right) - \ln\left(\frac{e}{3}\right)$$

$$= \ln 2 + \ln\sqrt{e} - \frac{1}{3}\left(\ln 8 - \ln e\right) - \left(\ln e - \ln 3\right)$$

$$= \cancel{\ln 2} + \frac{1}{2}\ln e - \cancel{\ln 2} + \frac{1}{3} - 1 + \ln 3$$

$$= \frac{-2}{3} + \frac{1}{2} + \ln 3 = \frac{-1}{6} + \ln 3$$ (Answer)

13.
$$\begin{aligned}
x^{\frac{1}{3}} - 2x^{\frac{-1}{3}} &= 1 \\
x^{\frac{1}{3}} - \frac{2}{x^{\frac{1}{3}}} &= 1 \\
y - \frac{2}{y} &= 1 \\
y^2 - 2 &= y \\
y^2 - y - 2 &= 0 \\
(y-2)(y+1) &= 0 \\
\therefore \quad y &= 2 \quad \text{or} \quad -1
\end{aligned}$$

$$x^{\frac{1}{3}} = 2 \quad \text{or} \quad x^{\frac{1}{3}} = -1$$

$$x = 2^3 = 8 \qquad x = -1$$ (Answer)

14.

$$e^x - 2e^{-x} = 1$$

$$e^x - \frac{2}{e^x} = 1$$

Let $u = e^x$

$$\therefore \quad u - \frac{2}{u} = 1$$

$$u^2 - 2 = u$$

$$u^2 - u - 2 = 0$$

$$(u-2)(u+1) = 0$$

$u = 2$ or -1

$e^x = 2$ or $e^x = -1$ (not defined)

(Answer)

$$x = \ln 2$$

15.

$$x + y = 1$$

$x = 1 - y$ putting this value of x in the 2nd equation.

$$\therefore \quad 2^x = 3^y$$

$$2^{1-y} = 3^y$$

$$(1-y)\ln 2 = y\ln 3$$

$$(1-y) \times \frac{\ln 2}{\ln 3} = y$$

$$\ln 2 - y\ln 2 = y\ln 3$$

$$y\ln 3 + y\ln 2 = \ln 2$$

$$y(\ln 3 + \ln 2) = \ln 2$$

$$y = \frac{\ln 2}{(\ln 3 + \ln 2)}$$

Since $x = 1 - y$

$$\therefore \quad x = 1 - \frac{\ln 2}{(\ln 3 + \ln 2)}$$

$$= \frac{\ln 3 + \ln 2 - \ln 2}{\ln 3 + \ln 2}$$

$$= \frac{\ln 3}{\ln 6}$$

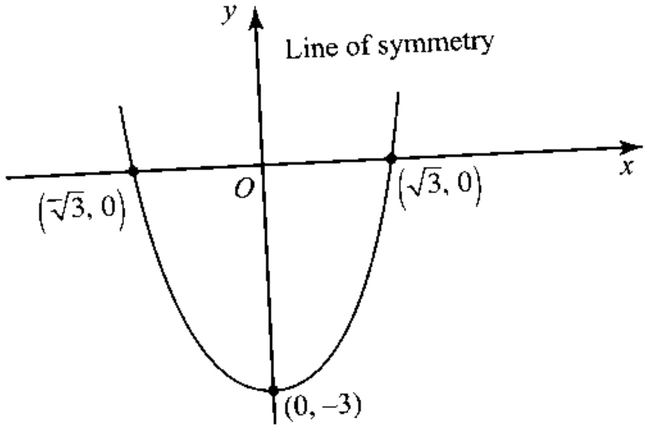
(Answer)

UNIT 4

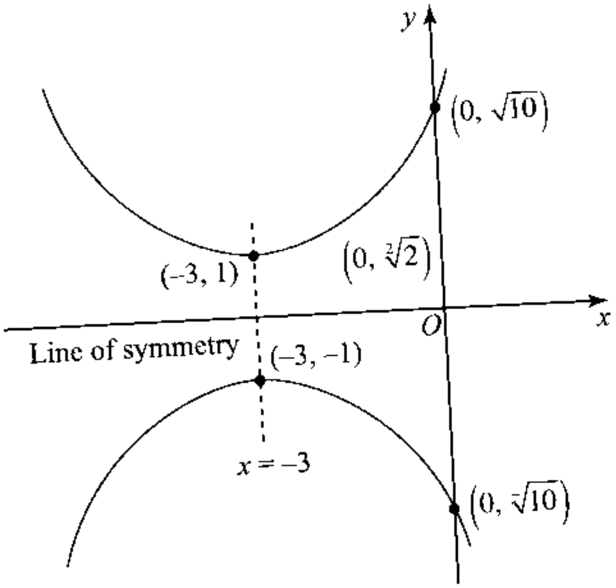
QUADRATIC EQUATIONS AND EXPRESSIONS

Suggested Solutions

1. (a) $y = x^2 - 3$



(b) $y^4 = (x + 3)^2 + 1$



2. $$3x^2 - 5x + 1 = 3\left\{x^2 - \frac{5}{3}x + \frac{1}{3}\right\}$$

$$= 3\left\{x^2 - \frac{5}{3}x + \left(\frac{5}{6}\right)^2 + \frac{1}{3} - \left(\frac{5}{6}\right)^2\right\}$$

$$= 3\left\{\left(x - \frac{5}{6}\right)^2 + \frac{1}{3} - \frac{25}{36}\right\}$$

$$= 3\left\{\left(x - \frac{5}{6}\right)^2 + \frac{12}{36} - \frac{25}{36}\right\}$$

$$= 3\left\{\left(x - \frac{5}{6}\right)^2 - \frac{13}{36}\right\}$$

$$= 3\left(x - \frac{5}{6}\right)^2 - \frac{13}{12}$$

$$\therefore \quad a = 3, \quad b = -\frac{5}{6} \quad \text{and} \quad c = -\frac{13}{12}$$

3. $$y = 4x^2 + 12x + 1$$

$$= 4\left\{x^2 + 3x + \frac{1}{4}\right\}$$

$$= 4\left\{x^2 + 3x + \left(\frac{3}{2}\right)^2 + \frac{1}{4} - \left(\frac{3}{2}\right)^2\right\}$$

$$= 4\left\{\left(x + \frac{3}{2}\right)^2 + \frac{1}{4} - \frac{9}{4}\right\}$$

$$= 4\left\{\left(x + \frac{3}{2}\right)^2 - \frac{8}{4}\right\}$$

$$= 4\left(x + \frac{3}{2}\right)^2 - 8$$

$$= (2x + 3)^2 - 8$$

$$2x + 3 = 0 \Rightarrow 2x = -3 \Rightarrow x = -\frac{3}{2}$$

Minimum point has coordinates

$$\left(-\frac{3}{2}, -8\right)$$

4. Let $f(x) = 9x^2 - 36x + 52$

$$9x^2 - 36x + 52 = 9\left\{x^2 - 4x + \frac{52}{9}\right\}$$

$$= 9\left\{x^2 - 4x + (2)^2 + \frac{52}{9} - (2)^2\right\}$$

$$= 9\left\{(x-2)^2 + \frac{52}{9} - 4\right\}$$

$$= 9\left\{(x-2)^2 + \frac{52-36}{9}\right\}$$

$$= 9\left\{(x-2)^2 + \frac{16}{9}\right\}$$

$$= 9(x-2)^2 + 16$$

$$= (3x-6)^2 + 16$$

$\therefore \quad a = 3, \; b = 6, \; c = 16.$

The minimum value of $f(x)$ is 16.

$\therefore$ Range of $f(x) \geqslant 16$. (Answer)

5. $2x^2 - 4x - 5$

$$= 2\left\{x^2 - 2x - \frac{5}{2}\right\}$$

$$= 2\left\{x^2 - 2x + 1 - \frac{5}{2} - 1\right\}$$

$$= 2\left\{(x-1)^2 - \frac{7}{2}\right\}$$

$$= 2(x-1)^2 - 7.$$

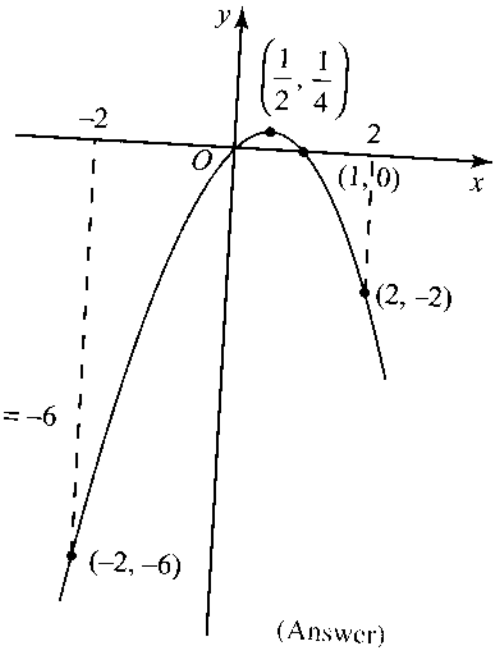
$\therefore \quad A = 2, \; B = -1$ and $C = -7$

Range of $2x^2 - 4x - 5$ is $\geqslant -7$.

(Answer)

6.

$$\begin{aligned} y &= x(1-x) \\ &= x - x^2 \\ &= -x^2 + x \\ &= -\{x^2 - x\} \\ &= -\left\{x^2 - x + \frac{1}{4} - \frac{1}{4}\right\} \\ &= -\left\{\left(x - \frac{1}{2}\right)^2 - \frac{1}{4}\right\} \\ &= -\left(x - \frac{1}{2}\right)^2 + \frac{1}{4} \end{aligned}$$



When $x = -2$, $y = -2(1 - -2) = -2(1 + 2) = -6$

When $x = 2$, $y = 2(1 - 2) = -2$

$\therefore$ Max. value $= \frac{1}{4}$

Min. value $= -6$

(Answer)

7.

$$\begin{aligned} \text{Let } f(x) &= 25x^2 - 20x + 11 \\ &= 25\left\{x^2 - \frac{20}{25}x + \frac{11}{25}\right\} \\ &= 25\left\{x^2 - \frac{\overset{4}{\cancel{20}}}{\underset{5}{\cancel{25}}}x + \left(\frac{2}{5}\right)^2 + \frac{11}{25} - \left(\frac{2}{5}\right)^2\right\} \\ &= 25\left\{\left(x - \frac{2}{5}\right)^2 + \frac{11}{25} - \frac{4}{25}\right\} \\ &= 25\left\{\left(x - \frac{2}{5}\right)^2 + \frac{7}{25}\right\} \end{aligned}$$

$$= 25\left(x-\frac{2}{5}\right)^2 + 7$$

$$= (5x-2)^2 + 7$$

$\therefore \quad p = 5,\ q = -2$ and $r = 7.$ (Answer)

$$25x^2 - 20x + 11 > 23$$
$$25x^2 - 20x + 11 - 23 > 0$$
$$25x^2 - 20x - 12 > 0$$
$$(5x-6)(5x+2) > 0$$
$$x > \frac{6}{5} \quad \text{or} \quad x < -\frac{2}{5}$$

(Answer)

8. $3x^2 - 7x + 1$

$$= 3\left\{x^2 - \frac{7}{3}x + \frac{1}{3}\right\}$$

$$= 3\left\{x^2 - \frac{7}{3}x + \left(\frac{7}{6}\right)^2 + \frac{1}{3} - \left(\frac{7}{6}\right)^2\right\}$$

$$= 3\left\{\left(x-\frac{7}{6}\right)^2 + \frac{1}{3} - \frac{49}{36}\right\}$$

$$= 3\left\{\left(x-\frac{7}{6}\right)^2 + \frac{12-49}{36}\right\}$$

$$= 3\left\{\left(x-\frac{7}{6}\right)^2 - \frac{37}{36}\right\}$$

$$= 3\left(x-\frac{7}{6}\right)^2 - \frac{37}{12}$$

$\therefore \quad$ minimum value $= -\dfrac{37}{12}$

when $x = \dfrac{7}{6}$ (Answer)

---

9. Let f (x) $= 2x^2 - 7x + 5$

$$= 2\left\{x^2 - \frac{7}{2}x + \frac{5}{2}\right\}$$

$$= 2\left\{x^2 - \frac{7}{2}x + \left(\frac{7}{4}\right)^2 + \frac{5}{2} - \left(\frac{7}{4}\right)^2\right\}$$

$$= 2\left\{\left(x - \frac{7}{4}\right)^2 + \frac{5}{2} - \frac{49}{16}\right\}$$

$$= 2\left\{\left(x - \frac{7}{4}\right)^2 + \frac{40 - 49}{16}\right\}$$

$$= 2\left\{\left(x - \frac{7}{4}\right)^2 - \frac{9}{16}\right\}$$

$$= 2\left(x - \frac{7}{4}\right)^2 - \frac{9}{8}$$

$\therefore$ $a = 2$, $b = -\frac{7}{4}$ and $c = -\frac{9}{8}$.

$2x^2 - 7x + 5$

$= (2x - 5)(x - 1)$

$\therefore$ critical values of x are 1 and $\frac{5}{2}$ for f (x) to be greater than 0,

$x > \frac{5}{2}$ or $x < 1$. (Answer)

10. Form a quadratic equation in y, we have

$y^2 + 16\,xy + 14\,y + 4x^2 + 16x + 13 = 0$

$y^2 + y(16x + 14) + (4x^2 + 16x + 13) = 0$

for real roots, $b^2 - 4ac > 0$

$(16x + 14)^2 - 4(1)(4x^2 + 16x + 13\,) > 0$

$256x^2 + 448x + 196 - 16x^2 - 64x - 52 > 0$

$240x^2 + 384x + 144 > 0$

$5x^2 + 8x + 3 > 0$

$(5x+3)(x+1) > 0$ (Answer)

$x < -1$ or $x > -\frac{3}{5}$ (Answer)

By substituting $x < -1$ or $x > -\frac{3}{5}$, values of y follow.

11. $x^2 - 4x + 9$

$= x^2 - 4x + 4 + 9 - 4$

$= (x-2)^2 + 5$ (Answer)

which is always positive

$x^3 + 2x^2 + x + 14 > (x+1)(x^2+5)$

$x^3 + 2x^2 + x + 14 > x^3 + x^2 + 5x + 5$

$x^2 - 4x + 9 > 0$

Already proved for any value of x. (Answer)

$\therefore$ x has any real value

12. $2x^2 - 3x + 7$

$$= 2\left\{x^2 - \frac{3}{2}x + \frac{7}{2}\right\}$$

$$= 2\left\{x^2 - \frac{3}{2}x + \left(\frac{3}{4}\right)^2 + \frac{7}{2} - \left(\frac{3}{4}\right)^2\right\}$$

$$= 2\left\{\left(x - \frac{3}{4}\right)^2 + \frac{7}{2} - \frac{9}{16}\right\}$$

$$= 2\left(x - \frac{3}{4}\right)^2 + \left(\frac{56-9}{16}\right) \times 2$$

$$= 2\left(x - \frac{3}{4}\right)^2 + \frac{47}{8}$$

(Answer

$\therefore$ Min. value of $f(x) = \frac{47}{8}$ when $x = \frac{3}{4}$

13. If $(2x+1)$ is a factor of $f(x)$

then when $2x + 1 = 0$, i.e. $x = -\frac{1}{2}$

$f(x) = 0$

$$f\left(-\frac{1}{2}\right) = 2\left(-\frac{1}{2}\right)^3 + a\left(-\frac{1}{2}\right)^2 + 16\left(-\frac{1}{2}\right) + 6 = 0$$

$$2\left(-\frac{1}{8}\right) + \frac{a}{4} - 8 + 6 = 0$$

$$-\frac{1}{4} + \frac{a}{4} - 2 = 0$$

$$\frac{a}{4} = \frac{9}{4}$$

$\therefore \quad a = 9$ (Answer)

(a) By division

$$\begin{array}{r|l}
 & \phantom{2x^3+{}}x^2 + 4x + 6 \\
\hline
2x + 1 & 2x^3 + 9x^2 + 16x + 6 \quad - \\
 & \underline{2x^3 + x^2} \\
 & \phantom{2x^3+{}}8x^2 + 16x + 6 \quad - \\
 & \phantom{2x^3+{}}\underline{8x^2 + 4x} \\
 & \phantom{2x^3+9x^2+{}}12x + 6 \quad - \\
 & \phantom{2x^3+9x^2+{}}\underline{12x + 6} \\
 & \phantom{2x^3+9x^2+{}}\underline{-\quad -}
\end{array}$$

$\therefore$ the quadratic factor is $x^2 + 4x + 6$ (Answer)

(b) $x^2 + 4x + 6$

$= x^2 + 4x + 4 + 6 - 4$

$= (x + 2)^2 + 2$

which is always positive (Answer)

14. $(2a - 1)x^2 + (3a + 1)x + a + 1 = 0$

Evaluate $b^2 - 4ac$, the discriminant, we obtain

$$\begin{aligned} &(3a+1)^2 - 4(2a-1)(a+1) \\ &= 9a^2 + 6a + 1 - 4(2a^2 + a - 1) \\ &= 9a^2 + 6a + 1 - 8a^2 - 4a + 4 \\ &= a^2 + 2a + 5 \\ &= a^2 + 2a + 1 + 5 - 1 \\ &= (a + 1)^2 + 4 \end{aligned}$$

which is positive

(Answer)

$\therefore$ the given equation has 2 real and distinct roots.

15. $x^2 - 3x + 1 = 2x + a$

$x^2 - 3x - 2x + 1 - a = 0$

$x^2 - 5x + 1 - a = 0$

for 2 real and distinct roots,

$$\begin{aligned} b^2 - 4ac &> 0 \\ 25 - 4(1)(1 - a) &> 0 \\ 25 - 4 + 4a &> 0 \\ 21 + 4a &> 0 \\ 4a &> -21 \\ a &> -\frac{21}{4} \\ a &> -5\frac{1}{4} \end{aligned}$$

(Answer)

$\therefore \quad a = -5$

16. $\mathrm{f}(x) = (k + 1)x^2 + (k + 2)x - (k - 1)$

$$
\begin{aligned}
b^2 - 4ac &> 0 \\
(k + 2)^2 + 4(k + 1)(k - 1) &> 0 \\
k^2 + 4k + 4 + 4(k^2 - 1) &> 0 \\
k^2 + 4k + 4 + 4k^2 - 4 &> 0 \\
5k^2 + 4k &> 0 \\
k(5k + 4) &> 0
\end{aligned}
$$

$$k > 0 \quad \text{or} \quad k < -\frac{4}{5}$$ (Answer)

UNIT 5

ARITHMETIC AND GEOMETRIC PROGRESSIONS

Suggested Solutions

1. $S_{20} = 1\ 220$

$S_{40} - S_{20} = 3\ 620$

Let a = the first term,

d = the common difference,

n = the number of terms.

(a) Then $S_{40} - S_{20} = 3\ 620$

$$\frac{n}{2}\left\{2a + (n-1)d\right\} - 1\ 220 = 3\ 620$$

$$\frac{40}{2}\{2a + 39d\} = 3\ 620 + 1\ 220$$

$$20(2a + 39d) = 4\ 840$$

$$2a + 39d = 242 \text{ ------------------------ } ①$$

$$S_{20} = \frac{n}{2}\left\{2a + (n-1)d\right\}$$

$$= \frac{20}{2}\{2a + 19d\} = 1\ 220$$

$$10(2a + 19d) = 1\ 220$$

$$2a + 19d = 122 \text{ ------------------------ } ②$$

$$2a + 39d = 242 \text{ ------------------------ } ①$$

Substracting ① from ②

$$20d = 120$$

$$d = \frac{120}{20}$$

$$= 6$$

Replacing $d = 6$ in ②

$2a + 19d = 122$ -- ②

$2a + 19 \times 6 = 122$

$2a + 114 = 122$

$2a = 122 - 114$

$2a = 8$

$a = \frac{8}{2} = 4$ (Answer)

(b) $d = 6$ (Answer)

(c) $S_{18} = \frac{n}{2}\{2a + (n-1)d\}$

$= \frac{18}{2}\{2 \times 4 + 17 \times 6\}$

$= 9(8 + 102)$

$= 9 \times 110$

$= 990$ (Answer)

(a) $a = 3$,

21^{st} term = $a + 20d = 8$

$3 + 20d = 8$

$20d = 5$

$d = \frac{5}{20} = \frac{1}{4}$ (Answer)

(b) $S_n = \frac{n}{2}\left\{2a + \left(n-1\right)d\right\} > 20$

$$\frac{n}{2}\left\{6 + \left(n-1\right) \times \frac{1}{4}\right\} > 20$$

$$\frac{n}{2}\left(6 + \frac{1}{4}n - \frac{1}{4}\right) > 20$$

$$n\left(\frac{23}{4} + \frac{1}{4}n\right) > 40$$

$$23n + n^2 > 160$$

$$n^2 + 23n - 160 > 0$$

$$n = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \text{ or } n = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-23 + \sqrt{529 - 4(1)(-160)}}{2} \text{ or } n = \frac{-23 - \sqrt{529 + 640}}{2}$$

$$= \frac{-23 + \sqrt{529 + 640}}{2} \text{ or } n = \frac{-23 - \sqrt{529 + 640}}{2}$$

$$= \frac{-23 + 34.2}{2} \text{ or } \frac{-23 - 34.2}{2}$$

$$= \frac{11.2}{2} \text{ or } \frac{-57.2}{2}$$

Let f $(n) = n^2 + 23n - 160$

When $n = -28.6$ 5.6

f $(n) = 0.16$ 0.16

n is therefore greater than 5.6 i.e. $n = 6$ (Answer)

3. $a = 10$

common ratio = r

$r > 0$

$$S_7 = \frac{a(r^7 - 1)}{r - 1}$$

8^{th} term = ar^7

9^{th} term = ar^8

$$\frac{a(r^7 - 1)}{(r - 1)} = 3(ar^7 + ar^8)$$

$$10\frac{(r^7 - 1)}{(r - 1)} = 30(r^7 + r^8)$$

$$\frac{(r^7 - 1)}{(r - 1)} = 3r^7(r + 1)$$

$$r^7 - 1 = 3r^7(r^2 - 1)$$
$$r^7 = 1 + 3r^7(r^2 - 1)$$
$$r^7 - 3r^7(r^2 - 1) = 1$$
$$r^7(1 - 3r^2 + 3) = 1$$
$$r^7(4 - 3r^2) = 1 \text{ (proved)}$$

(Answer)

4. 7.23, 7.28, 7.33, 7.38, …, 9.68

(a) $d = 7.28 - 7.23 = 0.05$ (Answer)

(b)
$$l = 9.68$$
$$a + (n - 1)d = 9.68$$
$$7.23 + (n - 1) \times 0.05 = 9.68$$
$$(n - 1) \times \frac{1}{20} = 9.68 - 7.23$$
$$\frac{1}{20}(n - 1) = 2.45$$
$$n - 1 = 2.45 \times 20$$
$$n - 1 = 49$$
$$n = 49 + 1$$
$$n = 50$$

(Answer)

(c)
$$S_{50} = \frac{n}{2}(a + l)$$
$$= \frac{50}{2}(7.23 + 9.68)$$
$$= \frac{50 \times 16.91}{2}$$
$$= 25 \times 16.91$$
$$= 422.75$$

(Answer)

5. (a)

$$\begin{aligned} 11^{\text{th}} \text{ term} = a + 10d &= y \\ x + 10d &= y \\ 10d &= y - x \\ d &= \frac{y - x}{10} \end{aligned}$$

(Answer)

(b)

$$\begin{aligned} a_3 = \text{the } 4^{\text{th}} \text{ term} &= a + 3d \\ &= x + 3\left(\frac{y - x}{10}\right) \\ &= \frac{10x + 3(y - x)}{10} \\ &= \frac{10x + 3y - 3x}{10} \\ &= \frac{7x + 3y}{10} \end{aligned}$$

(Shown)

6. (a) the arithmetic mean of a and $c = b$

$\therefore \quad b = \frac{a + c}{2}$ (Answer)

(b) the geometric mean between p and $r = q$

$\therefore \quad q = \sqrt{pr}$ (Answer)

7. (a)

$$\begin{aligned} S_n &= \frac{a(1 - r^n)}{1 - r} \\ S_\infty &= \frac{a}{1 - r} \\ \frac{S_n}{S_\infty} &= \frac{a(1 - r^n)}{1 - r} \div \frac{a}{(1 - r)} \\ &= \frac{a(1 - r^n)}{1 - r} \times \frac{(1 - r)}{a} \\ &= 1 - r^n \end{aligned}$$

$$r^n = 1 - \frac{S_n}{S_\infty}$$

$$\therefore \quad r = \sqrt[n]{\left(\frac{S_\infty - S_n}{S_\infty}\right)}$$ (Answer)

(b) $$r^{2n} = 1 - \frac{S_{2n}}{S_\infty}$$ (replace n by $2n$)

$$r^{2n} = \frac{S_\infty - S_{2n}}{S_\infty}$$

$$S_\infty r^{2n} = S_\infty - S_{2n}$$

$$S_{2n} = S_\infty - S_\infty r^{2n}$$

$$= S_\infty (1 - r^{2n})$$ (Answer)

8. $$S_5 = \frac{5}{2}\{2a + (n-1)d\} = 48$$

$$\frac{5}{2}(2a + 4d) = 48$$

$$10a + 20d = 96$$

$$5a + 10d = 48 \text{ ------------------------ } ①$$

$$S_6 = \frac{6}{2}\{2a + 5d\} = 64.5$$

$$6a + 15d = 64.5$$

$$2a + 5d = 21.5 \text{ ------------------------ } ②$$

$$5a + 10d = 48 \text{ ------------------------ } ①$$

Solve simultaneously, we have

$$10a + 25d = 107.5 \text{ ------------------------ } ② \times 5$$

$$10a + 20d = 96 \text{ ------------------------ } ① \times 2$$

$$5d = 11.5$$

$$d = \frac{11.5}{5} = 2.3$$

$$2a + 5d = 21.5$$

$$2a + 11.5 = 21.5$$

$$2a = 10$$

$$a = 5$$

90th term $= a + 89d = 5 + (89 \times 2.3)$

$= 5 + 204.7 = 209.7$ (Answer)

9. $a = 15$

5th term $= ar^4$

$$= 15 \times r^4 = \frac{5}{27}$$

$$r^4 = \frac{5}{27} \times \frac{1}{15}$$

$$r^4 = \frac{1}{81}$$

$$r = \frac{1}{3}$$

$$S_{20} = \frac{a\left(1 - r^n\right)}{1 - r}$$

$$= \frac{15\left(1 - r^{20}\right)}{1 - \frac{1}{3}}$$

$$= \frac{15\left(1 - \left(\frac{1}{3}\right)^{20}\right)}{\frac{2}{3}}$$

$$= \frac{45}{2}\left(1 - \left(\frac{1}{3}\right)^{20}\right)$$ (Answer)

$$S_\infty = \frac{a}{1-r} = \frac{15}{1-\frac{1}{3}}$$

$$= \frac{15}{\frac{2}{3}}$$

$$= 15 \times \frac{3}{2}$$

$$= \frac{45}{2} = 22.5$$ (Answer)

$$S_n = \frac{a\left(1-r^n\right)}{1-r} > 22$$

$$15\left(1-\left(\frac{1}{3}\right)^n\right) > 22 \times \frac{2}{3}$$

$$1-\left(\frac{1}{3}\right)^n > \frac{44}{3} \times \frac{1}{15}$$

$$1-\left(\frac{1}{3}\right)^n > \frac{44}{45}$$

$$1-\left(\frac{1}{3}\right)^n > 0.98$$

$$-\left(\frac{1}{3}\right)^n > -0.02$$

$$\left(\frac{1}{3}\right)^n < 0.02$$

$$n\lg\left(\frac{1}{3}\right)^n < \lg 0.02$$

$$-n\lg 3 < \lg 0.02$$

$$-n(0.4771) < -1.699$$

$$n(0.4771) > 1.699$$

$$n > \frac{1.699}{0.4771}$$

$$n > 3.561$$

least $n = 4$ (Answer)

10. $S_{70} = \frac{70}{2}\{2a + 69d\} = 6\ 387.5$

$$35\{2a + 69d\} = 6\ 387.5$$

$$2a + 69d = \frac{6\ 387.5}{35}$$

$$2a + 69d = 182.5 \text{ ------------------------------ } ①$$

1^{st} term $= a$ $=$ 1^{st} term of G. P.

3^{rd} term $= a + 2d$ $=$ 2^{nd} term of G. P.

7^{th} term $= a + 6d$ $=$ 3^{rd} term of G. P.

$$\therefore \frac{a + 2d}{a} = \frac{a + 6d}{a + 2d}$$

$$(a + 2d)^2 = a(a + 6d)$$

$$a^2 + 4ad + 4d^2 = a^2 + 6ad$$

$$4ad + 4d^2 = 6ad$$

$$4d^2 = 2ad$$

$$2a = 4d$$

$$a = 2d \text{ ------------------------------ } ②$$

$$2a + 69d = 182.5 \text{ ------------------------------ } ①$$

$4d + 69d = 182.5$ (Replacing a by $2d$)

$$73d = 182.5$$

$$d = \frac{182.5}{73} = 2.5$$

(a) $a = 2d = 2 \times 2.5 = 5$ (Answer)

(b) Common difference of the A.P. $= d = 2.5$. (Answer)

(c) Replacing a by $2d$, we obtain

$$a = 2d$$

$$a + 2d = 4d$$

$$a + 6d = 8d$$

$$r = \frac{4d}{2d} = \frac{8d}{4d} = 2$$

11. $a = 1$

$r = \frac{1}{e^x}$, which is less than 1.

$\therefore$ this G.P has a sum to infinity.

$$S_\infty = \frac{a}{1-r} = \frac{1}{1-\frac{1}{e^x}} = \frac{e^x}{e^x - 1}$$

12.

$$S_n = \frac{a\left(1 - r^n\right)}{1 - r}$$

$$= \frac{1\left(1 - e^{-xn}\right)}{1 - e^{-x}}$$

$$= \frac{1 - e^{-xn}}{1 - e^{-x}}$$

$$= \frac{1 - \frac{1}{e^{xn}}}{1 - \frac{1}{e^x}}$$

$$= \frac{e^{xn} - 1}{e^{xn}} \times \frac{e^x}{e^x - 1}$$

Put $x = 5$,

$$= \frac{e^{5n} - 1 \times e^5}{e^{5n}\left(e^5 - 1\right)}$$

$$= \frac{\left(e^{5n} - 1\right)}{e^{5n-5}\left(e^{5} - 1\right)}$$ (Answer)

$$S_{\infty} = \frac{a}{1 - r} = \frac{1}{1 - \frac{1}{e^{x}}}$$

$$= \frac{e^{x}}{e^{x} - 1}$$

$$= \frac{e^{5}}{e^{5} - 1}$$ (Answer)

By division, or

$$= 1 + \frac{1}{e^{5} - 1}$$ (Answer)

$$S_{\infty} - S_{n} = \frac{e^{5}}{e^{5} - 1} - \frac{(e^{5n} - 1)}{e^{5n-5}(e^{5} - 1)}$$

$$= \frac{e^{5n} - e^{5n} + 1}{e^{5n-5}(e^{5} - 1)}$$

$$= \frac{1}{e^{5n-5}(e^{5} - 1)}$$ (Answer)

13. $S_{\infty} = \frac{a}{1 - r} = \frac{x}{1 - \frac{1}{\sqrt{3}}} = \frac{\sqrt{3}x}{\sqrt{3} - 1}$

$$\frac{\sqrt{3}x}{\sqrt{3} - 1} = 1$$

$$\sqrt{3}x = \sqrt{3} - 1$$

$$x = \frac{\sqrt{3} - 1}{\sqrt{3}}$$

$$x = 1 - \frac{1}{\sqrt{3}}$$

$$x = 1 - \frac{\sqrt{3}}{3}$$ (Answer)

14. $S_\propto = \dfrac{a}{1 - r} = 3\left(2 + \sqrt{2}\right)$

Given $a = 3$,

$$\frac{3}{1 - r} = 3\left(2 + \sqrt{2}\right)$$

$$\frac{1}{1 - r} = 2 + \sqrt{2}$$

$$1 = \left(2 + \sqrt{2}\right)(1 - r)$$

$$1 = 2 - 2r + \sqrt{2} - \sqrt{2}r$$

$$2r + \sqrt{2}r = 2 - 1 + \sqrt{2}$$

$$r\left(2 + \sqrt{2}\right) = 1 + \sqrt{2}$$

$$r = \frac{1 + \sqrt{2}}{2 + \sqrt{2}} \times \frac{2 - \sqrt{2}}{2 - \sqrt{2}}$$

$$= \frac{2 - \sqrt{2} + 2\sqrt{2} - 2}{2} = \frac{\sqrt{2}}{2}$$

$$\text{or } \frac{1}{\sqrt{2}}$$ (Answer)

15. $$S_3 = \frac{a\left(1 - r^3\right)}{1 - r}$$

$$S_{\alpha} = \frac{a}{1 - r}$$

$$\frac{a\left(1 - r^3\right)}{1 - r} = \frac{1}{4}\left(\frac{a}{1 - r}\right)$$

$$a\left(1 - r^3\right) = \frac{1}{4}a$$

$$1 - r^3 = \frac{1}{4}$$

$$-r^3 = -\frac{3}{4}$$

$$r^3 = \frac{3}{4}$$

(Answer)

$$r = 0.909$$

$$\begin{aligned} 10^{\text{th}}\text{ term} &= ar^9 \\ &= 2 \times (0.909)^9 \\ &= 0.85 \end{aligned}$$

(Answer)

16. $$S_n = \frac{n}{2}\left\{2a + \left(n - 1\right)d\right\} = 5\,000$$

$$n\{2a + (n - 1)5\} = 10\,000$$

$$2an + 5n^2 - 5n = 10\,000$$

$$2an = 10\,000 + 5n - 5n^2$$

$$a = \frac{10\,000}{2n} + \frac{5n}{2n} - \frac{5n^2}{2n}$$

$$a = \frac{5\,000}{n} + \frac{5}{2} - \frac{5}{2}n$$

(Answer)

$$
\begin{aligned}
n^{\text{th}} \text{ term } &= a + (n-1)d \\
&= \frac{5\ 000}{n} + \frac{5}{2} - \frac{5}{2}n + \left(n - 1\right)5 \\
&= \frac{5\ 000}{n} + \frac{5}{2} - \frac{5}{2}n + 5n - 5 \\
&= \frac{5\ 000}{n} + \frac{5n}{2} - \frac{5}{2} \qquad \text{(Answer)}
\end{aligned}
$$

UNIT 6

THE BINOMIAL THEOREM

Suggested Solutions

1. (a) General term = ${}^{n}C_{r}\, a^{n-r}\, b^{r}$

$= {}^{-1}C_{r}\,(1)^{-1-r}\,(x^{2})^{r}$

$= {}^{-1}C_{r}\,(1)^{-1-r}\,(x^{2r})$

Since we have to find the cœfficient of x^8,

$\therefore\quad 2r = 8$

$r = 4$

$\therefore$ the required term = ${}^{-1}C_{4}\,(1)^{-1-4}\,(x^{2})^{4}$

$$= \frac{-1.\,-2.-3.-4}{1.\,2.\,3.\,4} \times (1)^{-5} \times x^{8}$$

$= 1 \times x^{8}$

$= x^{8}$

$\therefore$ cœfficient of x^8 $= 1$ (Answer)

(b) General term $= {}^{n}C_{r}\, a^{n-r}\, b^{r}$ Expansion of $\left(x + \frac{3}{x}\right)^{10}$.

$$= {}^{10}C_{r}\,(x)^{10-r}\left(\frac{3}{x}\right)^{r}$$

$10 - r = r$

$2r = 10$

$r = 5$

$$\therefore \text{ the term independent of } x = {}^{10}C_{5}\,(x)^{10-5}\left(\frac{3}{x}\right)^{5}$$

$$= 252\; x^{5} \times \frac{243}{x^{5}}$$

$= 61\ 236$ (Answer)

2. (a)

$$(1+x)^{\frac{1}{2}} = {}^{\frac{1}{2}}C_0 (1)^{\frac{1}{2}-0} (x)^0 + {}^{\frac{1}{2}}C_1 (1)^{\frac{1}{2}-1} (x)^1 + {}^{\frac{1}{2}}C_2 (1)^{\frac{1}{2}-2} (x)^2$$

$$= (1 \times 1 \times 1) + \left(\frac{1}{2} \times 1 \times x\right) + \left(\frac{\frac{1}{2} \times -\frac{1}{2}}{1 \times 2}\right) \times (1)^{-\frac{3}{2}} (x)^2$$

$$= 1 + \frac{1}{2}x + -\frac{1}{8}x^2$$

$$= 1 + \frac{1}{2}x - \frac{1}{8}x^2$$

(Answer)

(b) $(1.02)^{\frac{1}{2}} = (1 + 0.02)^{\frac{1}{2}}$

So that the expansion of $(1.02)^{\frac{1}{2}} = 1 + \frac{1}{2}x - \frac{1}{8}x^2$

where $x = 0.02$

value of $(1.02)^{\frac{1}{2}} = 1 + \frac{1}{2}(0.02) - \frac{1}{8}(0.02)^2$

$$= 1 + 0.01 - \frac{1}{8} \times 0.0004$$

$$= 1.01 - 0.000\,05$$

$$\approx 1.01 \text{ (to 2 d.p.)}$$

(Answer)

3.

$$y = \frac{1}{\sqrt{(1+2x)} + \sqrt{(1+x)}}$$

$$= \frac{1}{\sqrt{(1+2x)} + \sqrt{(1+x)}} \times \frac{\sqrt{(1+2x)} - \sqrt{(1+x)}}{\sqrt{(1+2x)} - \sqrt{(1+x)}}$$

$$= \frac{\sqrt{(1+2x)} - \sqrt{(1+x)}}{\left\{\sqrt{(1+2x)}\right\}^2 - \left\{\sqrt{(1+x)}\right\}^2}$$

$$= \frac{\sqrt{(1+2x)} - \sqrt{(1+x)}}{1 + 2x - 1 - x}$$

$$= \frac{\sqrt{(1 + 2x)} - \sqrt{(1 + x)}}{x}$$

$$= \frac{1}{x}\left\{\sqrt{(1 + 2x)} - \sqrt{(1 + x)}\right\}, \text{ provided } x \neq 0. \qquad \text{(Shown)}$$

$$\sqrt{(1 + 2x)} = (1 + 2x)^{\frac{1}{2}} = {}^{\frac{1}{2}}C_0\ (1)^{\frac{1}{2}}\ (2x)^0 + {}^{\frac{1}{2}}C_1\ (1)^{-\frac{1}{2}}\ (2x)^1 + {}^{\frac{1}{2}}C_2\ (1)^{-\frac{3}{2}}\ (2x)^2$$

$$+ {}^{\frac{1}{2}}C_3\ (1)^{-\frac{5}{2}}\ (2x)^3$$

$$= (1 \times 1 \times 1) + \left(\frac{1}{2} \times 1 \times 2x\right) + \left(\frac{\frac{1}{2} \times -\frac{1}{2}}{2 \times 1} \times 1 \times 4x^2\right)$$

$$+ \left(\frac{\frac{1}{2} \times -\frac{1}{2} \times -\frac{3}{2}}{3 \times 2 \times 1} \times 1 \times 8x^3\right)$$

$$= 1 + x - \frac{1}{2}x^2 + \frac{x^3}{2} - \ldots$$

We expand up to x^3, $\because$ we have to divide by x in the end.

$$\sqrt{(1 + x)} = (1 + x)^{\frac{1}{2}} = {}^{\frac{1}{2}}C_0 (1)^{\frac{1}{2}} (x)^0 + {}^{\frac{1}{2}}C_1 (1)^{-\frac{1}{2}} (x)^1 + {}^{\frac{1}{2}}C_2 (1)^{-\frac{3}{2}} (x)^2$$

$$+ {}^{\frac{1}{2}}C_3 (1)^{-\frac{5}{2}} (x)^3$$

$$= (1 \times 1 \times 1) + \left(\frac{1}{2} \times 1 \times x\right) + \left(\frac{\frac{1}{2} \times -\frac{1}{2}}{1 \times 2} \times 1 \times x^2\right)$$

$$+ \left(\frac{\frac{1}{2} \times -\frac{1}{2} \times -\frac{3}{2}}{1 \times 2 \times 3} \times 1 \times x^3\right)$$

$$= 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{x^3}{16}$$

$$= \frac{1}{x}\left\{\left(1 + x - \frac{1}{2}x^2 + \frac{x^3}{2}\right) - \left(1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{x^3}{16}\right)\right\}$$

$$= \frac{1}{x}\left(1 + x - \frac{1}{2}x^2 + \frac{x^3}{8} - 1 - \frac{1}{2}x + \frac{1}{8}x^2 - \frac{x^3}{16}\right)$$

$$= \frac{1}{x}\left(\frac{1}{2}x - \frac{3}{8}x^2 + \frac{7}{16}x^3\right)$$

$$= \frac{1}{2} - \frac{3}{8}x + \frac{7}{16}x^2 \qquad \text{(Answer)}$$

$$\frac{1}{2} - \frac{3}{8} \times 0.01 + \frac{7}{16} \times (0.01)^2$$

$$= \frac{1}{2} - \frac{3}{800} + \frac{7}{160\,000}$$

$$= \frac{80\,000 - 600 + 7}{160\,000}$$

$$= \frac{79\,407}{160\,000} \qquad \text{(Proved)}$$

4. $$(1 + x)^{\frac{1}{3}} = {}^{\frac{1}{3}}C_0 (1)^{\frac{1}{3}} (x)^0 + {}^{\frac{1}{3}}C_1 (1)^{-\frac{2}{3}} (x)^1 + {}^{\frac{1}{3}}C_2 (1)^{-\frac{5}{3}} (x)^2 + {}^{\frac{1}{3}}C_3 (1)^{-\frac{7}{3}} (x)^3$$

$$= (1 \times 1 \times 1) + \left(\frac{1}{3} \times 1 \times x\right) + \left(\frac{\frac{1}{3} \times -\frac{2}{3}}{2 \times 1} \times 1 \times x^2\right) + \left(\frac{\frac{1}{3} \times -\frac{2}{3} \times -\frac{5}{3}}{1 \times 2 \times 3} \times 1 \times x^3\right)$$

$$= 1 + \frac{1}{3}x - \frac{x^2}{9} + \frac{5}{81}x^3 \qquad \text{(Answer)}$$

5. $(3 - x)^{\frac{1}{3}}$

Consider the general term ${}^nC_r\, a^{n-r}\, b^r$

$$= {}^{\frac{1}{3}}C_r (3)^{\frac{1}{3} - r} (-x)^r$$

for x^2 $\qquad r = 2$

$$\therefore \quad {}^{\frac{1}{3}}C_2 (3)^{\frac{1}{3} - 2} (-x)^2$$

$$^{\frac{1}{3}}C_2 = \frac{\frac{1}{3}\ !}{2\ !\left(\frac{-5}{3}\right)!} = \frac{\left(\frac{1}{3}\right).\left(-\frac{2}{3}\right)}{2.1} = -\frac{2}{9} \times \frac{1}{2} = -\frac{1}{9}$$

$$\frac{-\frac{1}{3} \times -\frac{4}{3}}{2 \times 1} \times (3)^{-\frac{5}{3}}(x)^2$$

$$= \left(\frac{1}{3} \times \frac{4}{3} \times \frac{1}{2}\right) \times \frac{1}{3^{\frac{5}{3}}} \times x^2$$

$$= \frac{2}{9} \times \frac{1}{3^{\frac{5}{3}}} x^2$$

$$= \frac{2}{3^2 \times 3^{\frac{5}{3}}} x^2$$

$$= \frac{2}{3^{\frac{11}{3}}} x^2$$

∴ Cœfficient of $x^2 = 2 \times 3^{-\frac{11}{3}}$ (Answer)

Validity = $(x) < 3$ (Answer)

6. $$(2 + x)^{-\frac{1}{2}} = {}^{-\frac{1}{2}}C_0 (2)^{-\frac{1}{2}}(x^0) + {}^{-\frac{1}{2}}C_1 (2)^{-\frac{3}{2}}(x^1) + {}^{-\frac{1}{2}}C_2 (2)^{-\frac{5}{2}}(x^2)$$

$$+ {}^{-\frac{1}{2}}C_3 (2)^{-\frac{7}{2}}(x^3) + {}^{-\frac{1}{2}}C_4 (2)^{-\frac{9}{2}}(x^4)$$

$$= \left(1 \times \frac{1}{2^{\frac{1}{2}}} \times 1\right) + \left(-\frac{1}{2} \times \frac{1}{2^{\frac{2}{3}}} \times x\right) + \left(\frac{-\frac{1}{2} \times -\frac{3}{2}}{1 \times 2} \times \frac{1}{2^{\frac{5}{2}}} \times x^2\right)$$

$$+ \left(\frac{-\frac{1}{2} \times -\frac{3}{2} \times -\frac{5}{2}}{1 \times 2 \times 3} \times \frac{1}{2^{\frac{7}{2}}} \times x^3\right) + \left(\frac{-\frac{1}{2} \times -\frac{3}{2} \times -\frac{5}{2} \times -\frac{7}{2}}{1 \times 2 \times 3 \times 4} \times \frac{1}{2^{\frac{9}{2}}} x^4\right)$$

$$= \frac{1}{2^{\frac{1}{2}}} - \frac{1}{2^{\frac{5}{2}}} x + \left(\frac{3}{8} \times \frac{1}{2^{\frac{5}{2}}} x^2\right) - \left(\frac{5}{16} \times \frac{1}{2^{\frac{7}{2}}} x^3\right) + \left(\frac{35}{128} \times \frac{1}{2^{\frac{9}{2}}} x^4\right)$$

$$= \frac{1}{2^{\frac{1}{2}}} - \frac{1}{2^{\frac{5}{2}}}x + \left(\frac{3}{8} \times \frac{1}{4} \times \frac{1}{2^{\frac{1}{2}}}x^2\right) - \left(\frac{5}{128} \times \frac{1}{2^{\frac{1}{2}}}x^3\right) + \left(\frac{35}{2\ 048} \times \frac{1}{2^{\frac{1}{2}}}x^4\right)$$

$$= \left(\frac{1}{2^{\frac{1}{2}}}\right) - \left(\frac{1}{4} \times \frac{1}{2^{\frac{1}{2}}}\right)x + \left(\frac{3}{32} \times \frac{1}{2^{\frac{1}{2}}}\right)x^2 - \left(\frac{5}{128} \times \frac{1}{2^{\frac{1}{2}}}\right)x^3 + \left(\frac{35}{2\ 048} \times \frac{1}{2^{\frac{1}{2}}}\right)x^4$$

$$= \frac{1}{2^{\frac{1}{2}}}\left(1 - \frac{1}{4}x + \frac{3}{32}x^2 - \frac{5}{128}x^3 + \frac{35}{2\ 048}x^4\right)$$ (Answer)

(Answer)

Expansion is valid when $|x| < 2$

7. $$(1+x)^{\frac{1}{2}} = {}^{\frac{1}{2}}C_0 (1)^{\frac{1}{2}} (x)^0 + {}^{\frac{1}{2}}C_1 (1)^{-\frac{1}{2}} (x)^1 + {}^{\frac{1}{2}}C_2 (1)^{-\frac{3}{2}} (x)^2$$

$$= (1 \times 1 \times 1) + \left(\frac{1}{2} \times 1 \times x\right) + \left(\frac{\frac{1}{2} \times -\frac{1}{2}}{1 \times 2} \times 1 \times x^2\right)$$

$$= 1 + \frac{x}{2} - \frac{1}{8}x^2$$ (Answer)

$$(1+x)^{\frac{1}{2}} = \left(1 + \frac{1}{36}\right)^{\frac{1}{2}} = \left(\frac{36+1}{36}\right)^{\frac{1}{2}}$$

$$= \frac{\sqrt{37}}{6}$$

Replacing x by $\frac{1}{36}$ in the expansion, we have

$$\frac{\sqrt{37}}{6} = 1 + \left(\frac{1}{2} \times \frac{1}{36}\right) - \frac{1}{8} \times \left(\frac{1}{36}\right)^2$$

$$\frac{\sqrt{37}}{6} = 1 + \frac{1}{72} - \frac{1}{10\ 368}$$

$$\sqrt{37} = 6 + \frac{1}{12} - \frac{1}{1\ 728}$$

$$= \frac{10\,368 + 144 - 1}{1\,728}$$

$$= \frac{10\,511}{1\,728}$$

$$= 6\frac{143}{1\,728}$$

(Answer)

$$= 6.083$$

8. (a)

$$(x+1)^7 = {}^7C_0\,x^7 \times 1^0 + {}^7C_1\,x^6 \times 1^1 + {}^7C_2\,x^5 \times 1^2 + {}^7C_3\,x^4 \times 1^3 + {}^7C_4\,x^3 \times 1^4 + {}^7C_5\,x^2 \times 1^5 + {}^7C_6\,x^1 \times 1^6 + {}^7C_7\,x^0 \times 1^7$$

$$= (1 \times x^7 \times 1) + (7 \times x^6 \times 1) + (21 \times x^5 \times 1) + (35 \times x^4 \times 1) + (35 \times x^3 \times 1) + (21x^2 \times 1) + (7x \times 1) + (1 \times 1 \times 1)$$

$$= x^7 + 7x^6 + 21x^5 + 35x^4 + 35x^3 + 21x^2 + 7x + 1$$ (Answer)

(b) $(x-1)^7 = x^7 - 7x^6 + 21x^5 - 35x^4 + 35x^3 - 21x^2 + 7x - 1$ (Answer)

(c) $(x+1)^7 - (x-1)^7 = 14x^6 + 70x^4 + 42x^2 + 2$

Replace x by $\sqrt{3}$.

Then

$$\left(\sqrt{3}+1\right)^7 - \left(\sqrt{3}-1\right)^7 = 14\left(\sqrt{3}\right)^6 + 70\left(\sqrt{3}\right)^4 + 42\left(\sqrt{3}\right)^2 + 2$$

$$= 378 + 630 + 126 + 2$$

(Answer)

$$= 1\,136$$

9.

$$(1+3x)^{\frac{1}{3}} = {}^{\frac{1}{3}}C_0(1)^{\frac{1}{3}}(3x)^0 + {}^{\frac{1}{3}}C_1(1)^{-\frac{2}{3}}(3x)^1 + {}^{\frac{1}{3}}C_2(1)^{-\frac{5}{3}}(3x)^2 + {}^{\frac{1}{3}}C_3(1)^{-\frac{8}{3}}(3x)^3 + {}^{\frac{1}{3}}C_4(1)^{-\frac{11}{3}}(3x)^4$$

$$= (1 \times 1 \times 1) + \left(\frac{1}{3} \times 1 \times 3x\right) + \left(\frac{\frac{1}{3} \times -\frac{2}{3}}{1 \times 2} \times 1 \times 9x^2\right) + \left(\frac{\frac{1}{3} \times -\frac{2}{3} \times -\frac{5}{3}}{1 \times 2 \times 3} \times 1 \times 27x^3\right) + \left(\frac{\frac{1}{3} \times -\frac{2}{3} \times -\frac{5}{3} \times -\frac{8}{3}}{1 \times 2 \times 3 \times 4} \times 1 \times 81x^4\right)$$

$$= 1 + x + \left(-\frac{1}{9} \times 9x^2\right) + \left(\frac{5}{81} \times 27x^3\right) + \left(-\frac{10}{243} \times 81x^4\right)$$

$$= 1 + x - x^2 + \frac{5}{3}x^3 - \frac{10}{3}x^4$$

(Answer)

10. $$(1 + 3x^2)^{-2} = {}^{-2}C_0(1)^{-2}(3x^2)^0 + {}^{-2}C_1(1)^{-3}(3x^2)^1 + {}^{-2}C_2(1)^{-4}(3x^2)^2 + {}^{-2}C_3(1)^{-5}(3x^2)^3$$

$$= (1 \times 1 \times 1) + (-2 \times 1 \times 3x^2) + \left(\frac{-2 \times -3}{1 \times 2} \times 1 \times 9x^4\right) + \frac{-2 \times -3 \times -4}{1 \times 2 \times 3} \times 1 \times 27x^6$$

$$= 1 - 6x^2 + 27x^4 - 108x^6$$

(Answer)

1. $$\left(1 + \frac{1}{x}\right)^{\frac{1}{2}} = {}^{\frac{1}{2}}C_0\,(1)^{\frac{1}{2}}\left(\frac{1}{x}\right)^0 + {}^{\frac{1}{2}}C_1\,(1)^{-\frac{1}{2}}\left(\frac{1}{x}\right)^1 + {}^{\frac{1}{2}}C_2\,(1)^{-\frac{3}{2}}\left(\frac{1}{x}\right)^2$$

$$= (1 \times 1 \times 1) + \left(\frac{1}{2} \times 1 \times \frac{1}{x}\right) + \frac{\frac{1}{2} \times -\frac{1}{2}}{1 \times 2} \times 1 \times \left(\frac{1}{x}\right)^2$$

$$= 1 + \frac{1}{2x} - \frac{1}{8x^2}$$

(Answer)

$${}^{\frac{1}{2}}C_3(1)^{-\frac{5}{2}}\left(\frac{1}{x}\right)^3 = \frac{\frac{1}{2} \times -\frac{1}{2} \times -\frac{3}{2}}{1 \times 2 \times 3} \times 1 \times \frac{1}{x^3}$$

$$= \frac{1}{16}\left(\frac{1}{x}\right)^3$$

(Answer)

$$(1 - 3x)^{-\frac{1}{2}} = {}^{-\frac{1}{2}}C_0(1)^{-\frac{1}{2}}(-3x)^0 + {}^{-\frac{1}{2}}C_1(1)^{-\frac{3}{2}}(-3x)^1 + {}^{-\frac{1}{2}}C_2(1)^{-\frac{5}{2}}(-3x)^2 + {}^{-\frac{1}{2}}C_3(1)^{-\frac{7}{2}}(-3x)^3$$

$$= (1 \times 1 \times 1) + \left(-\frac{1}{2} \times 1 \times -3x\right) + \left(\frac{-\frac{1}{2} \times -\frac{3}{2}}{1 \times 2} \times 1 \times 9x^2\right) +$$

$$\frac{-\frac{1}{2} \times -\frac{3}{2} \times -\frac{5}{2}}{1 \times 2 \times 3} \times 1 \times -27x^3$$

$$= 1 + \frac{3}{2}x + \frac{27}{8}x^2 + \frac{135}{16}x^3$$ (Answer)

$$|x| < \frac{1}{3}$$ (Answer)

13. $$\frac{1}{\sqrt{1-3x}} = (1-3x)^{-\frac{1}{2}} = {}^{-\frac{1}{2}}C_0(1)^{-\frac{1}{2}}(-3x)^0 + {}^{-\frac{1}{2}}C_1(1)^{-\frac{3}{2}}(-3x)^1 + {}^{-\frac{1}{2}}C_2(1)^{-\frac{5}{2}}(-3x)^2$$

$$+ {}^{-\frac{1}{2}}C_3(1)^{-\frac{7}{2}}(-3x)^3$$

$$= (1 \times 1 \times 1) + \left(-\frac{1}{2} \times 1 \times -3x\right) + \left(\frac{-\frac{1}{2} \times -\frac{3}{2}}{1 \times 2} \times 1 \times 9x^2\right) +$$

$$\left(\frac{-\frac{1}{2} \times -\frac{3}{2} \times -\frac{5}{2}}{1 \times 2 \times 3} \times 27x^3\right)$$

$$= 1 + \frac{3}{2}x + \frac{27}{8}x^2 + \frac{135}{16}x^3$$ (Answer)

14. $$(1-x)^{-3} = {}^{-3}C_0(1)^{-3}(-x)^0 + {}^{-3}C_1(1)^{-4}(-x)^1 + {}^{-3}C_2(1)^{-5}(-x)^2 + {}^{-3}C_3(1)^{-6}(-x)^3$$

$$= (1 \times 1 \times 1) + (-3 \times 1 \times -x) + \left(\frac{-3 \times -4}{1 \times 2} \times 1 \times x^2\right) +$$

$$\left(\frac{-3 \times -4 \times -5}{1 \times 2 \times 3} \times 1 \times -x^3\right)$$

$$= 1 + 3x + 6x^2 + 10x^3$$

$(1-x)^{-3} = 1 + 3x + 6x^2 + 10x^3$

$(1+x)^{-3} = 1 - 3x + 6x^2 - 10x^3$

$\therefore \quad (1-x)^{-3} + (1+x)^{-3} = 2 + 12x^2 = 2(1 + 6x^2)$ (Answer)

15. $$(7-2x)^{\frac{1}{2}} = {}^{\frac{1}{2}}C_0\ (7)^{\frac{1}{2}}\ (-2x)^0 + {}^{\frac{1}{2}}C_1\ (7)^{\frac{-1}{2}}\ (-2x)^1 + {}^{\frac{1}{2}}C_2\ (7)^{-\frac{3}{2}}\ (-2x)^2 +$$

$$+ {}^{\frac{1}{2}}C_3\ (7)^{-\frac{5}{2}}\ (-2x)^3 + {}^{\frac{1}{2}}C_4\ (7)^{-\frac{7}{2}}\ (-2x)^4$$

$$= \left(1 \times 7^{\frac{1}{2}} \times 1\right) + \left(\frac{1}{2} \times 7^{-\frac{1}{2}} \times -2x\right) + \left(\frac{\frac{1}{2} \times -\frac{1}{2}}{1 \times 2} \times 7^{-\frac{3}{2}} \times 4x^2\right) +$$

$$\frac{\frac{1}{2} \times -\frac{1}{2} \times -\frac{3}{2}}{1 \times 2 \times 3} \times 7^{-\frac{5}{2}}\left(-8x^3\right) + \frac{\frac{1}{2} \times -\frac{1}{2} \times -\frac{3}{2} \times -\frac{5}{2}}{1 \times 2 \times 3 \times 4} \times 7^{-\frac{7}{2}} \times 1 \times 16x^4$$

$$= 7^{\frac{1}{2}} - \left(7^{-\frac{1}{2}}\right)x - \frac{1}{2}\left(7^{-\frac{3}{2}}\right)x^2 - \frac{1}{2}\left(7^{-\frac{5}{2}}\right)x^3 - \frac{5}{8}\left(7^{-\frac{7}{2}}\right)x^4$$

$$= 7^{\frac{1}{2}} - \frac{x}{7^{\frac{1}{2}}} - \frac{x^2}{2\left(7^{\frac{3}{2}}\right)} - \frac{x^3}{2\left(7^{\frac{5}{2}}\right)} - \frac{5x^4}{8\left(7^{\frac{7}{2}}\right)}$$

$$= \frac{1}{7^{\frac{7}{2}}}\left\{7^4 - \frac{x}{7^3} - \frac{x^2}{2\left(7^2\right)} - \frac{x^3}{2(7)} - \frac{5x^4}{8}\right\}$$

$$= \frac{1}{7^{\frac{7}{2}}(8)\left(7^3\right)}\left\{\left(7^4 \times 8 \times 7^3\right) - 8x(7) - 4\left(x^3\right)7^2 - 5 \times 7^3 x^4\right\}$$

$$= \frac{1}{8\left(7^{\frac{13}{2}}\right)}\left\{\left(8 \times 7^7\right) - 56x - 196x^3 - 1\,715x^4\right\}$$ (Answer)

$$|x| < \frac{7}{2}$$ (Answer)

16. $(1-4x)^{\ 2} = {}^{-2}C_0(1)^{-2}(-4x)^0 + {}^{-2}C_1\,(1)^{-3}\,(-4x)^1 + {}^{-2}C_2\,(1)^{-4}\,(-4x)^2 + {}^{-2}C_3\,(1)^{-5}(-4x)^3$

$$= \left(1 \times 1 \times 1\right) + \left(-2 \times 1 \times -4x\right) + \left(\frac{-2 \times -3}{1 \times 2} \times 1 \times 16x^2\right) +$$

$$\left(\frac{-2 \times -3 \times -4}{1 \times 2 \times 3}\right) \times 1 \times -64x^3$$

$$= 1 + 8x + 48x^2 + 256x^3$$

(Answer)

UNIT 7

INEQUALITIES

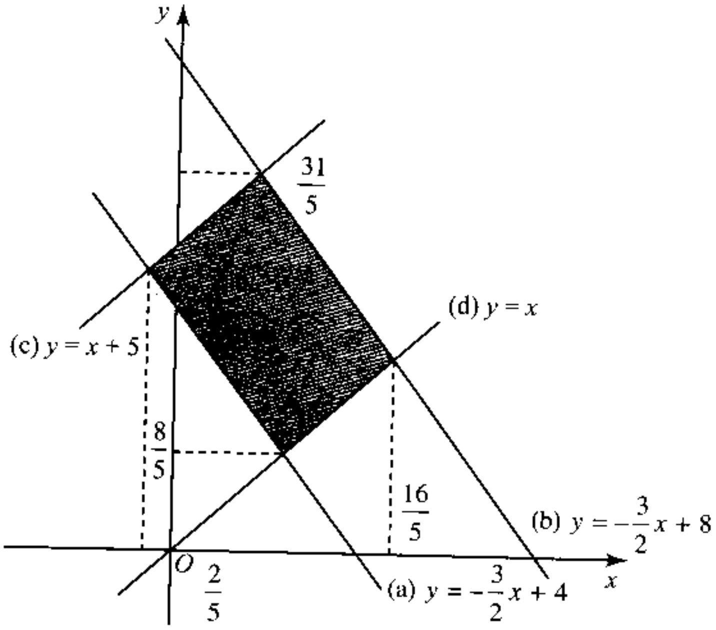
Suggested Solutions

We have to draw four lines and four regions:

(a)
$$\begin{aligned} 3x + 2y &\geqslant 8 \\ 2y &\geqslant -3x + 8 \\ y &\geqslant \frac{-3}{2}x + 4. \end{aligned}$$

(b)
$$\begin{aligned} 3x + 2y &\leqslant 16 \\ 2y &\leqslant -3x + 16 \\ y &\leqslant \frac{-3}{2}x + 8. \end{aligned}$$

(c)
$$\begin{aligned} x - y &\geqslant -5 \\ -y &\geqslant -x - 5 \\ y &\leqslant x + 5 \end{aligned}$$



(d)

$$\begin{aligned} x - y &\leqslant 0 \\ -y &\leqslant -x \\ y &\geqslant x \end{aligned}$$

The shaded part is the required region where

$$-\frac{2}{5} \leq x \leq \frac{16}{5}$$

and $\frac{8}{5} \leq y \leq \frac{31}{5}$ (Answer)

2. If $x - 1 = 0$ or if $x + 2 = 0$

i.e. if $x = 1$ or -2, the inequality will be impossible to solve.

$$\begin{aligned} \frac{3}{x-1} &> \frac{5}{x+2} \\ 3(x+2) &> 5(x-1) \\ 3x + 6 &> 5x - 5 \\ 3x - 5x &> -5 - 6 \\ -2x &> -11 \\ 2x &< 11 \\ x &< \frac{11}{2} \\ \text{or} \quad x &< 5\frac{1}{2} \end{aligned}$$

(Answer)

3. Critical values of x are:

$0, \; 2$ or -3

when

$x = \quad -3 \quad 0 \quad 2$

$f(x) = \; - \quad 0 \quad + \quad 0 \quad - \quad 0 \quad +$

Since $f(x) > 0$,

$\therefore \quad -3 < x < 0$ or $x > 2$. (Answer)

4.

$$x - \frac{5}{x} < \frac{7}{3}$$

$$x - \frac{5}{x} - \frac{7}{3} < 0$$

$$\frac{1}{3x}\left\{3x^2 - 15 - 7x\right\} < 0$$

Critical values of x are

$$\frac{1}{3x} = 0$$

impossible

$$3x^2 - 7x - 15 = 0$$

$$x^2 - \frac{7}{3}x - 5 = 0 \text{ when } a = 1, b = -\frac{7}{3}, c = -5$$

$$x = \frac{\frac{7}{3} + \sqrt{\left(\frac{49}{9}\right) - (4 \times 1 \times -5)}}{2} \quad \text{or} \quad \frac{\frac{7}{3} - \sqrt{\left(\frac{49}{9}\right) - (4 \times 1 \times -5)}}{2}$$

$$x = \frac{\frac{7}{3} + \sqrt{\frac{229}{9}}}{2} \quad \text{or} \quad \frac{\frac{7}{3} - \sqrt{\frac{229}{9}}}{2}$$

$$x = \frac{\frac{7}{3} + 5.04}{2} \quad \text{or} \quad \frac{\frac{7}{3} - 5.04}{2}$$

$= 3.69$ or -1.35 (2 d.p.)

or use formula again when $a = 3$, $b = -7$ and $c = -15$.

$x =$ $\frown$ -1.35 $\frown$ 3.69 $\frown$

$f(x) =$ $+$ 0 $-$ 0 $+$

$-1.35 < x < 3.69$

(Answer)

5. $x - 2y = 12$ -- ①

$x = 12 + 2y$

Substituting ① in $x^2 + y^2 = 29$ ---------------- ②

$$(12 + 2y)^2 + y^2 = 29$$
$$144 + 48y + 4y^2 + y^2 = 29$$
$$5y^2 + 48y + 115 = 0$$
$$(5y + 23)(y + 5) = 0$$
$$y = -\frac{23}{5} \text{ or } -5$$

Replacing these two values of y in ①

$$x = 12 + 2\left(-\frac{23}{5}\right) = 12 - \frac{46}{5} = \frac{60 - 46}{5} = \frac{14}{5}$$

or

$$x = 12 + 2(-5) = 12 - 10 = 2$$

when x =	2	$2\frac{4}{5}$
and y =	-5	$-\frac{23}{5}$

$x - 2y = 2 + 10 = 12$

$x^2 + y^2 = 4 + 25 = 29$

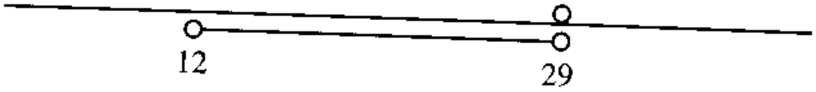
$x > 2$

$y > -5$

or $x < \frac{14}{5}$

$y < -\frac{23}{5}$

(Answer)



6. (a)

$$\frac{x}{x+3} < 0$$

$$\frac{x}{x+3} - 8 < 0$$

$$\frac{1}{x+3}\left[(x-8)(x+3)\right] < 0$$

$$x - 8x - 24 < 0$$

$$-7x < 24$$

$$x > -\left(\frac{24}{7}\right)$$

(Answer)

(b) $x(x + 3) < 8$

$x^2 + 3x - 8 < 0$

C.V. of $x =$

$$x = \frac{-3 + \sqrt{9 - (4 \times 1 \times -8)}}{2} \quad \text{or} \quad \frac{-3 - \sqrt{9 - (4 \times 1 \times -8)}}{2}$$

$$x = \frac{-3 + \sqrt{9 + 32}}{2} \quad \text{or} \quad \frac{-3 - \sqrt{9 + 32}}{2}$$

$$x = \frac{-3 + 6.4}{2} \quad \text{or} \quad x = \frac{-3 - 6.4}{2}$$

$$x = \frac{3.4}{2} \quad \text{or} \quad -\frac{9.4}{2}$$

$$x = 1.7 \quad \text{or} \quad -4.7$$

When $x =$		-4.7		1.7	
f $(x) =$	+	0	−	0	+

$\therefore \quad -4.7 < x < 1.7$

(Answer)

(c) $|x| < 7|x + 3|$

Squaring both sides, we have

$$x^2 < 49(x + 3)^3$$

$$49(x + 3)^2 - x^2 > 0$$

$$49(x^2 + 6x + 9) - x^2 > 0$$

$$49x^2 + 294x + 441 - x^2 > 0$$

$$48x^2 + 294x + 441 > 0$$

$$(6x + 21)(8x + 21) > 0$$

C.V. of $x = -\frac{21}{6}$ or $-\frac{21}{8}$

$= -\frac{7}{2}$ or $-\frac{21}{8}$

When $x =$ $\frown$ $-\frac{7}{2}$ $\frown$ $-\frac{21}{8}$ $\frown$

f $(x) =$ + 0 – 0 +

$\therefore$ $x > -\frac{21}{8}$ or $x < -\frac{7}{2}$ (Answer)

7. (a) $\frac{x + 2}{x - 2} < 5$

$$x + 2 < 5x - 10$$

$$-4x < -12$$

$$4x > 12$$

$$x > 3$$ (Answer)

(b) $\frac{|x| + 2}{|x| - 2} < 5$

$$|x| + 2 < 5|x| - 10$$

$$-4|x| < -12$$

$4|x| > 12$

$|x| > 3$

$\therefore \quad x > 3 \quad \text{or} \quad x < -3$ (Answer)

(c) $\left|\frac{x+2}{x-2}\right| < 5$

$$\left(\frac{x+2}{x-2}\right)^2 < 25$$

$$(x+2)^2 < 25(x-2)^2$$

$$x^2 + 4x + 4 < 25(x^2 - 4x + 4)$$

$$x^2 + 4x + 4 < 25x^2 - 100x + 100$$

$$25x^2 - x^2 - 100x - 4x + 100 - 4 > 0$$

$$24x^2 - 104x + 96 > 0$$

$$6x^2 - 26x + 24 > 0$$

$$3x^2 - 13x + 12 > 0$$

$$(3x-4)(x-3) > 0$$

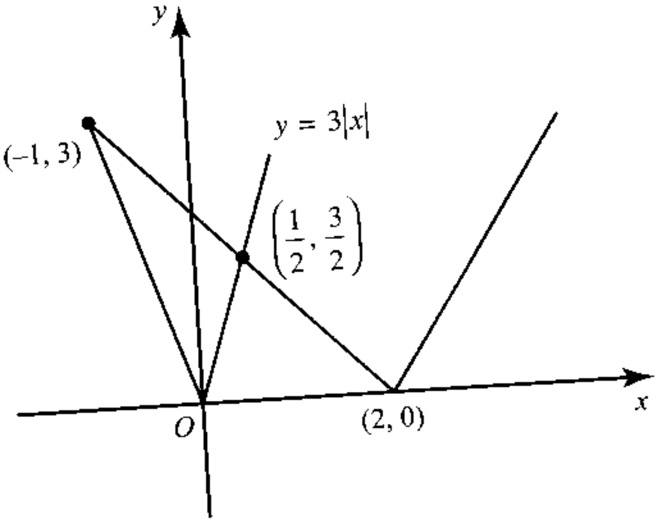
CV of $x = \frac{4}{3}$ or 3

When $x = \frown \; \frac{4}{3} \; \frown \; 3 \; \frown$

$f(x) = \quad + \quad 0 \quad - \quad 0 \quad +$

$\therefore \quad x > 3 \quad \text{or} \quad x < \frac{4}{3}$ (Answer)

8. (a)



(b) Hence $3|x| = |x - 2|$

Checking answer

Squaring both sides, we have:

$$9x^2 = x^2 - 4x + 4$$

$$8x^2 + 4x - 4 = 0$$

$$2x^2 + x - 1 = 0$$

$$(2x - 1)(x + 1) = 0$$

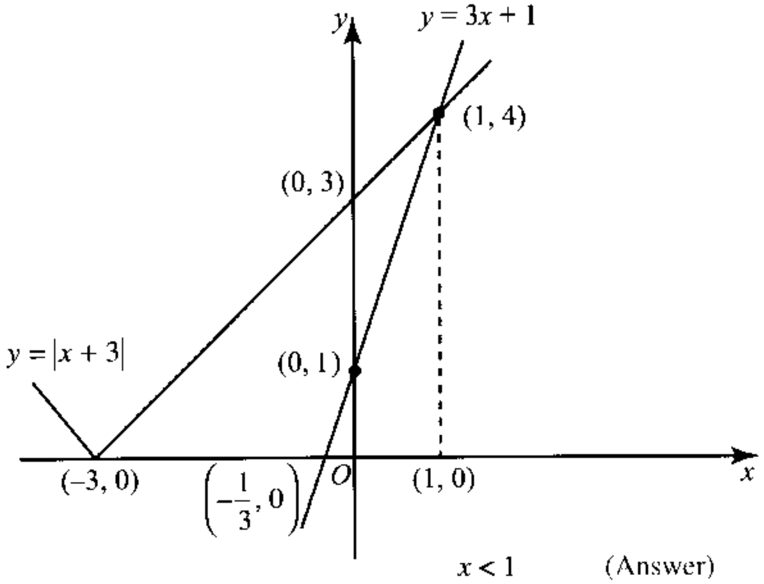
$$x = \frac{1}{2} \text{ or } -1$$

(Answer)

(c) $-1 < x < \frac{1}{2}$

(Answer)

9.



$x < 1$ (Answer)

10. $x^3 > 5x^2 - 6x$

$x^3 - 5x^2 + 6x > 0$

$x(x^2 - 5x + 6) > 0$

$x(x - 2)(x - 3) > 0$

C.V. of x are 0, 2 and 3.

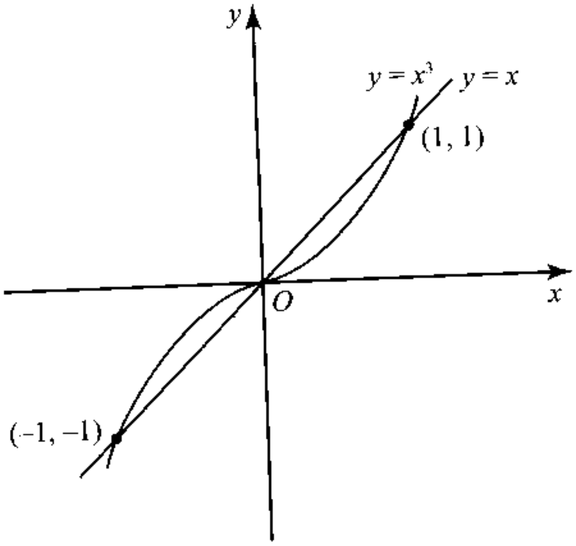
When x =		0		2		3	
f (x) =	−	0	+	0	−	0	+

$\therefore \quad 0 < x < 2$ or $x > 3$. (Answer)

11. $x^3 - x > 0$

$x(x^2 - 1) > 0$

$x(x + 1)(x - 1) > 0$



C.V. of x *are* $\qquad -1 \qquad 0 \qquad 1$

When $x = \qquad -1 \qquad 0 \qquad 1$

$\text{f}(x) = \quad - \quad 0 \quad + \quad 0 \quad - \quad 0 \quad +$

$\therefore \quad x > 1 \quad \text{or} \quad -1 < x < 0$ (Answer)

12. $\dfrac{x + 2}{5 - x} < 4$

$x + 2 < 4(5 - x)$

$x + 2 < 20 - 4x$

$5x < 18$

$x < \dfrac{18}{5}$ (Answer)

13. $| x - 3 | > 2 - 3x$

Squaring both sides, we have

$(x - 3)^2 > (2 - 3x)^2$

$x^2 - 6x + 9 > 4 - 12x + 9x^2$

$$-8x^2 + 6x + 5 > 0$$

$$8x^2 - 6x - 5 < 0$$

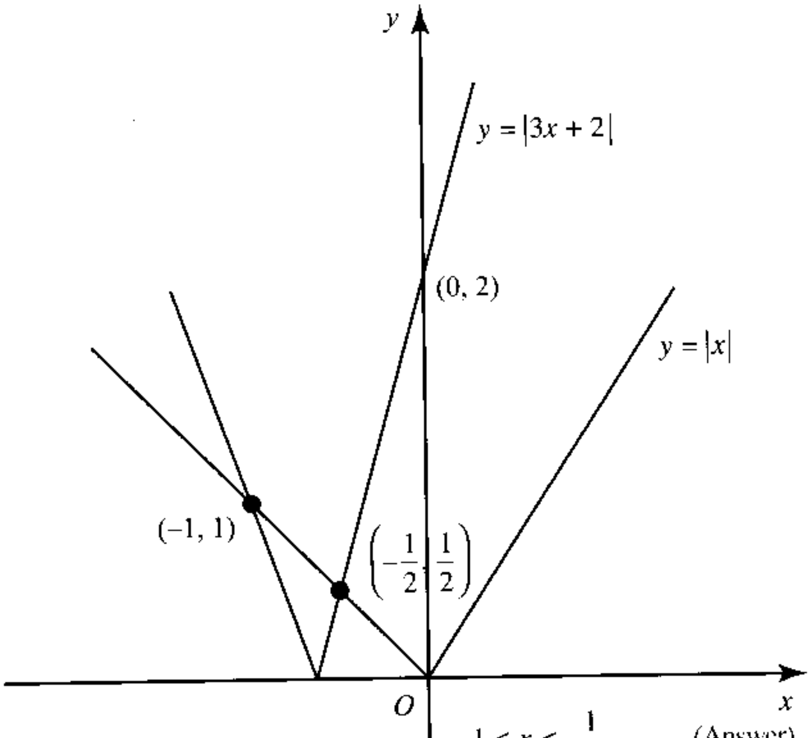
$$(2x + 1)(4x - 5) < 0$$

C.V. of $x = -\frac{1}{2}$ or $\frac{5}{4}$

$$\begin{array}{lccccc} \text{When } x = & \frown & -\frac{1}{2} & \frown & \frac{5}{4} & \frown \\ \text{f}(x) = & + & 0 & - & 0 & + \end{array}$$

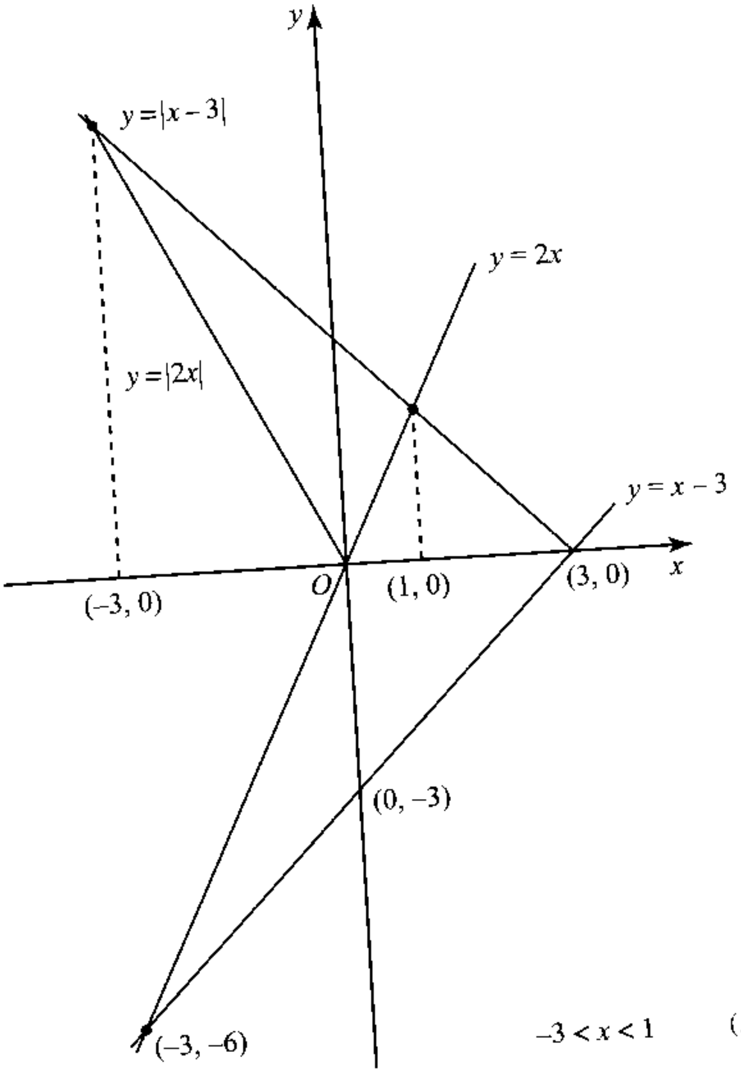
$\therefore \quad -\frac{1}{2} < x < \frac{5}{4}$ (Answer)

4.



$-1 < x < -\frac{1}{2}$ (Answer)

15.



$-3 < x < 1$ (Answer)

16. $|x + 3| > 2|5x - 2|$

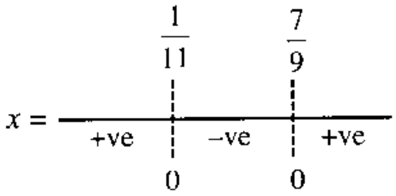
$(x + 3)^2 > 4(25x^2 - 20x + 4)$ squaring both sides

$x^2 + 6x + 9 > 100x^2 - 80x + 16$

$99x^2 - 86x + 7 < 0$

$(11x - 1)(9x - 7) < 0$

Critical values of x are $\frac{1}{11}$ or $\frac{7}{9}$



$$\therefore \quad \frac{1}{11} < x < \frac{7}{9}$$ (Answer)

UNIT 8

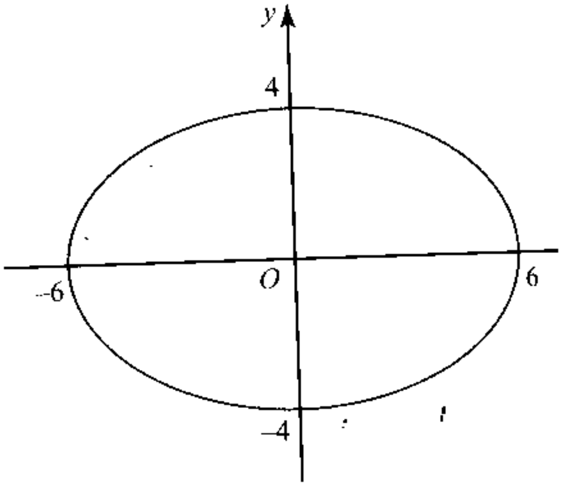
CURVE SKETCHING

Suggested Solutions

1. $4x^2 + 9y^2 = 144$

$$\frac{x^2}{\frac{144}{4}} + \frac{y^2}{\frac{144}{9}} = 1$$

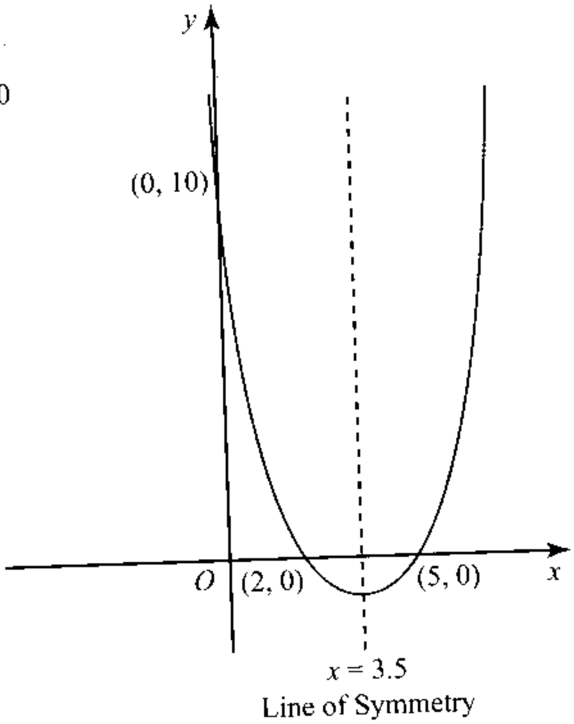
$$\left(\frac{x}{6}\right)^2 + \left(\frac{y}{4}\right)^2 = 1$$



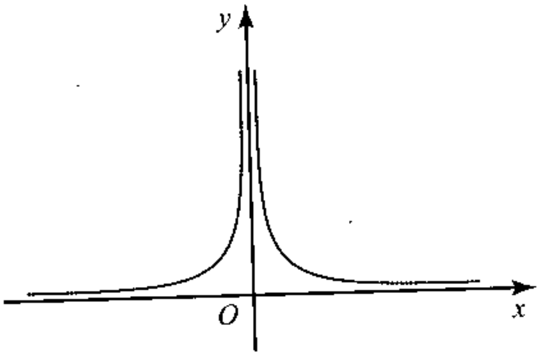
2. (a) $x^2 - 7x + 10 = 0$

$(x - 2)(x - 5) = 0$

$x = 2$ or 5.



(b) $y = \frac{1}{x^2}$



Line of symmetry

3. $x^2 = \sqrt{x}$

$x^4 = x$

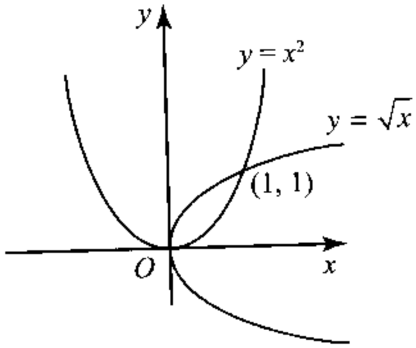
$x^4 - x = 0$

$x(x^3 - 1) = 0$

$x = 0$ or 1

$y = 0$ or 1

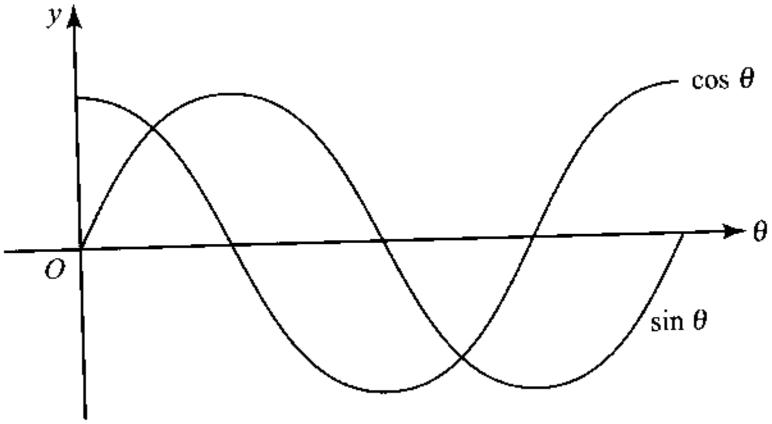
or (0, 0) and (1, 1).



4. $x^2 = \sin^2 \theta$

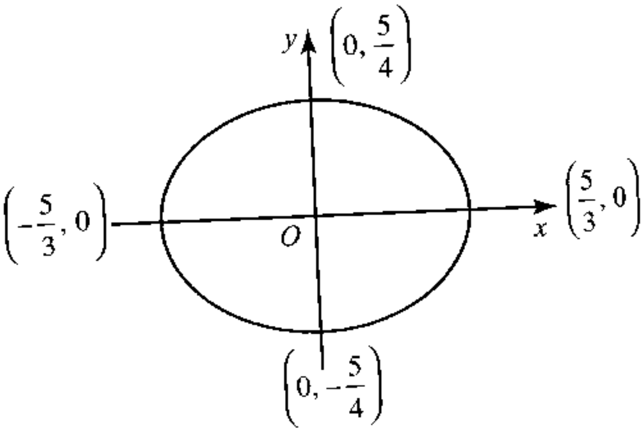
$y^2 = \cos^2 \theta$

$x^2 + y^2 = 1$

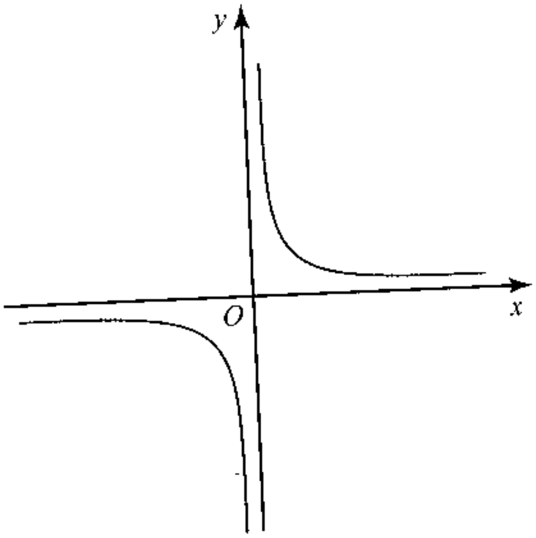


5. $(3x)^2 + (4y)^2 = 5^2$

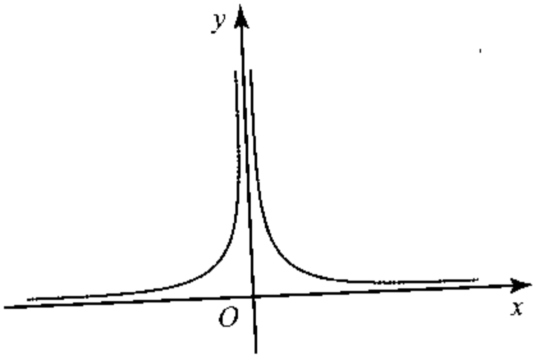
$$\left(\frac{x}{\frac{5}{3}}\right)^2 + \left(\frac{y}{\frac{5}{2}}\right)^2 = 1$$



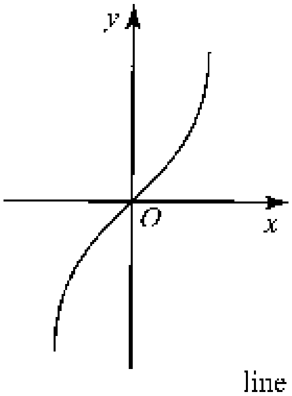
6. (a) $y = \frac{1}{x}$

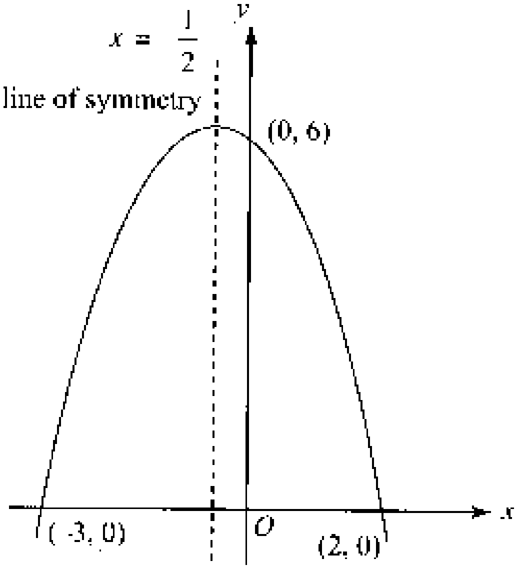


(b) $y = \frac{1}{x^2}$



(c) $y = x^3$



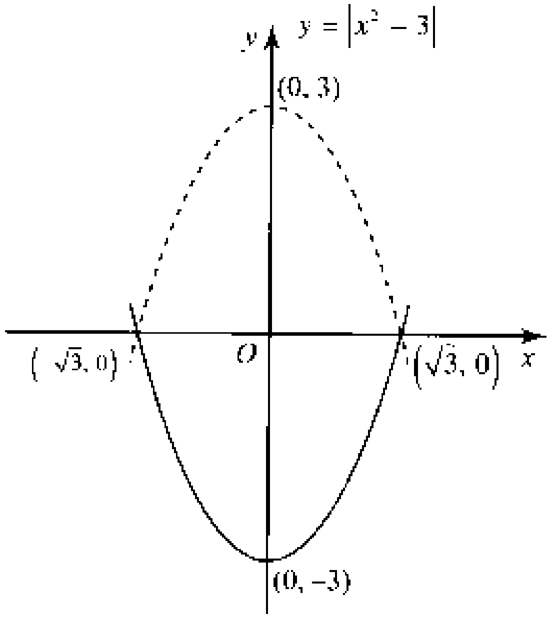


7. $y = 6 - x - x^2$

$y = (3 + x)(2 - x)$

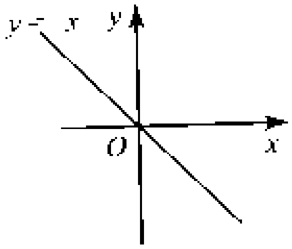
curve passes through

(−3, 0), (2, 0) and (0, 6)

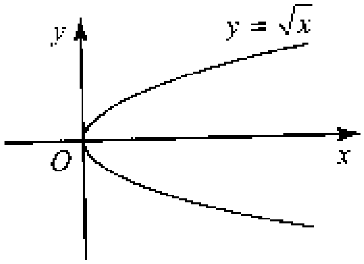


8. $y = x^2 - 3$

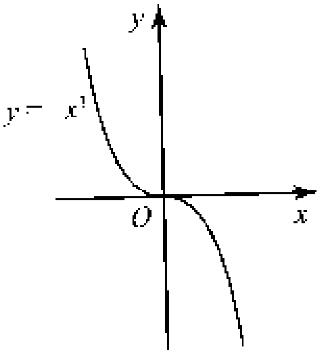
9. (a) $y = -x$



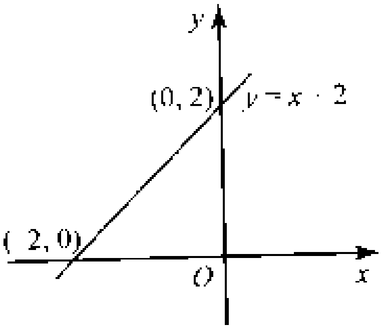
(b)



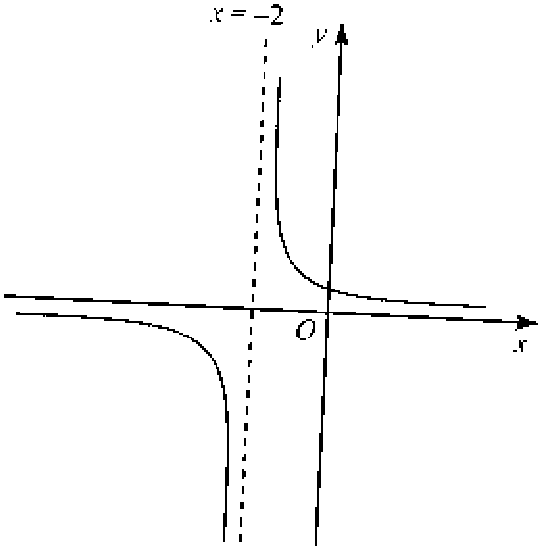
(c)



10. (a) $y = x + 2$



(b) $y = \dfrac{1}{x+2}$



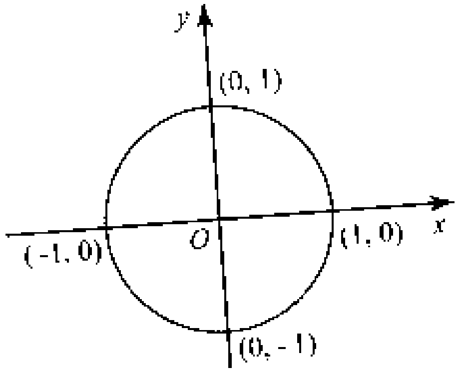
1. $y = \sin 2\theta$ $\qquad$ $x = \cos 2\theta$

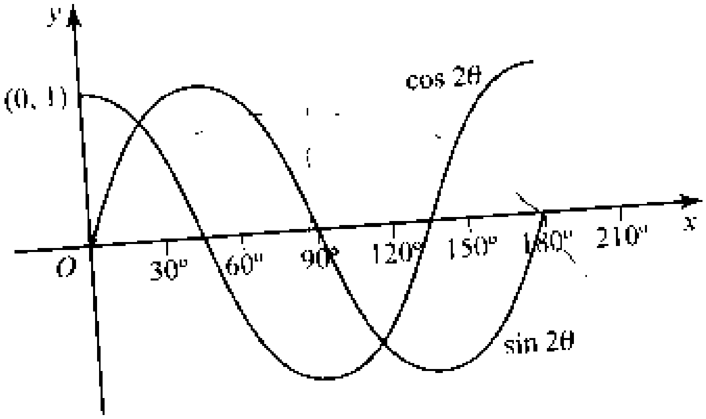
$y^2 = \sin^2 2\theta$ -- ---① $\qquad$ $x^2 = \cos^2 2\theta$--------②

Cartesian equation is obtained by adding ① and ②, thus

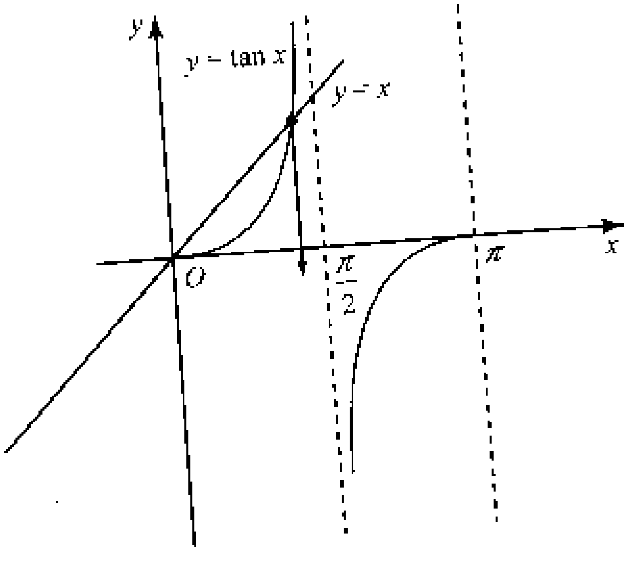
$\sin^2 2\theta + \cos^2 2\theta = 1$

$\therefore\ y^2 + x^2 = 1$


y
(0, 1)
x
(-1, 0)
O
(1, 0)
(0, -1)

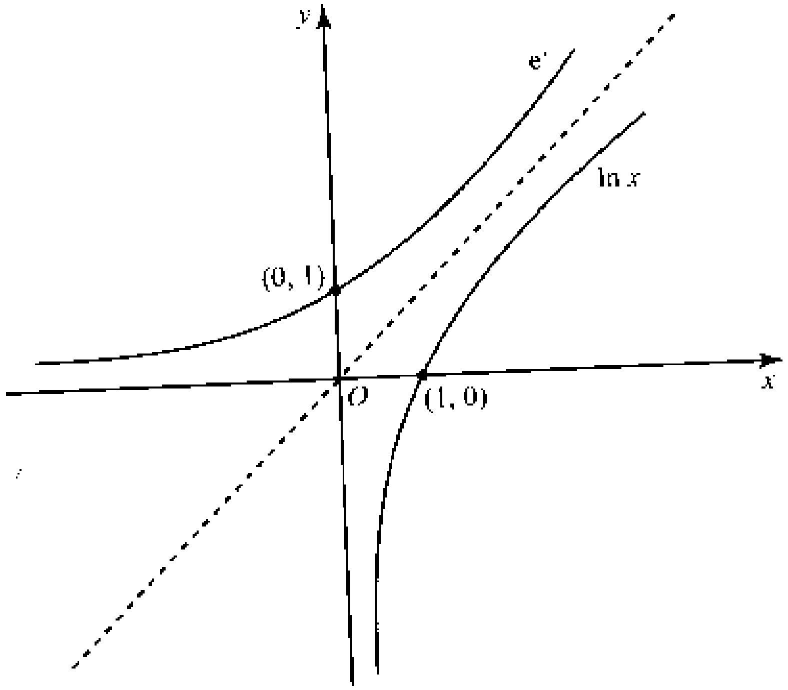

y
(0, 1)
cos 2θ
x
O
30°
60°
90°
120°
150°
180°
210°
sin 2θ

12.


y
y = tan x
y = x
x
O
$\frac{\pi}{2}$
π

13. $\ln x$ is the reflection of e^x in the line $y = x$.

In other words, the inverse of $\ln x$ is e^x.



14. $y = x^3 - 9x^2 + 23x - 15$

$f(x) = x^3 - 9x^2 + 23x - 15$

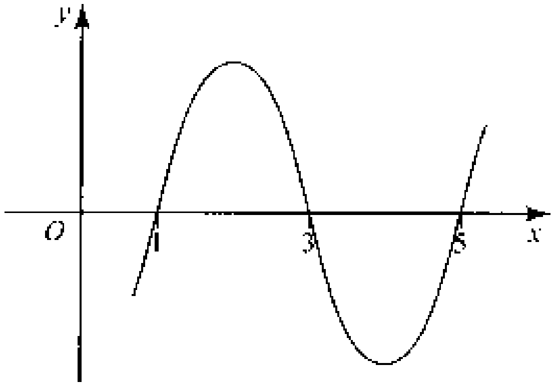
$f(0) = -15$

$f(1) = 1 - 9 + 23 - 15 = 0$ $\quad\therefore\quad (x - 1)$ is a factor

$f(2) = 8 - 36 + 46 - 15 \neq 0$

$f(3) = 27 - 81 + 69 - 15 = 0$ $\quad\therefore\quad (x - 3)$ is another factor

So that, $f(x) = (x - 1)(x - 3)(x - 5)$.



∴ the curve is positive where $1 < x < 3$ or when $x > 5$.

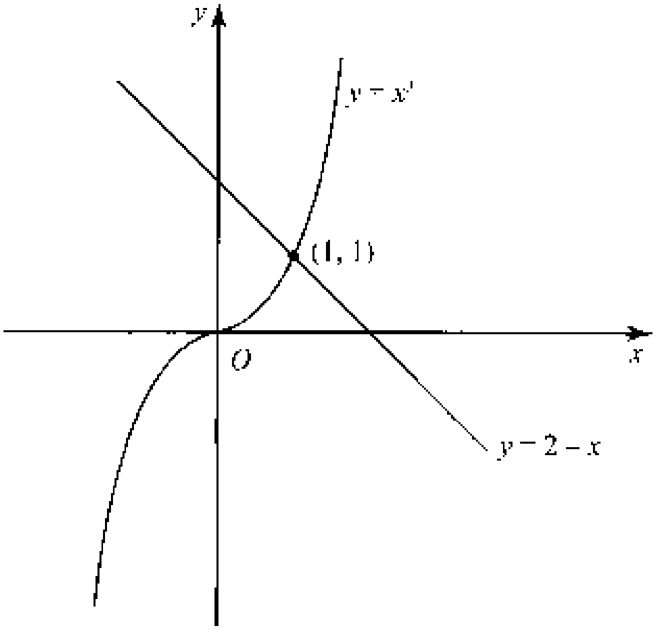
15. $x^3 + x \quad 2 = 0$

$x^3 = 2 \quad x$

the 2 graphs to sketch are

$y = x^3$

and $y - 2 \quad x$



UNIT 9

PERMUTATIONS AND COMBINATIONS

Suggested Solutions

1. (a) Number of different ways $= \dfrac{8!}{2!3!}$

$= \dfrac{40\ 320}{2 \times 6}$

$= 3\ 360$ (Answer)

(b) Number of codes that will begin and end with

$C = \dfrac{6!}{3!} = \dfrac{720}{6} = 120$ (Answer)

2. (a) No. of ways $= {}^{10}C_3 = \dfrac{10.9.8}{1.2.3} = 120$ (Answer)

(b) No. of arrangements of three $= {}^{7}P_3 = 210$ (Answer)

3. No. of ways $= {}^{9}C_3 + {}^{6}C_3 + {}^{3}C_3$

$= 84 + 20 + 1$

$= 105$ (Answer)

4. No. of ways $= {}^{12}C_5 = 792$ (Answer)

5. No. of ways $= {}^{25}C_7 = 480\ 700$ (Answer)

6. *p* can be correctly placed in *P* in only one way. After that *p* has been correctly placed, we have *q*, *r* and *s* left to be put in containers *Q*, *R* and *S*.

No. of ways in which q can be incorrectly placed is 2, i.e. q in R or S.
Since there are 4 objects, required number of ways $= 2 \times 4 = 8$. (Answer)

7. first $=$ 10 possibilities
second $=$ 9 possibilities
third $=$ 8 possibilities
$\therefore$ total no. of ways to have a first, a second and a third $= 10 \times 9 \times 8 = 720$. (Answer)

8. Total no. of ways

$$= (1B + 4G) + (2B + 3G) + (3B + 2G) + (4B + 1G)$$
$$= ({}^{20}C_1 \times {}^{13}C_4) + ({}^{20}C_2 \times {}^{13}C_3) + ({}^{20}C_3 \times {}^{13}C_2) + ({}^{20}C_4 \times {}^{13}C_1)$$
$$= (14\,300) + (190 \times 286) + (1\,140 \times 78) + (4\,845 \times 13)$$
$$= 14\,300 + 54\,340 + 88\,920 + 62\,985$$
$$= 220\,545.$$ (Answer)

9. No. of ways to seat 12 persons round a circular table

$$= (12 - 1)!$$
$$= 11!$$
$$= 39\,916\,800.$$ (Answer)

10. The 20 guests can be seated in
$(20 - 1)!$ ways
$= 19!$ ways.

11. The first digit has to be 6.
The last digit has to be 2 or 4
$\therefore$ 6 - - - - 2 $= 4! = 24$ ways
and 6 - - - - 4 $= 4! = 24$ ways
$\therefore$ total no. of ways $= 48$. (Answer)

12. $$\frac{n!}{(n-r)!r!}-\frac{n!}{(n-r+2)!(n-r-1)!}$$

$$=\frac{(n!)(n-r+1)(n-r+2)-n!(n-r+2)}{r!(n-r+2)!}$$

$$=\frac{n!(n-r+2)\{(n-r+1)-1\}}{r!(n-r+2)!}$$

$$=\frac{n!(n-r+2)(n-r)}{r!(n-r+2)!}$$ (Answer)

13. No. of arrangements $= \frac{9!}{2!2!}$

$= 90\ 720.$ (Answer)

14. No. of ways $= {}^{8}C_{6} \times {}^{5}C_{3} \times {}^{4}C_{2}$

$= 1\ 680.$ (Answer)

15. **(a)** $${}^{n+1}C_{n-1} = \frac{(n+1)!}{(n-1)!\{(n+1)-(n-1)\}!}$$

$$=\frac{(n+1)!}{(n-1)!(n+1-n+1)!}$$

$$=\frac{(n+1)!}{(n-1)!2!}$$

$$=\frac{(n+1)(n)}{2!}$$

$$=\frac{1}{2}n\,(n+1)$$ (Answer)

(b) $${}^{n-1}P_{n-1} = \frac{(n+1)!}{\{(n+1)-(n-1)\}!}$$

$$= \frac{(n+1)!}{(n+1-n+1)!}$$

$$= \frac{(n+1)!}{2!}$$

$$= \frac{1}{2}(n+1)$$ (Answer)

UNIT 10

FUNCTIONS

Suggested Solutions

$f(x) = x^2$

$g(x) = 3 - 2x$

(a) $fg(x) = f(3 - 2x)$

$= (3 - 2x)^2$ (Answer)

(b) $gf(x) = g(x^2)$

$= 3 - 2x^2$ (Answer)

(c) $f^{-1}x$: let $x^2 = y$

$x = \pm\sqrt{y}$

$\therefore \quad f^{-1}(x) = \pm\sqrt{x}$ (Answer)

(d) $g^{-1}(x)$: let $3 - 2x = y$

$-2x = y - 3$

$2x = 3 - y$

$x = \frac{3 - y}{2}$

$g^{-1}(x) = \frac{3 - x}{2}$ (Answer)

2. (a) $f(x) = e^x$

$f^2x = ff(x)$

$= f(e^x)$

$= e^{e^x}$ (Answer)

(b) $y = e^x$

$x = \ln y$

$f^{-1}(x) = \ln x$ (Answer)

(Answer)

(c) $f(-2) = e^{-2} = \frac{1}{e^2}$

3. $f(x) = e^{3x}$

$g(x) = \sqrt{x}$

(Answer)

(a) $fg(x) = f\left(\sqrt{x}\right) = e^{3\sqrt{x}}$

(b) $(fg)^{-1}(x) = \left(\frac{1}{3}\ln x\right)^2$ let $e^{3\sqrt{x}} = y$

$\ln y = 3\sqrt{x}$

$\sqrt{x} = \frac{1}{3}\ln y$

$x = \left(\frac{1}{3}\ln y\right)^2$ (Answer)

4. $g(x) = 1 + e^x$

let $1 + e^x = y$

$e^x = y - 1$

$\ln(y - 1) = x$ (Answer)

$\therefore\ g^{-1}(x) = \ln(x - 1)$

5. $f(x) = \ln x$

$g(x) = x + 1$

(a) (i) $f(x) = \ln x = y$

$e^y = x$ (Answer)

$f^{-1}(x) = e^x$

(ii) $g(x) = x + 1 = y$

$x = y - 1$ (Answer)

$g^{-1}(x) = (x - 1)$

(b) $(fg)^{-1}(x) = g^{-1}f^{-1}(x)$

$= g^{-1}(e^x)$ (Answer)

$= (e^x - 1)$

6. $h(x) = x^2 - 9x$

let $x^2 - 9x = y$

$x^2 - 9x - y = 0$

$$x = \frac{9 + \sqrt{81 + 4y}}{2} \quad \text{or} \quad \frac{9 - \sqrt{81 + 4y}}{2}$$

$$\therefore \quad h^{-1}(x) = 9 \pm \frac{\sqrt{81 + 4x}}{2}$$

(Answer)

7. (a) $f(x) = g(x)$

$x + 4 = 3x$

$2x = 4$

$x = 2$

(Answer)

(b) $f^{-1}(x) = g^{-1}(x)$

$$x - 4 = \frac{1}{3}x$$

$3x - 12 = x$

$2x = 12$

$x = 6$ (Answer)

$f^{-1}(x)$:

let $x + 4 = y$

$x = y - 4$

$\therefore \quad f^{-1}(x) = x - 4$

$g^{-1}(x)$:

let $3x = y$

$$x = \frac{1}{3}y$$

$$\therefore \quad g^{-1}(x) = \frac{1}{3}x$$

(Answer)

8. $h(x) = 2x^2 + 3$

$$k(x) = \sqrt{\frac{x - 3}{2}}$$

$$hk(x) = h\left(\sqrt{\frac{x - 3}{2}}\right) = 2\left(\frac{x - 3}{2}\right) + 3$$

$= x - 3 + 3$

$= x$

(Answer)

9. (a) (i) $f(x) = x - 1$

let $x - 1 = y$

$x = y + 1$

$\therefore \quad f^{-1}(x) = x + 1$ (Answer)

(ii) $g(x) = x^2$

let $x^2 = y$

$x = \pm\sqrt{y}$

$g^{-1}(x) = \pm\sqrt{x}$ (Answer)

(b)

$$fg(x) = gf(x)$$
$$f(x^2) = g(x - 1)$$
$$x^2 - 1 = (x - 1)^2$$
$$x^2 - 1 = x^2 - 2x + 1$$
$$-2x = -2$$
$$= 1$$

(Answer)

10. $g(x) = |x|$

$h(x) = \sqrt{x}$

let $|x| = y$

$x = y$

$\therefore \quad g^{-1}(x) = x$

let $\sqrt{x} = y$

$x = y^2$

$\therefore \quad h^{-1}(x) = x^2$

$$g^{-1}h^{-1}(x) = h^{-1}g^{-1}(x)$$
$$g^{-1}(x^2) = h^{-1}(x)$$
$$x^2 = x^2$$
$$x \in \mathbb{R}$$

(Answer)

11. $f(x) = \ln x$

(a) $f(2) = \ln 2$ (Answer)

(b) $f^{-1}(x) = e^x$

let $\ln x = y$

$x = e^y$ (Answer)

(c) $f^2(x) = ff(x) = f(\ln x)$

$= \ln(\ln x)$ (Answer)

12. $f(x) = x^2$

$g(x) = 3 - 2x$

(a) $fg(x)$

$= f(3 - 2x)$

$= (3 - 2x)^2$ (Answer)

(b) $f^{-1}x = \pm\sqrt{x}$

(Answer)

let $x^2 = y$

$\therefore \quad x = \pm\sqrt{y}$ (Answer)

(c) $g^{-1}(x) = \frac{3 - x}{2}$

(Answer)

let $3 - 2x = y$

$-2x = y - 3$

$2x = 3 - y$

$\therefore \quad x = \frac{3 - y}{2}$ (Answer)

(d) $(fg)^{-1}(x) = \frac{1}{2}\left(3 \pm \sqrt{x}\right)$

(Answer)

let $(3 - 2x)^2 = y$

$3 - 2x = \pm\sqrt{y}$

$-2x = \pm\sqrt{y} - 3$

$2x = 3 \pm \sqrt{y}$

$\therefore \quad x = \frac{1}{2}\left(3 + \sqrt{y}\right)$ (Answer)

13.

$$f(x) = 3x^{\frac{1}{2}} + 7 = y$$

$$3x^{\frac{1}{2}} = y - 7$$

$$x^{\frac{1}{2}} = \frac{y-7}{3}$$

$$x = \frac{(y-7)^2}{9}$$

$$f^{-1}(x) = \frac{(x-7)^2}{9}$$

$$f^{-2}(x) = f^{-1}f^{-1}(x)$$

$$= f^{-1}\frac{(x-7)^2}{9}$$

$$= \frac{\left\{\frac{(x-7)^2}{9} - 7\right\}^2}{9}$$

$$= \frac{\left(\frac{(x-7)^2 - 63}{9}\right)^2}{9}$$

$$= \frac{(x^2 - 14x + 49 - 63)^2}{81} \times \frac{1}{9}$$

$$= \frac{(x^2 - 14x - 14)^2}{729}$$

(Answer)

14. (a) $f(\theta) = \sin\theta = 0.5$

$$\theta = \frac{\pi}{6} \text{ or } \frac{5}{6}\pi$$

(Answer)

(b) $g(\theta) = \cos\theta = \frac{1}{\sqrt{2}}$

$$\theta = \frac{\pi}{4}$$

(Answer)

(c) $\mathrm{fg}\left(\frac{\pi}{2}\right)$

$= \mathrm{f}\left(\cos\frac{\pi}{2}\right)$

$= \mathrm{f}(0) = \sin 0$

$= 0$ (Answer)

15. $\mathrm{g}(x) = (x - 1)(x - 2)$

$\mathrm{h}(x) = (x^2 - 1)$

(a) $\mathrm{gh}(x)$

$= \mathrm{g}(x^2 - 1)$

$= \{(x^2 - 1) - 1\}\{(x^2 - 1) - 2\}$

$= (x^2 - 2)(x^2 - 3)$ (Answer)

(b) $\mathrm{gh}(x) = 0$

$(x^2 - 2)(x^2 - 3) = 0$

$\therefore \quad x^2 = 2 \text{ or } 3$

i.e. $x = \pm\sqrt{2} \quad \text{or} \quad +\sqrt{3}$ (Answer)

16. $\mathrm{f}(x) = 16 - x^2$

let $16 - x^2 = y$

$-x^2 = y - 16$

$x^2 = 16 - y$

$x = \pm\sqrt{(16 - y)}$

$\mathrm{f}^{-1}(x) = \pm\sqrt{(16 - x)}$ (Answer)

(b) $\mathrm{f}^2(x) = \mathrm{ff}(x)$

$= \mathrm{f}(16 - x^2)$

$= 16 - (16 - x^2)^2$

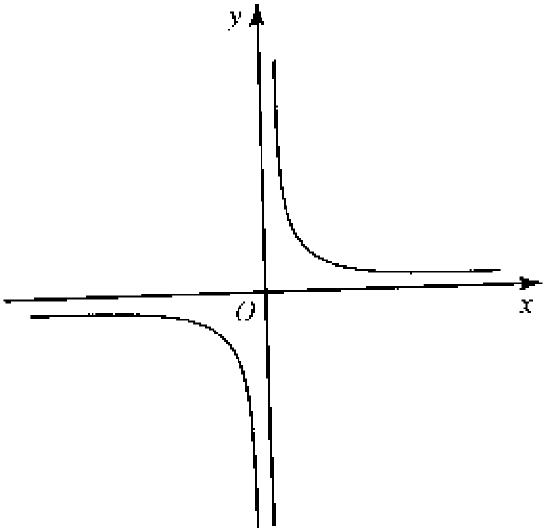
$= 16 - (256 - 32x^2 + x^4)$

$= 16 - 256 + 32x^2 - x^4$

$$= -240 + 32x^2 - x^4$$

$$= -(240 - 32x^2 + x^4)$$ (Answer)

17. (a) $f(x) = \frac{1}{x}$



Range of f (x) :

$f(x) \neq 0$ (Answer)

(b) $g(x) = \frac{1}{x^2}$

let $\frac{1}{x^2} = y$

$$x^2 = \frac{1}{y}$$

$$x = \pm\sqrt{\left(\frac{1}{y}\right)}$$

$$\therefore \quad g^{-1}(x) = \pm\sqrt{\left(\frac{1}{x}\right)}$$

$$\text{i.e.} \quad g^{-1}(2) = \pm\sqrt{\left(\frac{1}{2}\right)}$$

$$= \pm\frac{1}{\sqrt{2}}$$

$$= \pm\left(\frac{\sqrt{2}}{2}\right)$$

(Answer)

18. $f(x) = 3x - 1$

$g(x) = 2x + 5$

(a)

$$\begin{aligned} fg(x) &= f(2x+5) \\ &= 3(2x+5) - 1 \\ &= 6x + 15 - 1 \\ &= 6x + 14 \\ &= 2(3x+7) \end{aligned}$$

(Answer)

(b)

$$\begin{aligned} gf(x) &= g(3x-1) \\ &= 2(3x-1) + 5 \\ &= 6x - 2 + 5 \\ &= 6x + 3 \\ &= 3(2x+1) \end{aligned}$$

(Answer)

(c)

$$\begin{aligned} fg(x) &= 2gf(x) \\ 2(3x+7) &= 2 \times 3(2x+1) \\ 3x + 7 &= 3(2x+1) \\ 3x + 7 &= 6x + 3 \\ 3x - 6x &= 3 - 7 \\ -3x &= -4 \\ 3x &= 4 \\ x &= \frac{4}{3} \end{aligned}$$

(Answer)

UNIT 11

TRIGONOMETRY

Suggested Solutions

1.
$$7\sin^2\theta = \frac{1}{2}(8 + 5\cos\theta)$$
$$14\sin^2\theta = 8 + 5\cos\theta$$
$$14(1-\cos^2\theta) = 8 + 5\cos\theta$$
$$14 - 14\cos^2\theta = 8 + 5\cos\theta$$
$$14\cos^2\theta + 5\cos\theta - 6 = 0$$

(Answer)

$$\cos\theta = \frac{-5 \pm \sqrt{25-4(14)(-6)}}{28}$$
$$= \frac{-5 \pm \sqrt{25+336}}{28}$$
$$= \frac{-5 \pm \sqrt{361}}{28}$$
$$= \frac{-5 \pm 19}{28}$$
$$= \frac{-5+19}{28} \text{ or } \frac{-5-19}{28}$$
$$= \frac{14}{28} \text{ or } -\frac{24}{28}$$
$$= \frac{1}{2} \text{ or } -\frac{6}{7}$$

or factorise

$$(7\cos\theta + 6)(2\cos\theta - 1) = 0$$
$$\cos\theta = -\frac{6}{7} \text{ or } \frac{1}{2}$$

$\cos\theta = -\frac{6}{7}$ $\qquad$ $\cos\theta = \frac{1}{2}$

$\theta = 149°$ or $211°$ $\qquad$ $\theta = 60°, 300°$

$\therefore \quad \theta = 60°,\ 149°,\ 211°$ or $300°$. (Answer)

2. (a) $\tan\theta - \cot\theta$

$$\text{L. H. S.} = \frac{\sin\theta}{\cos\theta} - \frac{\cos\theta}{\sin\theta}$$

$$= \frac{\sin^2\theta - \cos^2\theta}{\sin\theta\cos\theta}$$

$$= \frac{-(\cos^2\theta - \sin^2\theta)}{\sin\theta\cos\theta}$$

$$= \frac{-2(\cos^2\theta - \sin^2\theta)}{2\sin\theta\cos\theta}$$

$$= \frac{-2\cos 2\theta}{\sin 2\theta}$$

$$= -2\cot 2\theta$$

(Answer)

(b) $\tan\theta - \cot\theta = 5$

$$-2\cot 2\theta = 5$$

$$\cot 2\theta = -\frac{5}{2}$$

$$\tan 2\theta = -\frac{2}{5}$$

$$\tan 2\theta = -0.4$$

$$2\theta = 158.2°,\ 338.2°,\ 518.2°,\ 698.2°$$

$$\theta = 79.1°,\ 169.1°,\ 259.2°,\ 349.1°.$$

(Answer)

3. Let

$$5\sin\theta - 12\cos\theta = R\sin(\theta - \alpha)$$

$$5\sin\theta - 12\cos\theta = R\{\sin\theta\cos\alpha - \sin\alpha\cos\theta\}$$

$$5\sin\theta - 12\cos\theta = R\cos\alpha\sin\theta - R\sin\alpha\cos\theta$$

Equating coefficients, we have

$R\cos\alpha = 5$ ------------------------------- ①

and $R\sin\alpha = 12$ ------------------------------- ②

Dividing ② by ①, we get

$$\tan\alpha = \frac{12}{5}$$

$$\alpha = 67.4°$$

$$R\sin\alpha = 12$$

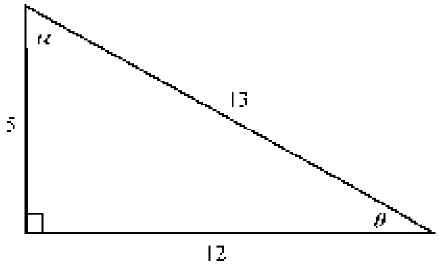
$$R\,\alpha\,0.923 = 12$$

$$R = 13.$$

$\therefore \quad 5\sin\theta \quad 12\cos\theta = 13\sin(\theta - 67.4°)$ --------- ① (Answer)

$5\sin\theta - 12\cos\theta = 5$

Using ① above, we have:



$\therefore \quad 13\sin(\theta - 67.4°) = 5$

$$\sin(\theta \quad 67.4°) = \frac{5}{13}$$

$$\sin(\theta - 67.4°) = 0.38$$

$$\theta - 67.4° = 22.6°$$

$\theta = 89.7°$ or $90.3°$ $(180° - 89.7°)$ (Answer)

4. (a)

$$\cos x\cot x = \cos x \times \frac{\cos x}{\sin x}$$

$$= \frac{\cos^2 x}{\sin x}$$

$$= \frac{1-\sin^2 x}{\sin x}$$

(Answer)

(b)

$$2\cos x\cot x = 3$$

$$2\left(\frac{1-\sin^2 x}{\sin x}\right) = 3$$

$$2 - 2\sin^2 x = 3\sin x$$

$$2\sin^2 x + 3\sin x - 2 = 0$$
$$(2\sin x - 1)(\sin x + 2) = 0$$
$$\sin x = \frac{1}{2} \text{ or } -2 \text{ rejected}$$
$$x = 30°,\ 150°.$$

(Answer)

5. (a)

$$2\cos\theta - 3\sin\theta = R\cos(\theta + \alpha)$$
$$2\cos\theta - 3\sin\theta = R\{\cos\theta\cos\alpha - \sin\theta\sin\alpha\}$$
$$2\cos\theta - 3\sin\theta = R\cos\alpha\cos\theta - R\sin\alpha\sin\theta$$

equating coefficients, we obtain

$R\sin\alpha = 3$ .... ①

$R\cos\alpha = 2$ .... ②

divide ① by ②, we have

$\tan\alpha = 1.5$

$\alpha = 56.3°$ $\qquad R = \sqrt{2^2 + 3^2} = \sqrt{13}$

$\therefore \quad 2\cos\theta - 3\sin\theta = \sqrt{13}\cos(\theta + 56.3°)$

(b)

$$2\cos\theta - 3\sin\theta = 3$$
$$R\cos(\theta + \alpha) = 3$$
$$\sqrt{13}\cos(\theta + 56.3°) = 3$$
$$\cos(\theta + 56.3°) = \frac{3}{\sqrt{13}}$$
$$\cos(\theta + 56.3°) = 0.8$$
$$\cos(\theta + 56.3°) = 0.8$$

$\theta + 56.3° = 36.9°$ or $323.1°$

36.9° is impossible and hence, is rejected

$\therefore \quad \theta + 56.3° = 323.1°.$

$\theta = 266.8°$

(Answer)

(c) $y = 2\cos\theta - 3\sin\theta$

$$\frac{dy}{dx} = -2\sin\theta - 3\cos\theta = 0 \text{ (for a stationary point)}$$

$$2\sin\theta + 3\cos\theta = 0$$
$$2\sin\theta = -3\cos\theta$$
$$2\tan\theta = -3$$
$$\tan\theta = -\frac{3}{2}$$
$$\theta = 123.7^\circ \qquad \left(123.7^\circ ; -\frac{3}{2}\right)$$

(Answer)

6. $\cot\theta + \tan\theta$

$$= \frac{\cos\theta}{\sin\theta} + \frac{\sin\theta}{\cos\theta}$$
$$= \frac{\cos^2\theta + \sin^2\theta}{\sin\theta\cos\theta}$$
$$= \frac{1}{\sin\theta\cos\theta} \times 2$$
$$= \frac{2}{2\sin\theta\cos\theta}$$
$$= \frac{2}{\sin 2\theta}$$
$$= 2\operatorname{cosec} 2\theta$$

(Answer)

7. (a)

$$3\cot\theta = 2\sin\theta$$
$$3\cot\theta - 2\sin\theta = 0$$
$$\frac{3\cos\theta}{\sin\theta} - 2\sin\theta = 0$$
$$3\cos\theta - 2\sin^2\theta = 0$$
$$3\cos\theta - 2\left(1 - \cos^2\theta\right) = 0$$
$$3\cos\theta - 2 + 2\cos^2\theta = 0$$
$$2\cos^2\theta + 3\cos\theta - 2 = 0$$

(Answer)

(b) $3\cot\theta = 2\sin\theta$

is the same as

$2\cos^2\theta + 3\cos\theta - 2 = 0$

$(2\cos\theta - 1)(\cos\theta + 2) = 0$

$\cos\theta = \frac{1}{2}$ or -2 (rejected)

$\therefore\ \theta = 60°$ or $300°$. (Answer)

8. (a)

$$\sin(x + 30°) = 2\sin(x + 60°)$$

$$\sin x\cos 30° + \sin 30°\cos x = 2(\sin x\cos 60° + \sin 60°\cos x)$$

$$\frac{\sqrt{3}}{2}\sin x + \frac{1}{2}\cos x = 2\left(\frac{1}{2}\sin x + \frac{\sqrt{3}}{2}\cos x\right)$$

$$\frac{\sqrt{3}}{2}\sin x + \frac{1}{2}\cos x = \sin x + \sqrt{3}\cos x$$

$$\frac{\sqrt{3}}{2}\sin x - \sin x + \frac{1}{2}\cos x - \sqrt{3}\cos x = 0$$

$$\sin x\left(\frac{\sqrt{3}}{2} - 1\right) + \cos x\left(\frac{1}{2} - \sqrt{3}\right) = 0$$

$$\sin x\left(\frac{\sqrt{3}}{2} - 1\right) = \left(\sqrt{3} - \frac{1}{2}\right)\cos x$$

$$\frac{\sin x}{\cos x} = \frac{\sqrt{3} - \frac{1}{2}}{\frac{\sqrt{3}}{2} - 1}$$

$$\tan x = \frac{2\sqrt{3} - 1}{\sqrt{3} - 2}$$

$$\tan x = \left(\frac{2\sqrt{3} - 1}{\sqrt{3} - 2}\right) \times \left(\frac{\sqrt{3} + 2}{\sqrt{3} + 2}\right)$$

$$= \frac{6 + 4\sqrt{3} - \sqrt{3} - 2}{3 - 4}$$

$$= \frac{4 + 3\sqrt{3}}{-1}$$

$$= -\left(4 + 3\sqrt{3}\right)$$ (Answer)

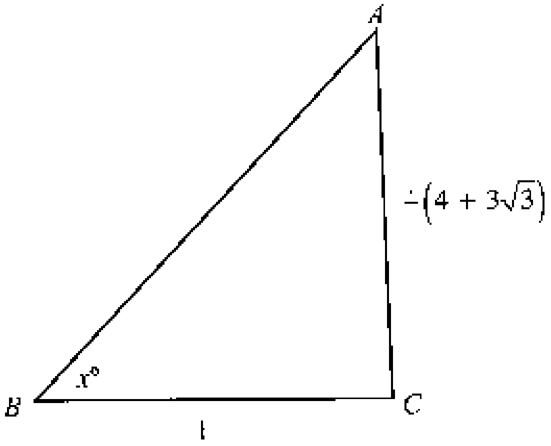
(b) $\tan x = -\left(4 + 3\sqrt{3}\right)$

$\tan x = -(4 + 5.196)$

$\tan x = 9.196$

$\therefore \quad x = 96.2°$ or $276.2°$

(c)



$$AB = \sqrt{\left(4 + 3\sqrt{3}\right)^2 + 1^2}$$

$$= \sqrt{16 + 24\sqrt{3} + 27 + 1}$$

$$= \sqrt{44 + 24\sqrt{3}}$$

$$= 2\sqrt{\left(11 + 6\sqrt{3}\right)}$$ (Answer)

9. (a) $3\sin\theta + 4\cos\theta = R\sin(\theta + \alpha)$

$R = \sqrt{3^2 + 4^2} = 5$

$\therefore \quad 3\sin\theta + 4\cos\theta = 5\sin(\theta + \alpha)$

$= 5(\sin\theta\cos\alpha + \sin\alpha\cos\theta)$

$$3\sin\theta + 4\cos\theta = 5\sin\theta\cos\alpha + 5\sin\alpha\cos\theta$$

$5\cos\alpha = 3$ $\qquad$ $5\sin\alpha = 4$

$\cos\alpha = \frac{3}{5}$ $\qquad$ $\sin\alpha = \frac{4}{5}$

$\alpha = 53.1°$

$\therefore\quad 3\sin\theta + 4\cos\theta = 5\sin(\theta + 53.1°)$ (Answer)

(b) $3\sin\theta + 4\cos\theta = 3$

$5\sin(\theta + 53.1°) = 3$

$\sin(\theta + 53.1°) = 0.6$

$\theta + 53.1° = \underset{\substack{\uparrow \\ \text{rejected}}}{36.9°} \text{ or } 143.1°$

$\therefore\quad \theta = 143.1° - 53.1° = 90.0°$ (Answer)

(c) Maximum value of $\dfrac{1}{3\sin\theta + 4\cos\theta - 3}$

will occur when

$3\sin\theta + 4\cos\theta - 3$ is a minimum.

i.e. when $\theta = 90°$

$(3 \times 1) + (4 \times 0) - 3$

$= 3 + 0 - 3$

$= 0$

Max. value $= \frac{1}{0} = \infty$ (Answer)

$\cos 3\theta + 2\sin 3\theta = 0$

$2\sin 3\theta = -\cos 3\theta$

$\dfrac{2\sin 3\theta}{\cos 3\theta} = -1$

$\tan 3\theta = -\frac{1}{2}$

$3\theta = 153.4°, 333.4°, 513.4°, 693.4°, 873.4°, 1\,053.4°, \ldots$

$\therefore\quad \theta = 51.1°, 111.1°, 171.1°, 231.1°, 291.1°, 351.1°.$ (Answer)

11. (a)

$\sin(30° - x) = 3\cos(30° - x)$

$\sin 30° \cos x - \sin x \cos 30° = 3(\cos 30° \cos x + \sin 30° \sin x)$

$$\frac{1}{2}\cos x - \frac{\sqrt{3}}{2}\sin x = 3\left(\frac{\sqrt{3}}{2}\cos x + \frac{1}{2}\sin x\right)$$

$$\frac{1}{2}\cos x - \frac{\sqrt{3}}{2}\sin x = \frac{3\sqrt{3}}{2}\cos x + \frac{3}{2}\sin x$$

multiply by 2

$$\cos x - \sqrt{3}\sin x = 3\sqrt{3}\cos x + 3\sin x$$

$$3\sin x + \sqrt{3}\sin x = \cos x - 3\sqrt{3}\cos x$$

$$\sin x\left(3 + \sqrt{3}\right) = \cos x\left(1 - 3\sqrt{3}\right)$$

$$\tan x = \frac{1 - 3\sqrt{3}}{3 + \sqrt{3}}$$

rationalising

$$\tan x = \frac{1 - 3\sqrt{3}}{3 + \sqrt{3}} \times \frac{3 - 3\sqrt{3}}{3 - \sqrt{3}} = \frac{3 - 3\sqrt{3} - 9\sqrt{3} + 27}{9 - 3}$$

$$= \frac{30 - 12\sqrt{3}}{6}$$

$$= \frac{6\left(5 - 2\sqrt{3}\right)}{6} = 5 - 2\sqrt{3}$$ (Answer)

(b) $\tan x = 5 - 2\sqrt{3}$ (depending on 11(a))

$$x = \tan^{-1}\left(5 - 2\sqrt{3}\right)$$

$= \tan^{-1}(5 - 3.464)$

$= \tan^{-1}(1.536)$

$= 56.9°$ or $236.9°$ (Answer)

12. (a)

$\cos(x + 30°) + \sin(x + 30°) = 0$

$\cos x \cos 30° - \sin x \sin 30° + \sin x \cos 30° + \sin 30° \cos x = 0$

$$\frac{\sqrt{3}}{2}\cos x - \frac{1}{2}\sin x + \frac{\sqrt{3}}{2}\sin x + \frac{1}{2}\cos x = 0$$

multiply by 2

$$\sqrt{3}\cos x - \sin x + \sqrt{3}\sin x + \cos x = 0$$

$$\cos x\left(\sqrt{3} + 1\right) + \sin x\left(\sqrt{3} - 1\right) = 0$$

$$\sin x\left(\sqrt{3} - 1\right) = -\cos x\left(\sqrt{3} + 1\right)$$

$$\tan x = \frac{-\left(\sqrt{3} + 1\right)}{\left(\sqrt{3} - 1\right)} \times \frac{\left(\sqrt{3} + 1\right)}{\left(\sqrt{3} + 1\right)}$$

$$\tan x = \frac{-\left(3 + 2\sqrt{3} + 1\right)}{3 - 1}$$

$$= \frac{-\left(4 + 2\sqrt{3}\right)}{2}$$

$$= \frac{-4 - 2\sqrt{3}}{2} = -2 - \sqrt{3}$$ (Answer)

(b) $\cos(x + 30°) + \sin(x + 30°) = 0$

$\sin x\left(\sqrt{3} - 1\right) + \cos x\left(\sqrt{3} + 1\right) = 0$ from part (a),

$$\left(\sqrt{3} - 1\right)\sin x = -\left(\sqrt{3} + 1\right)\cos x$$

$$\tan x = \frac{-\left(\sqrt{3} + 1\right)}{\left(\sqrt{3} - 1\right)} \times \frac{\left(\sqrt{3} + 1\right)}{\left(\sqrt{3} + 1\right)}$$

$$= \left(3 + \sqrt{3} + \sqrt{3} + 1\right)$$

$$= \frac{-4 - 2\sqrt{3}}{3 - 1}$$

$$= -2 - \sqrt{3}$$

$$= 3.732$$

$$\therefore \quad x = 105°$$ (Answer)

13. $\cot x - \operatorname{cosec} 2x$

$$\text{LHS} = \frac{\cos x}{\sin x} - \frac{1}{\sin 2x}$$

$$= \frac{\cos x}{\sin x} - \frac{1}{2\sin x \cos x}$$

$$= \frac{2\cos^2 x - 1}{2\sin x \cos x}$$

$$= \frac{\cos 2x}{\sin 2x}$$

$$= \cot 2x$$

(Answer)

14. $2x^2 + 3x - 2 = 0$

$(2x - 1)(x + 2) = 0$

$x = \frac{1}{2}$ or -2

$\therefore \quad \cos^2 x = \frac{1}{2}$ or $\underset{\substack{\uparrow \\ \text{rejected}}}{-2}$

$\cos x = \pm \frac{1}{\sqrt{2}}$

i.e. $x = 45°, 135°, 225°, 315°$

(Answer)

15. (a) $\sqrt{3}\cos\theta + \sin\theta = R\cos(\theta - \alpha)$

$$R = \sqrt{(3)^2 + 1^2}$$

$$= \sqrt{4}$$

$$= 2$$

$$\sqrt{3}\cos\theta + \sin\theta = 2(\cos\theta\cos\alpha + \sin\theta\sin\alpha)$$

$$\sqrt{3}\cos\theta + \sin\theta = 2\cos\theta\cos\alpha + 2\sin\theta\sin\alpha$$

$$2\cos\alpha = \sqrt{3}$$

$$\cos\alpha = \frac{\sqrt{3}}{2}$$

$$\alpha = \frac{\pi}{6}$$

$$\therefore \quad R\cos(\theta - \alpha) = 2\cos(\theta \quad \frac{\pi}{6})$$ (Answer)

(b) $$\sqrt{3}\cos\theta + \sin\theta = \sqrt{3}$$

$$2\cos\left(\theta - \frac{\pi}{6}\right) = \sqrt{3}$$

$$\cos\left(\theta - \frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}$$

$$\theta - \frac{\pi}{6} = \frac{\pi}{6}$$

$$\theta = \frac{\pi}{6} + \frac{\pi}{6} = \frac{\pi}{3}$$

or $$\theta = 2\pi - \frac{\pi}{3} = \frac{5}{3}\pi$$ (Answer)

16. $\sin 2x \quad \cos x = 0$

$2\sin x\cos x - \cos x = 0$

$\cos x(2\sin x - 1) = 0$

$\cos x = 0$

$\therefore \quad x = 90°$ or $270°$

or $2\sin x - 1 = 0$

$\sin x = \frac{1}{2}$

$x = 30°, 150°$

$\therefore \quad x = 30°, 90°, 150°, 270°$ (Answer)

17. (a) $4\cos^2\theta - \sin^2\theta \quad 3\sin\theta\cos\theta = 0$

$\div$ throughout by $\sin\theta\cos\theta$

$$\frac{4\cos^2\theta}{\sin\theta\cos\theta} - \frac{\sin^2\theta}{\sin\theta\cos\theta} - \frac{3\sin\theta\cos\theta}{\sin\theta\cos\theta}$$

$$\frac{4\cos\theta}{\sin\theta} - \frac{\sin\theta}{\cos\theta} - 3 = 0$$

$$\frac{4}{\tan\theta} - \tan\theta - 3 = 0$$

multiply by tan θ

$$4 - \tan^2\theta - 3\tan\theta = 0$$

$$\tan^2\theta + 3\tan\theta - 4 = 0$$

(Answer)

(b) $4\cos^2\theta - 3\sin\theta\cos\theta \quad \sin^2\theta = 0$

$(4\cos\theta + \sin\theta)(\cos\theta \quad \sin\theta) = 0$

Hence

$4\cos\theta + \sin\theta = 0$ or $\cos\theta - \sin\theta = 0$

$4\cos\theta = -\sin\theta$

$\tan\theta = 4$

$\tan\theta = \quad 4$

$\theta = 104°$

$\sin\theta = \cos\theta$

$\tan\theta = 1$

$\theta = 45°$

$\therefore \quad \theta = 45°$ or $104°$

(Answer)

UNIT 12

DIFFERENTIATION

Suggested Solutions

1. (a) $y = \sqrt{(x^3 - 1)}$

$y = (x^3 - 1)^{\frac{1}{2}}$

$$\frac{dy}{dx} = \frac{1}{2}(x^3 - 1)^{-\frac{1}{2}} \times 3x^2$$

$$= \frac{3}{2}x^2 \times \frac{1}{(x^3 - 1)^{\frac{1}{2}}}$$

$$= \frac{3x^2}{2\sqrt{(x^3 - 1)}}$$

(Answer)

(b) $y = 2x^3 - 3x^2$

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$y = 2x^3 - 3x^2$

$\frac{dy}{dx} = 6x^2 - 6x$

$0 = 6x(x-1)$

(i) $\frac{dy}{dx} = 3x^2 - 6x = 0$ (for turning points)

$3x(x - 2) = 0$

$\therefore \quad x = 0$ or 2

$y = 0$ or 4

$\therefore$ the 2 stationary points have coordinates (0, 0) and (2, 4).

(Answer)

(ii) $\frac{d^2y}{dx^2} = 6x - 6$

when $x = 0$

$\frac{d^2y}{dx^2} = -6$

$\therefore$ point (0, 0) is a maximum

(Answer)

when $x = 2$

$\frac{d^2y}{dx^2} = 12 - 6 = 6$ $\therefore$ point (2, 4) is a minimum. (Answer)

(iii) for an increasing function $\frac{dy}{dx}$ must be positive.

i.e. $(3x^2 - 6x) > 0$

$3x(x - 2) > 0$

$\therefore \quad x > 2$

or $\quad x < 0$ (Answer)

2. (a) $y = x^2 - 8x + 7$

$\frac{dy}{dx} = 2x - 8$ (Answer)

(b) $m = 2x - 8 = 2(1) - 8 = -6$

when $x = 1$, $y = 1 - 8 + 7 = 0$

$(1, 0)$, $m = -6$

$\frac{y - 0}{x - 1} = -6$

$y = -6x + 6$ (Answer)

3. $y = 3x^2 - 2x - 1$

(a) $\frac{dy}{dx} = 6x - 2 = 0$

$6x = 2$

$x = \frac{1}{3}$

$y = 3\left(\frac{1}{3}\right)^2 - 2\left(\frac{1}{3}\right) - 1$

$= \frac{1}{3} - \frac{2}{3} - 1$

$= -\frac{4}{3}$

$\therefore$ coordinates of turning point = $\left(\frac{1}{3}, -\frac{4}{3}\right)$ (Answer)

$$\frac{d^2y}{dx^2} = 6 \quad \text{(positive)}$$

$\therefore$ curve has a minimum point at $\left(\frac{1}{3}, -\frac{4}{3}\right)$ (Answer)

(b) $X = (0, -1)$

$$m = \frac{dy}{dx} = 2$$

gradient of normal = $\frac{1}{2}$

$\therefore$ Equation of normal at X is

$$\frac{y+1}{x-0} = \frac{1}{2}$$

$$2y + 2 = x$$

$$2y = x - 2$$

(Answer)

4. (a) $y = (3 - x)\ln x$

when $y = 0$ $\quad (3 - x)\ln x = 0$

$3 - x = 0$ or $\ln x = 0$

$x = 3$ $\quad x = 1$

$\therefore$ Coordinates of P are $(1, 0)$ and of Q are $(3, 0)$ (Answer)

(b) $$\frac{dy}{dx} = (3 - x) \times \frac{1}{x} + \ln x(-1)$$

$$= \frac{3 - x}{x} - \ln x = 0$$

(for stationary point)

$$3 - x - x\ln x = 0$$

$$x + x\ln x = 3$$

$$x(1 + \ln x) = 3$$

$$x = \frac{3}{(1 + \ln x)}$$

(Answer)

5. (a)

$$\text{T.S.A} = 2\pi rh + \pi r^2$$
$$96\pi = 2\pi rh + \pi r^2$$
$$2rh + r^2 = 96$$
$$2rh = 96 - r^2$$
$$h = \frac{96 - r^2}{2r}$$

(Answer)

(b)

$$V = \pi r^2 h$$
$$= \pi r^2 \times \left(\frac{96 - r^2}{2r}\right)$$
$$= \frac{1}{2}\pi r(96 - r^2)$$

(Answer)

(c)

$$\frac{dv}{dr} = 48\pi - \frac{3}{2}\pi r^2 = 0, \qquad V = 48\pi r - \frac{1}{2}\pi r^3$$

for a stationary value

$$48 - \frac{3}{2}r^2 = 0$$
$$\frac{3}{2}r^2 = 48$$
$$r^2 = 48 \times \frac{2}{3} = 32$$
$$r = 5.66 \text{ cm}$$

(Answer)

or $\sqrt{32}$

or $4\sqrt{2}$

(d)

$$\text{Stationary value} = \frac{1}{2}\pi \times 4\sqrt{2}(96 - 32)$$
$$= 2\sqrt{2}\pi \times 64$$
$$= 128\sqrt{2}\pi$$

(Answer)

$\frac{d^2V}{dr^2} = -3\pi r$, which is negative

$\therefore$ Stationary value is a maximum (Answer)

6. (a) Since curve passes through, (−1, 0), (2, 0) and (5, 0)

$\therefore$ $y = (x + 1)(x - 2)(x - 5)$

$y = (x^2 - x - 2)(x - 5)$

$y = x^3 - x^2 - 2x - 5x^2 + 5x + 10$

$y = x^3 - 6x^2 + 3x + 10$

$\therefore$ $a = 1, b = -6, c = 3$ and $d = 10$ (Answer)

(b) At A, $x = 0$

$\therefore$ $y = 0 - 0 + 0 + 10$

i.e. $y = 10$

$\therefore$ The coordinates of A are (0, 10) (Answer)

(c) For a minimum point,

$$\frac{dy}{dx} = 0$$

i.e. $3x^2 - 12x + 3 = 0$

Divide by 3

$x^2 - 4x + 1 = 0$

$$x = \frac{4 + \sqrt{16 - 4}}{2} \quad \text{or} \quad \frac{4 - \sqrt{16 - 4}}{2}$$

$$= \frac{4 + \sqrt{12}}{2} \quad \text{or} \quad \frac{4 - \sqrt{12}}{2}$$

$$= \frac{4 + 2\sqrt{3}}{2} \quad \text{or} \quad \frac{4 - 2\sqrt{3}}{2}$$

$$= 2 + \sqrt{3} \quad \text{or} \quad 2 - \sqrt{3}$$

x lies between 2 and 5

$\therefore$ $2 - \sqrt{3}$ is rejected

Hence the x-coordinate of P is $\left(2 + \sqrt{3}\right)$ (Answer)

7. (a) $y = xe^x$

$$\frac{dy}{dx} = xe^x + e^x$$

(Answer)

(b) For a stationary value

$$\frac{dy}{dx} = 0$$

$$\therefore \quad xe^x + e^x = 0$$

$$e^x(x + 1) = 0$$

$e^x = 0$ or $x = -1$

↑

impossible

when $x = -1$

$$y = (-1)e^{(-1)}$$

$$= -\frac{1}{e}$$

$\therefore$ coordinates of stationary point are $\left(-1, \frac{-1}{e}\right)$

(Answer)

8. $y = \cos 2x + 2 \sin x$

For stationary points,

$$\frac{dy}{dx} = 0$$

$$(-\sin 2x) \times 2 + 2\cos x = 0$$

$$-2\sin 2x + 2\cos x = 0$$

$$-\sin 2x + \cos x = 0$$

$$-2\sin x \cos x + \cos x = 0$$

$$\cos x(1 - 2\sin x) = 0$$

$\cos x = 0$ or $\sin x = \frac{1}{2}$

$x = \frac{\pi}{2}$ $\quad$ $x = \frac{\pi}{6}$

$\therefore$ $x = \frac{\pi}{6}$ or $x = \frac{\pi}{2}$

$$\frac{d^2y}{dx^2} = -4\cos 2x - 2\sin x$$

$$= -4\cos\frac{\pi}{3} - 2\sin\frac{\pi}{6}$$

$$= -4 \times \frac{1}{2} - 2 \times \frac{1}{2}$$

$$= -2 - 1$$

$= -3$, which is negative

$\therefore$ at $x = \frac{\pi}{6}$, stationary point is maximum (Answer)

at $x = \frac{\pi}{2}$,

$$\frac{d^2y}{dx^2} = -4\cos 2x - 2\sin x$$

$$= -4\cos \pi - 2\sin \frac{\pi}{2}$$

$$= (-4 \times -1) - (2 \times 1)$$

$= 2$, which is positive

$\therefore$ at $x = \frac{\pi}{2}$, stationary point has minimum value. (Answer)

9. (a) $y = x^3 - x^2 - 8x + k$

$$\frac{dy}{dx} = 3x^2 - 2x - 8$$ (Answer)

(b) for turning points

$$\frac{dy}{dx} = 0$$

$\therefore$ $3x^2 - 2x - 8 = 0$

$(3x + 4)(x - 2) = 0$

$x = \frac{-4}{3}$ or 2 (Answer)

(c)

$$\begin{array}{r}
x^2 + x - 6 \\
x - 2 \,\overline{\smash{)}\, x^3 - x^2 - 8x + k} \\
\underline{x^3 - 2x^2 } \\
x^2 - 8x + k \\
\underline{x^2 \quad 2x } \\
-6x + k \\
\underline{-6x + 12} \\
k - 12 = 0 \\
k = 12
\end{array}$$

(Answer)

Similarly, for $x = \frac{-4}{3}$

10. $y = x^3 \quad 3x + 2 + k$

(a) $\frac{dy}{dx} = 3x^2 - 3$ (Answer)

(b) $3x^2 - 3 = 0$

$3x^2 = 3$

$x^2 = 1$

$x = \pm 1$

$\therefore$ Coordinates of 2 turning points are

$x = 1$	and $x = \quad 1$
$y = 1 - 3 + 2 = 0$	$y = -1 + 3 + 2$
$(1, 0)$	$y = 4$
	$(-1, 4)$

(Answer)

(c)

$$\begin{array}{r}
x^2 + x - 2 \\
x - 1 \,\overline{\smash{)}\, \cancel{x^3} - 3x + 2 + k} \\
\underline{\cancel{x^3} - x^2 } \\
\cancel{x^2} - 3x + 2 + k \\
\underline{\cancel{x^2} - x } \\
-2\cancel{x} + \cancel{2} + k \\
\underline{\cancel{-2x} + \cancel{2} } \\
k = 0
\end{array}$$

or $\frac{(x^3 - 3x + 2 + k)}{(x + 1)}$ gives $k = 4$ (Answer)

11. $3x^2 - 2y^2 + xy = 5$

(a) Differentiating with respect to x,

$$6x - 4y\frac{dy}{dx} + x\frac{dy}{dx} + y = 0$$

$$\frac{dy}{dx}(x - 4y) = -6x - y$$

$$\frac{dy}{dx} = \frac{-6x - y}{x - 4y} = \frac{6x + y}{4y - x}$$

(Answer)

(b) parallel to x-axis mean

$$\frac{dy}{dx} = 0$$

$$\frac{6x + y}{4y - x} = 0$$

$$6x + y = 0$$

$$y = -6x$$

Substituting y by $-6x$ in given equation,

we have

$$3x^2 - 2(-6x)^2 - x(-6x) = 5$$

$$3x^2 - 72x^2 + 6x^2 = 5$$

$$-63x^2 = 5$$

$$x^2 = \frac{5}{-63}$$

No solution

(Answer)

12. $y = 3e^{-x} + 2e^{x}$

(a) $\frac{dy}{dx} = 0$ for stationary point.

$$\frac{dy}{dx} = -3e^{-x} + 2e^{x} = 0$$

$$\frac{-3}{e^{x}} + 2e^{x} = 0$$

$$-3 + 2e^{2x} = 0$$

$$2e^{2x} = 3$$

$$2x = \ln \frac{3}{2}$$

$$x = \frac{1}{2}\ln \frac{3}{2} = \frac{1}{2} \times 0.4055 = 0.2027$$ (Answer)

(b) $$\frac{d^2y}{dx^2} = 3e^{-x} + 2e^{x}$$

$$= \frac{3}{e^{x}} + 2e^{x}$$, which is positive

$\therefore$ it has a minimum value. (Answer)

13. $$y = x\ln x$$

(a) $$\frac{dy}{dx} = x \times \frac{1}{x} + \ln x = 1 + \ln x = 0$$

$$\ln x = -1$$

$$e^{1} = x$$

$$x = 0.3679$$ (Answer)

(b) $$\frac{d^2y}{dx^2} = \frac{1}{x}$$, which is +ve, $\therefore$ minimum (Answer)

14. $y = \dfrac{1}{1 - \tan x}$ by changing $\tan x$ into $\dfrac{\sin x}{\cos x}$ and simplify to obtain

(a) $$y = \frac{\cos x}{\cos x \quad \sin x}$$

Let $u = \cos x$ and let $v = \cos x - \sin x$

then $\dfrac{du}{dx} = -\sin x$ then $\dfrac{dv}{dx} = -\sin x - \cos x$

$$= -(\sin x + \cos x)$$

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

$$= \frac{(\cos x - \sin x) \times (-\sin x) + \cos x\,(\sin x + \cos x)}{(\cos x - \sin x)^2}$$

$$= \frac{\cancel{-\sin x \cos x} + \sin^2 x + \cancel{\sin x \cos x} + \cos^2 x}{(\cos x - \sin x)^2}$$

$$= \frac{1}{(\cos x - \sin x)^2}$$ (Answer)

(Answer)

(b) +ve

15. (a) Volume of cuboid = $3x \times 2x \times y$

$144 = 6x^2y$

$\therefore \quad y = \frac{144}{6x^2} = \frac{24}{x^2}$ (Answer)

(b) $\text{T.S.A} = (2 \times 3x \times 2x) + (2 \times y \times 3x) + (2 \times y \times 2x)$

$= \left(12x^2\right) + \left(6x \times \frac{24}{x^2}\right) + \left(4x \times \frac{24}{x^2}\right)$

$= 12x^2 + \frac{144}{x} + \frac{96}{x}$

$= 12x^2 + \frac{240}{x}$

$= 12\left(x^2 + \frac{20}{x}\right)$ cm^2 (Answer)

(c) $A = 12x^2 + 240x^{-1}$

$\frac{dA}{dx} = 24x - 240x^{-2} = 0$

$x - \frac{10}{x^2} = 10$

$x^3 - 10 = 0$

$x^3 = 10$

$\therefore \quad x = \sqrt[3]{10} = 2.15$ (Answer)

(d) $A = 12x^2 + \dfrac{240}{x}$

$= \dfrac{12x^3 + 240}{x}$, since $x^3 = 10$

$= \dfrac{120 + 240}{x}$

$= \dfrac{360}{x}$, which is +ve.

(Answer)

$\therefore$ A has a minimum value.

UNIT 13

INTEGRATION

Suggested Solutions

1. (a) $y = x$ -- ①

$x = \frac{y^2}{4}$ -- ②

$4x = y^2$

$4x = x^2$ replacing ① in ②

$x^2 - 4x = 0$

$x(x - 4) = 0$

$\therefore \quad x = 0$ or 4

$\therefore$ Coordinates are $O(0, 0)$ and $A(4, 4)$
Thus coordinates of $B = (4, 0)$ (Answer)

(b) Area of $OAB = \int_0^4 y\,dx$

$\int_0^4 \sqrt{4x}\,dx$

$x = \frac{y^2}{4}$

$y^2 = 4x$

$y = \sqrt{4x} = 2\sqrt{x}$

$= 2x^{\frac{1}{2}}$

$= \int_0^4 2x^{\frac{1}{2}}\,dx$

$= \left[2x^{\frac{3}{2}} \times \frac{2}{3}\right]_0^4$

$= \left[\frac{4}{3}x^{\frac{3}{2}}\right]_0^4$

$= \frac{4}{3}\left(4^{\frac{3}{2}}\right) - \frac{4}{3}\left(0^{\frac{3}{2}}\right)$

$= \left(\frac{4}{3} \times 2^3\right) - 0$

$= \frac{32}{3} = 10\frac{2}{3}$ sq. units (Answer)

(c) Shaded area = Area of region $OAB - \Delta OAB$

$= 10\frac{2}{3} - \left(\frac{1}{2} \times 4 \times 4\right)$

$= 10\frac{2}{3} - 8$

$= 2\frac{2}{3}$ sq. units (Answer)

2. (a) $\int_0^{\frac{\pi}{3}} \sin 2x \, dx$.

$= \left[-\frac{1}{2}\cos 2x\right]_0^{\frac{\pi}{3}}$

$= \left(-\frac{1}{2}\cos\frac{2}{3}\pi\right) - \left(-\frac{1}{2}\cos 0\right)$

$= \left(-\frac{1}{2} \times -\frac{1}{2}\right) - \left(-\frac{1}{2}\right)$

$= \frac{1}{4} + \frac{1}{2}$

$= \frac{3}{4}$ (Answer)

(b) $\int_0^{\frac{\pi}{3}} \cos^2 x \, dx$

$= \int_0^{\frac{\pi}{3}} \frac{1}{2}(1 + \cos 2x) \, dx$

$$= \int_0^{\frac{\pi}{3}} \left(\frac{1}{2} + \frac{1}{2}\cos 2x\right) \mathrm{d}x$$

$$= \left[\frac{1}{2}x + \frac{1}{4}\sin 2x\right]_0^{\frac{\pi}{3}}$$

$$= \left(\frac{1}{2} \times \frac{\pi}{3} + \frac{1}{4} \times \sin\frac{2}{3}\pi\right) - \left(0 + \frac{1}{4}\sin 0\right)$$

$$= \frac{\pi}{6} + \frac{1}{4} \times \frac{\sqrt{3}}{2} - (0 + 0)$$

$$= \frac{\pi}{6} + \frac{\sqrt{3}}{8}$$

(Answer)

(c) Area

$$= \frac{\pi}{6}\left\{\frac{1}{2}\left(1 + \frac{1}{4}\right) + \frac{3}{4}\right\}$$

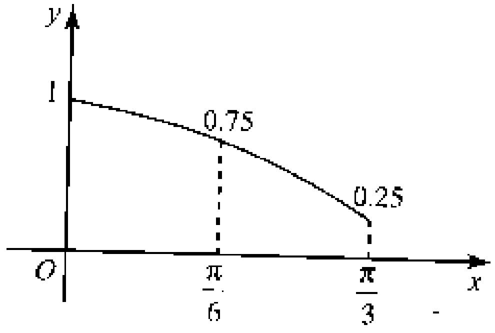
$$= \frac{\pi}{6}\left\{\frac{5}{8} + \frac{3}{4}\right\}$$

$$= \frac{11}{48}\pi = 0.720 \text{ sq. units}$$

$x =$	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$
$\cos^2 x =$	1	$\frac{3}{4}$	$\frac{1}{4}$

(Answer)

(d)



Trapezium rule gives an over-estimate of the area under curve.

(Answer)

3. $$f(x) = \frac{2x}{(x-1)(2x+1)^2}$$

Change into partial fractions, it gives

$$f(x) = \frac{2}{9(x-1)} - \frac{4}{9(2x+1)} + \frac{2}{3(2x+1)^2}$$

$$\therefore \int_1^2 f(x)\,dx = \left[\frac{2}{9}\ln(x-1) - \frac{4}{9 \times 2}\ln(2x+1) - \frac{1}{3}(2x+1)^{-1}\right]_1^2$$

$$= \left(\frac{2}{9}\ln 1 - \frac{2}{9}\ln 5 - \frac{1}{3 \times 5}\right) - \left(\frac{2}{9}\ln 0 - \frac{2}{9}\ln 3 - \frac{1}{3 \times 3}\right)$$

$$= \left(\frac{2}{9} \times 0 - \frac{2}{9}\ln 5 - \frac{1}{15}\right) - \left(0 - \frac{2}{9}\ln 3 - \frac{1}{9}\right)$$

$$= \left(0 - \frac{2}{9}\ln 5 - \frac{1}{15}\right) - \left(\frac{-2}{9}\ln 3 - \frac{1}{9}\right)$$

$$= \frac{-2}{9}\ln 5 - \frac{1}{15} + \frac{2}{9}\ln 3 + \frac{1}{9}$$

$$= \frac{2}{9}\ln 3 - \frac{2}{9}\ln 5 + \frac{2}{45}$$ (Answer)

4. (a) $y = (\ln 2x)^2$

At P

$y = 0$

$\ln (2x)^2 = 0$

$(2x)^2 = 1$

$2x = \pm 1$

$x = \frac{1}{2}$ (+ve only) (Answer)

(b) (i) $$\frac{dy}{dx} = 2(\ln 2x) \times \left(\frac{1}{2x}\right) \times 2$$

$$= \frac{2}{x}\ln 2x = 0$$

$\ln 2x = 0$

$2x = 1$

$x = \frac{1}{2}$

(ii) $$\frac{dy}{dx} = \frac{2}{x}\ln 2x$$

$$\frac{d^2y}{dx^2} = \left(\frac{2}{x} \times \frac{2}{2x}\right) + \ln 2x \times \left(\frac{-2}{x^2}\right)$$

$$= \frac{2}{x^2} - \frac{2}{x^2}\ln 2x$$

$$= \frac{2}{x^2}(1 - \ln 2x)$$

(Answer)

(c) $\int_{\frac{1}{2}}^{e}(\ln 2x)^2 \, dx$

let $x = e^u$

$$= \int_{\ln(0.5)}^{1}(\ln 2e^u)^2 e^u \, du$$

$\frac{dx}{du} = e^u$

$$= \int_{\ln(0.5)}^{1} 2\ln 2e^u \times e^u \, du$$

when $x = e$

$e^u = e$

$u = 1$

$$= \int_{\ln(0.5)}^{1} 2e^u(\ln 2 + \ln e^u) \, du$$

$$= \int_{\ln(0.5)}^{1} 2e^u(\ln 2 + u) \, du$$

$e^u =$

$$= \int_{\ln(0.5)}^{e} 2\ln 2(e^u) + 2ue^u \, du$$

$u = \ln(0.5)$

(Answer)

5. (a) $\frac{dy}{dx} = \sqrt{1 + 3x}$

$$\frac{dy}{dx} = (1 + 3x)^{\frac{1}{2}}$$

$$dy = (1 + 3x)^{\frac{1}{2}} \, dx$$

Integrating both sides, we obtain

$$y = \frac{2}{3} \times \frac{(1 + 3x)^{\frac{3}{2}}}{3} + C$$

$$y = \frac{2}{9}(1 + 3x)^{\frac{3}{2}} + C$$

When $x = 1$, $y = -1$

$$-1 = \frac{2}{9}(1 + 3)^{\frac{3}{2}} + C$$

$$-1 = \frac{2}{9} \times 8 + C$$

$$C = 1 - \frac{16}{9} = -\frac{25}{9}$$

$\therefore$ equation of curve is

$$y = \frac{2}{9}(1 + 3x)^{\frac{3}{2}} - \frac{25}{9}$$ (Answer)

(b) When $x = 0$, $y = \frac{2}{9} - \frac{25}{9} = -\frac{23}{19}$

$\left(0, -\frac{23}{29}\right)$ coordinates of point of intersection between the curve and the y-axis.

When $y = 0$,

$$y = \frac{2}{9}(1 + 3x)^{\frac{3}{2}} - \frac{25}{9}$$

$$0 = 2(1 + 3x)^{\frac{3}{2}} - 25$$

$$2(1 + 3x)^{\frac{3}{2}} = 25$$

$(1 + 3x)^{\frac{3}{2}} = \frac{25}{2}$ squaring both sides

$$(1 + 3x)^3 = \frac{625}{4}$$

$$1 + 3x = \sqrt[3]{\frac{625}{4}}$$

$$3x = \sqrt[3]{\frac{625}{4}} - 1$$

$$x = \frac{1}{3}\left\{\sqrt[3]{\frac{625}{4}} - 1\right\}$$

$$= \frac{1}{3}(5.386 - 1)$$

$$= \frac{1}{3} \times 4.386$$

$$= 1.462$$

[redacted]

Point of intersection with the x-axis = (1.462, 0) (Answer)

6. (a) $x = \sqrt{y - 2}$

Squaring both sides, we have

$x^2 = y \quad 2$

$\therefore \quad y = x^2 + 2$

Coordinates of M, when $x = 0$

$y = 0 + 2 = 2$

$\therefore$ Coordinates of $M = (0, 2)$ (Answer)

When $x = 3$

$y = 9 + 2 = 11$

$\therefore$ Coordinates of $N = (3, 11)$ (Answer)

(b) Area of $MOPN = \int_0^3 y \, dx$

$$= \int_0^3 \left(x^2 + 2\right) dx$$

$$= \left[\frac{x^3}{3} + 2x\right]_0^3$$

$$= (9 + 6) - (0 + 0)$$

$= 15$ square units (Answer)

(c) Volume $= \int_0^3 \pi y^2 \, dx$

$$= \pi \int_0^3 \left(x^2 + 2\right)^2 dx$$

$$= \pi \int_0^3 \left(x^4 + 4x^2 + 4\right) dx$$

$$= \pi \left\{\left(3^4 + 4 \times 3^2 + 4\right) - \left(0^4 + 4 \times 0^2 + 4\right)\right\}$$

$$= \pi \left\{(81 + 36 + 4) \quad 4\right\}$$

$= 117\pi$ units^3 (Answer)

7. (a) $\int_0^{\frac{\pi}{6}} (\sin x + \cos 2x)\, dx$

$= \left[-\cos x + \frac{1}{2}\sin 2x\right]_0^{\frac{\pi}{6}}$

$= \left(-\frac{\sqrt{3}}{2} + \frac{1}{2} \times \frac{\sqrt{3}}{2}\right) - (-1 + 0)$

$= \frac{\sqrt{3}}{4} - \frac{\sqrt{3}}{2} + 1$

$= \frac{\sqrt{3} - 2\sqrt{3} + 4}{4}$

$= \frac{4 - \sqrt{3}}{4} = 1 - \frac{1}{4}\sqrt{3}$ (Answer)

(b) (i) Coordinates of C are $(1, e)$ and
Coordinates of D are $(3, e^3)$ (Answer)

(ii) Area $= \int_1^3 y\, dx$

$= \int_1^3 e^x\, dx$

$= \left[e^x\right]_1^3$

$= e^3 - e = 20.09 - 2.718$

$= 17.37$ sq. units (Answer)

(iii) Volume $= \int_1^3 \pi y^2\, dx$

$= \pi \int_1^3 e^{2x}\, dx$

$= \pi \left[\frac{e^{2x}}{2}\right]_1^3$

$= \pi \left\{\frac{e^6}{2} - \frac{e^2}{2}\right\}$

$= \pi \{201.7 - 3.69\}$

$= 622$ cubic units (Answer)

8. $\int_1^3 x^2 \ln x \, dx = \int_1^3 \ln x . \, x^2$

Using integration by parts

Let $\frac{dv}{dx} = x^2$ and let $u = \ln x$

then $v = \frac{1}{3} x^3$ then $\frac{du}{dx} = \frac{1}{x}$

Applying $\int u \frac{dv}{dx} \, dx = uv - \int v \frac{du}{dx} \, dx$

We obtain,

$$\int_1^3 x^2 \ln x \, dx = \left[uv - \int v \frac{du}{dx} \, dx\right]_1^3$$

$$= \left[\frac{1}{3} x^3 \ln x - \int \frac{1}{3} x^3 \times \frac{1}{x} \, dx\right]_1^3$$

$$= \left[\frac{1}{3} x^3 \ln x - \int \frac{1}{3} x^2 \, dx\right]_1^3$$

$$= \left[\frac{1}{3} x^3 \ln x - \frac{1}{9} x^3\right]_1^3$$

$$= (9 \ln 3 - 3) - \left(0 - \frac{1}{9}\right)$$

$$= 9 \ln 3 - 3 + \frac{1}{9}$$

$$= 9 \ln 3 - \frac{26}{9}$$

(Answer)

(a) $\int 3x - \frac{2}{x^2} \, dx$

$$= \int \left(3x - 2x^{-2}\right) dx$$

$$= \frac{3x^2}{2} + 2x^{-1} + c$$

$$= \frac{3}{2} x^2 + \frac{2}{x} + c$$

(Answer)

(b) $y = \sqrt{2x + 3}$

$y = (2x + 3)^{\frac{1}{2}}$

(i) $\dfrac{dy}{dx} = \dfrac{1}{2}(2x + 3)^{\frac{1}{2}} \times 2 = \dfrac{1}{(2x + 3)^{\frac{1}{2}}} = \dfrac{1}{\sqrt{4 + 3}}$, when $x = 2$

$$= \frac{1}{\sqrt{7}} = \frac{\sqrt{7}}{7}$$ (Answer)

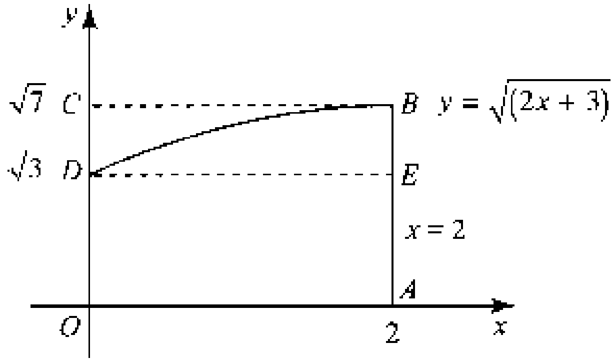
(ii) $\dfrac{dx}{dt} = 0.02$

$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$$

$$\frac{\sqrt{7}}{7} = \frac{dy}{dt} \times \frac{1}{0.02}$$

$$\frac{dy}{dt} = \frac{0.02 \times \sqrt{7}}{7} = 0.0076 \text{ units / sec.}$$ (Answer)

(iii)



$$\text{Area} = \int_0^2 y \, dx$$

$$= \int_0^2 (2x+3)^{\frac{1}{2}} \, dx$$

$$= \frac{2}{3}(2x+3)^{\frac{3}{2}} \times \frac{1}{2}$$

$$= \left[\frac{1}{3}(2x+3)^{\frac{3}{2}}\right]_0^2$$

$$= \frac{1}{3}(7)^{\frac{3}{2}} - \frac{1}{3}(3)^{\frac{3}{2}}$$

$$= 6.17 - 1.73$$

$$= 4.44 \text{ units}^2$$

(Answer)

(iv) Volume when *OAED* is rotated about

$$y\text{-axis} = \pi r^2 h \quad \left(r = 2 \text{ and } h = \sqrt{3}\right)$$

$$= \pi \times 4 \times \sqrt{3}$$

$$= 4\sqrt{3}\pi \text{ cu. units}$$

Volume of *BDE* = *DEBC* - *DBC*

$$= \pi r^2 h - \int_{\sqrt{3}}^{\sqrt{7}} \pi x^2 \, dy$$

$$= \pi \times 4 \times \left(\sqrt{7} - \sqrt{3}\right) - \pi \int_{\sqrt{3}}^{\sqrt{7}} \left(y^2 - 3\right)^2 dy$$

$$= 4\pi\left(\sqrt{7} - \sqrt{3}\right) - \frac{\pi}{4}\int_{\sqrt{3}}^{\sqrt{7}} \left(y^4 - 6y^2 + 9\right) dy$$

$$= 4\pi\left(\sqrt{7} - \sqrt{3}\right) - \frac{\pi}{4}\left[\frac{y^5}{5} - \frac{6y^3}{3} + 9y\right]_{\sqrt{3}}^{\sqrt{7}}$$

$$= 4\pi\left(\sqrt{7} - \sqrt{3}\right) - \frac{\pi}{4}\left\{\left(\frac{7^{\frac{5}{2}}}{5} - 2(7)^{\frac{3}{2}} + 9\sqrt{7}\right)\right.$$

$$\left. - \left(\frac{3^{\frac{5}{2}}}{5} - 2(3)^{\frac{3}{2}} + 9\sqrt{3}\right)\right\}$$

$$= 4\pi\left(\sqrt{7} - \sqrt{3}\right) - \frac{\pi}{4}\left\{(25.9 - 37 + 23.8) - (3.1 - 10.4 + 15.6)\right\}$$

$$= 11.5 - \frac{\pi}{4}\left\{12.7 - 8.3\right\}$$

$= 11.5 - 3.5 = 8$ cu. units (Answer)

Therefore volume generated by this region is $\left(4\sqrt{3}\pi + 8\right)$ units^3.

10. (a) At A, $x = 3, y = e^{-3}$

$\therefore$ coordinates of $A = (3, e^{-3})$ (Answer)

At B, $x = 3, y = 0$

$\therefore$ coordinates of $B = (3, 0)$ (Answer)

At C, $x = 0, y = e^{-0} = e^{0} = 1$

$\therefore$ coordinates of $C = (0, 1)$ (Answer)

(b) Area $= \int_0^3 y \, dx$

$$= \int_0^3 e^{-x} \, dx$$

$$= \left[-e^{-x}\right]_0^3$$

$$= \left(-e^{-3}\right) - \left(-e^{-0}\right)$$

$$= -\frac{1}{e^3} - \left(-e^{0}\right)$$

$$= -\frac{1}{e^3} - -1$$

$$= \left(-\frac{1}{e^3} + 1\right) \quad \text{sq. units}$$ (Answer)

(c) Volume generated $= \pi\int_0^3 y^2\,dx$

$$= \pi\int_0^3 e^{-2x}\,dx$$

$$= \pi\left[-\frac{e^{-2x}}{2}\right]_0^3$$

$$= -\frac{1}{2}\pi\left[e^{-2x}\right]_0^3$$

$$= -\frac{1}{2}\pi\left[e^{-6} - e^{-2(0)}\right]$$

$$= -\frac{1}{2}\pi\left[e^{-6} - e^{0}\right]$$

$$= -\frac{1}{2}\pi\left[e^{-6} - 1\right]$$

$$= 1.57$$ (Answer)

11. (a)

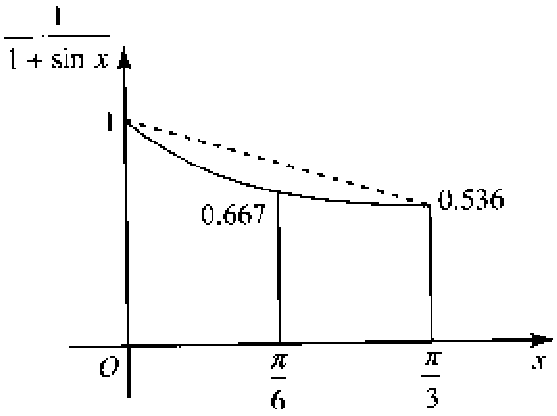
When x	=	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$
$\frac{1}{1 + \sin x}$	=	1	$\frac{2}{3}$	$\frac{2}{2 + \sqrt{3}}$
	=	1	0.667	0.536

Estimated area $= \frac{\pi}{6}\left\{\frac{1}{2}\left(1 + 0.536\right) + 0.667\right\}$

$$= \frac{\pi}{6}\{1.435\}$$

$$= 0.751 \text{ sq. units}$$ (Answer)

(b)



The trapezium rule gives an over-estimate of the true value because the trapezium includes a segment which does not form part of the area under the curve.

(Answer)

12. $\int_0^3 2xe^{3x}dx$ using integration by parts

Let $u = 2x$ and let $\frac{dv}{dx} = e^{3x}$

then $\frac{du}{dx} = 2$ then $V = \frac{1}{3}e^{3x}$

$$\int_0^3 2xe^{3x}dx = [uv]_0^3 - \int_0^3 \frac{du}{dx}\,dx$$

$$= \left[\frac{2}{3}xe^{3x}\right]_0^3 - \int_0^3 \frac{2}{3}e^{3x}dx$$

$$= \left[\frac{2}{3}xe^{3x} - \frac{2}{9}e^{3x}\right]_0^3$$

$$= \left(\frac{2}{3} \times 3 \times e^9 - \frac{2}{9}e^9\right) - \left(\frac{2}{3} \times 0 \times e^0 - \frac{2}{9}e^0\right)$$

$$= 2e^9 - \frac{2}{9}e^9 - 0 + \frac{2}{9}$$

$$= 16\,206 - 1\,801 + \frac{2}{9}$$

$$= 14\,405\frac{2}{9}$$

(Answer)

[redacted]

13. (a) $\frac{dy}{dx} = 2x^2 - 3x + 1$

$dy = (2x^2 - 3x + 1)\,dx$

Integrating both sides, we have

$y = \frac{2}{3}x^3 - \frac{3}{2}x^2 + x + c$

When $x = 0$, $y = 3$

$\therefore \quad c = 3$

hence equation of curve is

$y = \frac{2}{3}x^3 - \frac{3}{2}x^2 + x + 3$ (Answer)

(b) If the gradient is positive,

$\frac{dy}{dx} > 0$

i.e. $(2x^2 - 3x + 1) > 0$

$(2x - 1)(x - 1) > 0$

CV of $x = \frac{1}{2}$ or 1

When x =		$\frac{1}{2}$		1	
f (x) =	$+$	0	$-$	0	$+$

$\therefore \quad x > 1$ or $x < \frac{1}{2}$ (Answer)

14. (a) $y = \frac{9}{2x + 3}$

$y = 9(2x + 3)^{1}$

$\frac{dy}{dx} = -9(2x + 3)^{-2} \times 2$

$= -18(2x + 3)^{2}$

At R, $x = 3$

$$\therefore \quad \frac{dy}{dx} = \frac{18}{9^2} = \frac{-18}{81} = \frac{-2}{9}$$

Equation of tangent at R (3, 1) is

$$\frac{y - 1}{x - 3} = \frac{-2}{9}$$

$$9y - 9 = -2x + 6$$

$$9y = -2x + 15$$

(Answer)

(b) Volume $OPRQ$ = Volume $OARQ$ + Volume APR

Volume $OPRQ$

$2xy + 3y = 9$

$2xy = 9 - 3y$

$x = \frac{9 - 3y}{2y}$

$$= \text{Vol } OARQ + \int_1^3 \pi x^2 dy$$

$$= (\pi \times 9 \times 1) + \pi \int_1^3 \left(\frac{9 - 3y}{2y}\right)^2 dy$$

$$= 9\pi + \pi \int_1^3 \left(\frac{9}{2y} - \frac{3}{2}\right)^2 dy$$

$$= 9\pi + \pi \int_1^3 \left(\frac{81}{4y^2} - \frac{27}{2y} + \frac{9}{4}\right) dy$$

$$= 9\pi + \pi \int_1^3 \frac{81}{4} y^{-2} - \frac{27}{2} y^{-1} + \frac{9}{4} dy$$

$$= 9\pi + \pi \left[\frac{-81}{4} y^{-1} - \frac{27}{2} \ln y + \frac{9}{4} y\right]_1^3$$

$$= 9\pi + \pi \left[\frac{-81}{4y} - \frac{27}{2} \ln y + \frac{9}{4} y\right]_1^3$$

$$= 9\pi + \pi \left[\left(\frac{-81}{4(3)} - \frac{27}{2} \ln 3 + \frac{9}{4} \times 3\right) - \left(\frac{-81}{4} - 0 + \frac{9}{4}\right)\right]$$

$$= 9\pi + \pi \left(\frac{-81}{12} + \frac{27}{4} - \frac{27}{2} \ln 3 + \frac{81}{4} - \frac{9}{4}\right)$$

$$= 9\pi + \pi\left(\frac{18}{4} - \frac{27}{2}\ln 3\right)$$

$$= 9\pi + \frac{18\pi}{4} - \frac{27}{2}\pi\ln 3$$

$$= \frac{27}{2}\pi - \frac{27}{2}\pi\ln 3$$

$$= \frac{27}{2}\pi\left(1 - \ln 3\right)$$

(Answer)

15. (a) Let $u = \sin x$ and $y = \frac{\sin x}{\cos x} = \frac{u}{v}$

and $v = \cos x$

$$\frac{du}{dx} = \cos x \qquad \frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

$$\frac{dv}{dx} = -\sin x \qquad = \frac{\cos x \times \cos x - \sin x(-\sin x)}{\cos^2 x}$$

$$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x}$$

$$= \frac{1}{\cos^2 x}$$

$$= \sec^2 x$$

(Answer)

(b) $$\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \sec^2 x \, dx$$

$$= \left[\tan x\right]_{\frac{\pi}{4}}^{\frac{\pi}{3}}$$

$$= \left(\sqrt{3} - 1\right)$$

(Answer)

(c) $y = \cot x$

$$y = \frac{\cos x}{\sin x}$$

Let $u = \cos x$ and let $v = \sin x$

then $\frac{du}{dx} = -\sin x$ then $\frac{dv}{dx} = \cos x$

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

$$= \frac{(\sin x \times -\sin x) - (\cos x \times \cos x)}{\sin^2 x}$$

$$= \frac{-\sin^2 x - \cos^2 x}{\sin^2 x}$$

$$= \frac{-(\sin^2 x + \cos^2 x)}{\sin^2 x}$$

$$= \frac{-1}{\sin^2 x}$$

$$= -\operatorname{cosec}^2 x$$ (Answer)

UNIT 14

VECTORS

Suggested Solutions

1. For line a :

(a) $\mathbf{r} = \mathbf{i} - 2\mathbf{k} + \lambda(2\mathbf{i} + \mathbf{j} + 3\mathbf{k})$

$$\mathbf{r} = \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 + 2\lambda \\ \lambda \\ 2 + 3\lambda \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$x = 1 + 2\lambda$$
$$2\lambda = x - 1$$
$$\lambda = \frac{x - 1}{2}$$

$$\lambda = y$$
$$z = 2 + 3\lambda$$

For line b :

$\mathbf{r} = 6\mathbf{i} - 5\mathbf{j} + 4\mathbf{k} + \varphi(\mathbf{i} - 2\mathbf{j} + \mathbf{k})$

$$\mathbf{r} = \begin{pmatrix} 6 \\ -5 \\ 4 \end{pmatrix} + \varphi \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} = \begin{pmatrix} 6 + \varphi \\ -5 - 2\varphi \\ 4 + \varphi \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

Comparing x, y and z from line a and b.

$\therefore \quad 1 + 2\lambda = 6 + \varphi \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad \ldots$ ①

$2\lambda - \varphi = 6 - 1 \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad \ldots$ ①

$2\lambda - \varphi = 5 \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad \ldots$ ①

$\lambda = -5 - 2\varphi$

$\lambda + 2\varphi = -5 \quad \ldots\ldots \quad \ldots\ldots$ ②

$z = 4 + \varphi = -2 + 3\lambda$

$3\lambda - \varphi = 6$ -- ③

solving (1) and (2), we obtain

$2\lambda - \varphi = 5$ -- ①

$\lambda + 2\varphi = -5$ -- ②

$$4\lambda - \cancel{2\varphi} = 10 \quad \text{-----------} \quad ① \times 2$$

$$+ \quad \underline{\lambda + \cancel{2\varphi} = -5} \quad \text{-----------} \quad ②$$

$5\lambda = 5$

$\lambda = 1$ -- Back in ②

$1 + 2\varphi = -5$

$2\varphi = -6$

$\varphi = -3$

Substituting in (3), we have

$3\lambda - \varphi = 3 \times 1 - -3$

$= 3 + 3 = 6$, which is true.

∴ lines a and b intersects, as the equations are consistent. (Answer)

(b) The point of intersection is when $\lambda = 1$ and $\varphi = -3$

Since $x = 1 + 2\lambda = 3$

$y = \lambda = 1$

and $z = -2 + 3\lambda = 1$

i.e. point of intersection has position vector $\begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix}$ (Answer)

(c) Let normal to the plane be **n** and let $\mathbf{n} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$

$(2\mathbf{i} + \mathbf{j} + 3\mathbf{k})$ is the direction of line a.

$$\therefore \quad \mathbf{n} \cdot \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} = 0$$

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} = 0$$

$$2a + b + 3c = 0 \quad \text{-------------------------------------} \quad ①$$

$(\mathbf{i} - 2\mathbf{j} + \mathbf{k})$ is the direction of line b.

$$\therefore \quad \mathbf{n} \cdot \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} = 0$$

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} = 0$$

i.e., $$a - 2b + c = 0 \quad \text{-------------------------------------} \quad ②$$

Solving (1) and (2), we get

$a = -\frac{7}{5}c$, $b = -\frac{1}{5}c$, and $c = c$

$$\therefore \quad \mathbf{n} = \begin{pmatrix} -\frac{7}{5} \\ -\frac{1}{5} \\ 1 \end{pmatrix}$$

Multiply by -5, we have

$$\mathbf{n} = \begin{pmatrix} 7 \\ 1 \\ -5 \end{pmatrix}$$

$\therefore$ Cartesian equation of plane =

$$\begin{pmatrix} 7 \\ 1 \\ -5 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} = d$$

$$7x + y - 5z = d$$

Taking any point on either line a or line b, i.e. (1, 0, −2)

$7 \times 1 + 0 - 5 \times -2 = d$

$d = 7 + 10 = 17$

$\therefore$ Required equation $= 7x + y - 5z = 17$ (Answer)

2. $\overline{\mathbf{OP}} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$, $\overline{\mathbf{OQ}} = \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix}$, $\overline{\mathbf{OR}} = \begin{pmatrix} 3 \\ 1 \\ p \end{pmatrix}$ and $\overline{\mathbf{OS}} = \begin{pmatrix} 1 \\ 0 \\ q \end{pmatrix}$

(a) $\overline{\mathbf{PQ}} = \mathbf{q} - \mathbf{p}$

$$= \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -3 \\ 1 \end{pmatrix}$$

$$|\overline{\mathbf{PQ}}| = \sqrt{(1)^2 + (-3)^2 + (1)^2} = \sqrt{1+9+1} = \sqrt{11}$$

$\therefore$ unit vector in the direction of $\overline{\mathbf{PQ}}$ is

$$\frac{1}{\sqrt{11}}(\mathbf{i} - 3\mathbf{j} + \mathbf{k})$$ (Answer)

(b) $\overline{\mathbf{PO}} \,.\, \overline{\mathbf{OR}} = \begin{pmatrix} -1 \\ -2 \\ -1 \end{pmatrix} . \begin{pmatrix} 3 \\ 1 \\ p \end{pmatrix} = 0$

$$-3 - 2 - p = 0$$

$$-p = 5$$

$$p = -5$$ (Answer)

(c) $\overline{\mathbf{OS}} = \begin{pmatrix} 1 \\ 0 \\ q \end{pmatrix}$

$$|\overline{\mathbf{OS}}| = \sqrt{(1)^2 + (0)^2 + (q)^2} = \sqrt{5}$$

$$\sqrt{1+q^2} = \sqrt{5}$$

$$1 + q^2 = 5$$

$$q^2 = 4$$

$$q = \pm 2$$

(Answer)

3. (a)

$$\overline{\mathbf{OX}} = \overline{\mathbf{OP}} + \overline{\mathbf{PS}} + \overline{\mathbf{SX}}$$

$$= 8\mathbf{i} + 8\mathbf{k} + 4\mathbf{j}$$

$$= 8\mathbf{i} + 4\mathbf{j} + 8\mathbf{k}$$

$$= 4(2\mathbf{i} + \mathbf{j} + 2\mathbf{k})$$

(Answer)

(b)

$$\mathbf{O}\overline{\mathbf{Y}} = \mathbf{O}\overline{\mathbf{V}} + \mathbf{V}\overline{\mathbf{Y}}$$

$$= 8\mathbf{k} + 4\mathbf{i}$$

$$= 4\mathbf{i} + 8\mathbf{k}$$

$$= 4(\mathbf{i} + 2\mathbf{k})$$

(Answer)

(c)

$$\mathbf{X}\overline{\mathbf{O}} \;.\; \mathbf{O}\vec{\mathbf{Y}} = \begin{pmatrix} 8 \\ -4 \\ -8 \end{pmatrix} . \begin{pmatrix} 4 \\ 0 \\ 8 \end{pmatrix} = -32 + 0 + 64 = 96$$

$$\mathbf{X}\overline{\mathbf{O}} \;.\; \mathbf{O}\vec{\mathbf{Y}} = |\mathbf{X}\overline{\mathbf{O}}| \times |\mathbf{O}\overline{\mathbf{Y}}| \times \cos\theta$$

$$= \sqrt{(64+16+64)} \times \sqrt{(16+64)} \times \cos\theta = -96$$

$$= \sqrt{144} \times \sqrt{80} \times \cos\theta = -96$$

$$\cos\theta = \frac{-96}{12 \times \sqrt{80}}$$

$$= \frac{-8}{\sqrt{80}} = \frac{-2}{\sqrt{5}}$$

$$\therefore \quad \theta = 26.6°$$

(Answer)

4. $\overline{\mathbf{OP}} = 10\mathbf{i} + 3\mathbf{k}$

$\overline{\mathbf{OQ}} = 7\mathbf{i} - 3\mathbf{j} + 6\mathbf{k}$

(a) Direction of $\overline{\mathbf{PQ}} = \mathbf{q} - \mathbf{p}$

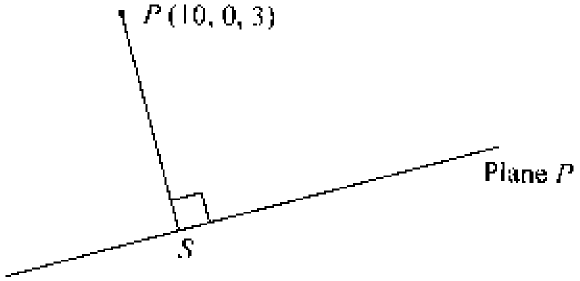
$$= -3\mathbf{i} - 3\mathbf{j} + 3\mathbf{k}$$

$$\bar{\mathbf{PQ}} \,.\, \mathbf{n} = \begin{pmatrix} -3 \\ -3 \\ 3 \end{pmatrix} . \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix} = -9 + 3 + 6 = 0$$

Since **n** is perpendicular to both $\overline{\mathbf{PQ}}$ and the plane.

$\therefore$ $\overline{\mathbf{PQ}}$ is parallel to the plane. (Answer)

(b)



Line *PS* has equation $\mathbf{r} = 10\mathbf{i} + 3\mathbf{k} + \lambda(3\mathbf{i} - \mathbf{j} + 2\mathbf{k})$

$\therefore$ at *S*, $3(10 + 3\lambda) + -1(0 - \lambda) + 2(3 + 2\lambda) = 8$

$$30 + 9\lambda + \lambda + 6 + 4\lambda = 8$$

$$14\lambda = 8 - 36$$

$$\lambda = \frac{-28}{14} = -2$$

Hence position vector of $S = \begin{pmatrix} 10 + 3\lambda \\ -\lambda \\ 3 + 2\lambda \end{pmatrix} = \begin{pmatrix} 4 \\ 2 \\ -1 \end{pmatrix}$

$\therefore$ vector equation of the line that passes through *S* and which is parallel to *PQ* is

$$\mathbf{r} = (4\mathbf{i} + 2\mathbf{j} - \mathbf{k}) + \mu(-\mathbf{i} - \mathbf{j} + \mathbf{k})$$

or $\mathbf{r} = (4\mathbf{i} + 2\mathbf{j} - \mathbf{k}) + \mu(\mathbf{i} + \mathbf{j} - \mathbf{k})$ (Answer)

5. (a) $\overline{\mathbf{XS}} = \overline{\mathbf{XQ}} + \overline{\mathbf{QS}}$

$= 18\mathbf{j} + 12\mathbf{i}$

$= 12\mathbf{i} + 18\mathbf{j}$ (Answer)

$\overline{\mathbf{XO}} = \overline{\mathbf{XL}} + \overline{\mathbf{LO}}$

$= -18\mathbf{j} + 6\mathbf{i}$

$= 6\mathbf{i} - 18\mathbf{j}$ (Answer)

(b) $$\overline{\mathbf{XS}} \,.\, \overline{\mathbf{XO}} = \begin{pmatrix} 12 \\ 18 \\ 0 \end{pmatrix} . \begin{pmatrix} 6 \\ -18 \\ 0 \end{pmatrix}$$

$= -72 - 324 + 0 = -396$ (Answer)

$$\overline{\mathbf{XS}} \,.\, \overline{\mathbf{XO}} = \sqrt{|\overline{\mathbf{XS}}|} \times \sqrt{|\overline{\mathbf{XO}}|} \times \cos\theta = -396$$

$$= \sqrt{468} \times \sqrt{360} \times \cos\theta = -396$$

$$\cos\theta = \frac{-396}{\sqrt{468} \times \sqrt{360}} = -0.964$$

$\theta = 164.6°$ (Answer)

6. Plane $p : x + 3y - 2z = 3$

$\therefore$ normal $\mathbf{n}$ to the plane $= \begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix}$

(a) Equation of line **XY** :

$$\mathbf{r} = \begin{pmatrix} 4 \\ -1 \\ 3 \end{pmatrix} + t\begin{pmatrix} -3 \\ 1 \\ -2 \end{pmatrix}$$

Any point on the line **XY** has coordinates $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4 \quad 3t \\ 1 + t \\ 3 - 2t \end{pmatrix}$

If this point lies in the plane, then

$$\begin{aligned}(4-3t)+3(-1+t)-2(3-2t) &= 3\\ 4 \quad 3t-3+3t-6+4t &= 3\\ 4t-5 &= 3\\ 4t &= 8\\ t &= 2\end{aligned}$$

$\therefore$ Point of intersection $M = \begin{pmatrix}4-3t\\-1+t\\3-2t\end{pmatrix} = \begin{pmatrix}-2\\1\\-1\end{pmatrix}$

hence position vector of $M = -2\mathbf{i} + \mathbf{j} - \mathbf{k}$ (Answer)

(b) If $\mathbf{XY} = \begin{pmatrix}3\\1\\-2\end{pmatrix}$ and $\mathbf{n}$ to p is $\begin{pmatrix}1\\3\\-2\end{pmatrix}$,

$\begin{pmatrix}-3\\1\\2\end{pmatrix}$ and $\begin{pmatrix}1\\3\\-2\end{pmatrix}$ are non-zero, non parallel vectors then

$$\begin{pmatrix}1\times-2 - -2\times3\\ -2\times1 - -3\times-2\\ -3\times3 \quad 1\times1\end{pmatrix} = \begin{pmatrix}-2+6\\-2 \quad 6\\9-1\end{pmatrix} = \begin{pmatrix}4\\-8\\-10\end{pmatrix} = -2\begin{pmatrix}-2\\4\\5\end{pmatrix}$$

$\therefore$ $\begin{pmatrix}-2\\4\\5\end{pmatrix}$ is non-zero and perpendicular to both.

i.e. $\begin{pmatrix}x\\y\\z\end{pmatrix} \cdot \begin{pmatrix}-2\\4\\5\end{pmatrix} = \begin{pmatrix}4\\1\\3\end{pmatrix} \cdot \begin{pmatrix}-2\\4\\5\end{pmatrix}$, since $\mathbf{r} \cdot \mathbf{n} = \mathbf{a} \cdot \mathbf{n}$

$\therefore$ Cartesian equation of q is $-2x + 4y + 5z = 3$ (Answer)

7. $\mathbf{p} = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}, \ \mathbf{q} = \begin{pmatrix} 3 \\ -2 \\ 5 \end{pmatrix}, \ \mathbf{r} = \begin{pmatrix} x \\ -x \\ x-1 \end{pmatrix}$

(a)

$$\mathbf{p} \cdot \mathbf{q} = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ -2 \\ 5 \end{pmatrix} = 3 - 4 - 5 = -6$$

$$= |\mathbf{p}| \times |\mathbf{q}| \times \cos\theta = \sqrt{6} \times \sqrt{38} \cos\theta$$

$$\cos\theta = \frac{-6}{\sqrt{6} \times \sqrt{38}} = -\frac{\sqrt{6}}{\sqrt{38}}$$

$$\theta = 66.6°$$

(Answer)

(b)

$$\mathbf{q} \cdot \mathbf{r} = \begin{pmatrix} 3 \\ -2 \\ 5 \end{pmatrix} \cdot \begin{pmatrix} x \\ -x \\ x-1 \end{pmatrix} = (3x + 2x + 5x - 5)$$

$$= (10x - 5)$$

$$\mathbf{q} \cdot \mathbf{r} = |\mathbf{q}| \times |\mathbf{r}| \times \cos\theta$$

$$\therefore \quad \theta = \cos^{-1}\left\{\frac{(10x-5)}{|\mathbf{q}| \times |\mathbf{r}|}\right\}$$

(Answer)

(c) If **q** and **r** are perpendicular, then $\theta = 90°$

$$\therefore \quad 10x - 5 = 0$$

$$x = \frac{1}{2}$$

(Answer)

8. (a) $\overline{\mathbf{AC}} = \mathbf{c} - \mathbf{a} = \begin{pmatrix} 0 \\ -1 \\ 3 \end{pmatrix}$

$\overline{\mathbf{BD}} = \mathbf{d} - \mathbf{b} = \begin{pmatrix} -3 \\ -4 \\ 3 \end{pmatrix}$

$$\overline{\mathbf{AC}} \,.\, \overline{\mathbf{BD}} = \begin{pmatrix} 0 \\ -1 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} -3 \\ -4 \\ 3 \end{pmatrix} = 0 + 4 + 9 = 13$$

$$= \sqrt{1+9} \times \sqrt{9+16+9} \times \cos\theta$$

$$\cos\theta = \frac{13}{\sqrt{10} \times \sqrt{34}} = 0.705$$

$$\theta = 45.2^\circ$$ (Answer)

(b) Equation of AB is $\mathbf{r} = \begin{pmatrix} 3 \\ 0 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 3 \\ -2 \end{pmatrix}$

Equation of CD is $\mathbf{r} = \begin{pmatrix} 3 \\ -1 \\ -1 \end{pmatrix} + \mu \begin{pmatrix} 4 \\ 0 \\ -2 \end{pmatrix}$

For AB, $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 - \mu \\ 3\lambda \\ -2 - 2\lambda \end{pmatrix}$

For CD, $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 - 4\mu \\ -1 \\ 1 - 2\mu \end{pmatrix}$

$3 - \lambda = 3 - 4\mu$

$4\mu - \lambda = 0$ ………………………………… ①

$3\lambda = -1$ ………………………………… ②

$1 \quad 2\mu = -2 - 2\lambda$

$-2\mu + 2\lambda = \quad 3$ ………………………………… ③

$3\lambda = \quad 1$ ………………………………… ②

$\lambda = -\frac{1}{3}$

$$4\mu - \lambda = 0 \quad \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \text{①}$$

$$4\mu + \frac{1}{3} = 0$$

$$4\mu = -\frac{1}{3}$$

$$\mu = -\frac{1}{12}$$

Substituting in (3), we have

$$-2\mu + 2\lambda$$

$$= -2 \times -\frac{1}{12} + 2 \times -\frac{1}{3}$$

$$= \frac{1}{6} - \frac{2}{3}$$

$$= -\frac{1}{2}$$

∴ the 2 lines AB and CD do not intersect. (Answer)

(c) $\overrightarrow{\mathbf{OM}} = 3\mathbf{i} + 2\mathbf{j} + \mathbf{k}$

$\overrightarrow{\mathbf{BC}} = \mathbf{i} - 4\mathbf{j} + 5\mathbf{k}$

Equation of $\overline{\mathbf{BC}}$:

$$\mathbf{r} = \begin{pmatrix} 2 \\ 3 \\ -4 \end{pmatrix} + s\begin{pmatrix} 1 \\ -4 \\ 5 \end{pmatrix}$$

any point Q on $\overrightarrow{\mathbf{BC}}$ is $(2 + s)\mathbf{i} + (3 - 4s)\mathbf{j} + (-4 + 5s)\mathbf{k}$

$$\overline{\mathbf{MQ}} = \mathbf{q} - \mathbf{m}$$

$$= (2 + s - 3)\mathbf{i} + (3 - 4s - 2)\mathbf{j} + (-4 + 5s - 1)\mathbf{k}$$

$$= (s - 1)\mathbf{i} + (-4s + 1)\mathbf{j} + (5s - 5)\mathbf{k}$$

$$\overline{\mathbf{MQ}} \cdot (\mathbf{i} - 4\mathbf{j} + 5\mathbf{k}) = 0$$

$$\begin{pmatrix} s-1 \\ -4s+1 \\ 5s-5 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -4 \\ 5 \end{pmatrix} = 0$$

$$(s-1) \times 1 + (-4s+1) \times -4 + (5s-5) \times 5 = 0$$

$$s - 1 + 16s - 4 + 25s - 25 = 0$$

$$42s - 30 = 0$$

$$42s = 30$$

$$s = \frac{30}{42} = \frac{5}{7}$$

Hence $\overline{\mathbf{MQ}} = \left(\frac{5}{7} - 1\right)\mathbf{i} + \left(-\frac{20}{7} + 1\right)\mathbf{j} + \left(\frac{25}{7} - 5\right)\mathbf{k}$

$$= -\frac{2}{7}\mathbf{i} - \frac{13}{7}\mathbf{j} - \frac{10}{7}\mathbf{k}$$

$$\therefore \quad \left|\overline{\mathbf{MQ}}\right| = \sqrt{\left(\frac{-2}{7}\right)^2 + \left(\frac{-13}{7}\right)^2 + \left(\frac{-10}{7}\right)^2}$$

$$= \sqrt{\frac{4}{49} + \frac{169}{49} + \frac{100}{49}}$$

$$= \sqrt{\frac{273}{49}}$$

$$= \frac{\sqrt{273}}{7}$$ (Answer)

9. $\overline{\mathbf{OP}} = 2\mathbf{i} - 2\mathbf{j} + 5\mathbf{k}$

$\overline{\mathbf{OQ}} = 3\mathbf{i} + 2\mathbf{k}$

$= -2\mathbf{i} + 10\mathbf{j} + 7\mathbf{k}$

$= 2\mathbf{j} + 7\mathbf{k}$

(a) $$\overline{\mathbf{PQ}} = \mathbf{q} - \mathbf{p} = \begin{pmatrix}3\\0\\2\end{pmatrix} - \begin{pmatrix}2\\-2\\5\end{pmatrix} = \begin{pmatrix}1\\2\\-3\end{pmatrix}$$

$$\overline{\mathbf{PS}} = \mathbf{s} - \mathbf{p} = \begin{pmatrix}0\\2\\7\end{pmatrix} - \begin{pmatrix}2\\-2\\5\end{pmatrix} = \begin{pmatrix}-2\\4\\2\end{pmatrix}$$

$$\overline{\mathbf{PQ}} \,.\, \overline{\mathbf{PS}} = \begin{pmatrix}1\\2\\3\end{pmatrix} . \begin{pmatrix}-2\\4\\2\end{pmatrix} = 2 + 8 - 6 = 0$$

$\therefore$ $\overline{\mathbf{PQ}}$ is _ to $\mathbf{PS}$. (Answer)

(b) $$\overline{\mathbf{PS}} = \begin{pmatrix}-2\\4\\2\end{pmatrix} = 2\begin{pmatrix}-1\\2\\1\end{pmatrix}$$

$$\overline{\mathbf{QR}} = \mathbf{r} - \mathbf{q} = \begin{pmatrix}-2\\10\\7\end{pmatrix} - \begin{pmatrix}3\\0\\2\end{pmatrix} = \begin{pmatrix}-5\\10\\5\end{pmatrix} = 5\begin{pmatrix}-1\\2\\1\end{pmatrix}$$

Since both are represented by the same vector $\begin{pmatrix}-1\\2\\1\end{pmatrix}$.

$\therefore$ $\overline{\mathbf{PS}}$ and $\overline{\mathbf{QR}}$ are parallel. (Answer)

$$|\overline{\mathbf{PS}}| : |\overline{\mathbf{QR}}|$$

$$\sqrt{4+16+4} \; : \; \sqrt{25+100+25}$$

$$\sqrt{24} \; : \; \sqrt{150} \qquad (\div 6)$$

we obtain $\sqrt{4} \; : \; \sqrt{25}$

$= \; 2 \; : \; 5$ (Answer)

10. (a) p and q have equations

$$2x - y + 3z = 4 \quad ①$$

and $x + 2y - 4z = 6$ respectively. ②

Solve (1) and (2), by eliminating x, we get

$$y = \frac{8 + 11z}{5}$$

Solve (1) and (2), by eliminating y, we have

$$x = \frac{14 - 2z}{5}$$

When $z = t$, common point in both planes will be $\left(\frac{14-2t}{5}, \frac{8+11t}{5}, t\right)$

or $(14 - 2t, 8 + 11t, 5t)$

When $t = 0$, common point in both planes will also be (14, 8, 0), called B.

And when $t = 1$, the point will be (12, 19, 5), called B.

Equation of common line is

$$\mathbf{r} = \begin{pmatrix} 14 \\ 8 \\ 0 \end{pmatrix} + t\begin{pmatrix} -2 \\ 11 \\ 5 \end{pmatrix}$$

(Answer)

(b) Angle between the 2 planes = angle between the 2 normals.
i.e. between $(2\mathbf{i} - \mathbf{j} + 3\mathbf{k})$ and $(\mathbf{i} + 2\mathbf{j} - 4\mathbf{k})$

$$\cos\theta = \frac{2 - 2 - 12}{\sqrt{4+1+9} \times \sqrt{1+4+16}}$$

$$= \frac{-12}{\sqrt{14} \times \sqrt{21}} = -0.6999$$

(Answer)

$\theta = 45.6°$ or $134.4°$

11. (a) $\overline{\mathbf{YZ}} = \overline{\mathbf{YA}} + \overline{\mathbf{AC}} + \overline{\mathbf{CZ}}$

$$= -10\mathbf{k} - 5\mathbf{i} + \frac{15}{2}\mathbf{k}$$

$$= -5\mathbf{i} - \frac{5}{2}\mathbf{k}$$

(Answer)

$\overline{\mathbf{XZ}} = \overline{\mathbf{XB}} + \overline{\mathbf{BC}} + \overline{\mathbf{CZ}}$

$$= -5\mathbf{k} - 5\mathbf{j} + \frac{15}{2}\mathbf{k}$$

$$= -5\mathbf{j} + \frac{5}{2}\mathbf{k}$$

(Answer)

(b)
$$\overline{\mathbf{YZ}} \,.\, \overline{\mathbf{XZ}} = \begin{pmatrix} -5 \\ 0 \\ -\frac{5}{2} \end{pmatrix} \cdot \begin{pmatrix} 0 \\ -5 \\ \frac{5}{2} \end{pmatrix} = 0 + 0 + \frac{-25}{4}$$

$$= \frac{25}{4} = -6\tfrac{1}{4}$$

(Answer)

$$\cos X\hat{Z}Y = \frac{-6\frac{1}{4}}{\sqrt{25+0+\frac{25}{4}} \times \sqrt{0+25+\frac{25}{4}}}$$

$$= \frac{-6\frac{1}{4}}{\frac{125}{4}}$$

$$= \frac{25}{4} \times \frac{4}{125} = \frac{1}{5} = -0.2$$

$\therefore \quad X\hat{Z}Y = 101.5°$

(Answer)

12. $\mathbf{O\overline{P}} = 2\mathbf{i} - 3\mathbf{j} + \mathbf{k} = \mathbf{p}$

$\mathbf{O\overline{Q}} = 3\mathbf{i} + 4\mathbf{j} - 2\mathbf{k} = \mathbf{q}$

(a) $$\cos P\hat{O}Q = \frac{6 - 12 - 2}{\sqrt{4+9+1} \times \sqrt{9+16+4}}$$

$$= \frac{-8}{\sqrt{14} \times \sqrt{29}} = -0.397$$

$\therefore \quad P\hat{O}Q = 66.6°$ (Answer)

(b) $\mathbf{\overline{PQ}} = 3\,\mathbf{\overline{QR}}$

$\mathbf{q} - \mathbf{p} = 3(\mathbf{r} - \mathbf{q})$

$$\begin{pmatrix} 3 \\ 4 \\ -2 \end{pmatrix} - \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} = 3\mathbf{r} - 3\begin{pmatrix} 3 \\ 4 \\ -2 \end{pmatrix}$$

$$\begin{pmatrix} 1 \\ 7 \\ -3 \end{pmatrix} = 3\mathbf{r} - \begin{pmatrix} 9 \\ 12 \\ -6 \end{pmatrix}$$

$$3\mathbf{r} = \begin{pmatrix} 1 \\ 7 \\ -3 \end{pmatrix} + \begin{pmatrix} 9 \\ 12 \\ -6 \end{pmatrix}$$

$$3\mathbf{r} = \begin{pmatrix} 10 \\ 19 \\ -9 \end{pmatrix}$$

$$\mathbf{r} = \begin{pmatrix} \frac{10}{3} \\ \frac{19}{3} \\ -3 \end{pmatrix}$$

$\therefore \quad = \frac{10}{3}\mathbf{i} + \frac{19}{3}\mathbf{j} - 3\mathbf{k}$ (Answer)

13. Let for l, $\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-5}{1} = t$

then $\frac{x-1}{2} = t,\ x - 1 = 2t,\ x = 2t + 1$

$\frac{y-2}{3} = t,\ y - 2 = 3t,\ y = 3t + 2$

$\frac{z-5}{1} = t,\ z - 5 = t,\ z = t + 5$

For m, let $\frac{x-2}{1} = \frac{y-3}{5} = \frac{z-1}{2} = s$

then $\frac{x-2}{1} = s,\ x - 2 = s,\ x = s + 2$

$\frac{y-3}{5} = s,\ y - 3 = 5s,\ y = 5s + 3$

$\frac{z-1}{2} = s,\ z - 1 = 2s,\ z = 2s + 1$

Working with x, $s + 2 = 2t + 1$ -- ①

with y, $5s + 3 = 3t + 2$ -- ②

with z, $2s + 1 = t + 5$ -- ③

Solving (1) and (2), we have

$s + 2 = 2t + 1$. $\qquad$ $5s + 3 = 3t + 2$

$s - 2t = -1\ (\times 5)$ (1a) $\qquad$ $5s - 3t = -1$ (2a)

$5s - 3t = -1$ from (2a)

5(1a) − (2a)

$$\begin{array}{l} \hline 5s - 10t = -5 \\ 5s - 3t = -1 \\ \hline \end{array}$$

By substraction

$-7t = -4$

$t = \frac{4}{7}$

Substituting in (1a)

$$s - 2t = -1$$

$$s - \frac{8}{7} = 1$$

$$s = 1 + \frac{8}{7} = \frac{15}{7}$$

Substituting s and t in (3), we have

$$2s + 1 \qquad\qquad t + 5$$

$$= \frac{10}{7} + 1 \qquad\qquad = \frac{4}{7} + 5$$

$$= \frac{17}{7} \qquad\qquad = \frac{39}{7}$$

These two are not equal, (Answer)

$\therefore$ the two lines do not intersect

(b) For l, equation is

$$\mathbf{r} = \begin{pmatrix} 1 \\ 2 \\ 5 \end{pmatrix} + s\begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}$$

$$X = \begin{matrix} x = 2s + 1 \\ y = 3s + 2 \\ z = s + 5 \end{matrix}$$

$$Y = \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix}$$

$$\therefore \quad \overline{\mathbf{X}\mathbf{Y}} = \mathbf{y} - \mathbf{x}$$

$$= \begin{pmatrix} -2s \\ -4 - 3s \\ -2 - s \end{pmatrix}$$

$$\overline{\mathbf{X}\mathbf{Y}} \, . \, l = 0$$

$$\begin{pmatrix} -2s \\ -4 - 3s \\ -2 - s \end{pmatrix} . \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} = 0$$

$$s = -1$$

$$\therefore \quad \overline{\mathbf{OX}} = \begin{pmatrix} 1 \\ -1 \\ 4 \end{pmatrix}$$ (Answer)

(c) $m : x = 2 + s$

$y = 3 + 5s$

$z = 1 + 2s$

$$\vec{\mathbf{OY}} = \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix}$$

if y lies on m, then

$2 + s = 1$

$3 + 5s = - 2$

$1 + 2s = 3$

which is not so.

$\therefore$ y does not lie on m.

$$\mathbf{XY}. \begin{pmatrix} 1 \\ 5 \\ 2 \end{pmatrix} \neq 0, \ \therefore \ \mathbf{XY} \text{ is not } \perp \text{ to } m.$$ (Answer)

14. For a, $\mathbf{r} = \begin{pmatrix} 2 \\ -1 \\ 4 \end{pmatrix} + s\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$

$x = s + 2$

$y = s - 1$

$z = -s + 4$

For b, $\mathbf{r} = \quad + \ t$

$x = -2t - 2$

$y = t + 2$

$z = t + 1$

Equating x and y.

$$s + 2 = -2t - 2$$
$$s - 1 = t + 2$$

by substraction

$$3 = -3t - 4$$
$$-3t = 7$$
$$3t = -7$$
$$t = \frac{-7}{3}$$

$$s - 1 = \frac{-7}{3} + 2$$
$$s = \frac{-7}{3} + 3 = \frac{2}{3}$$
$$-s + 4 = t + 1$$

Replacing s by $\frac{2}{3}$ and t by $\frac{-7}{3}$

$$\frac{-2}{3} + 4 = \frac{-7}{3} + 1$$
$$\frac{10}{3} = \frac{-4}{3}$$

$\therefore$ the 2 lines a and b do not intersect. (Answer)

(b) $$\cos\theta = \frac{-2+1-1}{\sqrt{1+1+1} \times \sqrt{4+1+1}}$$
$$= \frac{-2}{\sqrt{3} \times \sqrt{6}} = -0.471$$

$\therefore$ $\theta = 61.9°$ (Answer)

15. (a) $$\overrightarrow{\mathbf{OS}} = \begin{pmatrix} 4 \\ -3 \\ 2 \end{pmatrix}$$

$$\overrightarrow{\mathbf{OT}} = \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix}$$

$$\cos S\hat{O}T = \frac{8 - 9 - 2}{\sqrt{16+9+4} \times \sqrt{4+9+1}}$$

$$= \frac{-3}{\sqrt{29} \times \sqrt{14}} = -0.149$$

$$\therefore \quad S\hat{O}T = 81.4^\circ$$

(Answer)

(b) $$\overrightarrow{\mathbf{ST}} = \mathbf{t} - \mathbf{s}$$

$$= \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix} - \begin{pmatrix} 4 \\ -3 \\ 2 \end{pmatrix}$$

$$= \begin{pmatrix} -2 \\ 6 \\ -3 \end{pmatrix} = -2\mathbf{i} + 6\mathbf{j} - 3\mathbf{k}$$

(Answer)

(c) Unit vector in the direction of

$$\overrightarrow{\mathbf{ST}} = \frac{1}{\sqrt{4+36+9}} (-2\mathbf{i} + 6\mathbf{j} - 3\mathbf{k})$$

$$= \frac{1}{7}(-2\mathbf{i} + 6\mathbf{j} - 3\mathbf{k})$$

$$= \frac{-2}{7}\,\mathbf{i} + \frac{6}{7}\,\mathbf{j} - \frac{3}{7}\,\mathbf{k}$$

(Answer)

UNIT 15

PARAMETERS

Suggested Solutions

1. (a)

$$x = \frac{1+\sin\theta}{\cos\theta}$$

$$\frac{dx}{d\theta} = \frac{\cos\theta\left(\cos\theta\right)-\left(1+\sin\theta\right)\left(-\sin\theta\right)}{\cos^2\theta}$$

$$= \frac{\cos^2\theta+\sin\theta+\sin^2\theta}{\cos^2\theta}$$

$$= \frac{1+\sin\theta}{\cos^2\theta}$$

$$y = \frac{1}{\cos\theta} = \sec\theta$$

$$\frac{dy}{d\theta} = \sec\theta\tan\theta$$

$$\therefore \frac{dy}{dx} = \frac{dy}{d\theta}\times\frac{d\theta}{dx}$$

$$= \sec\theta\tan\theta\times\frac{\cos^2\theta}{1+\sin\theta}$$

$$= \frac{1}{\cos\theta}\times\frac{\sin\theta}{\cos\theta}\times\frac{\cos^2\theta}{1+\sin\theta}$$

$$= \frac{\sin\theta}{1+\sin\theta}$$

(Answer)

(b) Where the gradient $= \frac{1}{2}$

$$\frac{\sin\theta}{1+\sin\theta} = \frac{1}{2}$$

$$2\sin\theta = 1+\sin\theta$$

$\sin\theta = 1$

$\theta = \frac{\pi}{2}$ or 90°

(Answer)

2. Given $x = \theta + \sin\theta$

$\therefore \quad \frac{dx}{d\theta} = 1 + \cos\theta$

and $\quad y = 1 - \cos\theta$

$\frac{dy}{d\theta} = \sin\theta$

$\therefore \quad \frac{dy}{dx} = \frac{dy}{d\theta} \times \frac{d\theta}{dx}$

$= \sin\theta \times \frac{1}{1 + \cos\theta}$

$= \frac{\sin\theta}{1 + \cos\theta}$

(Answer)

3. Given $x = t^2$

and $y = 1 - t$

(a) $x = t^2$ -- ①

$1 - t = y$

$-t = y - 1$

$t = 1 - y$

$\therefore \quad x = (1 - y)^2$ Replace t in eq. (1) above (Answer)

(b) $\frac{dx}{dt} = 2t$

$\frac{dy}{dt} = -1$

$\therefore \quad \frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$

$$= -1 \times \frac{1}{2t}$$

$$= \frac{-1}{2t}$$ (Answer)

4. $x = 2\theta + \cos 2\theta$

$$\frac{dx}{d\theta} = 2 - 2\sin 2\theta$$

$$y = 1 - \sin 2\theta$$

$$\frac{dy}{d\theta} = -2\cos 2\theta$$

$$\frac{dy}{dx} = \frac{dy}{d\theta} \times \frac{d\theta}{dx}$$

$$= -2\cos 2\theta \times \frac{1}{2 - 2\sin 2\theta} \qquad (\div 2)$$

$$= \frac{-\cos 2\theta}{1 - \sin 2\theta}$$

$$= \frac{\cos 2\theta}{\sin 2\theta - 1}$$ (Answer)

5. $x = t^2 - 1$

$y = 2t + \ln (t + 1)$

(a) $\frac{dx}{dt} = 2t$ and $\frac{dy}{dt} = 2 + \frac{1}{(t+1)}$

$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$$

$$= \left\{2 + \frac{1}{(t+1)}\right\} \times \frac{1}{2t}$$

$$= \frac{(2t + 2 + 1)}{(t+1)} \times \frac{1}{2t}$$

$$= \frac{(2t + 3)}{2t(t+1)}$$ (Answer)

(b) When the gradient is -4

$$\frac{(2t+3)}{2t(t+1)} = -4$$

$$2t + 3 = -8t(t+1)$$

$$2t + 3 = -8t^2 - 8t$$

$$8t^2 + 10t + 3 = 0$$

$$(2t+1)(4t+3) = 0$$

When $t = \frac{1}{2}$

$$x = \left(\frac{-1}{2}\right)^2 - 1 \qquad y = 2\left(\frac{-1}{2}\right)\ln\left(\frac{1}{2}\right)$$

$$x = -\frac{3}{4} \qquad y = \ln\frac{1}{2} - 1$$

When $t = -\frac{3}{4}$

$$x = \left(\frac{-3}{4}\right)^2 - 1 \qquad y = 2\left(\frac{3}{4}\right)\ln\left(\frac{1}{4}\right)$$

$$= -\frac{7}{16} \qquad y = \ln\frac{1}{4} - \frac{3}{2}$$

$\therefore$ coordinates are

$$\left(\frac{-3}{4}, \ln\frac{1}{2} - 1\right) \text{ and}$$

$$\left(\frac{-7}{16}, \ln\frac{1}{4} - \frac{3}{2}\right)$$

(Answer)

6. (a) $x = \cos 2\theta - 1$

$y = 1 - 2\sin 2\theta$

$$\frac{dx}{d\theta} = -2\sin 2\theta$$

$$\frac{dy}{d\theta} = -4\cos 2\theta$$

$$\frac{dy}{dx} = \frac{-4\cos 2\theta}{-2\sin 2\theta} = 2\cot 2\theta$$ (Answer)

(b) $x = \cos 2\theta - 1$

$\therefore \quad \cos 2\theta = x + 1$

$y = 1 - 2\sin 2\theta$

$2\sin 2\theta = 1 - y$

$\sin 2\theta = \frac{1 - y}{2}$

Take $\cos 2\theta = x + 1$ ---------------- ①

and $\sin 2\theta = \frac{1 - y}{2}$ ---------------- ②

Squaring everywhere, $\cos^2 2\theta = (x + 1)^2$ ---------------- ①

and $\sin^2 2\theta = \left(\frac{1 - y}{2}\right)^2$ ---------------- ②

By addition, we have

$$1 = (x + 1)^2 + \left(\frac{1 - y}{2}\right)^2$$

$$1 = (x + 1)^2 + \frac{(1 - y)^2}{4}$$

$$4 = 4(x + 1)^2 + (1 - y)^2$$ (Answer)

7. $y = 2\sin\theta$ and $x = 1 - 3\cos 2\theta$

$\frac{dy}{d\theta} = 2\cos\theta$ $\qquad \frac{dx}{d\theta} = 6\sin 2\theta$

$$\frac{dy}{dx} = \frac{2\cos\theta}{6\sin 2\theta} = \frac{\cancel{2}\cancel{\cos\theta}}{6 \times \cancel{2}\sin\theta\cancel{\cos\theta}}$$

$$= \frac{1}{6\sin\theta}$$

$$= \frac{1}{6}\text{cosec }\theta$$ (Answer)

8. $x = a\cos^2 t$

$y = a\sin^2 t$

(a) (i) $$\frac{dx}{dt} = -2a\cos t \sin t = -a\sin 2t$$

(Answer)

(ii) $$\frac{dy}{dt} = 2a\sin t \cos t = a\sin 2t$$

(Answer)

(iii) $$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$$

$$= \frac{a\sin 2t}{-a\sin 2t} = -1$$

(Answer)

(b) By addition, $x + y = a(\cos^2 t + \sin^2 t)$

$x + y = a$

(Answer)

9. $x = b\sin^3 \theta$

$y = b\cos^3 \theta$

(a) $$\frac{dx}{d\theta} = 3b\sin^2 \theta \cos \theta$$

$$\frac{dy}{d\theta} = -3b\cos^2 \theta \sin \theta$$

$$\frac{dy}{dx} = \frac{\cancel{-3b}\,\overset{\cos\theta}{\cancel{\cos^2\theta}}\,\cancel{\sin\theta}}{\cancel{3b}\,\underset{\sin\theta}{\cancel{\sin^2\theta}}\,\cancel{\cos\theta}}$$

$$= \frac{-\cos\theta}{\sin\theta}$$

$$= -\cot\theta$$

(Answer)

(b) Gradient at any point and the curve is $-\cot\theta$

point on curve $= (x, y) = (b\sin^3 \theta, b\cos^3 \theta)$

$$\frac{y - b\cos^1 \theta}{x - b\sin^3 \theta} = -\cot\theta$$

$$y - b\cos^3 \theta = -\cot\theta\,(x - b\sin^3 \theta)$$

$$y = b\cos^3 \theta - x\cot\theta + b\sin^3 \theta \cot\theta$$

$y = -x\cot\theta + b\cos^3\theta + b\sin^2\theta\cos\theta$

$y = -x\cot\theta + b\cos\theta\,(\cos^2\theta + \sin^2\theta)$ (Answer)

$\therefore\quad y = -x\cot\theta + b\cos\theta$

10. $x = \lambda\sin^3\theta$

$y = \lambda\cos^3\theta$

(a) $\dfrac{dx}{d\theta} = 3\lambda\sin^2\theta\cos\theta$

$\dfrac{dy}{d\theta} = -3\lambda\cos^2\theta\sin\theta$

$\dfrac{dy}{dx} = \dfrac{-3\lambda\cos^2\theta\sin\theta}{3\lambda\sin^2\theta\cos\theta}$

$= -\cot\theta$

Gradient of tangent $= -\cot\theta$

Point on curve $= (x, y) = (\lambda\sin^3\theta, \lambda\cos^3\theta)$

$\dfrac{y - \lambda\cos^3\theta}{x - \lambda\sin^3\theta} = -\cot\theta$

$y - \lambda\cos^3\theta = -\cot\theta\,(x - \lambda\sin^3\theta)$

$y = \lambda\cos^3\theta - x\cot\theta + \lambda\sin^3\theta\cot\theta$

$y = -x\cot\theta + \lambda\cos^3\theta + \lambda\sin^2\theta\cos\theta$

$y = -x\cot\theta + \lambda\cos\theta\,(\cos^2 + \sin^2\theta)$ (Answer)

$\therefore\quad y = -x\cot\theta + \lambda\cos\theta$

(b) When $y = 0$, $0 = -x\cot\theta + \lambda\cos\theta$

$x\cot\theta = \lambda\cos\theta$

$x = \dfrac{\lambda\cos\theta}{\cot\theta}$

$\therefore\quad x = \lambda\sin\theta$

$\therefore$ A has coordinates $(+\lambda\sin\theta, 0)$

and when $x = 0$,

$y = 0 + \lambda\cos\theta$

$\therefore\quad y = \lambda\cos\theta$

∴ Coordinates of $B = (0, \lambda\cos\theta)$

$$\text{Length of } AB = \sqrt{(\lambda\cos\theta - 0)^2 + (0 - \lambda\sin\theta)^2}$$

$$= \sqrt{\lambda^2\cos^2\theta + \lambda^2\sin^2\theta}$$

$$= \sqrt{\lambda^2(\cos^2\theta + \sin^2\theta)}$$

$$= \sqrt{\lambda^2}$$

$$= \lambda$$

(Answer)

11. $x = t + e^t$

$y = t - e^t$

(a) $\dfrac{dx}{dt} = 1 + e^t$

$\dfrac{dy}{dt} = 1 - e^t$

$\dfrac{dy}{dx} = \dfrac{1 - e^t}{1 + e^t}$ (Answer)

(b) $\text{gradient} = \dfrac{dy}{dx} = \dfrac{1 - e^t}{1 + e^t} = \dfrac{1}{2}$

$$2 - 2e^t = 1 + e^t$$

$$3e^t = +2 - 1$$

$$e^t = +\frac{1}{3}$$

$$t = \ln\frac{1}{3}$$

(Answer)

12. $x = 4 - e^{2t}$

$y = 5 + e^{2t}$

(a) By addition, $x + y = 9$ (Answer)

(b) $\frac{dx}{dt} = -2e^{2t}$

$\frac{dy}{dt} = 2e^{2t}$

$\frac{dy}{dx} = \frac{2e^{2t}}{-2e^{2t}} = -1$ (Answer)

For a greatest value, $\frac{dy}{dx} = 0$, which is impossible. (Answer)

13. (a) $x = a\sin t$

$\frac{dx}{dt} = a\cos t$

$y = at\cos t$

$\frac{dy}{dt} = atx - \sin t + a\cos t$

$= a\cos t - at\sin t$

$\therefore \quad \frac{dy}{dx} = \frac{a\cos t - at\sin t}{a\cos t}$

$= \frac{\cos t - t\sin t}{\cos t}$ (Answer)

(b) when $\frac{dy}{dx} = 0$

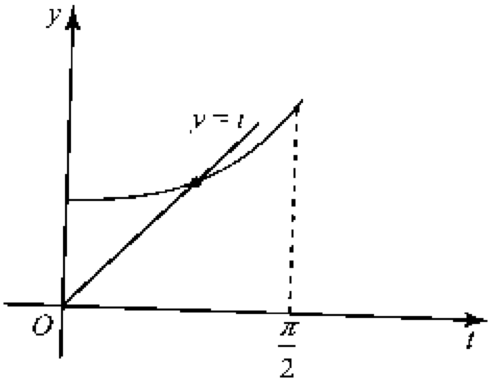
$\cos t - t\sin t = 0$

$t\sin t = \cos t$

$\therefore \quad t = \frac{\cos t}{\sin t} = \cot t$ (Answer)

(c) For $t = \cot t$, we draw the two graphs

$y = t$ and $y = \cot t$



Only one root.

(Answer)

14. $x = \theta + 2\ln\theta$

$y = 2\theta - \ln\theta$

(a) $\frac{dx}{d\theta} = 1 + \frac{2}{\theta}$

$\frac{dy}{d\theta} = 2 - \frac{1}{\theta}$

since $\frac{dy}{dx} = \frac{dy}{\cancel{d\theta}} \cdot \frac{\cancel{d\theta}}{dx}$ or $\frac{dy}{dx} = \frac{dy}{d\theta} \div \frac{dx}{d\theta}$

$$\therefore \quad \frac{dy}{dx} = \frac{2 - \frac{1}{\theta}}{1 + \frac{2}{\theta}} = \frac{2\theta - 1}{\theta + 2}$$

(Answer)

(b) When $\theta = 1$, gradient $= \frac{2 - 1}{1 + 2} = \frac{1}{3}$

Point on curve has coordinates

$(\theta + 2\ln\theta, 2\theta - \ln\theta)$

$= (1 + 0, 2 - 0) = (1, 2)$

Equation $\frac{y - 2}{x - 1} = \frac{1}{3}$

$3y - 6 = x - 1$

$3y = x + 5$

(Answer)

(c) At the stationary point, $\frac{dy}{dx} = 0$

i.e. $\frac{2\theta - 1}{\theta + 2} = 0$

$$2\theta - 1 = 0$$

$$\theta = \frac{1}{2}$$

$$y = 2\theta - \ln \theta$$

$$\therefore \quad y = 1 - \ln\left(\frac{1}{2}\right)$$

Coordinates of stationary point =

$(\theta + 2\ln \theta, 2\theta - \ln \theta)$ where $\theta = \frac{1}{2}$

$$\left(\frac{1}{2} + 2\ln\frac{1}{2}, 1 - \ln\frac{1}{2}\right)$$

$$\frac{d^2x}{d\theta^2} = \frac{-2}{\theta^2}$$

and $\frac{d^2y}{d\theta^2} = \frac{1}{\theta^2}$

$\therefore \quad \frac{d^2y}{dx^2} = \frac{1}{\theta^2} \times \frac{\theta^2}{-2}$ = negative value

hence this value of y will be a maximum. (Answer)

15. $x = at^2$

$y = 2at$

(a) $\frac{dx}{dt} = 2at$

$$\frac{dy}{dt} = 2a$$

$\frac{dy}{dx} = \frac{2a}{2at} = \frac{1}{t}$ (Answer)

(b) gradient $= \frac{1}{t}$

Any point on curve has coordinates $(at^2, 2at)$

$\therefore$ equation of tangent is

$$\frac{y - 2at}{x - at^2} = \frac{1}{t}$$

$$yt - 2at^2 = x - at^2$$

$$yt = x + at^3$$

At A, $y = 0$

$$\therefore \quad 0 = x + at^2$$

$$x = -at^2$$

Equation of any tangent to the curve is $yt = x + at^2$ (Answer)

16.

$$x = a\cos^3\theta$$

$$\frac{dx}{d\theta} = -3a\cos^2\theta\sin\theta$$

$$y = a\sin^3\theta$$

$$\frac{dy}{d\theta} = 3a\sin^2\theta\cos\theta$$

$$\therefore \quad \frac{dy}{dx} = \frac{3a\sin^2\theta\cos\theta}{-3a\cos^2\theta\sin\theta}$$

$$= -\tan\theta$$

Equation of tangent

$$\frac{y - a\sin^3\theta}{x - a\cos^3\theta} = -\tan\theta$$

$$y - a\sin^3\theta = -x\tan\theta + a\tan\theta\cos^3\theta$$

$$y = -x\tan\theta + a\tan\theta\cos^3\theta + a\sin^3\theta$$

At A, $y = 0$

$$\therefore \quad 0 = -x\tan\theta + a\tan\theta\cos^3\theta + a\sin^3\theta$$

$$x\tan\theta = a \times \frac{\sin\theta}{\cos\theta} \times \cos^3\theta + a\sin^3\theta$$

$$x\tan\ \theta = a\sin\ \theta \cos^2 \theta + a\sin^3 \theta$$

$$x\tan\ \theta = a\sin\ \theta\,(\cos^2 \theta + \sin^2 \theta)$$

$$\therefore \quad x = \frac{a\sin \theta}{\tan \theta} = a\cos\ \theta$$

Similarly $y = a\sin\ \theta$

Hence the length of AB

$$= \sqrt{(0 - a\cos\theta)^2 + (a\sin\theta - 0)^2}$$

$$= \sqrt{a^2}$$

$= a$ which is a constant (Answer)

UNIT 16

NUMERICAL SOLUTION OF EQUATIONS

Suggested Solutions

1. (a) $x = \frac{2x}{3} + \frac{7}{x^3}$

$= \frac{2x^4 + 21}{3x^3}$

Where $x = 2,$ $\quad f(x) = 2.2083$

$x = 2.2083,$ $\quad f(x) = 2.1222$

$x = 2.1222,$ $\quad f(x) = 2.1472$

$x = 2.1472,$ $\quad f(x) = 2.1386$

$x = 2.1386,$ $\quad f(x) = 2.1414$

$x = 2.1414,$ $\quad f(x) = 2.1405$

$x = 2.1405,$ $\quad f(x) = 2.1408$

$x = 2.1408,$ $\quad f(x) = 2.1407$

$x = 2.1407,$ $\quad f(x) = 2.1407$

$\therefore \quad x = 2.141$ (Answer)

(b) $x = \frac{2x^4 + 21}{3x^3}$

$3x^4 = 2x^4 + 21$

$x^4 = 21$

$x = \sqrt[4]{21}$ (Answer)

2. (a) $A = \int_0^a y\ dx$

$$= \int_0^a e^{-x}\ dx$$

$$= \left[-e^{-x}\right]_0^a$$

$$= -e^{-a} - -e^{-0}$$

$$= \frac{-1}{e^a} + 1$$

$$= \frac{-1 + e^a}{e^a}$$

$$= \frac{e^a - 1}{e^a} \text{ sq. units}$$

(Answer)

(b) $\frac{e^a - 1}{e^a} = \frac{1}{2}a$

$$2e^a - 2 = ae^a$$

$$a = \frac{2\left(e^a - 1\right)}{e^a}$$

(Answer)

(c) $a = 2 - 2e^{-a}$

When $a = 2.5$, $f(a) = 1.8358$

$a = 1.8358$, $f(a) = 1.6810$

$a = 1.6810$, $f(a) = 1.6276$

$a = 1.6276$, $f(a) = 1.6072$

$a = 1.6072$, $f(a) = 1.5991$

$a = 1.5991$, $f(a) = 1.5958$

$a = 1.5958$, $f(a) = 1.5995$

$a = 1.5945$, $f(a) = 1.5940$

(Answer)

$\therefore \quad a = 1.59$

3. Let $f(x) = x^3 - 2x^2 + x - 2$

(a) $f(0) = 0 - 0 + 0 - 2 = -2$

$f(1) = 1 - 2 + 1 - 2 = 2$

$f(2) = 8 - 8 + 2 - 2 = 0$

$f(3) = 27 - 18 + 3 - 2 = 10$

$\therefore \quad x = 2$ (Answer)

(b) $x = \sqrt[3]{2x^2 - x + 2}$

$x = 1$ $\quad f(x) = \sqrt[3]{2 - 1 + 2} = \sqrt[3]{3} = 1.4422$

$x = 1.4422$ $\quad f(x) = 1.6772$

$x = 1.6772$ $\quad f(x) = 1.8119$

Not convergent.

When $x = 2$, $\quad f(x) = 0$ $\quad$ Only 1 root. (Answer)

4. $f(x) = x^3 - 2x - 2$

(a) $f(1) = 1 - 2 - 2 = \ 3$

$f(2) = 8 - 4 - 2 = 2$

Sign changes

$\therefore$ there is a root between 1 and 2 (Answer)

(b) $x = \dfrac{2x^3 + 2}{3x^2 - 2}$

When $x = 1$, $\quad f(x) = 4$

$x = 4$, $\quad f(x) = 2.8261$

$x = 2.8261$, $\quad f(x) = 2.1467$

$x = 2.1467$, $\quad f(x) = 1.8423$

$x = 1.8423$, $\quad f(x) = 1.7728$

$x = 1.7728$, $\quad f(x) = 1.7693$

$x = 1.7693$, $\quad f(x) = 1.7693$

$x = 1.7693$

$\therefore$ This root $= 1.7693$ (Answer)

(Answer)

(c) $x = 1.77$

5. (a) $y = xe^x$

$$\frac{dy}{dx} = xe^x + e^x$$

for a minimum value

$$\frac{dy}{dx} = 0$$

$\therefore$ $xe^x + e^x = 0$

$e^x(x + 1) = 0$

i.e. $e^x = 0$, which is impossible

and $x + 1 = 0$

i.e. $x = -1$

When $x = -1$

$y = xe^x$

$= -xe^{-1}$

$= \frac{-1}{e}$

$\therefore$ Coordinates of $x = \left(-1, \frac{-1}{e}\right)$ (Answer)

(b) $y = xe^x$

$y = 7$

$\therefore$ Point of intersection is

When $7 = xe^x$

$x = \frac{7}{e^x}$

or $e^x = \frac{7}{x}$

i.e. $\ln\left(\frac{7}{x}\right) = x$ (Answer)

(c) $x = \ln\left(\frac{7}{x}\right)$

When $x = 1.5$, $f(x) = 1.5404$

$x = 1.5404$, $f(x) = 1.5139$

$x = 1.5139$, $f(x) = 1.5312$

$x = 1.5312$, $f(x) = 1.5199$

$x = 1.5199$, $f(x) = 1.5273$

$x = 1.5273$, $f(x) = 1.5224$

$x = 1.5224$, $f(x) = 1.5256$

$x = 1.5256$, $f(x) = 1.5235$

$x = 1.5235$, $f(x) = 1.5248$

$\therefore$ the root of $x = 1.52$ (Answer)

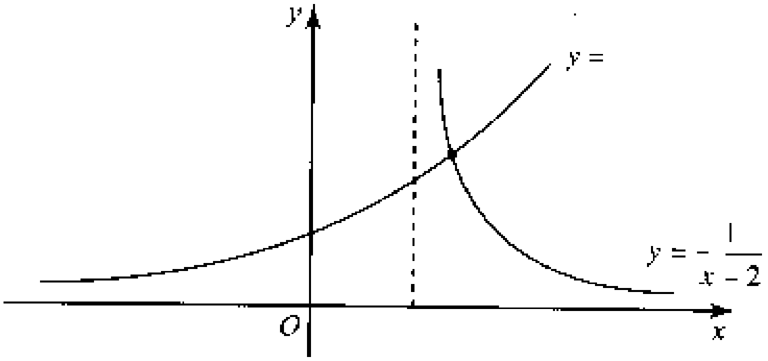
6. (a) $e^{\frac{1}{2}x}(x - 2) - 1$

$$e^{\frac{1}{2}x} = \frac{1}{x - 2}$$

$\therefore$ the graphs are

$y = e^{\frac{1}{2}x}$

and $y = \frac{1}{x\ \ 2}$



(Answer)

(b) $f(x) = e^{\frac{1}{2}x}(x - 2) - 1$

$f(2) = e^{1}(2 - 2) - 1 = 0 - 1 = -1$

$f(2.5) = e^{1.25}(2.5 - 2) - 1 = e^{1.25}(0.5) - 1 = 0.745$

Sign changes.

$\therefore$ there is a root between 2 and 2.5 (Answer)

(c) $f(x) = 2 + e^{-\frac{1}{2}x}$

When $x = 2$, $f(x) = 2.3679$
$x = 2.379$, $f(x) = 2.3061$
$x = 2.3061$, $f(x) = 2.3157$
$x = 2.3157$, $f(x) = 2.3142$
$x = 2.3142$, $f(x) = 2.3144$
$x = 2.3144$, $f(x) = 2.3144$

$\therefore$ $x = 2.31$ (2 d.p.) (Answer)

7. $x = x^3 - 0.74x^2 + 0.74$

$f(x) = x^3 - 0.74x^2 + 0.74$

$x = 0$, $f(x) = 0.74$
$x = 1$, $f(x) = 1 - 0.74 + 0.74 = 1$

Converges to 1.

$x = 0.7$, $f(x) = 0.7204$
$x = 0.7204$, $f(x) = 0.7298$
$x = 0.7298$, $f(x) = 0.7346$
When $x = 1$, $f(x) = 1$ (Answer)

8. $$x^2 = \frac{1}{\sin x}$$

$$\sin x = \frac{1}{x^2}$$

The 2 graphs are:

$$y = \sin x$$

and $$y = \frac{1}{x^2}$$

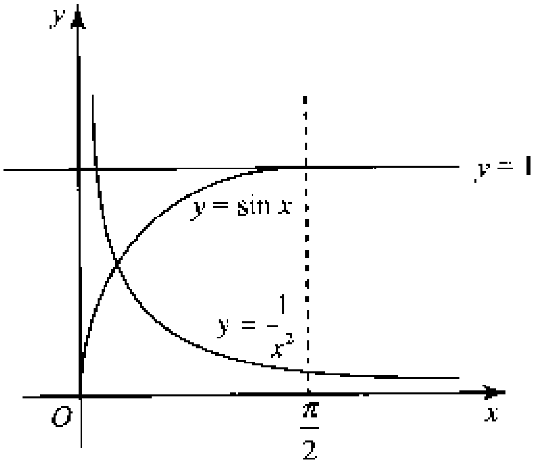
$$f(x) = x^2\sin x \quad 1$$

$$f(0) = 0^2\sin(0) - 1 = -1$$

$$f\left(\frac{\pi}{2}\right) = \left(\frac{\pi}{2}\right)^2 \sin\left(\frac{\pi}{2}\right) - 1$$

$$= \left(\frac{\pi}{2}\right)^2 - 1 = +\text{ve}$$

The change of signs means that a root lies between 0 and $\left(\frac{\pi}{2}\right)$



0 to $\left(\frac{\pi}{2}\right)$ = 0 to 1.732

Take $x = 0.8$, then $f(x) = 1.1807$

$x = 1.1807$, $f(x) = 1.0398$

$x = 1.0398$, $f(x) = 1.0769$

$x = 1.0769$, $f(x) = 1.0657$

$x = 1.0657$, $f(x) = 1.0689$

$x = 1.0689$, $f(x) = 1.06803$

$\therefore$ $x = 1.068$ (to 3 d.p.)

≈ 1.07 (to 3 s.f.) (Answer)

9. $x = \frac{2}{3}x + \frac{5}{3x}$

(a) When $x = 1.3$, $\quad f(x) = 2.1488$

$x = 2.1488$, $\quad f(x) = 2.2082$

$x = 2.2082$, $\quad f(x) = 2.2268$

$x = 2.2268$, $\quad f(x) = 2.2330$

$x = 2.2330$, $\quad f(x) = 2.2351$

$x = 2.2351$, $\quad f(x) = 2.2358$

$\therefore \quad \alpha = 2.24$ (Answer)

(b) $\alpha = \frac{2}{3}\alpha + \frac{5}{3\alpha}$

$3\alpha^2 = 2\alpha^2 + 5$

$\alpha^2 = 5$

$\alpha = \pm\sqrt{5}$ (Answer)

10. (a) $f(x) = x^3 - 2x - 2$

$f(1) = 1 - 2 - 2 = \ 3$

$f(2) = 8 - 4 - 2 = 2$

Signs changes.

$\therefore$ there is a root between 1 and 2 (Answer)

(b) $x = \sqrt[3]{2x + 2}$

When $x = 1.5$, $\quad f(x) = 1.7100$

$x = 1.7100$, $\quad f(x) = 1.7566$

$x = 1.7566$, $\quad f(x) = 1.7666$

$x = 1.7666$, $\quad f(x) = 1.7687$ (Answer)

(c) $\therefore \quad x = 1.77$ (Answer)

11. (a) $x^3 + 2x - 1 = 0$

$f(x) = x^3 + 2x - 1$

$f(0) = 0 + 0 - 1 = -1$

$f(1) = 1 + 2 - 1 = 2$

Signs changes.

∴ root lies between 0 and 1 (Answer)

(b)
$$x = \frac{3x^2 - 1}{2x + 1}$$

$2x^2 + x = 3x^2 - 1$

$x^2 - x - 1 = 0$

$$x = \frac{1 \pm \sqrt{1 + 4}}{2}$$

$$= \frac{1 + 2.24}{2} \text{ or } \frac{1 - 2.24}{2}$$

$= 1.62$ or -0.62

∴ x lies between 1 and 2. (Answer)

(c) $x^3 + 2x - 1 = 0$

$2x = 1 - x^3$

$$x = \frac{1 - x^3}{2}$$

When $x = 0.5$, $f(x) = 0.4375$

$x = 0.4375$, $f(x) = 0.4581$

$x = 0.4581$, $f(x) = 0.4519$

$x = 0.4519$, $f(x) = 0.4539$

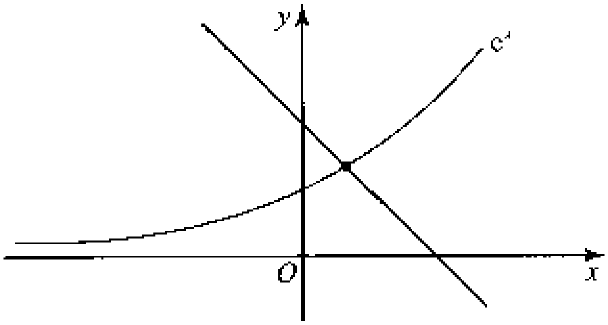
∴ root = 0.45 (Answer)

12. (a) $e^x + x - 2 = 0$

$e^x = 2 - x$

$y = e^x$

$y = 2 - x$



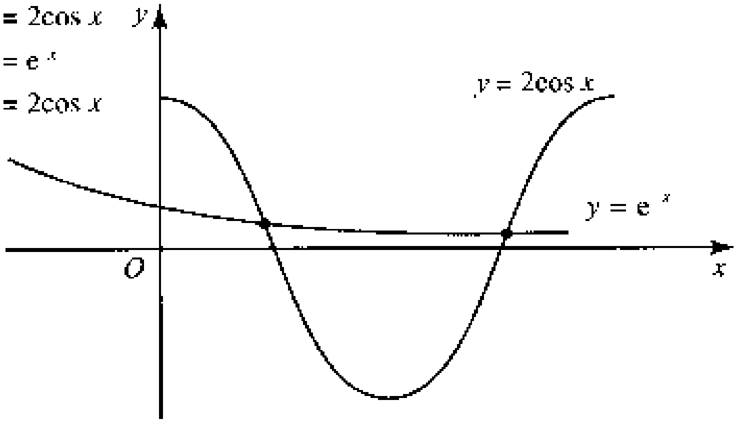
1 root (Answer)

(b) $e^{-x} - 2\cos x = 0$

$e^{-x} = 2\cos x$

$y = e^{-x}$

$y = 2\cos x$



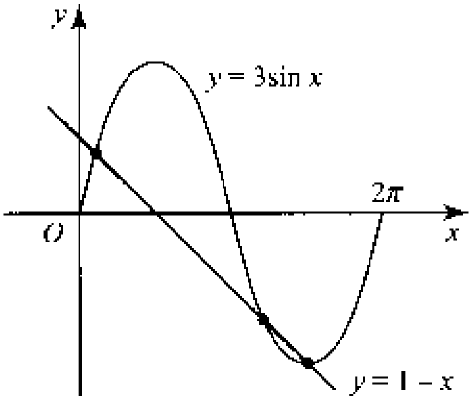
2 roots (Answer)

(c) $3\sin x + x - 1 = 0$

$3\sin x = 1 - x$

$y = 3\sin x$

$y = 1 - x$



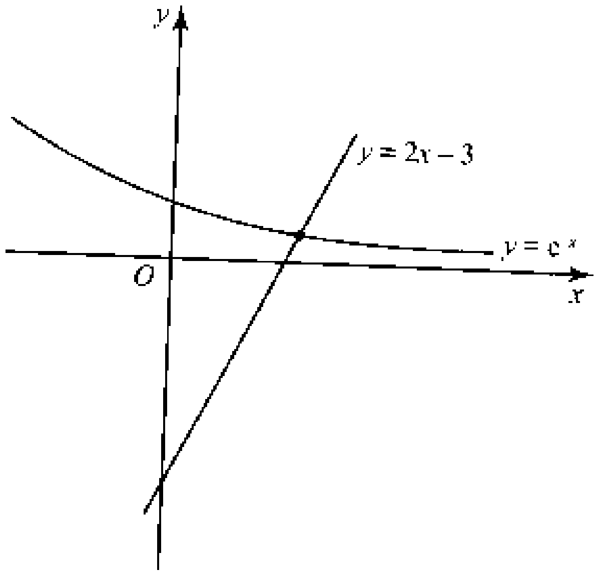
3 roots (Answer)

13. $e^{-x} - 2x + 3 = 0$

$e^{-x} = 2x - 3$

$y = e^{-x}$

$y = 2x - 3$



1 point of intersection

i.e. 1 root

The root is > 0

$x = \frac{e^{-x}}{2} + 1.5$

$x = 0.5$, $f(x) = 1.8033$

$x = 1.8033$, $f(x) = 1.5824$

$x = 1.5824$, $f(x) = 1.6027$

$x = 1.6027$, $f(x) = 1.6007$

$x = 1.6007$, $f(x) = 1.6009$

$x = 1.6009$, $f(x) = 1.6009$

$x = 1.6$ (1 d.p.)

(Answer)

14. $x_{n+1} = \frac{2}{x_n^2} + \frac{3}{5}x_n$

Since $f(0) = \alpha$

$f(1) = 2.6$

$f(2) = 1.7$

$\therefore$ there is a root between 1 and 2,

starting with $x = 1$, $f(x) = 2.6$

$x = 2.6$, $f(x) = 1.8559$

$x = 1.8559$, $f(x) = 1.6942$

$x = 1.6942$, $f(x) = 1.7133$

$x = 1.7133$, $f(x) = 1.7093$

$x = 1.7093$, $f(x) = 1.7101$

$x = 1.7101$

$x = 1.7100,$

$x = 1.7100$

$\therefore$ the root of $x = 1.71$ (Answer)

15. (a) (i) $x = \sqrt[3]{3x^2 - 5}$ (Answer)

(ii) $x = \sqrt{\dfrac{x^3 + 5}{3}}$ (Answer)

(b) (i) $x = \sqrt[3]{\dfrac{3x - 7}{2}}$ (Answer)

(ii) $x = \dfrac{2x^3 + 7}{3}$ (Answer)

(c) $x = 2\sin x - e^{x}$ (Answer)

(d) (i) $x = \dfrac{x^2 + 4 - \ln x}{4}$ (Answer)

(ii) $x = \sqrt{4x + \ln x - 4}$ (Answer)

16. $x = \dfrac{2}{x^3} + \dfrac{3}{4}x$

Let $f(x) = \dfrac{2}{x^3} + \dfrac{3}{4}x - x$

When $x = 0,$ $f(0) = \infty$

$x = 1$ $f(1) = \dfrac{7}{4}$

$x = 2,$ $f(2) = \dfrac{-1}{4}$

Since sign changes, there is a root between 1 and 2. (Answer)

So for $x = 1$, $\mathrm{f}(x) = \dfrac{2}{x^3} + \dfrac{3}{4}x$

$= 2 + \dfrac{3}{4} = 2.75$

$x = 2.75$, $\mathrm{f}(x) = 2.1587$

$x = 1.75$, $\mathrm{f}(x) = 1.6857$

$x = 1.6857$, $\mathrm{f}(x) = 1.6818$

$x = 1.6818$, $\mathrm{f}(x) = 1.6818$

$\therefore$ root of $x = 1.68$ (to 2 d.p.) (Answer)

17. (a) At A and B, $y = 0$

$\therefore$ $(x - 6)(\ln x) = 0$

i.e. $x - 6 = 0$ or $\ln x = 0$

hence $x = 6$, or $x = 1$

$\therefore$ Coordinates of $A = (1, 0)$ (Answer)

and coordinates of $B = (6, 0)$. (Answer)

(b) If P is a minimum point,

then $\dfrac{dy}{dx} = 0$

i.e. $(x - 6) \times \dfrac{1}{x} + (\ln x) \times 1 = 0$

$1 - \dfrac{6}{x} + \ln x = 0$ (multiply by x)

$x + x\ln x = 6$

$x(1 + \ln x) = 6$

$x = \dfrac{6}{(1 + \ln x)}$ (Hence Shown) (Answer)

(c) When $x = 3$, $\dfrac{6}{(1 + \ln 3)} = 2.8590$

$x = 2.8590$, $\mathrm{f}(x) = 2.9262$

$x = 2.9262,\quad f(x) = 2.8934$

$x = 2.8934,\quad f(x) = 2.9092$

$x = 2.9092,\quad f(x) = 2.9015$

$x = 2.9015,\quad f(x) = 2.9052$

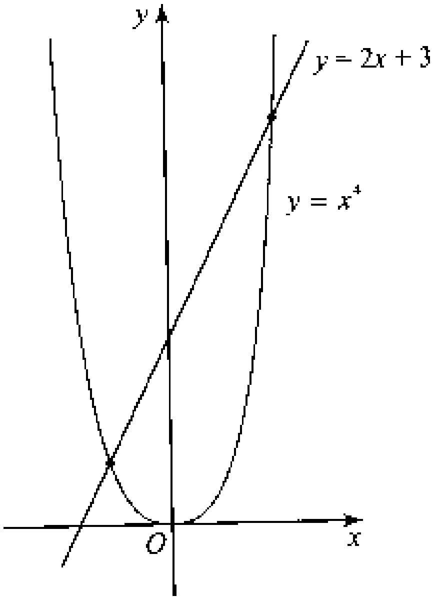
$\therefore\quad x = 2.9$ (Answer)

18. $x^4 - 2x - 3 = 0$

$x^4 = 2x + 3$

$y = x^4$

$y = 2x + 3$



The sketch shows that there are only 2 real roots. (Shown) (Answer)

The greater root is greater than 0.

$x^4 = 2x + 3$

$x = \sqrt[4]{(2x + 3)}$

When $x = 1.4,\quad f(x) = 1.5519$

$x = 1.5645\quad 1.5718$

$x = 1.5734\quad 1.5744$

$x = 1.5746\quad 1.5747$

$\therefore$ the greater root is 1.57 (Answer)

19. When $x = 1,$ $\ln(4\sin x) = 1.2137$

$x_2 = 1.2137$ $f(x) = 1.3211$

$x_3 = 1.3548$

$x_4 = 1.3628$

$x_5 = 1.3645$

$x_6 = 1.3648$

$\therefore$ root of $x = 1.36$ (Answer)

20.

$$x_{n+1} = \frac{1}{4}\left(x_n^3 + 1\right)$$

$$x = \frac{1}{4}\left(x^3 + 1\right)$$

When $x = 1,$ $f(x) = 0.5$

$x = 0.5,$ $f(x) = 0.2813$

$x = 0.2813,$ $f(x) = 0.2556$

$x = 0.2556,$ $f(x) = 0.2542$

$x = 0.2541,$ $f(x) = 0.2541$

$\therefore$ root of $x = 0.25$ (Answer)

UNIT 17

COMPLEX NUMBERS

Suggested Solutions

1. Given $z_1 = 3 - 2i$

and $z_2 = -4 + 5i$

(a) $3z_1 + iz_2 = 3(3 - 2i) + i(-4 + 5i)$

$= 9 - 6i - 4i - 5$

$= 4 - 10i$ (Answer)

(b) $iz_1 - z_2 = i(3 - 2i) - (-4 + 5i)$

$= 3i + 2 + 4 - 5i$

$= 6 - 2i$ (Answer)

(c) $z_1 z_2 = (3 - 2i)(-4 + 5i)$

$= -12 + 15i + 8i - 10i^2$

$= -12 + 10 + 23i$

$= -2 + 23i$ (Answer)

(d) $z_2^2 = (-4 + 5i)^2$

$= 16 - 40i - 25$

$= -9 - 40i$ (Answer)

(e) $z_2 \div z_1 = \dfrac{-4 + 5i}{3 - 2i} \times \dfrac{3 + 2i}{3 + 2i}$

$$= \frac{(-4 + 5i)(3 + 2i)}{(3 - 2i)(3 + 2i)}$$

$$= \frac{-12 - 8i + 15i - 10}{9 + 4} = \frac{-22 + 7i}{13} = \frac{-22}{13} + \frac{7i}{13}$$ (Answer)

2. Let $\sqrt{-16 - 30i} = a + bi$.

Then $-16 - 30i = (a + bi)^2$. (a & $b \in \mathbb{R}$)

$-16 - 30i = a^2 + 2abi - b^2$

Equating coefficients,

real parts = real parts

and imagining parts = imaginary parts.

we obtain $a^2 - b^2 = -16$ ------ ----- ----- ----- ------ ----- ①

and $2ab = -30$

$a = \frac{-15}{b}$ ----- ------ ----- ----- ----- ----- ②

Substituting (2) in (1), we have

$$-16 = a^2 - b^2$$

$$-16 = \left(\frac{-15}{b}\right)^2 - b^2$$

$$-16 = \frac{225}{b^2} - b^2$$

Now let $b^2 = y$, then

$$16 = \frac{225}{y} - y$$

$$-16y = 225 - y^2$$

$$y^2 - 16y - 225 = 0$$

$$(y - 25)(y + 9) = 0$$

$y = 25$ or $y = -9$

$b^2 = 25$ $\quad$ $b^2 = -9$

$b = \pm 5$ $\quad$ no solution

Case 1 $\quad$ **Case 2**

$b = 5$ or $b = -5$

$a = \frac{-15}{5} = -3$ or $a = \frac{-15}{-5} = 3$

$\therefore$ the 2 roots are $(-3 + 5i)$ and $(3 - 5i)$ (Answer)

3. (a) One root is $(2 - i)$,

$\therefore$ the other root is its conjugate,

i.e. $(2 + i)$ (Answer)

(b) $[x - (2 + i)]\,[x - (2 - i)] = 0$

$x^2 - x(2 - i) - x(2 + i) + (2 + i)(2 - i) = 0$

$x^2 - 2x + xi \quad 2x - xi + 4 - i^2 = 0$

$x^2 - 4x + 5 = 0$ (Answer)

$\therefore \quad a = -4$

(c) $b = 5$ (Answer)

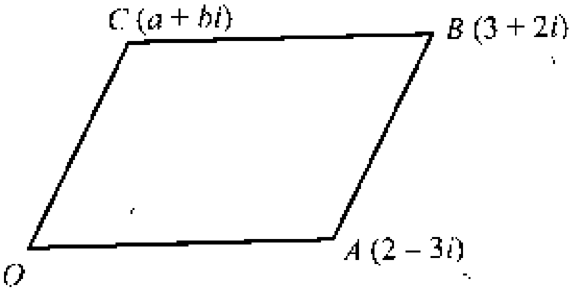
4. Given A is the point where $z_1 = 2 - 3i$

B is the point where $z_2 = 3 + 2i$

and C is the point where $z = a + bi$

(a) If $OABC$ is a par^m,

then



Let C be $z = a + bi$

then $|OC| = |AB| \Rightarrow OC^2 = AB^2$

$\left(\sqrt{a^2 + b^2}\right)^2 = \left|(1 + 5i)\right|$ $\qquad A\vec{B} = \underset{\sim}{b} - \underset{\sim}{a}$

$= 1 + 5i$

$a^2 + b^2 = 1 + 25$

$a^2 + b^2 = 26$ -- ①

Also $\overline{OA}^2 = \overline{C}B^2$

$$(4 + 9) = (3 - a)^2 + (2 - b)^2$$

$$13 = 9 - 6a + a^2 + 4 - 4b + b^2$$

$$a^2 + b^2 - 6a - 4b = 0$$

$$26 - 6a - 4b = 0$$

$$6a + 4b = 26$$

$$3a + 2b = 13 \quad \text{---} ②$$

$$a = \frac{13 - 2b}{3}$$

Replacing in (1) above, we have

$$\left(\frac{13 - 2b}{3}\right)^2 + b^2 = 26$$

$$b^2 - 4b - 5 = 0$$

$$b = 5 \text{ or } -1$$

and $a = 1$ or 5

or mid-point or $AC = \dfrac{z_1 + z}{2}$

mid-point or $OB = \dfrac{z_2}{2}$

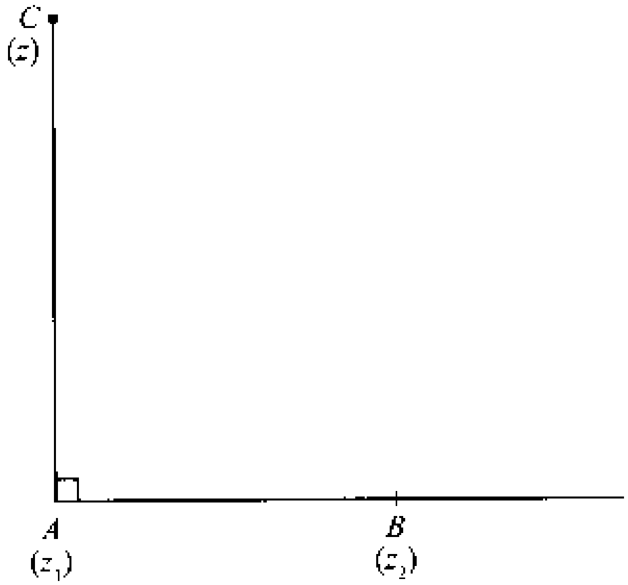
Since $OABC$ is a par^{m}

$$z_1 + z = z_2$$

$$\therefore \quad z = z_2 - z_1 = 1 + 5i$$

$$\therefore \quad z = (1 + 5i) \quad \text{or} \quad (5 - 1)$$ (Answer)

(b)



$$
\begin{aligned}
2AB &= AC \\
2(\underset{\sim}{b} - \underset{\sim}{a}) &= (\underset{\sim}{c} - \underset{\sim}{a}) \\
2(z_2 - z_1) &= z - z_1 \\
2z_2 - 2z_1 &= z - z_1 \\
z &= 2z_2 - z_1 \\
&= 2(3 + 2i) - (2 \quad 3i) \\
&= 6 + 4i - 2 + 3i \\
&= 4 + 7i
\end{aligned}
$$

5. Given $z_1 = 3 + 2i$

and $z_2 = 5 \quad 3i$

(a) $z_1 \quad z_2 = (3 + 2i) - (5 - 3i)$

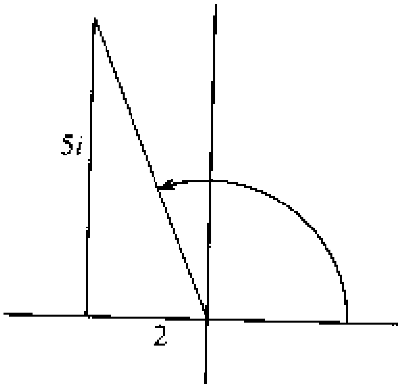
$= 3 + 2i - 5 + 3i$

$= -2 + 5i$

$|z_1 - z_2| = \sqrt{4 + 25} = \sqrt{29}$ (Answer)

(b) $\arg(z_1 - z_2) = \arg(-2 + 5i) = \tan^{-1}\left(\frac{-5}{2}\right)$

$= 1.95$ rad (Answer)



6. (a) Given that $z = 2 - 3i$

$\therefore \quad z^3 = (2 - 3i)^3 = (4 - 12i - 9)(2 - 3i)$

$= (-5 - 12i)(2 - 3i)$

$= -10 - 24i + 15i + 36i^2$

$= -46 - 9i$

$\therefore \quad |z^3| = \sqrt{(-46)^2 + (-9)^2}$

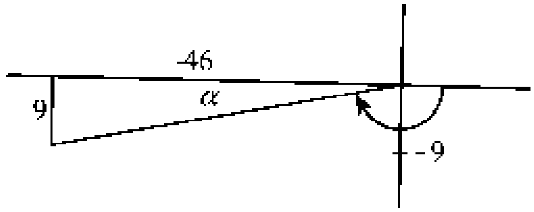
$= \sqrt{2116 + 81}$

$= \sqrt{2197}$

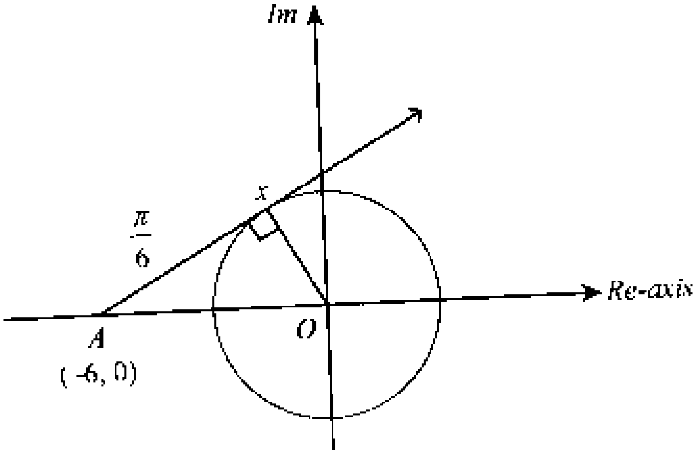
$= 46.87$ (Answer)

(b) $\arg z^3 = \tan^{-1}\left(\frac{9}{46}\right)$

$= -2.948$ rad. (Answer)



7.



The least value of $|z| = OX$

In $\Delta\, OAX = \sin\frac{\pi}{6} = \frac{OX}{OA}$

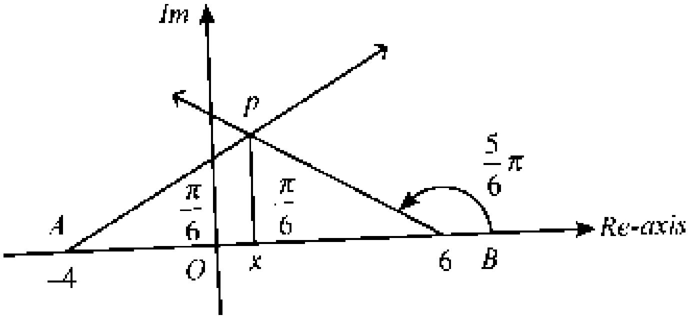
$$\frac{1}{2} = \frac{OX}{6}$$

$$OX = 3$$

$\therefore$ Least value of $|z| = 3$ units. (Answer)

8. Given $\arg(z+4) = \frac{\pi}{6}$

And $\arg(z - 6) = \frac{5}{6}\pi$



Arg $(z + 4)$ and arg $(z - 6)$ meet at P.

PX is $\perp$ to the real axis

$\therefore$ X is the mid-point of AB

i.e. $OX = 1$ and $AX = 5$ units.

$$\tan\frac{\pi}{6} = \frac{PX}{AX}$$

$$PX = AX \times \tan\frac{\pi}{6}$$

$$= 5 \times \frac{1}{\sqrt{3}} = \frac{5}{\sqrt{3}}$$

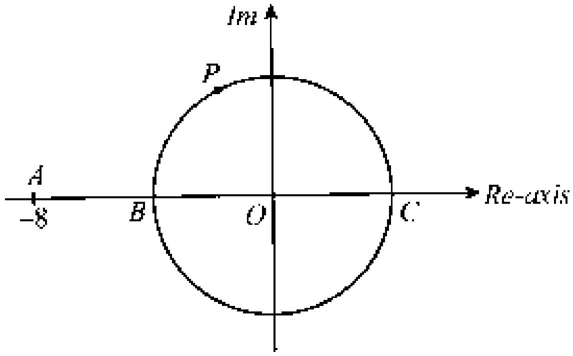
$$|z| = |OP| = \sqrt{OX^2 + PX^2}$$

$$= \sqrt{1 + \frac{25}{3}}$$

$$= \sqrt{\frac{28}{3}}$$

(Answer)

9. (a) $|z + 8| = |z - (\ 8)| = AP$ (as shown)



P is any point on the circumference of the circle, centre O, radius 4 units.

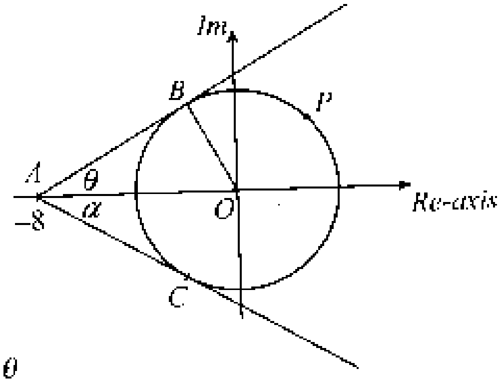
$\therefore$ Least value of $AP = AB = 8 - 4$ units (Answer)

and Greatest value of $AP = AC = 8 + 4 = 12$ units (Answer)

(b) Arg $(z + 8)$ $= \arg(z - (-8))$
$= \angle PAO$, where P is any point on the circumference of the circle.

Greatest value of $\angle PAO$
$= \angle BAO$

and least value of $\angle PAO$
$= \angle CAO$



$\therefore$ Greatest value of $\arg(z + 8) = \theta$

$$\sin\theta = \frac{OB}{OA} = \frac{4}{8} = \frac{1}{2}$$

$\therefore \quad \theta = \frac{\pi}{6}$ (Answer)

and least value of $\arg(z + 8) = \alpha = \frac{-\pi}{6}$ (Answer)

10. (a) $|2 + 3i| = \sqrt{2^2 + 3^2} = \sqrt{4 + 9} = \sqrt{13}$

$\arg(2 + 3i) = \tan^{-1}\left(\frac{3}{2}\right) = 1$ rad (to nearest integer)

$\therefore \quad 2 + 3i = \sqrt{13}\, e^{i}$ (Answer)

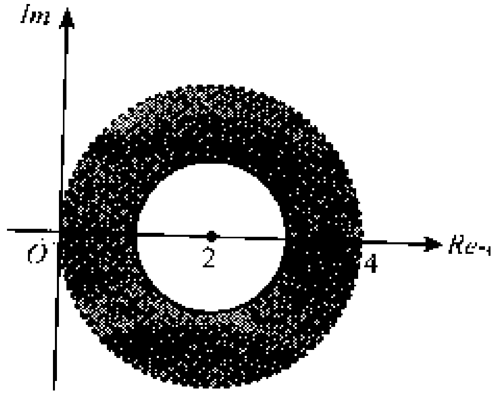
(b) $|3 - 5i| = \sqrt{3^2 + (-5)^2} = \sqrt{9 + 25} = \sqrt{34}$

$\arg(3 - 5i) = \tan^{-1}\left(\frac{-5}{3}\right) = -1$ rad (to the nearest integer).

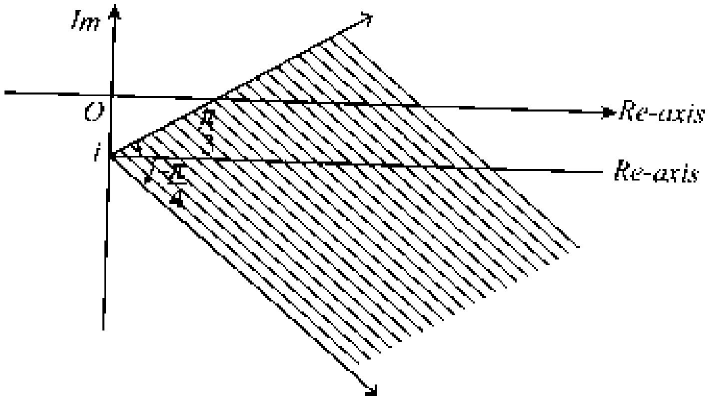
$\therefore \quad 3 - 5i = \sqrt{34}\, e^{i}$ (Answer)

11. (a) $\{1 \le |z-2| < 2\}$

The shaded region is the required set.



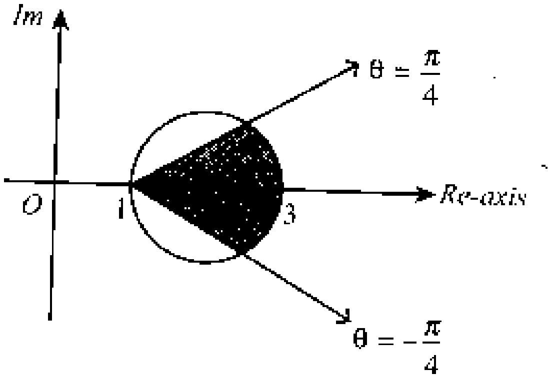
(b) $\frac{-\pi}{4} \le \arg(z+i) \le \frac{\pi}{3}$



The shaded part is the required region.

(c) $|(z-2)| \le 1$

$\frac{-\pi}{4} \le \arg(z-1) \le \frac{\pi}{4}$

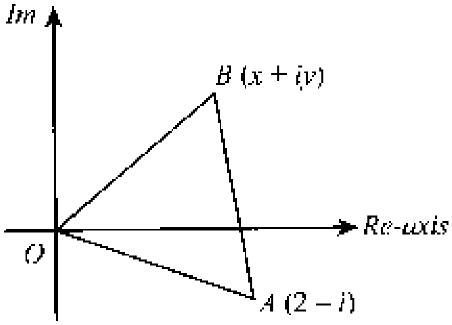


The 2 given inequalities are represented by the shaded region.

12. Since B is the complex number $x + iy$,

$\therefore$ $OB = \sqrt{x^2 + y^2}$

and $OA = \sqrt{2^2 + (-1)^2} = \sqrt{5}$



$\therefore$ $x^2 + y^2 = 5$ since OAB is equilateral

Also $AB = \sqrt{(2 - x)^2 + (-1 - y)^2}$

AB^2 also will be equal to 5.

$\therefore$ $(2 - x)^2 + (-1 - y)^2 = 5$

$4 - 4x + x^2 + 1 + 2y + y^2 = 5$

$4 - 4x + 1 + 2y + x^2 + y^2 = 5$

i.e. $4 - 4x + 1 + 2y + 5 = 5$

hence $2y - 4x = -5$

$$y = \frac{4x - 5}{2}$$

Substituting in $x^2 + y^2 = 5$

We obtain

$$x = 1 + \frac{1}{2}\sqrt{3}$$ (Answer)

and $$y = \frac{-1}{2} + \sqrt{3}$$ (Answer)

13. $$z = \frac{1}{2 + ix} \times \frac{2 - ix}{2 - ix}$$

$$z = \frac{(2 - ix)}{(4 + x^2)}$$

$$z = \frac{2}{(4 + x^2)} - \frac{ix}{(4 + x^2)}$$

$$\therefore \qquad z^* = \frac{2}{(4 + x^2)} + \frac{ix}{(4 + x^2)}$$

Then $z + z^* = \frac{4}{(4 + x^2)}$

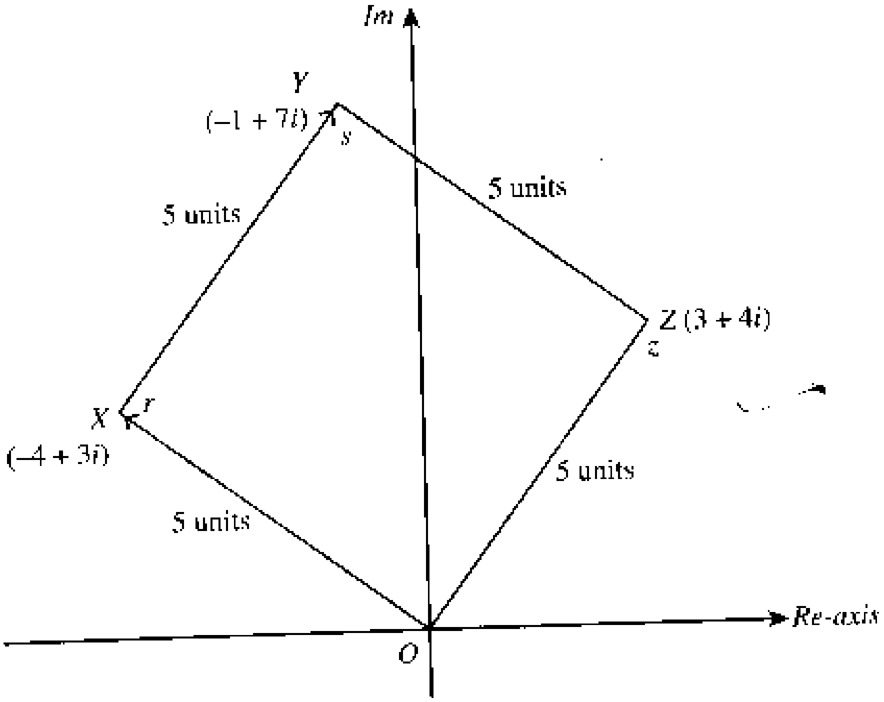
and $4z\,z^* = \left\{\frac{4}{(4 + x^2)^2} + \frac{x^2}{(4 + x^2)^2}\right\} \times 4$

$$= \frac{(4 + x^2)}{(4 + x^2)^2} \times 4$$

$$= \frac{4}{(4 + x^2)} = z + z^*$$

As X varies, the modulus of z will always be $\frac{1}{(4 + x^2)}$, which means that the locus of x will be the circumference of a circle

14.

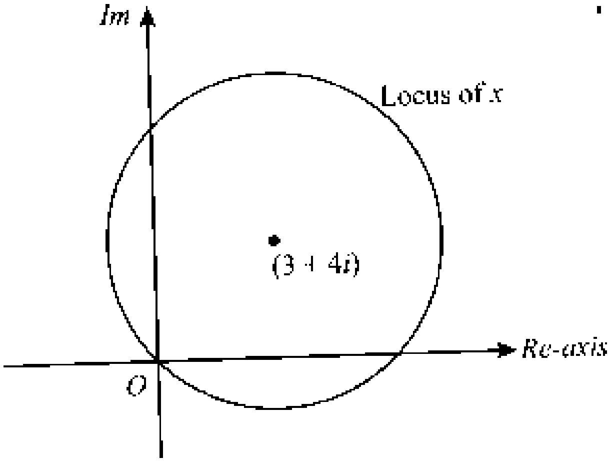


(a) $r = -4 + 3i$ (Answer)

$s = -1 + 7i$ (Answer)

(b) (i) $|z - 3 - 4i| = 5$

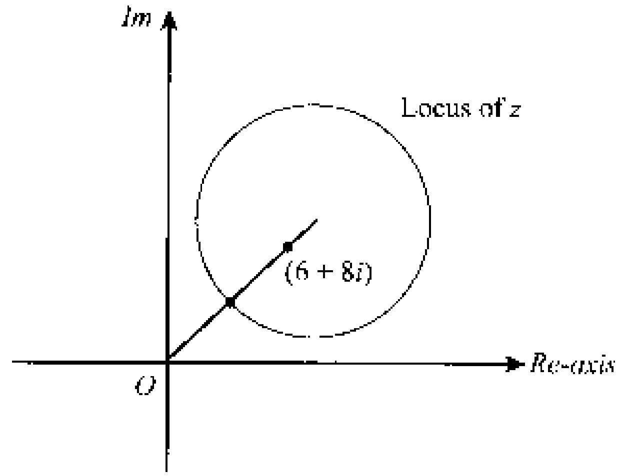
$|z - (3 + 4i)| = 5$



(Answer)

(ii) $|z| = |z - 6 - 8i|$

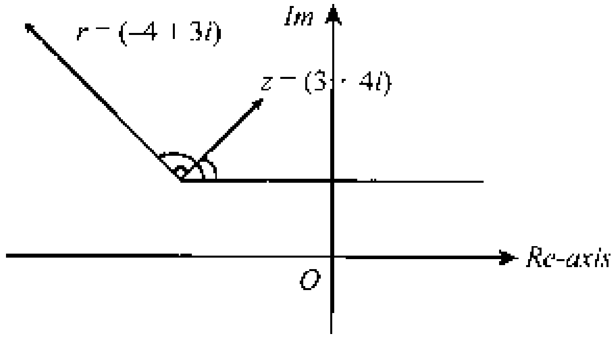
$|z| = |z - (6 + 8i)|$



(Answer)

(iii) $\arg(z - \mathbf{r}) \quad \arg z = \frac{1}{2}\pi$

$\arg(z - (-4 + 3i)) - \arg(3 + 4i)$



(Answer)

15. Given that

$u = (1 - i)$

(a) $x^3 - 5x^2 + 8x - k$

$f(1 - i) = (1 - i)^3 - 5(1 - i)^2 + 8(1 - i) - k = 0$

$= -2 - 2i - 5(-2i) + 8 - 8i - k = 0$

$-2 - 2i + 10i + 8 - 8i - k = 0$

$-k + 6 = 0$

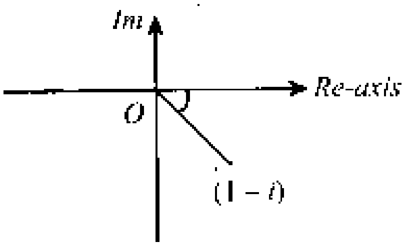
$k = 6$ (Answer)

(b) The other complex root is $(1 + i)$ (Answer)

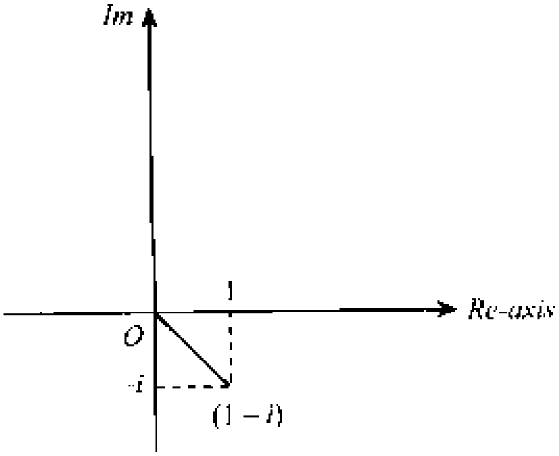
(c) $|u| = \sqrt{1^2 + (-1)^2} = \sqrt{1 + 1} = \sqrt{2}$ (Answer)

$\arg u = \tan^{-1}(-1)$

$= \dfrac{-\pi}{4}$ (Answer)



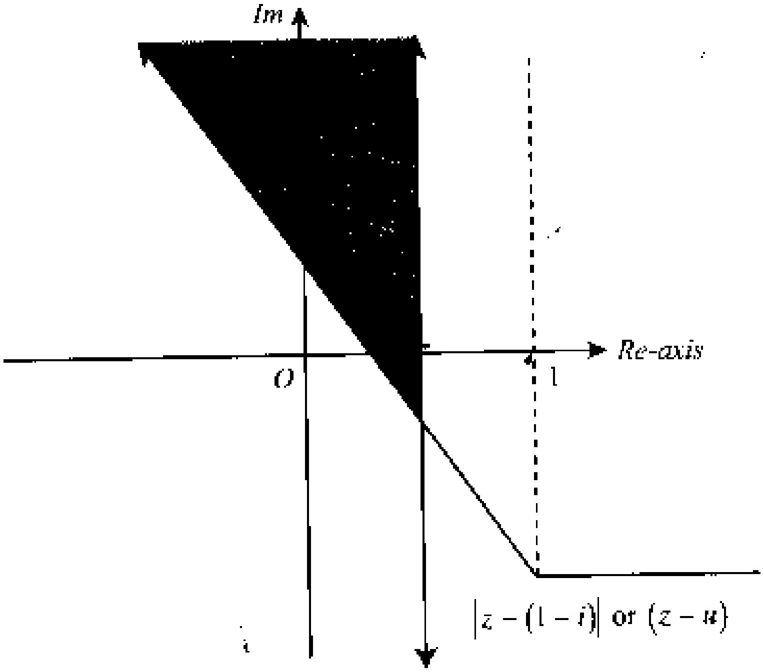
(d)



(Answer)

(e) $|z| < |z - 1|$

and $o < \arg(z - u) < \frac{2}{3}\pi$



$|z| < |z - 1|$ (Answer)

16. (a) $x^3 + 3x^2 + x + 3 = 0$

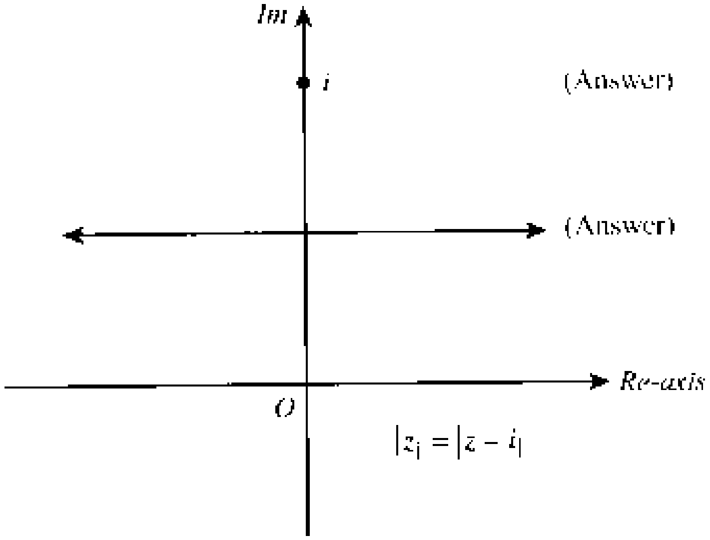
$f(i) = i^3 + 3i^2 + i + 3$

$= -i - 3 + i + 3 = 0$

$\therefore$ i is a complex root of the equation. (Answer)

(b) The other complex root is $-i$. (Answer)

(c)



17. Given that $u = 1 + 2i$,

(a) and (i) $w = 3\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$

$= 3\left(\frac{1}{2} + i \times \frac{\sqrt{3}}{2}\right)$

$= \frac{3}{2} + \frac{\sqrt{3}}{2}i$ (Answer)

(ii) $uw = (1 + 2i)\left(\frac{3}{2} + \frac{\sqrt{3}}{2}i\right)$

$= \frac{3}{2} + \frac{\sqrt{3}}{2}i + 3i + \sqrt{3}\,i^2$

$= \frac{3}{2} + i\left(3 + \frac{\sqrt{3}}{2}\right) - \sqrt{3}$

$= \left(\frac{3}{2} - \sqrt{3}\right) + i\left(3 + \frac{\sqrt{3}}{2}\right)$ (Answer)

(iii) $$\frac{u}{w} = \frac{i + 2i}{\frac{3}{2} + \frac{\sqrt{3}}{2}i}$$

$$= \frac{2 + 4i}{3 + \sqrt{3}\,i} \times \frac{3 - \sqrt{3}\,i}{3 - \sqrt{3}\,i}$$

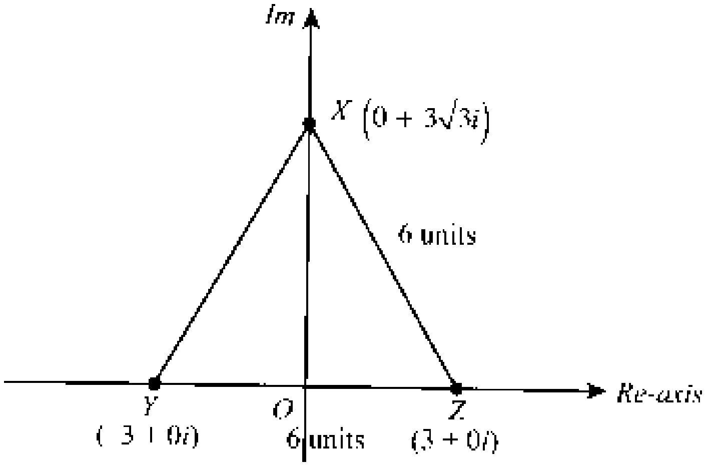
$$= \frac{6 - 2\sqrt{3}i + 12i + 4\sqrt{3}}{9 + 3}$$

$$= \frac{\left(6 + 4\sqrt{3}\right)}{12} + \frac{\left(12 \quad 2\sqrt{3}\right)i}{12}$$

$$= \frac{\left(3 + 2\sqrt{3}\right)}{6} + \frac{\left(6 \quad \sqrt{3}\right)i}{6}$$

(Answer)

(b)



(Answer)

(c) ΔXYZ is equilateral. (Answer)

18. Given that $u = 1 + 2i$
and $v = 3 - i$

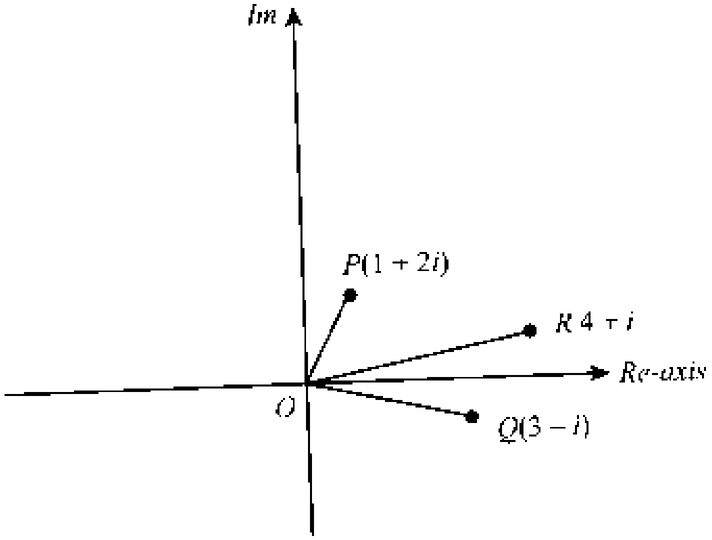
(a) $u + v = (1 + 2i) + (3 - i)$
$= 1 + 2i + 3 - i$
$= 4 + i$

(Answer)

$$uv = (1 + 2i)(3 - i) = 3 - i + 6i + 2$$
$$= 5 + 5i$$
(Answer)

(b) $$\arg(uv) = \arg(5 + 5i) = \tan^{-1}(1)$$
$$= \frac{\pi}{4}$$
(Answer)

(c)



$$P\hat{O}R = \arg OP - \arg OR$$
$$= \tan^{-1}(2) - \tan^{-1}\left(\frac{1}{4}\right)$$
$$= 1.107 - 0.245$$
$$= 0.862$$
(Answer)

$$Q\hat{O}R = \arg OQ + \arg OR$$
$$= \tan^{-1}\left(-\frac{1}{3}\right) + \tan^{-1}\left(\frac{1}{4}\right)$$
$$= 0.322 + 0.245$$
$$= 0.567$$
(Answer)

19. (a) $z^2 + 3z + (4i + 2) = 0$

$$z = \frac{-3 \pm \sqrt{9 - 4(4i + 2)}}{2}$$

$$= \frac{-3 \pm \sqrt{9 - 16i - 8}}{2}$$

$$= \frac{-3 \pm \sqrt{1 - 16i}}{2}$$

Let $\sqrt{1 - 16i} = x + iy$

then $1 - 16i = x^2 - y^2 + 2i\,xy$

$$x^2 - y^2 = 1 \qquad\qquad 2xy = -16$$

$$x^2 - \left(\frac{-8}{x}\right)^2 = 1 \qquad\qquad xy = -8$$

$$y = \frac{-8}{x}$$

$$x^2 - \left(\frac{64}{x^2}\right) = 1$$

$$x^4 - 64 = x^2$$

$$x^4 - x^2 - 64 = 0$$

Replacing x^2 by y, we obtain

$$y^2 - y - 64 = 0$$

$$y = \frac{1 \pm \sqrt{1 + 256}}{2}$$

$$y = \frac{1 + \sqrt{257}}{2} \quad \text{or} \quad \frac{1 - \sqrt{257}}{2}$$

$$y = \frac{1 + 16.03}{2} \quad \text{or} \quad \frac{1 - 16.03}{2}$$

$$y = \frac{17.03}{22} \quad \text{or} \quad \frac{-15.03}{2}$$

$$y = 8.52 \quad \text{or} \quad -7.52$$

$$x^2 = 8.52 \quad (\text{or} \quad -7.52) \text{ rejected}$$

$$x = \pm\sqrt{8.52} = \pm 2.919$$

$$y = \frac{-8}{2.919} \quad \text{or} \quad \frac{-8}{-2.919}$$

$$y = -2.741 \quad \text{or} \quad 2.741$$ (Answer)

But $$z = \frac{-3 + \sqrt{1-16i}}{2} \quad \text{or} \quad \frac{-3 \quad \sqrt{1-16i}}{2}$$

i.e. $$z = \frac{3 + 2.191 - 2.741i}{2} \quad \text{or} \quad \frac{-3 - 2.919 + 2.741i}{2}$$

$$z = \frac{0.081 - 2.741i}{2}$$

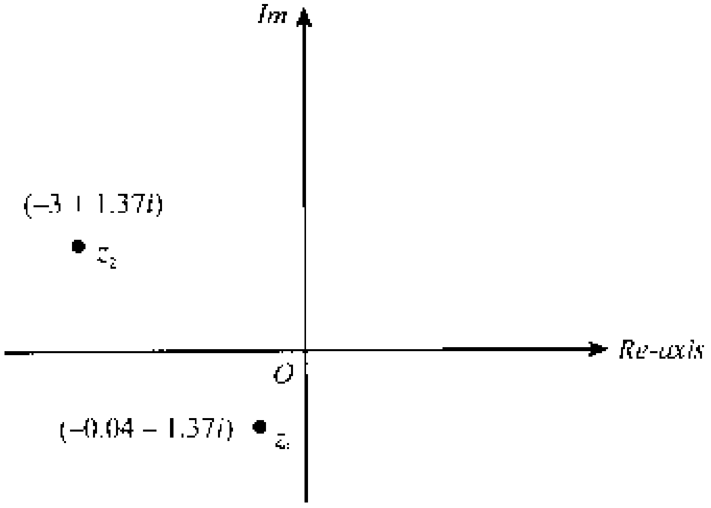
$$= \left(-0.0405 - 1.3705i\right) \quad \text{or} \quad \frac{-5.919 + 2.741i}{2}$$

$$z = (-0.0405 - 1.3705i) \quad \text{or} \quad (-2.9595 + 1.3705i)$$ (Answer)

(b) modulus of $z = |z| = \sqrt{1.8799} = 1.371$ (Answer)

$\arg z = \tan^{-1}(33.8)$

(c) $z = -0.04 - 1.37i \quad \text{or} \quad z = 3 + 1.37i$



(Answer)

20. $$u = \frac{2+i}{3-i}$$

(a) $$\frac{2+i}{3-i} \times \frac{3+i}{3+1}$$

$$= \frac{6 + 2i + 3i \quad 1}{9+1}$$

$$= \frac{5+5i}{10}$$

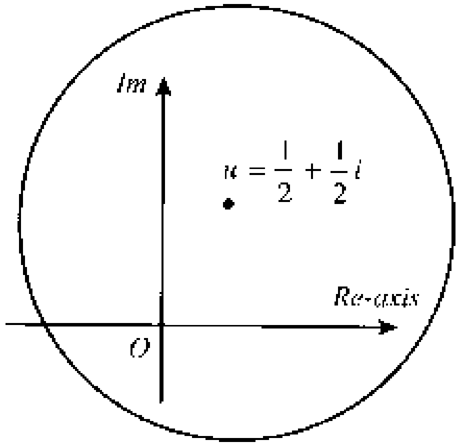
$$= \frac{1}{2} + \frac{1}{2}i$$ (Answer)

(b) Modules of $u = |u = \sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2} = \sqrt{\frac{1}{4} + \frac{1}{4}}$

$$= \sqrt{\frac{1}{2}} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$$ (Answer)

$$\arg u = \tan^{-1}\left(\frac{\frac{1}{2}}{\frac{1}{2}}\right) = \tan^{-1}(1) = \frac{\pi}{4}$$ (Answer)

(c)



(Answer)

(d) Circle drawn with centre u and radius 2.

UNIT 18

FIRST ORDER DIFFERENTIAL EQUATIONS

Suggested Solutions

1. (a) $y\frac{\mathrm{d}y}{\mathrm{d}x} = \cos x$

$y\,\mathrm{d}y = \cos x\,\mathrm{d}x$

$\frac{1}{2}y^2 = \sin x + c$ (Answer)

(b) $\frac{\mathrm{d}y}{\mathrm{d}x} = xy$

$\frac{1}{y}\mathrm{d}y = x\,\mathrm{d}x$

$\ln y = \frac{1}{2}x^2 + c$

$e^{\frac{1}{2}x^2 + c} = y$

$y = e^{\frac{1}{2}x^2} \times e^c$

$y = e^{\frac{1}{2}x^2} \times A$

$y = A\,e^{\frac{1}{2}x^2}$ (Answer)

e^c is a constant, which can be replaced by A

2. (a) $\frac{1}{x}\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y}{x^2 + 1}$

$\frac{1}{y}\,\mathrm{d}y = \frac{x}{x^2 + 1}\,\mathrm{d}x$

$$\int_2^y \frac{1}{y}\,\mathrm{d}y = \int_1^x \frac{x}{x^2+1}\,\mathrm{d}x$$

$$\left[\ln y\right]_2^y = \frac{1}{2}\left[\ln\left(x^2+1\right)\right]_1^x$$

$$\therefore \quad \ln y - \ln 2 = \frac{1}{2}\left(\ln\left(x^2+1\right) - \ln 2\right)$$

$$\ln y = \frac{1}{2}\left[\ln\left(x^2+1\right) - \ln 2\right] + \ln 2$$

$$\ln y = \frac{1}{2}\left(\ln\left(x^2+1\right) - \ln 2 + 2\ln 2\right)$$

$$\ln y = \frac{1}{2}\left(\ln\left(x^2+1\right) + \ln 2\right)$$

$$\ln y = \frac{1}{2}\ln(2)\left(x^2+1\right)$$

$$\ln y = \ln\left\{2\left(x^2+1\right)\right\}^{\frac{1}{2}}$$

$$y = \left\{2\left(x^2+1\right)\right\}^{\frac{1}{2}}$$

$$\therefore \quad y = \sqrt{2\left(x^2+1\right)} \qquad \text{(Answer)}$$

(b)

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 2y$$

$$\frac{1}{y}\,\mathrm{d}y = 2\,\mathrm{d}x$$

$$\int_3^y \frac{1}{y}\,\mathrm{d}y = 2\int_0^x \mathrm{d}x$$

$$\left[\ln y\right]_3^y = 2\left[x\right]_0^x$$

$$\ln y - \ln 3 = 2x$$

$$\ln\frac{y}{3} = 2x$$

$$e^{2x} = \frac{y}{3}$$

$$\therefore \quad y = 3e^{2x} \qquad \text{(Answer)}$$

3. Let x = no. of bacteria at time t hours

and let a = no. of bacteria at time $t = 0$

then $\frac{dx}{dt} \propto x$

i.e. $$\frac{dx}{dt} = kx$$

$$\frac{1}{x}\,dx = k\,dt$$

$$\int_a^{2a} \frac{1}{x}\,dx = \int_0^3 k\,dt$$

$$\left[\ln x\right]_a^{2a} = \left[kt\right]_0^3$$

$$\ln 2a - \ln a = 3k - 0$$

$\therefore$ $$k = \frac{1}{3}\ln\frac{2a}{a}$$

i.e. $$k = \frac{1}{3}\ln 2$$

When the bacteria will triple, $x = 3a$ when $t = T$

$\therefore$ $$\int_a^{3a} \frac{1}{x}\,dx = \int_0^T k\,dt$$

$$\left[\ln x\right]_a^{3a} = KT$$

$$\ln 3a \quad \ln a = T \times \frac{1}{3}\ln 2$$

$$\ln\frac{3}{a} = T \times \frac{1}{3}\ln 2$$

$$\ln 3 = \frac{1}{3}T\ln 2$$

$$T = \frac{3\ln 3}{\ln 2}$$

$\therefore$ $T = 3 \times 1.6 = 4.8$ hours (Answer)

4.

$$\frac{dy}{dx} = 2x^2 - 4x$$

$$dy = (2x^2 - 4x)\ dx$$

$$\int dy = \int \left(2x^2 - 4x\right) dx$$

$$y = \frac{2}{3}x^3 - 2x^2 + c$$

Since $y = 3$ when $x = 1$

$$3 = \frac{2}{3} - 2 + c$$

$$3 - \frac{2}{3} + 2 = c$$

$$c = 4\tfrac{1}{3} = \frac{13}{3}$$

$\therefore$ The particular solution is

$$y = \frac{2}{3}x^3 \quad 2x^2 + \frac{13}{3}$$

(Answer)

5.

$$\frac{ds}{dt} = t^2 - t + 4$$

$$ds = (t^2 - t + 4)\ dt$$

$$\int ds = \int \left(t^2 - t + 4\right) dt$$

$$s = \frac{1}{3}t^3 - \frac{1}{2}t^2 + 4t + c$$

$$250 = \frac{125}{3} - \frac{25}{2} + 20 + c$$

$$250 - \frac{125}{3} + \frac{25}{2} - 20 = c$$

$$c = \frac{1\,500 - 250 + 75 - 120}{6}$$

$$c = \frac{1\,205}{6}$$

$\therefore \quad s = \frac{1}{3}t^3 - \frac{1}{2}t^2 + 4t + \frac{1\,205}{6}$ (Answer)

initial displacement is when $t = 0$,

then $s = \frac{1\,205}{6} = 200\frac{5}{6}\ m$ (Answer)

6. (a) $$\frac{dy}{dx} = 2xy^2$$

$$\frac{1}{y^2}\ dy = 2x\ dx$$

$$\int y^{-2}\ dy = \int 2x\ dx$$

$$-y^{-1} = x^2 + c$$

$$\frac{-1}{y} = x^2 + c$$

$$\frac{1}{y} = Ax^2$$

$$\frac{1}{y} = Kx^2$$

where $K = -A$

$$y = \frac{1}{Kx^2}$$

$$y = \frac{1}{K}x^{-2}$$ (Answer)

(b) $$\frac{dy}{dx} = \frac{e^{2x}}{y}$$

$$y\ dy = e^{2x}\ dx$$

$$\int y\ dy = \int e^{2x}\ dx$$

$$\frac{y^2}{2} = \frac{e^{2x}}{2} + c$$

When $y = 3$, $x = 0$

$$\frac{9}{2} = \frac{e^{0}}{2} + c$$

$$\frac{9}{2} = \frac{1}{2} + c$$

$$c = 4$$

$$\therefore \quad \frac{y^2}{2} = \frac{e^{2x}}{2} + 4$$

$$y^2 = e^{2x} + 8$$

$$y = \pm \sqrt{e^{2x} + 8}$$

(Answer)

7.

$$\frac{dr}{dt} = \frac{0.1}{r}$$

$$r\,dr = 0.1\,dt$$

$$\int r\,dr = \int 0.1\,dt$$

$$\frac{r^2}{2} = 0.1\,t + c$$

when $t = 0$, $r = 0.4$ cm

$$\frac{(0.4)^2}{2} = 0 + c$$

$$c = \frac{0.16}{2} = 0.08$$

$$\therefore \quad \frac{r^2}{2} = 0.1t + 0.08$$

(Answer)

When $t = 2$,

$$\frac{r^2}{2} = 0.1 \times 2 + 0.08$$

$$r^2 = 0.4 + 0.16$$

$$r^2 = 0.56$$

$$\text{Area} = \pi r^2$$

$$= 0.56\,\pi \text{ cm}^2$$

(Answer)

8.

$$\frac{\mathrm{d}r}{\mathrm{d}t} = \frac{k}{r^2}$$ (Answer)

$$r^2\,\mathrm{d}r = k\,\mathrm{d}t$$

$$\int r^2\,\mathrm{d}r = \int k\,\mathrm{d}t$$

$$\frac{1}{3}r^3 = kt + c$$

When $r = 16, \quad t = 3$

$$\therefore \quad \frac{1}{3} \times (16)^3 = 3k + c$$

But when $r = 1, \; t = 0$

$$\therefore \quad \frac{1}{3} = 0 + c$$

and $c = \frac{1}{3}$

$$\frac{1}{3} \times (16)^3 = 3k + c$$

$$\frac{4\,096}{3} = 3k + \frac{1}{3}$$

$$9k = 4\,095$$

$$k = 455$$

(Answer)

$$\therefore \quad \frac{1}{3}r^3 = 455t + \frac{1}{3}$$

When $r = 20$

$$\frac{1}{3} \times (20)^3 = 455t + \frac{1}{3}$$

$$\frac{8\,000}{3} - \frac{1}{3} = 455t$$

$$\frac{7\,999}{3} = 455t$$

$$t = \frac{7\,999}{3 \times 455} = 5.9 \text{ sec}$$

(Answer)

9. (a) $$\frac{dy}{dt} = ky$$

$$\frac{1}{y}\,dy = k\,dt$$

$$\int \frac{1}{y}\,dy = \int k\,dt$$

$$\ln y = kt + c$$

When $t = 0,\ y = y_0$

$$\ln y_0 = c$$

$\therefore$ $$\ln y = kt + \ln y_0$$

$$\ln y - \ln y_0 = kt$$

$$\ln \frac{y}{y^0} = kt$$

$$e^{kt} = \frac{y}{y^0}$$

$$y = y_0 e^{kt}$$ (Answer)

(b) When $t = 5,\ y = 2y_0$.

So $$2y_0 = y_0 e^{k \times 5}$$

$$e^{5k} = 2$$

$$5k = \ln 2$$

$$k = \frac{1}{5}\ln 2$$

$\therefore$ $$y = y_0\, e^{(\frac{1}{5}\ln 2)\,t}$$

After a further 10 years, $t = 15$

So $$y = y_0\, e^{(\frac{1}{5}\ln 2)} \times 15$$

$$= y_0\, e^{\ln 8}$$

$$= 8\, y_0$$

$\therefore$ After 15 years, the population has increased by 8 times. (Answer)

10. $\frac{dx}{dt} = kx$ ($k > 0$, because there is an increase)

$\int \frac{1}{x}\, dx = \int k\, dt$

$\ln x = kt + c$

When $t = 0$, $x = 1\,000$

$\therefore$ $\ln 1\,000 = 0 + c$

$c = \ln 1\,000$

i.e. $\ln x = kt + \ln 1\,000$

When $t = 1$, $x = 1\,672$

$\ln 1\,672 = k \times 1 + \ln 1\,000$

$k = \ln 1.672$

$\therefore$ $\ln x = (\ln 1.672) \times t + \ln 1\,000$

Now when $x = 2\,000$,

$\ln 2\,000 = (\ln 1.672) \times t + \ln 1\,000$

$t \times (\ln 1.672) = \ln 2$

$t = \frac{\ln 2}{\ln 1.672} = 1.35$ hours

Time will be 13.35 or 1.35 p.m (Answer)

11. $\frac{dy}{dt} \propto y$

$\therefore$ $\frac{dy}{dt} = -ky$ (there is decay, the rate is negative)

$\int \frac{1}{y}\, dy = \int k\, dt$

$\ln y = -kt + c$

$y = Ae^{kt}$

When $t = 0$, let $y = y_0$,

Then $y_0 = Ae^0$

$y_0 = A$

i.e. $y = y_0 e^{kt}$

When $t = 2\,000$, $y = \frac{1}{2}\,y_0$

$$\therefore \qquad \frac{1}{2}\,y_0 = y_0\,e^{2\,000k}$$

$$2 = e^{2\,000k}$$

$$\ln 2 = 2\,000k$$

i.e. $\qquad k = \frac{\ln 2}{2\,000}$

$$\therefore \qquad y = y_0\,e^{\left(\frac{\ln 2}{2\,000}\right)t}$$

When $\quad t = 100$,

$$y = y_0\,e^{\left(\frac{\ln 2}{2\,000}\right) \times 100}$$

$$= 96.53\,y_0$$

$\therefore \qquad$ After 100 years, about 97% still remains. (Answer)

12. Let T° C be the temperature of the body at time t minutes after being brought into its surroundings.

Then $\quad \frac{dT}{dt} = -k\left(T - 17\right)$

$$\therefore \qquad \int \frac{1}{\left(T - 17\right)}\,dT = -\int k\,dt$$

$$\ln\left(T - 17\right) = -kt + c$$

$$T - 17 = Ae^{kt}$$

When $\quad t = 0$, $T = 60°$

$$\therefore \qquad 60° - 17° = A$$

$$A = 43°$$

$$T \quad 17 = 43\,e^{-kt}$$

When $\quad t = 10$, $T = 48°$

$$\therefore \qquad 48 - 17 = 43\,e^{10k}$$

$$31 = 43\,e^{-10k}$$

$$e^{\ 10k} = \frac{31}{43}$$

$$e^{10k} = \frac{43}{31}$$

$$10k = \ln\left(\frac{43}{31}\right)$$

$$k = \frac{1}{10}\ln\left(\frac{43}{31}\right)$$

When $t = 20$ (after a furthur 10 minutes)

$$T - 17 = 43\,e^{-\left(\frac{1}{10}\ln\frac{43}{31}\right)\times 20}$$

$$\begin{aligned} T &= 17 + 43\,e^{\ 2\ln\frac{43}{31}} \\ &= 17 + 43 \times 0.0443 \\ &= 17 + 1.9 \\ &= 18.9 \end{aligned}$$

(Answer)

13. (a) $V = \frac{4}{3}\,h^3$

$$\frac{dV}{dh} = 4h^2$$

$$\frac{dh}{dt} = \underset{\substack{\uparrow \\ \textbf{in}}}{h} \quad \underset{\substack{\uparrow \\ \textbf{out}}}{kh^2}$$

When $\frac{dh}{dt} = 10$, $h = 2$, we get

$$10 = 2 \quad 4k$$

$$k = -2$$

$$\therefore \quad \frac{dh}{dt} = h + 2h^2$$

(Answer)

(b) $$\frac{dh}{\left(h + 2h^2\right)} = dt$$

$$\frac{1}{h(1 + 2h)}\, dh = dt$$

using partial fractions,

$$\left\{\frac{1}{h} - \frac{2}{(1 + 2h)}\right\} dh = dt$$

$$\int\left\{\frac{1}{h} - \frac{2}{(1 + 2h)}\right\} dh = dt$$

$$\ln h - \ln (1 + 2h) = t + c$$

When $t = 0, h = 0,$

$\therefore \quad \ln h - \ln (1 + 2h) = t$

i.e. $$t = \ln\left(\frac{h}{1 + 2h}\right)$$ (Answer)

14. (a) $$\frac{dP}{dt} \propto \sqrt{(P - x)}$$

$$\frac{dP}{dt} = k\sqrt{(P - x)}$$

(b) $$\int \frac{1}{\sqrt{(P - x)}}\, dp = -kt + c$$

$$2(P - x)^{\frac{1}{2}} = -kt + c$$

When $t = 0, P = 3x$

$$2(3x - x)^{\frac{1}{2}} = c$$

$\therefore \quad 2\,(P - x)^{\frac{1}{2}} = -kt + 2(2x)^{\frac{1}{2}}$

When $P = x, t = 1$

$$2(0)^{\frac{1}{2}} = -k + 2(2x)^{\frac{1}{2}}$$

∴ $$k = 2 \times (2x)^{\frac{1}{2}}$$

Hence $2(P - x)^{\frac{1}{2}} = -2(2x)^{\frac{1}{2}} t + 2(2x)^{\frac{1}{2}}$

(c) When $P = 2x$

$$2(x)^{\frac{1}{2}} = -2(2x)^{\frac{1}{2}} t + 2(2x)^{\frac{1}{2}}$$

$$2 = -2(2)^{\frac{1}{2}} t + 2(2)^{\frac{1}{2}}$$

$$2\sqrt{2}\ t = 2\sqrt{2} - 2$$

$$\sqrt{2}\ t = \sqrt{2} - 1$$

$$t = 1 - \frac{1}{\sqrt{2}}$$ (Answer)

(d) $2(P - x)^{\frac{1}{2}} = -2(2x)^{\frac{1}{2}} t + 2(2x)^{\frac{1}{2}}$ divide by 2

$$(P - x)^{\frac{1}{2}} = -(2x)^{\frac{1}{2}} t + 2(2x)^{\frac{1}{2}}$$

$$(P - x)^{\frac{1}{2}} = (2x)^{\frac{1}{2}} \{1 - t\}$$ squaring both sides

$$P - x = 2x(1 - t)^2$$

$$P = 2x(1 - t)^2 + x$$

15. (a) $$\frac{d\alpha}{dt} = \frac{-k}{\alpha}$$ (Answer)

$$\alpha \ d\alpha = -k \ dt$$

$$\int \alpha \ d\alpha = -\int k \ dt$$

$$\frac{\alpha^2}{2} = -kt + c$$

$$\frac{\alpha^2}{2} = A\ kt$$

$$\alpha^2 = 2A\ kt$$

$$\alpha = \sqrt{2A\ kt}$$ (Answer)

(b) When $\alpha = 10$, $t = 10$.

$$10 = \sqrt{2A\ k \times 10}$$

$$\sqrt{2A\ k \times 10} = 10$$

$$2A\ k \times 10 = 100$$

$$k = \frac{100}{10 \times 2A} = \frac{5}{A}$$

$$\alpha^2 = 2A\ kt$$

$$\alpha^2 = 2A \times \frac{5}{A} \times 12$$

$$\alpha^2 = 120$$

$$\alpha = \sqrt{120}$$

$$\approx 11^\circ$$ (Answer)

16. $$\frac{dN}{dt} = \frac{1}{8}(N - 600)$$ (Answer)

$$\int \frac{1}{(N \quad 600)}\, dN = \int \frac{1}{8}\, dt$$

$$\ln(N - 600) = \frac{1}{8}t + c$$

$$\ln(1\ 000 - 600) = 0 + c$$

$$c = \ln 400$$

$$\ln(N - 600) = \frac{1}{8}t + c$$ (Answer)

When $N = 15\ 000$

$$\ln(15\ 000 - 600) = \frac{1}{8}t + \ln 400$$

$$\frac{1}{8}t = \ln 14\ 400 - \ln 400$$

$$\frac{1}{8}t = \ln\frac{14\ 400}{400}$$

$$\frac{1}{8}t = \ln 36$$

$$t \approx 29 \text{ days}$$ (Answer)

17. (a) $\frac{dr}{dt} \propto \frac{1}{r^2}$

$\therefore$ $$\frac{dr}{dt} = k \times \frac{1}{r^2}$$

$$r^2\,dr = k\,dt$$

$$\int r^2\,dr = \int k\,dt$$

$$\frac{r^3}{3} = kt + c$$

When $t = 0, r = 1$

$\therefore$ $$\frac{1}{3} = 0 + c$$

$$c = \frac{1}{3}$$

hence $\frac{r^3}{3} = kt + \frac{1}{3}$

When $t = 3, r = 16$

$\therefore$ $$\frac{16^3}{3} = 3k + \frac{1}{3}$$

$$3k = \frac{4\ 096}{3} - \frac{1}{3}$$

$$3k = \frac{4\ 095}{3}$$

$$k = \frac{4\ 095}{3 \times 3} = 455$$

$\therefore$ $$r^3 = 3 \times 455t + 1$$

When $r = 20$, we have

$$20^3 = 1\,365t + 1$$

$$t = \frac{8\,000 - 1}{1\,365} = 5.9 \text{ seconds (to 2 significant figures)}$$ (Answer)

18. (a)

$$\frac{dy}{dx} = \left(\frac{y}{x}\right)^2$$

$$\frac{dy}{dx} = \frac{y^2}{x^2}$$

$$\frac{1}{y^2}\,dy = \frac{1}{x^2}\,dx$$

$$\int y^{-2}\,dy = \int x^{-2}\,dx$$

$$y^{-1} = -x^{-1} + c$$

$$\frac{-1}{y} = \frac{-1}{x} + c$$

$$\frac{1}{y} = \frac{1}{x} - c$$

$$\frac{1}{y} = \frac{1 - xc}{x}$$

$$y = \frac{x}{1 - xc}$$ (Answer)

(b) **When** $x = 1$, $y = 2$

$$2 = \frac{1}{1 - c}$$

$$2(1 - c) = 1$$

$$2 - 2c = 1$$

$$2c = 1$$

$$c = \frac{1}{}$$

$$\therefore \quad y = \frac{x}{1 - \frac{1}{2}x}$$

$$y = \frac{2x}{2 - x}$$ (Answer)

19. (a) $\frac{dx}{dt} \propto x$

$\therefore \quad \frac{dx}{dt} = -kx \qquad (\therefore$ there is decay)

$$\frac{1}{x}\,dx = -k\,dt$$

$$\int \frac{1}{x}\,dx = \int -k\,dt$$

$$\ln x = -kt + c$$

When $t = 0, x = M$

$\therefore \quad \ln M = 0 + c$

$$c = \ln M$$

i.e. $\quad \ln x = -kt + \ln M$ (Answer)

(b) $\quad x = e^{kt - \ln M}$

$$x = e^{kt} \times e^{\ln M}$$

$\therefore \quad x = Ae^{-kt}$ (Answer)

But $A = e^{\ln M}$

$\therefore \quad \ln A = \ln M$

So $A = M$ (Answer)

When $x = \frac{1}{2}M, t = 20$

$$\frac{1}{2}M = Me^{20k} \qquad\qquad e^{20k} = 2$$

$$\frac{1}{2} = e^{20k} \qquad\qquad \ln 2 = 20k$$

$$\ln\frac{1}{2} = -20k \qquad\qquad k = \frac{1}{20}\ln 2$$

$$k = -\frac{1}{20}\ln\frac{1}{2}$$

So that $\ln x = \left(-\frac{1}{20}\ln 2\right)t + \ln M$

When $x = \frac{1}{3}M$

$$\ln \frac{1}{3}M = \left(-\frac{1}{20}\ln 2\right)t + \ln M$$

$$\left(\frac{1}{20}\ln 2\right)t = \ln M - \ln \frac{1}{3}M$$

$$\left(\frac{1}{20}\ln 2\right)t = \ln \frac{M}{\frac{1}{3}M}$$

$$\left(\frac{1}{20}\ln 2\right)t = \ln 3$$

$$t = \frac{\ln 3}{\frac{1}{20}\ln 2}$$

$$t = \frac{20\ln 3}{\ln 2} = 20 \times 1.585 = 31.7 \text{ hours}$$ (Answer)

20. $\frac{dS}{dt} \propto S$

$$\frac{dS}{dt} = -kS \quad \text{(decrease in temperature)}$$

$$\frac{1}{S}\,dS = -kdS$$

$$\int \frac{1}{S}\,dt = \int -k\,dt$$

$$\ln S = -kt + c$$

When $t = 0$, $S = (70° \quad 18°)$. By substitution, we have

$$= 52°$$

$$\ln 52° = c$$

When $t = 10$, $S = 50° \quad 18° = 32°C$

$$\ln 32 = -10k + \ln 52$$

$$-10k = \ln \frac{32}{52}$$

$$-k = \frac{1}{10} \ln \left(\frac{32}{52}\right)$$

$$-k = \frac{1}{10} \ln \left(\frac{8}{13}\right)$$

$$\therefore \qquad \ln S = \frac{t}{10} \ln \left(\frac{8}{13}\right) + \ln 52$$

$$\ln S = \ln \left(\frac{8}{13}\right)^{\frac{t}{10}} + \ln 52$$

$$\ln S = \ln \left\{\left(\frac{8}{13}\right)^{\frac{t}{10}} \times 52\right\}$$

$$S = 52 \times \left(\frac{8}{13}\right)^{\frac{t}{10}}$$

After 10 more minutes, $t = 20$

$$S = 52 \times \left(\frac{8}{13}\right)^{\frac{20}{10}}$$

$$= 52 \times \left(\frac{8}{13}\right)^{2} = \frac{52 \times 8 \times 8}{13 \times 13}$$

$$= \frac{256}{13} = 19.7 \text{ minutes}$$

$\therefore$ Final temperature = 19.7°C + 18°C

= 37.7°C (Answer)