'A' LEVEL PURE MATHEMATICS

Theory-Practice Nexus

First EDITION

Zimsec Syllabus-Specific

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PREFACE

This book came as a result of rigorous work and extensive consultation with 'A' Level Mathematics teachers, former and current students. After successfully publishing the students' guide series in 'A' Level Mechanics, Pure Mathematics and Statistics for the Cambridge International Examination 'A' level syllabus, the authors saw it necessary to also provide relevant study material to their constituency as a way of contributing to the development of the human in Africa through education.

This book is the first and only mathematics text book authored, printed and published in Zimbabwe by Zimbabweans for Zimbabwean 'A' Level mathematics students. Most, if not all text books available on the market fail to adequately tackle the Zimsec syllabus well as they are generally authored to suit any syllabus hence the authors of this book saw a knowledge gap which resulted in the production of this book.

Students equipped with this book need not worry about consulting other books as it provides a permanent solution to the Zimsec syllabus. It focuses on paper 1 which is a compulsory paper written by all students sitting for the Zimsec 'A' level syllabus.

The book was developed using the 'doing by learning' and 'learning by doing' approaches. At the beginning of each chapter, the book gives a brief, but complete theoretical framework meant to orient students to the topic at hand. This 'lead-in' theory is specifically designed to give students a learning platform before putting the concepts into practice, thus 'doing by learning'. To cement on the concepts learnt, detailed worked examples from past exam papers are incorporated into the text coupled with diagrams, tables and hints. We believe that plunging a student in the practice zone enables one to have a feel of the terrain. As such, a pool of challenging past exam questions with answers at the back are part of every topic to ensure 'learning by doing'. The fusion of the two approaches is an attempt to 'hit two birds with one stone' thereby providing a theory-practice nexus. While covering every topic within the syllabus, the book places concepts in an order that is incremental in nature with linking devices called **adverts** strategically placed between sections. The adverts are used as introductory units to a pool of topics that apply the same concept.

We have no doubt that not only students will find this book useful hence it is important that every school provides it to both students and teachers if they are serious about producing quality results.

We would like to acknowledge the Cambridge International Examinations (CIE) and the Zimbabwe School Examinations Council (ZIMSEC) for the past exam questions used in this text book. Responsibility for the answers to the questions is the authors' alone.

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Acronyms and Abbreviations

ZIMSEC	Zimbabwe Schools Examination Council
CIE	Cambridge International Examinations
(Z)	Zimsec
(C)	Cambridge
(ZOAM)	Zimsec 'O' Level Additional Mathematics
(ZSP)	Zimsec Specimen Paper
z	Complex Number

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Chapter One: Quadratics

"A bold onset is half the battle."

– Giuseppe Garibaldi

This topic deals with algebraic problems in higher order power two. Equations are named after their highest order power. For example:

$ax^1 + b = 0$	Linear equation
$\boldsymbol{ax^2 + bx + c = 0}$	**Quadratic equation**
$ax^3 + bx^2 + cx + d = 0$	Cubic equation and so on

Quadratic equations are solved using either the method of factorisation or the quadratic formula. Of particular interest is the quadratic formula which states that:

$$\boldsymbol{x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}}$$

At this stage, much emphasis is placed on the discriminant; that is the value under the square root sign. A discriminant is a tool used to draw up a conclusion on the nature of roots or solutions.

$$\textbf{Discriminant} = \boldsymbol{b^2 - 4ac}$$

The discriminant gives birth to three conditions outlined immediately below:

1. $\boldsymbol{b^2 - 4ac = 0}$

   This condition gives rise to two identical roots or solutions commonly known as one *repeated* root. Two identical roots occur at the point of intersection of a **curve** and a **tangent** as shown by the diagram below:

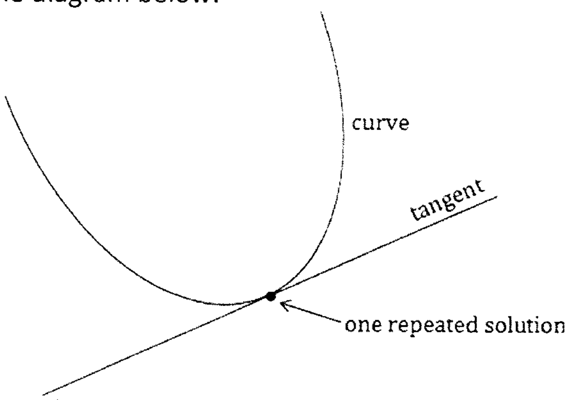


Fig. 1.1

If the discriminant is zero, the quadratic formula reduces to

$$x = \frac{-b \pm \sqrt{0}}{2a}$$

$$\Rightarrow x = \frac{-b+0}{2a} \qquad or \qquad x = \frac{-b-0}{2a}$$

$$\therefore \boldsymbol{x = -\frac{b}{2a}} \qquad or \qquad \boldsymbol{x = -\frac{b}{2a}} \qquad \begin{bmatrix} two\ identical\ roots \\ or\ one\ repeated\ root \end{bmatrix}$$

2. $\boldsymbol{b^2 - 4ac > 0}$

This condition leads to two different or *distinct* roots. It implies that the two graphs intersect at two different points as shown in Fig.1.2:

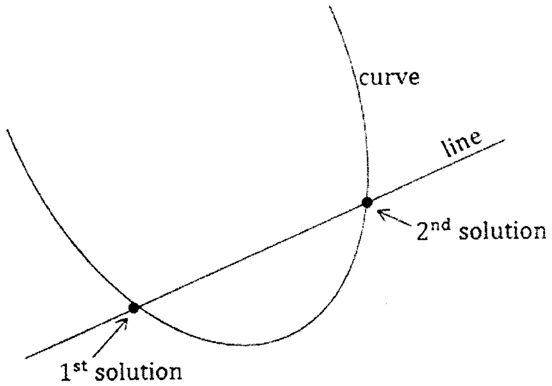


Fig. 1.2

Assuming that the discriminant is, for arguments sake, 9, the quadratic formula reduces to:

$$x = \frac{-b \pm \sqrt{9}}{2a}$$

$$\therefore \boldsymbol{x = \frac{-b+3}{2a}} \qquad \boldsymbol{or} \qquad \boldsymbol{x = \frac{-b-3}{2a}} \qquad \begin{bmatrix} \boldsymbol{two\ different} \\ \boldsymbol{solutions} \end{bmatrix}$$

3. $\boldsymbol{b^2 - 4ac < 0}$

This condition gives rise to non-existence of real roots. Diagrammatically, the two graphs do not intersect as shown in Fig. 1.3:

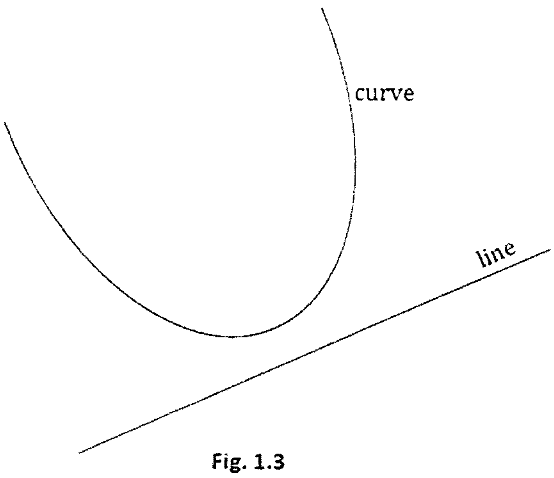


Fig. 1.3

Assuming that the discriminant is –9, the quadratic formula reduces to:

$$x = \frac{-b \pm \sqrt{-9}}{2a}$$

$\Rightarrow x$ is undefined because the square root of a negative number cannot be evaluated in real terms.

NB:

- To use any of the three conditions outlined above, one has to combine the two equations in question and reduce them into a general quadratic equation. It is from this general equation that one can pull out a, b and c.
- Conditions two and three make use of inequalities.

The nature of roots is inspired by the location of the discriminant on the number line. A discriminant is ether positive, negative or neutral (zero). Below is a number line summary of conditions hinged to the discriminant.

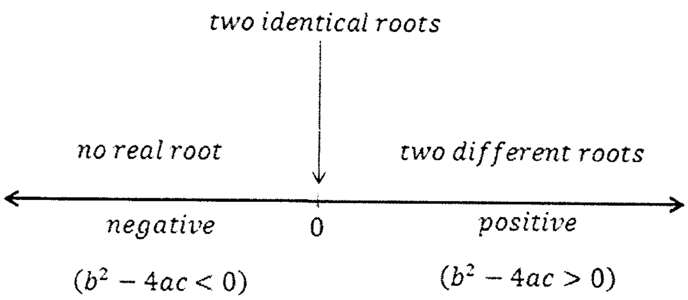


Fig. 1.4

Method of Completing Square

This technique is used to summarise a quadratic expression and/or equation. As the name implies, completing the square is inspired by 'perfect squares'. In this case, a quadratic expression is manipulated into a prefect square by considering the following steps in a standard quadratic expression, $ax^2 + bx + c$:

- Factoring out the coefficient of the term in degree power two with the view of making the coefficient positive one $(+1)$.

$$a\left(x^2 + \frac{b}{a}x + \frac{c}{a}\right)$$

- Adding and immediately subtracting the square of half the coefficient of the term in x.

$$a\left[x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 - \left(\frac{b}{2a}\right)^2 + \frac{c}{a}\right]$$

- Expressing the first and third terms as a factor summarises the first three terms into a perfect square. The last two terms remain the same.

$$a\left[\left(x + \frac{b}{2a}\right)^2 - \left(\frac{b}{2a}\right)^2 + \frac{c}{a}\right]$$

$$a\left[\left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a^2} + \frac{c}{a}\right]$$

- Extending the effect of the overall multiplier to all the terms inside the major pair of brackets.

$$a\left(x + \frac{b}{2a}\right)^2 + c - \frac{b^2}{4a}$$

NB: Once the quadratic expression has been successfully summarised, one can easily determine the coordinates at a turning point (whether maximum or minimum). The x value is given by equating the value inside the pair of brackets to zero and the corresponding y value is the part free of the brackets.

In this case,

$$x + \frac{b}{2a} = 0 \qquad and \qquad y = c - \frac{b^2}{4a}$$

$$x = -\frac{b}{2a}$$

$\therefore$ The coordinates at a turning point are $\left(-\frac{b}{2a};\ c - \frac{b^2}{4a}\right)$

Example 1

$if\ f(x) = 2x^2 - 12x + 13$

$\Rightarrow f(x) = 2\left[x^2 - 6x + \frac{13}{2}\right]$

$\Rightarrow f(x) = 2\left[x^2 - 6x + (-3)^2 - (-3)^2 + \frac{13}{2}\right]$

$\Rightarrow f(x) = 2\left[(x-3)^2 - 9 + \frac{13}{2}\right]$

$\Rightarrow f(x) = 2\left[(x-3)^2 - \frac{5}{2}\right]$

$\therefore \boldsymbol{f(x) = 2(x-3)^2 - 5}$

$\boldsymbol{where\ (3,\ -5)\ are\ the\ coordinates\ of a\ turning\ point}$

Example 2

$if\ f(x) = 2x^2 - 12x + 7$

$\Rightarrow f(x) = 2\left[x^2 - 6x + \frac{7}{2}\right]$

$\Rightarrow f(x) = 2\left[x^2 - 6x + (-3)^2 - (-3)^2 + \frac{7}{2}\right]$

$\Rightarrow f(x) = 2\left[(x-3)^2 - 9 + \frac{7}{2}\right]$

$\Rightarrow f(x) = 2\left[(x-3)^2 - \frac{11}{2}\right]$

$\therefore \boldsymbol{f(x) = 2(x-3)^2 - 11}$

$\boldsymbol{where\ (3,\ -11)\ are\ the\ coordinates\ of a\ turning\ point}$

Example 3

$h(x) = 6x - x^2$

$\Rightarrow h(x) = -1[x^2 - 6x]$

$\Rightarrow h(x) = -1[x^2 - 6x + (-3)^2 - (-3)^2]$

$\Rightarrow h(x) = -1[(x-3)^2 - 9]$

$\therefore \boldsymbol{h(x) = 9 - (x-3)^2}$

where* (3, 9) *are the coordinates of a turning point

Revision Exercise

1 The equation of a curve is $y = 8x - x^2$.

(i) Express $8x - x^2$ in the form $a - (x + b)^2$, stating the numerical values of a and b. [3]

(ii) Hence, or otherwise, find the coordinates of the stationary point of the curve. [2]

2 The function f is defined by $\mathrm{f} : x \mapsto 2x^2 - 8x + 11$ for $x \in \mathbb{R}$.

Express $\mathrm{f}(x)$ in the form $a(x + b)^2 + c$, where a, b and c are constants. [3]

3 Express $2x^2 - 4x + 1$ in the form $a(x + b)^2 + c$ and hence state the coordinates of the minimum point, A, on the curve $y = 2x^2 - 4x + 1$. [4]

4 Express $2x^2 + 8x - 10$ in the form $a(x + b)^2 + c$. [3]

5 The function $\mathrm{f} : x \mapsto x^2 - 4x + k$ is defined for the domain $x \geqslant p$, where k and p are constants.

Express $\mathrm{f}(x)$ in the form $(x + a)^2 + b + k$, where a and b are constants. [2]

Worked Examination Questions on Quadratics

Question (Cambridge, June 2007 qp.1)

1 Find the value of the constant c for which the line $y = 2x + c$ is a tangent to the curve $y^2 = 4x$. [4]

Solution

$y = 2x + c \longrightarrow 1$

$y^2 = 4x \longrightarrow 2$

by combining (1) *and* (2)

$(2x + c)^2 = 4x$

$\Rightarrow (2x + c)(2x + c) = 4x$

$\Rightarrow 4x^2 + 2cx + 2cx + c^2 = 4x$

$\Rightarrow 4x^2 + 4cx - 4x + c^2 = 0$

$\Rightarrow 4x^2 + (4c - 4)x + c^2 = 0$

where $a = 4;\ b = (4c - 4);\ c = c^2$

using $b^2 - 4ac = 0$

$\Rightarrow (4c - 4)^2 - 4(4)(c^2) = 0$

$\Rightarrow 16c^2 - 32c + 16 - 16c^2 = 0$

$\Rightarrow -32c + 16 = 0$

$\Rightarrow -32c = -16$

$\therefore \boldsymbol{c = \frac{1}{2}}$

Question (Cambridge, June 2009 qp.1)

2 Find the set of values of k for which the line $y = kx - 4$ intersects the curve $y = x^2 - 2x$ at two distinct points. [4]

Solution

$y = kx - 4 \longrightarrow 1$

$y = x^2 - 2x \longrightarrow 2$

by combining (1) *and* (2)

$kx - 4 = x^2 - 2x$

$\Rightarrow -x^2 + kx + 2x - 4 = 0$

$\Rightarrow -x^2 + (k + 2)x - 4 = 0$

where $a = -1;\ b = (k + 2);\ c = -4$

using $b^2 - 4ac > 0$

$\Rightarrow (k + 2)^2 - 4(-1)(-4) > 0$

$\Rightarrow k^2 + 4k + 4 + 4(-4) > 0$

$\Rightarrow k^2 + 4k + 4 - 16 > 0$

$\Rightarrow k^2 + 4k - 12 > 0$

$\Rightarrow (k + 6)(k - 2) > 0$

$\Rightarrow -6$ and 2 are critical values of k

NB: A quadratic graph is ∪-shaped if the coefficient of the term in order power 2 is positive and ∩-shaped when the coefficient of the term in order power 2 is negative. In this case,

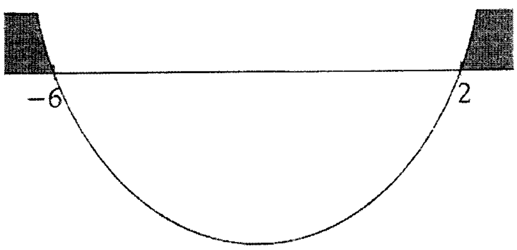


Fig. 1.5

Since the region that satisfies the inequality, $\mathbf{k^2 + 4k - 12 > 0}$, *is positive, we shade the area above the* $x -$ *axis. This area must be bound by the* $x -$ *axis and the curve.*

$\therefore \mathbf{k < -6 \text{ and } k > 2}$

Question (Cambridge, November 2005 qp.1)

9 The equation of a curve is $xy = 12$ and the equation of a line l is $2x + y = k$, where k is a constant.

(ii) Find the set of values of k for which l does not intersect the curve. [4]

Solution

$xy = 12 \longrightarrow 1$

$2x + y = k \longrightarrow 2$

by combining (1) *and* (2)

$x(k - 2x) = 12$

$\Rightarrow kx - 2x^2 = 12$

$\Rightarrow -2x^2 + kx - 12 = 0$

where $a = -2;\ b = k;\ c = -12$

using $b^2 - 4ac < 0$

$\Rightarrow k^2 - 4(-2)(-12) < 0$

$\Rightarrow k^2 + 8(-12) < 0$

$\Rightarrow k^2 - 96 < 0$

$\Rightarrow (k - \sqrt{96})(k + \sqrt{96}) < 0$

$\Rightarrow \sqrt{96}$ and $-\sqrt{96}$ are critical values of k

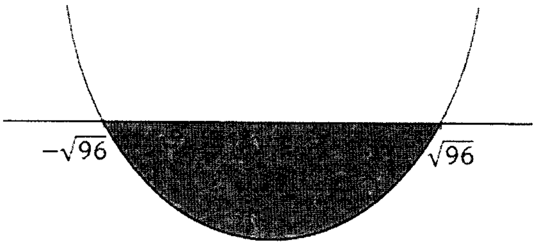


Fig. 1.6

Since the region that satisfies the inequality, $\mathbf{k^2 - 96 < 0}$, *is negative, we shade the area below the* $x-$ *axis. This area must be bound by the* $x-$ *axis and the curve.*

$\therefore -\sqrt{96} < k < \sqrt{96}$

Revision Questions on Quadratics

Question (Unknown source)

Find the set of value of k for which the line $y = 2x + k$ cuts the curve $y = x^2 + kx + 5$ at two distinct points. [6]

November 2002 qp.1 (Zimsec)

1. (a) Find the set of values of k, for which the equation $kx^2 - 3x = k - 3$ has real roots. [2]

November 2000 qp.3 (Cambridge)

1. In the quadratic equation $kx^2 + 2(k+1)x + (k-1) = 0$, k is constant.
   i. Solve the equation in the case when $k = 5$. [2]
   ii. Find the set of values of k for which the equation has distinct real roots. [3]

November 2002 qp.1 (Zimsec, O Level Additional Mathematics)

8. (a) Find the range of values of k for which the equation

$$2x^2 + (4k-2)x + (2k-1) = 0 \text{ has real roots.}$$ [4]

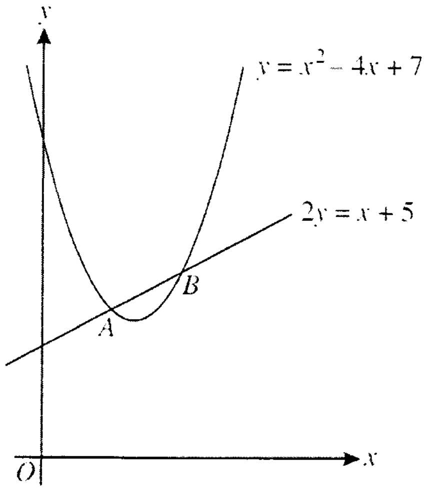
(b) Find the range of values of c for which the curve $y = x^2 - 3x$ intersects the straight line $y = x + c$ at two distinct points. [4]

November 2007 qp.1 (Cambridge)

1 Determine the set of values of the constant k for which the line $y = 4x + k$ does not intersect the curve $y = x^2$. [3]

November 2009 qp.12 (Cambridge)

10



(ii) Determine the set of values of k for which the line $2y = x + k$ does not intersect the curve $y = x^2 - 4x + 7$. [4]

June 2011 qp.13 (Cambridge)

2 Find the set of values of m for which the line $y = mx + 4$ intersects the curve $y = 3x^2 - 4x + 7$ at two distinct points. [5]

June 2012 qp.13 (Cambridge)

10 The equation of a line is $2y + x = k$, where k is a constant, and the equation of a curve is $xy = 6$.

(ii) Find the set of values of k for which the line $2y + x = k$ intersects the curve $xy = 6$ at two distinct points. [3]

Chapter Two: Polynomials

"Nothing will work unless you do."

– John Wooden

Polynomials deal with problems concerning algebraic functions with much emphasis on the order of powers. This topic analyses the relationship between factors and multiples. On one hand, a factor is a lower order expression that gets into a higher order expression (in this case, a polynomial) without leaving a remainder. On the other hand, a multiple (that is, the polynomial) accommodates lower order expressions.

For example, if the polynomial $f(x)$ is such that,

$f(x) = 2x^2 + 3x + 1,$

$(x+1)$ and $(2x+1)$ are factors of $f(x)$ because they get into $f(x)$ without leaving a remainder.

Much of the work in polynomials is centred on the analysis and application of the **remainder** and **factor theorems**.

The **factor theorem** states that when a polynomial is divided by a factor, the remainder is zero (0).

The **remainder theorem** states that when a polynomial is divided by a lower order expression, it leaves a remainder.

Factors versus Roots

A factor is an *expression* whereas a root is a *solution*.

For example, if $(x+1)$ is a factor then $x = -1$ is a root. This implies a factor can be transformed into a root or vice versa. This feeds from the fact that,

$If\ (x+1)(3x-2) = 0$

$either\ \ (x+1) = 0 \qquad or \qquad (3x-2) = 0$

$\Rightarrow x = -1 \qquad or \qquad x = \frac{2}{3}$

As such, $(x+1)$ and $(3x-2)$ are factors, and $x = -1$ and $x = \frac{2}{3}$ the corresponding roots.
Using linear factors, $P(roots) = 0$ and $P(non-root) = remainder$

In short, if a root and a non-root are substituted in place of a variable in a polynomial, the result is zero and a remainder respectively.

For example, given that, $(x+1)$ is a factor of $P(x)$ where

$$P(x) = x^4 - 2x^2 - 3x - 2$$

$\Rightarrow x = -1 \; is \; a \; root \; of \; P(x),$

$\Rightarrow P(-1) = 0$

$\Rightarrow P(-1) = (-1)^4 - 2(-1)^2 - 3(-1) - 2$

$\Rightarrow P(-1) = 1 - 2 + 3 - 2$

$\Rightarrow \mathbf{P(-1) = 0} \; (\boldsymbol{shown})$

If the same polynomial $P(x) = x^4 - 2x^2 - 3x - 2$ is divided by a non-factor $(x+2)$,

$\Rightarrow P(-2) = (-2)^4 - 2(-2)^2 - 3(-2) - 2$

$\Rightarrow P(-2) = 16 - 8 + 6 - 2$

$\Rightarrow P(-2) = 12$

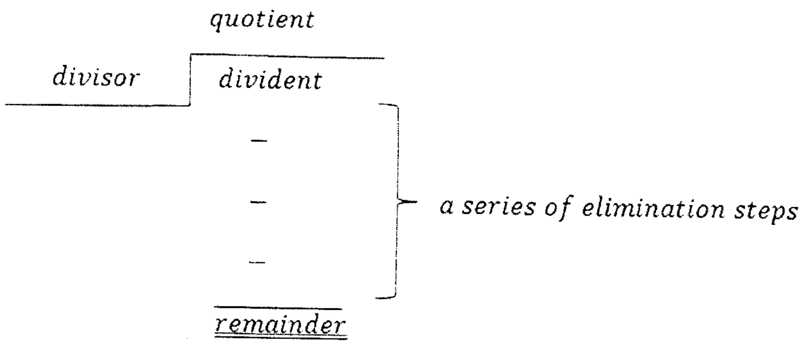
$\therefore \mathbf{12} \; \boldsymbol{is \; the \; remainder \; when \; P(x) \; is \; divided \; by \; (x+2).}$

NB: The strategy outlined immediately above only works when using linear factors.

When using quadratic and other higher order factors, the **only** workable way out is to make use of long division. Since long division is ideal for all types of factors (linear, quadratic, cubic and so on) it is mainly reserved for non-linear factors because the method outlined above provides a less laborious way to account for linear factors.

The Long Division

This technique is made up of four component parts as shown below:



Using the example from the previous section, $P(x) = x^4 - 2x^2 - 3x - 2$ and $(x+1)$ is a factor of $P(x)$. To find the other factor of $P(x)$, one has to use long division.

$$
\begin{array}{r|l}
 & \quad x^3 - x^2 - x - 2 \\
\hline
(x+1) & x^4 + 0x^3 - 2x^2 - 3x - 2 \\
 & \underline{-(x^4 + x^3)} \\
 & \qquad -x^3 - 2x^2 - 3x - 2 \\
 & \qquad \underline{-(-x^3 - x^2)} \\
 & \qquad\qquad -x^2 - 3x - 2 \\
 & \qquad\qquad \underline{-(-x^2 - x)} \\
 & \qquad\qquad\qquad -2x - 2 \\
 & \qquad\qquad\qquad \underline{-(-2x - 2)}
\end{array}
$$

In this case,

- $(x+1)$ is the divisor;
- $x^4 - 2x^2 - 3x - 2$ is the dividend;
- $x^3 - x^2 - x - 2$ is the quotient;
- $zero\ (0)$ is the remainder.

As such, $P(x) = x^4 - 2x^2 - 3x - 2$ can be written as a product of its factors, that is

$P(x) = (x+1)(x^3 - x^2 - x - 2)$.

Worked Examination Questions on Polynomials

Question (Cambridge, June 2010 qp.32)

5 The polynomial $2x^3 + 5x^2 + ax + b$, where a and b are constants, is denoted by $p(x)$. It is given that $(2x+1)$ is a factor of $p(x)$ and that when $p(x)$ is divided by $(x-2)$ the remainder is 9.

(i) Find the values of a and b. [5]

(ii) When a and b have these values, factorise $p(x)$ completely. [3]

Solution

(i) *Given that,*

$$p(x) = 2x^3 + 5x^2 + ax + b$$

- To solve for the unknown values, one has to form a pair of simultaneous equations using the fact that $(2x + 1)$ is a factor of $p(x)$ and that when $p(x)$ is divided by $(x + 2)$ the remainder is 9.

$\Rightarrow p\left(-\frac{1}{2}\right) = 0 \text{ and } p(-2) = 9$

$\Rightarrow 2\left(-\frac{1}{2}\right)^3 + 5\left(-\frac{1}{2}\right)^2 + a\left(-\frac{1}{2}\right) + b = 0$

$\Rightarrow -\frac{1}{4} + \frac{5}{4} - \frac{a}{2} + b = 0$

$\Rightarrow 1 - \frac{a}{2} + b = 0$

$\Rightarrow -a + 2b = -2 \longrightarrow 1$

$\text{and} \quad 2(-2)^3 + 5(-2)^2 + a(-2) + b = 9$

$\Rightarrow -16 + 20 - 2a + b = 9$

$\Rightarrow 4 - 2a + b = 9$

$\Rightarrow b = 5 + 2a \longrightarrow 2$

$\text{by substituting } b \text{ in } (1)$

$\Rightarrow -a + 2(5 + 2a) = -2$

$\Rightarrow -a + 10 + 4a = -2$

$\Rightarrow 3a = -12$

$\Rightarrow a = -4$

$\text{by substituting } a \text{ in } (2)$

$\Rightarrow b = 5 + 2(-4)$

$\Rightarrow b = -3$

$\therefore \boldsymbol{a = -4 \text{ and } b = -3}$

(ii) *Since* $p(x) = 2x^3 + 5x^2 - 4x - 3$

by long division,

$$\begin{array}{r|l} & x^2 + 2x - 3 \\ \hline (2x+1) & 2x^3 + 5x^2 - 4x - 3 \\ & -(2x^3 + x^2) \\ & \quad 4x^2 - 4x - 3 \\ & \quad -(4x^2 + 2x) \\ & \qquad -6x - 3 \\ & \qquad (-6x - 3) \end{array}$$

$\Rightarrow p(x) = (2x+1)(x^2 + 2x - 3)$

where $x^2 + 2x - 3 = (x+3)(x-1)$

$\therefore \boldsymbol{p(x) = (2x+1)(x+3)(x-1)}$

Question (Cambridge, November 2008 qp.3)

5 The polynomial $4x^3 - 4x^2 + 3x + a$, where a is a constant, is denoted by $p(x)$. It is given that $p(x)$ is divisible by $2x^2 - 3x + 3$.

(i) Find the value of a. [3]

(ii) When a has this value, solve the inequality $p(x) < 0$, justifying your answer. [3]

Solution

(i) *Given that,*

$$p(x) = 4x^3 - 4x^2 + 3x + a$$

using long division,

$$\begin{array}{r|l} & 2x + 1 \\ \hline (2x^2 - 3x + 3) & 4x^3 - 4x^2 + 3x + a \\ & -(4x^3 - 6x^2 + 6x) \\ & \quad 2x^2 - 3x + a \\ & \quad -(2x^2 - 3x + 3) \\ & \qquad a - 3 \end{array}$$

where $a - 3 = 0$

$\therefore \boldsymbol{a = 3}$

(ii) $Now,\ p(x) = (2x^2 - 3x + 3)(2x + 1)$

$if\ \ p(x) < 0$

$then\ \ (2x^2 - 3x + 3)(2x + 1) < 0$

$either\ \ (2x^2 - 3x + 3) < 0 \qquad or \qquad (2x + 1) < 0$

$\Rightarrow 2x < -1$

$\Rightarrow x < -\frac{1}{2}\ only\ because\ \ 2x^2 - 3x + 3 < 0\ has\ no\ real\ roots\ i.e.\ \ b^2 - 4ac < 0$

$In\ this\ case,\ \ b^2 - 4ac = (-3)^2 - 4(2)(3)$

$\Rightarrow b^2 - 4ac = 9 - 24$

$\Rightarrow b^2 - 4ac = -15 < 0$

$\therefore \boldsymbol{x < -\frac{1}{2}\ only}$

Revision Questions on Polynomials

November 2003 qp.1 (Zimsec, O Level Additional Mathematics)

1. (a) The remainder when $x^3 - x^2 + 5x + a$ is divided by $x + 2$ is twice the remainder when it is divided by $x - 1$. Find the value of a [5]

(b) Solve the equation

$$2x^3 - 5x^2 + x + 2 = 0$$ [5]

(c) Find the values of p and q for which $x^2 - 2x - 3$ is a factor of

$$2x^3 + px^2 - 12x + q$$ [6]

November 2006 qp.1 (Zimsec)

1. The polynomial $x^3 + px^2 + qx - 81$, where p and q are constants, has factors $(x + 1)$ and $(x - 3)$. Calculate the value of p and the value of q. [4]

June 2007 qp.3 (Cambridge)

2 The polynomial $x^3 - 2x + a$, where a is a constant, is denoted by $p(x)$. It is given that $(x - 2)$ is a factor of $p(x)$.

(i) Find the value of a. [2]

(ii) When a has this value, find the quadratic factor of $p(x)$. [2]

November 2007 qp.1 (Zimsec, O Level Additional Mathematics)

13. (a) The polynomial $P(x) = x^4 + ax^3 + bx^2 - 2x - 4$ has factors $(x - 1)$ and $(x + 2)$

(i). Show that $a = 3$ and $b = 2$ [4]

(ii). Find the other quadratic factor of $P(x)$ and show that this factor is positive for all real values of x [4]

(b) Find the range of values of x for which $x^2 + x - 6 > 0$ [4]

November 2007 qp.3 (Cambridge)

2 The polynomial $x^4 + 3x^2 + a$, where a is a constant, is denoted by $p(x)$. It is given that $x^2 + x + 2$ is a factor of $p(x)$. Find the value of a and the other quadratic factor of $p(x)$. [4]

November 2007 qp.1 (Zimsec)

1. Find the values of a, b and c such that

$$2x^4 + 6x^3 + 7x^2 + 15x + 5 = (x^2 + 3x + 1)(ax^2 + bx + c)$$

for all values of x [3]

June 2011 qp.31 (Cambridge)

4 The polynomial $f(x)$ is defined by

$$f(x) = 12x^3 + 25x^2 - 4x - 12.$$

(i) Show that $f(-2) = 0$ and factorise $f(x)$ completely. [4]

(ii) Given that

$$12 \times 27^y + 25 \times 9^y - 4 \times 3^y - 12 = 0,$$

state the value of 3^y and hence find y correct to 3 significant figures. [3]

June 2011 qp.33 (Cambridge)

5 The polynomial $ax^3 + bx^2 + 5x - 2$, where a and b are constants, is denoted by $p(x)$. It is given that $(2x - 1)$ is a factor of $p(x)$ and that when $p(x)$ is divided by $(x - 2)$ the remainder is 12.

(i) Find the values of a and b. [5]

(ii) When a and b have these values, find the quadratic factor of $p(x)$. [2]

November 2011 qp.31 (Cambridge)

3 The polynomial $x^4 + 3x^3 + ax + 3$ is denoted by $p(x)$. It is given that $p(x)$ is divisible by $x^2 - x + 1$.

(i) Find the value of a. [4]

(ii) When a has this value, find the real roots of the equation $p(x) = 0$. [2]

November 2011 qp.33 (Cambridge)

7 The polynomial $p(x)$ is defined by

$$p(x) = ax^3 - x^2 + 4x - a,$$

where a is a constant. It is given that $(2x - 1)$ is a factor of $p(x)$.

(i) Find the value of a and hence factorise $p(x)$. [4]

(ii) When a has the value found in part **(i)**, express $\dfrac{8x - 13}{p(x)}$ in partial fractions. [5]

June 2012 qp.31 (Cambridge)

3 The polynomial $p(x)$ is defined by

$$p(x) = x^3 - 3ax + 4a,$$

where a is a constant.

(i) Given that $(x - 2)$ is a factor of $p(x)$, find the value of a. [2]

(ii) When a has this value,

(a) factorise $p(x)$ completely. [3]

Chapter Three: Analytical Geometry

"I like to pick things apart, analyse them and put them back in a better order than they had been in before."

– Jessica Thompson

Widely known as **coordinate geometry**, **analytical geometry** deals with problems concerning the location of points in space. The topic revolves around properties of shapes and lines. It must be emphasised that of all the plane shapes with straight edges, rhombus and kite are two shapes of particular interest in analytical geometry. These two shapes are special because their diagonals meet at 90° (see Fig.3.1).

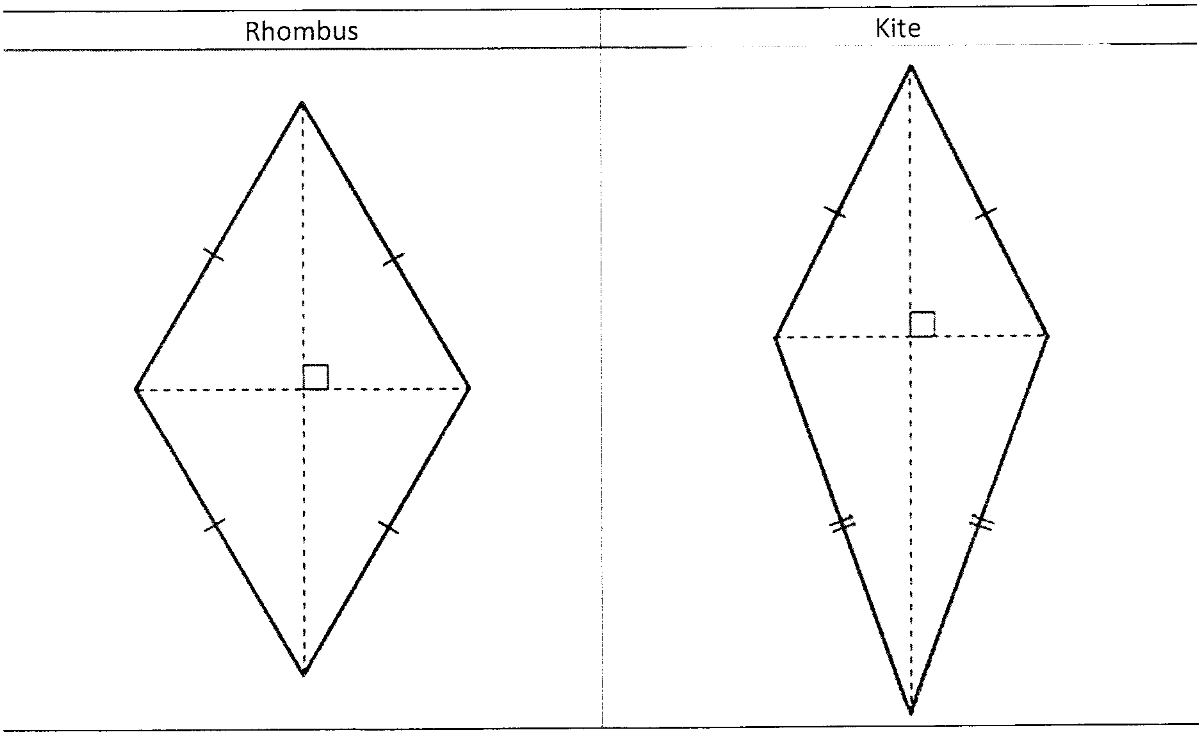


Fig. 3.1

The geometry of shapes with straight edges feeds from six concepts outlined below:

1. <u>Parallel Lines</u>
   Parallel lines travel in the same direction. As such they share the same gradient $(m_1 = m_2)$.

2. <u>Perpendicular Lines</u>
   These are lines that meet at 90°. The product of gradients of two perpendicular lines is −1. $(m_1 \times m_2 = -1)$.

3. Midpoint

   This is used to describe a point that is half-way through two given points. It is given mathematically as follows: $\left(\frac{x_1+x_2}{2}\right);\left(\frac{y_1+y_2}{2}\right)$

4. Distance between two points

   Distance is a measure of size of the path joining two given points. It is given by:

$$\text{distance} = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

5. Points on coordinate axis
   - All points on the x-axis correspond to a y-value of zero.
   - All points on the y-axis correspond to an x -value of zero.

6. Points of intersection

   A point of intersection is common to the two graphs in question. To determine the coordinates at the point of intersection, one has to solve the two equations simultaneously.

The Circle

The circle is another unique shape because it is the only plane shape without any straight edge. A circle is defined by two general equations:

I. $\boldsymbol{(x - a)^2 + (y - b)^2 = r^2}$

where $(a,\ b)$ is the centre and r is the radius. For example, given that a circle has equation $(x-2)^2 + (y+7)^2 = 8$; $(2,-7)$ is the centre and $\sqrt{8}$ is the radius.

NB: the centre is given by switching the signs of the values associated with x and y, and the radius is given by the square-root of the stand-alone value free from x and y.

II. $\boldsymbol{x^2 + y^2 + 2gx + 2fy + c = 0}$

where $(-g,\ -f)$ is the centre and $\sqrt{g^2 + f^2 - c}$ is the radius. For example, given that a circle has equation $x^2 + y^2 + 4x - 3y - 7 = 0$.

In this case;

$2g = 4$ *and* $2f = -3$

$\Rightarrow g = 2$ $\quad\quad$ $\Rightarrow f = -\frac{3}{2}$

NB: g and f are given by dividing the coefficients of x and y by 2

$$\Rightarrow \textbf{centre} = \left(-2,\ \frac{3}{2}\right)$$

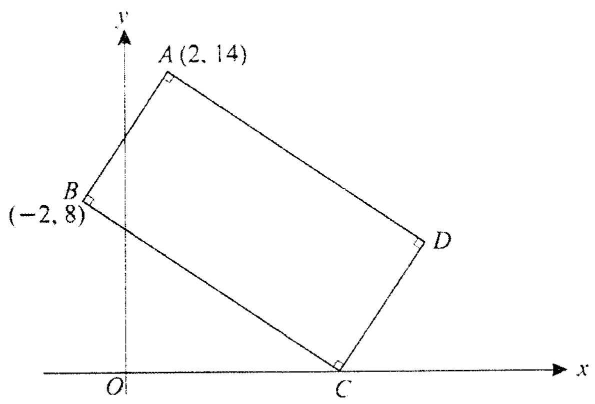
$$\text{radius} = \sqrt{(2)^2 + \left(-\frac{3}{2}\right)^2 - (-7)}$$

$$\textit{and}\ \ \textbf{radius} = \frac{\sqrt{53}}{2}\,\textbf{units}$$

Worked Examination Questions on Analytical Geometry

Question (Cambridge, June 2007 qp.1)

6



The diagram shows a rectangle $ABCD$. The point A is $(2, 14)$, B is $(-2, 8)$ and C lies on the x-axis. Find

(i) the equation of BC, [4]

(ii) the coordinates of C and D. [3]

Solution

(i). $Grad\ (AB) = \frac{14-8}{2-(-2)}$

$$\Rightarrow Grad\ (AB) = \frac{3}{2}$$

$$BC \perp AB$$

$$\Rightarrow m_1 \times m_2 = -1$$

$$\Rightarrow \frac{3}{2} \times m_2 = -1$$

$$\Rightarrow m_2 = \frac{-2}{3}$$

$$using\ y = mx + c,$$
$$\Rightarrow 8 = \frac{-2}{3}(-2) + c$$
$$\Rightarrow c = \frac{20}{3}$$
$$\therefore \boldsymbol{y = -\frac{2}{3}x + \frac{20}{3}}$$

(ii). $C\ is\ a\ point\ on\ the\ x-axis,$

$$\Rightarrow at\ C,\ \ y = 0$$
$$\Rightarrow 0 = \frac{-2}{3}x + \frac{20}{3}$$
$$\Rightarrow \frac{2}{3}x = \frac{20}{3}$$
$$\Rightarrow x = 10$$
$$\therefore \boldsymbol{C\ (10,\ \ 0)}$$

- *The translation vector which maps B onto A is the same as the translation vector which maps C onto D*

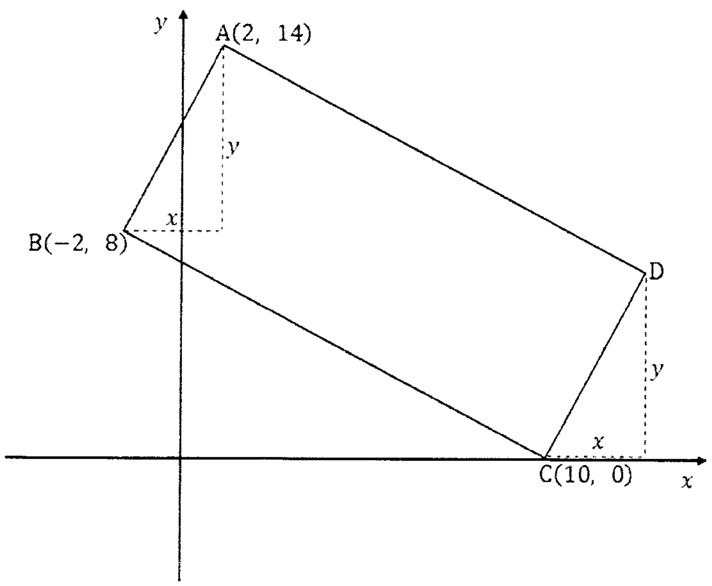


Fig. 3.2

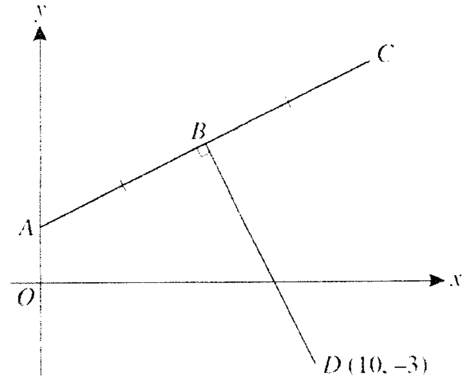
$$Translation\ vector\ \overrightarrow{BA} = \overrightarrow{CD} = \begin{pmatrix}4\\6\end{pmatrix}$$
$$\Rightarrow \overrightarrow{OD} = \begin{pmatrix}10\\0\end{pmatrix} + \begin{pmatrix}4\\6\end{pmatrix}$$

$$\Rightarrow \overrightarrow{OD} = \begin{pmatrix} 14 \\ 6 \end{pmatrix}$$

$$\therefore \boldsymbol{D}\ (\mathbf{14},\ \mathbf{6})$$

Question (Cambridge, June 2009 qp.1)

8



The diagram shows points A, B and C lying on the line $2y = x + 4$. The point A lies on the y-axis and $AB = BC$. The line from $D\ (10, -3)$ to B is perpendicular to AC. Calculate the coordinates of B and C. [7]

Solution

$AC: 2y = x + 4$

$\Rightarrow y = \frac{1}{2}x + 2$

$\Rightarrow Grad(AC) = \frac{1}{2}$

$BD \perp AC$

$\Rightarrow m_1 \times m_2 = -1$

$\Rightarrow \frac{1}{2} \times m_2 = -1$

$\Rightarrow m_2 = -2$

$using\ y = mx + c,$

$\Rightarrow -3 = -2(10) + c$

$\Rightarrow c = 17$

$\Rightarrow\ y = -2x + 17$

Lines BD and AC intersect at B,

$AC: y = \frac{1}{2}x + 2 \longrightarrow 1$

$BD: y = -2x + 17 \longrightarrow 2$

by combining (1) and (2),

$\Rightarrow \frac{1}{2}x + 2 = -2x + 17$

$\Rightarrow \frac{5}{2}x = 15$

$\Rightarrow x = 6$

by substituting x in (2)

$\Rightarrow y = -2(6) + 17$

$\Rightarrow y = 5$

$\therefore \boldsymbol{B\ (6,\ \ 5)}$

The translation which maps A onto B is the same as the one which maps B onto C

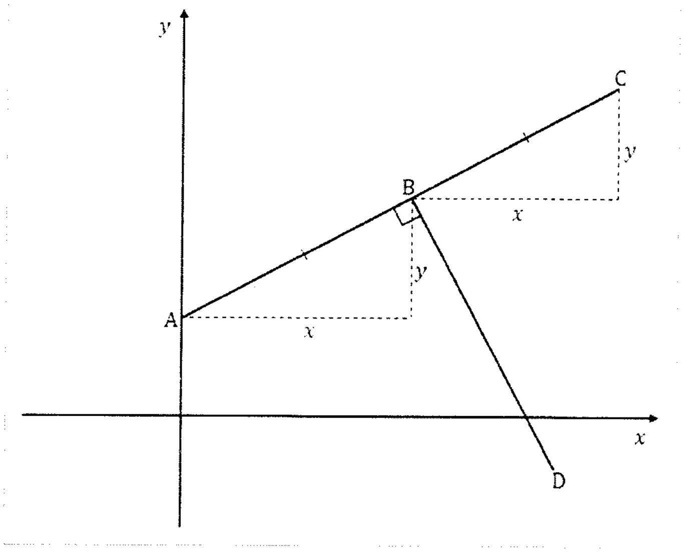


Fig. 3.3

$$\left\{\begin{array}{l} At\ A,\ \ x = 0 \\ \Rightarrow y = \frac{1}{2}(0) + 2 \\ \Rightarrow y = 2 \\ \Rightarrow A(0,\ \ 2) \end{array}\right\}$$

Translation vector $\overrightarrow{AB} = \overrightarrow{BC} = \begin{pmatrix} 6 \\ 3 \end{pmatrix}$

$\overrightarrow{OC} = \binom{6}{5} + \binom{6}{3}$

$\overrightarrow{OC} = \binom{12}{8}$

$\therefore \boldsymbol{C = (12,\ 8)}$

Question (Zimsec, June 2003 qp.1)

15. The equation of a circle is $x^2 + y^2 + 6x + 2y - 6 = 0$.

Find

(i). The equation of the diameter of the circle which passes through the point $(1,\ 4)$. [4]

(ii). The exact values of k for which the line $y = x - k$ is a tangent to the circle. [6]

Solution

(i). $Given\ the\ equation\ x^2 + y^2 + 6x + 2y - 6 = 0$

$2g = 6$ $\quad and \quad$ $2f = 2$

$\Rightarrow g = 3$ $\quad\quad$ $\Rightarrow f = 1$

$centre\ is\ given\ by\ (-g;\ -f)$

$\Rightarrow centre = (-3,\ -1)$

$Since\ the\ diameter\ passes\ through\ point\ (1,\ 4)\ and\ centre\ (-3,\ -1),$

$gradient = \frac{4 - -1}{1 - -3}$

$\Rightarrow gradient = \frac{5}{4}$

$using\ y = mx + c$

$\Rightarrow 4 = \frac{5}{4}(1) + c$

$\Rightarrow c = \frac{11}{4}$

$\therefore \boldsymbol{y = \frac{5}{4}x + \frac{11}{4}}$

(ii). *Given the two equations:*

$x^2 + y^2 + 6x + 2y - 6 = 0 \longrightarrow 1$

$y = x - k \longrightarrow 2$

by substituting (2) *in* (1),

$\Rightarrow x^2 + (x-k)^2 + 6x + 2(x-k) - 6 = 0$

$\Rightarrow x^2 + x^2 - 2kx + k^2 + 6x + 2x - 2k - 6 = 0$

$\Rightarrow 2x^2 + (8-2k)x + (k^2 - 2k - 6) = 0$

where $a = 2; b = (8-2k); c = (k^2 - 2k - 6)$

using $b^2 - 4ac = 0$

$\Rightarrow (8-2k)^2 - 4(2)(k^2 - 2k - 6) = 0$

$\Rightarrow 64 - 32k + 4k^2 - 8k^2 + 16k + 48 = 0$

$\Rightarrow -4k^2 - 16k + 112 = 0$

$\therefore \boldsymbol{k = -2 - 4\sqrt{2}}$ ***or*** $\boldsymbol{k = -2 + 4\sqrt{2}}$

Question (Zimsec, June 2010 qp.1)

7. Write down the equation of the circle with centre $(-3;\ 2)$ and radius $\sqrt{10}$. [1]

Show that the point $A(-2;\ -1)$ lies on the circle, and find the coordinates of B, the other end of the diameter through A. [4]

Solution

The general equation states that:

$(x-a)^2 + (y-b)^2 = r^2$

where $(a,\ b)$ *is the centre and* r *is the radius.*

$\Rightarrow a = -3,\ \ b = 2,\ \ r = \sqrt{10}$

$\Rightarrow [x - (-3)]^2 + [y - 2]^2 = [\sqrt{10}]^2$

$\therefore \boldsymbol{(x+3)^2 + (y-2)^2 = 10}$ ***is the equation of the circle***

NB: All the points on a graph satisfy the equation of that graph. By substituting -2 and -1 for x and y respectively:

$\Rightarrow (-2+3)^2 + (-1-2)^2 = 10$

$\Rightarrow (1)^2 + (-3)^2 = 10$

$\therefore \mathbf{10 = 10}\ (\boldsymbol{shown})$

Using the sketch diagram below

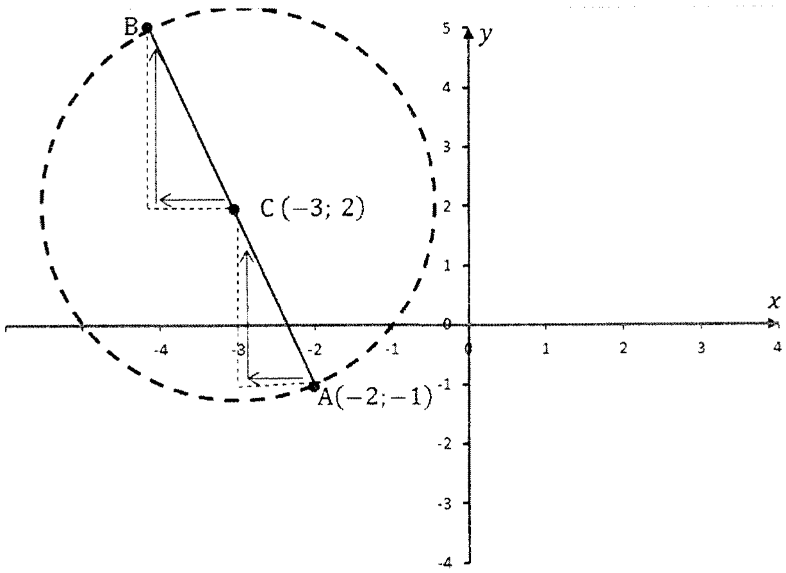


Fig 3.4

NB: by vector move, the translation vector which maps A onto C, $\begin{pmatrix} -1 \\ 3 \end{pmatrix}$, is the same as the translation vector which maps C onto B.

$\Rightarrow \overrightarrow{OB} = \overrightarrow{OC} + translation\ vector$

$\Rightarrow \overrightarrow{OB} = \begin{pmatrix} -3 \\ 2 \end{pmatrix} + \begin{pmatrix} -1 \\ 3 \end{pmatrix}$

$\Rightarrow \overrightarrow{OB} = \begin{pmatrix} -4 \\ 5 \end{pmatrix}$

$\therefore \boldsymbol{B} = (\mathbf{-4,\quad 5})$

Question (Cambridge, June 2011 qp.12)

7 The line L_1 passes through the points $A(2, 5)$ and $B(10, 9)$. The line L_2 is parallel to L_1 and passes through the origin. The point C lies on L_2 such that AC is perpendicular to L_2. Find

(i) the coordinates of C. [5]

(ii) the distance AC. [2]

Solution

For questions without diagrams, it is important to come up with an imaginative picture capturing the relationship connecting the given facts

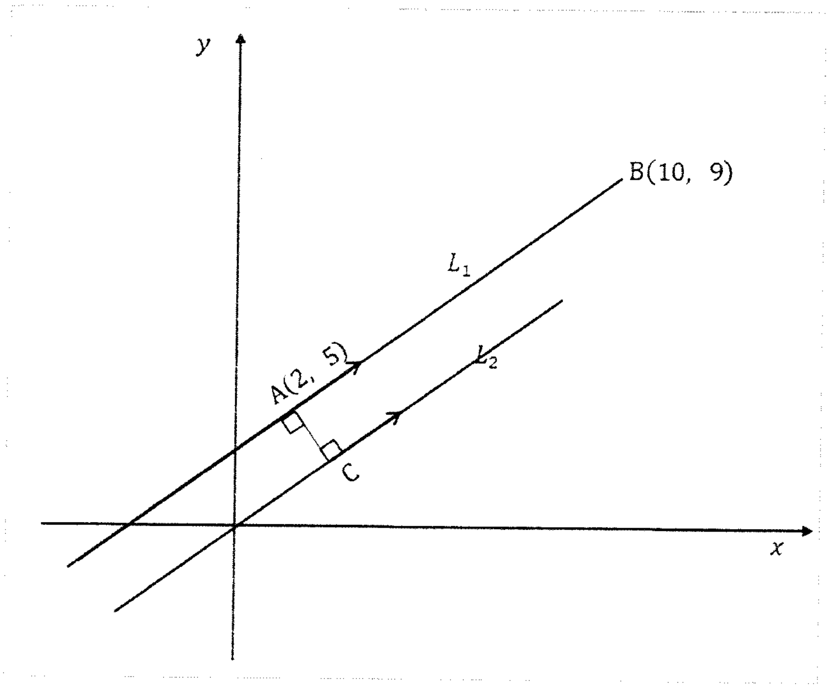


Fig. 3.5

(i). $Grad\ (AB) = \frac{9-5}{10-2}$

$\Rightarrow Grad\ (AB) = \frac{1}{2}$

$AB \perp AC$

$\Rightarrow m_1 \times m_2 = -1$

$\Rightarrow \frac{1}{2} \times m_2 = -1$

$\Rightarrow m_2 = -2$

$using\ y = mx + c,$

$\Rightarrow 5 = -2(2) + c$

$\Rightarrow c = 9$

$\Rightarrow y = -2x + 9$

$L_2: y = \frac{1}{2}x \begin{pmatrix} c = 0\ because\ it\ passes\ through\ the\ origin \\ and\ m = \frac{1}{2}\ because\ L_2\ is\ parallel\ to\ L_1 \end{pmatrix}$

$C\ is\ a\ point\ of\ intersection\ of\ L_2\ and$ AC

$L_2: y = \frac{1}{2}x \longrightarrow 1$

$AC: y = -2x + 9 \longrightarrow 2$

$by\ combining\ (1)\ and\ (2)$

$\Rightarrow \frac{1}{2}x = -2x + 9$

$\Rightarrow \frac{5}{2}x = 9$

$\Rightarrow x = \frac{18}{5}$

$by\ substituting\ x\ in\ (1)$

$\Rightarrow y = \frac{1}{2}\left(\frac{18}{5}\right)$

$\Rightarrow y = \frac{9}{5}$

$\therefore \boldsymbol{C}\left(\frac{\mathbf{18}}{\mathbf{5}}, \frac{\mathbf{9}}{\mathbf{5}}\right)$

(ii). $|AC| = \sqrt{\left(2 - \frac{18}{5}\right)^2 + \left(5 - \frac{9}{5}\right)^2}$

$\Rightarrow |AC| = \frac{\mathbf{8\sqrt{5}}}{\mathbf{5}}\ \boldsymbol{units}$

<u>Revision Questions on Analytical Geometry</u>

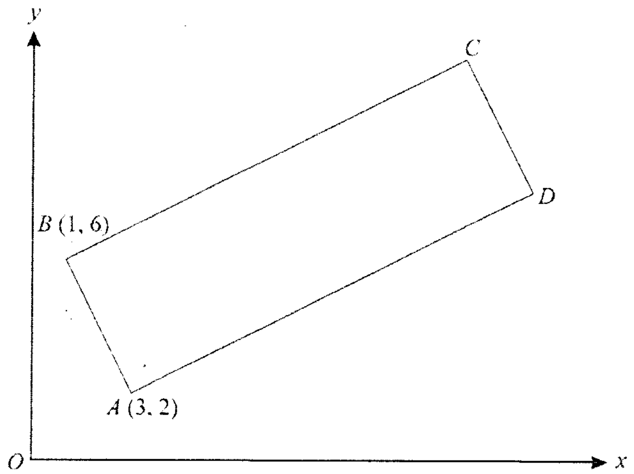
November 2001 qp.1 (Zimsec)

1. The points P, Q have coordinates $(2,\ -1)$, $(4,\ 5)$ respectively. The line L passes through the mid-point of PQ and is parallel to the line with the equation $2x + y + 1 = 0$. Find the equation of L, giving your answer in the form $y = mx + c$. [3]

2. A circle has centre at the point with coordinates $(-1,\ 2)$ and has a radius 6. Find the equation of the circle, giving your answer in the form $x^2 + y^2 + ax + by + c = 0$. [3]

November 2002 qp.1 (Cambridge)

9



The diagram shows a rectangle $ABCD$, where A is $(3, 2)$ and B is $(1, 6)$.

(i) Find the equation of BC. [4]

Given that the equation of AC is $y = x - 1$, find

(ii) the coordinates of C, [2]

(iii) the perimeter of the rectangle $ABCD$. [3]

November 2002 qp.1 (Zimsec, O level Additional Mathematics)

1. Solve the following equations simultaneously

$$y = x + 1$$

$$y^2 + xy = 3$$ [4]

3. The perpendicular bisector of the line joining the points $(3,\ 4)$ and $(7,\ 6)$ meets the y −axis at $(0,\ h)$. Calculate h. [5]

June 2003 qp.1 (Cambridge)

7 The line L_1 has equation $2x + y = 8$. The line L_2 passes through the point A (7, 4) and is perpendicular to L_1.

(i) Find the equation of L_2. [4]

(ii) Given that the lines L_1 and L_2 intersect at the point B, find the length of AB. [4]

November 2003 qp.1 (Zimsec)

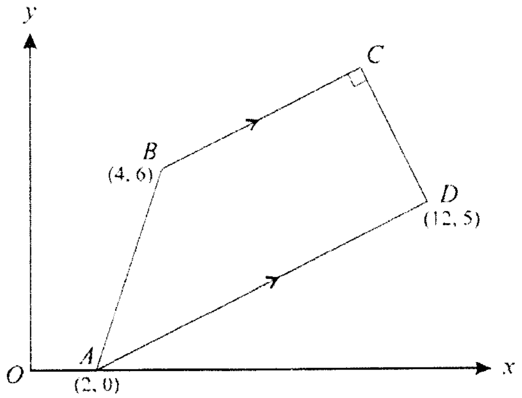
14. A circle touches the line $y = \frac{3}{4}x$ at the point $(4,\ 3)$ and passes through the point $(-12,\ 11)$. Find:

(i). The equation of the perpendicular bisector of the line passing through the points $(4,\ 3)$ and $(-12,\ 11)$ [4]

(ii). The equation of the circle [8]

November 2003 qp.1 (Cambridge)

5



The diagram shows a trapezium $ABCD$ in which BC is parallel to AD and angle $BCD = 90°$. The coordinates of A, B and D are (2, 0), (4, 6) and (12, 5) respectively.

(i) Find the equations of BC and CD. [5]

(ii) Calculate the coordinates of C. [2]

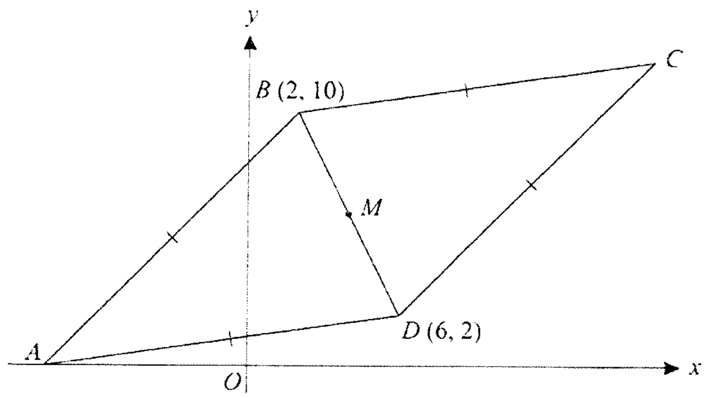
November 2004 qp.1 (Zimsec)

12. Find the points of intersection of the circle $x^2 + y^2 - 6x + 2y - 17 = 0$ and the line $x - y + 2 = 0$. [5]

Hence show that an equation of the circle which has these points as the ends of a diameter is $x^2 + y^2 - 4y - 5 = 0$. [4]

June 2005 qp.1 (Cambridge)

5



The diagram shows a rhombus $ABCD$. The points B and D have coordinates $(2, 10)$ and $(6, 2)$ respectively, and A lies on the x-axis. The mid-point of BD is M. Find, by calculation, the coordinates of each of M, A and C. [6]

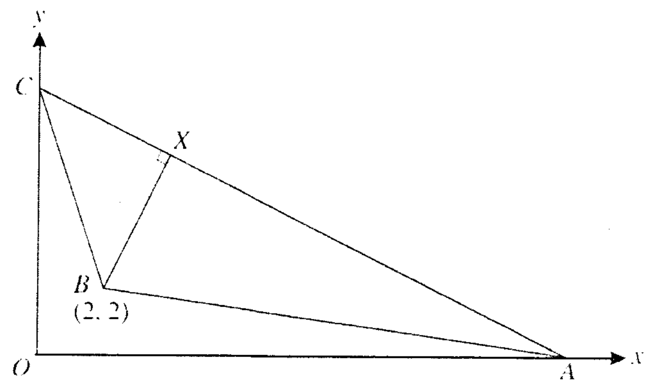
November 2007 qp.1 (Zimsec, O Level Additional Maths)

2. Solve the simultaneous equations

$$y - x = 3 \text{ and } xy = 4$$ [4]

June 2008 qp.1 (Cambridge)

11



In the diagram, the points A and C lie on the x- and y-axes respectively and the equation of AC is $2y + x = 16$. The point B has coordinates $(2, 2)$. The perpendicular from B to AC meets AC at the point X.

(i) Find the coordinates of X. [4]

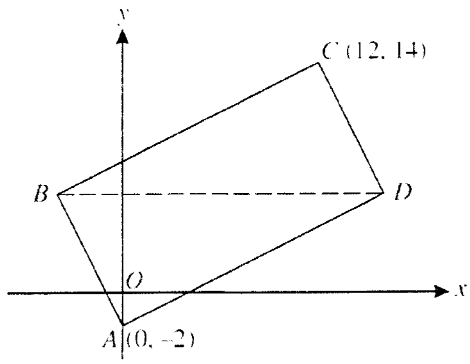
The point D is such that the quadrilateral $ABCD$ has AC as a line of symmetry.

(ii) Find the coordinates of D. [2]

(iii) Find, correct to 1 decimal place, the perimeter of $ABCD$. [3]

November 2009 qp.12 (Cambridge)

9



The diagram shows a rectangle $ABCD$. The point A is $(0, -2)$ and C is $(12, 14)$. The diagonal BD is parallel to the x-axis.

(i) Explain why the y-coordinate of D is 6. [1]

The x-coordinate of D is h.

(ii) Express the gradients of AD and CD in terms of h. [3]

(iii) Calculate the x-coordinates of D and B. [4]

(iv) Calculate the area of the rectangle $ABCD$. [3]

November 2010 qp.1 (Zimsec)

10 (i) Write down the equation of a circle with centre $(4; -3)$ and radius 5 [1]

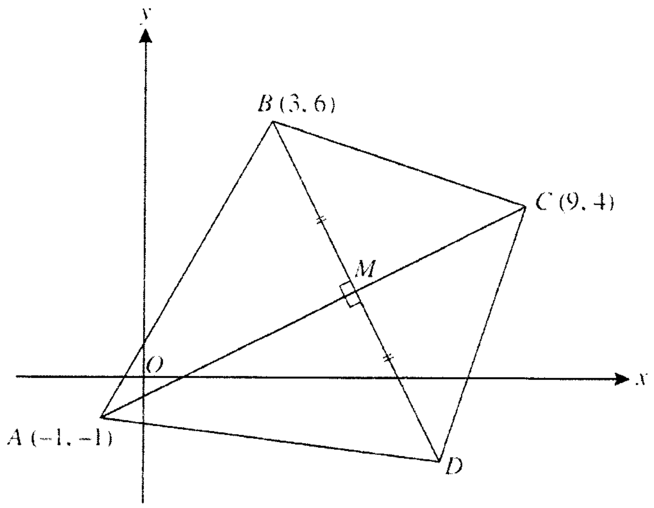
(ii) State the condition satisfied by the point $(x;\ y)$ inside this circle. [1]

(iii) Sketch this circle and the line $2x + y = 3$ on the same diagram with the line intersecting the circle at two points [2]

(iv) Find the range of values of x such that the point inside the circle lies on the line $2x + y = 3$ [3]

November 2011 qp.12 (Cambridge)

9



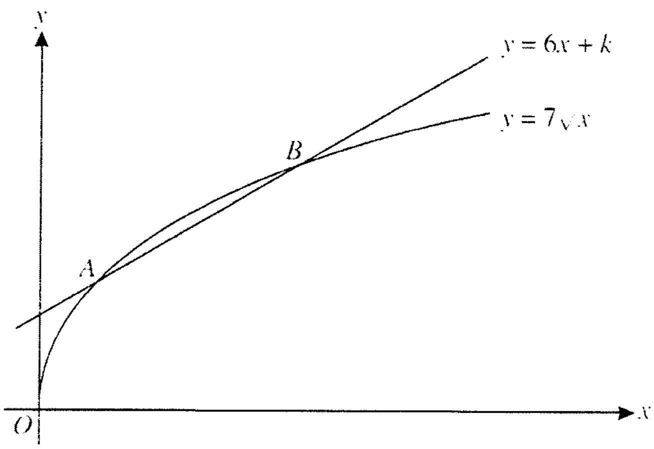
The diagram shows a quadrilateral $ABCD$ in which the point A is $(-1, -1)$, the point B is $(3, 6)$ and the point C is $(9, 4)$. The diagonals AC and BD intersect at M. Angle $BMA = 90°$ and $BM = MD$. Calculate

(i) the coordinates of M and D, [7]

(ii) the ratio $AM : MC$. [2]

June 2012 qp.11 (Cambridge)

5



The diagram shows the curve $y = 7\sqrt{x}$ and the line $y = 6x - k$, where k is a constant. The curve and the line intersect at the points A and B.

(i) For the case where $k = 2$, find the x-coordinates of A and B. [4]

(ii) Find the value of k for which $y = 6x + k$ is a tangent to the curve $y = 7\sqrt{x}$. [2]

November 2012 qp.13 (Cambridge)

10 A straight line has equation $y = -2x + k$, where k is a constant, and a curve has equation $y = \frac{2}{x-3}$.

(i) Show that the x-coordinates of any points of intersection of the line and curve are given by the equation $2x^2 - (6 + k)x + (2 + 3k) = 0$. [1]

(ii) Find the two values of k for which the line is a tangent to the curve. [3]

The two tangents, given by the values of k found in part **(ii)**, touch the curve at points A and B.

(iii) Find the coordinates of A and B and the equation of the line AB. [6]

Chapter Four: Logarithmic and Exponential Functions

"I found out that if you are going to win games, you had better be ready to adapt."
– Scotty Bowman, Hockey Coach

Theoretical Framework

This topic revolves around the laws of indices and logarithms, with much emphasis on the relationship between indices and logarithms.

Indices

A number in index form is made up of three parts, namely:

- Base
- Power/Logarithm
- Number

The relationship connecting the three is such that a base, B, raised to a power, P, gives a number, N, as shown below:

$$\boldsymbol{B^P = N}$$

If the power is an unknown variable, the index number of that form is known as an exponential function, for example, $2^x, 3^x, 10^x$. Since all members of the exponential family behave in the same way, base e is used to represent the exponential family. Below is a snapshot on the laws of indices:

- $a^x \times a^y = a^{x+y}$ $\quad - sum\ of\ powers$
- $a^x \div a^y = a^{x-y}$ $\quad - difference\ of\ powers$
- $(a^x)^y = a^{xy}$ $\quad - product\ of\ powers$
- $a^{-x} = \frac{1}{a^x}$ $\quad - negative\ power$
- $a^{\frac{x}{y}} = \sqrt[y]{a^x}\ or\ \left(\sqrt[y]{a}\right)^x$ $\quad - fractional\ power$
- $a^0 = 1$ $\quad - power\ of\ zero$
- $a^1 = a$ $\quad - power\ of\ one$

Logarithms

A logarithm is made up of three parts, namely:

- Base
- Power/Logarithm
- Number

The relationship connecting the three is such that the logarithm of a number, N, to a given base, B, gives the power, P, as shown below:

$$\log_B N = P$$

As such,

$$\log_B N = P \Leftrightarrow B^P = N$$

Laws of logarithms are used as the guiding framework to the study of logarithms. Below is a breakdown of the laws of logarithms:

- $\log_a x + \log_a y = \log_a xy$
- $\log_a x - \log_a y = \log_a \left(\frac{x}{y}\right)$
- $\log_a x^y = y \log_a x$
- $\log_a a = 1$

NB: Logarithms are defined for positive real numbers only.

Types of Logarithms

1) Common Logarithms

   These are logarithms that use a real number as the base, for example, $\log_{10} 2$; $\log_3 2$; $\log_5 10$ and so on. If a question remains silent on the base, it is believed that the problem is a logarithm to base 10.

2) Natural Logarithms

   These are logarithms to base e where e is a special base that represents members of the exponential family. A special notation, ln, is used to represent natural logarithms.

As such,

$$\log_e a = \ln a$$

$\Rightarrow \log_e e^1 = \ln e^1 = 1$

$\Rightarrow \ln e^x = x \quad and \quad e^{\ln x} = x$

Since this topic is abounding in restrictive conditions, the following conditions have to be observed when using indices and logarithms:

- When introducing e or $\ln$, extend the effects of e and $\ln$ to both the left hand side (LHS) and right hand side (RHS).
- e and $\ln$ are not **distributive** in nature; they affect the LHS and RHS as a whole, and not individual terms.

  For example, given that $y = 3 + 2x$,

  by introducing e and $\ln$

  $\Rightarrow e^y = e^{(3+2x)} \quad and \quad \ln y = \ln(3 + 2x)$ respectively.

  It is an error of principle to take it as

  $e^y = e^3 + e^{2x} \quad and \quad \ln y = \ln 3 + \ln 2x$ respectively.

- When introducing e to eliminate $\ln$ and introducing $\ln$ to eliminate e, make sure the coefficient of the term in $\ln$ and e respectively is one (1). That elimination process only works when coefficient of the function to be eliminated is one (1).

Linear Law

This is a technique used to reduce a given equation into linear form using logarithms. This is done by way of introducing logarithms to both sides of the equation and re-arranging the equation to liken it to the general equation of a straight line, $y = mx + c$.

Questions on this topic test the ability of students to identify the gradient and y −intercept from the reduced equation. For example, given that

$$y^2 = bx^3$$

By introducing ln to both sides,

$$\ln y^2 = \ln bx^3$$

$\Rightarrow 2\ln y = \ln b + \ln x^3$

$\Rightarrow 2\ln y = 3\ln x + \ln b$

$\Rightarrow \ln y = \frac{3}{2}\ln x + \frac{1}{2}\ln b$ *which is similar to*

$y = mx + c$

In this case, $y \equiv \ln y$; $m \equiv \frac{3}{2}$; $x \equiv \ln x$ *and* $c \equiv \frac{1}{2}\ln b$.

Worked Examination Questions on Exponential Functions

Question (Cambridge, June 2008 qp.3)

2 Solve, correct to 3 significant figures, the equation

$$e^x + e^{2x} = e^{3x}.$$ [5]

Solution

Given that, $e^x + e^{2x} = e^{3x}$

let $y = e^x$

$\Rightarrow y + y^2 = y^3$

$\Rightarrow y^3 - y^2 - y = 0$

$\Rightarrow y(y^2 - y - 1) = 0$

either $y = 0$ *or* $y^2 - y - 1 = 0$

$\Rightarrow y = 0$ *or* 1.61803 *or* -0.61803

where $e^x = 1.61803$ *only* $\left(\begin{array}{c}\textit{in this case, } 0 \textit{ and } -0.61803 \textit{ are invalid since} \\ \textit{logarithms are defined for positive real numbers only}\end{array}\right)$

$\Rightarrow \ln e^x = \ln 1.61803$

$\therefore \boldsymbol{x = 0.481}$

Question (Cambridge, June 2011 qp.33)

1 Use logarithms to solve the equation $5^{2x-1} = 2(3^x)$, giving your answer correct to 3 significant figures. [4]

Solution

Given that,

$$5^{2x-1} = 2(3^x)$$

by introducing ln *to both sides of the equation,*

$\ln 5^{2x-1} = \ln 2(3^x)$

$\Rightarrow (2x-1)\ln 5 = \ln 2 + \ln 3^x$

$\Rightarrow 2x\ln 5 - \ln 5 = \ln 2 + x\ln 3$

$\Rightarrow 2x\ln 5 - x\ln 3 = \ln 2 + \ln 5$

$\Rightarrow x(2\ln 5 - \ln 3) = \ln 10$

$\Rightarrow x = \dfrac{\ln 10}{(2\ln 5 - \ln 3)}$

$\therefore \boldsymbol{x = 1.09}$

Revision Questions on Exponential Functions

June 2003 qp.1 (Zimsec)

1. Solve the equation

$$e^{3-x} = 2e^{-3x}$$

giving your answer exactly in terms of logarithms. [3]

June 2004 qp.3 (Cambridge)

4 **(i)** Show that if $y = 2^x$, then the equation

$$2^x - 2^{-x} = 1$$

can be written as a quadratic equation in y. [2]

(ii) Hence solve the equation

$$2^x - 2^{-x} = 1.$$ [4]

June 2006 qp.3 (Cambridge)

1 Given that $x = 4(3^{-y})$, express y in terms of x. [3]

November 2011 qp.31 (Cambridge)

1 Using the substitution $u = e^x$, or otherwise, solve the equation

$$e^x = 1 + 6e^{-x},$$

giving your answer correct to 3 significant figures. [4]

November 2007 qp.1 (Zimsec, O Level Additional Maths)

1. (b) Find the real solution of the equation

$$e^{3x} - 2e^x - 3e^{-x} = 0$$ [4]

June 2007 qp.3 (Cambridge)

4 Using the substitution $u = 3^x$, or otherwise, solve, correct to 3 significant figures, the equation

$$3^x = 2 + 3^{-x}.$$ [6]

June 2010 qp.32 (Cambridge)

1 Solve the equation

$$\frac{2^x + 1}{2^x - 1} = 5,$$

giving your answer correct to 3 significant figures. [4]

November 2009 qp.31 (Cambridge)

2 Solve the equation $3^{x+2} = 3^x + 3^2$, giving your answer correct to 3 significant figures. [4]

November 2012 qp.31 (Cambridge)

2 Solve the equation

$$5^{x-1} = 5^x - 5,$$

giving your answer correct to 3 significant figures. [4]

Worked Examination Questions on Natural Logarithms

Question (Cambridge, June 2009 qp.3)

1 Solve the equation $\ln(2 + e^{-x}) = 2$, giving your answer correct to 2 decimal places. [4]

Solution

Given that,

$$\ln(2 + e^{-x}) = 2$$

by introducing e on both sides of the equation,

$e^{\ln(2+e^{-x})=}e^2$

$\Rightarrow 2 + e^{-x} = e^2$

$\Rightarrow e^{-x} = e^2 - 2$

by introducing ln *on both sides of the equation,*

$\Rightarrow \ln e^{-x} = \ln(e^2 - 2)$

$\Rightarrow -x = \ln(e^2 - 2)$

$\Rightarrow x = -\ln(e^2 - 2)$

$\therefore \boldsymbol{x = -1.68}$

Question (Cambridge, November 2010 qp.31)

2 Solve the equation

$$\ln(1 + x^2) = 1 + 2\ln x,$$

giving your answer correct to 3 significant figures. [4]

Solution

Given that,

$$\ln(1 + x^2) = 1 + 2\ln x$$

by collecting like terms,

$\Rightarrow \ln(1 + x^2) - \ln x^2 = 1$

$\Rightarrow \ln\left(\frac{1 + x^2}{x^2}\right) = 1$

by introducing e on both sides of the equation,

$\Rightarrow e^{\ln\left(\frac{1+x^2}{x^2}\right)} = e^1$

$\Rightarrow \frac{1 + x^2}{x^2} = e^1$

$\Rightarrow 1 + x^2 = x^2 e^1$

$\Rightarrow x^2 - x^2 e^1 = -1$

$\Rightarrow x^2(1 - e^1) = -1$

$\Rightarrow x^2 = -\frac{1}{1 - e^1}$

$\Rightarrow x = \sqrt{-\frac{1}{1 - e^1}}$

$\therefore \boldsymbol{x = 0.763}$

Revision Questions on Natural Logarithms

November 2008 qp.3 (Cambridge)

1 Solve the equation

$$\ln(x + 2) = 2 + \ln x,$$

giving your answer correct to 3 decimal places. [3]

November 2009 qp.32 (Cambridge)

1 Solve the equation

$$\ln(5 - x) = \ln 5 - \ln x,$$

giving your answers correct to 3 significant figures. [4]

June 2012 qp.32 (Cambridge)

1 Solve the equation

$$\ln(3x + 4) = 2\ln(x + 1),$$

giving your answer correct to 3 significant figures. [4]

June 2012 qp.33 (Cambridge)

2 Solve the equation $\ln(2x + 3) = 2\ln x + \ln 3$, giving your answer correct to 3 significant figures. [4]

November 2012 qp.33 (Cambridge)

1 Solve the equation

$$\ln(x + 5) = 1 + \ln x,$$

giving your answer in terms of e. [3]

Worked Examination Question on Common Logarithms

Question (Cambridge, June 2011 qp.32)

2 (i) Show that the equation

$$\log_2(x + 5) = 5 - \log_2 x$$

can be written as a quadratic equation in x. [3]

(ii) Hence solve the equation

$$\log_2(x + 5) = 5 - \log_2 x.$$ [2]

Solution

(i) *Given that,*

$$\log_2(x+5) = 5 - \log_2 x$$

by collecting like terms,

$\Rightarrow \log_2(x+5) + \log_2 x = 5$

$\Rightarrow \log_2[(x+5)(x)] = 5$

$\Rightarrow \log_2(x^2+5x) = 5$

by transforming the logarithm into index form,

$\Rightarrow 2^5 = x^2 + 5x$

$\therefore \boldsymbol{x^2 + 5x - 32 = 0}$

(ii) *Since* $\log_2(x+5) = 5 - \log_2 x$ *can be written as* $x^2 + 5x - 32 = 0$, *the problem can be solved using the quadratic formula.*

$\Rightarrow x = 3.68 \ \ or \ -8.68$

Since x can never be negative,

$\therefore \boldsymbol{x = 3.68\ only}$

Revision Questions on Common Logarithms

November 2002 qp.3 (Cambridge)

3 **(i)** Show that the equation

$$\log_{10}(x+5) = 2 - \log_{10} x$$

may be written as a quadratic equation in x. [3]

(ii) Hence find the value of x satisfying the equation

$$\log_{10}(x+5) = 2 - \log_{10} x. \quad [2]$$

November 2007 qp.1 (Zimsec, O Level Additional Mathematics)

1. (a) Express as a single logarithm in its simplest form

$$\log 2 + 2\log 18 - \frac{3}{2}\log 36$$

[3]

November 2010 qp.1 (Zimsec)

1. If a and b are positive real numbers, $a \neq b$ and $\log_a b + \frac{2}{\log_a b} = 3$, express b in terms of a. [4]

Worked Examination Question on Linear Law

Question (Cambridge, June 2010 qp.33)

2 The variables x and y satisfy the equation $y^3 = Ae^{2x}$, where A is a constant. The graph of $\ln y$ against x is a straight line.

(i) Find the gradient of this line. [2]

(ii) Given that the line intersects the axis of $\ln y$ at the point where $\ln y = 0.5$, find the value of A correct to 2 decimal places. [2]

Solution

(i) *Given that,* $y^3 = Ae^{2x}$

by reducing the equation to linear form,

$\Rightarrow \ln y^3 = \ln Ae^{2x}$

$\Rightarrow 3\ln y = \ln A + \ln e^{2x}$

$\Rightarrow 3\ln y = \ln A + 2x$

$\Rightarrow \ln y = \frac{2}{3}x + \frac{1}{3}\ln A$

by making a comparative assessment with the general equation of a line,

$y = mx + c$, ***the gradient is*** $\mathbf{\frac{2}{3}}$.

(ii) *Since the* $y -$ *intercept is* 0.5,

$\Rightarrow \frac{1}{3}\ln A = 0.5$

$\Rightarrow \ln A = 1.5$

by introducing e on both sides of the equation,

$\Rightarrow e^{\ln A} = e^{1.5}$

$\therefore \boldsymbol{A = 4.48}$

Revision Questions on Linear Law

November 2003 qp.2 (Zimsec, O Level Additional Maths)

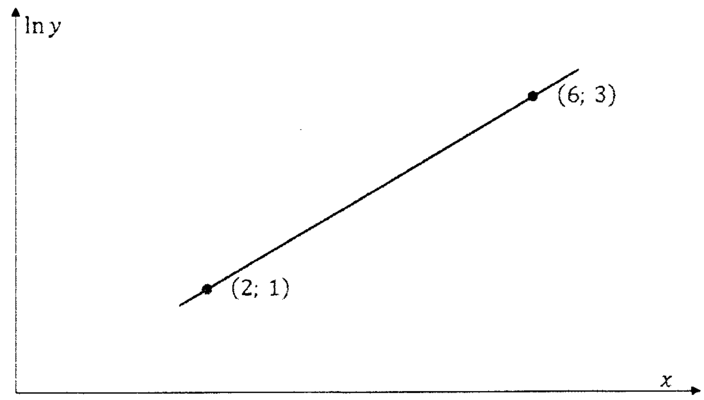
3. (a) The table shows experimental values of two variables x and y.

x	1	2	3	4	5
y	1.9	5.0	9.3	15.2	22.0

The variables are related by an equation of the form $py + qx^2 = x$. Using the given data draw the graph of $\frac{y}{x}$ against x and use it to estimate values of p and q.

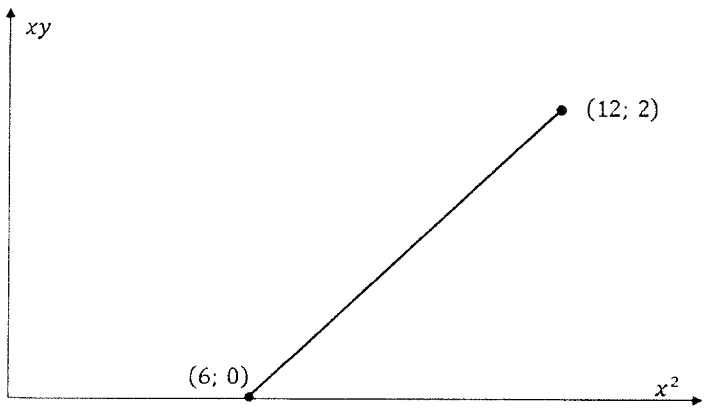
[8]

(b) The graph below shows part of a straight line graph obtained by plotting $\ln y$ against x.



Express y in terms of x. [4]

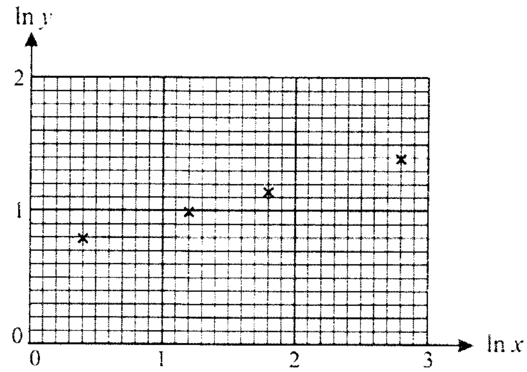
(c) The diagram below shows part of a straight line graph drawn to represent the equation $x + \frac{a}{x} = by$.



Calculate the values of a and b. [4]

November 2005 qp.3 (Cambridge)

2



Two variable quantities x and y are related by the equation $y = Ax^n$, where A and n are constants. The diagram shows the result of plotting $\ln y$ against $\ln x$ for four pairs of values of x and y. Use the diagram to estimate the values of A and n. [5]

June 2010 qp.31 (Cambridge)

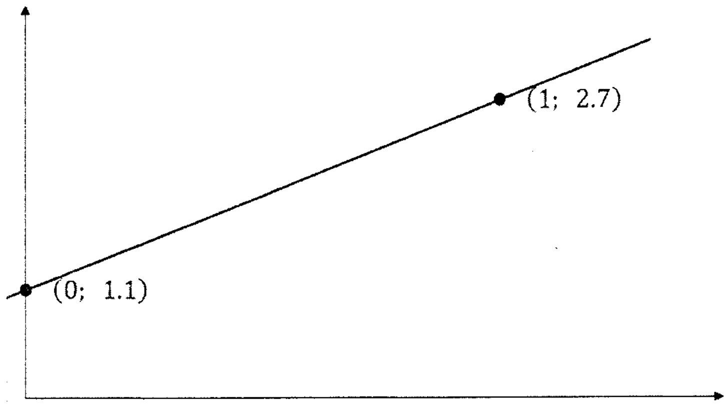
3 The variables x and y satisfy the equation $x^n y = C$, where n and C are constants. When $x = 1.10$, $y = 5.20$, and when $x = 3.20$, $y = 1.05$.

(i) Find the values of n and C. [5]

(ii) Explain why the graph of $\ln y$ against $\ln x$ is a straight line. [1]

June 2010 qp.1 (Zimsec)

5. A mathematician working with an exponential relation $y = ab^x$ reduced it to linear form and came out with the graph shown in the diagram below.



(i). State the label on each axis. [2]
(ii). Calculate the value of a and the value of b. [3]

Chapter Five: Modulus Functions and Inequalities

"It's a mathematical fact two negatives make a positive so even under adverse circumstances think positively."

– Amit Abraham

A modulus sign is used to cushion against a negative sign.

For example,

$|x| = 10\ means\ x = 10\ or\ x = -10\quad and\ |x| < 10\ means - 10 < x < 10.$

Inequalities are either linear or quadratic in nature. Questions on this topic can be grouped into three. This form of grouping is inspired by the techniques used to solve the problems. The section immediately below gives a detailed outline of the three instruments used in question analysis.

Analytical Tools

1. **Simple Interpretation**

This technique can only be used when there is a modulus sign on one side of the equation or inequality and there is a real number on the other side.

For example, given that,

$$|x - 3| < 5$$

$\Rightarrow -5 < x - 3 < 5$

For easy analysis, the inequality can be broken down into two:

$\Rightarrow -5 < x - 3$	*and*	$x - 3 < 5$
$\Rightarrow -5 + 3 < x$		$x < 5 + 3$
$\Rightarrow -2 < x$		$x < 8$

By combining the two results,

$\therefore \mathbf{-2 < x < 8}$

2. The Graphical Method

This method is used to solve problems where the modulus sign is affecting only one side of the equation or inequality and there is an unknown variable on the other side. As such, the graphical method has a bias towards the first technique. The only difference is that the graphical method factors in a graph to aid the decision-making process. Problems solved using this technique give rise to two values of x where only one of the values can be used as a critical value.

For example, given that,

$$|x - 6| < 3 - 2x$$

It is strongly encouraged to construct a pair or graphs to help in the determination of a critical value. In this case,

$let\ y = LHS$ and $y = RHS$

$\Rightarrow y = |x - 6|$ $\Rightarrow y = 3 - 2x$

NB: Modulus graphs cannot be extended to the region below the x −axis. To draw a modulus graph, act as if you are drawing a normal line, but when the line hits the x −axis, reflect it using the x −axis as the mirror.

$y = |x - 6|$

x	0	6
y	−6	0

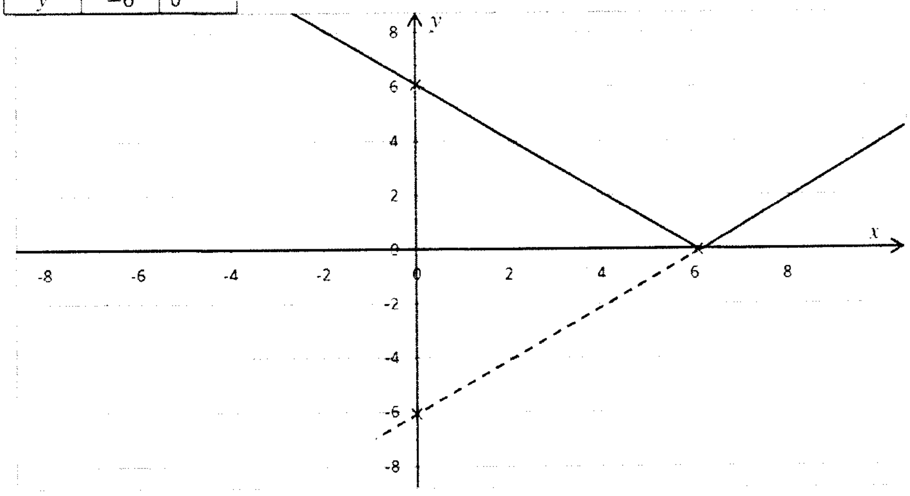


Fig. 5.1

NB: the section below the x −axis has to be reflected to make it positive.

$y = 3 - 2x$

x	0	$\frac{3}{2}$
y	3	0

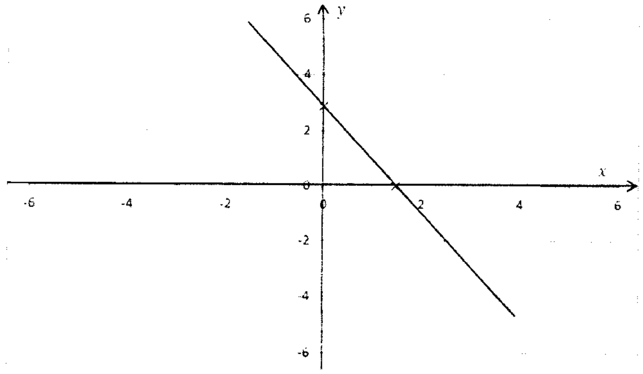


Fig. 5.2

When the two graphs have been shown on a single diagram, they lead to the following image (this diagram is what should be shown to the examiner).

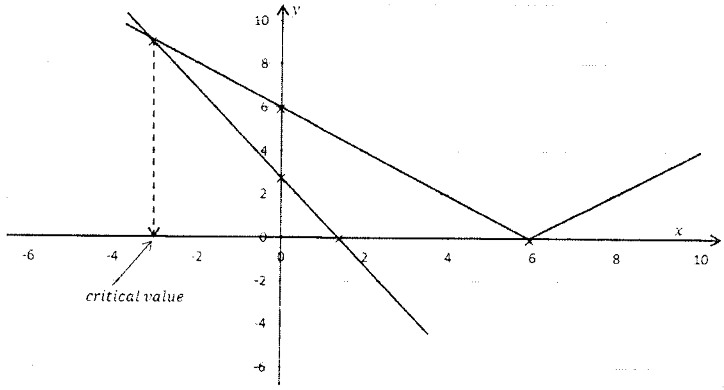


Fig. 5.3

Now, to find the critical value, use the simple interpretation technique and merge the results with the graph above.

$\Rightarrow |x - 6| < 3 - 2x$

$\Rightarrow -(3 - 2x) < (x - 6) < 3 - 2x$

$\Rightarrow -3 + 2x < x - 6$ *and* $x - 6 < 3 - 2x$

$\Rightarrow 2x - x < -6 + 3$ *and* $3x < 3 + 6$

$\Rightarrow x < -3$ *and* $x < 3$ $(discard)$

Since $x = -3$ *is a critical value,*

$\therefore \boldsymbol{x < -3}$ ***only***

3. Quadratic Inequality Approach

This technique is mainly used when the modulus sign is affecting both sides of the inequality or equation. In cases where there is a special multiplier to a modulus sign, it is best to extend the effect of the multiplier to the terms cushioned by the modulus sign. Problems of this nature give rise to quadratic inequalities.

For example, given that,

$$3|x+1| < |x+2|$$

$\Rightarrow (3x+3)^2 < (x+2)^2$

$\Rightarrow 9x^2 + 18x + 9 < x^2 + 4x + 4$

$\Rightarrow 8x^2 + 14x + 5 < 0$

Using the quadratic formula or otherwise, $-\frac{1}{2}$ *and* $-\frac{5}{4}$ *are critical values.*

Using the graphical interpretation (see Chapter 1 on quadratics).

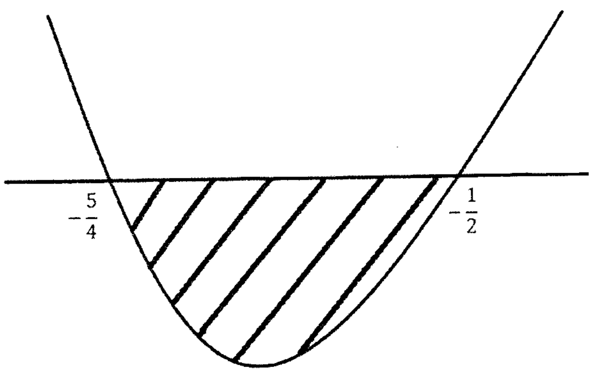


Fig. 5.4

$\therefore -\frac{5}{4} < x < -\frac{1}{2}$

Worked Examination Question on Simple Interpretation

Question (Cambridge, June 2012 qp.31)

1 Solve the equation $|4 - 2^x| = 10$, giving your answer correct to 3 significant figures. [3]

Solution

$|4 - 2^x| = 10$

either $4 - 2^x = 10$ *or* $4 - 2^x = -10$

$\Rightarrow 2^x = -6$

$\Rightarrow \ln 2^x = \ln -6$

$\Rightarrow x \ln 2 = \ln -6$

$\Rightarrow x$ *is undefined because* $\ln(-6)$ *is indescribable*

$\Rightarrow 2^x = 14$

$\Rightarrow \ln 2^x = \ln 14$

$\Rightarrow x \ln 2 = \ln 14$

$\Rightarrow x = \frac{\ln 14}{\ln 2}$

$\therefore \boldsymbol{x = 3.81}$

Revision Question on Simple Interpretation

November 2006 qp.3 (Cambridge)

1 Find the set of values of x satisfying the inequality $|3^x - 8| < 0.5$, giving 3 significant figures in your answer. [4]

Worked Examination Question on Graphical Method

Question (Cambridge, November 2009 qp.31)

1 Solve the inequality $2 - 3x < |x - 3|$. [4]

Solution

$2 - 3x < |x - 3|$

$let\ y = LHS$ $\quad and \quad$ $y = RHS$

$\Rightarrow y = 2 - 3x$ $\quad and \quad$ $y = |x - 3|$

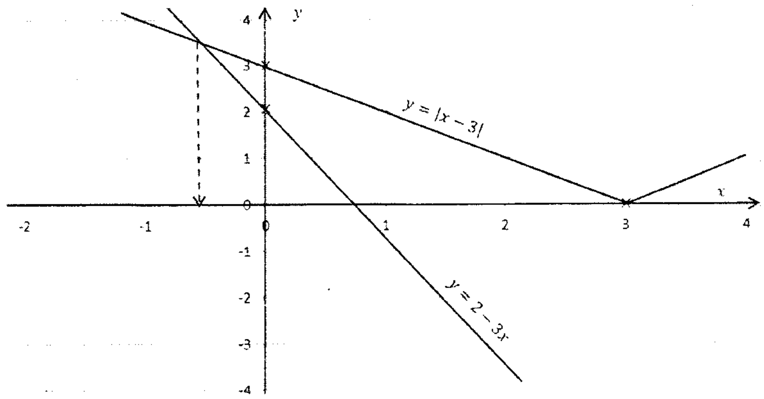


Fig. 5.5

$2 - 3x < (x - 3) < -(2 - 3x)$

$\Rightarrow 2 - 3x < x - 3$ $\quad and \quad$ $x - 3 < -2 + 3x$

$\Rightarrow -4x < -5$ $\qquad$ $\Rightarrow -2x < 1$

$\Rightarrow x > \frac{5}{4}$ $\qquad$ $\Rightarrow x > -\frac{1}{2}$

$Since\ -\frac{1}{2}\ is\ a\ critical\ value,$

$\therefore \boldsymbol{x > -\frac{1}{2}\ only}$

Revision Questions on Graphical Method

June 2006 qp.3 (Cambridge)

2 Solve the inequality $2x > |x - 1|$. [4]

November 2007 qp.1 (Zimsec)

3. Sketch, on the same axes, the graphs of $y = |2x - 3|$ and $y = x + 1$. Hence or otherwise, solve the inequality $|2x - 3| < x + 1$ [4]

<u>Worked Examination Question on Quadratic Inequalities</u>

Question (Cambridge, June 2010 qp.31)

1 Solve the inequality $|x + 3a| > 2|x - 2a|$, where a is a positive constant. [4]

Solution

$|x + 3a| > 2|x - 2a|$

$\Rightarrow |x + 3a| > |2x - 4a|$

$\Rightarrow (x + 3a)^2 > (2x - 4a)^2$

$\Rightarrow x^2 + 6ax + 9a^2 > (4x^2 - 16ax + 16a^2)$

$\Rightarrow -3x^2 + 22ax - 7a^2 > 0$

using the quadratic formula,

$$x = \frac{-22a \pm \sqrt{(22a)^2 - 4(-3)(-7a^2)}}{2(-3)}$$

$$\Rightarrow x = \frac{-22a \pm \sqrt{484a^2 - 84a^2}}{-6}$$

$$\Rightarrow x = \frac{-22a \pm \sqrt{400a^2}}{-6}$$

$$\Rightarrow x = \frac{-22a \pm 20a}{-6}$$

$$\Rightarrow x = \frac{-2a}{-6} \quad or \quad x = \frac{-42a}{-6}$$

$\frac{1}{3}a$ *and* $7a$ *are critical values*

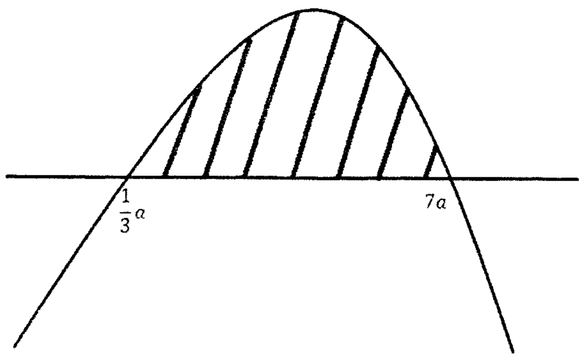


Fig. 5.6

$\therefore \frac{1}{3}a < x < 7a$

Revision Questions on Quadratic Inequality Approach

June 2008 qp.3 (Cambridge)

1 Solve the inequality $|x - 2| > 3|2x + 1|$. [4]

June 2010 qp.1 (Zimsec)

3. Solve the inequality $|3x + 1| \geq 2|x - 2|$ [4]

June 2010 qp.33 (Cambridge)

1 Solve the inequality $|x - 3| > 2|x + 1|$. [4]

June 2011 qp.32 (Cambridge)

1 Solve the inequality $|x| < |5 + 2x|$. [3]

November 2010 qp.31 (Cambridge)

1 Solve the inequality $2|x - 3| > |3x + 1|$. [4]

Chapter Six: Vectors

"If A is a success in life, then A is three dimensional in nature, that is A equals x plus y plus z. Work is x; y is play; and z is keeping your mouth shut."

– Albert Einstein

The location of points in space depends on the **size** and **direction** of travel from the point of origin. This form of analysis raises two quantities: vector and scalar.

Vector versus Scalar Quantities

A scalar quantity is component with magnitude or size only. For example, covering a distance of 3km in an unspecified direction.

A vector quantity is a component with both magnitude and direction. For example, covering a distance of 3km *due south*. In this case, much emphasis is placed on the direction of travel.

Types of Vectors

(i). Position Vector

A position vector describes a quantity that emanates or feeds from the point of origin. As such, the source point is always the origin. For example, $\overrightarrow{OA}$; $\overrightarrow{OQ}$; $\overrightarrow{OZ}$; $\overrightarrow{OY}$; $\overrightarrow{OX}$ and so on.

(ii). Displacement Vector

A displacement vector is used to describe any vector that is not 'referred to' from the point of origin. That is, vector in space. For example, $\overrightarrow{AB}$; $\overrightarrow{QX}$; $\overrightarrow{ZB}$ and so on.

NB: The difference between the vectors is best explained by the breakdown below:

Position Vector

- All coordinates are used to describe position vectors because they are 'referred to' from the point of origin.
- For example, given that $A(2;\ 6)$ and $B(5;\ -3)$ are points in space, diagrammatically the set-up is as follows:

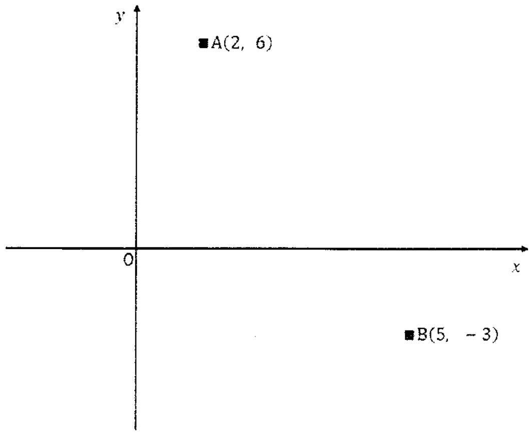


Fig.6.1

$\longmapsto$

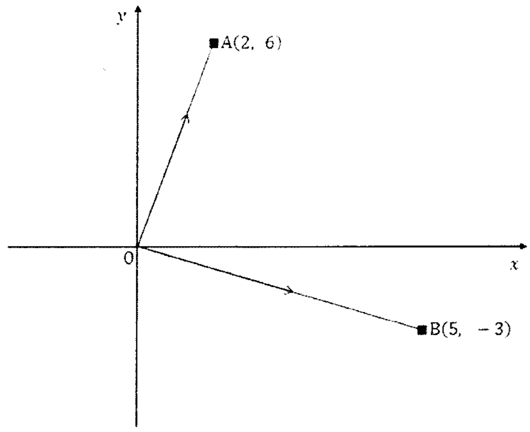


Fig.6.2

- A pair of coordinates shows the location of a point in space whereas a vector shows the movement for one point to another.
- As such,

  $A(2;\ 6) \longmapsto \overrightarrow{OA} = \begin{pmatrix} 2 \\ 6 \end{pmatrix}$

  $B(5;\ -3) \longmapsto \overrightarrow{OB} = \begin{pmatrix} 5 \\ -3 \end{pmatrix}$

- Coordinates are expressed in row form and vectors are expressed in column form.

Displacement Vector

- There are two approaches to the analysis and interpretation of displacement vectors: diagrammatic approach and position vector approach.

  (i). Diagrammatic Approach

  This approach makes use of a diagram to analyse the movement from one point to another. Any reversal in direction should be compensated by a switch in sign. For example if

  $\overrightarrow{AB} = \begin{pmatrix} -3 \\ 7 \end{pmatrix}$

  then $\overrightarrow{BA} = -\begin{pmatrix} -3 \\ 7 \end{pmatrix}$

  $\Rightarrow \overrightarrow{BA} = \begin{pmatrix} 3 \\ -7 \end{pmatrix}$

using the example in Fig. 6.1 *and Fig* 6.2 *above,*

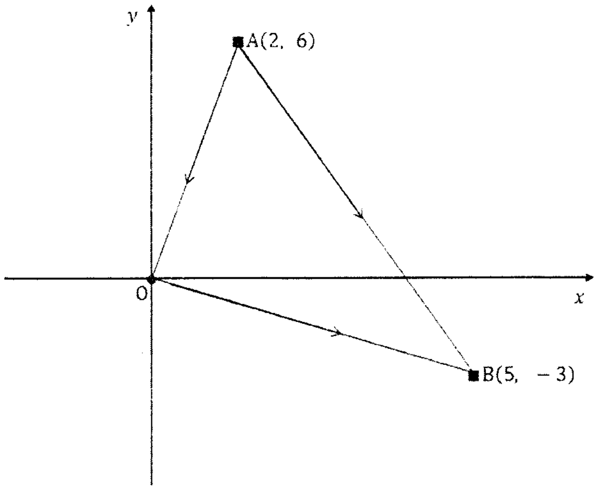


Fig.6.3

$\overrightarrow{AB} = \overrightarrow{AO} + \overrightarrow{OB}$

$\Rightarrow \overrightarrow{AB} = -\binom{2}{6} + \binom{5}{-3}$

$\Rightarrow \overrightarrow{AB} = \binom{-2}{-6} + \binom{5}{-3}$

$\therefore \overrightarrow{\mathbf{AB}} = \binom{\mathbf{3}}{\mathbf{-9}}$

(ii). <u>Position Vector Approach</u>

This technique is best in analysing problems without diagrams. The approach states that:

$$\boldsymbol{displacement\ vector = destination\ point - source\ point}$$

using $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$

$\Rightarrow \overrightarrow{AB} = \binom{5}{-3} - \binom{2}{6}$

$\therefore \overrightarrow{\mathbf{AB}} = \binom{\mathbf{3}}{\mathbf{-9}}$

Addition and Subtraction of Vectors

All the examples we have reviewed so far are meant to put the reader in context of vectors. Much of the analysis at this stage makes use of 3-dimensional vectors (x, y and z) and not 2-dimensional vectors (x and y only) as in the examples above. Using arbitrary vectors:

$$\overrightarrow{OA} = \begin{pmatrix} 3 \\ -2 \\ 6 \end{pmatrix} \; and \; \overrightarrow{OB} = \begin{pmatrix} -1 \\ 3 \\ -2 \end{pmatrix},$$

$$\overrightarrow{OA} + 3\overrightarrow{OB} = \begin{pmatrix} 3 \\ -2 \\ 6 \end{pmatrix} + 3\begin{pmatrix} -1 \\ 3 \\ -2 \end{pmatrix}$$

$$\Rightarrow \overrightarrow{OA} + 3\overrightarrow{OB} = \begin{pmatrix} 3 \\ -2 \\ 6 \end{pmatrix} + \begin{pmatrix} -3 \\ 9 \\ -6 \end{pmatrix}$$

$$\therefore \mathbf{\overrightarrow{OA} + 3\overrightarrow{OB}} = \begin{pmatrix} \mathbf{0} \\ \mathbf{7} \\ \mathbf{0} \end{pmatrix}$$

and

$$2\overrightarrow{OA} - 3\overrightarrow{OB} = 2\begin{pmatrix} 3 \\ -2 \\ 6 \end{pmatrix} - 3\begin{pmatrix} -1 \\ 3 \\ -2 \end{pmatrix}$$

$$\Rightarrow 2\overrightarrow{OA} - 3\overrightarrow{OB} = \begin{pmatrix} 6 \\ -4 \\ 12 \end{pmatrix} - \begin{pmatrix} -3 \\ 9 \\ -6 \end{pmatrix}$$

$$\therefore \mathbf{2\overrightarrow{OA} - 3\overrightarrow{OB}} = \begin{pmatrix} \mathbf{9} \\ \mathbf{-13} \\ \mathbf{18} \end{pmatrix}$$

Dot/Scalar Product

The sum of products of corresponding directions is known as dot product. Conceptually,

$$\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} \cdot \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} = (x_1x_2) + (y_1y_2) + (z_1z_2)$$

$$For\ example, given\ that\ \overrightarrow{OA} = \begin{pmatrix} 3 \\ -2 \\ 6 \end{pmatrix} \; and \; \overrightarrow{OB} = \begin{pmatrix} -1 \\ 3 \\ -2 \end{pmatrix}$$

$$\Rightarrow \overrightarrow{OA} \cdot \overrightarrow{OB} = (3 \times -1) + (-2 \times 3) + (6 \times -2)$$

$$\Rightarrow \overrightarrow{OA} \cdot \overrightarrow{OB} = -3 - 6 - 12$$

$$\therefore \mathbf{\overrightarrow{OA} \cdot \overrightarrow{OB} = -21}$$

Dot product is used to draw up a conclusion on the size of an angle between two vectors. The table below summarises the relationship between dot product and angle size.

Table 6.1 – Dot product and angle size

Dot product	Type of angle
Negative	Obtuse angle
Zero	90°
Positive	Acute angle

Magnitude/ Modulus of Vector

Modulus is a measure of the size of the path joining two points. Conceptually,

$$\left|\begin{matrix} x \\ y \\ z \end{matrix}\right| = \sqrt{x^2 + y^2 + z^2}$$

$For\ example,\ \ given\ that\ \overrightarrow{OA} = \begin{pmatrix} 3 \\ -2 \\ 6 \end{pmatrix}\ and\ \overrightarrow{OB} = \begin{pmatrix} -1 \\ 3 \\ -2 \end{pmatrix}$

$|\overrightarrow{OA}| = \sqrt{(3)^2 + (-2)^2 + (6)^2}$ *and* $\overrightarrow{OB} = \sqrt{(-1)^2 + (3)^2 + (-2)^2}$

$\Rightarrow |\overrightarrow{OA}| = \sqrt{9 + 4 + 36}$ $\Rightarrow \overrightarrow{OB} = \sqrt{1 + 9 + 4}$

$\Rightarrow |\overrightarrow{OA}| = \sqrt{49}$ $\therefore \overrightarrow{\mathbf{OB}} = \sqrt{\mathbf{14}}$

$\therefore |\overrightarrow{\mathbf{OA}}| = \mathbf{7}$

Unit Vector

This is used to describe a vector whose magnitude is one (1). To transform a vector into a unit vector, one has to divide a vector by its modulus.

$$\boldsymbol{Unit\ vector} = \frac{\boldsymbol{Vector}}{\boldsymbol{Modulus}}$$

For example:

$$\widehat{OA} = \frac{\overrightarrow{OA}}{|\overrightarrow{OA}|}$$

$$\Rightarrow \widehat{OA} = \frac{\begin{pmatrix} 3 \\ -2 \\ 6 \end{pmatrix}}{7}$$

$$\therefore \widehat{\mathbf{OA}} = \frac{\mathbf{1}}{\mathbf{7}}\begin{pmatrix} \mathbf{3} \\ \mathbf{-2} \\ \mathbf{6} \end{pmatrix}$$

Angle between Two Vectors

The scalar product approach states that,

$$\cos\theta = \frac{\textbf{Dot Product}}{\textbf{Moduli Product}}$$

where θ is the angle between the two directions in question. This technique makes use of converging or diverging directions. For example, angle AOB is given by:

$\overleftarrow{A}O\overleftarrow{B}$ } $\overrightarrow{AO}$ and $\overrightarrow{BO}$ are converging directions.

Or

$\overleftarrow{A}O\overrightarrow{B}$ } $\overrightarrow{OA}$ and $\overrightarrow{OB}$ are diverging directions

$$\cos AOB = \frac{\overrightarrow{OA}\cdot\overrightarrow{OB}}{|\overrightarrow{OA}||\overrightarrow{OB}|}$$

$where\ \overrightarrow{OA}\cdot\overrightarrow{OB} = -21$

$and\ |\overrightarrow{OA}| \times |\overrightarrow{OB}| = 7\sqrt{14}$

$\left\{ \begin{array}{c} see\ how\ to\ obtain\ dot\ product\ and \\ moduli\ from\ sections\ above \end{array} \right\}$

$$\cos AOB = \frac{-21}{7\sqrt{14}}$$

$$\Rightarrow AOB = \cos^{-1}\left(\frac{-21}{7\sqrt{14}}\right)$$

$$\therefore \boldsymbol{AOB = 143.3^\circ}$$

Forms of Vector Expression

1. Coordinate Form

   The three directions x, y and z are expressed in row form. For example, $\mathrm{A}(3,\ -2,\ 6)$.

2. Column Form

   Three directions x, y and z are expressed in column form. For example, $\overrightarrow{\mathrm{OA}} = \begin{pmatrix} 3 \\ -2 \\ 6 \end{pmatrix}$.

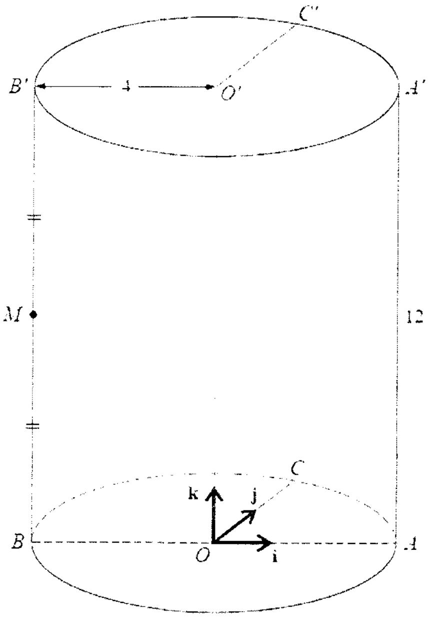
3. Vector Form

The three directions x, y and z are associated by a plus or minus sign using the coefficients i, j and k respectively. For example, $\overrightarrow{OA} = 3i - 2j + 6k$.

Worked Examination Questions on Vectors

Question (Cambridge, June 2002 qp.1)

5



The diagram shows a solid cylinder standing on a horizontal circular base, centre O and radius 4 units. The line BA is a diameter and the radius OC is at 90° to OA. Points O', A', B' and C' lie on the upper surface of the cylinder such that OO', AA', BB' and CC' are all vertical and of length 12 units. The mid-point of BB' is M.

Unit vectors **i**, **j** and **k** are parallel to OA, OC and OO' respectively.

(i) Express each of the vectors $\overrightarrow{MO}$ and $\overrightarrow{MC'}$ in terms of **i**, **j** and **k**. [3]

(ii) Hence find the angle OMC'. [4]

Solution

(i). $\overrightarrow{MO} = \overrightarrow{MB} + \overrightarrow{BO}$

$\Rightarrow \overrightarrow{MO} = -6k + 4i$

$\therefore \boldsymbol{\overrightarrow{MO} = 4i - 6k}$

$\overrightarrow{MC'} = \overrightarrow{MB'} + \overrightarrow{B'O'} + \overrightarrow{O'C'}$

$\Rightarrow \overrightarrow{MC'} = 6k + 4i + 4j$

$\therefore \boldsymbol{\overrightarrow{MC'} = 4i + 4j + 6k}$

(ii). $\cos\theta = \dfrac{\overrightarrow{MO}\cdot\overrightarrow{MC'}}{|\overrightarrow{MO}||\overrightarrow{MC'}|}$

$$\Rightarrow \cos\theta = \frac{\begin{pmatrix}4\\0\\-6\end{pmatrix}}{\sqrt{52}}\cdot\frac{\begin{pmatrix}4\\4\\6\end{pmatrix}}{\sqrt{68}}$$

$$\Rightarrow \cos\theta = \frac{-20}{\sqrt{3536}}$$

$$\Rightarrow \theta = \cos^{-1}\left(\frac{-20}{\sqrt{3536}}\right)$$

$$\therefore \boldsymbol{\theta = 109.7^0}$$

Question (Cambridge, June 2003 qp.1)

8 The points A, B, C and D have position vectors $3\mathbf{i} + 2\mathbf{k}$, $2\mathbf{i} - 2\mathbf{j} + 5\mathbf{k}$, $2\mathbf{j} + 7\mathbf{k}$ and $-2\mathbf{i} + 10\mathbf{j} + 7\mathbf{k}$ respectively.

(i) Use a scalar product to show that BA and BC are perpendicular. [4]

(ii) Show that BC and AD are parallel and find the ratio of the length of BC to the length of AD. [4]

Solution

(i). *Given that;*

$$\overrightarrow{OA} = \begin{pmatrix}3\\0\\2\end{pmatrix},\ \overrightarrow{OB} = \begin{pmatrix}2\\-2\\5\end{pmatrix},\ \overrightarrow{OC} = \begin{pmatrix}0\\2\\7\end{pmatrix},\ and\ \overrightarrow{OD} = \begin{pmatrix}-2\\10\\7\end{pmatrix}$$

$\Rightarrow \overrightarrow{BA} = \overrightarrow{OA} - \overrightarrow{OB}$ *and* $\overrightarrow{BC} = \overrightarrow{OC} - \overrightarrow{OB}$

$$\Rightarrow \overrightarrow{BA} = \begin{pmatrix}3\\0\\2\end{pmatrix} - \begin{pmatrix}2\\-2\\5\end{pmatrix} \qquad \Rightarrow \overrightarrow{BC} = \begin{pmatrix}0\\2\\7\end{pmatrix} - \begin{pmatrix}2\\-2\\5\end{pmatrix}$$

$$\Rightarrow \overrightarrow{BA} = \begin{pmatrix}1\\2\\-3\end{pmatrix} \qquad \Rightarrow \overrightarrow{BC} = \begin{pmatrix}-2\\4\\2\end{pmatrix}$$

The scalar product of perpendicular vectors is zero.

$\Rightarrow \ \overrightarrow{BA} \cdot \overrightarrow{BC} = 0$

$\Rightarrow -2 + 8 - 6 \ = 0$

$\Rightarrow$ *Scalar product is zero (condition satisfied)*

$\therefore$ ***BA and BC are perpendicular***

(ii). $\overrightarrow{BC} = \begin{pmatrix} -2 \\ 4 \\ 2 \end{pmatrix}$

$\Rightarrow \overrightarrow{AD} = \overrightarrow{OD} - \overrightarrow{OA}$

$\Rightarrow \overrightarrow{AD} = \begin{pmatrix} -2 \\ 10 \\ 7 \end{pmatrix} - \begin{pmatrix} 3 \\ 0 \\ 2 \end{pmatrix}$

$\Rightarrow \overrightarrow{AD} = \begin{pmatrix} -5 \\ 10 \\ 5 \end{pmatrix}$

$\Rightarrow \overrightarrow{BC}$ *and* $\overrightarrow{AD}$ *are parallel if and only if they have the same direction vector.*

$\Rightarrow \overrightarrow{BC} = \begin{pmatrix} -2 \\ 4 \\ 2 \end{pmatrix}$ $\qquad \overrightarrow{AD} = \begin{pmatrix} -5 \\ 10 \\ 5 \end{pmatrix}$

$\Rightarrow \overrightarrow{BC} = 2\begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}$ $\qquad \overrightarrow{AD} = 5\begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}$

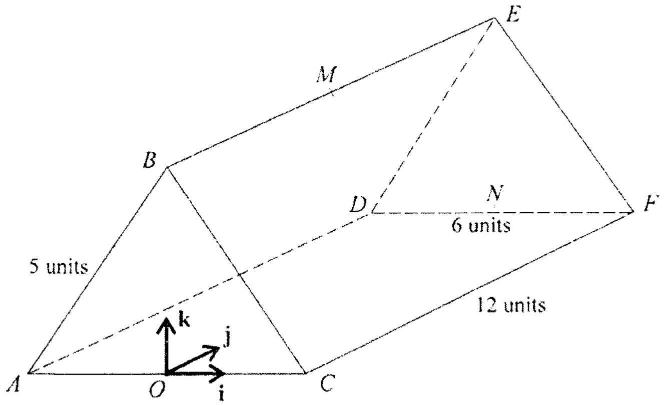
Since $\overrightarrow{BC}$ *and* $\overrightarrow{AD}$ *have the same direction vector,* $\begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}$, ***they are parallel***

Now, $2\begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix} \parallel 5\begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}$

$\therefore$ ***Ratio of*** $\overrightarrow{BC} : \overrightarrow{AD} = \mathbf{2:5}$

Question (Cambridge, November 2003 qp.1)

7



The diagram shows a triangular prism with a horizontal rectangular base $ADFC$, where $CF = 12$ units and $DF = 6$ units. The vertical ends ABC and DEF are isosceles triangles with $AB = BC = 5$ units. The mid-points of BE and DF are M and N respectively. The origin O is at the mid-point of AC.

Unit vectors **i**, **j** and **k** are parallel to OC, ON and OB respectively.

(i) Find the length of OB. [1]

(ii) Express each of the vectors $\overrightarrow{MC}$ and $\overrightarrow{MN}$ in terms of **i**, **j** and **k**. [3]

(iii) Evaluate $\overrightarrow{MC}.\overrightarrow{MN}$ and hence find angle CMN, giving your answer correct to the nearest degree. [4]

Solution

(i). *by Pythagoras theorem,*

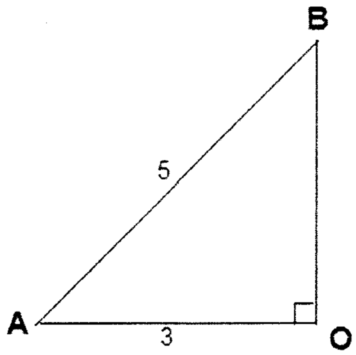
$\Rightarrow (AB)^2 = (OA)^2 + (OB)^2$

$\Rightarrow 5^2 = 3^2 + (OB)^2$

$\Rightarrow 25 - 9 = (OB)^2$

$\Rightarrow (OB)^2 = \sqrt{16}$

$\therefore \boldsymbol{OB = 4\ units}$



(ii). $\overrightarrow{MC} = \overrightarrow{MB} + \overrightarrow{BO} + \overrightarrow{OC}$

$\Rightarrow \overrightarrow{MC} = -6j - 4k + 3i$

$\therefore \boldsymbol{\overrightarrow{MC} = 3i - 6j - 4k,\ and}$

$\Rightarrow \overrightarrow{MN} = \overrightarrow{ME} + \overrightarrow{EN}$

$\therefore \overrightarrow{\boldsymbol{MN}} = \boldsymbol{6j - 4k}$

(iii). $\overrightarrow{MC} \cdot \overrightarrow{MN} = \begin{pmatrix} 3 \\ -6 \\ -4 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 6 \\ -4 \end{pmatrix}$

$\Rightarrow \overrightarrow{MC} \cdot \overrightarrow{MN} = 0 - 36 + 16$

$\therefore \overrightarrow{\boldsymbol{MC}} \cdot \overrightarrow{\boldsymbol{MN}} = \boldsymbol{-20}$

$using \ \cos CMN = \dfrac{\overrightarrow{MC} \cdot \overrightarrow{MN}}{|\overrightarrow{MC}|\,|\overrightarrow{MN}|}$

$Now, \ \cos CMN = \dfrac{-20}{\sqrt{61} \times \sqrt{52}}$

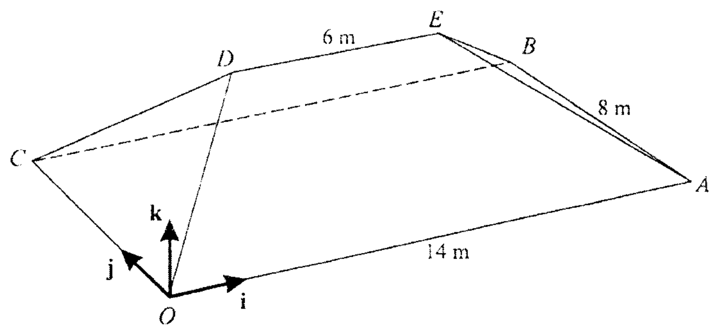
$\Rightarrow CMN = \cos^{-1}\left(\dfrac{-20}{\sqrt{3172}}\right)$

$\Rightarrow CMN = 110.8°$

$\therefore \boldsymbol{CMN} = \boldsymbol{111°}$

Question (Cambridge, June 2006 qp.1)

8



The diagram shows the roof of a house. The base of the roof, $OABC$, is rectangular and horizontal with $OA = CB = 14$ m and $OC = AB = 8$ m. The top of the roof DE is 5 m above the base and $DE = 6$ m. The sloping edges OD, CD, AE and BE are all equal in length.

Unit vectors **i** and **j** are parallel to OA and OC respectively and the unit vector **k** is vertically upwards.

(i) Express the vector $\overrightarrow{OD}$ in terms of **i**, **j** and **k**, and find its magnitude. [4]

(ii) Use a scalar product to find angle DOB. [4]

Solution

(i). *Let Q be a point on the base vertically below D,*

$$\Rightarrow \overrightarrow{OD} = \frac{1}{2}\overrightarrow{OC} + \left(4\,units \parallel \overrightarrow{OA}\right) + \overrightarrow{QD}$$

$$\Rightarrow \overrightarrow{OD} = 4j + 4i + 5k$$

$$\therefore \boldsymbol{\overrightarrow{OD} = 4i + 4j + 5k}$$

$$Now,\ \left|\overrightarrow{OD}\right| = \sqrt{4^2 + 4^2 + 5^2}$$

$$\Rightarrow \left|\overrightarrow{OD}\right| = \sqrt{16 + 16 + 25}$$

$$\therefore \left|\boldsymbol{\overrightarrow{OD}}\right| = \sqrt{\mathbf{57}}\ \boldsymbol{units}$$

(ii). $\cos\theta = \frac{\overrightarrow{OD}\cdot\overrightarrow{OB}}{\left|\overrightarrow{OD}\right|\left|\overrightarrow{OB}\right|}$

$$but\ \overrightarrow{OB} = \overrightarrow{OA} + \overrightarrow{AB}$$

$$\Rightarrow \overrightarrow{OB} = 14i + 8j$$

$$\Rightarrow \cos\theta = \frac{\begin{pmatrix}4\\4\\5\end{pmatrix}}{\sqrt{57}} \cdot \frac{\begin{pmatrix}14\\8\\0\end{pmatrix}}{\sqrt{260}}$$

$$\Rightarrow \cos\theta = \frac{88}{\sqrt{14820}}$$

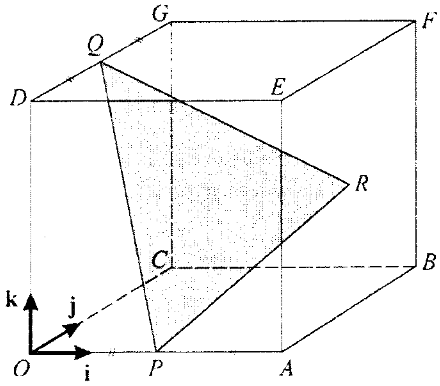
$$\Rightarrow \theta = \cos^{-1}\left(\frac{88}{\sqrt{14820}}\right)$$

$$\Rightarrow \theta = 43.7^0$$

$$\therefore \boldsymbol{DOB = 43.7^0}$$

Question (Cambridge, November 2007 qp.1)

10



The diagram shows a cube *OABCDEFG* in which the length of each side is 4 units. The unit vectors **i**, **j** and **k** are parallel to $\overrightarrow{OA}$, $\overrightarrow{OC}$ and $\overrightarrow{OD}$ respectively. The mid-points of *OA* and *DG* are *P* and *Q* respectively and *R* is the centre of the square face *ABFE*.

(i) Express each of the vectors $\overrightarrow{PR}$ and $\overrightarrow{PQ}$ in terms of **i**, **j** and **k**. [3]

(ii) Use a scalar product to find angle *QPR*. [4]

(iii) Find the perimeter of triangle *PQR*, giving your answer correct to 1 decimal place. [3]

Solution

(i). $\overrightarrow{PR} = \overrightarrow{PA} + \frac{1}{2}\overrightarrow{AB} + \frac{1}{2}\overrightarrow{AE}$

$\therefore \boldsymbol{\overrightarrow{PR} = 2i + 2j + 2k}$

$\overrightarrow{PQ} = \overrightarrow{PO} + \overrightarrow{OD} + \overrightarrow{DQ}$

$\Rightarrow \overrightarrow{PQ} = -2i + 4k + 2j$

$\therefore \boldsymbol{\overrightarrow{PQ} = -2i + 2j + 4k}$

(ii). $\cos\theta = \frac{\overrightarrow{PQ}\,.\,\overrightarrow{PR}}{|\overrightarrow{PQ}||\overrightarrow{PR}|}$

$$\Rightarrow \cos\theta = \frac{\begin{pmatrix}2\\2\\2\end{pmatrix}}{\sqrt{12}} \cdot \frac{\begin{pmatrix}-2\\2\\4\end{pmatrix}}{\sqrt{24}}$$

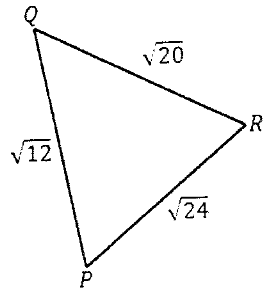
$$\Rightarrow \cos\theta = \frac{8}{\sqrt{288}}$$

$$\Rightarrow \theta = \cos^{-1}\left(\frac{8}{\sqrt{288}}\right)$$

$$\therefore \boldsymbol{\theta = 61.9^\circ}$$

(iii). $Perimeter\ of\ triangle\ PQR = |\overrightarrow{PQ}| + |\overrightarrow{PR}| + |\overrightarrow{QR}|$

$where\ |\overrightarrow{PQ}| = \sqrt{12},\ |\overrightarrow{PR}| = \sqrt{24},\ and$



$$\overrightarrow{QR} = \overrightarrow{QG} + \overrightarrow{GF} + \frac{1}{2}\overrightarrow{FB} + \frac{1}{2}\overrightarrow{BA}$$

$$\Rightarrow \overrightarrow{QR} = 2j + 4i - 2k - 2j$$

$$\Rightarrow \overrightarrow{QR} = 4i - 2k$$

$$\Rightarrow |\overrightarrow{QR}| = \sqrt{[(4)^2 + (-2)^2]}$$

$$\Rightarrow |\overrightarrow{QR}| = \sqrt{20}$$

$Now,\ perimeter\ of\ triangle\ PQR = \sqrt{12} + \sqrt{24} + \sqrt{20}$

$\therefore$ ***Perimeter of triangle PQR* = 12.8 *units***

Revision Questions on Vectors

November 2012 qp.12 (Cambridge)

7 The position vectors of the points A and B, relative to an origin O, are given by

$$\overrightarrow{OA} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} \quad \text{and} \quad \overrightarrow{OB} = \begin{pmatrix} k \\ -k \\ 2k \end{pmatrix},$$

where k is a constant.

(i) In the case where $k = 2$, calculate angle AOB. [4]

(ii) Find the values of k for which $\overrightarrow{AB}$ is a unit vector. [4]

November 2012 qp.13 (Cambridge)

9 The position vectors of points A and B relative to an origin O are given by

$$\overrightarrow{OA} = \begin{pmatrix} p \\ 1 \\ 1 \end{pmatrix} \quad \text{and} \quad \overrightarrow{OB} = \begin{pmatrix} 4 \\ 2 \\ p \end{pmatrix},$$

where p is a constant.

(i) In the case where OAB is a straight line, state the value of p and find the unit vector in the direction of $\overrightarrow{OA}$. [3]

(ii) In the case where OA is perpendicular to AB, find the possible values of p. [5]

(iii) In the case where $p = 3$, the point C is such that $OABC$ is a parallelogram. Find the position vector of C. [2]

November 2011 qp.11 (Cambridge)

8 Relative to an origin O, the point A has position vector $4\mathbf{i} + 7\mathbf{j} - p\mathbf{k}$ and the point B has position vector $8\mathbf{i} - \mathbf{j} - p\mathbf{k}$, where p is a constant.

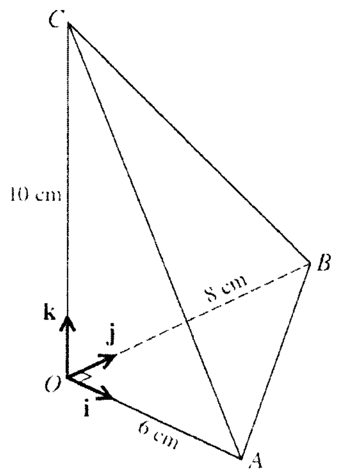
(i) Find $\overrightarrow{OA}.\overrightarrow{OB}$. [2]

(ii) Hence show that there are no real values of p for which OA and OB are perpendicular to each other. [1]

(iii) Find the values of p for which angle $AOB = 60°$. [4]

November 2010 qp.11 (Cambridge)

5



The diagram shows a pyramid $OABC$ with a horizontal base OAB where $OA = 6$ cm, $OB = 8$ cm and angle $AOB = 90°$. The point C is vertically above O and $OC = 10$ cm. Unit vectors **i**, **j** and **k** are parallel to OA, OB and OC as shown.

Use a scalar product to find angle ACB. [6]

November 2006 qp.1 (Zimsec)

12. Given that the position vectors of points A, B and C are $(6i + 2j + 6k)$; $(2i + 3j + k)$ and $(14i + 16k)$ respectively,

    i. Find $\overrightarrow{AB}$ and $\overrightarrow{AC}$ and state the exact value of $|\overrightarrow{AB}|$, [4]

    ii. State a precise relationship between vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$.
    Hence draw a sketch to show the relative arrangement of points A, B and C in space. [3]

June 2003 qp.1 (Zimsec)

12. The points A, B and C have position vectors

$$\begin{pmatrix}1\\2\\3\end{pmatrix}, \begin{pmatrix}2\\2\\0\end{pmatrix} \text{ and } \begin{pmatrix}4\\0\\0\end{pmatrix} \text{ respectively.}$$

The point O is the origin and the point M is the mid-point of AB

i. Find the vectors $\overrightarrow{OM}$ and $\overrightarrow{CM}$ [2]

ii. Calculate $O\hat{M}C$. Hence find the area of triangle OMC. [5]

November 2010 qp.1 (Zimsec)

5. The position vectors of points A and B with respect to the origin O, are given by

$$\overrightarrow{OA} = i + 3j + 3k,$$
$$\overrightarrow{OB} = -4i + 5j + 3k.$$

Show that $\cos(A\hat{O}B) = \frac{4}{\sqrt{38}}$ [2]

Hence, or otherwise, find the position vector of the point P on OB such that AP is perpendicular to OB. [4]

November 2007 qp.1 (Zimsec)

8. Two birds, P and Q fly such that their position vectors with respect to an origin O are given by

$$\overrightarrow{OP} = (2t+3)i + (t-1)j + 3tk \text{ and}$$

$$\overrightarrow{OQ} = (t-2)i + (3t+1)j + (t+2)k$$

for $0 \leq t \leq 10$, where $i, j\ and\ k$ are unit vectors of magnitude 1 metre in the x, y and z directions respectively.

(a) For the time $t = 0$,

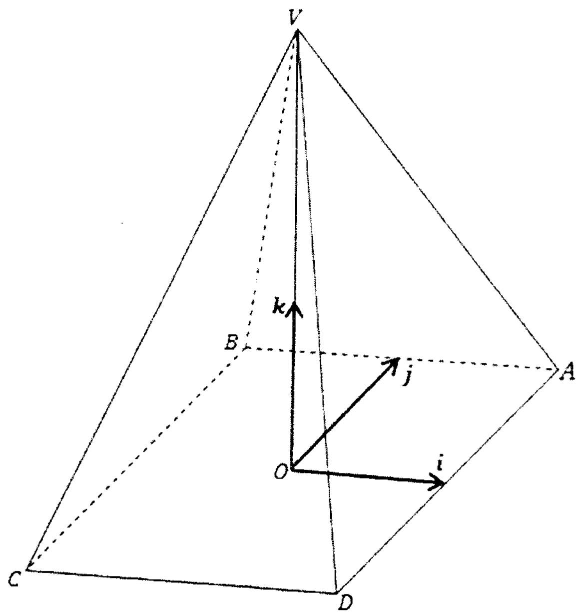
(i). calculate the distance between the two birds, [3]

(ii). Find the position vector of the point mid-way between the two birds. [1]

(b) Find the value of t for which $P\hat{O}Q = 90°$, giving your answer to 2 significant figures. [3]

June 2001 qp.1 (Cambridge)

7.



With respect to the origin O, the corners A, B, C of the square base $ABCD$ of a pyramid have position vectors $\boldsymbol{i}+\boldsymbol{j}, -\boldsymbol{i}+\boldsymbol{j}, -\boldsymbol{i}-\boldsymbol{j}$ respectively (see diagram). Write down the position vector of D. [1]

The vertex V of the pyramid has position vector $(\sqrt{6})\boldsymbol{k}$.

(i). Express the vectors $\overrightarrow{AV}$ and $\overrightarrow{CV}$ in terms of $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$. [3]

(ii). Use a scalar product to show that angle $AVC = 60°$. [2]

Advert One: The Concept of Partial Fractions

"A man is like a fraction whose numerator is what he is and whose denominator is what he thinks of himself. The larger the denominator, the smaller the fraction."

– Leo Tolstoy

The concept of partial fractions is used to breakdown a combined fraction into its component fractions. A combined fraction is one in which the denominator is expressed as a product of factors. Partial fractions are used as a 'lead-in' concept to questions on binomial expansion and integration. The choice of a partialising technique is inspired by the nature of the combined denominator. There are four techniques used to breakdown a consolidated denominator:

- Linear-factor approach
- Quadratic-factor approach
- Repeated-factor approach
- Improper-fractions approach

➢ Linear-Factor Approach

A factor is said to be linear if the highest order power of the unknown variable is one (1). For example, $(x+1)$ and $(x-3)$ are linear factors. Below is an example outlining the breakdown of a combined fraction using the linear-factor approach.

$$Given\ that\ \ f(x) = \frac{3x-2}{(x-1)(x+2)},$$

NB: Each factor is assigned to a constant which assumes the position of the numerator as shown below:

$$\Rightarrow \frac{3x-2}{(x-1)(x+2)} = \frac{A}{(x-1)} + \frac{B}{(x+2)}$$

$$\Rightarrow 3x-2 = A(x+2) + B(x-1)$$

$let\ x = 1$

$1 = 3A$

$\Rightarrow A = \frac{1}{3}$

$let\ x = -2$

$-8 = -3B$

$\Rightarrow B = \frac{8}{3}$

$$\Rightarrow \frac{3x-2}{(x-1)(x+2)} = \frac{\frac{1}{3}}{(x-1)} + \frac{\frac{8}{3}}{(x+2)}$$

$$\therefore \boldsymbol{f(x) = \frac{1}{3(x-1)} + \frac{8}{3(x+2)}}$$

- **Quadratic-Factor Approach**

A factor is said to be quadratic if the highest order power of the unknown variable is two (2) where the power directly affects the unknown variable. For example, (x^2+3) is a quadratic factor.

$$If\ f(x) = \frac{2x^2-1}{(x+1)(x^2+3)}$$

NB:

- A linear factor is assigned a constant that assumes the position of the numerator;
- A quadratic factor is assigned two constants to assume the numerator. One of the constants is attached to a variable x and the other is a stand-alone constant.

$$then\ \frac{2x^2-1}{(x+1)(x^2+2)} = \frac{A}{(x+1)} + \frac{(Bx+C)}{(x^2+2)}$$

$$\Rightarrow 2x^2-1 = A(x^2+2)+(Bx+C)(x+1)$$

$let x = -1$

$1 = 3A$

$\Rightarrow A = \frac{1}{3}$

$let\ x = 0$

$-1 = 2A + C$

$\Rightarrow -1 = 2\left(\frac{1}{3}\right) + C$

$\Rightarrow C = -\frac{5}{3}$

$let\ x = 1$

$1 = 3A + 2B + 2C$

$1 = 3\left(\frac{1}{3}\right) + 2B + 2\left(-\frac{5}{3}\right)$

$\Rightarrow B = \frac{5}{3}$

$$\frac{2x^2-1}{(x+1)(x^2+3)} = \frac{\frac{1}{3}}{(x+1)} + \frac{\left(\frac{5}{3}x-\frac{5}{3}\right)}{(x^2+2)}$$

$$\therefore \boldsymbol{f(x) = \frac{1}{3(x+1)} + \frac{(5x-5)}{3(x^2+2)}}$$

- **Repeated-Factor Approach**

Factors are regarded as repeated if there is a power affecting all the terms to the factor. For example, $(x+2)^2$ and $(x-3)^3$ are repeated factors because the powers are not affecting individual terms. A repeated factor is broken down by assigning a constant to a factor to the power 1, and a constant to a factor to the power 2, and a constant to the power 3, and so on.

$$For\ example,\quad \frac{1}{(x+2)^4}=\frac{A}{(x+2)}+\frac{B}{(x+2)^2}+\frac{C}{(x+2)^3}+\frac{D}{(x+2)^4}$$

$$If\ f(x)=\frac{4x^2-3x-1}{(x-2)(x+1)^2}$$

$$then\quad \frac{4x^2-3x-1}{(x-2)(x+1)^2}=\frac{A}{(x-2)}+\frac{B}{(x+1)}+\frac{C}{(x+1)^2}$$

$$\Rightarrow 4x^2-3x-1=A(x+1)^2+B(x-2)(x+1)+C(x-2)$$

$let\ x=2$

$9=9A$

$\Rightarrow A=1$

$let\ x=-1$

$6=-3C$

$\Rightarrow C=-2$

$let\ x=0$

$-1=A-2B-2C$

$-1=1-2B-2(-2)$

$\Rightarrow B=-1$

$$\therefore \boldsymbol{f(x)=\frac{1}{(x-2)}-\frac{1}{(x+1)}-\frac{2}{(x+1)^2}}$$

- **Improper-Fractions Approach**

A fraction is improper if the highest order power in the numerator is exactly equal to or greater than the highest order power in the denominator. When the highest order power in the numerator is **exactly equal** to the highest order power in the denominator, a fraction is broken down by way of introducing a constant that is free of the denominator in addition to the normal breakdown process. For example,

$$If\ f(x)=\frac{2x^2+2x+2}{(x+1)(x+3)}$$

NB: the highest order power in the numerator is 2, which is exactly equal to the highest order power in the denominator when expanded.

$$then\ \frac{2x^2+2x+2}{(x+1)(x+3)} = A + \frac{B}{(x+1)} + \frac{C}{(x+3)}$$

$$2x^2+2x+2 = A(x+1)(x+3) + B(x+3) + C(x+1)$$

$let\ x=-1$	$let\ x=-3$	$let\ x=0$
$\Rightarrow 2=2B$	$\Rightarrow 14=-2C$	$\Rightarrow 2=3A+3B+C$
$\Rightarrow B=1$	$\Rightarrow C=-7$	$\Rightarrow 2=3A+3(1)-7$
		$\Rightarrow A=2$

$$\therefore \boldsymbol{f(x) = 2 + \frac{1}{(x+1)} - \frac{7}{(x+3)}}$$

When the highest order power in the numerator is **greater** than the highest order power in the denominator, we employ the **long division** process. For example,

$$If\ f(x) = \frac{3x^3-2x^2-16x+20}{(x-2)(x+2)}$$

$$\Rightarrow \frac{3x^3-2x^2-16x+20}{(x-2)(x+2)} =$$

$$\begin{array}{r|l} & 3x-2 \\ \hline (x^2-4) & 3x^3-2x^2-16x+20 \\ & -(3x^3+0x^2-12x) \\ \hline & \quad -2x^2-4x+20 \\ & \quad -(-2x^2+0x+8) \\ \hline & \qquad -4x+12 \end{array}$$

$$Since\ the\ combined\ fraction = quotient + \frac{remainder}{divisor}$$

$$\Rightarrow \frac{3x^3-2x^2-16x+20}{(x-2)(x+2)} = (3x-2) + \frac{-4x+12}{x^2-4}$$

$$\Rightarrow \frac{3x^3-2x^2-16x+20}{(x-2)(x+2)} = (3x-2) + \frac{-4x+12}{(x-2)(x+2)}$$

$$where \quad \frac{-4x+12}{(x-2)(x+2)} = \frac{A}{(x-2)} + \frac{B}{(x+2)}$$

$$\Rightarrow -4x+12 = A(x+2) + B(x-2)$$

$let\ x = 2$

$4 = 4A$

$A = 1$

$let\ x = -2$

$20 = -4B$

$B = -5$

$$\Rightarrow \frac{-4x+12}{(x-2)(x+2)} = \frac{1}{(x-2)} - \frac{5}{(x+2)}$$

$$\therefore \frac{\mathbf{3x^3 - 2x^2 - 16x + 20}}{\mathbf{(x-2)(x+2)}} = \mathbf{3x - 2 + \frac{1}{(x-2)} - \frac{5}{(x+2)}}$$

Long division is universal in that it applies to cases when the highest order power in the numerator is **equal to** or **greater** than the highest order power in the denominator. Students are, therefore, encouraged to master this technique.

Revision Questions on Partial Fractions

Express the following as partial fractions:

Linear-Factor Approach

1. $\frac{x^2+7x-6}{(x-1)(x-2)(x+1)}$
2. $\frac{3x}{(x+1)(x-2)}$
3. $\frac{1}{y(4-y)}$

Quadratic-Factor Approach

4. $\frac{10}{(2-x)(1+x^2)}$
5. $\frac{6+7x}{(2-x)(1+x^2)}$
6. $\frac{4x}{(x+4)(x^2+3)}$
7. $\frac{3x^2+x}{(x+2)(x^2+1)}$

Repeated-Factor Approach

8. $\frac{4x}{(3x+1)(x+1)^2}$
9. $\frac{9x^2+4}{(2x+1)(x-2)^2}$
10. $\frac{7x+4}{(2x+1)(x+1)^2}$

Improper-Fractions Approach

11. $\frac{x^3-x-2}{(x-1)(x^2+1)}$
12. $\frac{x^3-2x^2-4x-4}{x^2+x-2}$
13. $\frac{4x^2+5x+3}{2x^2+5x+2}$

Chapter Seven: Sequences and Series

"I am tomorrow, or some future day, what I establish today. I am today what I established yesterday or some previous day."

– James Joyce

Doing mathematics involves finding patterns and crafting beautiful and meaningful explanations out of the patterns. Sequences and series are jointly used to study number patterns from different perspectives.

Definition of Terms

1. **Sequence**
   This is used to describe an array of numbers following a specific order or pattern. For example: $2;\ 4;\ 6;\ 8;\ 10\ldots$

2. **Series**
   This refers to the sum of numbers in a sequence. For example: $2 + 4 + 6 + 8 + 10\ldots$

Notation used in the computation of sequences and series

- $U_n - the\ n^{th}\ term$
- $a - first\ term$
- $n - position\ of\ the\ term\ in\ the\ sequence$
- $d - common\ difference$
- $r - common\ ratio$
- $l - last\ term$

Types of Sequences

1. <u>**Arithmetic Progression (AP)**</u>
   An AP is a sequence in which the **difference** between any pair of successive terms is uniform. For example:

$$\underbrace{100 \quad ; \quad 95}_{-5} \quad \underbrace{\quad ; \quad 90}_{-5} \quad \underbrace{\quad ; \quad 85}_{-5}$$

$$\boldsymbol{Common\ Difference = Proceeding\ Term - Preceding\ Term}$$

a) <u>The n^{th} term</u>

$$\boldsymbol{U_n = a + (n-1)d}$$

For example, the sixth term in the above sequence is giving as follows:

$\Rightarrow U_6 = 100 + (6-1)(-5)$

$\therefore \boldsymbol{U_6 = 75}$

b) <u>Sum of n terms</u>

$$\boldsymbol{S_n = \frac{n}{2}[2a + (n-1)d] \quad or \quad S_n = \frac{n}{2}[a + l]}$$

For example, the sum of the first four terms in the above sequence is given as follows:

$S_4 = \frac{4}{2}[2(100) + (4-1)(-5)]$

$\therefore \boldsymbol{S_4 = 370}$

2. <u>Geometric Progression (GP)</u>

A GP describes a sequence in which any pair of successive terms gives a uniform **ratio.** For example,

$$\underbrace{2 \quad ; \quad 4}_{\times 2} \quad \underbrace{\quad ; \quad 8}_{\times 2} \quad \underbrace{\quad ; \quad 16}_{\times 2}$$

$$\boldsymbol{Common\ Ratio = \frac{Proceeding\ Term}{Preceding\ Term}}$$

a) <u>The n^{th} term</u>

$$U_n = ar^{n-1}$$

For example the sixth term in the above sequence is given by:

$U_6 = (2)(2)^{6-1}$

$\therefore U_6 = 64$

b) <u>Sum of n terms</u>

$$S_n = \frac{a(1-r^n)}{(1-r)}$$

For example, the sum of the first four terms in the above sequence is given by:

$S_6 = \frac{2(1-2^4)}{(1-2)}$

$\therefore S_6 = 30$

c) <u>Sum to Infinity</u>

This concept can only be employed to convergent sequences where the common ratio lies between negative one and positive one exclusive .That is, $-1 < r < 1$.

It is given mathematically as follows:

$$S_\infty = \frac{a}{(1-r)}$$

NB: questions on sequences and series are centred on the manipulation of statements in theory into mathematical expressions. This transformation process leads to an equation which should be subsequently solved.

Series Expansion

Series expansion, widely known as binominal theorem, is a predetermined pattern used to expand the relationship connecting the sum or difference of two terms. For **positive whole number values of n**, the general formula states that:

$$(a+b)^n = {}^nC_0(a)^{n-0}(b)^0 + {}^nC_1(a)^{n-1}(b)^1 + {}^nC_2(a)^{n-2}(b)^2 + {}^nC_3(a)^{n-3}(b)^3 \ldots$$

For example, the first three terms the expansion of $(2-x)^6$ are given as follows:

$$(2-x)^6 = {}^6C_0(2)^{6-0}(-x)^0 + {}^6C_1(2)^{6-1}(-x)^1 + {}^6C_2(2)^{6-2}(-x)^2$$

$$\therefore (2-x)^6 = 64 - 192x + 120x^2$$

Students are encouraged to master the universal expansion that applies to **all values of n**:

$$(1+x)^n = 1 + \frac{(n)(x)}{1!} + \frac{(n)(n-1)(x)^2}{2!} + \frac{(n)(n-1)(n-2)(x)^3}{3!} \ldots$$

This formula only works when the first term is positive one (1). If the first term is not 1, factor out the limiting term. Remember to pull out the power as well when factoring out that term.

For example,

$$(-2+x)^4 = (-2)^4\left(1-\frac{x}{2}\right)^4$$

$$\Rightarrow (-2+x)^4 = 16\left[1 + \frac{4\left(-\frac{x}{2}\right)}{1!} + \frac{(4)(4-1)\left(-\frac{x}{2}\right)^2}{2!} \ldots\right]$$

$$\Rightarrow (-2+x)^4 = 16\left[1 - 2x + \frac{3}{2}x^2 \ldots\right]$$

$$\therefore (-2+x)^4 = 16 - 32x + 24x^2 \ldots$$

In most cases, questions on the binomial theorem are hinged to partial fractions. For problems of this nature, the combined fraction has to be broken down into its component parts before bringing the denominators up.

If the terms to the factors are arranged in such a way that the first term is not real, re-arrange the terms to start with the real number.

For example, given that,

$$f(x) = \frac{3x+1}{(x^2-4)(x+1)}$$

Obtain an expansion $f(x)$ in ascending powers of x, up to and including the term in x^2.

$$\frac{(3x+1)}{(x^2-4)(x+1)} = \frac{(Ax+B)}{(x^2-4)} + \frac{C}{(x+1)}$$

$\Rightarrow 3x + 1 = (Ax+B)(x+1) + C(x^2-4)$

$let\ x = -1$

$-2 = -3C$

$\Rightarrow C = \frac{2}{3}$

$let\ x = 0$

$1 = B - 4C$

$1 = B - 4\left(\frac{2}{3}\right)$

$\Rightarrow B = \frac{11}{3}$

$let\ x = 2$

$7 = (2A+B)(3)$

$7 = 6A + 3\left(\frac{11}{3}\right)$

$\Rightarrow A = -\frac{2}{3}$

$$\therefore \boldsymbol{f(x)} = \frac{\left(-\frac{2}{3}x + \frac{11}{3}\right)}{(x^2-4)} + \frac{\frac{2}{3}}{(x+1)}$$

$$Now,\ f(x) = \left(-\frac{2}{3}x + \frac{11}{3}\right)(-4+x^2)^{-1} + \frac{2}{3}(1+x)^{-1}$$

$$where\ (-4+x^2)^{-1} = (-4)^{-1}\left(1 - \frac{x^2}{4}\right)^{-1}$$

$$\Rightarrow (-4+x^2)^{-1} = -\frac{1}{4}\left(1 - \frac{x^2}{4}\right)^{-1}$$

$$\Rightarrow (-4+x^2)^{-1} = -\frac{1}{4}\left[1 + \frac{(-1)\left(-\frac{x^2}{4}\right)^1}{1!} + \cdots\right]$$

$$\Rightarrow (-4+x^2)^{-1} = -\frac{1}{4}\left(1 + \frac{x^2}{4}\right)$$

$$\Rightarrow (-4+x^2)^{-1} = -\frac{1}{4} - \frac{x^2}{16}$$

$$\Rightarrow \left(-\frac{2}{3}x + \frac{11}{3}\right)(-4 + x^2)^{-1} = \left(-\frac{2}{3}x + \frac{11}{3}\right)\left(-\frac{1}{4} - \frac{x^2}{16}\right)$$

$$\Rightarrow \left(-\frac{2}{3}x + \frac{11}{3}\right)(-4 + x^2)^{-1} = \frac{1}{6}x - \frac{11}{12} - \frac{11}{48}x^2$$

$$\Rightarrow \left(-\frac{2}{3}x + \frac{11}{3}\right)(-4 + x^2)^{-1} = -\frac{11}{12} + \frac{1}{6}x - \frac{11}{48}x^2$$

$$and\ \frac{2}{3}(1 + x)^{-1} = \frac{2}{3}\left[1 + \frac{(-1)(x)}{1!} + \frac{(-1)(-1-1)(x)^2}{2!} + \cdots\right]$$

$$\Rightarrow \frac{2}{3}(1 + x)^{-1} = \frac{2}{3}[1 - x + x^2]$$

$$\Rightarrow \frac{2}{3}(1 + x)^{-1} = \frac{2}{3} - \frac{2}{3}x + \frac{2}{3}x^2$$

$$f(x) = \left[-\frac{11}{12} + \frac{1}{6}x - \frac{11}{48}x^2\right] + \left[\frac{2}{3} - \frac{2}{3}x + \frac{2}{3}x^2\right]$$

$$\therefore f(x) = -\frac{1}{4} - \frac{1}{2}x + \frac{7}{16}x^2$$

Behaviour of Sequences

A sequence may exhibit any of the following four attributes:

- Periodicity
- Oscillation
- Convergence
- Divergence

1. Periodic Sequence

This attribute manifests itself in a sequence that repeats the same pattern after a particular number of terms. Periodic sequences are cyclic in nature. For example:

$$3;\ 7;\ 11;\ 3;\ 7;\ 11;\ 3;\ 7;\ 11\ \ldots$$

2. Oscillating Sequence

An oscillating sequence is inspired by the action of pendulum. It swings back and forth around a particular value. For example:

$$1;\ -x;\ x^2;\ -x^3;\ x^4;\ -x^5 \ldots$$

In this case, the terms are alternating the signs as shown by an illustrative diagram below:

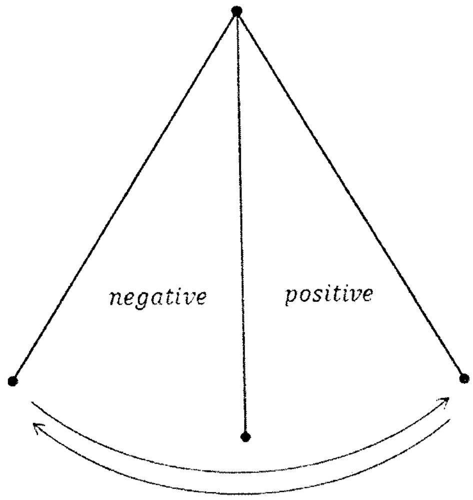


Fig.7.1

3. Converging Sequencing

This is used to describe a sequence where terms progress in such a way that they reduce to a particular value. A convergent sequence is one that has a sum to infinity. For example:

$$V_n = 3 - \left(\frac{1}{4}\right)^n$$

- $\Rightarrow V_1 = \frac{11}{4} = 2{,}75$
- $V_2 = \frac{47}{16} = 2{,}984375$
- $V_3 = \frac{191}{64} = 2{,}984375$
- $V_4 = \frac{767}{250} = 2{,}99609375$

  and so on

In this case, the terms converge to 3.

4. Diverging Sequence

This is a sequence that springs up from a certain value and accelerates in one direction. For example:

$$U_n = (n-3)(n+2)$$

- $U_1 = (-2)(3) = -6$
- $U_2 = (-1)(4) = -4$
- $U_3 = (0)(5) = 0$
- $U_4 = (1)(6) = 6$
- $U_5 = (2)(7) = 14$

  and so on

In this case, the sequence stretches from – 6 to positive infinity.

The Sigma Notation

The sigma notation, $\sum$, is a symbol used to represent the **' sum of '**. For example,

$$\sum_{r=1}^{r=4} r^2 \ means\ 1^2 + 2^2 + 3^2 + 4^2,$$

That is the sum of all terms of the form r^2 from $r = 1$ to $r = 4$.

- If the sequence is never ending, that is if its stretching to infinity, it is expressed as follows:

  $$\sum_{r=1}^{\infty} r^2$$

  implying the sum of all terms of the form r^2 from $r = 1$ to positive infinity.

- This notation is mainly used to summarise a series.

Worked Examination Questions on Progressions

Question (Zimsec, November 2006 qp.1)

13. (a) Given that

$$\sum_{r=1}^{n}(2r-3)=255,$$

find the value of n. [5]

(b). After running a 40 km marathon race, an athlete "trains down" by running 80% of the distance run the previous day, starting the day after the competition.

Find (i). the distance run on the tenth day after the marathon race, [2]

(ii). the first day on which the athlete will have run a total of more than 155km after the marathon race. [5]

Solution

a) *Given that*

$$\sum_{r=1}^{n}(2r-3)=255$$

by taking a snapshot of the first few terms:

$when\ r=1;\ u_1=-1$

$when\ r=2;\ u_2=1$

$when\ r=3;\ u_3=3$

$when\ r=4;\ u_4=5\ and\ so\ on$

From the analysis, the progression is following an arithmetic progression (AP) with the first term of -1 and common difference of 2.

$\sum_{r=1}^{n}(2r-3)$ *represents sum of all terms*

$\Rightarrow S_n=255$

$\Rightarrow \frac{n}{2}[2a+(n-1)d]=255$

$\Rightarrow \frac{n}{2}[2(-1)+(n-1)(2)]=255$

$\Rightarrow \frac{n}{2}[-2 + 2n - 2] = 255$

$\Rightarrow \frac{n}{2}[-4 + 2n] = 255$

$\Rightarrow -2n + n^2 = 255$

$\Rightarrow n^2 - 2n - 255 = 0$

$\Rightarrow n = 17 \; or -15$

$\therefore \boldsymbol{n = 17 \; only \; since \; n \; is \; never \; negative}$

b) (i) The distance run is following a geometric progression (GP) because the ratio between any pair of succesive terms is 0.8. In this case,

$a = 80\% \; of \; 40$

$\Rightarrow a = 32$

$and \; r = 0.8$

$using \; U_n = ar^{n-1}$

$\Rightarrow U_{10} = 32(0.8)^{10-1}$

$\therefore \boldsymbol{U_{10} = 4.29 \; km}$

(ii) $Using \; S_n = \frac{a(1-r^n)}{(1-r)}$

$\Rightarrow 155 = \frac{32(1 - 0.8^n)}{(1 - 0.8)}$

$\Rightarrow 155 = 160(1 - 0.8^n)$

$\Rightarrow \frac{155}{160} = 1 - 0.8^n$

$\Rightarrow 0.8^n = 1 - \frac{155}{160}$

$\Rightarrow 0.8^n = \frac{1}{32}$

$by \; taking \; logarithms \; of \; the \; LHS \; and \; RHS$

$\Rightarrow \log 0.8^n = \log \frac{1}{32}$

$$\Rightarrow n\log 0.8 = \log\frac{1}{32}$$

$$\Rightarrow n = \frac{\log\frac{1}{32}}{\log 0.8}$$

$$\Rightarrow n = 15.5$$

$$\therefore \boldsymbol{n = 16^{th}\ day}$$

Question (Cambridge, November 1993 qp.1)

15. (a) A geometric progression G has positive first term a, common ratio r and sum to infinity S. The sum to infinity of the even-numbered terms of G, i.e. the second, fourth, sixth,...terms, is $-\frac{1}{2}S$.
Find the value of r. [3]
   (i) Given that the third term of G is 2, show that the sum to infinity of the odd-numbered terms of G, i.e. the first, third, fifth, ... terms, is $\frac{81}{4}$. [3]
   (ii) In another Geometric progression H, each term is the modulus of the odd-numbered terms of G. Show that the sum to infinity of H is $2S$. [2]

(b) The sum of the first hundred terms of an arithmetic progression with first term a and common difference d is T. The sum of the first 50 odd-numbered terms i.e. the first, third, fifth, ... ninety-ninth, is $\frac{1}{2}T - 1000$. Find the value of d. [4]

Solution

a) In a geometric progression (GP), the terms in the sequence are given by $U_n = ar^{n-1}$. The snapshot of the first few terms is as follows:

$$a;\ ar;\ ar^2;\ ar^3;\ ar^4;\ ar^5;\ ar^6 \ldots$$

$Even - numbered\ sequence$: $ar; ar^3; ar^5 \ldots$

Where the first term is ar and common ratio is r^2 $\left(that\ is\ \frac{ar^3}{ar}\right)$

$$S_\infty = \frac{a}{(1-r)}$$

$$\Rightarrow -\frac{1}{2}S = \frac{ar}{(1-r^2)}$$

$$\Rightarrow -\frac{1}{2}\left(\frac{a}{1-r}\right) = \frac{ar}{(1-r^2)}$$

$$\Rightarrow \frac{-a}{2(1-r)} = \frac{ar}{(1-r)(1+r)}$$

$$\Rightarrow -a(1-r)(1+r) = 2ar(1-r)$$

$$\Rightarrow -(1+r) = 2r$$

$$\Rightarrow -1 - r = 2r$$

$$\Rightarrow -1 = 3r$$

$$\therefore \boldsymbol{r = -\frac{1}{3}}$$

(i). $Odd - numbered\ sequence: a;\ ar^2;\ ar^4;\ ar^4;\ ar^6\ ...$

Where the first term is a and common ratio is r^2 $\left(that\ is\ \frac{ar^2}{a}\right)$

$Since\ the\ third\ term\ of\ G\ is\ 2$

$$\Rightarrow ar^2 = 2$$

$$\Rightarrow a\left(-\frac{1}{3}\right)^2 = 2$$

$$\Rightarrow a\left(\frac{1}{9}\right) = 2$$

$$\Rightarrow a = 18$$

$Now,\ S_\infty\ of\ odd - numbered\ sequence\ is\ given\ by;$

$S_\infty = \frac{a}{1-r^2}$ {since the first term is a and common ratio is r^2}

$$S_\infty = \frac{18}{1-\left(-\frac{1}{3}\right)^2}$$

$$\Rightarrow S_\infty = \frac{18}{\left(\frac{8}{9}\right)}$$

$$\therefore \boldsymbol{S_\infty = \frac{81}{4}\ (shown)}$$

(ii). Now that the values of a and r are known to be 18 and $-\frac{1}{3}$ respectively,

$G: a;\ ar;\ ar^2;\ ar^3;\ ar^4\ ...$

$\Rightarrow G: 18;\ -6;\ 2;\ -\frac{2}{3};\ \frac{2}{9}\dots$

Since the modulus sign is used to cushion against a negative sign.

$\Rightarrow H: 18;\ 6;\ 2;\ \frac{2}{3};\ \frac{2}{9}\ \dots$

$S_{\infty\ of\ H} = \frac{18}{1-\frac{1}{3}}$

$\Rightarrow S_{\infty of H} = \frac{18}{\frac{2}{3}}$

$\Rightarrow S_{\infty\ of\ H} = 27$

and $S_{\infty of G} = S = \frac{18}{1--\frac{1}{3}}$

$\Rightarrow S = \frac{18}{\frac{4}{3}}$

$\Rightarrow S = \frac{27}{2}$

$Now,\ S_{\infty\ of\ H} = 2S$

$\Rightarrow 27 = 2\left(\frac{27}{2}\right)$

$\therefore \mathbf{27 = 27\ (\textit{shown})}$

b) For an Arithmetic Progression (AP)with first term a and common difference d, the snapshot of the first few terms is as follows:

$$a; (a+d); (a+2d); (a+3d); (a+4d)\dots$$

$S_n = \frac{n}{2}[2a+(n-1)d]$

$\Rightarrow S_{100} = T = \frac{100}{2}[2a+(100-1)d]$

$\Rightarrow T = 50(2a+99d)$

$\Rightarrow T = 100a + 4\,950d$

Odd − numbered sequence: $a; (a+2d); (a+4d)\dots$

where the first term is a and common difference is $2d$

$S_{50} = \frac{50}{2}[2a+(50-1)(2d)]$

$\Rightarrow \frac{1}{2}T - 1\,000 = 25[2a+98d]$

$\Rightarrow \frac{1}{2}[100a+4\,950d] - 1\,000 = 50a + 2\,450d$

$\Rightarrow 50a + 2\,475d - 1000 = 50a + 2\,450d$

$\Rightarrow 2\,475d - 2\,450d = 1\,000$

$\Rightarrow 25d = 1\,000$

$\therefore \boldsymbol{d = 40}$

Question (Zimsec, June 2003 qp.1)

17. (a). Evaluate

$$\sum_{r=1}^{200}(2r-3)$$

[3]

(b). Find the least number of terms for which

$$\sum_{r=0}^{n}\left(\frac{1}{3}\right)^{r}$$

differs from its sum to infinity by less than 0,001. [6]

(c). Two sequences are defined for $n = 1.2.3 \ldots\ldots\ldots$ as follows

$$U_n = (n+1)(n+2);\ V_n = 3 - \left(\frac{1}{4}\right)^{n}$$

(i). Describe the behaviour of each of the sequences as $n \longrightarrow \infty$. [2]

(ii). Express $U_{n+1} + U_n$ in terms of n simplifying your answer. [2]

Solution

a) *Given the series*

$$\sum_{r=1}^{200}(2r-3)$$

The first step is to determine the first few terms of the sequence with the view of establishing the type of the progression (that is whether it is an AP or a GP).

$when\ \ r = 1;\ U_1 = -1$

$when\ r = 2;\ U_2 = 1$

$when\ r = 3;\ U_3 = 3$

$when\ r = 4;\ U_4 = 5\ \ and\ so\ on.$

As such, the progression is arithmetic in nature with a first term of -1 and a common difference of 2.

$Now,\ S_n = \frac{n}{2}[2a + (n-1)d]$

$\Rightarrow S_{200} = \frac{200}{2}[2(-1) + (200-1)(2)]$

$\therefore\ \boldsymbol{S_{200} = 39\,600}$

b) Due to the fact that the progression $\left(\frac{1}{3}\right)^r$ has a sum to infinity, it implies that it is a geometric progression (GP). Below is a snapshot of the progression:

$when\ r = 0;\ U_1 = 1$

$when\ r = 1;\ U_2 = \frac{1}{3}$

$when\ r = 2;\ U_3 = \frac{1}{9}\ and\ so\ on.$

$when\ r = n;\ U_{n+1} = \left(\frac{1}{3}\right)^n$

In this case, $a = 1$ and $r = \frac{1}{3}$

$$\sum_{r=0}^{n}\left(\frac{1}{3}\right)^r = S_{n+1} \qquad and \qquad S_\infty = \frac{a}{(1-r)}$$

$$\Rightarrow S_{n+1} = \frac{a(1-r^{n+1})}{(1-r)}$$

$$\Rightarrow S_{n+1} = \frac{1\left[1-\left(\frac{1}{3}\right)^{n+1}\right]}{1-{}^{1}/_{3}}$$

$$\Rightarrow S_{n+1} = \frac{\left[1-\left(\frac{1}{3}\right)^{n+1}\right]}{{}^{2}/_{3}}$$

$$\Rightarrow S_\infty = \frac{1}{1-\frac{1}{3}}$$

$$\Rightarrow S_\infty = \frac{1}{\left(\frac{2}{3}\right)}$$

$$\Rightarrow S_\infty = \frac{3}{2}$$

$Now,\ S_\infty - S_{n+1} < 0.001$

$$\Rightarrow \frac{3}{2} - \frac{\left[1-\left(\frac{1}{3}\right)^{n+1}\right]}{2/3} < 0.001$$

$$\Rightarrow \frac{3}{2} - 0{,}001 < \frac{\left[1-\left(\frac{1}{3}\right)^{n+1}\right]}{2/3}$$

$$\Rightarrow \frac{1\,499}{1\,000} < \frac{1-\left(\frac{1}{3}\right)^{n+1}}{2/3}$$

$$\Rightarrow \left(\frac{1}{3}\right)^{n+1} < 1 - \frac{1\,499}{1\,500}$$

$$\Rightarrow \left(\frac{1}{3}\right)^{n+1} < \frac{1}{1\,500}$$

by taking logarithms to both sides of the inequality

$$\Rightarrow \log\left(\frac{1}{3}\right)^{n+1} < \log\left(\frac{1}{1\,500}\right)$$

$$\Rightarrow n+1 > \frac{\log\left(\frac{1}{1\,500}\right)}{\log\left(\frac{1}{3}\right)} \qquad \left\{\begin{matrix}\textit{remember to switch the inequality} \\ \textit{sign when dividing with a negative number}\end{matrix}\right\}$$

$$\Rightarrow n > \frac{\log\left(\frac{1}{1\,500}\right)}{\log\left(\frac{1}{3}\right)} - 1$$

$$\Rightarrow n > 5{,}66$$

$$\therefore \boldsymbol{n = 6}$$

c) (i) $U_n = (n+1)(n+2)$

$U_1 = 6$

$U_2 = 12$

$U_3 = 20$

$U_4 = 30$

$U_5 = 42$

and so on

$V_n = 3 - \left(\frac{1}{4}\right)^n$

$V_1 = \frac{11}{4} = 2{,}75$

$V_2 = \frac{47}{16} = 2{,}9375$

$V_3 = \frac{191}{64} = 2{,}984375$

$V_4 = \frac{767}{256} = 2{,}99609375$

and so on

From the above snapshot,

$\boldsymbol{U_n = (n+1)(n+2)}$ ***is a diverging sequence, and***

$\boldsymbol{V_n = 3 - \left(\frac{1}{4}\right)^n}$ ***is a converging sequence.***

(ii) $U_{n+1} + U_n = ([\![n+1]\!]+1)([\![n+1]\!]+2) + (n+1)(n+2)$

$\Rightarrow U_{n+1} + U_n = (n+2)(n+3) + (n+1)(n+2)$

$\Rightarrow U_{n+1} + U_n = (n+2)[(n+3)+(n+1)]$

$\therefore \boldsymbol{U_{n+1} + U_n = (n+2)(2n+4)}$

Question (Cambridge, November 2012 qp.1)

8 **(a)** In a geometric progression, all the terms are positive, the second term is 24 and the fourth term is $13\frac{1}{2}$. Find

(i) the first term. [3]

(ii) the sum to infinity of the progression. [2]

(b) A circle is divided into n sectors in such a way that the angles of the sectors are in arithmetic progression. The smallest two angles are $3°$ and $5°$. Find the value of n. [4]

Solution

(i). *Giving that,*

$U_2 = 24$

$ar = 24 \longrightarrow 1$

$U_4 = 13\frac{1}{2}$

$ar^3 = \frac{27}{2} \longrightarrow 2$

by solving (1) *and* (2) *simultaneously*

$\Rightarrow \frac{ar^3}{ar} = \frac{\frac{27}{2}}{24}$

$\Rightarrow r^2 = \frac{9}{16}$

$\Rightarrow r = \frac{3}{4}$ *only since all terms are positive*

by substituting r in (1)

$$\Rightarrow a\left(\frac{3}{4}\right) = 24$$

$$\therefore \boldsymbol{a = 32}$$

(ii). $S_\infty = \frac{a}{1-r}$

$$\Rightarrow S_\infty = \frac{32}{1-\frac{3}{4}}$$

$$\Rightarrow S_\infty = \frac{32}{\frac{1}{4}}$$

$$\therefore \boldsymbol{S_\infty = 128}$$

Since the angles are in arithmetic progression, $a = 3$ and $d = 2$.

All sectors form a circle which is 360°, that is a complete revolution.

$$\Rightarrow S_n = 360$$

$$\Rightarrow \frac{n}{2}[2a + (n-1)d] = 360$$

$$\Rightarrow \frac{n}{2}[2(3) + (n-1)(2)] = 360$$

$$\Rightarrow \frac{n}{2}[6 + 2n - 2] = 360$$

$$\Rightarrow \frac{n}{2}[4 + 2n] = 360$$

$$\Rightarrow 2n + n^2 = 360$$

$$\Rightarrow n^2 + 2n - 360 = 0$$

$$\Rightarrow n = 18 \; or - 20$$

$\therefore$ **$n = 18$ only since n can never be negative**.

<u>Revision Questions on Progressions</u>

November 2003 qp.2 (Zimsec, O Level Additional Mathematics)

2. (a) The eighth term of an arithmetic progression is 150 and the fifty-third is –30. Determine
   - (i). the first term and the common difference, [4]
   - (ii). the number of terms whose sum is zero. [3]

(b). The sum of an infinite geometric progression is 500. Given that the common ratio is 0.8, calculate

(i). The first term, [2]

(ii). The twentieth term, [2]

(iii). The least number of terms of the progression whose sum exceeds 499. [5]

November 2003 qp.1 (Zimsec)

12. (a). The sequence $U_1, U_2 \ldots U_n \ldots$ is such that $U_{r+1} = \frac{27}{U_r}$ for $r \geq 1$.

Given also that $U_1 + U_2 = 12$, find the possible values of U_1. [3]

For each value of U_1, describe the behaviour of the sequence as n tends to infinity. [2]

(b). Given that $a_n = 4 + (0.1)^n$, show that

$$\sum_{n=N+1}^{2N} a_n = 4N + \frac{(0.1)^N}{9}(1 - (0.1)^N)$$

[5]

November 2002 qp.1 (Zimsec)

8. A sequence U_r is defined by $U_r = (n - 3r)$.

(i). Write down the first 3 terms of the sequence. [1]

(ii). Find in terms of n a formula for

$$\sum_{r=n}^{2n} U_r$$

[5]

November 2001 qp.1 (Zimsec)

16. (a). Three sequences are defined below, for $n = 1, 2, 3, \ldots$ .Describe the behaviour of each sequence as n tends to infinity.

i. $a_n = (-1)^n$

ii. $b_n = 2^{-n}$

iii. $c_n = (-1)^n + 3n$ [3]

(b). A sequence $U_1,\ U_2,\ U_3$ ... is defined by

$$U_1 = 2 \text{ and } U_{n+1} = U_n + 3 \text{ for } n \geq 1$$

(i). Write down the first four terms of the sequence. [1]

(ii). State what type of sequence it is, and express U_n in terms of n. [2]

(c) A geometric progression of positive terms is such that the sum of its first two terms is 24 and the third term is 2. Find the common ratio and the sum to infinity of this progression. [6]

June 2008 qp.1 (Cambridge)

7 The first term of a geometric progression is 81 and the fourth term is 24. Find

(i) the common ratio of the progression. [2]

(ii) the sum to infinity of the progression. [2]

The second and third terms of this geometric progression are the first and fourth terms respectively of an arithmetic progression.

(iii) Find the sum of the first ten terms of the arithmetic progression. [3]

June 2011 qp.13 (Cambridge)

6 **(a)** A geometric progression has a third term of 20 and a sum to infinity which is three times the first term. Find the first term. [4]

(b) An arithmetic progression is such that the eighth term is three times the third term. Show that the sum of the first eight terms is four times the sum of the first four terms. [4]

November 2005 qp.1 (Cambridge)

6 A small trading company made a profit of \$250 000 in the year 2000. The company considered two different plans, plan *A* and plan *B*, for increasing its profits.

Under plan *A*, the annual profit would increase each year by 5% of its value in the preceding year. Find, for plan *A*,

(i) the profit for the year 2008, [3]

(ii) the total profit for the 10 years 2000 to 2009 inclusive. [2]

Under plan *B*, the annual profit would increase each year by a constant amount \$*D*.

(iii) Find the value of *D* for which the total profit for the 10 years 2000 to 2009 inclusive would be the same for both plans. [3]

November 2007 qp.1 (Cambridge)

4 The 1st term of an arithmetic progression is a and the common difference is d, where $d \neq 0$.

(i) Write down expressions, in terms of a and d, for the 5th term and the 15th term. [1]

The 1st term, the 5th term and the 15th term of the arithmetic progression are the first three terms of a geometric progression.

(ii) Show that $3a = 8d$. [3]

(iii) Find the common ratio of the geometric progression. [2]

Worked Examination Questions on Series Expansion

Question (Cambridge, June 2012 qp.32)

3 Expand $\sqrt{\left(\frac{1-x}{1+x}\right)}$ in ascending powers of x, up to and including the term in x^2, simplifying the coefficients. [5]

Solution

$$Let\ f(x) = \sqrt{\left(\frac{1-x}{1+x}\right)}$$

$$\Rightarrow f(x) = \left(\frac{1-x}{1+x}\right)^{\frac{1}{2}}$$

$$\Rightarrow f(x) = \frac{(1-x)^{\frac{1}{2}}}{(1+x)^{\frac{1}{2}}}$$

$$\Rightarrow f(x) = (1-x)^{\frac{1}{2}}(1+x)^{-\frac{1}{2}}$$

$$where\ (1-x)^{\frac{1}{2}} = 1 + \frac{\left(\frac{1}{2}\right)(-x)}{1!} + \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)(-x)^2}{2!} + \cdots$$

$$\Rightarrow \mathbf{(1-x)^{\frac{1}{2}} = 1 - \frac{x}{2} - \frac{x^2}{8}}$$

$$and\ \ (1+x)^{-\frac{1}{2}} = 1 + \frac{\left(-\frac{1}{2}\right)(x)}{1!} + \frac{-\frac{1}{2}\left(-\frac{1}{2}-1\right)(x)^2}{2!} + \cdots$$

$$\Rightarrow \boldsymbol{(1+x)^{-\frac{1}{2}} = 1 - \frac{x}{2} + \frac{3x^2}{8}}$$

$$Now,\ \ f(x) = \left(1 - \frac{x}{2} - \frac{x^2}{8}\right)\left(1 - \frac{x}{2} + \frac{3x^2}{8}\right)$$

$$\Rightarrow f(x) = 1 - \frac{x}{2} + \frac{3x^2}{8} - \frac{x}{2} + \frac{x^2}{4} - \frac{x^2}{8} + \cdots$$

$$\therefore \boldsymbol{f(x) = 1 - x + \frac{x^2}{2}}$$

Question (Cambridge, November 2011 qp.33)

1 Expand $\frac{16}{(2+x)^2}$ in ascending powers of x, up to and including the term in x^2, simplifying the coefficients. [4]

Solution

$$Let\ \ f(x) = \frac{16}{(2+x)^2}$$

$$\Rightarrow f(x) = 16(2+x)^{-2}$$

$$\Rightarrow f(x) = 16(2)^{-2}\left(1 + \frac{x}{2}\right)^{-2}$$

$$\Rightarrow f(x) = 4\left[1 + \frac{(-2)\left(\frac{x}{2}\right)}{1!} + \frac{(-2)(-2-1)\left(\frac{x}{2}\right)^2}{2!} + \cdots\right]$$

$$\Rightarrow f(x) = 4\left[1 - x + \frac{3x^2}{4}\right]$$

$$\therefore \boldsymbol{f(x) = 4 - 4x + 3x^2}$$

Question (Cambridge, November 2009 qp.31)

8 (i) Express $\dfrac{5x+3}{(x+1)^2(3x+2)}$ in partial fractions. [5]

(ii) Hence obtain the expansion of $\dfrac{5x+3}{(x+1)^2(3x+2)}$ in ascending powers of x, up to and including the term in x^2, simplifying the coefficients. [5]

Solution

(i) $$\frac{5x+3}{(x+1)^2(3x+2)} = \frac{A}{(x+1)} + \frac{B}{(x+1)^2} + \frac{C}{(3x+2)}$$

$$\Rightarrow 5x+3 = A(x+1)(3x+2) + B(3x+2) + C(x+1)^2$$

$let\ \ x = -1$

$\Rightarrow -2 = -B$

$\Rightarrow B = 2$

$let\ \ x = -\dfrac{2}{3}$

$\Rightarrow -\dfrac{1}{3} = \dfrac{1}{9}C$

$\Rightarrow C = -3$

$let\ \ x = 0$

$\Rightarrow 3 = 2A + 2B + C$

$\Rightarrow 3 = 2A + 2(2) - 3$

$\Rightarrow 2A = 2$

$\Rightarrow A = 1$

$$\therefore \frac{\mathbf{5x+3}}{\mathbf{(x+1)^2(3x+2)}} = \frac{\mathbf{1}}{\mathbf{(x+1)}} + \frac{\mathbf{2}}{\mathbf{(x+1)^2}} - \frac{\mathbf{3}}{\mathbf{(3x+2)}}$$

(ii) $$f(x) = \frac{5x+3}{(x+1)^2(3x+2)}$$

$$\Rightarrow f(x) = \frac{1}{(x+1)} + \frac{2}{(x+1)^2} - \frac{3}{(3x+2)}$$

$$\Rightarrow f(x) = (1+x)^{-1} + 2(1+x)^{-2} - 3(2+3x)^{-1}$$

$$where\ \ (1+x)^{-1} = 1 + \frac{(-1)(x)}{1!} + \frac{(-1)(-1-1)(x)^2}{2!} + \cdots$$

$$\Rightarrow \mathbf{(1+x)^{-1} = 1 - x + x^2},$$

$$and\ \ 2(1+x)^{-2} = 2\left[1 + \frac{(-2)(x)}{1!} + \frac{(-2)(-2-1)(x)^2}{2!} + \cdots\right]$$

$$\Rightarrow 2(1+x)^{-2} = 2[1 - 2x + 3x^2]$$

$$\Rightarrow \mathbf{2(1+x)^{-2} = 2 - 4x + 6x^2},$$

$$and\ -3(2+3x)^{-1} = (-3)(2)^{-1}\left(1+\frac{3x}{2}\right)^{-1}$$

$$\Rightarrow -3(2+3x)^{-1} = -\frac{3}{2}\left[1+\frac{(-1)\left(\frac{3x}{2}\right)}{1!}+\frac{(-1)(-1-1)\left(\frac{3x}{2}\right)^2}{2!}+\cdots\right]$$

$$\Rightarrow -3(2+3x)^{-1} = -\frac{3}{2}\left[1-\frac{3x}{2}+\frac{9x^2}{4}\right]$$

$$\Rightarrow \mathbf{-3(2+3x)^{-1} = -\frac{3}{2}+\frac{9x}{4}-\frac{27x^2}{8}}$$

$$Now,\ f(x) = [1-x+x^2]+[2-4x+6x^2]+\left[-\frac{3}{2}+\frac{9x}{4}-\frac{27}{8}x^2\right]$$

$$\therefore \mathbf{f(x) = \frac{3}{2}-\frac{11}{4}x+\frac{29}{8}x^2}$$

Question (Cambridge, November 2012 qp.33)

9 (i) Express $\dfrac{9-7x+8x^2}{(3-x)(1+x^2)}$ in partial fractions. [5]

(ii) Hence obtain the expansion of $\dfrac{9-7x+8x^2}{(3-x)(1+x^2)}$ in ascending powers of x, up to and including the term in x^3. [5]

Solution

(i) $$\frac{9-7x+8x^2}{(3-x)(1+x^2)} = \frac{A}{(3-x)}+\frac{Bx+C}{(1+x^2)}$$

$$\Rightarrow 9-7x+8x^2 = A(1+x^2)+(Bx+C)(3-x)$$

$let\ x=3$	$let\ x=0$	$let\ x=1$
$\Rightarrow 60 = 10A$	$\Rightarrow 9 = A+3C$	$\Rightarrow 10 = 2A+2B+2C$
$\Rightarrow A = 6$	$\Rightarrow 9 = 6+3C$	$\Rightarrow 10 = 2(6)+2B+2$
	$\Rightarrow C = 1$	$\Rightarrow B = -2$

$$\therefore \mathbf{\frac{9-7x+8x^2}{(3-x)(1+x^2)} = \frac{6}{(3-x)}+\frac{(-2x+1)}{(1+x^2)}}$$

(ii) $let\ f(x) = \dfrac{9 - 7x + 8x^2}{(3 - x)(1 + x^2)}$

$$\Rightarrow f(x) = \frac{6}{(3 - x)} + \frac{(-2x + 1)}{(1 + x^2)}$$

$$\Rightarrow f(x) = 6(3 - x)^{-1} + (-2x + 1)(1 + x^2)^{-1}$$

$$where\ 6(3 - x)^{-1} = 6(3)^{-1}\left(1 - \frac{x}{3}\right)^{-1}$$

$$\Rightarrow 6(3 - x)^{-1} = 2\left[1 + \frac{(-1)\left(-\frac{x}{3}\right)}{1!} + \frac{(-1)(-1 - 1)\left(-\frac{x}{3}\right)^2}{2!} + \frac{(-1)(-1 - 1)(-1 - 2)\left(-\frac{x}{3}\right)^3}{3!} + \cdots\right]$$

$$\Rightarrow 6(3 - x)^{-1} = 2\left[1 + \frac{x}{3} + \frac{x^2}{9} + \frac{x^3}{27}\right]$$

$$\Rightarrow \mathbf{6(3 - x)^{-1} = 2 + \frac{2x}{3} + \frac{2x^2}{9} + \frac{2x^3}{27}}$$

$$and\ (-2x + 1)(1 + x^2)^{-1} = (-2x + 1)\left[1 + \frac{(-1)(x^2)^1}{1!} + \cdots\right]$$

$$\Rightarrow (-2x + 1)(1 + x^2)^{-1} = (-2x + 1)(1 - x^2)$$

$$\Rightarrow (-2x + 1)(1 + x^2)^{-1} = -2x + 2x^3 + 1 - x^2$$

$$\Rightarrow \mathbf{(-2x + 1)(1 + x^2)^{-1} = 1 - 2x - x^2 + 2x^3}$$

$$Now,\ f(x) = \left[2 + \frac{2x}{3} + \frac{2x^2}{9} + \frac{2x^3}{27}\right] + [1 - 2x - x^2 + 2x^3]$$

$$\therefore \mathbf{f(x) = 3 - \frac{4}{3}x - \frac{7}{9}x^2 + \frac{56}{27}x^3}$$

<u>Revision Questions on Series Expansion</u>

November 2010 qp.33 (Cambridge)

1 Expand $(1 + 2x)^{-3}$ in ascending powers of x, up to and including the term in x^2, simplifying the coefficients. [3]

June 2011 qp.31 (Cambridge)

1 Expand $\sqrt[3]{(1-6x)}$ in ascending powers of x up to and including the term in x^3, simplifying the coefficients. [4]

June 2012 qp.31 (Cambridge)

2 (i) Expand $\frac{1}{\sqrt{(1-4x)}}$ in ascending powers of x, up to and including the term in x^2, simplifying the coefficients. [3]

(ii) Hence find the coefficient of x^2 in the expansion of $\frac{1+2x}{\sqrt{(4-16x)}}$. [2]

June 2012 qp.33 (Cambridge)

1 Expand $\frac{1}{\sqrt{(4+3x)}}$ in ascending powers of x, up to and including the term in x^2, simplifying the coefficients. [4]

November 2012 qp.31 (Cambridge)

4 When $(1+ax)^{-2}$, where a is a positive constant, is expanded in ascending powers of x, the coefficients of x and x^3 are equal.

(i) Find the exact value of a. [4]

(ii) When a has this value, obtain the expansion up to and including the term in x^2, simplifying the coefficients. [3]

November 2009 qp.32 (Cambridge)

8 (i) Express $\frac{1+x}{(1-x)(2+x^2)}$ in partial fractions. [5]

(ii) Hence obtain the expansion of $\frac{1+x}{(1-x)(2+x^2)}$ in ascending powers of x, up to and including the term in x^2. [5]

June 2010 qp.33 (Cambridge)

9 **(i)** Express $\frac{4 + 5x - x^2}{(1 - 2x)(2 + x)^2}$ in partial fractions. [5]

(ii) Hence obtain the expansion of $\frac{4 + 5x - x^2}{(1 - 2x)(2 + x)^2}$ in ascending powers of x, up to and including the term in x^2. [5]

November 2010 qp.31 (Cambridge)

8 Let $f(x) = \frac{3x}{(1 + x)(1 + 2x^2)}$.

(i) Express $f(x)$ in partial fractions. [5]

(ii) Hence obtain the expansion of $f(x)$ in ascending powers of x, up to and including the term in x^3. [5]

June 2011 qp.32 (Cambridge)

8 **(i)** Express $\frac{5x - x^2}{(1 + x)(2 + x^2)}$ in partial fractions. [5]

(ii) Hence obtain the expansion of $\frac{5x - x^2}{(1 + x)(2 + x^2)}$ in ascending powers of x, up to and including the term in x^3. [5]

November 2007 qp.1 (Cambridge)

3 **(i)** Find the first three terms in the expansion of $(2 + u)^5$ in ascending powers of u. [3]

(ii) Use the substitution $u = x + x^2$ in your answer to part **(i)** to find the coefficient of x^2 in the expansion of $(2 + x + x^2)^5$. [2]

June 2012 qp.12 (Cambridge)

3 The coefficient of x^3 in the expansion of $(a + x)^5 + (2 - x)^6$ is 90. Find the value of the positive constant a. [5]

Chapter Eight: Trigonometry

"Today I am going to give you two examinations, one in trigonometry and one in honesty. I hope you will pass them both, but if you must fail one, let it be trigonometry, for there are many good [people] in this world today who cannot pass an examination in trigonometry, but there are no good [people] in the world who cannot pass an examination in honesty."

– Madison Sarratt

Trigonometry is a branch of mathematics that deals with problems concerning angles and distances. 'A' Level trigonometry focuses on two components: trig identities and trig equations.

Trigonometrical Identities

These are trigonometrical statements that resemble a state of balance where the LHS is exactly equal to the RHS. Trig-identities are grouped into five families as outlined below:

A. Standard Identities

- $\tan\theta \equiv \frac{\sin\theta}{\cos\theta}$
- $\sec\theta \equiv \frac{1}{\cos\theta}$
- $\operatorname{cosec}\theta \equiv \frac{1}{\sin\theta}$
- $\cot\theta \equiv \frac{1}{\tan\theta} \equiv \frac{\cos\theta}{\sin\theta}$

B. The Pythagorean Group

- $\cos^2\theta + \sin^2\theta \equiv 1$
- $1 + \tan^2\theta \equiv \sec^2\theta$
- $1 + \cot^2\theta \equiv \operatorname{cosec}^2\theta$

C. Double Angle Formulae

- $\sin 2\theta \equiv 2\sin\theta\cos\theta$
- $\cos 2\theta \equiv \cos^2\theta - \sin^2\theta \equiv 2\cos^2\theta - 1 \equiv 1 - 2\sin^2\theta$
- $\tan 2\theta \equiv \frac{2\tan\theta}{1-\tan^2\theta}$

D. Addition Formulae

- $\sin(A \pm B) \equiv \sin A \cos B \pm \cos A \sin B$
- $\cos(A \pm B) \equiv \cos A \cos B \mp \sin A \sin B$
- $\tan(A \pm B) \equiv \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$

E. Small Angle Formulae

- $\sin\theta \simeq \theta$
- $\cos\theta \simeq 1 - \frac{\theta^2}{2}$
- $\tan\theta \simeq \theta$

It is an undisputed fact that a lot of students make an attempt to memorise the identities, yet still fail to solve some problems using these identities. It is against this background that students are encouraged to be flexible enough to adjust or manipulate identities. An identity is just a framework that is open to any adjustment; *as long as the adjustment does not lead to breach of mathematical principles*. A snapshot of some **valid** adjustments is outlined below:

A. Standard Identities

$Framework: \tan(\ ) \equiv \frac{\sin(\ )}{\cos(\ )}$

- $\tan 4\theta \equiv \frac{\sin 4\theta}{\cos 4\theta}$
- $\tan^3 2\theta \equiv \frac{\sin^3 2\theta}{\cos^3 2\theta}$

$Framework: \sec(\ ) \equiv \frac{1}{\cos(\ )}$

- $\sec 5x \equiv \frac{1}{\cos 5x}$
- $\sec^2 x \equiv \frac{1}{\cos^2 x}$

B. Pythagorean Group

$Framework: \cos^2(\ ) + \sin^2(\ ) \equiv 1$

- $\cos^2 3\theta + \sin^2 3\theta \equiv 1$

NB: all the adjustments should retain the squared ratio, that is, both sin( ) and cos( ) should be squared quantities and the supporting value (angle) has to be the same, for the relationship to reduce to 1.

C. Double Angle Formulae

Framework: $\sin 2\theta \equiv 2\sin\theta\cos\theta$

- $\sin 4\theta \equiv 2\sin 2\theta\cos 2\theta$
- $\sin 10\theta \equiv 2\sin 5\theta\cos 5\theta$
- $\sin^2 2\theta \equiv (2\sin\theta\cos\theta)^2$

NB: make sure the angle to the LHS is a double to the angles on the RHS.

Framework: $\cos 2\theta \equiv 2\cos^2\theta - 1 \equiv 1 - 2\sin^2\theta$

This is the most widely used identity in 'A' Level mathematics. Adjustment can only be extended to angles, but the framework remains the same:

- $\cos(4\theta) \equiv 2\cos^2(2\theta) - 1 \equiv 1 - 2\sin^2(2\theta)$
- $\cos(6\theta) \equiv 2\cos^2(3\theta) - 1 \equiv 1 - 2\sin^2(3\theta)$
- $\cos^2 2\theta \equiv (2\cos^2\theta - 1)^2 \equiv (1 - 2\sin^2\theta)^2$

Since this identity can be modelled along the lines of $\cos\theta$ or $\sin\theta$, the choice of a particular path is influenced by the desired conclusion. If the desired end has a bias towards $\cos\theta$, one has to use the identity in $\cos\theta$ and vice versa.

Revision Questions on Pure Identities

June 2007 qp.1 (Cambridge)

3 Prove the identity $\frac{1-\tan^2 x}{1+\tan^2 x} \equiv 1 - 2\sin^2 x$. [4]

June 2009 qp.1 (Cambridge)

1 Prove the identity $\frac{\sin x}{1-\sin x} - \frac{\sin x}{1+\sin x} \equiv 2\tan^2 x$. [3]

June 2011 qp.12 (Cambridge)

5 **(i)** Prove the ~~identity~~ $\dfrac{\cos\theta}{\tan\theta(1-\sin\theta)} \equiv 1 + \dfrac{1}{\sin\theta}$. [3]

June 2011 qp.13 (Cambridge)

8 **(i)** Prove the identity $\left(\dfrac{1}{\sin\theta} - \dfrac{1}{\tan\theta}\right)^2 \equiv \dfrac{1-\cos\theta}{1+\cos\theta}$. [3]

June 2012 qp.13 (Cambridge)

1 **(i)** Prove the identity $\tan^2\theta - \sin^2\theta \equiv \tan^2\theta\sin^2\theta$. [3]

November 2008 qp.1 (Cambridge)

2 Prove the identity

$$\frac{1+\sin x}{\cos x} + \frac{\cos x}{1+\sin x} \equiv \frac{2}{\cos x}.$$ [4]

November 2010 qp.11 (Cambridge)

4 **(i)** Prove the identity $\dfrac{\sin x\tan x}{1-\cos x} \equiv 1 + \dfrac{1}{\cos x}$. [3]

Unknown source

2. Prove that:

$$\sqrt{\frac{1-\cos\theta}{1+\cos\theta}} \equiv \operatorname{cosec}\theta - \cot\theta$$

Unknown source

3. Prove that:

$$\sec 2\theta + \tan 2\theta \equiv \frac{\cos\theta + \sin\theta}{\cos\theta - \sin\theta}$$

November 2002 qp.1 (Zimsec, O Level Additional Mathematics)

6. Prove the identity

$$(\operatorname{cosec} A + \sin A)(\sec A + \cos A) \equiv \frac{2+\sin^2 A \ \cos^2 A}{\sin A \ \cos A} \quad [4]$$

June 2002 qp.3 (Cambridge)

1 Prove the identity

$$\cot\theta - \tan\theta \equiv 2\cot 2\theta. \quad [3]$$

Unknown source

Express $\sin 4\theta$ in terms of $\sin 2\theta$ and $\cos 2\theta$,and hence express $\frac{\sin 4\theta}{\cos\theta}$ in terms of $\cos\theta$ only. [4]

Trigonometrical Equations

In most cases, an equation has to be simplified into a digestible form using one of the identities from the five families outlined above.

For example,

$$\sec\theta + 2\cos\theta = 1$$

$\Rightarrow \dfrac{1}{\cos\theta} + 2\cos\theta = 1$

$\Rightarrow 1 + 2\cos^2\theta = \cos\theta$

$\Rightarrow 2\cos^2\theta - \cos\theta + 1 = 0$

When a trig equation is solved, it will lead to one solution or two solutions in cases of quadratic equations. This solution is known as the **principal** or **primary value** ($\boldsymbol{PV}$). There are three main methods used to develop a **PV** into secondary and tertiary values. These are:

i. Graphical Method

ii. Quadrant Diagram

iii. General-Solution Approach

An Overview of General Solutions

$\mathbf{\sin(\theta): \theta = (-1)^n \cdot PV + 180n}$

$\mathbf{\cos(\theta): \theta = \pm PV + 360n}$

$\mathbf{\tan(\theta): \theta = PV + 180n}$

where n is an integer.

If the equation is in radians, manipulate the general solutions to radians. Since n is an integer, use the different values of n up to a certain stage where all the values in the specified range have been exhausted.

For example, given that,

$$\tan 2\theta = 1, \ \ for \ \ 0° < \theta < 180°$$

$\Rightarrow 2\theta = \tan^{-1}(1)$

$\Rightarrow 2\theta = 45°(PV)$

NB: do **not** prematurely make θ the subject of the formula; this can **only** be done when the general solution has been introduced.

$2\theta = 45 + 180n$

$\theta = \dfrac{45 + 180n}{2}$

$when\, n = 0; \ \ \theta = 22.5°$

$when\, n = 1; \ \theta = 112.5°$

$when\, n = 2; \ \theta = 202.5° \ (out\ of\ range)$

$\therefore \boldsymbol{\theta = 22.5° \ and \ 112.5°}$

Worked Examination Questions on Equations of the form $R\cos(\theta \pm \alpha)\,/R\sin(\theta \pm \alpha)$

Question (Cambridge, November 2008 qp.3)

6 (i) Express $5\sin x + 12\cos x$ in the form $R\sin(x + \alpha)$, where $R > 0$ and $0° < \alpha < 90°$, giving the value of α correct to 2 decimal places. [3]

(ii) Hence solve the equation

$$5\sin 2\theta + 12\cos 2\theta = 11,$$

giving all solutions in the interval $0° < \theta < 180°$. [5]

Solution

(i) $5\sin x + 12\cos x = R\sin(x + \alpha)$

using the addition formula $\sin(A \pm B)$,

$\Rightarrow 5\sin x + 12\cos x = R[\sin x\cos\alpha + \cos x\sin\alpha]$

$\Rightarrow 5\sin x + 12\cos x = R\sin x\cos\alpha + R\cos x\sin\alpha$

by comparing coefficients,

$R\cos\alpha = 5 \longrightarrow 1$

and $R\sin\alpha = 12 \longrightarrow 2$

by dividing (2) *by* (1),

$$\Rightarrow \frac{R\sin\alpha}{R\cos\alpha} = \frac{12}{5}$$

$$\Rightarrow \tan\alpha = \frac{12}{5}$$

$$\Rightarrow \alpha = \tan^{-1}\left(\frac{12}{5}\right)$$

$\Rightarrow \alpha = 67.38°$

by substituting α *in* (1),

$$\Rightarrow R = \frac{5}{\cos\tan^{-1}\left(\frac{12}{5}\right)}$$

$\Rightarrow R = 13$

$\therefore \mathbf{5\sin x + 12\cos x = 13\sin(x + 67.38°)}$

(ii) $5\sin 2\theta + 12\cos 2\theta = 11$

Since the LHS is the same as the expression $13\sin(x + 67.38°)$ *in part* (i) *where* $2\theta = x$,

$\Rightarrow 13\sin(2\theta + 67.38°) = 11$

$\Rightarrow \sin(2\theta + 67.38°) = \dfrac{11}{13}$

$\Rightarrow (2\theta + 67.38°) = \sin^{-1}\left(\dfrac{11}{13}\right)$

$\Rightarrow (2\theta + 67.38°) = 57.8°(PV)$

using the general solution for $\sin(2\theta + 67.38°)$,

$(2\theta + 67.38°) = (-1)^n \cdot 57.8 + 180n$

$\Rightarrow \theta = \dfrac{(-1)^n \cdot 57.8 + 180n - 67.38}{2}$

when $n = 0$; $\theta =$ *out of range*

when $n = 1$; $\theta = 27.4°$

when $n = 2$; $\theta = 175.2°$

$\therefore \boldsymbol{\theta = 27.4°}$ ***and*** $\boldsymbol{175.2°}$

Revision Questions on Equations of the form $\boldsymbol{R\ cos(\theta \pm \alpha)\ /R\ sin(\theta \pm \alpha)}$

November 1999 qp.1 (Cambridge)

6. Express $4\sin\theta + 2\cos\theta$ in the form $R\sin(\theta + \alpha)$, where R is positive and α is an acute angle, giving the exact values of R and $\tan\alpha$. [2]

Hence solve the equation

$$4\sin\theta + 2\cos\theta = 3,$$

for $0° \leq \theta \leq 360°$, giving your answers in degrees correct to 1 decimal place. [4]

June 1994 qp.1 (Cambridge)

6. Express $3\cos\theta - 5\sin\theta$ in the form $R\cos(\theta + \alpha)$, where $R > 0$ and $0 < \alpha < 90°$. [2]

Hence, or otherwise, find the general solution of the equation $3\cos\theta - 5\sin\theta = 2$, giving your answer correct to the nearest 0.1°. [4]

June 1997 qp.1 (Cambridge)

10. Given that

$$3\cos x - 4\sin x \equiv R\cos(x + \alpha),$$

where $R > 0$ and $0° < \alpha < 90°$, find the values of R and α, giving the value of α correct to two decimal places. [2]

Hence solve the equation $3\cos 2\theta - 4\sin 2\theta = 2$

for $0° < \theta < 360°$, giving your answers correct to one decimal place. [6]

November 2003 qp.2 (Zimsec, O Level Additional Mathematics)

4. (a) Given that $3\cos\theta + \sin\theta \equiv R\cos(\theta - \alpha)$, where R is positive and α acute, evaluate R and α. Hence solve the equation

$$3\cos\theta + \sin\theta = 2 \text{ for } 0° < \theta < 360°.$$ [8]

(b). Given that $\sin\alpha = \frac{4}{5}$, where $90° < \alpha < 180°$,
and that $\cos\beta = -\frac{5}{13}$, where $180° < \beta < 270°$,
calculate, without using tables or calculator

(i). $\sin(\alpha - \beta)$,
(ii). $\cos(2\alpha)$,
(iii). $\sin(2\beta)$,
(iv). $\tan(2\alpha)$. [8]

November 1990 qp.1 (Cambridge)

5. Express $5\cos\theta - 12\sin\theta$ in the form $R\cos(\theta + \alpha)$, where $R \geq 0$ and $0 \leq \alpha < 360°$, stating the value of R and giving the value of α correct to 0.1°. [4]

November 2010 qp.33 (Cambridge)

8 (i) Express $(\sqrt{6})\cos\theta + (\sqrt{10})\sin\theta$ in the form $R\cos(\theta - \alpha)$, where $R > 0$ and $0° < \alpha < 90°$. Give the value of α correct to 2 decimal places. [3]

(ii) Hence, in each of the following cases, find the smallest positive angle θ which satisfies the equation

(a) $(\sqrt{6})\cos\theta + (\sqrt{10})\sin\theta = -4$. [2]

(b) $(\sqrt{6})\cos\frac{1}{2}\theta + (\sqrt{10})\sin\frac{1}{2}\theta = 3$. [4]

November 2011 qp.31 (Cambridge)

6 (i) Express $\cos x + 3\sin x$ in the form $R\cos(x - \alpha)$, where $R > 0$ and $0° < \alpha < 90°$, giving the exact value of R and the value of α correct to 2 decimal places. [3]

(ii) Hence solve the equation $\cos 2\theta + 3\sin 2\theta = 2$, for $0° < \theta < 90°$. [5]

November 2011 qp.33 (Cambridge)

3 (i) Express $8\cos\theta + 15\sin\theta$ in the form $R\cos(\theta - \alpha)$, where $R > 0$ and $0° < \alpha < 90°$. Give the value of α correct to 2 decimal places. [3]

(ii) Hence solve the equation $8\cos\theta + 15\sin\theta = 12$, giving all solutions in the interval $0° < \theta < 360°$. [4]

November 2012 qp.33 (Cambridge)

2 (i) Express $24\sin\theta - 7\cos\theta$ in the form $R\sin(\theta - \alpha)$, where $R > 0$ and $0° < \alpha < 90°$. Give the value of α correct to 2 decimal places. [3]

(ii) Hence find the smallest positive value of θ satisfying the equation

$$24\sin\theta - 7\cos\theta = 17.$$ [2]

Worked Examination Questions on General Trig Equations

Question (Cambridge, June 2010 qp.32)

3 It is given that $\cos a = \frac{3}{5}$, where $0° < a < 90°$. Showing your working and without using a calculator to evaluate a,

(i) find the exact value of $\sin(a - 30°)$, [3]

(ii) find the exact value of $\tan 2a$, and hence find the exact value of $\tan 3a$. [4]

Solution

(i) $\cos a = \frac{3}{5}$,

using a right angled triangle,

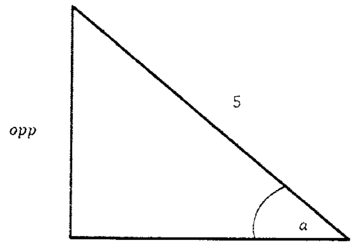


Fig. 8.1

$$5^2 = opp^2 + 3^2$$
$$opp = \sqrt{25 - 9}$$
$$opp = 4$$

$$\Rightarrow \sin a = \frac{4}{5} \textit{ and } \tan a = \frac{4}{3}$$

$$Now, \quad \sin(a - 30) = \sin a \cos 30 - \cos a \sin 30$$

$$\Rightarrow \sin(a - 30) = \sin a \left(\frac{\sqrt{3}}{2}\right) - \cos a \left(\frac{1}{2}\right)$$

$$\Rightarrow \sin(a - 30) = \left(\frac{4}{5}\right)\left(\frac{\sqrt{3}}{2}\right) - \left(\frac{3}{5}\right)\left(\frac{1}{2}\right)$$

$$\Rightarrow \sin(a - 30) = \frac{2\sqrt{3}}{5} - \frac{3}{10}$$

$$\therefore \mathbf{\sin(a - 30) = \frac{4\sqrt{3} - 3}{10}}$$

(ii) $\tan 2a = \dfrac{2\tan a}{1 - \tan^2 a}$

$$\Rightarrow \tan 2a = \frac{2\left(\frac{4}{3}\right)}{1 - \left(\frac{4}{3}\right)^2}$$

$$\Rightarrow \tan 2a = \frac{8}{3} \div \left(1 - \frac{16}{9}\right)$$

$$\Rightarrow \tan 2a = \frac{8}{3} \times \frac{-9}{7}$$

$$\Rightarrow \tan 2a = -\frac{24}{7}$$

$$Now, \quad \tan 3a = \tan(2a + a)$$

$$\Rightarrow \tan 3a = \frac{\tan 2a + \tan a}{1 - \tan 2a \tan a}$$

$$\Rightarrow \tan 3a = \frac{-\frac{24}{7} + \frac{4}{3}}{1 - \left(-\frac{24}{7}\right)\left(\frac{4}{3}\right)}$$

$$\Rightarrow \tan 3a = -\frac{44}{21} \div \left(1 + \frac{32}{7}\right)$$

$$\Rightarrow \tan 3a = -\frac{44}{21} \div \frac{39}{7}$$

$$\Rightarrow \tan 3a = -\frac{44}{21} \times \frac{7}{39}$$

$$\therefore \mathbf{\tan 3a = -\frac{44}{117}}$$

Question (Cambridge, June 2008 qp.3)

4 (i) Show that the equation $\tan(30° + \theta) = 2\tan(60° - \theta)$ can be written in the form

$$\tan^2\theta + (6\sqrt{3})\tan\theta - 5 = 0.$$ [4]

(ii) Hence, or otherwise, solve the equation

$$\tan(30° + \theta) = 2\tan(60° - \theta),$$

for $0° \leqslant \theta \leqslant 180°$. [3]

Solution

(i) $\tan(30° + \theta) = 2\tan(60° - \theta)$

$$where \ \tan(30° + \theta) = \frac{\tan 30 + \tan\theta}{1 - \tan 30 \tan\theta}$$

$$\Rightarrow \tan(30° + \theta) = \frac{\frac{\sqrt{3}}{3} + \tan\theta}{1 - \frac{\sqrt{3}}{3}\tan\theta}$$

$$and \ \tan(60° - \theta) = \frac{\tan 60 - \tan\theta}{1 + \tan 60 \tan\theta}$$

$$\Rightarrow \tan(60° - \theta) = \frac{\sqrt{3} - \tan\theta}{1 + \sqrt{3}\tan\theta}$$

$$Now,\quad \frac{\frac{\sqrt{3}}{3}+\tan\theta}{1-\frac{\sqrt{3}}{3}\tan\theta} = 2\left[\frac{\sqrt{3}-\tan\theta}{1+\sqrt{3}\tan\theta}\right]$$

$$\Rightarrow \frac{\frac{\sqrt{3}}{3}+\tan\theta}{1-\frac{\sqrt{3}}{3}\tan\theta} = \frac{2\sqrt{3}-2\tan\theta}{1+\sqrt{3}\tan\theta}$$

$$\Rightarrow \left(\frac{\sqrt{3}}{3}+\tan\theta\right)\left(1+\sqrt{3}\tan\theta\right) = \left(2\sqrt{3}-2\tan\theta\right)\left(1-\frac{\sqrt{3}}{3}\tan\theta\right)$$

$$\Rightarrow \frac{\sqrt{3}}{3}+\tan\theta+\tan\theta+\sqrt{3}\tan^2\theta = 2\sqrt{3}-2\tan\theta-2\tan\theta+\frac{2\sqrt{3}}{3}\tan^2\theta$$

$$\Rightarrow \frac{\sqrt{3}}{3}+2\tan\theta+\sqrt{3}\tan^2\theta = 2\sqrt{3}-4\tan\theta+\frac{2\sqrt{3}}{3}\tan^2\theta$$

$$\Rightarrow \frac{\sqrt{3}}{3}\tan^2\theta+6\tan\theta-\frac{5\sqrt{3}}{3} = 0$$

$$\therefore \mathbf{\tan^2\theta+\left(6\sqrt{3}\right)\tan\theta-5=0}\ (\textit{shown})$$

(ii) $Since\ \ \tan(30°+\theta) = 2\tan(60°-\theta),$

$can\ be\ written\ as,$

$$\tan^2\theta+\left(6\sqrt{3}\right)\tan\theta-5=0$$

$let\ x = \tan\theta,$

$\Rightarrow x^2+6\sqrt{3}x-5=0$

$using\ the\ quadratic\ formula,$

$\Rightarrow x = 0.461\ or -10.853$

$but\ \tan\theta = 0.461$ $\quad or \quad$ $\tan\theta = -10.85$

$\Rightarrow \theta = \tan^{-1}(0.461)$ $\qquad$ $\theta = \tan^{-1}(-10.853)$

$\Rightarrow \theta = 24.7°(PV_1)$ $\qquad$ $\theta = -84.7°(PV_2)$

$using\ the\ general\ solution\ for\ \tan\theta,$

$$\theta = PV + 180n$$

$\theta = 24.7 + 180n$

$when\, n = 0;\ \theta = 24.7$

$when\, n = 1;\ \theta = out\ of\ range$

$\theta = -87.4 + 180n$

$when\, n = 0;\ \theta = out\ of\ range$

$when\, n = 1;\ \theta = 95.3$

$\therefore\ \boldsymbol{\theta = 24.7° \text{ and } 95.3°}$

Question (Cambridge, June 2009 qp.3)

3 (i) Prove the identity $\operatorname{cosec} 2\theta + \cot 2\theta \equiv \cot\theta$. [3]

(ii) Hence solve the equation $\operatorname{cosec} 2\theta + \cot 2\theta = 2$, for $0° \leqslant \theta \leqslant 360°$. [2]

Solution

(i) $\operatorname{cosec} 2\theta + \cot 2\theta \equiv \cot\theta$

$where\ LHS = \operatorname{cosec} 2\theta + \cot 2\theta$

$$\Rightarrow LHS = \frac{1}{\sin 2\theta} + \frac{\cos 2\theta}{\sin 2\theta}$$

$$\Rightarrow LHS = \frac{1 + \cos 2\theta}{\sin 2\theta}$$

$$\Rightarrow LHS = \frac{1 + 2\cos^2\theta - 1}{2\sin\theta\cos\theta}$$

$$\Rightarrow LHS = \frac{2\cos^2\theta}{2\sin\theta\cos\theta}$$

$$\Rightarrow LHS = \frac{\cos\theta}{\sin\theta}$$

$\therefore\ \boldsymbol{LHS = \cot\theta \equiv RHS\ (shown)}$

(ii) $\operatorname{cosec} 2\theta + \cot 2\theta \equiv 2$

$\Rightarrow \cot\theta = 2$

$$\Rightarrow \frac{1}{\tan\theta} = 2$$

$\Rightarrow 1 = 2\tan\theta$

$\Rightarrow \tan\theta = \dfrac{1}{2}$

$\Rightarrow \theta = \tan^{-1}\left(\dfrac{1}{2}\right)$

$\Rightarrow \theta = 26.6° \ (PV)$

$using\ the\ general\ solution\ for \tan\theta,$

$$\theta = PV + 180n$$

$\theta = 26.6 + 180n$

$when\ n = 0;\ \theta = 26.6°$

$when\ n = 1;\ \theta = 206.6°$

$\therefore \boldsymbol{\theta = 26.6°\ and\ 206.6°}$

Revision Questions on General Trig Equations

June 2003 qp.1 (Zimsec)

14. Prove the identity $\sin\left(x+\frac{\pi}{3}\right)\cos\left(x-\frac{\pi}{3}\right) \equiv \frac{1}{4}\left(2\sin 2x + \sqrt{3}\right)$ [5]

Hence, or otherwise, find values of x in the range $0 \le x \le 2\pi$ such that

$\sin\left(x+\dfrac{\pi}{3}\right)\cos\left(x-\dfrac{\pi}{3}\right) = \dfrac{\sqrt{3}}{8}$

giving your answers in radians to 3 significant figures. [3]

June 2010 qp.1 (Zimsec)

1. Given that $\cos\theta = -\frac{1}{5}$ and $-180° < \theta < -90°$, find the exact value of $\cot\theta$. [3]

November 1997 qp.1 (Cambridge)

1. Find all values of x for which $0° < x < 360°$ that satisfy the equation

$$\sin\left(\tfrac{1}{2}x\right) = \tfrac{1}{4}.$$ [3]

November 2002 qp.1 (Zimsec, O Level Additional Mathematics)

14. (a) Solve the following equations for $0° < x < 360°$.

(i). $2\tan x - 2\cot x = 3$. [4]

(ii). $12\sin^2 x + 2\cos x = 10$. [4]

(b). Given that $\tan\alpha = -\frac{1}{\sqrt{3}}$ and that α is reflex, find $\sin\alpha$ and $\cos\alpha$. [4]

June 1991 qp.1 (Cambridge)

6. Find all values of θ such that $0° < \theta < 360°$ for which $2\cos 2\theta = 3 - 2\cos\theta$, giving your answers correct to $0.1°$. [5]

Unknown Source

7 (i). Given that $15\cos^2\theta + 2\sin^2\theta = 7$, show that $\tan^2\theta = \frac{8}{5}$. [4]

(ii). Solve $15\cos^2\theta + 2\sin^2\theta = 7$ for $0 \leq \theta \leq \pi$ rad. [3]

November 2006 qp.1 (Zimsec)

3. Express $(\tan\theta + \cot\theta)$ in terms of cosec 2θ. [3]

Hence, or otherwise, solve the equation

$$\tan\theta + \cot\theta = \frac{4}{\sqrt{3}}$$

for $0° < \theta < 90°$ [3]

November 2007 qp.3 (Cambridge)

5 **(i)** Show that the equation

$$\tan(45° + x) - \tan x = 2$$

can be written in the form

$$\tan^2 x + 2\tan x - 1 = 0.$$ [3]

(ii) Hence solve the equation

$$\tan(45° + x) - \tan x = 2,$$

giving all solutions in the interval $0° \leqslant x \leqslant 180°$. [4]

November 2009 qp.32 (Cambridge)

4 The angles α and β lie in the interval $0° < x < 180°$, and are such that

$$\tan\alpha = 2\tan\beta \quad \text{and} \quad \tan(\alpha + \beta) = 3.$$

Find the possible values of α and β. [6]

June 2010 qp.31 (Cambridge)

2 Solve the equation

$$\sin\theta = 2\cos 2\theta + 1,$$

giving all solutions in the interval $0° \leqslant \theta \leqslant 360°$. [6]

June 2010 qp.33 (Cambridge)

3 Solve the equation

$$\tan(45° - x) = 2\tan x,$$

giving all solutions in the interval $0° < x < 180°$. [5]

November 2010 qp.31 (Cambridge)

3 Solve the equation

$$\cos(\theta + 60°) = 2\sin\theta,$$

giving all solutions in the interval $0° \leqslant \theta \leqslant 360°$. [5]

June 2011 qp.32 (Cambridge)

3 Solve the equation

$$\cos\theta + 4\cos 2\theta = 3,$$

giving all solutions in the interval $0° \leqslant \theta \leqslant 180°$. [5]

June 2011 qp.33 (Cambridge)

4 **(i)** Show that the equation

$$\tan(60° + \theta) + \tan(60° - \theta) = k$$

can be written in the form

$$(2\sqrt{3})(1 + \tan^2\theta) = k(1 - 3\tan^2\theta).$$ [4]

(ii) Hence solve the equation

$$\tan(60° + \theta) + \tan(60° - \theta) = 3\sqrt{3},$$

giving all solutions in the interval $0° \leqslant \theta \leqslant 180°$. [3]

June 2012 qp.32 (Cambridge)

4 Solve the equation

$$\operatorname{cosec} 2\theta = \sec\theta + \cot\theta,$$

giving all solutions in the interval $0° < \theta < 360°$. [6]

June 2012 qp.33 (Cambridge)

6 It is given that $\tan 3x = k\tan x$, where k is a constant and $\tan x \neq 0$.

(i) By first expanding $\tan(2x + x)$, show that

$$(3k - 1)\tan^2 x = k - 3.$$ [4]

(ii) Hence solve the equation $\tan 3x = k\tan x$ when $k = 4$, giving all solutions in the interval $0° < x < 180°$. [3]

(iii) Show that the equation $\tan 3x = k\tan x$ has no root in the interval $0° < x < 180°$ when $k = 2$. [1]

November 2012 qp.31 (Cambridge)

3 Solve the equation

$$\sin(\theta + 45°) = 2\cos(\theta - 30°),$$

giving all solutions in the interval $0° < \theta < 180°$. [5]

Chapter Nine: Circular Measure

"You don't just luck into things... You build step by step, whether its friendship or opportunities."
– Barbara Bush

This topic deals with problems concerning <u>perimeter</u> and <u>area</u> of plane shapes with much emphasis on sectors. As such, circular measure feeds from mensuration, circle geometry and trigonometry.

- Mensuration forms the basis of the topic as it gives a detailed breakdown of properties of shapes, and how to determine their perimeters and areas.
- Circle geometry provides the theorems that are used for analysis. One of the most widely used theorems states that: At the point of contact of a tangent and line that passes through the centre, a 90° angle is formed. This is best explained by a diagram below:

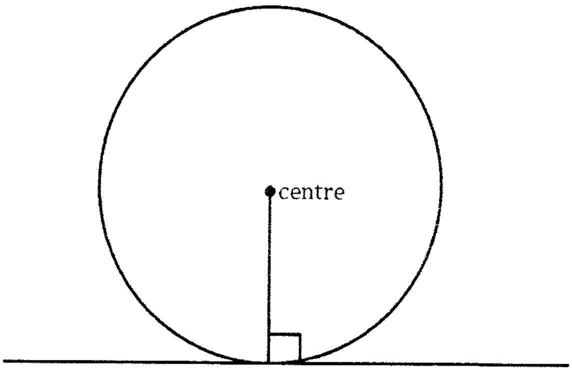


Fig. 9.1

- Trigonometry is used in the analysis of questions centred on triangles. The breakdown to trigonometry is as follows:

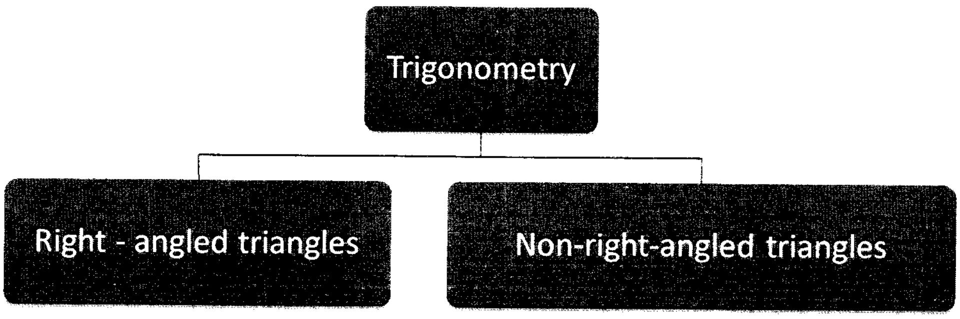


Fig. 9.2

- Pythagoras theorem
- Trigonometrical (trig) ratios (SohCahToa)

- Sine rule
- Cosine rule

NB:

1. All right angled triangles are analysed using either the Pythagoras theorem or the trig-ratios approach.
2. All non-right-angled triangles are analysed using the sine rule or cosine rule depending on the situation.

The Circle

This is a curved circular shape without straight edges (see Fig. 9.3).

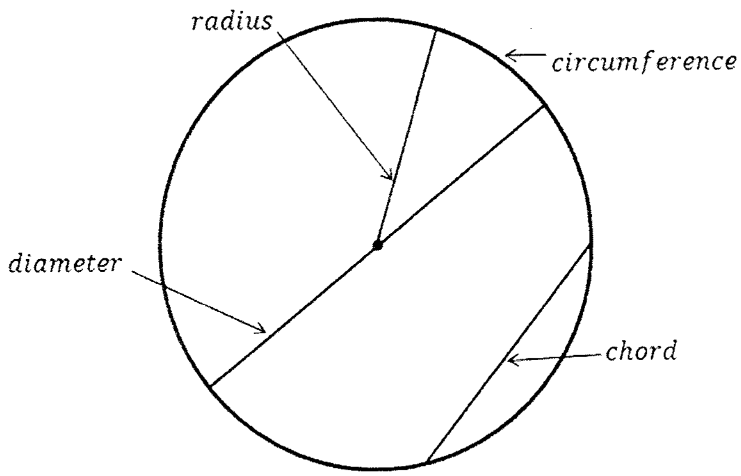


Fig. 9.3

- Circumference measures the distance right round the circle (perimeter).
- Radius refers to the distance measured from the centre to any point on the circumference.
- Chord is used to describe a line joining any two points on the circumference.
- Diameter is a special type of chord passing through the centre.

Segment versus Sector

- Segment is a region bound by a chord and part of the circumference. A chord has an effect of dividing a circle into two segments as shown in Fig. 9.4.

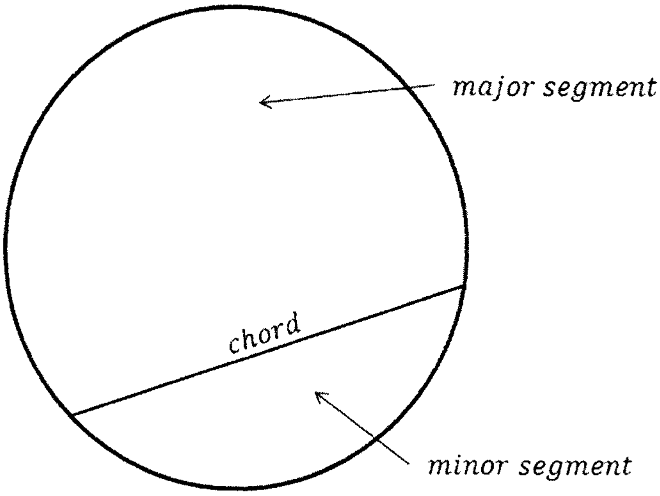


Fig. 9.4

- Sector is a region bound by two radii and part of the circumference. As such, a sector is hinged to the centre of the circle:

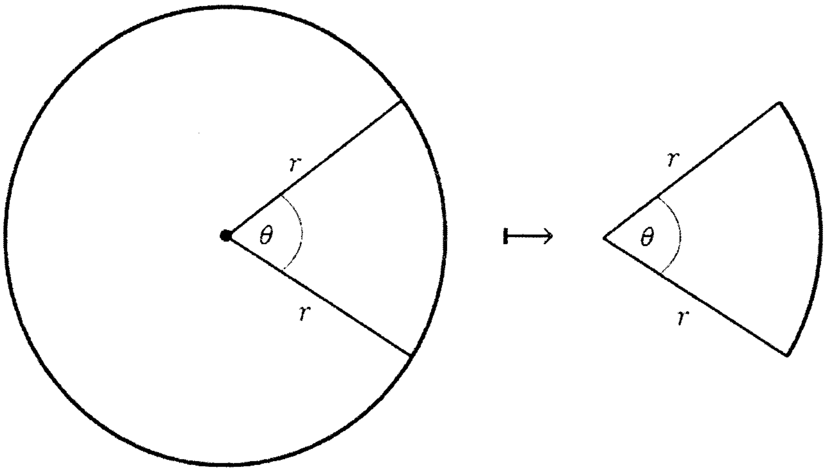


Fig. 9.5

Sectorial Analysis

Two important concepts in sectorial analysis are:

- Area of a sector, and
- Arc length of a sector.

These two measures can be calculated in degrees or radians, but much of the analysis at this stage is done in radians.

Table 9.1

	Degrees	Radians
Area	$\frac{\theta}{360} \times \pi r^2$	$\frac{1}{2} r^2 \theta$
Arc length	$\frac{\theta}{360} \times 2\pi r$	$r\theta$

Where θ is the size of the angle that the sector takes away from the full circle and

$$360° = 2\pi\, rad$$

$$180° = \pi\, rad$$

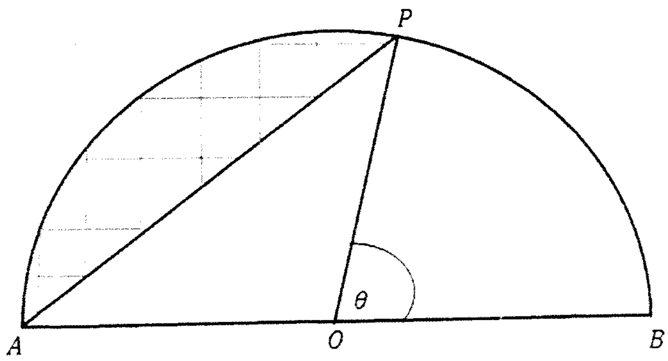
$$90° = \frac{\pi}{2}\, rad$$

And so on.

Worked Examination Questions on Circular Measure

Question (Cambridge, June 1997 qp.1)

2.



The diagram shows a semicircle APB on AB as diameter. The mid-point of AB is O. The point P on the semicircle is such that the area of the sector POB is equal to twice the area of the shaded segment. Given that angle POB is θ radians, show that

$$3\theta = 2(\pi - \sin\theta)$$

Solution

$Area\ of\ sector\ POB = 2[Area\ of\ shaded\ region]$

$where,\ \ area\ of\ sector\ POB = \frac{1}{2}r^2\theta,$

$and\ area\ of\ sector\ POA = \frac{1}{2}r^2(\pi - \theta),$

$and\ area\ of\ triangle\ POA = \frac{1}{2}r^2\sin(\pi - \theta),$

$\Rightarrow area\ of\ shaded\ region = \frac{1}{2}r^2(\pi - \theta) - \frac{1}{2}r^2\sin(\pi - \theta)$

$\Rightarrow area\ of\ shaded\ region = \frac{1}{2}r^2[(\pi - \theta) - \sin(\pi - \theta)]$

$Now,\ \ \frac{1}{2}r^2\theta = 2\left\{\frac{1}{2}r^2[(\pi - \theta) - \sin(\pi - \theta)]\right\}$

$\Rightarrow \frac{1}{2}r^2\theta = r^2[(\pi - \theta) - \sin(\pi - \theta)]$

$\Rightarrow \frac{1}{2}\theta = [\pi - \theta - \sin(\pi - \theta)]$

$\Rightarrow \theta = 2\pi - 2\theta - 2\sin(\pi - \theta)$

$\Rightarrow \theta + 2\theta = 2\pi - 2[\sin\pi\cos\theta - \cos\pi\sin\theta]$

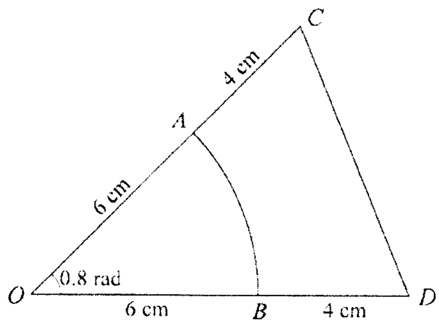
$\Rightarrow 3\theta = 2\pi - 2[(0)\cos\theta - (-1)\sin\theta]$

$\Rightarrow 3\theta = 2\pi - 2\sin\theta$

$\therefore \boldsymbol{3\theta = 2(\pi - \sin\theta)}$

Question (Cambridge, June2004 qp.1)

5



In the diagram, OCD is an isosceles triangle with $OC = OD = 10$ cm and angle $COD = 0.8$ radians. The points A and B, on OC and OD respectively, are joined by an arc of a circle with centre O and radius 6 cm. Find

(i) the area of the shaded region. [3]

(ii) the perimeter of the shaded region. [4]

Solution

i. $Area\ of\ shaded\ region\ =\ Area\ of\ triangle - Area\ of\ sector$

$where,\ \ area\ of\ triangle = \frac{1}{2}(10)^2 \sin 0.8$

$\Rightarrow area\ of\ triangle = 35.86780454$

$and\ area\ of\ sector = \frac{1}{2}(6)^2(0.8)$

$\Rightarrow area\ of\ sector = 1.4$

$Now,\ \ area\ of\ shaded\ region = 35.86780454 - 14.4$

$\therefore \boldsymbol{area\ of\ shaded\ region\ = 21.5\text{cm}^2}$

ii. $Perimeter\ of\ shaded\ region = AC + CD + DB + BA$

$where,\ \ AC = DB = 4$ cm,

$and,\ \ BA = 6(0.8)$

$\Rightarrow BA = 4.8$ cm

$and\ (CD)^2 = 10^2 + 10^2 - 2(10)(10)\cos(0.8)$

$\Rightarrow CD = \sqrt{200 - 200\cos(0.8)}$

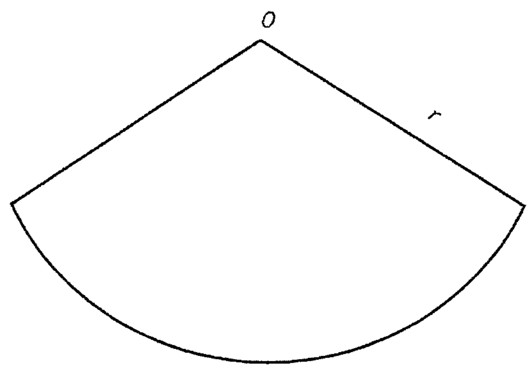
$\Rightarrow CD = 7.788366846$

$Now,\ \ Perimeter = 4 + 4 + 4.8 + 7.788366846$

$\therefore \boldsymbol{Perimeter = 46.2\ \text{cm}}$

Question (Cambridge, June 1994 qp.1)

4.



The diagram shows a sector of a circle, with centre O and radius r. The length of the arc is equal to half the perimeter of the sector. Find the area of the sector in terms of r. [3]

Solution

$Arc\ length = \frac{1}{2}[Perimeter\ of\ sector]$

$where,\ \ arc\ length = r\theta,$

$and\ perimeter = r + r + r\theta$

$\Rightarrow\ perimeter = 2r + r\theta$

$Now,\ \ r\theta = \frac{1}{2}[2r + r\theta]$

$\Rightarrow 2r\theta = 2r + r\theta$

$\Rightarrow r\theta = 2r$

$\Rightarrow \theta = 2$

$Now,\ \ Area\ of\ sector = \frac{1}{2}r^2(2)$

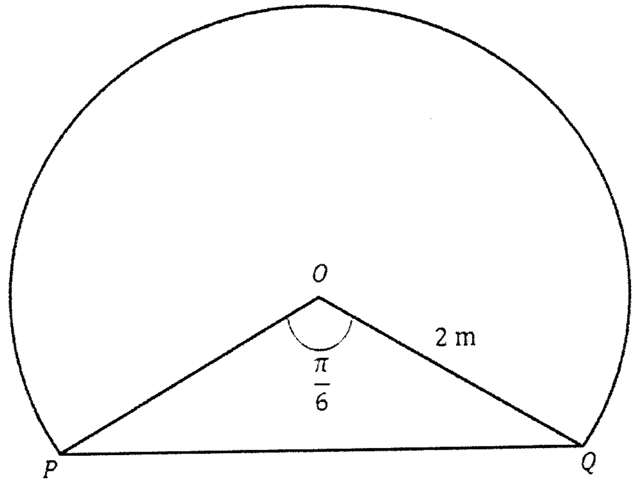
$\therefore\ \boldsymbol{Area\ of\ sector = r^2\ \mathbf{units}^2}$

Revision Questions on Circular Measure

November 2002 qp.1 (Zimsec, O Level Additional Mathematics)

12. (a) A section of a railway track is a circular arc of length 50 m and radius 300 m. Find the angle through which the direction of the track turns. [2]

(b)

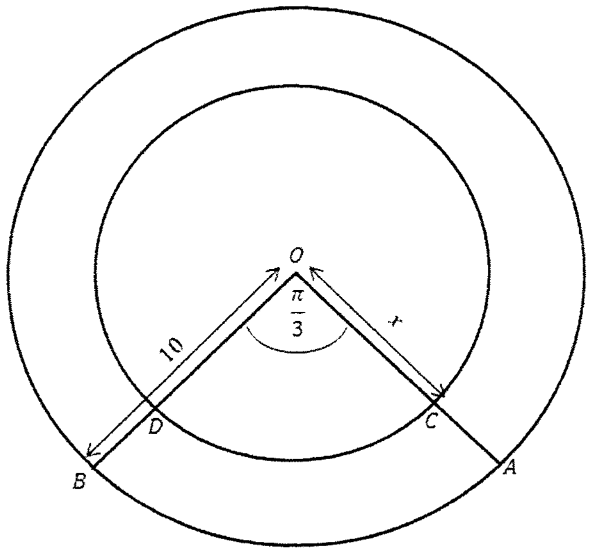


The diagram above shows the cross-section of a tunnel which has the shape of a segment of a circle O. The radius of the circle is 2 m and the size of the angle POQ is $\frac{\pi}{6}$ radians. Calculate the perimeter of the cross-section. [4]

(c) A chord AB of length 8a is drawn in a circle of radius 12a. The tangents to the circle at A and B meet at C. Find the area enclosed by AC, BC and the minor arc AB. [6]

June 2003 qp.1 (Zimsec)

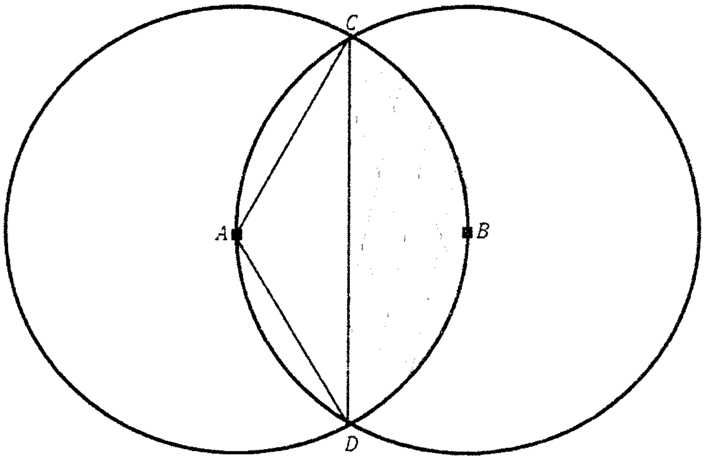
9.



The diagram above shows two circles with centre O. $OA = OB = 10$ cm and $OC = OD = x$ cm. The angle AOB is $\frac{\pi}{3}$. The difference between the area of the minor sectors OAB and OCD is 6π. Find the value of x. [5]

June 1991 qp.1 (Cambridge)

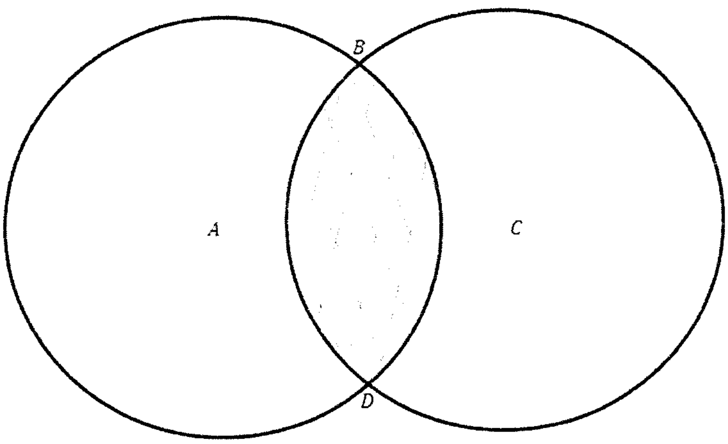
5.



The diagram shows two circles, with centres A and B, intersecting at C and D in such a way that the centre of each lies on the circumference of the other. The radius of each circle is 1 unit. Write down the size of angle CAD and calculate the area of the shaded region (bounded by the arc CBD and the straight line CD). Hence show that the area of the region common to the interiors of the two circles is approximately 39% of the area of one circle. [4]

November 2004 qp.1 (Zimsec)

13.



Two equal circles with centres A and C and radius 2 cm intersect at B and D (see diagram).

The angle subtended by the common chord at the centre of each circle is 3θ.

a) Find

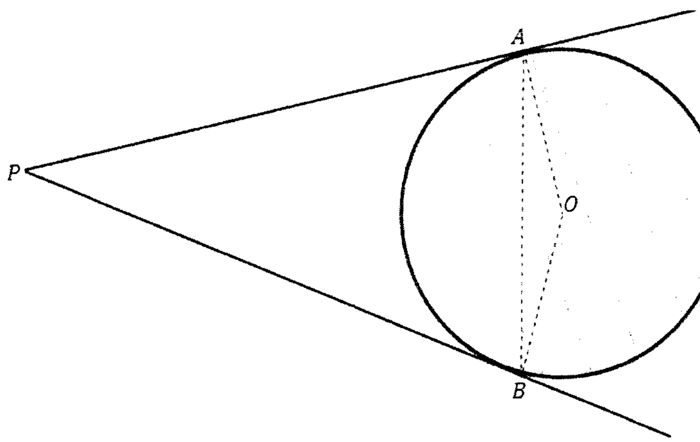
(i) an expression for the shaded area in terms of θ. [2]

(ii) the area of the quadrilateral ABCD in terms of θ. [2]

b) Given that the shaded area is equal to one quarter of the area of one of the circles, show that $12\theta - 4\sin\theta(3\cos^2\theta - \sin^2\theta) = \pi$. [5]

June 2010 qp.1 (Zimsec)

13. The diagram below shows a circle centre O and two tangents AP and BP drawn from a point P.



Given that AP $= 20$ cm and AB $= 12$ cm,

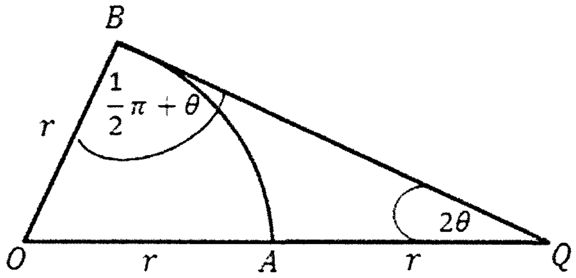
(i) Show that the obtuse angle AOB $= 2.532$ radians, [3]

(ii) Calculate the radius of the circle, [2]

(iii) Calculate the area of the shaded segment AB [3]

November 1996 qp.1 (Cambridge)

6.



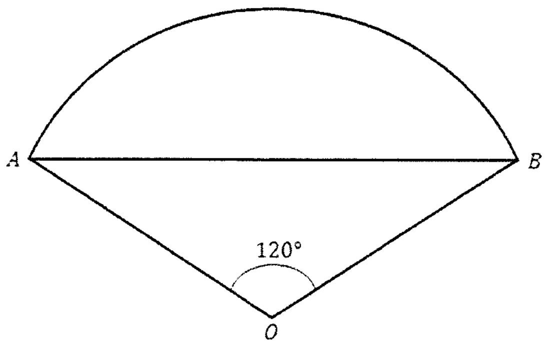
In the diagram, OAB is a sector of a circle with centre O and radius r cm. The point A is the mid-point of OQ and angle $OQB = 2\theta$, $OBQ = \frac{1}{2}\pi + \theta$. By using the sine rule, show that $\sin\theta = \frac{1}{4}$. [3]

The area, in cm^2, of the sector OAB is numerically equal to the perimeter, in cm, of the sector OAB.

Find r, correct to three significant figures. [3]

November 1995 qp.1 (Cambridge)

2.



The diagram shows a sector OAB of a circle centre O, in which angle $AOB = 120°$.

(i) Show that

$$\frac{\text{Area of sector } OAB}{\text{Area of triangle } OAB} = \frac{4\pi}{3\sqrt{3}}$$

[3]

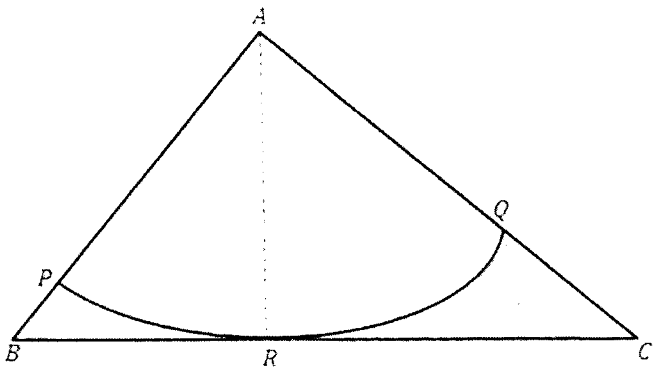
(ii) Find the exact value of

$$\frac{\text{Length of arc } AB}{\text{Length of chord } AB}$$

[3]

November 1997 qp.1 (Cambridge)

14.



Triangle ABC is such that $AB = 5$ cm, $BC = 7$cm and $CA = 8$ cm. The point R is the foot of the perpendicular from A to BC. With centre A and radius AR, a circular arc is drawn, from a point P on AB to a point Q on AC, touching the line BC at R. (See diagram).

(i). Show that angle $BAC = \frac{1}{3}\pi\ radians$ and that $\sin B = \frac{4}{7}\sqrt{3}$ [4]

(ii). Show that the area of the shaded region, which lies inside triangle ABC but outside sector APQ is

$$\left(10\sqrt{3} - \frac{200}{49}\pi\right)\text{cm}^2$$

[5]

Chapter Ten: Differentiation

"We identify in our experience a differentiation between what we do and what happens to us."
– Alan Watts

Differentiation is a breakdown process meant to decompose a function, an equation or an expression into a gradient function. The process gives a result which measures the change in one variable with respect to the other (that is, the gradient). This topic analyses the ten examinable **concepts** and four **applications** of differentiation.

Concepts in Differentiation

1. **Differentiation of a Constant**
   A constant breaks down to zero.
   For example, given that, $y = 3$
   $\Rightarrow \frac{dy}{dx} = 0.$

2. **Differentiation of Simple Algebraic Expressions**

   A simple expression is broken down by way of dropping the power and use the power as a multiplier to the term in question before reducing the power by one. The general rule states that if $y = x^n$

   $$\text{then } \frac{dy}{dx} = nx^{n-1}.$$

   For example, given that, $y = 7x^3 - 2x^2 + 4x - 4,$

   $\Rightarrow \frac{dy}{dx} = 21x^2 - 4x + 4$

3. **Differentiation of Complex Algebraic Expressions**

   Complex expressions are broken down by way of using the $'PA - strategy'$ where:

   - $P\ stands\ for\ the\ derivative\ with\ respect\ to\ \mathbf{power},$
   - $A\ stands\ for\ the\ derivative\ with\ respect\ to\ \mathbf{algebraic}\ expression.$

The general rule states that if,

$$y = (ax + b)^n,$$

then, $\dfrac{dy}{dx} = n(ax + b)^{n-1} \times a$

For example, given that, $y = (2x + 5)^5$

$\Rightarrow \dfrac{dy}{dx} = 5(2x + 5)^4 \times 2$

$\therefore \boldsymbol{\dfrac{dy}{dx} = 10(2x + 5)^4}$

4. **Differentiation of Exponential Functions**

The derivative of exponential functions is given by the product of the original function and its inner derivative, that is,

When $y = e^{ax}$

then, $\dfrac{dy}{dx} = a \times e^{ax}$

For example, given that, $y = 10e^{5x}$

$\Rightarrow \dfrac{dy}{dx} = 5 \times 10e^{5x}$

$\therefore \boldsymbol{\dfrac{dy}{dx} = 50e^{5x}}$

5. **Differentiation of Logarithmic Functions**

The general rule states that,

If $y = \ln(ax + b)$

then, $\dfrac{dy}{dx} = \dfrac{1}{(ax + b)} \times a$

$\Rightarrow \dfrac{dy}{dx} = \dfrac{a}{(ax + b)}$

For example, given that, $y = \ln(5 - 2x)$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{(5-2x)} \times -2$$

$$\therefore \boldsymbol{\frac{dy}{dx} = \frac{-2}{(5-2x)}}$$

6. **Differentiation of Trig Functions**

Trig functions have a predetermined set of results that are quotable. Table 10.1 below gives the standard results.

Functions	Derivative
$\sin\theta$	$\cos\theta$
$\cos\theta$	$-\sin\theta$
$\tan\theta$	$\sec^2\theta$
$\operatorname{cosec}\theta$	$-\operatorname{cosec}\theta\cot\theta$
$\sec\theta$	$\sec\theta\tan\theta$
$\cot\theta$	$-\operatorname{cosec}^2\theta$

Table 10.1

The variations in the differentiation of trig functions are accounted for using the $PTA - strategy$ where:

- $P\ stands\ for\ the\ derivative\ with\ respect\ to\ \mathbf{power},$
- $T\ stands\ for\ the\ derivative\ with\ respect\ to\ \mathbf{trig}\ function, and$
- $A\ stands\ for\ derivative\ with\ respect\ to\ \mathbf{algebraic}\ expression.$

Examples below outline the concept:

i) Given that, $y = \sin^4 7\theta$

$$\Rightarrow \frac{dy}{d\theta} = 4\sin^3 7\theta \times \cos 7\theta \times 7$$

$$\therefore \boldsymbol{\frac{dy}{d\theta} = 28\sin^3 7\theta \cos 7\theta}$$

ii) Given that, $y = \tan^5\theta$

$$\Rightarrow \frac{dy}{d\theta} = 5\tan^4\theta \times \sec^2\theta \times 1$$

$$\therefore \boldsymbol{\frac{dy}{d\theta} = 5\tan^4\theta \sec^2\theta}$$

iii) Given that, $y = \operatorname{cosec} 2\theta$

Since there is no power, ignore $\boldsymbol{P}$ and account for $\boldsymbol{T}$ & $\boldsymbol{A}$ only as shown below,

$$\Rightarrow \frac{dy}{d\theta} = -\operatorname{cosec} 2\theta \cot 2\theta \times 2$$

$$\therefore \boldsymbol{\frac{dy}{d\theta} = -2\operatorname{cosec} 2\theta \cot 2\theta}$$

7. **Differentiation of Products (The Product Rule)**

All products are broken down using the product rule which states that:

$$\boldsymbol{\frac{d}{dx}(uv) = v\frac{du}{dx} + u\frac{dv}{dx}}$$

where u is the first term and v is the second term.

For example, given that, $y = 7\ln(2x+1)\cos x$

$$\Rightarrow \frac{dy}{dx} = \cos x\left[7 \times \frac{1}{(2x+1)} \times 2\right] + [7\ln(2x+1)](-\sin x)$$

$$\therefore \boldsymbol{\frac{dy}{dx} = \frac{14\cos x}{(2x+1)} - 7\sin x\ln(2x+1)}$$

8. Differentiation of Fractions (The Quotient Rule)

All fractions are broken down using the quotient rule which states that:

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

where u is the numerator and v is the denominator.

For example, given that,

$$y = \frac{3x^4 - 5x}{e^{3x}},$$

$$\Rightarrow \frac{dy}{dx} = \frac{e^{3x}(12x^3 - 5) - (3x^4 - 5x)3e^{3x}}{(e^{3x})^2}$$

$$\Rightarrow \frac{dy}{dx} = \frac{e^{3x}[(12x^3 - 5) - 3(3x^4 - 5x)]}{e^{6x}}$$

$$\Rightarrow \frac{dy}{dx} = \frac{e^{3x}[12x^3 - 5 - 9x^4 + 15x]}{e^{6x}}$$

$$\therefore \frac{dy}{dx} = \frac{e^{3x}[-9x^4 + 12x^3 + 15x - 5]}{e^{6x}}$$

9. Parametric Differentiation

If a curve is defined by a pair of parametric equations, the two equations have to be individually differentiated with respect to the parameter and then use the **chain rule** to combine the two results.

For example, given that a curve is defined by the following pair of parametric equations:

$$y = 2\cos 3\theta \quad and \quad x = \sin 3\theta.$$

$y = 2\cos 3\theta$

$$\Rightarrow \frac{dy}{d\theta} = 2(-\sin 3\theta) \times 3$$

$$\Rightarrow \frac{dy}{d\theta} = -6\sin 3\theta$$

$x = \sin 3\theta$

$$\Rightarrow \frac{dx}{d\theta} = (\cos 3\theta) \times 3$$

$$\Rightarrow \frac{dx}{d\theta} = 3\cos 3\theta$$

Using the chain rule,

$$\frac{dy}{dx} = \frac{dy}{d\theta} \div \frac{dx}{d\theta}$$

$$\Rightarrow \frac{dy}{dx} = \frac{-6\sin 3\theta}{3\cos 3\theta}$$

$$\therefore \frac{dy}{dx} = -2\tan 3\theta$$

10. **Implicit Differentiation**

This technique is used to differentiate equations where either x *or* y cannot be easily expressed as a stand-alone item. Implicit differentiation is based on the notion that all terms are differentiated with respect to x. In the case where the term is in terms of y, a differential coefficient $\left(\text{that is, } \frac{dy}{dx}\right)$ is attached to the derivative. For example, the derivative of $4y^3$ is $12y^2\frac{dy}{dx}$. The product rule is used to account for the derivative of a product, if any exists.

For example, given that,

$$3x^3 - 2x^2y + 5y^4 - 4 = 0$$

$$\Rightarrow 9x^2 - \left[y(4x) + 2x^2(1)\frac{dy}{dx}\right] + 20y^3\frac{dy}{dx} - 0 = 0$$

$$\Rightarrow 9x^2 - 4xy - 2x^2\frac{dy}{dx} + 20y^3\frac{dy}{dx} = 0$$

$$\Rightarrow -2x^2\frac{dy}{dx} + 20y^3\frac{dy}{dx} = -9x^2 + 4xy$$

$$\Rightarrow \frac{dy}{dx}(20y^3 - 2x^2) = 4xy - 9x^2$$

$$\therefore \frac{dy}{dx} = \frac{4xy - 9x^2}{20y^3 - 2x^2}$$

Applications of Differentiation

1. Tangents and Normals

- A tangent is a line that touches, but not crosses a curve. This implies that a curve and tangent intersect at exactly one point.

- A normal is a line perpendicular to a tangent at the point of contact with a curve.
- The relationship connecting a curve, a tangent and a normal is illustrated by the diagram below:

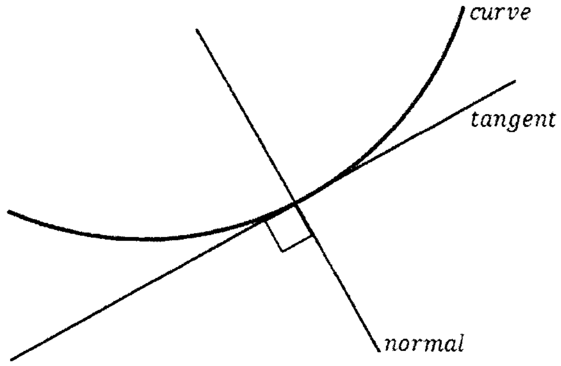


Fig.10.1

- Conceptually, the product of gradients of a tangent and normal is negative 1. That is $m \times n = -1$, where m is the gradient of tangent and n is the gradient of normal.

- A curve and a tangent share the same gradient at the point of contact. The gradient of a tangent is given by substituting the x value at the point of contact into the gradient function of a curve.

- Gradient of a normal is given by $m \times n = -1$
  For example, given that the gradient function of a curve is, $\frac{dy}{dx} = 2x + 1$ and point $\mathrm{P}(2, 3)$ is a point on the curve.
  Gradient of the tangent $= 2(2) + 1 = 5$
  Gradient of the normal: $5 \times n = -1$

$$n = -\frac{1}{5}$$

2. Stationary Points

A *stationary point*, also known as a *turning point*, occurs at a point where a curve changes its nature from being an increasing function into a decreasing function or vice-versa. At a turning point, gradient is equal to zero. Table 10.2 summarises the relationship between nature of gradient and the resultant sign.

Table 10.2

Nature of gradient	Sign
Increasing function	$\frac{dy}{dx} > 0$ i.e. gradient is positive
Decreasing function	$\frac{dy}{dx} < 0$ i.e. gradient is negative
Constant function (stationary point)	$\frac{dy}{dx} = 0$

- To find the x value at a turning point, one has to equate the gradient function to zero and solve the resultant equation.
- The corresponding y-value is given by substituting the x-value in the original equation/function.
- The nature of a turning point (that is maximum or minimum) is given by substituting the x-value at a stationary point in the second derivative $\left(\text{that is } \frac{d^2y}{dx^2} \text{ or } f''(x)\right)$. If the second derivative is positive, the turning point is a minimum and if the second derivative is negative, the turning point is a maximum.

For example,

Given that $y = 2x^2 - 8x$,

$$\frac{dy}{dx} = 4x - 8$$

Using the fact that $\frac{dy}{dx} = 0$ at a turning point

$4x - 8 = 0$	$and \;\; y = 2x^2 - 8x$
$x = 8$	$y = 2(2)^2 - 8(2)$
$x = 2$	$y = -8$

$\therefore$ ***the coordinates at turning point are*** $(\mathbf{2, -8})$.

Since $y = 2x^2 - 8x$

And $\frac{dy}{dx} = 4x - 8$

$$\Rightarrow \frac{d^2y}{dx^2} = 4$$

In this case the turning point is a minimum because the second derivative is positive.

3. Rate of Change

- Rate is a measure of the change in one variable **with respect to time**. For example, the rate of change in volume with respect to time and rate of change in radius with respect to time is given by $\frac{dv}{dt}$ and $\frac{dr}{dt}$ respectively.
- Questions on rate of change test the ability of students to relate three (3) differential facts. Two of the facts are exposed leading to the determination of the third fact.
- The two exposed facts are combined using the 'chain rule' with the view of eliminating an unwanted variable. This process gives the third differential fact.

  For example, $\frac{dy}{dt} = \frac{dy}{dx} \times \frac{dx}{dt}$

  $\frac{dx}{dt} = \frac{dx}{dy} \times \frac{dy}{dt}$,

  $\frac{dv}{dt} = \frac{dv}{dr} \times \frac{dr}{dt}$, and so on.

4. Maclaurin's Series

Maclaurin's theorem is a hybrid of differentiation and series expansion that finds its application in different types of functions. It is really a predetermined framework which follows a specific pattern. The theorem states that:

$$f(x) = f(0) + \frac{xf'(0)}{1!} + \frac{x^2f''(0)}{2!} + \frac{x^3f'''(0)}{3!} \ldots$$

For example, the first three terms in the expansion of $f(x) = 8x^3 + 7x^2 + 3x - 2$ is given by: $f(x) = 8x^3 + 7x^2 + 3x - 2$

$f'(x) = 24x^2 + 14x + 3$

$f''(x) = 48x + 14$

Where $f(0) = -2$,

and $f'(0) = 3$,

and $f''(0) = 14$

$$\Rightarrow f(x) = -2 + \frac{x(3)}{1!} + \frac{x^2(14)}{2!}$$

$$\therefore \boldsymbol{f(x) = -2 + 3x + 7x^2}$$

NB: For some widely used functions, the results of the Maclaurin's expansion are known and, therefore, quotable. Below is a breakdown of some prominent expansions:

- $\cos\theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} \ldots$
- $\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} \ldots$
- $e^\theta = 1 + \theta + \frac{\theta^2}{2!} + \frac{\theta^3}{3!} + \frac{\theta^4}{4!} \ldots$
- $\ln(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} \ldots$

Students are strongly encouraged to know the general formula and not overload an already overloaded mind by trying to memorise expansions of different functions. The main advantage of knowing the general formula is that it can be used to derive expansions for these functions.

Worked Examination Questions on Differentiation of Simple and Complex Algebraic Expressions

Question (Cambridge, June 2005 qp.1)

2 Find the gradient of the curve $y = \frac{12}{x^2 - 4x}$ at the point where $x = 3$. [4]

Solution

$$Given\ that\ y = \frac{12}{x^2 - 4x}$$

$$\Rightarrow y = 12(x^2 - 4x)^{-1}$$

$$\Rightarrow \frac{dy}{dx} = (-1)(12)(x^2 - 4x)^{-2}(2x - 4)$$

$$\Rightarrow \frac{dy}{dx} = \frac{-12(2x - 4)}{(x^2 - 4x)^2}$$

$when\ x = 3,$

$$\Rightarrow \frac{dy}{dx} = \frac{-12[2(3) - 4]}{[(3)^2 - 4(3)]^2}$$

$$\Rightarrow \frac{dy}{dx} = \frac{(-12)(2)}{(9 - 12)^2}$$

$$\Rightarrow \frac{dy}{dx} = \frac{-24}{9}$$

$$\therefore \boldsymbol{Gradient = -\frac{8}{3}}$$

Question (Cambridge, November 2002 qp.1)

8 A curve has equation $y = x^3 + 3x^2 - 9x + k$, where k is a constant.

(i) Write down an expression for $\frac{dy}{dx}$. [2]

(ii) Find the x-coordinates of the two stationary points on the curve. [2]

(iii) Hence find the two values of k for which the curve has a stationary point on the x-axis. [3]

Solution

(i) $Given\ that\ y = x^3 + 3x^2 - 9x + k$

$$\therefore \boldsymbol{\frac{dy}{dx} = 3x^2 + 6x - 9}$$

(ii) $At\ a\ stationary\ point,$

$$\frac{dy}{dx} = 0$$

$$\Rightarrow 3x^2 + 6x - 9 = 0$$

$$\Rightarrow 3x^2 + 9x - 3x - 9 = 0$$

$$\Rightarrow 3x(x + 3) - 3(x + 3) = 0$$

$\Rightarrow (3x-3)(x+3) = 0$

either $3x - 3 = 0$ *or* $x + 3 = 0$

$\therefore \boldsymbol{x = 1}$ ***or*** $\boldsymbol{x = -3}$

(iii) *Given that the stationary point lies on the* $x-$ *axis,*

$\Rightarrow y = 0$

using $y = x^3 + 3x^2 - 9x + k,$

when $x = 1$

$\Rightarrow 0 = (1)^3 + 3(1)^2 - 9(1) + k$

$\Rightarrow 0 = 1 + 3 - 9 + k$

$\Rightarrow 0 = -5 + k$

$\therefore \boldsymbol{k = 5}$

when $x = -3$

$\Rightarrow 0 = (-3)^3 + 3(-3)^2 - 9(-3) + k$

$\Rightarrow 0 = -27 + 27 + 27 + k$

$\Rightarrow 0 = 27 + k$

$\therefore \boldsymbol{k = -27}$

Question (Cambridge, November 2007 qp.1)

8 The equation of a curve is $y = (2x-3)^3 - 6x$.

(i) Express $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ in terms of x. [3]

(ii) Find the x-coordinates of the two stationary points and determine the nature of each stationary point. [5]

Solution

(i) *Given that* $y = (2x-3)^3 - 6x$

$\Rightarrow \frac{dy}{dx} = 3(2)(2x-3)^2 - 6$

$\therefore \boldsymbol{\frac{dy}{dx} = 6(2x-3)^2 - 6}$

$\frac{d^2y}{dx^2} = 6(2)(2)(2x-3)$

$\therefore \boldsymbol{\frac{d^2y}{dx^2} = 24(2x-3)}$

(ii) $At\ a\ stationary\ point,\ \frac{dy}{dx} = 0$

$\Rightarrow 6(2x - 3)^2 - 6 = 0$

$\Rightarrow 6(2x - 3)^2 = 6$

$\Rightarrow (2x - 3)^2 = 1$

$\Rightarrow (2x - 3) = \pm 1$

$\Rightarrow 2x = \pm 1 + 3$

$\Rightarrow x = \frac{2}{2}\ or\ \frac{4}{2}$

$\therefore \boldsymbol{x = 1\ or\ x = 2}$

$From\ \text{(i)}\ above,\ \frac{d^2y}{dx^2} = 24(2x - 3)$

$\Rightarrow \frac{d^2y}{dx^2} = -24\ \ when\ x = 1$

$\Rightarrow \frac{d^2y}{dx^2} = 24\ \ when\ x = 2$

when x = 1, the stationary value is a maximum,

and when x = 2, the stationary value is a minimum.

<u>Revision Questions on Simple and Complex Algebraic Expressions</u>

November 1992 qp.1 (Cambridge)

9. Find, by differentiation, the coordinates of the turning points on the curve

$$y = x^3 - 2x^2 - 4x + 5,$$

stating the nature of each turning point. [5]

November 2002 qp.1 (Zimsec)

18. Given that $y = x^3 - x^2 - 5x + 5$

Find,

(i). $\frac{dy}{dx}$, [1]

(ii). The equation of the tangent to the curve at the point where $x = 2$, [3]

(iii). The coordinates of the turning points, determining whether each is a maximum or a minimum. [6]

Hence sketch the curve. [2]

November 2006 qp.1 (Zimsec)

8. The point P$(-2;\ 6)$ lies on the curve $y = 2 - x^2 - x^3$.

(i). Find the equation of the tangent and the equation of the normal at the point P, each in the form $y = mx + c$. [6]

(ii). Given that the tangent meets the y –axis at A and that the normal meets the y –axis at B, show that the length of AB is $16\frac{1}{4}$units. [2]

November 2002 qp.1 (Zimsec, O Level Additional Mathematics)

2. A curve has the equation $y = 2(3x - 5)^2$

Find

(i). $\frac{dy}{dx}$ [1]

(ii). The equation of the normal to the curve at the point where $x = 2$. [3]

June 2006 qp.1 (Cambridge)

1 A curve has equation $y = \frac{k}{x}$. Given that the gradient of the curve is -3 when $x = 2$, find the value of the constant k. [3]

June 2008 qp.1 (Cambridge)

4 The equation of a curve C is $y = 2x^2 - 8x + 9$ and the equation of a line L is $x + y = 3$.

(i) Find the x-coordinates of the points of intersection of L and C. [4]

(ii) Show that one of these points is also the stationary point of C. [3]

June 2010 qp.12 (Cambridge)

10 The equation of a curve is $y = \frac{1}{6}(2x - 3)^3 - 4x$.

(i) Find $\frac{dy}{dx}$. [3]

(ii) Find the equation of the tangent to the curve at the point where the curve intersects the y-axis. [3]

(iii) Find the set of values of x for which $\frac{1}{6}(2x - 3)^3 - 4x$ is an increasing function of x. [3]

November 2008 qp.1 (Cambridge)

8 The equation of a curve is $y = 5 - \frac{8}{x}$.

(i) Show that the equation of the normal to the curve at the point $P(2, 1)$ is $2y + x = 4$. [4]

This normal meets the curve again at the point Q.

(ii) Find the coordinates of Q. [3]

(iii) Find the length of PQ. [2]

November 2010 qp.13 (Cambridge)

5 A curve has equation $y = \frac{1}{x - 3} + x$.

(i) Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$. [2]

(ii) Find the coordinates of the maximum point A and the minimum point B on the curve. [5]

November 2011 qp.11 (Cambridge)

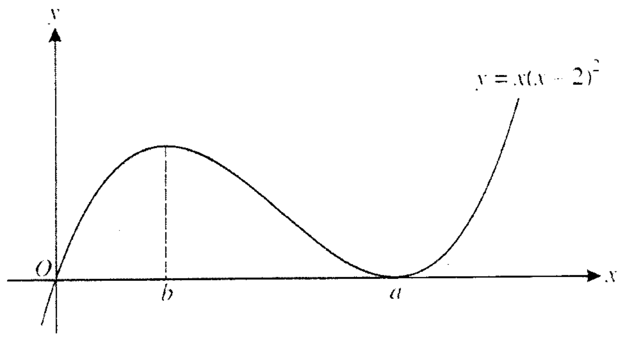
2 A curve has equation $y = 3x^3 - 6x^2 + 4x + 2$. Show that the gradient of the curve is never negative. [3]

November 2012 qp.11 (Cambridge)

5 A curve has equation $y = 2x + \frac{1}{(x-1)^2}$. Verify that the curve has a stationary point at $x = 2$ and determine its nature. [5]

November 2012 qp.13 (Cambridge)

11



The diagram shows the curve with equation $y = x(x-2)^2$. The minimum point on the curve has coordinates $(a, 0)$ and the x-coordinate of the maximum point is b, where a and b are constants.

(i) State the value of a. [1]

(ii) Find the value of b. [4]

(iii) Find the area of the shaded region. [4]

(iv) The gradient, $\frac{dy}{dx}$, of the curve has a minimum value m. Find the value of m. [4]

<u>Worked Examination Questions on Differentiation and Mensuration</u>

This section analyses questions on differentiation inclined to mensuration (perimeter, area and volume).

Question (Cambridge, June 2002 qp.1)

8 A hollow circular cylinder, open at one end, is constructed of thin sheet metal. The total external surface area of the cylinder is 192π cm². The cylinder has a radius of r cm and a height of h cm.

(i) Express h in terms of r and show that the volume, V cm³, of the cylinder is given by

$$V = \tfrac{1}{2}\pi(192r - r^3).$$ [4]

Given that r can vary,

(ii) find the value of r for which V has a stationary value. [3]

(iii) find this stationary value and determine whether it is a maximum or a minimum. [3]

Solution

8. (i) $Let\ the\ total\ surface\ area\ (TSR) = 192\pi$

$$TSR = Curved\ surface\ area + Base\ area$$

$$\Rightarrow 192\pi = 2\pi rh + \pi r^2$$

$$\Rightarrow 2\pi rh = 192\pi - \pi r^2$$

$$\Rightarrow h = \frac{\pi(192 - r^2)}{2\pi r}$$

$$\Rightarrow h = \frac{192 - r^2}{2r}$$

$$Now,\ V = \pi r^2 h$$

$$\Rightarrow V = \pi r^2\left(\frac{192 - r^2}{2r}\right)$$

$$\Rightarrow V = \frac{\pi r}{2}(192 - r^2)$$

$$\therefore \boldsymbol{V = \frac{\pi}{2}(192r - r^3)\ (shown)}$$

(ii) $At\ a\ stationary\ value,\ \frac{dV}{dr} = 0$

$$using\ V = \frac{\pi}{2}(192r - r^3)$$

$$\Rightarrow \frac{dV}{dr} = \frac{\pi}{2}[192 - 3r^2]$$

$$\Rightarrow \frac{\pi}{2}[192 - 3r^2] = 0$$

$$\Rightarrow 192 - 3r^2 = 0$$

$$\Rightarrow 3r^2 = 192$$

$$\Rightarrow r^2 = 64$$

$$\Rightarrow r = \pm 8$$

$$\therefore \boldsymbol{r = 8\ cm \quad Since\ r\ cannot\ be\ negative}$$

(iii) $When\ r = 8,$

$$\Rightarrow V = \frac{\pi}{2}[192(8) - (8)^3]$$

$$\Rightarrow V = \frac{\pi}{2}(1536 - 512)$$

$$\Rightarrow V = \frac{\pi}{2}(1024)$$

$$\therefore \boldsymbol{V = 512\pi\ cm^3}$$

$$Since\ \frac{dV}{dr} = \frac{\pi}{2}(192 - 3r^2)$$

$$\Rightarrow \frac{d^2V}{dr^2} = \frac{\pi}{2}(-6r)$$

$$\Rightarrow \frac{d^2V}{dr^2} = -3\pi r$$

$$\Rightarrow \frac{d^2V}{dr^2} = -24\pi\ when\ r = 8$$

$\therefore$ ***The stationary value is a maximum***

Question (Cambridge, November 2003 qp.1)

8 A solid rectangular block has a base which measures $2x$ cm by x cm. The height of the block is y cm and the volume of the block is 72 cm^3.

(i) Express y in terms of x and show that the total surface area, A cm^2, of the block is given by

$$A = 4x^2 + \frac{216}{x}.$$ [3]

Given that x can vary,

(ii) find the value of x for which A has a stationary value. [3]

(iii) find this stationary value and determine whether it is a maximum or a minimum. [3]

Solution

(i). $V = lbh$

$$\Rightarrow 72 = (2x)(x)(y)$$

$$\Rightarrow 72 = 2x^2y$$

$$\Rightarrow y = \frac{72}{2x^2}$$

$$\Rightarrow y = \frac{36}{x^2}\ cm$$

$$A = 2(lb) + 2(lh) + 2(bh)$$

$$\Rightarrow A = 2(2x)(x) + 2(2x)(y) + 2(x)(y)$$

$$\Rightarrow A = 4x^2 + 4xy + 2xy \longrightarrow 1$$

by substituting y into equation (1),

$$\Rightarrow A = 4x^2 + 4x\left(\frac{36}{x^2}\right) + 2x\left(\frac{36}{x^2}\right)$$

$$\Rightarrow A = 4x^2 + \frac{144x}{x^2} + \frac{72x}{x^2}$$

$$\Rightarrow A = 4x^2 + \frac{144}{x} + \frac{72}{x}$$

$$\Rightarrow A = 4x^2 + \frac{144 + 72}{x}$$

$$\therefore \boldsymbol{A = 4x^2 + \frac{216}{x}}\ \boldsymbol{(shown)}$$

(ii) *From* (i) *above*, $A = 4x^2 + \frac{216}{x}$,

$$\Rightarrow A = 4x^2 + 216x^{-1}$$

$$\Rightarrow \frac{dA}{dx} = 8x - 216x^{-2}$$

A has a stationary value when $\frac{dA}{dx} = 0$

$$\Rightarrow 8x - 216x^{-2} = 0$$

$$\Rightarrow 8x - \frac{216}{x^2} = 0$$

$$\Rightarrow 8x^3 - 216 = 0$$

$$\Rightarrow 8x^3 = 216$$

$$\Rightarrow x^3 = 27$$

$$\Rightarrow x = \sqrt[3]{27}$$

$$\therefore \boldsymbol{x = 3}$$

(iii) $A = 4x^2 + \frac{216}{x}$

$when\ x = 3$

$\Rightarrow A = 4(3)^2 + \frac{216}{3}$

$\Rightarrow \boldsymbol{A = 108\ cm^2}$

$using\ \frac{dA}{dx} = 8x - 216x^{-2}$

$\Rightarrow \frac{d^2A}{dx^2} = 8 + 432x^{-3}$

$when\ x = 3$

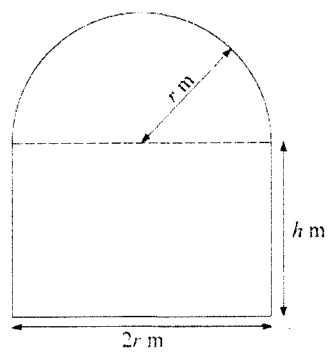
$\Rightarrow \frac{d^2A}{dx^2} = 8 + 432(3)^{-3}$

$\Rightarrow \frac{d^2A}{dx^2} = 24\ \ (positive)$

$\therefore$ ***The stationary value is a minimum***

Question (Cambridge, June 2004 qp.1)

8



The diagram shows a glass window consisting of a rectangle of height h m and width $2r$ m and a semicircle of radius r m. The perimeter of the window is 8 m.

(i) Express h in terms of r. [2]

(ii) Show that the area of the window, A m², is given by

$$A = 8r - 2r^2 - \frac{1}{2}\pi r^2.$$ [2]

Given that r can vary,

(iii) find the value of r for which A has a stationary value. [4]

(iv) determine whether this stationary value is a maximum or a minimum. [2]

Solution

(i). $Perimeter = 2h + 2r + \pi r$

$\Rightarrow 8 = 2h + 2r + \pi r$

$\Rightarrow 2h = 8 - 2r - \pi r$

$\therefore \boldsymbol{h} = \left(\frac{\mathbf{8 - 2r - \pi r}}{\mathbf{2}}\right)$

(ii). $A = area\ of\ rectangle + area\ of\ semi\ circle,$

$\Rightarrow A = 2rh + \frac{1}{2}\pi r^2,\ \ but\ \ h = \left(\frac{8 - 2r - \pi r}{2}\right)$

$\Rightarrow A = 2r\left[\frac{8 - 2r - \pi r}{2}\right] + \frac{1}{2}\pi r^2$

$\Rightarrow A = 8r - 2r^2 - \pi r^2 + \frac{1}{2}\pi r^2$

$\therefore \mathbf{A = 8r - 2r^2 - \frac{1}{2}\pi r^2}$ $\boldsymbol{(shown)}$

(iii). $A = 8r - 2r^2 - \frac{1}{2}\pi r^2$

$\Rightarrow \frac{dA}{dr} = 8 - 4r - \pi r$

$At\ a\ turning\ point,\ \ \frac{dA}{dr} = 0,$

$0 = 8 - 4r - \pi r$

$\Rightarrow 4r + \pi r = 8$

$\Rightarrow r(4 + \pi) = 8$

$\therefore \boldsymbol{r} = \left(\frac{\mathbf{8}}{\mathbf{4 + \pi}}\right)$

(iv). $\frac{dA}{dr} = 8 - 4r - \pi r$

$$\Rightarrow \frac{d^2A}{dr^2} = -4 - \pi$$

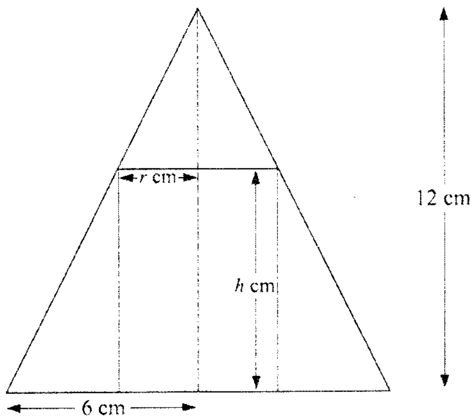
$$\Rightarrow \frac{d^2A}{dr^2} = -1(4 + \pi)[negative]$$

$\therefore$ ***The stationary value is a maximum***

Revision Questions on Differentiation and Mensuration

November 2005 qp.1 (Cambridge)

5



The diagram shows the cross-section of a hollow cone and a circular cylinder. The cone has radius 6 cm and height 12 cm, and the cylinder has radius r cm and height h cm. The cylinder just fits inside the cone with all of its upper edge touching the surface of the cone.

(i) Express h in terms of r and hence show that the volume, V cm^3, of the cylinder is given by

$$V = 12\pi r^2 - 2\pi r^3.$$ [3]

(ii) Given that r varies, find the stationary value of V. [4]

June 2010 qp.12 (Cambridge)

8 A solid rectangular block has a square base of side x cm. The height of the block is h cm and the total surface area of the block is 96 cm^2.

(i) Express h in terms of x and show that the volume, $V\text{ cm}^3$, of the block is given by

$$V = 24x - \tfrac{1}{2}x^3.$$ [3]

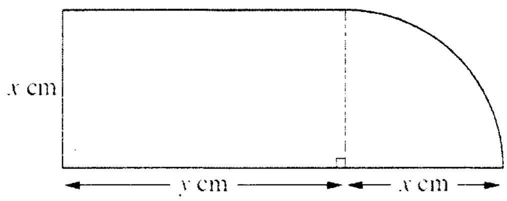
Given that x can vary,

(ii) find the stationary value of V, [3]

(iii) determine whether this stationary value is a maximum or a minimum. [2]

November 2010 qp.11 (Cambridge)

8



The diagram shows a metal plate consisting of a rectangle with sides x cm and y cm and a quarter-circle of radius x cm. The perimeter of the plate is 60 cm.

(i) Express y in terms of x. [2]

(ii) Show that the area of the plate, $A\text{ cm}^2$, is given by $A = 30x - x^2$. [2]

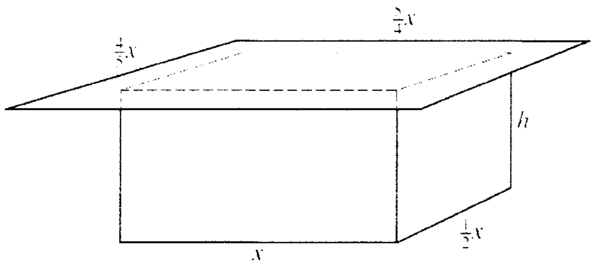
Given that x can vary,

(iii) find the value of x at which A is stationary, [2]

(iv) find this stationary value of A, and determine whether it is a maximum or a minimum value. [2]

November 2010 qp.12 (Cambridge)

10



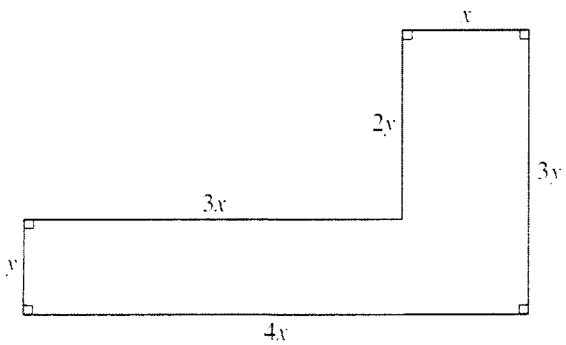
The diagram shows an open rectangular tank of height h metres covered with a lid. The base of the tank has sides of length x metres and $\frac{1}{2}x$ metres and the lid is a rectangle with sides of length $\frac{5}{4}x$ metres and $\frac{4}{5}x$ metres. When full the tank holds $4\ m^3$ of water. The material from which the tank is made is of negligible thickness. The external surface area of the tank together with the area of the top of the lid is $A\ m^2$.

(i) Express h in terms of x and hence show that $A = \frac{3}{2}x^2 + \dfrac{24}{x}$. [5]

(ii) Given that x can vary, find the value of x for which A is a minimum, showing clearly that A is a minimum and not a maximum. [5]

November 2011 qp.11 (Cambridge)

7



The diagram shows the dimensions in metres of an L-shaped garden. The perimeter of the garden is 48 m.

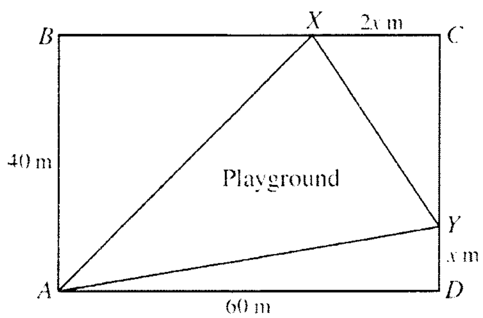
(i) Find an expression for y in terms of x. [1]

(ii) Given that the area of the garden is $A\ m^2$, show that $A = 48x - 8x^2$. [2]

(iii) Given that x can vary, find the maximum area of the garden, showing that this is a maximum value rather than a minimum value. [4]

November 2012 qp.12 (Cambridge)

3



The diagram shows a plan for a rectangular park $ABCD$, in which $AB = 40$ m and $AD = 60$ m. Points X and Y lie on BC and CD respectively and AX, XY and YA are paths that surround a triangular playground. The length of DY is x m and the length of XC is $2x$ m.

(i) Show that the area, A m^2, of the playground is given by

$$A = x^2 - 30x + 1200.$$ [2]

(ii) Given that x can vary, find the minimum area of the playground. [3]

Worked Examination Questions on Implicit Differentiation

Question (Cambridge, November 2012 qp.31)

7 The equation of a curve is $\ln(xy) - y^3 = 1$.

(i) Show that $\frac{dy}{dx} = \frac{y}{x(3y^3 - 1)}$. [4]

(ii) Find the coordinates of the point where the tangent to the curve is parallel to the y-axis, giving each coordinate correct to 3 significant figures. [4]

Solution

(i) *Given that,*

$$\ln(xy) - y^3 = 1$$

using laws of logarithms,

$$\ln x + \ln y - y^3 = 1$$

$$\Rightarrow \frac{1}{x} + \frac{1}{y}\frac{dy}{dx} - 3y^2\frac{dy}{dx} = 0$$

$$\Rightarrow \frac{1}{x} = 3y^2\frac{dy}{dx} - \frac{1}{y}\frac{dy}{dx}$$

$$\Rightarrow \frac{1}{x} = \frac{dy}{dx}\left(3y^2 - \frac{1}{y}\right)$$

$$\Rightarrow \frac{1}{x} = \frac{dy}{dx}\left(\frac{3y^3 - 1}{y}\right)$$

$$\Rightarrow \frac{1}{x} \div \left(\frac{3y^3 - 1}{y}\right) = \frac{dy}{dx}$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{x} \times \left(\frac{y}{3y^3 - 1}\right)$$

$$\therefore \frac{\boldsymbol{dy}}{\boldsymbol{dx}} = \frac{\boldsymbol{y}}{\boldsymbol{x(3y^3 - 1)}} (\boldsymbol{shown})$$

(ii) *A line is parallel to the y − axis if the denominator of the gradient function (i.e. dx) is equal to zero.*

In this case, $x(3y^3 - 1) = 0$

either $x = 0$ *or* $3y^3 - 1 = 0$

but $x = 0$ *is undefined*

$$\Rightarrow 3y^3 - 1 = 0$$

$$\Rightarrow y^3 = \frac{1}{3}$$

$$\Rightarrow y = \sqrt[3]{\frac{1}{3}}$$

$$\Rightarrow y = 0.693$$

by substituting y in the equation of the curve,

$$\ln x + \ln(0.693) - (0.693)^3 = 1$$

$$\Rightarrow \ln x = 1 + \frac{1}{3} - \ln(0.693)$$

$$\Rightarrow x = e^{\left(\frac{4}{3} - \ln 0.693\right)}$$

$$\Rightarrow x = 5.47$$

$\therefore$ ***the tangent is*** $\parallel$ ***to the y − axis at the point,*** $(\mathbf{5.47\,;0.693})$

Question (Zimsec, November 2004 qp.1)

7. A curve is given by the equation $3x^2 - 7xy + 4y^2 = 16$

(i). Show that $\frac{dy}{dx} = \frac{6x-7y}{7x-8y}$ [3]

(ii). Hence show that the gradient of the curve cannot be equal to 1 [3]

Solution

(i). $3x^2 - 7xy + 4y^2 = 16$

$$\Rightarrow 6x - 7\left[y(1) + x\left(\frac{dy}{dx}\right)\right] + 8y\left(\frac{dy}{dx}\right) = 0$$

$$\Rightarrow 6x - 7y - 7x\frac{dy}{dx} + 8y\frac{dy}{dx} = 0$$

$$\Rightarrow 6x - 7y = 7x\frac{dy}{dx} - 8y\frac{dy}{dx}$$

$$\Rightarrow 6x - 7y = \frac{dy}{dx}(7x - 8y)$$

$$\therefore \boldsymbol{\frac{dy}{dx} = \frac{6x - 7y}{7x - 8y}} \ \ (\boldsymbol{shown})$$

(ii). **NB:** if the gradient is not equal to one, the result from the gradient function will not satisfy the equation of the curve.

$$\frac{6x - 7y}{7x - 8y} = 1$$

$$\Rightarrow 6x - 7y = 7x - 8y$$

$$\Rightarrow 8y - 7y = 7x - 6x$$

$$\Rightarrow y = x$$

by substituting y in the original equation,

$$\Rightarrow 3x^2 - 7x(x) + 4(x)^2 \neq 16$$

$$\Rightarrow 3x^2 - 7x^2 + 4x^2 \neq 16$$

$$\Rightarrow 0 \neq 16 \ \ (condition\ satisfied)$$

$$\therefore \boldsymbol{the\ gradient\ cannot\ equal\ one}$$

Revision Questions on Implicit Differentiation

November 1996 qp.1 (Cambridge)

10. Find the gradient of the curve

$$y^3 - 2xy^2 + 3x^2 - 3 = 0$$

at the point (2, 3). [5]

June 2001 qp.1 (Cambridge)

8. A curve has equation $x^2 + xy + y^2 = 3$.

i. Show that $\frac{dy}{dx} = -\frac{2x+y}{x+2y}$. [3]

ii. Find the coordinates of the points on this curve where the gradient is zero. [4]

June 2008 qp.3 (Cambridge)

6 The equation of a curve is $xy(x + y) = 2a^3$, where a is a non-zero constant. Show that there is only one point on the curve at which the tangent is parallel to the x-axis, and find the coordinates of this point. [8]

November 2009 qp.32 (Cambridge)

3 The equation of a curve is $x^3 - x^2y - y^3 = 3$.

(i) Find $\frac{dy}{dx}$ in terms of x and y. [4]

(ii) Find the equation of the tangent to the curve at the point (2, 1), giving your answer in the form $ax + by + c = 0$. [2]

June 2010 qp.32 (Cambridge)

6 The equation of a curve is

$$x \ln y = 2x + 1.$$

(i) Show that $\frac{dy}{dx} = -\frac{y}{x^2}$. [4]

(ii) Find the equation of the tangent to the curve at the point where $y = 1$, giving your answer in the form $ax + by + c = 0$. [4]

June 2011 qp.31

5 The curve with equation

$$6e^{2x} + ke^{y} + e^{2y} = c,$$

where k and c are constants, passes through the point P with coordinates $(\ln 3, \ln 2)$.

(i) Show that $58 + 2k = c$. [2]

(ii) Given also that the gradient of the curve at P is -6, find the values of k and c. [5]

June 2012 qp.31

6 The equation of a curve is $3x^2 - 4xy + y^2 = 45$.

(i) Find the gradient of the curve at the point $(2, -3)$. [4]

(ii) Show that there are no points on the curve at which the gradient is 1. [3]

Worked Examination Questions on Parametric Differentiation

Question (Cambridge, November 2008 qp.3)

4 The parametric equations of a curve are

$$x = a(2\theta - \sin 2\theta), \qquad y = a(1 - \cos 2\theta).$$

Show that $\frac{dy}{dx} = \cot\theta$. [5]

Solution

Given that, $x = a(2\theta - \sin 2\theta)$ *and* $y = a(1 - \cos 2\theta)$

NB*: a is a constant, as such, it will not affect the differentiation process*

$x = a(2\theta - \sin 2\theta)$ $\qquad$ $y = a(1 - \cos 2\theta)$

$\frac{dx}{d\theta} = a(2 - 2\cos 2\theta)$ $\qquad$ $\frac{dy}{d\theta} = a(2\sin 2\theta)$

using the chain rule,

$$\frac{dy}{dx} = \frac{dy}{d\theta} \div \frac{dx}{d\theta}$$

$$\Rightarrow \frac{dy}{dx} = \frac{a(2\sin 2\theta)}{a(2 - 2\cos 2\theta)}$$

$$\Rightarrow \frac{dy}{dx} = \frac{2\sin 2\theta}{2(1 - \cos 2\theta)}$$

$$\Rightarrow \frac{dy}{dx} = \frac{(2\sin\theta\cos\theta)}{[1 - (1 - 2\sin^2\theta)]}$$

$$\Rightarrow \frac{dy}{dx} = \frac{2\sin\theta\cos\theta}{(1 - 1 + 2\sin^2\theta)}$$

$$\Rightarrow \frac{dy}{dx} = \frac{2\sin\theta\cos\theta}{2\sin^2\theta}$$

$$\Rightarrow \frac{dy}{dx} = \frac{\cos\theta}{\sin\theta}$$

$$\therefore \boldsymbol{\frac{dy}{dx} = \cot\theta \ (shown)}$$

Question (Zimsec, November 2010 qp.1)

11. Given that $x = \ln(3 + 2t)$ and $y = e^{3t^2}$

   (i). Find $\frac{dy}{dx}$ in terms of t [3]

   (ii). Show that the curve has only one turning point and write down the coordinates of the turning point [4]

Solution

(i). *Given that* $x = \ln(3 + 2t)$ *and* $y = e^{3t^2}$

$$\Rightarrow \frac{dx}{dt} = \frac{1}{(3 + 2t)} \times 2 \quad and \quad \frac{dy}{dt} = \left(e^{3t^2}\right) \times 6t$$

$$\Rightarrow \frac{dx}{dt} = \frac{2}{(3 + 2t)} \qquad \Rightarrow \frac{dy}{dt} = 6te^{3t^2}$$

using the chain rule,

$$\frac{dy}{dx} = \frac{dy}{dt} \div \frac{dx}{dt}$$

$$\Rightarrow \frac{dy}{dx} = 6te^{3t^2} \div \frac{2}{(3+2t)}$$

$$\Rightarrow \frac{dy}{dx} = 6te^{3t^2} \times \frac{(3+2t)}{2}$$

$$\therefore \boldsymbol{\frac{dy}{dx} = 3te^{3t^2}(3+2t)}$$

(ii). *At a turning point,* $\frac{dy}{dx} = 0$

$\Rightarrow 3te^{3t^2}(3+2t) = 0$

either $3t = 0$ *or* $e^{3t^2} = 0$ *or* $3 + 2t = 0$

$\Rightarrow t = 0$ *or* $3t^2 = \ln 0$ $\quad 2t = -3$

$\left[\text{when } t = -\frac{3}{2},\ x = \ln 0, x = undefined\right]$

~~$\Rightarrow t = 0$ *only because* $\ln 0$ *is undefined*~~

$\Rightarrow x = \ln[3 + 2(0)]$ *and* $y = e^{3(0)^2}$

$\Rightarrow x = \ln 3$ $\quad y = 1$

$\therefore$ ***the coordinates at the turning point are*** $(\mathbf{\ln 3};\ \mathbf{1})$

Question (Cambridge, November 2010 qp.33)

2 The parametric equations of a curve are

$$x = \frac{t}{2t+3}, \qquad y = e^{-2t}.$$

Find the gradient of the curve at the point for which $t = 0$. [5]

Solution

Given that, $x = \frac{t}{2t+3}$

using the quotient rule,

$$\frac{dx}{dt} = \frac{(2t+3)(1)-(t)(2)}{(2t+3)^2}$$

$$\Rightarrow \frac{dx}{dt} = \frac{2t+3-2t}{(2t+3)^2}$$

$$\Rightarrow \frac{dx}{dt} = \frac{3}{(2t+3)^2}$$

$and\ y = e^{-2t}$

$$\Rightarrow \frac{dy}{dt} = -2e^{-2t}$$

using the chain rule,

$$\frac{dy}{dx} = \frac{dy}{dt} \div \frac{dx}{dt}$$

$$\Rightarrow \frac{dy}{dx} = (-2e^{-2t}) \div \frac{3}{(2t+3)^2}$$

$$\Rightarrow \frac{dy}{dx} = -2e^{-2t} \times \frac{(2t+3)^2}{3}$$

$$\Rightarrow \frac{dy}{dx} = -\frac{2}{3}e^{-2t}(2t+3)^2$$

$when\ t = 0;\ \ \frac{dy}{dx} = -\frac{2}{3}e^{-2(0)}[2(0)+3]^2$

$$\Rightarrow \frac{dy}{dx} = -\frac{2}{3}(1)(3)^2$$

$$\therefore \boldsymbol{\frac{dy}{dx} = -6}$$

Revision Questions on Parametric Differentiation

June 2003 qp.1 (Zimsec)

6. Given that $x = \theta - \cos\theta$ and $y = 1 - \sin\theta$

Show that $\frac{dy}{dx} = -\sec\theta + \tan\theta$ [5]

June 2010 qp.1 (Zimsec)

9. Given that $x = \sin^2 t$ and $y = \cos 2t$, find $\frac{dy}{dx}$ in its simplest form. [4]

Hence or otherwise describe the shape of the graph of y against x. [1]

November 2001 qp.1 (Zimsec)

8. A curve has parametric equations

$$x = t^2 \text{ and } y = (2 - t)^2$$

where t takes all real values

i. Show that $\frac{dy}{dx} = 1 - \frac{2}{t}$ [3]
ii. Find the coordinates of the points on the curve where the tangent to the curve is
   a) Horizontal
   b) Vertical [3]

June 1994 qp.1 (Cambridge)

7. The parametric equations of a curve C are $x = t + e^t$, $y = t + e^{-t}$. Find $\frac{dy}{dx}$ in terms of t, and hence find the coordinates of the stationary point of C. [5]

June 2009 qp.3 (Cambridge)

6 The parametric equations of a curve are

$$x = a\cos^3 t, \quad y = a\sin^3 t,$$

where a is a positive constant and $0 < t < \frac{1}{2}\pi$.

(i) Express $\frac{dy}{dx}$ in terms of t. [3]

(ii) Show that the equation of the tangent to the curve at the point with parameter t is

$$x \sin t + y \cos t = a \sin t \cos t.$$ [3]

(iii) Hence show that, if this tangent meets the x-axis at X and the y-axis at Y, then the length of XY is always equal to a. [2]

June 2011 qp.32 (Cambridge)

5 The parametric equations of a curve are

$$x = \ln(\tan t), \quad y = \sin^2 t,$$

where $0 < t < \frac{1}{2}\pi$.

(i) Express $\frac{dy}{dx}$ in terms of t. [4]

(ii) Find the equation of the tangent to the curve at the point where $x = 0$. [3]

November 2011 qp.31 (Cambridge)

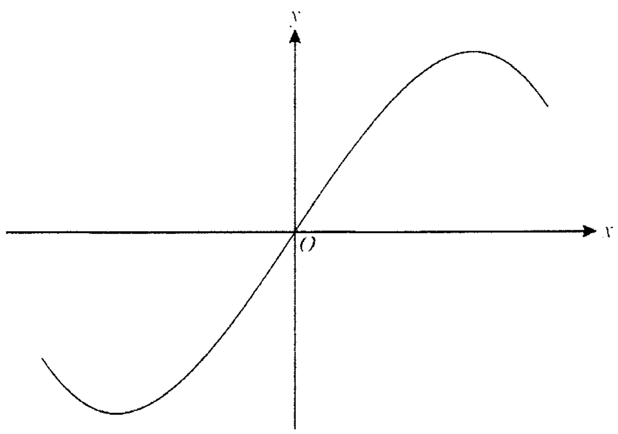
2 The parametric equations of a curve are

$$x = 3(1 + \sin^2 t), \quad y = 2\cos^3 t.$$

Find $\frac{dy}{dx}$ in terms of t, simplifying your answer as far as possible. [5]

November 2011 qp.33 (Cambridge)

8



The diagram shows the curve with parametric equations

$$x = \sin t + \cos t, \quad y = \sin^3 t + \cos^3 t,$$

for $\frac{1}{4}\pi < t < \frac{5}{4}\pi$.

(i) Show that $\frac{dy}{dx} = -3\sin t\cos t$. [3]

(ii) Find the gradient of the curve at the origin. [2]

(iii) Find the values of t for which the gradient of the curve is 1, giving your answers correct to 2 significant figures. [4]

June 2012 qp.33 (Cambridge)

3 The parametric equations of a curve are

$$x = \sin 2\theta - \theta, \quad y = \cos 2\theta + 2\sin\theta.$$

Show that $\frac{dy}{dx} = \frac{2\cos\theta}{1 + 2\sin\theta}$. [5]

November 2012 qp.33 (Cambridge)

3 The parametric equations of a curve are

$$x = \frac{4t}{2t+3}, \quad y = 2\ln(2t+3).$$

(i) Express $\frac{dy}{dx}$ in terms of t, simplifying your answer. [4]

(ii) Find the gradient of the curve at the point for which $x = 1$. [2]

Worked Examination Questions on Differentiation of Products

Question (Cambridge, November 2009 qp.31)

4 A curve has equation $y = e^{-3x}\tan x$. Find the x-coordinates of the stationary points on the curve in the interval $-\frac{1}{2}\pi < x < \frac{1}{2}\pi$. Give your answers correct to 3 decimal places. [6]

Solution

$At\ any\ statinary\ point, \frac{dy}{dx} = 0$

$Given\ that,\ \ y = e^{-3x}\tan x$

$\Rightarrow \frac{dy}{dx} = \tan x\,(-3e^{-3x}) + e^{-3x}(\sec^2 x)$

$\Rightarrow \frac{dy}{dx} = -3\tan x e^{-3x} + e^{-3x}\sec^2 x$

$\Rightarrow \frac{dy}{dx} = e^{-3x}(-3\tan x + \sec^2 x)$

$\Rightarrow e^{-3x}(-3\tan x + \sec^2 x) = 0$

$but\ \sec^2 x = 1 + \tan^2 x$

$\Rightarrow -3\tan x + 1 + \tan^2 x = 0$

$\Rightarrow \tan^2 x - 3\tan x + 1 = 0$

$let\ m = \tan x$

$\Rightarrow m^2 - 3m + 1 = 0$

$\Rightarrow m = 2.61803\ and\ 0.381966$

$Now,\ \ \tan x = 2.61803$ $\quad$ and $\quad$ $\tan x = 0.381966$

$\Rightarrow x = \tan^{-1}(2.61803)$ $\quad$ $\Rightarrow x = \tan^{-1}(0.381966)$

$\Rightarrow x = 1.206$ $\quad$ $\Rightarrow x = 0.365$

$\therefore \boldsymbol{x = 1.206\ and\ 0.365\ rad}$

Revision Questions on Differentiation of Products

June 2010 qp.1 (Zimsec)

2. Differentiate with respect to t

   i. $e^{-2t}\sin t,$
   ii. $\sec^2(3t - 100).$ [4]

November 1990 qp.1 (Cambridge)

9. (i) Differentiate $3u^2 \sin 5u$ with respect to u. [2]

June 2007 qp.3 (Cambridge)

3 The equation of a curve is $y = x \sin 2x$, where x is in radians. Find the equation of the tangent to the curve at the point where $x = \frac{1}{4}\pi$. [4]

November 2007 qp.3 (Cambridge)

4 The curve with equation $y = e^{-x} \sin x$ has one stationary point for which $0 \leqslant x \leqslant \pi$.

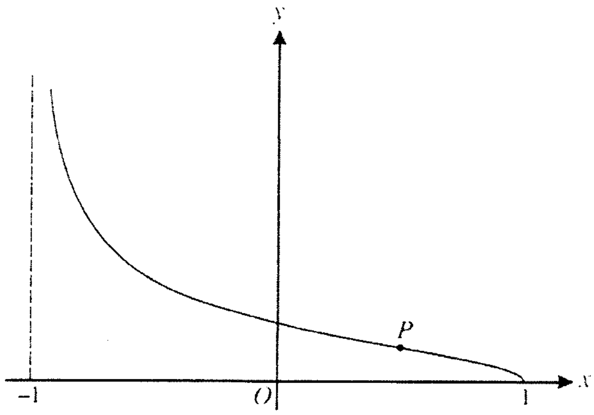
(i) Find the x-coordinate of this point. [4]

(ii) Determine whether this point is a maximum or a minimum point. [2]

Worked Examination Questions on Differentiation of Fractions

Question (Cambridge, June 2010 qp.31)

9



The diagram shows the curve $y = \sqrt{\left(\frac{1-x}{1+x}\right)}$.

(i) By first differentiating $\frac{1-x}{1+x}$, obtain an expression for $\frac{dy}{dx}$ in terms of x. Hence show that the gradient of the normal to the curve at the point (x, y) is $(1 + x)\sqrt{(1 - x^2)}$. [5]

(ii) The gradient of the normal to the curve has its maximum value at the point P shown in the diagram. Find, by differentiation, the x-coordinate of P. [4]

Solution

(i) $\quad Given\ that,\ y = \sqrt{\left(\frac{1-x}{1+x}\right)}$

$$let\ f(x) = \frac{1-x}{1+x},\ the\ value\ under\ the\ square\ root\ sign$$

using the quotient rule,

$$f'(x) = \frac{(1+x)(-1) - (1-x)(1)}{(1+x)^2}$$

$$\Rightarrow f'(x) = \frac{-1-x-1+x}{(1+x)^2}$$

$$\Rightarrow f'(x) = \frac{-2}{(1+x)^2}$$

$$Now,\ y = \left(\frac{1-x}{1+x}\right)^{\frac{1}{2}}$$

using differentiation of complex algebraic expressions,

$$\Rightarrow \frac{dy}{dx} = \frac{1}{2}\left(\frac{1-x}{1+x}\right)^{-\frac{1}{2}} \times f'(x)$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{2}\left(\frac{1-x}{1+x}\right)^{-\frac{1}{2}} \times \frac{-2}{(1+x)^2}$$

$$\Rightarrow \frac{dy}{dx} = \frac{-1}{(1+x)^2}\left(\frac{1-x}{1+x}\right)^{-\frac{1}{2}}$$

using laws of indices,

$$\Rightarrow \frac{dy}{dx} = \frac{-1}{(1+x)^2}\left(\frac{1+x}{1-x}\right)^{\frac{1}{2}}$$

$$\Rightarrow \frac{dy}{dx} = \frac{-1}{(1+x)^2} \times \frac{(1+x)^{\frac{1}{2}}}{(1-x)^{\frac{1}{2}}}$$

$$\Rightarrow \frac{dy}{dx} = \frac{-(1+x)^{\frac{1}{2}}}{(1+x)^2(1-x)^{\frac{1}{2}}}$$

$$\Rightarrow \frac{dy}{dx} = \frac{-(1+x)^{-\frac{3}{2}}}{(1-x)^{\frac{1}{2}}}$$

using the fact that,

$m_1 \times m_2 = -1$

where m_1 *is the gradient of the tangent and* m_2 *is the gradient of the normal,*

$$\Rightarrow \frac{-(1+x)^{-\frac{3}{2}}}{(1-x)^{\frac{1}{2}}} \times m_2 = -1$$

$$\Rightarrow -(1+x)^{-\frac{3}{2}} \times m_2 = -(1-x)^{\frac{1}{2}}$$

$$\Rightarrow m_2 = \frac{(1-x)^{\frac{1}{2}}}{(1+x)^{-\frac{3}{2}}}$$

$$\Rightarrow m_2 = (1-x)^{\frac{1}{2}}(1+x)^{\frac{3}{2}}$$

$$\Rightarrow m_2 = [(1-x)(1+x)^3]^{\frac{1}{2}}$$

$$\Rightarrow m_2 = [(1-x)(1+x)(1+x)^2]^{\frac{1}{2}}$$

$$\Rightarrow m_2 = [(1-x^2)(1+x)^2]^{\frac{1}{2}}$$

$$\Rightarrow m_2 = (1-x^2)^{\frac{1}{2}}(1+x)$$

$$\therefore \boldsymbol{m_2 = (1+x)\sqrt{(1-x^2)}}$$

(ii) *At any maximum or minimum point, gradient is zero.*

In this case, given that,

$$m_2 = (1+x)\sqrt{(1-x^2)}$$

$$\Rightarrow m_2 = (1+x)(1-x^2)^{\frac{1}{2}}$$

$$\Rightarrow \frac{dm_2}{dx} = (1-x^2)^{\frac{1}{2}}(1) + (1+x)\left[\left(\frac{1}{2}\right)(1-x^2)^{-\frac{1}{2}}\right](-2x)$$

$$\Rightarrow \frac{dm_2}{dx} = (1-x^2)^{\frac{1}{2}} + (1+x)\left[-x(1-x^2)^{-\frac{1}{2}}\right]$$

$$\Rightarrow \frac{dm_2}{dx} = (1-x^2)^{\frac{1}{2}} - \frac{x(1+x)}{(1-x^2)^{\frac{1}{2}}}$$

$$\Rightarrow (1-x^2)^{\frac{1}{2}} - \frac{x(1+x)}{(1-x^2)^{\frac{1}{2}}} = 0$$

$$\Rightarrow (1-x^2)^{\frac{1}{2}} = \frac{x(1+x)}{(1-x^2)^{\frac{1}{2}}}$$

$$\Rightarrow (1-x^2) = x(1+x)$$

$$\Rightarrow 1 - x^2 = x + x^2$$

$$\Rightarrow 2x^2 + x - 1 = 0$$

$$\Rightarrow x = 0.5 \ and - 1$$

$$\therefore \boldsymbol{x = 0.5 \ only}$$

Question (Cambridge, June 2012 qp.33)

4 The curve with equation $y = \frac{e^{2x}}{x^3}$ has one stationary point.

(i) Find the x-coordinate of this point. [4]

(ii) Determine whether this point is a maximum or a minimum point. [2]

Solution

(i) *At a turning point, the gradient is zero.*

$$Given\ that,\ \ y = \frac{e^{2x}}{x^3}$$

$$\Rightarrow \frac{dy}{dx} = \frac{x^3(2e^{2x}) - (e^{2x})(3x^2)}{(x^3)^2}$$

$$\Rightarrow \frac{dy}{dx} = \frac{2x^3e^{2x} - 3x^2e^{2x}}{x^6}$$

$$\Rightarrow \frac{dy}{dx} = \frac{x^2e^{2x}(2x-3)}{x^6}$$

$$\Rightarrow \frac{x^2e^{2x}(2x-3)}{x^6} = 0$$

$\Rightarrow x^2e^{2x}(2x-3)=0$

$\Rightarrow 2x-3=0$

$\therefore x=\frac{3}{2}$

(ii) *Using the nature of gradient, pick any two values of x where one of the values is lightly below $\frac{3}{2}$ and the other slightly above $\frac{3}{2}$. Substitute these values in the gradient function and determine the nature of the gradient: –*

$$\frac{dy}{dx}=\frac{x^2e^{2x}(2x-3)}{x^6}$$

Table 10.2

	$x<3/2$	$x=3/2$	$x>3/2$
dy/dx	$-$	0	$+$
slope	\	—	/

From Table 10.2,

when $x=\frac{3}{2}$, *the turning point is a minimum.*

Revision Questions on Differentiation of Fractions

November 1990 qp.1 (Cambridge)

9. (ii). Differentiate $\frac{3t}{t^2+1}$ with respect to t. [2]

November 1995 qp.1 (Cambridge)

6. Use differentiation to find the x-coordinates of the points at which the graph of $y=\frac{x^2}{2x-1}$ has either a maximum or minimum and distinguish between them. [7]

November 2008 qp.3 (Cambridge)

3 The curve $y = \dfrac{e^x}{\cos x}$, for $-\frac{1}{2}\pi < x < \frac{1}{2}\pi$, has one stationary point. Find the x-coordinate of this point. [5]

June 2011 qp.31 (Cambridge)

2 Find $\dfrac{dy}{dx}$ in each of the following cases:

(i) $y = \ln(1 + \sin 2x)$, [2]

(ii) $y = \dfrac{\tan x}{x}$. [2]

June 2011 qp.33 (Cambridge)

2 The curve $y = \dfrac{\ln x}{x^3}$ has one stationary point. Find the x-coordinate of this point. [4]

November 2011 qp.33 (Cambridge)

2 The equation of a curve is $y = \dfrac{e^{2x}}{1 + e^{2x}}$. Show that the gradient of the curve at the point for which $x = \ln 3$ is $\frac{9}{50}$. [4]

<u>Worked Examination Question on Rate of change</u>

Question (Cambridge, November 2006 qp.1)

8 The equation of a curve is $y = \dfrac{6}{5 - 2x}$.

(i) Calculate the gradient of the curve at the point where $x = 1$. [3]

(ii) A point with coordinates (x, y) moves along the curve in such a way that the rate of increase of y has a constant value of 0.02 units per second. Find the rate of increase of x when $x = 1$. [2]

Solution

(i) $y = \frac{6}{5-2x} \equiv 6(5 - 2x)^{-1}$

$$\Rightarrow \frac{dy}{dx} = (-6)(-2)(5 - 2x)^{-2}$$

$$\Rightarrow \frac{dy}{dx} = \frac{12}{(5-2x)^2}$$

$when\ x = 1$

$$\Rightarrow \frac{dy}{dx} = \frac{12}{[5-2(1)]^2}$$

$$\Rightarrow \frac{dy}{dx} = \frac{12}{9}$$

$$\therefore \frac{dy}{dx} = \frac{4}{3}$$

(ii) $Given\ that\ \frac{dy}{dx} = \frac{4}{3},\ \frac{dy}{dt} = 0.02,\ \frac{dx}{dt} = ?$

$$\Rightarrow \frac{dx}{dt} = \frac{dy}{dt} \times \frac{dx}{dy}$$

$$\Rightarrow \frac{dx}{dt} = \frac{3}{4} \times 0.02$$

$$\therefore \frac{dx}{dt} = 0.015\ units\ per\ second$$

Revision Questions on Rate of Change

November 2006 qp.1 (Zimsec)

10. A right circular metallic cylinder has radius r centimetres and height h centimetres. Given that the volume of the cylinder is 200cm^3, write down a formula for h in terms of r. [2]

The cylinder is melted and then rolled in a machine to form another cylinder. Given that at the time the radius is 4 cm the height h is increasing at a rate of 0.4cm/minute, find the rate of change of the radius giving your answer to 4 decimal places. [4]

The total surface area of the cylinder, S, is given by $S = 2\pi r^2 + \frac{400}{r}$

Find

(i). $\frac{dS}{dr}$ [1]

(ii). The rate at which the total surface area of the cylinder is changing when $r = 4$ cm and $\frac{dh}{dt} = 0.4$ cm. Comment on the result. [3]

June 2012 qp.12 (Cambridge)

2 The equation of a curve is $y = 4\sqrt{x} + \frac{2}{\sqrt{x}}$.

(i) Obtain an expression for $\frac{dy}{dx}$. [3]

(ii) A point is moving along the curve in such a way that the x-coordinate is increasing at a constant rate of 0.12 units per second. Find the rate of change of the y-coordinate when $x = 4$. [2]

June 2012 qp.11 (Cambridge)

4 A watermelon is assumed to be spherical in shape while it is growing. Its mass, M kg, and radius, r cm, are related by the formula $M = kr^3$, where k is a constant. It is also assumed that the radius is increasing at a constant rate of 0.1 centimetres per day. On a particular day the radius is 10 cm and the mass is 3.2 kg. Find the value of k and the rate at which the mass is increasing on this day. [5]

June 2011qp. 11 (Cambridge)

2 The volume of a spherical balloon is increasing at a constant rate of 50 cm^3 per second. Find the rate of increase of the radius when the radius is 10 cm. [Volume of a sphere = $\frac{4}{3}\pi r^3$.] [4]

<u>Worked Examination Question on Maclaurin Series</u>

Question (Zimsec, November 2002 qp.1)

11. Given that $f(x) = \ln(2 - 3x)$,

(i). Show that $f''(x) = \frac{-9}{(2-3x)^2}$, [2]

(ii). Find $f'''(x)$. [1]

Hence obtain a Maclaurin's series expansion for $\ln(2 - 3x)$ neglecting terms in x^4 and higher degree. [3]

State clearly what circumstance might justify neglecting these terms. [1]

Solution

(i). $Given\ that\ f(x) = \ln(2-3x)$

$$\Rightarrow f'(x) = \frac{1}{(2-3x)} \times -3$$

$$\Rightarrow f'(x) = \frac{-3}{(2-3x)}$$

$$\Rightarrow f'(x) = -3(2-3x)^{-1}$$

$$\Rightarrow f''(x) = 3(2-3x)^{-2} \times -3$$

$$\therefore \boldsymbol{f''(x) = \frac{-9}{(2-3x)^2}\ (shown)}$$

(ii). $Since\ f''(x) = -9(2-3x)^{-2}$

$$\Rightarrow f'''(x) = 18(2-3x)^{-3} \times -3$$

$$\therefore \boldsymbol{f'''(x) = \frac{-54}{(2-3x)^3}}$$

$using\ Maclaurin'series,$

$$f(x) = f(0) + \frac{xf'(0)}{1!} + \frac{x^2f''(0)}{2!} + \frac{x^3f'''(0)}{3!}$$

$$where\ f(0) = \ln 2;\quad f'(0) = -\frac{3}{2};\ f''(0) = -\frac{9}{4};\ f'''(0) = -\frac{27}{4}$$

$$\Rightarrow\ f(x) = \ln 2 + \frac{x\left(-\frac{3}{2}\right)}{1!} + \frac{x^2\left(-\frac{9}{4}\right)}{2!} + \frac{x^3\left(-\frac{27}{4}\right)}{3!}$$

$$\therefore \boldsymbol{f(x) = \ln 2 - \frac{3}{2}x - \frac{9}{8}x^2 - \frac{9}{8}x^3}$$

Terms in x^4 and higher degree are negligible because the series is convergent.

Revision Questions on Maclaurin Series

June 2010 qp.1 (Zimsec)

8. Solve $\frac{dy}{dx} = xy$ given $x = 0$ when $y = 1$.

Use the series expansion of e^x to write down the first two terms of Maclaurin series for the solution. [5]

June 2001 qp.1 (Zimsec)

15. It is given that $f(x) = e^x \cos(x\sqrt{3})$.

(i). Find $f'(x)$. [2]

(ii). Show that $f''(x) = -2e^x[\cos(x\sqrt{3}) + (\sqrt{3})\sin(x\sqrt{3})]$. [2]

(iii). Show that $f'''(x) = -8f(x)$. [2]

(iv). Hence obtain the Maclaurin series for $f(x)$ up to and including the term in x^3.[3]

Use the series for $\cos x$ given in the List of Formulae to verify that

$$\cos(x\sqrt{3}) = 1 - \frac{3x^2}{2} + \frac{3x^4}{8} - \cdots$$ [1]

By multiplying the series for e^x and the series for $\cos(x\sqrt{3})$, up to and including the terms in x^4, verify your answer to part (iv), and find the term in x^4 in the series expansion of $f(x)$. [1]

June 2003 qp.1 (Zimsec)

18. (a) Given that $y = e^{-3x}\sin 2x$ and

$$\frac{dy}{dx} = 2e^{-3x}\cos 2x - 3y$$

Show that $\frac{d^2y}{dx^2} = -13y - \frac{6dy}{dx}$. [4]

Hence find Maclaurin series expansion for y, in ascending powers of x, up to and including the term in x^3. [6]

November 2003 qp.1 (Zimsec)

11. Given that $y = \ln\left(e^{-x} - \frac{1}{2}\right)$, show that $\frac{dy}{dx} = -\frac{1}{2}(e^{-y} + 2)$.

By finding the second derivative, or otherwise, show that the series expansion of y in ascending powers of x, up to and including the term in x^2, is

$$-\ln 2 - 2x - x^2$$ [7]

November 2001 qp.1 (Zimsec)

11. Given that $y = e^{-x}\sin x$, find an expression for $\frac{dy}{dx}$ and deduce that

$$\frac{d^2y}{dx^2} = -2e^{-x}\cos x$$ [4]

By differentiating again, or otherwise, obtain the first three non-zero terms of the Maclaurin series for y. [4]

November 1996 qp.1 (Cambridge)

19. (b) Given that $y = \tan\left(3x + \frac{1}{4}\pi\right)$, show that

(i). $\frac{dy}{dx} = 3(1 + y^2)$, [1]

(ii). $\frac{d^3y}{dx^3} = 6y\left(\frac{d^2y}{dx^2}\right) + 6\left(\frac{dy}{dx}\right)^2$. [2]

Find Maclaurin's series for $\tan\left(3x + \frac{1}{4}\pi\right)$ up to and including the term in x^3. [4]

Chapter Eleven: Integration

"Wisdom is knowing what path to take...integrity is taking it."

– Unknown

Integration is a building up process which is the extreme opposite of differentiation. Integration is the reverse process of differentiation. This topic calls for critical analysis before drawing up a conclusion. Integration, I, is centred on answering the following questions:

- Is I a standard?
- Is I a product?
- Is I a fraction?

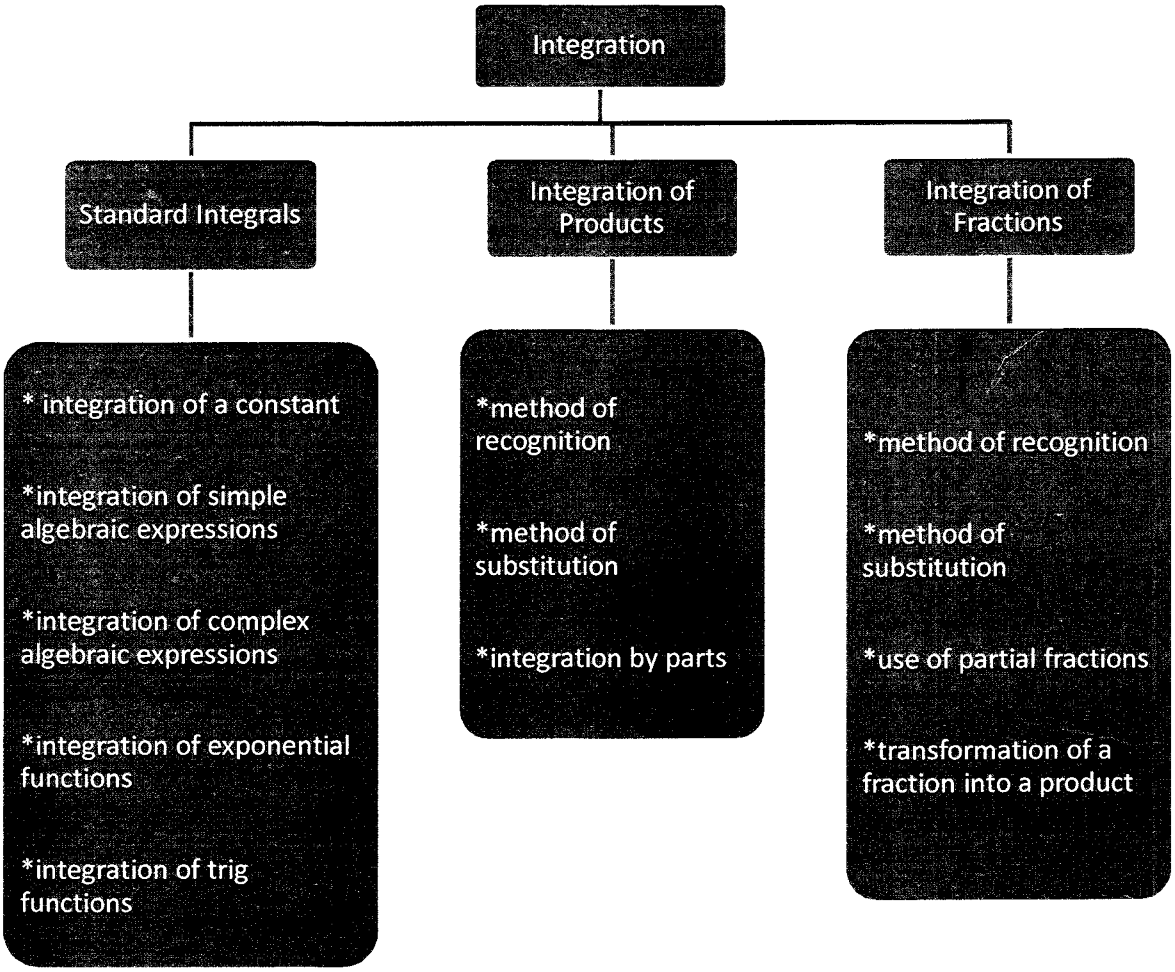


Fig. 11.1

Concepts in Integration

Concepts in integration are grouped into three sections: Standard Integrals; Integration of Products; and Integration of Fractions as shown in Fig. 11.1.

I. Standard Integrals

1. Integration of a Constant

- To build-up a constant, one has to attach a variable in question. The general formula states that:

$$\int a\,dx = ax + c$$

Where:

a is a constant,

dx means with respect to x,

c is a constant of integration.

For example,

- $\int 3\,dx = 3x + c$
- $\int -7\,dy = -7y + c$

2. Integration of Simple Algebraic Expressions

To integrate an algebraic expression, one has to add one to the existing power and divide the term in question by the new power. The general formula states that:

$$\int ax^n dx = \frac{ax^{(n+1)}}{(n+1)} + c$$

For example,

- $\int 3x^3 - 4x^2 + 10x - 2\,dx = \frac{3x^4}{4} - \frac{4x^3}{3} + \frac{10x^2}{2} - 2x + c$

$$\therefore \int 3x^3 - 4x^2 + 10x - 2\,dx = \frac{3}{4}x^4 - \frac{4}{3}x^3 + 5x^2 - 2x + c$$

- $\int 4x^2 - 6x + 1\ dx = \frac{4x^3}{3} - \frac{6x^2}{2} + x + c$

$$\therefore \int 4x^2 - 6x + 1\ dx = \frac{4}{3}x^3 - 3x^2 + x + c$$

3. Integration of Complex Algebraic Expressions

- An expression is said to be complex if there is a power affecting two or more terms. In such a case, one has to use $\frac{P}{A}$ strategy to integrate the expression.

  $\frac{P}{A}$ stands for:

$$\frac{P - integral\ with\ respect\ to\ \mathbf{power}}{A - derivative\ with\ respect\ to\ the\ \mathbf{algebraic}\ expression}$$

- In layman's terms, the expression has to undergo the normal integration process and immediately multiply the denominator by the derivative of the expression bound by a pair of brackets.
- The general formula states that:

$$\int [f(x)]^n dx = \frac{[f(x)]^{(n+1)}}{(n+1)\times f'(x)} + c$$

For example,

- $let\ I = \int (2x+7)^3 dx$

  $\Rightarrow I = \dfrac{(2x+7)^4}{4\times 2} + c$

  $\therefore \boldsymbol{I = \dfrac{(2x+7)^4}{8} + c}$

- $let\ I = \int \frac{21}{(3x-3)^5} dx$

  $\Rightarrow I = \displaystyle\int 21(3x-3)^{-5} dx$

  $\Rightarrow I = \dfrac{21(3x-3)^{-4}}{-4\times 3} + c$

  $\Rightarrow I = \dfrac{-7(3x-3)^{-4}}{4} + c$

  $\therefore \boldsymbol{I = \dfrac{-7}{4(3x-3)^4} + c}$

- $let\ I = \int \sqrt{2x^2+7} dx$

  $\Rightarrow I = \displaystyle\int (2x^2+7)^{\frac{1}{2}} dx$

$$\Rightarrow I = \frac{(2x^2+7)^{\frac{3}{2}}}{\frac{3}{2} \times 4x} + c$$

$$\therefore \boldsymbol{I = \frac{(2x^2+7)^{\frac{3}{2}}}{6x} + c}$$

- $let\ I = \int \frac{1}{\sqrt{2x+1}} dx$

$$\Rightarrow I = \int \frac{1}{(2x+1)^{\frac{1}{2}}} dx$$

$$\Rightarrow I = \int (2x+1)^{\frac{-1}{2}} dx$$

$$\Rightarrow I = \frac{(2x+1)^{\frac{1}{2}}}{\frac{1}{2} \times 2} + c$$

$$\therefore \boldsymbol{I = (2x+1)^{\frac{1}{2}} + c}$$

NB: Integration is either definite or indefinite. Definite integration is used to describe all problems with limits and indefinite is used to describe all problems without limits. In cases where limits are not given, a constant of integration has to be added to the answer and in cases where limits are given, the result has to be evaluated.

The definite integral is given by substituting the upper limit in the result and reducing it by the value given when the lower limit has been substituted in place of x.

For example,

Indefinite integration:

$$let\ I = \int 2x^2 + 3x\ dx$$

$$I = \frac{2x^3}{3} + \frac{3x^2}{2} + c$$

Definite integration:

$$I = \int_1^2 2x^2 + 3x\,dx$$

$$I = \left[\frac{2x^3}{3} + \frac{3x^2}{2}\right]_1^2$$

$$I = \left[\frac{2(2)^3}{3} + \frac{3(2)^2}{2}\right] - \left[\frac{2(1)^2}{3} + \frac{3(1)^2}{2}\right]$$

$$I \Rightarrow \left(\frac{34}{3}\right) - \left(\frac{13}{6}\right)$$

$$\therefore \boldsymbol{I} = \frac{\mathbf{55}}{\mathbf{6}}$$

4. Integration of Exponential Functions

To integrate an exponential function, quote the function and divide it by its inner derivative. The general formula states that

$$\int \boldsymbol{e^{f(x)}\,dx = \frac{1}{f'(x)}e^{f(x)} + c}$$

For example,

- $If\;\; y = e^{ax}$

$$\int y\,dx = \int e^{ax}\,dx$$

$$\Rightarrow \int y\,dx = \frac{e^{ax}}{a} + c$$

$$\therefore \int \boldsymbol{y\;dx = \frac{1}{a}e^{ax} + c}$$

- Given that, $y = 7e^{3x+2}$

$$\int y\,dx = \int 7e^{(3x+2)}\,dx$$

$$\Rightarrow \int y\,dx = \frac{7e^{(3x+2)}}{3} + c$$

$$\therefore \int \boldsymbol{y\,dx = \frac{7}{3}e^{(3x+2)} + c}$$

➢ Given that, $y = e^{3x^2}$

$$\Rightarrow \int y\,dx = \frac{e^{3x^2}}{6x} + c$$

$$\therefore \int \boldsymbol{y\,dx} = \frac{\boldsymbol{e^{3x^2}}}{\boldsymbol{6x}} + \boldsymbol{c}$$

5. Integration of Trig Functions

Trig functions make use of standard results as shown by the table below:

Function	Integral
$\sin\theta$	$-\cos\theta$
$\cos\theta$	$\sin\theta$
$\tan\theta$	$\ln(\sec\theta)$
$\operatorname{cosec}\theta$	$-\ln(\operatorname{cosec}\theta + \cot\theta)$
$\sec\theta$	$\ln(\sec\theta + \tan\theta)$
$\cot\theta$	$\ln(\sin\theta)$
$\sec\theta\tan\theta$	$\sec\theta$
$\sec^2\theta$	$\tan\theta$
$\operatorname{cosec}\theta\cot\theta$	$-\operatorname{cosec}\theta$
$\operatorname{cosec}^2\theta$	$-\cot\theta$

Table 11.1

NB: When integrating trig functions, quote the result and divide the standard result by its inner derivative.

Example 1

$$\int \sin(4\theta + 2)\ d\theta = -\frac{\cos(4\theta + 2)}{4} + c$$

Example 2

$$\int \cot 3\theta\, d\theta = \frac{\ln(\sin 3\theta)}{3} + c$$

Example 3

$$\int \sec 2\theta \tan 2\theta\, d\theta = \frac{\sec 2\theta}{2} + c$$

II. Integration of Products

1. Method of Recognition

This method is based on the notion that if the first term is a perfect derivative of the second term, quote the second term as the result.

$$\int \boldsymbol{f'(x)f(x)\,dx = f(x) + c}$$

For example,

$$\int \cos\theta \sin\theta\, d\theta = \sin\theta + c\ \boldsymbol{because\ \cos\theta\ \ is\ a\ perfect\ derivative\ of\ \sin\theta}.$$

2. Method of Substitution

This technique can only be used when the media for substitution is given (see *integration of fractions* section).

3. Integration by Parts

If the use of any of the two methods outlined immediately above is not feasible, the only workable method is integration by parts which states that:

$$\int \boldsymbol{u\frac{dv}{dx}dx = uv - \int v\frac{du}{dx}\,dx}$$

The LHS represents the question and the RHS represents the solution.

The term identified as u has to be differentiated to give $\frac{du}{dx}$ and the term identified as $\frac{dv}{dx}$ should be integrated to give v.

There are two possible conditions that lead to the choice of a term to differentiate:

- The term should be logarithmic in nature, for example $\ln x$ or;
- The term should have a potential to breakdown into a constant.

Example 1

Given that,

$$I = \int 2xe^{3x}\,dx$$

$let\ u = 2x$ $\qquad and \qquad$ $\frac{dv}{dx} = e^{3x}$

$\frac{du}{dx} = 2$ $\qquad\qquad$ $v = \frac{e^{3x}}{3}$

$$\Rightarrow I = 2x\left(\frac{e^{3x}}{3}\right) - \int 2\left(\frac{e^{3x}}{3}\right)dx$$

$$\Rightarrow I = \frac{2}{3}xe^{3x} - \int \frac{2}{3}e^{3x}\,dx$$

$$\Rightarrow I = \frac{2}{3}xe^{3x} - \frac{2}{9}e^{3x} + c$$

$$\therefore \boldsymbol{I = \frac{2}{3}e^{3x}\left(x - \frac{1}{3}\right) + c}$$

Example 2

Given that,

$$I = \int 3x^2 \ln x\,dx$$

$let\ u = \ln x$ $\qquad and \qquad$ $\frac{dv}{dx} = 3x^2$

$\frac{du}{dx} = \frac{1}{x}$ $\qquad\qquad$ $v = x^3$

$$\Rightarrow I = x^3 \ln x - \int x^3 \left(\frac{1}{x}\right) dx$$

$$\Rightarrow I = x^3 \ln x - \int x^2\, dx$$

$$\Rightarrow I = x^3 \ln x - \frac{x^3}{3} + c$$

$$\therefore \boldsymbol{I = x^3 \left(\ln x - \frac{1}{3}\right) + c}$$

NB: in cases where integration by parts is leading to another product that cannot be integrated with ease, use further integration by parts to solve the problem and substitute the results in the solution side of the original question (see Solution for *June 2010 qp.32 q.2 page 207)*.

III. <u>Integration of Fractions</u>

1. Method of Recognition

Recognize the natural logarithm of the denominator $[\ln(denominator)]$ as the result if the numerator is a perfect derivative of the denominator.

The general rule states that:

$$\boldsymbol{\int \frac{f'(x)}{f(x)} dx = \ln[f(x)] + c}$$

For example, given that,

$$I = \int \frac{2}{2x-3} dx$$

$\therefore \boldsymbol{I = \ln(2x - 3) + c}$

- **NB:** in the case where the numerator is not a perfect, but has a bias towards, the derivative of the denominator, adjust it into a perfect derivative and use method of recognition.

  If a special multiplier has been introduced to manipulate the numerator, a compensating move has to be considered to retain the status quo. For instance, $\frac{1}{2} \times 2 = 1$, which retains the original function and maintains the status quo.

For example, given that,

$$I = \int \frac{3x}{3x^2 + 1} dx$$

$$\Rightarrow I = \frac{1}{2} \int \frac{2 \times 3x}{3x^2 + 1} dx$$

$$\therefore \boldsymbol{I = \frac{1}{2} \ln(3x^2 + 1) + c}$$

2. **Method of Substitution**

   This technique is used to transform a given variable into a new variable using the media of substitution as specified in the question. When using the method of substitution, one has to consider the following steps:

   - Change the differential coefficient by way of differentiating the media for substitution. Express the differential coefficient to be eliminated as subject of the formula.
   - Change the limits, if any exist, using the given media of substitution.
   - Substitute the new limits and differential coefficient in the original question and simplify the problem to the specified conclusion. In cases where trig functions are involved, use relevant trig identities as far as possible to simplify the expression.

Example 1

J01/P1/Q16 (iii)

Use the substitution $s = \sin x$ to find the exact value of $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \sin^2 x \cos^3 x \, dx$ [6]

$$Let, \ I = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \sin^2 x \cos^3 x \, dx$$

- Change of differential coefficient

  $$s = \sin x$$

  $$\Rightarrow \frac{ds}{dx} = \cos x$$

$$\Rightarrow ds = \cos x \, dx$$
$$\Rightarrow dx = \frac{ds}{\cos x}$$

- Change of limits

Upper limit
$$s = \sin\frac{\pi}{3}$$
$$s = \frac{\sqrt{3}}{2}$$

Lower limit
$$s = \sin\frac{\pi}{6}$$
$$s = \frac{1}{2}$$

- Substitution of results in I

$$I = \int_{\frac{1}{2}}^{\frac{\sqrt{3}}{2}} s^2 \cos^3 x \cdot \frac{ds}{\cos x}$$
$$\Rightarrow I = \int_{\frac{1}{2}}^{\frac{\sqrt{3}}{2}} s^2 \cos^2 x \, ds$$
$$\Rightarrow I = \int_{\frac{1}{2}}^{\frac{\sqrt{3}}{2}} s^2 \left(1 - s^2\right) ds$$
$$\Rightarrow I = \int_{\frac{1}{2}}^{\frac{\sqrt{3}}{2}} s^2 - s^4 \, ds$$
$$\Rightarrow I = \left[\frac{s^3}{3} - \frac{s^5}{5}\right]_{\frac{1}{2}}^{\frac{\sqrt{3}}{2}}$$
$$\Rightarrow I = \left[\frac{\left(\frac{\sqrt{3}}{2}\right)^3}{3} - \frac{\left(\frac{\sqrt{3}}{2}\right)^5}{5}\right] - \left[\frac{\left(\frac{1}{2}\right)^3}{3} - \frac{\left(\frac{1}{2}\right)^5}{5}\right]$$
$$\Rightarrow I = \frac{20\sqrt{3} - 9\sqrt{3}}{160} - \frac{17}{480}$$
$$\Rightarrow I = \frac{11\sqrt{3}}{160} - \frac{17}{480}$$
$$\therefore \boldsymbol{I} = \left(\frac{\mathbf{33\sqrt{3} - 17}}{\mathbf{480}}\right)$$

Example 2

J91/P1/Q10

By means of the substitution $u = 1 + \sqrt{x}$ or otherwise find $\int \frac{1}{1+\sqrt{x}}\,dx$ giving your answer in terms of x **[5]**

- Change of differential coefficient

  $u = 1 + \sqrt{x}$

  $\Rightarrow u = 1 + x^{\frac{1}{2}}$

  $\Rightarrow \frac{du}{dx} = \frac{1}{2}x^{-\frac{1}{2}}$

  $\Rightarrow \frac{du}{dx} = \frac{1}{2\sqrt{x}}$

  $\Rightarrow dx = 2\sqrt{x}\,du$

- Change of limits

  Not relevant because there are no limits

- Substitution of the results in the original question

$$Let,\ \ I = \int \frac{1}{1+\sqrt{x}}\,dx$$
$$\Rightarrow I = \int \frac{1}{u}\cdot 2\sqrt{x}\ \,du$$
$$where\ \ u = 1 + \sqrt{x}$$
$$\Rightarrow \sqrt{x} = u - 1$$
$$\Rightarrow 2\sqrt{x} = 2u - 2$$
$$\Rightarrow I = \int \frac{2u-2}{u}\,du$$
$$\Rightarrow I = \int \left(\frac{2u}{u} - \frac{2}{u}\right)du$$
$$\Rightarrow I = \int 2\,du - 2\int \frac{1}{u}du$$
$$\Rightarrow I = 2u - 2\ln u + c$$
$$but\ u = 1 + \sqrt{x}$$
$$\therefore \boldsymbol{I = 2\left(1+\sqrt{x}\right) - 2\ln\left(1+\sqrt{x}\right) + c}$$

3. **Integration using Partial Fractions**

Partial fractions are used to breakdown a fraction into its component parts where the denominator is expressed as a product of factors. Use a relevant technique to breakdown a combined denominator into distinct or stand-alone factors.

For example

$$Given\ that,\quad I = \int \frac{2x^2 - 5}{(x+1)(2-x)}\,dx$$

$$using\ partial\ fractions,\quad \frac{2x^2 - 5}{(x+1)(2-x)} = A + \frac{B}{(x+1)} + \frac{C}{(2-x)}$$

$$\Rightarrow 2x^2 - 5 = A(x+1)(2-x) + B(2-x) + C(x+1)$$

$let\ x = 2$

$\Rightarrow 3 = 3C$

$\Rightarrow C = 1$

$let\ x = -1$

$\Rightarrow -3 = 3B$

$\Rightarrow B = -1$

$let\ x = 0$

$-5 = 2A + 2B + C$

$-5 = 2A + 2(-1) + 1$

$\Rightarrow A = -2$

$$\Rightarrow \frac{2x^2 - 5}{(x+1)(2-x)} = -2 - \frac{1}{(x+1)} + \frac{1}{(2-x)}$$

$$Now,\quad I = \int -2 - \frac{1}{(x+1)} + \frac{1}{(2-x)}\,dx$$

$$\Rightarrow I = \int -2\,dx - 1\int \frac{1}{(x+1)}\,dx + \int \frac{1}{(2-x)}\,dx$$

$$\Rightarrow I = -2x - \ln(x+1) - \ln(2-x) + c$$

$$\Rightarrow I = -2x - [\ln(x+1) + \ln(2-x)] + c$$

$$\therefore \boldsymbol{I = -2x - \ln[(x+1)(2-x)] + c}$$

4. **Transformation of a Fraction Into a Product**

This strategy can only be used when the given fraction cannot be integrated using any of the methods outlined immediately above. In such cases, the denominator has to be brought up to make the expression a product. Once the expression has been successfully transformed into a product, one can use integration by parts to solve the problem.

For example

Given that,

$$I = \int \frac{\ln x}{x^2}\, dx$$

$$\Rightarrow I = \int x^{-2} \ln x \, dx \text{ and use integration by parts.}$$

Applications of Integration

Integration is used to determine area under a curve and the volume generated when a particular region is rotated about the x axis or y axis.

1. **Area Under a Curve**
   Area is given by finding the definite integral of the curve in question. As such,

$$\boldsymbol{A = \int_{x_1}^{x_2} y\, dx}$$

where y is the equation of the curve, and x_1 and x_2 are limits.

For example, the area enclosed by the curve $y = \frac{4}{x^2} + 2x$, the x-axis and lines $x = 1$ and $x = 2$ is calculated as follows:

$$A = \int_1^2 \frac{4}{x^2} + 2x\, dx$$

$$\Rightarrow A = \int_1^2 4x^{-2} + 2x\, dx$$

$$\Rightarrow A = \left[\frac{4x^{-1}}{-1} + \frac{2x^2}{2}\right]_1^2$$

$$\Rightarrow A = \left[\frac{-4}{x} + x^2\right]_1^2$$

$$\Rightarrow A = \left[\frac{-4}{2} + 2^2\right] - \left[\frac{-4}{1} + 1^2\right]$$

$$\therefore \boldsymbol{A = 5\ units^2}$$

NB: If the region in question is bounded by two graphs, one has to calculate area under graph assuming the ceiling of the region and reduce it by area under the graph assuming the floor of the region.

For example, given the curve $y = 2x - x^2$ and the straight line $y = \frac{x}{2}$ intersecting at O and B as shown in the diagram below:

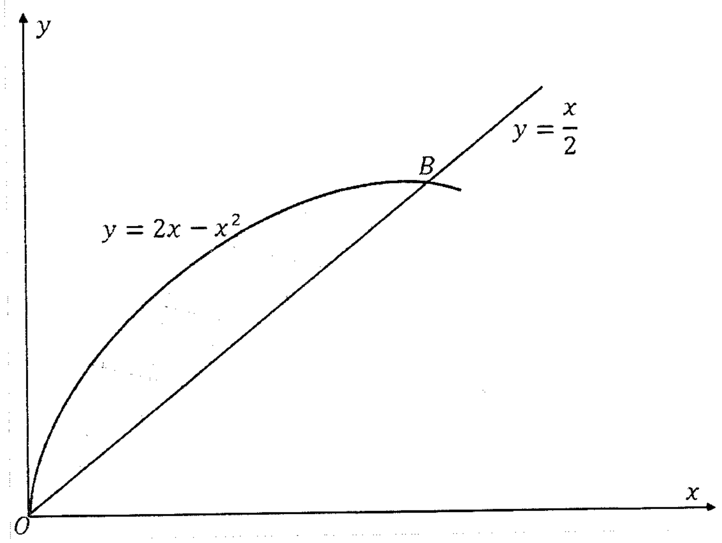


Fig. 11.2

- The pair coordinates of B is given by solving the two equations simultaneously;

$\Rightarrow \frac{x}{2} = 2x - x^2$

$\Rightarrow x = 4x - 2x^2$

$\Rightarrow 2x^2 - 4x + x = 0$

$\Rightarrow 2x^2 - 3x = 0$

$\Rightarrow x(2x - 3) = 0$

$either\ x = 0$

$or\ \ 2x - 3 = 0$

$x = \frac{3}{2}$

$when\ \ x = \frac{3}{2};\ \ y = \frac{\left(\frac{3}{2}\right)}{2}$

$\therefore \boldsymbol{B\ is\ \left(\frac{3}{2};\ \frac{3}{4}\right)}$

- Area below the upper bound, $y = 2x - x^2$, is given by:

$\Rightarrow A = \int_0^{\frac{3}{2}} 2x - x^2 dx$

$$\Rightarrow A = \left[\frac{2x^2}{2} - \frac{x^3}{3}\right]_0^{\frac{3}{2}}$$

$$\Rightarrow A = \left[\left(\frac{3}{2}\right)^2 - \frac{\left(\frac{3}{2}\right)^3}{3}\right] - \left[(0)^2 - \frac{(0)^3}{3}\right]$$

$$\Rightarrow \boldsymbol{A = \frac{9}{8} units^2}$$

- Area below the lower bound, $y = \frac{x}{2}$, is given by:

$$\Rightarrow A = \int_0^{\frac{3}{2}} \frac{x}{2}\, dx$$

$$\Rightarrow A = \left[\frac{x^2}{4}\right]_0^{\frac{3}{2}}$$

$$\Rightarrow A = \left[\frac{\left(\frac{3}{2}\right)^2}{4}\right] - \left[\frac{(0)^2}{4}\right]$$

$$\Rightarrow \boldsymbol{A = \frac{9}{16} units^2}$$

- *Area of the shaded region is , therefore, given by:*
  $A = area\ below\ the\ upper\ bound -\ area\ below\ the\ lower\ bound$

$$\Rightarrow A = \frac{9}{8} - \frac{9}{16}$$

$$\therefore \boldsymbol{A = \frac{9}{16} units^2}$$

2. Volume of Revolution

This concept is used to measure the capacity of the solid feature formed when a particular region is rotated about the x axis or y axis.

If the region is rotated about the $x-axis$, volume is given by:

$$V_{x-axis} = \pi \int_{x_1}^{x_2} y^2 dx,$$

$$\text{and if the region is rotated about the } y - \text{axis, volume is given by:}$$

$$V_{y-axis} = \pi \int_{y_1}^{y_2} x^2 dy$$

For example, the volume of the solid obtained when the shaded region is rotated through 360° about the x axis and y axis is given by:

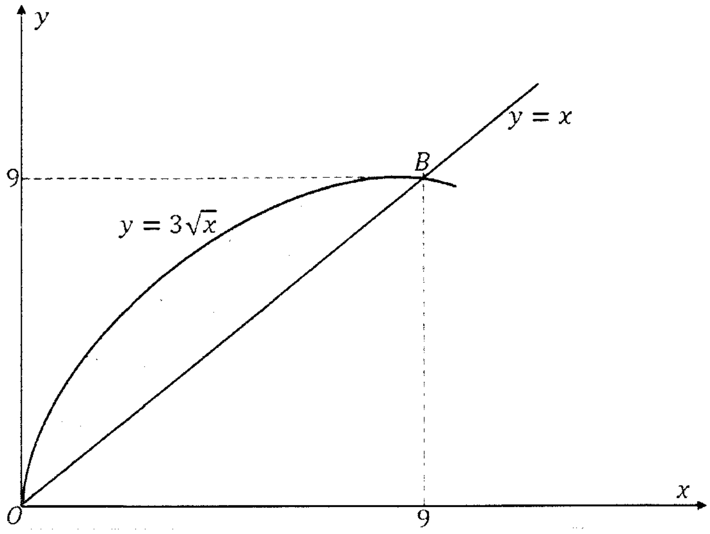


Fig. 11.3

Volume formed by the region bound by two graphs is given as follows:

$$V = V_{ceiling} - V_{floor}$$

Volume of the solid feature formed when the shaded region is rotated completely about the x axis:

Given that

$y = 3\sqrt{x}$ *and* $y = x$

$y^2 = 9x$ *and* $y^2 = x^2$

$V_{ceiiling} = \pi\int_0^9 9x\ dx$ and $V_{floor} = \pi\int_0^9 x^2\ dx$

$= \pi\left[\left(\frac{9x^2}{2}\right)\right]_0^9$ $\qquad = \pi\left[\left(\frac{x^3}{3}\right)\right]_0^9$

$= \frac{729\pi}{2}$ $\qquad = 243\pi$

$V = \frac{729\pi}{2} - 243\pi$

$\therefore \boldsymbol{V = \frac{243}{2}\pi\ units^3}$

Volume of the solid feature formed when the shaded region is rotated completely about the y axis:

$Given\ that\ \ y = 3\sqrt{x}$ and $y = x$

$y^2 = 9x$ $\qquad y^2 = x^2$

$x = \frac{y^2}{9}$ $\qquad x^2 = y^2$

$x^2 = \frac{y^4}{81}$

$V_{ceiiling} = \pi\int_0^9 y^2\ dy$ and $V_{floor} = \pi\int_0^9 \frac{y^4}{81}\ dy$

$= \pi\left[\left(\frac{y^3}{3}\right)\right]_0^9$ $\qquad = \pi\left[\left(\frac{y^5}{405}\right)\right]_0^9$

$= 243\pi$ $\qquad = \frac{729\pi}{5}$

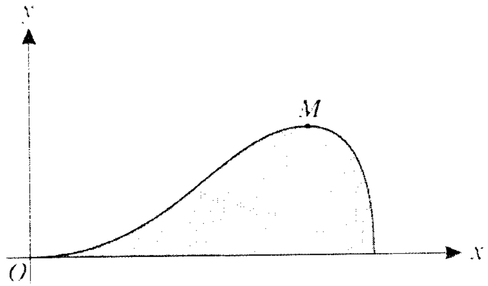
$V = 243\pi - \frac{729\pi}{5}$

$\therefore \boldsymbol{V = \frac{486}{5}\pi\ units^3}$

Worked Examination Questions on the Method of Substitution

Question (Cambridge, June 2009 qp.3)

10



The diagram shows the curve $y = x^2\sqrt{(1-x^2)}$ for $x \geqslant 0$ and its maximum point M.

(i) Find the exact value of the x-coordinate of M. [4]

(ii) Show, by means of the substitution $x = \sin\theta$, that the area A of the shaded region between the curve and the x-axis is given by

$$A = \frac{1}{4}\int_0^{\frac{1}{2}\pi} \sin^2 2\theta \, d\theta.$$ [3]

(iii) Hence obtain the exact value of A. [4]

Solution

(i) *Since m is a turning point,* $\frac{dy}{dx} = 0$

Given that,

$$y = x^2(1-x^2)^{\frac{1}{2}}$$

$$\Rightarrow \frac{dy}{dx} = (1-x^2)^{\frac{1}{2}}(2x) + (x^2)\left[\frac{1}{2}(1-x^2)^{-\frac{1}{2}}(-2x)\right]$$

$$\Rightarrow \frac{dy}{dx} = 2x(1-x^2)^{\frac{1}{2}} + x^2\left[-x(1-x^2)^{-\frac{1}{2}}\right]$$

$$\Rightarrow \frac{dy}{dx} = 2x(1-x^2)^{\frac{1}{2}} - x^3(1-x^2)^{-\frac{1}{2}}$$

$$\Rightarrow 2x(1-x^2)^{\frac{1}{2}} - x^3(1-x^2)^{-\frac{1}{2}} = 0$$

$$\Rightarrow 2x(1-x^2)^{\frac{1}{2}} = \frac{x^3}{(1-x^2)^{\frac{1}{2}}}$$

$$\Rightarrow 2x(1-x^2) = x^3$$

$$\Rightarrow 2x - 2x^3 = x^3$$

$$\Rightarrow 3x^3 - 2x = 0$$

$$\Rightarrow x(3x^2 - 2) = 0$$

$either\ \ x = 0\ or\ \ 3x^2 - 2 = 0$

$$\Rightarrow 3x^2 = 2$$

$$\Rightarrow x^2 = \frac{2}{3}$$

$$\Rightarrow \boldsymbol{x = \sqrt{\frac{2}{3}}\ only}$$

(ii) $Area\ under\ the\ graph\ is\ given\ by,$

$$A = \int_{x_1}^{x_2} y\,dx$$

$$\Rightarrow A = \int_{x_1}^{x_2} x^2\sqrt{(1-x^2)}\,dx$$

$where\ x_1 and\ x_2 :\quad y = 0$

$$\Rightarrow x^2(1-x^2)^{\frac{1}{2}} = 0$$

$either\ \ x^2 = 0\ or\ (1-x^2)^{\frac{1}{2}} = 0$

$$\Rightarrow x = 0$$

$and\ (1-x^2)^{\frac{1}{2}} = 0$

$$\Rightarrow 1 - x^2 = 0$$

$$\Rightarrow x^2 = 1$$

$$\Rightarrow x = \pm 1$$

$$\Rightarrow x = 0\ and\ x = 1$$

$So,\ \ A = \int_0^1 x^2\sqrt{(1-x^2)}\,dx$

- Change of the differential coefficient:

$x = \sin\theta$

$\frac{dx}{d\theta} = \cos\theta$

$dx = \cos\theta\, d\theta$

- Change of limits:

Upper limit

$x = \sin\theta$

$\Rightarrow 1 = \sin\theta$

$\Rightarrow \theta = \sin^{-1}(1)$

$\Rightarrow \theta = \frac{\pi}{2}$

Lower limit

$x = \sin\theta$

$\Rightarrow 0 = \sin\theta$

$\Rightarrow \theta = \sin^{-1}(0)$

$\Rightarrow \theta = 0$

- Substitution of results in A

$$A = \int_0^{\frac{\pi}{2}} \sin^2\theta\sqrt{(1-\sin^2\theta)} \cdot \cos\theta\, d\theta$$

$\Rightarrow A = \int_0^{\frac{\pi}{2}} \sin^2\theta\sqrt{(\cos^2\theta)} \cdot \cos\theta\, d\theta$

$\Rightarrow A = \int_0^{\frac{\pi}{2}} \sin^2\theta \cos\theta \cos\theta\, d\theta$

$\Rightarrow A = \int_0^{\frac{\pi}{2}} \sin^2\theta \cos^2\theta\, d\theta$

$\Rightarrow A = \int_0^{\frac{\pi}{2}} (\sin\theta\cos\theta)^2\, d\theta$

$* \, Remember \sin 2\theta = 2\sin\theta\cos\theta$

$\Rightarrow \sin\theta\cos\theta = \frac{1}{2}\sin 2\theta$

$\Rightarrow (\sin\theta\cos\theta)^2 = \left(\frac{1}{2}\sin 2\theta\right)^2$

$$\Rightarrow (\sin\theta\cos\theta)^2 = \frac{1}{4}\sin^2 2\theta$$

$$\Rightarrow A = \int_0^{\frac{\pi}{2}} \frac{1}{4}\sin^2 2\theta \, d\theta$$

$$\therefore \boldsymbol{A = \frac{1}{4}\int_0^{\frac{\pi}{2}} \sin^2 2\theta \, d\theta}$$

(iii) *to integrate the powers of* $\sin\theta$ *, use the double angle identity for* $\cos\theta$ *to eliminate the power of* $\sin\theta$*.*

$$* \cos 2\theta = 1 - 2\sin^2\theta$$

$$\Rightarrow \cos 4\theta = 1 - 2\sin^2 2\theta$$

$$\Rightarrow 2\sin^2 2\theta = 1 - \cos 4\theta$$

$$\Rightarrow \sin^2 2\theta = \frac{1}{2}(1 - \cos 4\theta)$$

$$Now, \; A = \frac{1}{4}\int_0^{\frac{\pi}{2}} \sin^2 2\theta \, d\theta$$

$$\Rightarrow A = \frac{1}{4}\int_0^{\frac{\pi}{2}} \frac{1}{2}(1 - \cos 4\theta) \, d\theta$$

$$\Rightarrow A = \frac{1}{8}\int_0^{\frac{\pi}{2}} 1 - \cos 4\theta \, d\theta$$

$$\Rightarrow A = \frac{1}{8}\left[\theta - \frac{\sin 4\theta}{4}\right]_0^{\frac{\pi}{2}}$$

$$\Rightarrow A = \frac{1}{8}\left[\left\{\frac{\pi}{2} - \frac{\sin\left(4 \times \frac{\pi}{2}\right)}{4}\right\} - \left\{0 - \frac{\sin(4 \times 0)}{4}\right\}\right]$$

$$\Rightarrow A = \frac{1}{8}\left[\frac{\pi}{2} - 0\right]$$

$$\therefore \boldsymbol{A = \frac{\pi}{16} units^2}$$

Question (Cambridge, November 2011 qp.33)

10 **(i)** Use the substitution $u = \tan x$ to show that, for $n \neq -1$,

$$\int_0^{\frac{1}{4}\pi} (\tan^{n+2} x + \tan^n x)\, dx = \frac{1}{n+1}.$$ [4]

(ii) Hence find the exact value of

(a) $\int_0^{\frac{1}{4}\pi} (\sec^4 x - \sec^2 x)\, dx$, [3]

(b) $\int_0^{\frac{1}{4}\pi} (\tan^9 x + 5\tan^7 x + 5\tan^5 x + \tan^3 x)\, dx$. [3]

Solution

(i) $let\ I = \int_0^{\frac{\pi}{4}} (tan^{n+2}\, x + tan^n x)dx$

- Change of differential coefficient:

  $u = \tan x$

  $\Rightarrow \frac{du}{dx} = sec^2 x$

  $\Rightarrow du = sec^2 x\ dx$

  $\Rightarrow dx = \frac{du}{sec^2 x}$

- Change of limits:

Upper limit	Lower limit
$u = \tan x$	$u = \tan x$
$\Rightarrow u = \tan {}^{\pi}/_{4}$	$\Rightarrow u = \tan 0$
$\Rightarrow u = 1$	$\Rightarrow u = 0$

- Substitution of results in I:

$$I = \int_0^1 (u^{n+2} + u^n) \frac{du}{sec^2 x}$$

  $*\ where\ sec^2 x = 1 + tan^2 x$

$$\Rightarrow \int_0^1 (u^{n+2} + u^n)\frac{du}{(1 + tan^2x)}$$

$$\Rightarrow I = \int_0^1 \frac{(u^{n+2} + u^n)}{(1 + u^2)}\, du$$

$$\Rightarrow I = \int_0^1 \frac{u^n(u^2 + 1)}{(u^2 + 1)}\, du$$

$$\Rightarrow I = \int_0^1 u^n\, du$$

$$\Rightarrow I = \left[\frac{u^{n+1}}{(n + 1)}\right]_0^1$$

$$\Rightarrow I = \left[\frac{1^{n+1}}{n + 1}\right] - \left[\frac{0^{n+1}}{n + 1}\right]$$

$$\therefore \boldsymbol{I = \frac{1}{n + 1}}\ \boldsymbol{(shown)}$$

(ii)(a) $let\ I = \int_0^{\frac{\pi}{4}} (sec^4x - sec^2x)dx$

by changing $\sec x$ *to* $\tan x$ *since the given reduction formula is in the terms* $\tan x$,

$sec^2x = 1 + tan^2x$

$\Rightarrow sec^4 = (1 + tan^2\, x)^2$

$\Rightarrow sec^4x = 1 + 2\, tan^2x + \ tan^4x$

$\Rightarrow (sec^4x - sec^2x) = (1 + 2tan^2x + tan^4x) - (1 + tan^2x)$

$\Rightarrow (sec^4x - sec^2x) = tan^4x + tan^2x$

$Now\ , I = \int_0^{\frac{\pi}{4}} tan^4x + \ tan^2x\ dx$

by quoting the reduction formula, where $n = 2$

$$\Rightarrow I = \frac{1}{2 + 1}$$

$$\therefore \boldsymbol{I = \frac{1}{3}}$$

(b) $let\ \tan x = t$

$$\Rightarrow I = \int_0^{\frac{\pi}{4}} t^9 + 5t^7 + 5t^5 + t^3\, dx$$

NB:

- For one to quote the reduction formula, the coefficients of corresponding terms must be similar;
- As such, some adjustments have to be done to liken the coefficients.

$$\Rightarrow I = \int_0^{\frac{\pi}{4}} t^9 + 5t^7 + \{-4t^7 + 4t^7\} + 5t^5 + \{-4t^5 + 4t^5\} + t^3\, dx$$

$$\Rightarrow I = \int_0^{\frac{\pi}{4}} t^9 + t^7 + 4t^7 + t^5 + 4t^5 + t^3\, dx$$

$$\Rightarrow I = \int_0^{\frac{\pi}{4}} t^9 + t^7 + 4t^7 + 4t^5 + t^5 + t^3\, dx$$

$$\Rightarrow I = \int_0^{\frac{\pi}{4}} t^9 + t^7 dx\ +\ 4\int_0^{\frac{\pi}{4}} t^7 + t^5 dx\ +\int_0^{\frac{\pi}{4}} t^5 + t^3 dx$$

$By\ quoting\ the\ reduction\ formula\ where\ n = 7,\ \ 5\ \ and\ 3\ respectively,$

$$\Rightarrow I = \frac{1}{(7+1)} + 4\left(\frac{1}{5+1}\right) + \frac{1}{(3+1)}$$

$$\Rightarrow I = \frac{1}{8} + \frac{4}{6} + \frac{1}{4}$$

$$\therefore \boldsymbol{I = \frac{25}{24}}$$

Revision Questions on the Method of Substitution

November 2001 qp.1 (Zimsec)

12. The value of

$$\int_0^{\frac{1}{2}\pi} \frac{\sin 2x}{\sin x + 2}\, dx$$

is denoted by I. Use the substitution $u = \sin x$ to show that

$$I = \int_0^1 \left(2 - \frac{4}{u+2}\right)\, du$$

[5]

Hence find the exact value of I. [3]

November 2010 qp.1 (Zimsec)

8. (a) By means of the substitution $u = \sin x$ or otherwise, find the exact value of

$$\int_0^{\frac{\pi}{2}} \cos x\sqrt{\sin x}\, dx.$$

[3]

(b) Solve the equation $\cos 3y = -\frac{1}{\sqrt{2}}$ for $0° < y < 270°$. [4]

June 2003 qp.1 (Zimsec)

7. By using the substitution $x^2 = u + 1$, or otherwise,

find

$$\int \frac{x^3}{\sqrt{x^2 - 1}}\, dx$$

giving your answer in terms of x. [5]

June 2010 qp.1 (Zimsec)

12. Use the substitution $x = a\sin\theta$ to evaluate

$$\int_0^a \sqrt{a^2 - x^2}\, dx.$$

[6]

November 1995 qp.1 (Cambridge)

7. Use the substitution $u = 2x + 3$ to find

$$\int \frac{x}{(2x+3)^3}\,dx.$$

[5]

November 1997 qp.1 (Cambridge)

12. The value of $\int_1^4 \frac{1}{x(1+\sqrt{x})}\,dx$ is denoted by I. Use the substitution $u = \sqrt{x}$ to show that

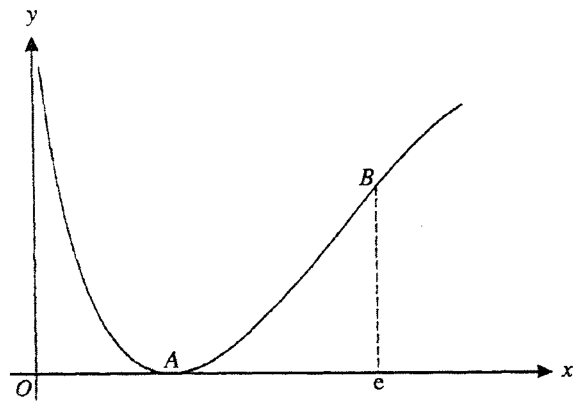
$$I = \int_1^2 \frac{2}{u(1+u)}\,du.$$

[3]

Hence, by using partial fractions, show that $I = \ln\left(\frac{16}{9}\right)$. [5]

June 2002 qp.3 (Cambridge)

10



The function f is defined by $f(x) = (\ln x)^2$ for $x > 0$. The diagram shows a sketch of the graph of $y = f(x)$. The minimum point of the graph is A. The point B has x-coordinate e.

(i) State the x-coordinate of A. [1]

(ii) Show that $f''(x) = 0$ at B. [4]

(iii) Use the substitution $x = e^u$ to show that the area of the region bounded by the x-axis, the line $x = e$, and the part of the curve between A and B is given by

$$\int_0^1 u^2 e^u\,du.$$ [3]

(iv) Hence, or otherwise, find the exact value of this area. [3]

June 2007 qp.3 (Cambridge)

7 Let $I = \int_1^4 \frac{1}{x(4-\sqrt{x})}\,dx$.

(i) Use the substitution $u = \sqrt{x}$ to show that $I = \int_1^2 \frac{2}{u(4-u)}\,du$. [3]

(ii) Hence show that $I = \frac{1}{2}\ln 3$. [6]

November 2009 qp.32 (Cambridge)

6 **(i)** Use the substitution $x = 2\tan\theta$ to show that

$$\int_0^2 \frac{8}{(4+x^2)^2}\,dx = \int_0^{\frac{1}{4}\pi} \cos^2\theta\,d\theta.$$ [4]

(ii) Hence find the exact value of

$$\int_0^2 \frac{8}{(4+x^2)^2}\,dx.$$ [4]

November 2010 qp.31 (Cambridge)

5 Let $I = \int_0^1 \frac{x^2}{\sqrt{(4-x^2)}}\,dx$.

(i) Using the substitution $x = 2\sin\theta$, show that

$$I = \int_0^{\frac{1}{6}\pi} 4\sin^2\theta\,d\theta.$$ [3]

(ii) Hence find the exact value of I. [4]

June 2011 qp.31 (Cambridge)

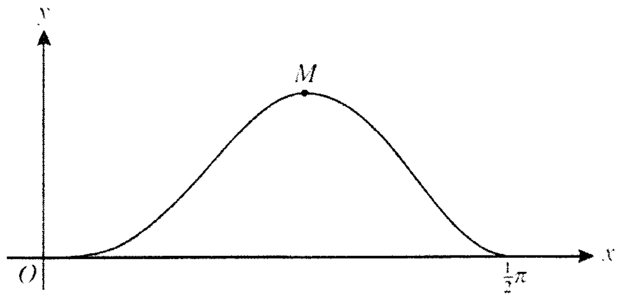
7 The integral I is defined by $I = \int_0^2 4t^3\ln(t^2+1)\,dt$.

(i) Use the substitution $x = t^2 + 1$ to show that $I = \int_1^5 (2x-2)\ln x\,dx$. [3]

(ii) Hence find the exact value of I. [5]

June 2011 qp.33 (Cambridge)

8



The diagram shows the curve $y = 5\sin^3 x \cos^2 x$ for $0 \leqslant x \leqslant \frac{1}{2}\pi$, and its maximum point M.

(i) Find the x-coordinate of M. [5]

(ii) Using the substitution $u = \cos x$, find by integration the area of the shaded region bounded by the curve and the x-axis. [5]

June 2012 qp.32 (Cambridge)

8 Let $I = \int_2^5 \frac{5}{x + \sqrt{(6 - x)}}\,dx$.

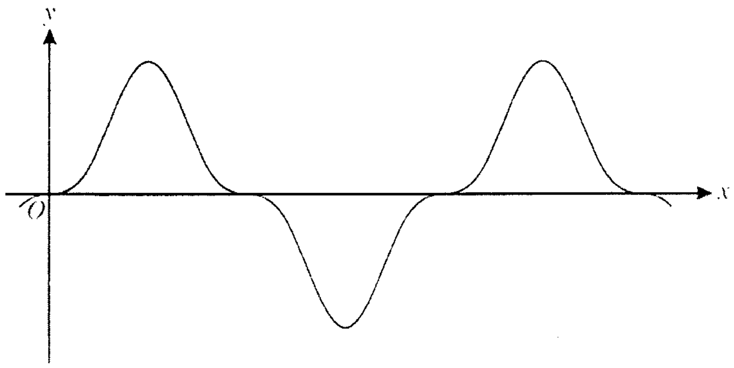
(i) Using the substitution $u = \sqrt{(6 - x)}$, show that

$$I = \int_1^2 \frac{10u}{(3 - u)(2 + u)}\,du.$$ [4]

(ii) Hence show that $I = 2\ln\left(\frac{9}{2}\right)$. [6]

November 2012 qp.33 (Cambridge)

7



The diagram shows part of the curve $y = \sin^3 2x \cos^3 2x$. The shaded region shown is bounded by the curve and the x-axis and its exact area is denoted by A.

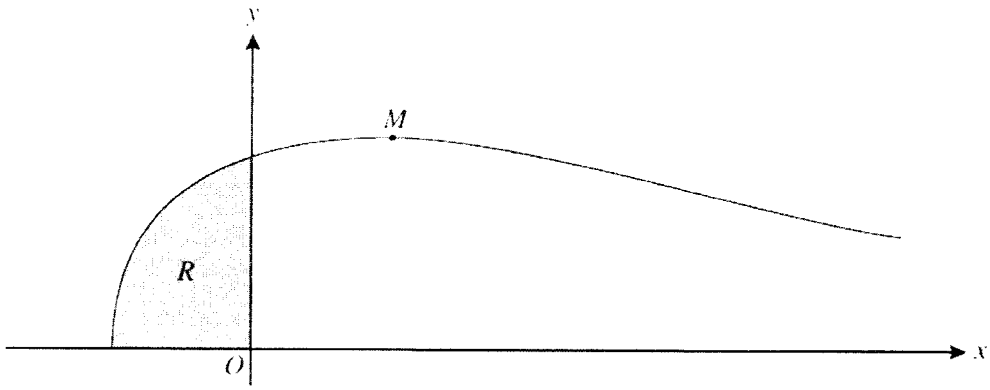
(i) Use the substitution $u = \sin 2x$ in a suitable integral to find the value of A. [6]

(ii) Given that $\int_0^{k\pi} |\sin^3 2x \cos^3 2x| \, dx = 40A$, find the value of the constant k. [2]

Worked Examination Questions on Integration by Parts

Question (Cambridge, June 2008 qp.3)

9



The diagram shows the curve $y = e^{-\frac{1}{2}x}\sqrt{(1 + 2x)}$ and its maximum point M. The shaded region between the curve and the axes is denoted by R.

(i) Find the x-coordinate of M. [4]

(ii) Find by integration the volume of the solid obtained when R is rotated completely about the x-axis. Give your answer in terms of π and e. [6]

Solution

(i) *Since m is a turning point,* $\frac{dy}{dx} = 0$

Given that, $y = e^{-\frac{1}{2}x}\sqrt{(1+2x)}$

$\Rightarrow y = e^{-\frac{1}{2}x}(1+2x)^{\frac{1}{2}}$

$\Rightarrow \frac{dy}{dx} = (1+2x)^{\frac{1}{2}}\left(-\frac{1}{2}e^{-\frac{1}{2}x}\right) + e^{-\frac{1}{2}x}\left[\left(\frac{1}{2}\right)(1+2x)^{-\frac{1}{2}}(2)\right]$

$$\Rightarrow \frac{dy}{dx} = -\frac{1}{2}e^{-\frac{1}{2}x}(1+2x)^{\frac{1}{2}} + e^{-\frac{1}{2}x}(1+2x)^{-\frac{1}{2}}$$

$$\Rightarrow -\frac{1}{2}e^{-\frac{1}{2}x}(1+2x)^{\frac{1}{2}} + e^{-\frac{1}{2}x}(1+2x)^{-\frac{1}{2}} = 0$$

$$\Rightarrow e^{-\frac{1}{2}x}(1+2x)^{-\frac{1}{2}} = \frac{1}{2}e^{-\frac{1}{2}x}(1+2x)^{\frac{1}{2}}$$

$$\Rightarrow \frac{1}{(1+2x)^{\frac{1}{2}}} = \frac{1}{2}(1+2x)^{\frac{1}{2}}$$

$$\Rightarrow 1 = \frac{1}{2}(1+2x)$$

$$\Rightarrow 2 = 1 + 2x$$

$$\Rightarrow 2x = 1$$

$$\therefore \boldsymbol{x = \frac{1}{2}}$$

(ii) $$V = \pi \int_{x_1}^{x_2} y^2\, dx$$

$where\ x_1 and\ x_2 : y = 0$

$$\Rightarrow e^{-\frac{1}{2}x}\sqrt{(1+2x)} = 0$$

$$\Rightarrow (1+2x)^{\frac{1}{2}} = 0$$

$$\Rightarrow 1 + 2x = 0$$

$$\Rightarrow 2x = -1$$

$$\Rightarrow x = -\frac{1}{2}$$

$$and\ \ y = e^{-\frac{1}{2}x}\sqrt{(1+2x)}$$

$$\Rightarrow y^2 = \left[e^{-\frac{1}{2}x}\sqrt{(1+2x)}\right]^2$$

$$\Rightarrow y^2 = e^{-x}(1+2x)$$

$$Now,\ \ V = \pi \int_{-\frac{1}{2}}^{0} e^{-x}(1+2x)\, dx$$

Since $(1+2x)$ has the potential to breakdown into a constant,

$let\ u = 1 + 2x$ $\quad and \quad$ $\frac{dv}{dx} = e^{-x}$

$\frac{du}{dx} = 2$ $\quad\quad v = -e^{-x}$

$\Rightarrow V = \pi\left[-e^{-x}(1+2x) - \int -2e^{-x}\ dx\right]$

$\Rightarrow V = \pi\left[-e^{-x}(1+2x) + 2\int e^{-x}\ dx\right]$

$\Rightarrow V = \pi[-e^{-x}(1+2x) - 2e^{-x}]$

$\Rightarrow V = \pi\{-e^{-x}[1 + 2x + 2]\}_{-\frac{1}{2}}^{0}$

$\Rightarrow V = \pi[-e^{0}(2(0)+3)] - \pi\left[-e^{\frac{1}{2}}\left(2\left(-\frac{1}{2}\right)+3\right)\right]$

$\Rightarrow V = -3\pi + 2\pi e^{\frac{1}{2}}$

$\therefore \boldsymbol{V = \pi\left(2e^{\frac{1}{2}} - 3\right)\ units^3}$

Question (Cambridge, June 2010 qp.32)

2 Show that $\int_0^{\pi} x^2 \sin x\, dx = \pi^2 - 4$. [5]

Solution

$Let\ I = \int_0^{\pi} x^2 \sin x\ dx$

Since x^2 has the potential to breakdown into a constant,

$let\ u = x^2$ $\quad and \quad$ $\frac{dv}{dx} = \sin x$

$\frac{du}{dx} = 2x$ $\quad\quad v = -\cos x$

$\Rightarrow I = -x^2\cos x - \int -2x\cos x\ dx$

$\Rightarrow I = -x^2\cos x + \int 2x\cos x\ dx$

$let\ R = \int 2x\cos x\ dx$

using further integration by parts,

$let\ u = 2x$ *and* $\frac{dv}{dx} = \cos x$

$\frac{du}{dx} = 2$ $v = \sin x$

$R = 2x \sin x - \int 2 \sin x \ dx$

$\Rightarrow R = 2x \sin x + 2 \cos x$

by substituting R in I,

$\Rightarrow I = -x^2 \cos x + [2x \sin x + 2 \cos x]$

$\Rightarrow I = [-x^2 \cos x + 2x \sin x + 2 \cos x]_0^{\pi}$

$\Rightarrow I = [-\pi^2 \cos \pi + 2\pi \sin \pi + 2 \cos \pi] - [-0^2 \cos 0 + 2(0) \sin 0 + 2 \cos 0]$

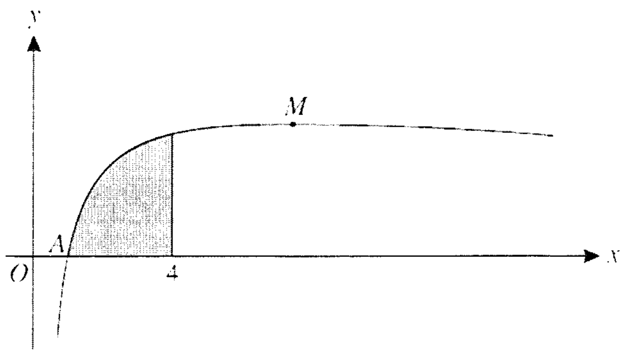
$\Rightarrow I = [-\pi^2(-1) + 2(-1)] - [2(1)]$

$\Rightarrow I = \pi^2 - 2 - 2$

$\therefore \boldsymbol{I = \pi^2 - 4\ (shown)}$

Question (Cambridge, November 2009 qp.31)

9



The diagram shows the curve $y = \frac{\ln x}{\sqrt{x}}$ and its maximum point M. The curve cuts the x-axis at the point A.

(i) State the coordinates of A. [1]

(ii) Find the exact value of the x-coordinate of M. [4]

(iii) Using integration by parts, show that the area of the shaded region bounded by the curve, the x-axis and the line $x = 4$ is equal to $8 \ln 2 - 4$. [5]

Solution

(i) $at\ A,\ \ y = 0$

$\Rightarrow x = 1$

$\therefore \boldsymbol{A\ (1, 0)}$

(ii) $At\ M,\ \dfrac{dy}{dx} = 0\ because\ M\ is\ a\ maximum\ point.$

$using\ the\ quotient\ rule,$

$$\frac{dy}{dx} = \frac{x^{\frac{1}{2}}\left(\frac{1}{x}\right) - (\ln x)(\frac{1}{2}x^{-\frac{1}{2}})}{\left(\sqrt{x}\right)^2}$$

$$\Rightarrow \frac{dy}{dx} = \frac{x^{-\frac{1}{2}} - \frac{1}{2}x^{-\frac{1}{2}}\ \ln x}{x}$$

$$\Rightarrow \frac{dy}{dx} = \frac{x^{-\frac{1}{2}}(1 - \frac{1}{2}\ \ln x)}{x}$$

$$\Rightarrow \frac{x^{-\frac{1}{2}}\left(1 - \frac{1}{2}\ \ln x\right)}{x} = 0$$

$$\Rightarrow x^{-\frac{1}{2}}(1 - \frac{1}{2}\ln x) = 0$$

$$\Rightarrow 1 - \frac{1}{2}\ln x\ = 0$$

$$\Rightarrow \frac{1}{2}\ln x = 1$$

$$\Rightarrow \ln x = 2$$

$$\therefore \boldsymbol{x = e^2}$$

(iii) *to use integration by parts, the fraction has to be first transformed into a product as shown:*

$Given\ that\ \ y = \dfrac{\ln x}{\sqrt{x}}$

$\Rightarrow y = \dfrac{\ln x}{x^{\frac{1}{2}}}$

$\Rightarrow y = x^{-\frac{1}{2}} \ln x$

$where\ \ area = \displaystyle\int_1^4 x^{-\frac{1}{2}} \ln x\, dx$

$Since\ \ \ln x\ \ cannot\ be\ integrated,$

$let\ u = \ln x$ $\quad and \quad$ $\dfrac{dv}{dx} = x^{-\frac{1}{2}}$

$\Rightarrow \dfrac{du}{dx} = \dfrac{1}{x}$ $\qquad$ $\Rightarrow v = \dfrac{x^{\frac{1}{2}}}{\frac{1}{2}}$

$\Rightarrow v = 2x^{\frac{1}{2}}$

$\Rightarrow area = 2x^{\frac{1}{2}} \ln x - \displaystyle\int \frac{1}{x} \cdot 2x^{\frac{1}{2}}\, dx$

$\Rightarrow area = 2x^{\frac{1}{2}} \ln x - \displaystyle\int 2x^{-\frac{1}{2}}\, dx$

$\Rightarrow area = 2x^{\frac{1}{2}}\ \ln x - \dfrac{2x^{\frac{1}{2}}}{\frac{1}{2}}$

$\Rightarrow area = 2x^{\frac{1}{2}} \ln x\ - 4x^{\frac{1}{2}}$

$\Rightarrow area = \left[2x^{\frac{1}{2}}(\ln x - 2)\right]_1^4$

$\Rightarrow area = \left[2(4)^{\frac{1}{2}}(\ln 4 - 2)\right] - \left[2(1)^{\frac{1}{2}}(\ln 1 - 2)\right]$

$\Rightarrow area = 4(\ln 4 - 2) - 2(-2)$

$\Rightarrow area = 4 \ln 4 - 8 + 4$

$\Rightarrow area = 4 \ln 2^2 - 4$

$\therefore \boldsymbol{area = 8 \ln 2 - 4 \quad (shown)}$

Revision Questions on Integration by Parts

November 1992 qp.1 (Cambridge)

11. (i) Show that

$$\int_0^2 xe^{2x}dx = \frac{1}{4}(3e^4 + 1)$$

[3]

(ii) Find the exact value of

$$\int_0^2 x^2e^{2x}dx$$

[2]

June 2003 qp.1 (Zimsec)

18 (b) Show that $\int_0^{\ln 4} x^2e^xdx = 2(\ln 4 - 1)^2 + 1$. [5]

November 1995 qp.1 (Cambridge)

8. Find $\int x\sin 2x dx$. [3]

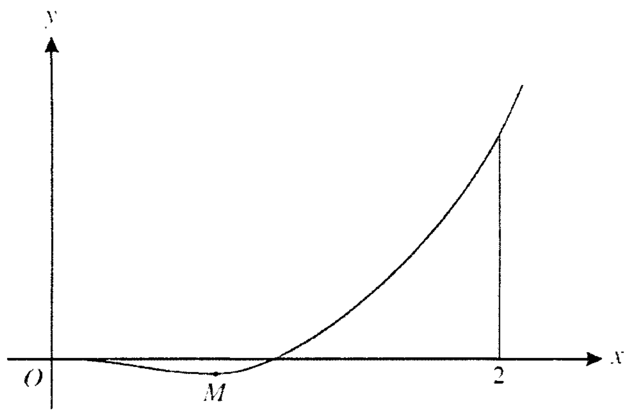
November 2007 qp.3 (Cambridge)

3 Use integration by parts to show that

$$\int_2^4 \ln x \, dx = 6\ln 2 - 2.$$ [4]

November 2010 qp.31 (Cambridge)

9



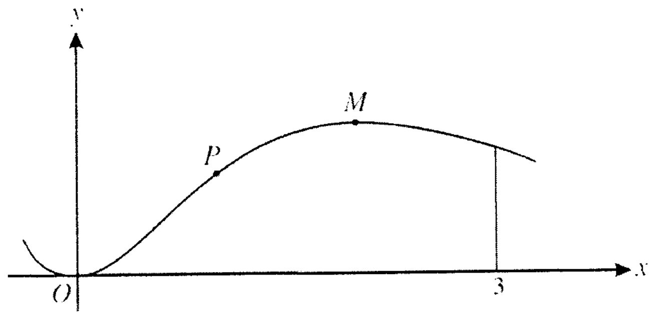
The diagram shows the curve $y = x^3 \ln x$ and its minimum point M.

(i) Find the exact coordinates of M. [5]

(ii) Find the exact area of the shaded region bounded by the curve, the x-axis and the line $x = 2$. [5]

June 2011 qp.32 (Cambridge)

10



The diagram shows the curve $y = x^2 e^{-x}$.

(i) Show that the area of the shaded region bounded by the curve, the x-axis and the line $x = 3$ is equal to $2 - \frac{17}{e^3}$. [5]

(ii) Find the x-coordinate of the maximum point M on the curve. [4]

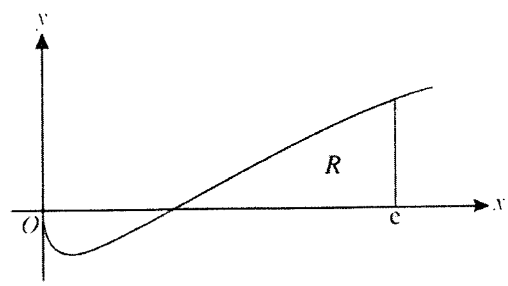
(iii) Find the x-coordinate of the point P at which the tangent to the curve passes through the origin. [2]

June 2011 qp.33 (Cambridge)

3 Show that $\int_0^1 (1 - x)e^{-\frac{1}{2}x}\, dx = 4e^{-\frac{1}{2}} - 2$. [5]

June 2012 qp.32 (Cambridge)

9



The diagram shows the curve $y = x^{\frac{1}{2}} \ln x$. The shaded region between the curve, the x-axis and the line $x = e$ is denoted by R.

(i) Find the equation of the tangent to the curve at the point where $x = 1$, giving your answer in the form $y = mx + c$. [4]

(ii) Find by integration the volume of the solid obtained when the region R is rotated completely about the x-axis. Give your answer in terms of π and e. [7]

November 2012 qp.31 (Cambridge)

5 (i) By differentiating $\frac{1}{\cos x}$, show that if $y = \sec x$ then $\frac{dy}{dx} = \sec x \tan x$. [2]

(ii) Show that $\frac{1}{\sec x - \tan x} \equiv \sec x + \tan x$. [1]

(iii) Deduce that $\frac{1}{(\sec x - \tan x)^2} \equiv 2\sec^2 x - 1 + 2\sec x \tan x$. [2]

(iv) Hence show that $\int_0^{\frac{1}{4}\pi} \frac{1}{(\sec x - \tan x)^2}\,dx = \frac{1}{4}(8\sqrt{2} - \pi)$. [3]

November 2012 qp.33 (Cambridge)

5 The expression $f(x)$ is defined by $f(x) = 3xe^{-2x}$.

(i) Find the exact value of $f'(-\frac{1}{2})$. [3]

(ii) Find the exact value of $\int_{-\frac{1}{2}}^{0} f(x)\,dx$. [5]

<u>Worked Examination Questions on Partial Fractions</u>

Question (June 2012 qp.33)

8 Let $f(x) = \frac{4x^2 - 7x - 1}{(x+1)(2x-3)}$.

(i) Express $f(x)$ in partial fractions. [5]

(ii) Show that $\int_2^6 f(x)\,dx = 8 - \ln(\frac{49}{3})$. [5]

Solution

(i) $f(x)$ is an improper fraction since the highest order power in the denominator is the same as the highest order power in the numerator.

As such,

$$\frac{4x^2-7x-1}{(x+1)(2x-3)} = A + \frac{B}{(x+1)} + \frac{C}{(2x-3)}$$

$$\Rightarrow 4x^2-7x-1 = A(x+1)(2x-3) + B(2x-3) + C(x+1)$$

$let\ x = -1$

$10 = -5B$

$\Rightarrow B = -2$

$let\ x = \frac{3}{2}$

$-\frac{5}{2} = \frac{5}{2}C$

$\Rightarrow C = -1$

$let\ x = 0$

$-1 = -3A - 3B + C$

$-1 = -3A - 3(-2) - 1$

$3A = 6$

$\Rightarrow A = 2$

$$\therefore \boldsymbol{f(x) = 2 - \frac{2}{(x+1)} - \frac{1}{(2x-3)}}$$

(ii) $Let\ I = \int_2^6 2 - \frac{2}{(x+1)} - \frac{1}{(2x-3)}\ dx$

$$\Rightarrow I = \int_2^6 2\ dx - 2\int_2^6 \frac{1}{(x+1)}\ dx - \int_2^6 \frac{1}{(2x-3)}\ dx$$

$$\Rightarrow I = 2x - 2\ln(x+1) - \frac{1}{2}\int \frac{2}{(2x-3)}\ dx$$

$$\Rightarrow I = 2x - 2\ln(x+1) - \frac{1}{2}\ln(2x-3)$$

$$\Rightarrow I = 2x - \left[2\ln(x+1) + \frac{1}{2}\ln(2x-3)\right]$$

$$\Rightarrow I = 2x - \left[\ln(x+1)^2 + \ln(2x-3)^{\frac{1}{2}}\right]$$

$$\Rightarrow I = \left\{2x - \ln\left[(x+1)^2(2x-3)^{\frac{1}{2}}\right]\right\}_2^6$$

$$\Rightarrow I = [2(6) - \ln(49 \times 3)] - [2(2) - \ln(9 \times 1)]$$

$\Rightarrow I = 12 - \ln 147 - 4 + \ln 9$

$\Rightarrow I = 8 + \ln 9 - \ln 147$

$\Rightarrow I = 8 + \ln\left(\frac{9}{147}\right)$

$\Rightarrow I = 8 + \ln\left(\frac{3}{49}\right)$

$\Rightarrow I = 8 + \ln\left(\frac{49}{3}\right)^{-1}$

$\therefore \boldsymbol{I = 8 - \ln\left(\frac{49}{3}\right)}$ *(shown)*

Revision Questions on Partial Fractions

November 2010 qp.1 (Zimsec)

14. (i) Show that $2x^3 - x^2 + 8x - 4 = (2x - 1)(x^2 + 4)$ [2]

(ii) Express $\frac{x^2+2x+20}{2x^3-x^2+8x-4}$ in partial fractions. [4]

(iii) Hence evaluate $\int_1^3 \frac{x^2+2x+20}{2x^3-x^2+8x-4}\,dx$, and give your answer correct to three significant figures. [4]

November 1999 qp.1 (Cambridge)

13. (i) Express $\frac{1}{x(x+1)}$ in partial fractions. [2]

(ii) The region bounded by the curve $= \frac{1}{x(x+1)}$, the x −axis, and the lines $x = 1$ and $x = 2$ is rotated completely about the x-axis to form a solid of revolution.

a) Use part (i) to show that y^2 may be expressed as

$$\frac{1}{x^2} - \frac{2}{x} + \frac{2}{x+1} + \frac{1}{(x+1)^2}.$$ [2]

b) Hence show that the volume of the solid of revolution is

$$\pi\left(\frac{2}{3} + 2\ln\frac{3}{4}\right)$$ [5]

June 2002 qp.3 (Cambridge)

6 Let $f(x) = \dfrac{4x}{(3x+1)(x+1)^2}$.

(i) Express $f(x)$ in partial fractions. [5]

(ii) Hence show that $\int_0^1 f(x)\,dx = 1 - \ln 2$. [5]

June 1994 qp.1 (Cambridge)

9. Express $\frac{1}{x^2(x-1)}$ in the form $\frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-1}$, where A, B and C are constants. [3]

Hence find $\int \frac{1}{x^2(x-1)} dx$. [2]

June 2008 qp.3 (Cambridge)

7 Let $f(x) \equiv \dfrac{x^2 + 3x + 3}{(x+1)(x+3)}$.

(i) Express $f(x)$ in partial fractions. [5]

(ii) Hence show that $\int_0^3 f(x)\,dx = 3 - \frac{1}{2}\ln 2$. [4]

June 2010 qp.31 (Cambridge)

8 (i) Express $\dfrac{2}{(x+1)(x+3)}$ in partial fractions. [2]

(ii) Using your answer to part (i), show that

$$\left(\frac{2}{(x+1)(x+3)}\right)^2 \equiv \frac{1}{(x+1)^2} - \frac{1}{x+1} + \frac{1}{x+3} + \frac{1}{(x+3)^2}.$$ [2]

(iii) Hence show that $\int_0^1 \dfrac{4}{(x+1)^2(x+3)^2}\,dx = \frac{7}{12} - \ln\frac{3}{2}$. [5]

June 2010 qp.32 (Cambridge)

10 (i) Find the values of the constants A, B, C and D such that

$$\frac{2x^3 - 1}{x^2(2x - 1)} \equiv A + \frac{B}{x} + \frac{C}{x^2} + \frac{D}{2x - 1}.$$ [5]

(ii) Hence show that

$$\int_1^2 \frac{2x^3 - 1}{x^2(2x - 1)}\,dx = \tfrac{3}{2} + \tfrac{1}{2}\ln\left(\tfrac{16}{27}\right).$$ [5]

November 2010 qp.33 (Cambridge)

5 Show that $\displaystyle\int_0^7 \frac{2x + 7}{(2x + 1)(x + 2)}\,dx = \ln 50.$ [7]

November 2011 qp.31 (Cambridge)

8 Let $\mathrm{f}(x) = \dfrac{12 + 8x - x^2}{(2 - x)(4 + x^2)}$.

(i) Express $\mathrm{f}(x)$ in the form $\dfrac{A}{2 - x} + \dfrac{Bx + C}{4 + x^2}$. [4]

(ii) Show that $\displaystyle\int_0^1 \mathrm{f}(x)\,dx = \ln\left(\tfrac{25}{2}\right)$. [5]

June 2012 qp.31 (Cambridge)

9 By first expressing $\dfrac{4x^2 + 5x + 3}{2x^2 + 5x + 2}$ in partial fractions, show that

$$\int_0^4 \frac{4x^2 + 5x + 3}{2x^2 + 5x + 2}\,dx = 8 - \ln 9.$$ [10]

Worked Examination Questions on Standard Integrals

Question (November 2009 qp.31)

5 (i) Prove the identity $\cos 4\theta - 4\cos 2\theta + 3 \equiv 8\sin^4\theta$. [4]

(ii) Using this result find, in simplified form, the exact value of

$$\int_{\frac{1}{6}\pi}^{\frac{1}{3}\pi} \sin^4\theta \, d\theta.$$ [4]

Solution

(i) $\cos 4\theta - 4\cos 2\theta + 3 \equiv 8\sin^4\theta$

$Let\ the\ LHS = \cos 4\theta - 4\cos 2\theta + 3$

$where\ \cos 2\theta = 1 - 2\sin^2\theta$

$and\ \cos 4\theta = 1 - 2\sin^2 2\theta$

$\Rightarrow LHS = 1 - 2\sin^2 2\theta - 4(1 - 2\sin^2\theta) + 3$

$\Rightarrow LHS = 1 - 2\sin^2 2\theta - 4 + 8\sin^2\theta + 3$

$\Rightarrow LHS = 8\sin^2\theta - 2\sin 2\theta \sin 2\theta$

$\Rightarrow LHS = 8\sin^2\theta - 2(2\sin\theta\cos\theta)(2\sin\theta\cos\theta)$

$\Rightarrow LHS = 8\sin^2\theta - 8\sin^2\theta\cos^2\theta$

$\Rightarrow LHS = 8\sin^2\theta - 8\sin^2\theta\,(1 - \sin^2\theta)$

$\Rightarrow LHS = 8\sin^2\theta - 8\sin^2\theta + 8\sin^4\theta$

$\therefore \boldsymbol{LHS = 8\sin^4\theta \equiv RHS\ (shown)}$

(ii) $let\ I = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \sin^4\theta\ d\theta$

$from\ the\ identity\ above,$

$\sin^4\theta = \frac{1}{8}[\cos 4\theta - 4\cos 2\theta + 3]$

$$\Rightarrow I = \frac{1}{8}\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \cos 4\theta - 4\cos 2\theta + 3 \; d\theta$$

$$\Rightarrow I = \frac{1}{8}\left[\frac{\sin 4\theta}{4} - \frac{4\sin 2\theta}{2} + 3\theta\right]_{\frac{\pi}{6}}^{\frac{\pi}{3}}$$

$$\Rightarrow I = \frac{1}{8}\left[\left\{\frac{\sin\left(4\times\frac{\pi}{3}\right)}{4} - \frac{4\sin\left(2\times\frac{\pi}{3}\right)}{2} + 3\left(\frac{\pi}{3}\right)\right\} - \left\{\frac{\sin\left(4\times\frac{\pi}{6}\right)}{4} - \frac{4\sin\left(2\times\frac{\pi}{6}\right)}{2} + 3\left(\frac{\pi}{6}\right)\right\}\right]$$

$$\Rightarrow I = \frac{1}{8}\left[\left(-\frac{\sqrt{3}}{8} - \sqrt{3} + \pi\right) - \left(\frac{\sqrt{3}}{8} - \sqrt{3} + \frac{\pi}{2}\right)\right]$$

$$\Rightarrow I = \frac{1}{8}\left[\left(-\frac{9\sqrt{3}}{8} + \pi\right) - \left(\frac{-7\sqrt{3}}{8} + \frac{\pi}{2}\right)\right]$$

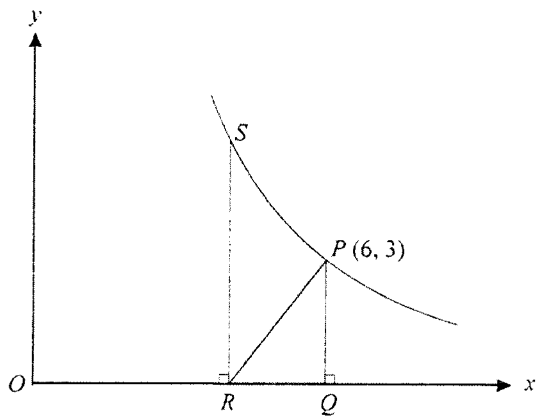
$$\Rightarrow I = \frac{1}{8}\left[-\frac{\sqrt{3}}{4} + \frac{\pi}{2}\right]$$

$$\therefore \boldsymbol{I = \frac{\pi}{16} - \frac{\sqrt{3}}{32}}$$

Revision Questions on Standard Integrals

June 2004 qp.1 (Cambridge)

7



The diagram shows part of the graph of $y = \frac{18}{x}$ and the normal to the curve at $P(6, 3)$. This normal meets the x-axis at R. The point Q on the x-axis and the point S on the curve are such that PQ and SR are parallel to the y-axis.

(i) Find the equation of the normal at P and show that R is the point $(4\frac{1}{2}, 0)$. [5]

(ii) Show that the volume of the solid obtained when the shaded region $PQRS$ is rotated through $360°$ about the x-axis is 18π. [4]

June 2001 qp.1 (Cambridge)

12.

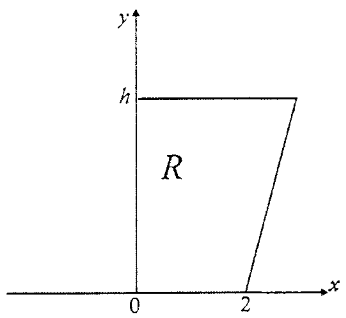


Fig.1

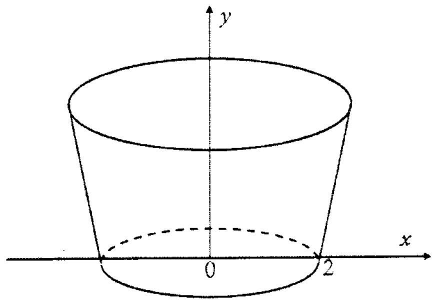


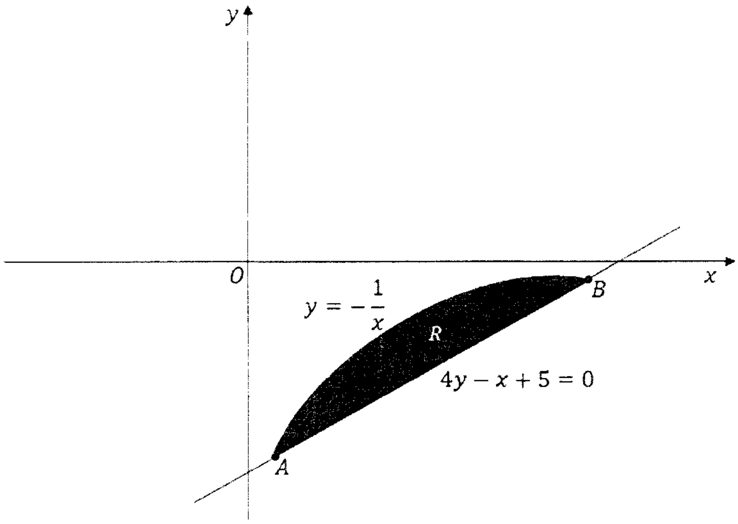
Fig.2

The region R shown in Fig. 1 is bounded by the line $y = 8(x - 2)$, the axes, and the line $y = h$. Find by integration, the volume formed when R is rotated through 360° about the y-axis (see diagram). [5]

A whisky glass has the shape indicated in Fig. 2, where the units are centimetres. A whisky taster holds the glass upright, and pours in whisky to a depth of 2cm. He then adds water to a further depth of 2 cm. Show that if he had poured in the water first and then the whisky, each to a depth of 2 cm, the glass would have contained approximately 25% more whisky that the first method. [3]

November 2004 qp.1 (Zimsec)

15. In the diagram below, the shaded region R is bounded by part of the graph $y = -\frac{1}{x}$ and the straight line $4y - x + 5 = 0$. The two graphs intersect at the points A and B.



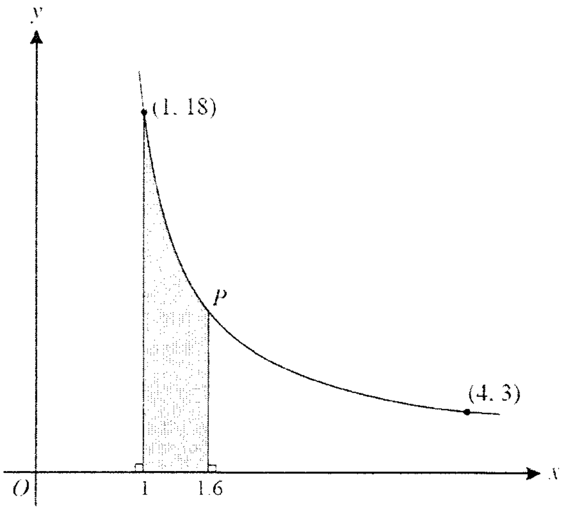
(i) Find the coordinates of the points A and B. [3]

(ii) Show that the area of the region R is $\frac{15}{8} - \ln 4$. [4]

(iii) Find the volume generated when the region R is rotated through 360° about the y axis, giving your answer in terms of π. [5]

June 2008 qp.1 (Cambridge)

9



The diagram shows a curve for which $\frac{dy}{dx} = -\frac{k}{x^3}$, where k is a constant. The curve passes through the points (1, 18) and (4, 3).

(i) Show, by integration, that the equation of the curve is $y = \frac{16}{x^2} + 2$. [4]

The point P lies on the curve and has x-coordinate 1.6.

(ii) Find the area of the shaded region. [4]

June 2007 qp.3 (Cambridge)

5 **(i)** Express $\cos\theta + (\sqrt{3})\sin\theta$ in the form $R\cos(\theta - \alpha)$, where $R > 0$ and $0 < \alpha < \frac{1}{2}\pi$, giving the exact values of R and α. [3]

(ii) Hence show that $\displaystyle\int_0^{\frac{1}{2}\pi} \frac{1}{(\cos\theta + (\sqrt{3})\sin\theta)^2}\,d\theta = \frac{1}{\sqrt{3}}$. [4]

November 2007 qp.3 (Cambridge)

1 Find the exact value of the constant k for which $\displaystyle\int_1^k \frac{1}{2x-1}\,dx = 1$. [4]

June 2010 qp.31 (Cambridge)

4 **(i)** Using the expansions of $\cos(3x - x)$ and $\cos(3x + x)$, prove that

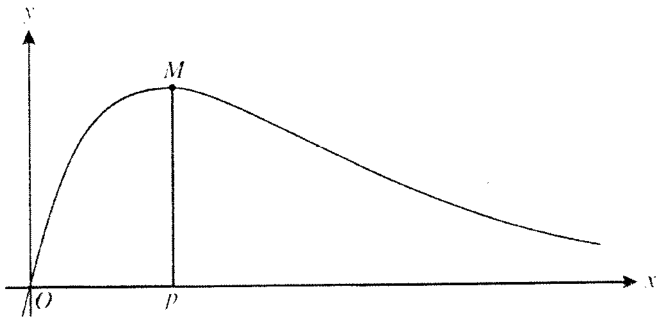
$$\tfrac{1}{2}(\cos 2x - \cos 4x) \equiv \sin 3x \sin x.$$ [3]

(ii) Hence show that

$$\int_{\frac{1}{6}\pi}^{\frac{1}{3}\pi} \sin 3x \sin x\,dx = \tfrac{1}{8}\sqrt{3}.$$ [3]

June 2010 qp.33 (Cambridge)

5



The diagram shows the curve $y = e^{-x} - e^{-2x}$ and its maximum point M. The x-coordinate of M is denoted by p.

(i) Find the exact value of p. [4]

(ii) Show that the area of the shaded region bounded by the curve, the x-axis and the line $x = p$ is equal to $\frac{1}{8}$. [4]

June 2010 qp.33 (Cambridge)

7 **(i)** Prove the identity $\cos 3\theta \equiv 4\cos^3\theta - 3\cos\theta$. [4]

(ii) Using this result, find the exact value of

$$\int_{\frac{1}{3}\pi}^{\frac{1}{2}\pi} \cos^3\theta \, d\theta.$$ [4]

June 2011 qp.31 (Cambridge)

9 **(i)** Prove the identity $\cos 4\theta + 4\cos 2\theta \equiv 8\cos^4\theta - 3$. [4]

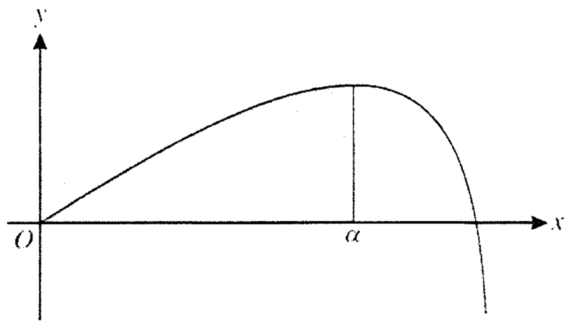
(ii) Hence

(a) solve the equation $\cos 4\theta + 4\cos 2\theta = 1$ for $-\frac{1}{2}\pi \leqslant \theta \leqslant \frac{1}{2}\pi$. [3]

(b) find the exact value of $\int_0^{\frac{1}{4}\pi} \cos^4\theta \, d\theta$. [3]

June 2012 qp.31 (Cambridge)

5



The diagram shows the curve

$$y = 8\sin\tfrac{1}{2}x - \tan\tfrac{1}{2}x$$

for $0 \leqslant x < \pi$. The x-coordinate of the maximum point is α and the shaded region is enclosed by the curve and the lines $x = \alpha$ and $y = 0$.

(i) Show that $\alpha = \frac{2}{3}\pi$. [3]

(ii) Find the exact value of the area of the shaded region. [4]

Advert Two: Proportion and Curve Sketching

"The elegance of a mathematical theorem is directly proportional to the number of independent ideas one can see in the theorem and inversely proportional to the effort it takes to see them."
– George Polya

Proportionality

Proportionality, also known as **variation**, is used to account for the relationship connecting two or more variables where the change in one variable triggers change in the other variable(s). This module only reviews three types of variation, that is **direct**, **inverse** and **joint** variation. The fourth type known as **partial** variation is beyond the scope of this syllabus.

Types of Variation

1. **Direct Variation**
   This type of variation analyses the positive relationship connecting two variables where the increase in one variable results in an increase in the other variable or vice-versa. For example given that x varies directly as y, the following notation is used to describe the relationship:

$$x \propto y \qquad \textit{where } \propto \textit{ is the proportionality sign}$$

$$\Rightarrow x = ky \qquad \begin{Bmatrix} \textit{to remove the proportional sign, introduce} \\ \textit{an equal sign and a constant of variation} \end{Bmatrix}$$

NB: The value of k is determined using the known values of x and y.

2. **Inverse Variation**
   Inverse variation is used to analyse the negative relationship connecting two variables where the increase in variable causes a decrease in the other variable or vice-versa. For example, given that x varies inversely as y, the following notation describes the relationship:

$$x \propto \frac{1}{y}$$

$$\Rightarrow x = k\left(\frac{1}{y}\right)$$

3. Joint Variation

This special type of variation analyses problems where one variable varies with two more variables at the same time. Joint variation can be experienced in three different scenarios where:

i. A variable varies directly with one variable and directly with another variable. For example, x is directly proportional to y and directly proportional to z. This can be written as:

$$x \propto (y)(z)$$
$$\Rightarrow x = kyz$$

ii. A variable varies inversely with one variable and inversely with another. For example, x is inversely proportional to y and inversely proportional to z. This can be written as:

$$x \propto \left(\frac{1}{y}\right)\left(\frac{1}{z}\right)$$
$$\Rightarrow x = k\left(\frac{1}{yz}\right)$$

iii. A variable varies directly with one variable and inversely with another. For example, x is directly proportional to y and inversely proportional to z. This relationship can be written as:

$$x \propto (y)\left(\frac{1}{z}\right)$$
$$\Rightarrow x = k\left(\frac{y}{z}\right)$$

<u>Worked Examination Question on Proportion</u>

Question (Cambridge, November 1998 qp.1)

3. The power, P kilowatts, needed from a car's engine to drive the car at its maximum speeds of $v\,\text{kmh}^{-1}$ on a level road is directly proportional to v^3. Calculate the percentage increase in power needed from the engine if the car's maximum speed is to be raised from 100 kmh^{-1} to 110 kmh^{-1}. [3]

Solution

3. $P \propto v^3$

$\Rightarrow P = kv^3$

$If\ the\ speed\ is\ to\ be\ raised\ from\ 100\ km/h\ to\ 110\ km/h\ implies\ the\ new$

$multiplier\ to\ speed\ is\ \frac{110}{100}\ resulting\ in\ a\ new\ speed\ of\ \left(\frac{110}{100}v\right).$

$Let\ P_1\ be\ the\ new\ power\ after\ the\ speed\ has\ increased.$

$\Rightarrow P_1 = k\left(\frac{110}{100}v\right)^3$

$\Rightarrow P_1 = k\left(\frac{1331}{1000}v^3\right)$

$\Rightarrow P_1 = \frac{1331}{1000}kv^3 \quad where\ kv^3\ is\ the\ original\ amount\ of\ P$

$\Rightarrow P_1 = 1{,}331P$

$Now,\ P_1\ is\ given\ by\ multiplying\ the\ original\ power\ by\ 1{,}331$

$\therefore\ \boldsymbol{the\ percentage\ increase\ in\ power}\ \mathbf{133\%}.$

Revision Questions on Proportion

November 1997 qp.1 (Cambridge)

4. A planet whose mean distance from the sun is Rkm completes one revolution of its orbit round the sun in T days. The relation between T and R is that T^2 is directly proportional to R^3. The mean distances of the Earth and the planet Saturn from the sun are 1.50×10^8 km and 1.43×10^9 km respectively. Assuming that the Earth takes 365 days to complete one revolution of its orbit, find the corresponding number of days taken by Saturn. [4]

June 2003 qp.1 (Zimsec)

3. The variables x and y are inversely proportional to each other. A third variable z is proportional to the square root of y. Express x in terms of z. [3]

November 2003 qp.1 (Zimsec Specimen Paper)

10. The volume of a cone varies jointly as the height and square of the base radius. Calculate the percentage change in volume if the base radius is increased by 70% and the height decreased by 10%. [5]

June 2001 qp.1 (Cambridge)

3. It is given that z varies directly as x^3 and inversely as y^2.

   i. Write down an equation expressing z in terms of x, y and a constant. [1]

   ii. Find the percentage change in z when x is increased by 20% and y is increased by 50%. State whether the change is an increase or a decrease. [3]

Curve Sketching

Questions on curve sketching test the ability of students to draw trig, logarithmic, exponential, and algebraic (quadratic and linear, for example) graphs. As such, students are strongly encouraged to consolidate curve sketching and transformations so that they have the concepts at the tips of their fingers.

1. **Trig Graphs**

   i. $y = \sin\theta$

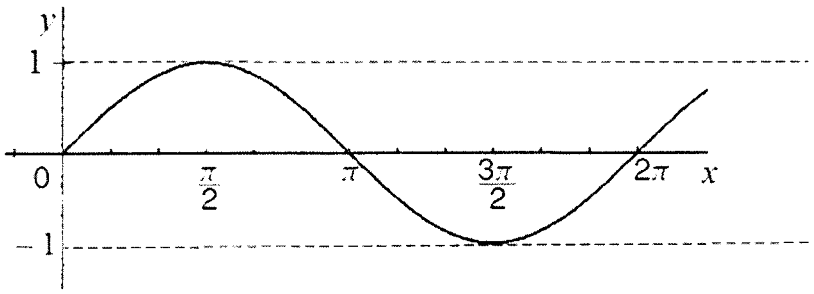


Fig. A2.1

   ii. $y = \cos\theta$

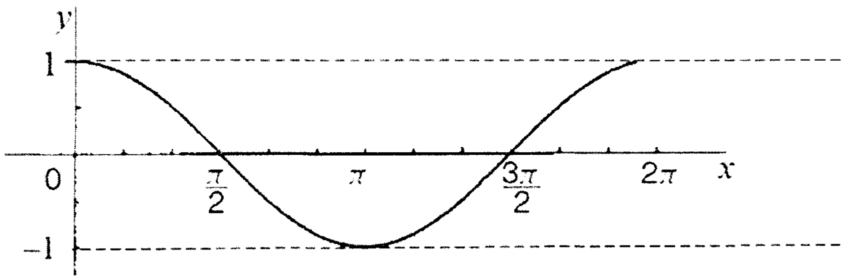


Fig. A2.2

iii. $y = \tan\theta$

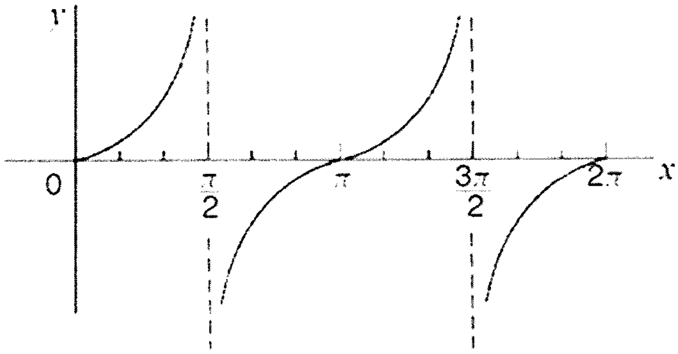


Fig. A2.3

iv. $y = \mathbf{cosec}\,\theta$

$y = \frac{1}{\sin\theta}$ is a reflection of the graph of $y = \sin\theta$ using the mirror lines $y = 1$ and $y = -1$. This graph is disjointed, with the asymptotes at the points where the graph of $y = \sin\theta$ crosses the x –axis (see Fig. A2.4).

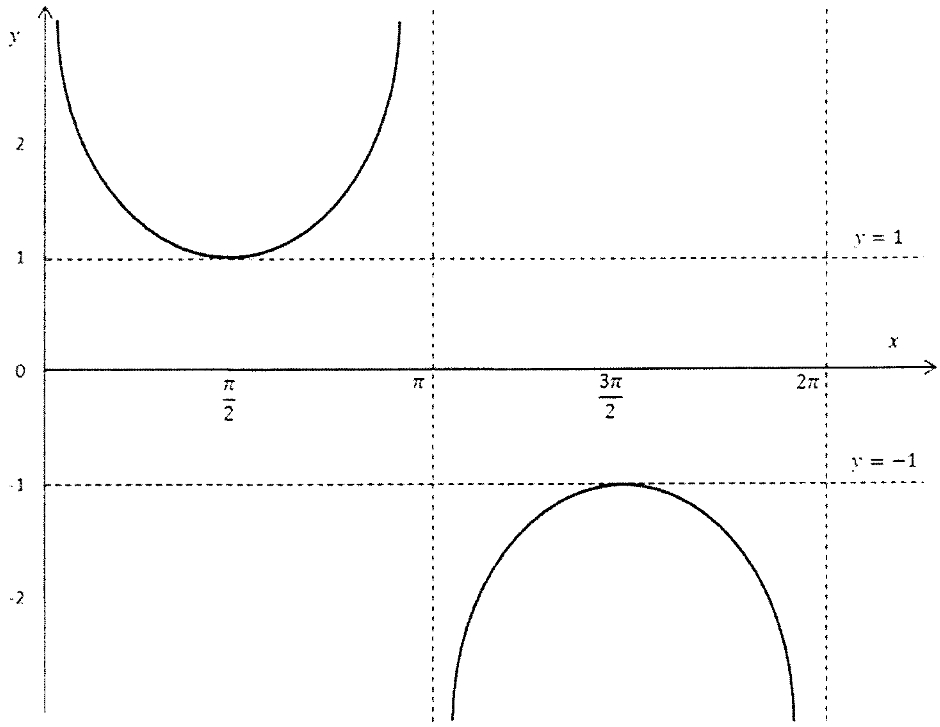


Fig. A2.4

v. $\boldsymbol{y = \sec\theta}$

$y = \frac{1}{\cos\theta}$ is a reflection of the graph of $y = \cos\theta$ using the mirror lines $y = 1$ and $y = -1$. The graph of $y = \sec\theta$ is not continuous. It has asymptotes at the points where the graph of $y = \cos\theta$ crosses the x –axis.

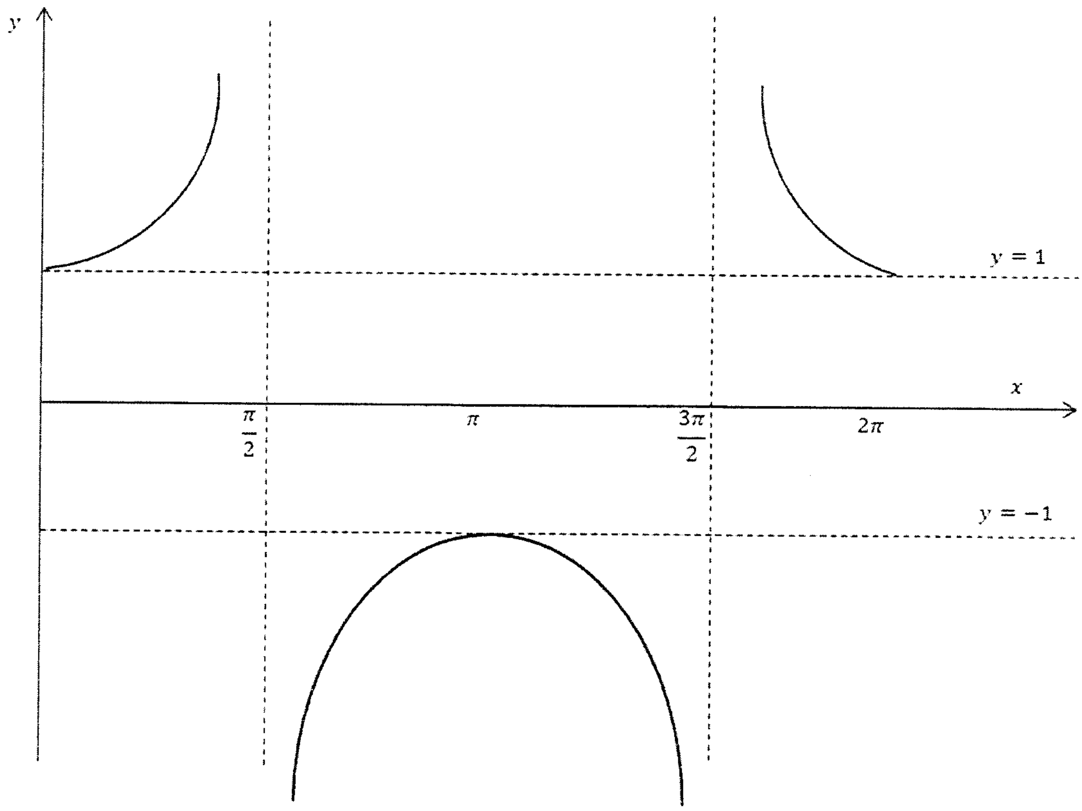


Fig. A2.5

vi. $\boldsymbol{y = \cot\theta}$

$y = \frac{1}{\tan\theta}$ is an adjustment to the graph of $y = \tan\theta$ with new set of asymptotes at the points where the graph of $y = \tan\theta$ crosses the x –axis. The graph of $y = \cot\theta$ moves from the left asymptote to the right asymptote through the midpoint to a pair of asymptotes.

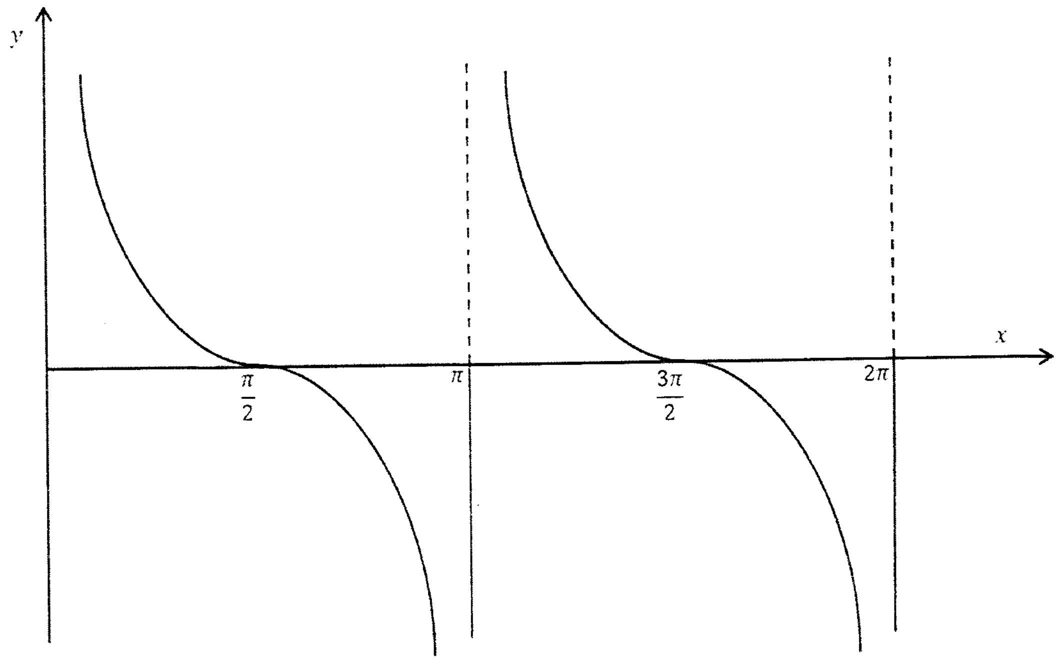


Fig. A2.6

2. Logarithmic Graphs

All logarithms are defined for positive values of x and cross the x −axis at 1 since $\ln(1) = 0$. The line $x = 0$ (y axis) is an asymptote of the graph $y = \ln x$. As x increases, y increases at a decreasing rate as shown by the curve in Fig. A2.7.

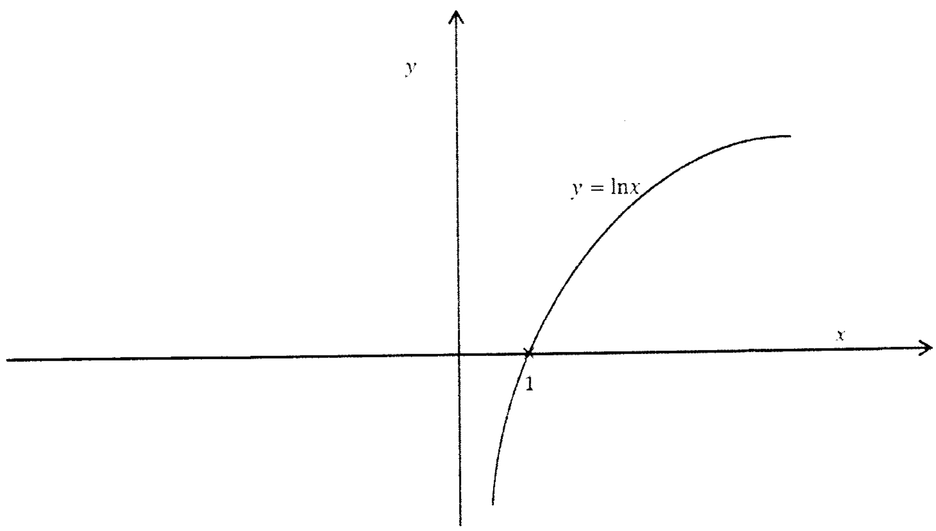


Fig. A2.7

3. Exponential Graphs

An exponential graph is a reflection of the graph $y = \ln(x)$ in the mirror line $y = x$. As such, it crosses the y −axis at 1 and uses the x −axis as the asymptote.

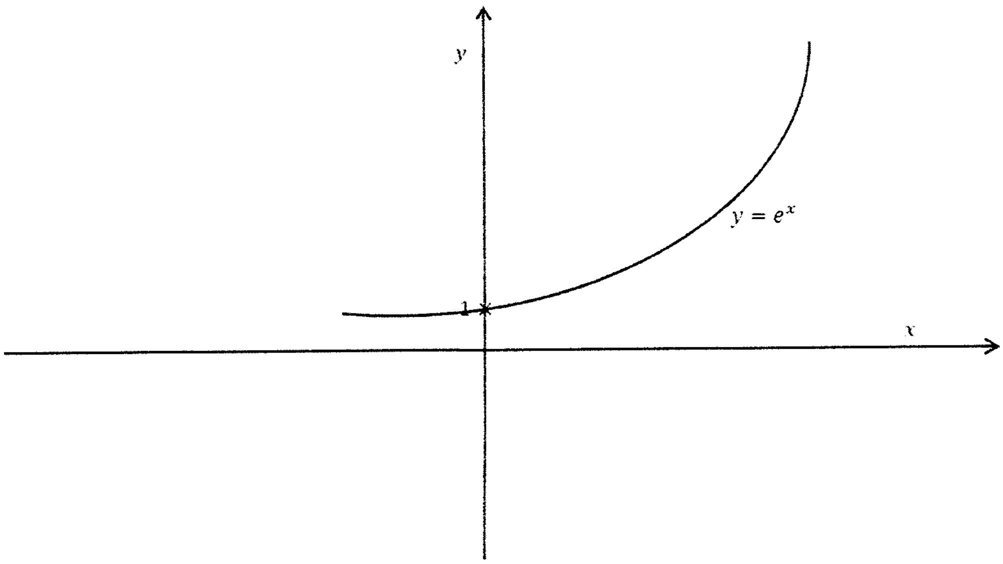


Fig. A2.8

Transformation of Graphs

'A' level transformation places specific emphasis on translation, stretch and reflection. One should be able to identify and describe a particular transformation, and effect the transformation. Transformation can be effected parallel to either the x-axis or y-axis. Table A2.1 summarises the three transformations:

Table A2.1

	y axis	x axis
Translation	$y = f(x) \pm a$	$y = f(x \pm a)$ Reverse translation
Stretch	$y = af(x)$	$y = f(ax)$ Reduction
Reflection	$y = f(-x)$	$y = -f(x)$

Translation

Translation refers to the linear movement of an object/shape. When a value has been 'added to' or 'subtracted from' the original function, an object has to be moved up or down (vertical shift). Addition denotes an upward shift and subtraction denotes a downward shift. When a value has been directly 'added to' or 'subtracted from' the x-value, an object has to be moved parallel to the x-axis. Critical to note is the fact that horizontal translation reverses the potential direction of motion. For example, adding a particular value is denoted by a movement to the left or vice versa.

Stretch

This is used to describe a multiplier or elastic effect. When a multiplier has been introduced to the whole function, an object has to be amplified by the scale factor in the y-direction. If a multiplier is directly affecting the x-value, then the object has to be reduced by the size of the multiplier in the direction parallel to the x-axis.

Reflection

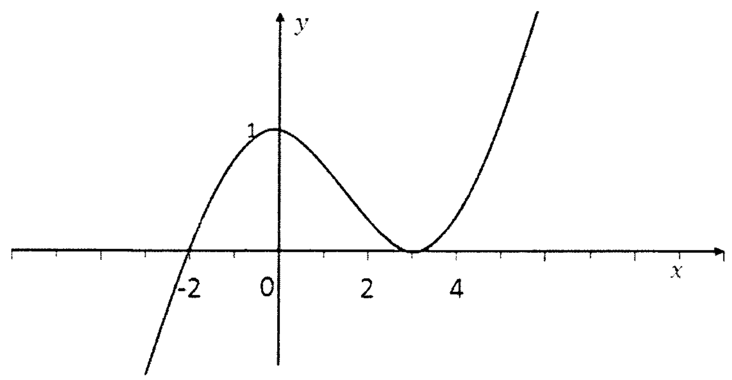
This is used to account for the mirror effect. A minus sign is used to describe a reflection. An object is reflected in the mirror line $x = 0$ (that is the y-axis), when the minus directly affects the x-value. In cases where the minus sign is affecting the function as a whole, the object is reflected in the mirror line $y = 0$ (that is the x-axis).

NB: to fully unravel the concept, practical examples will be used as an addendum to the theory above.

Worked Examination Questions on Curve Sketching

Question (Zimsec Specimen Paper, November 2003 qp.1)

7



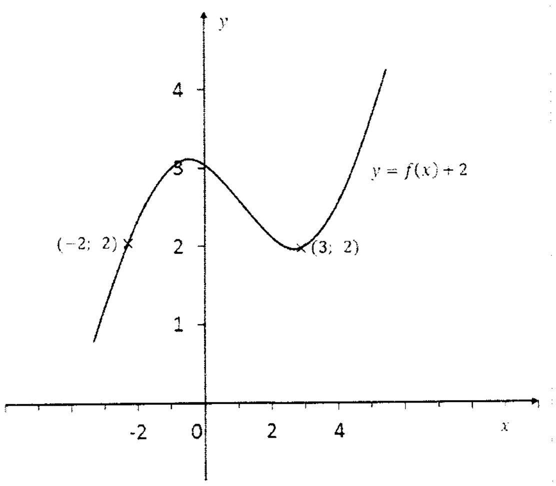
The graph of $y = f(x)$ is shown in the diagram above. On separate diagrams, sketch the graphs of

(i). $y = f(x) + 2$ showing coordinates of intersection with the y-axis, [2]

(ii). $y = 3f(x)$ showing coordinates of intersection with the axes. [2]

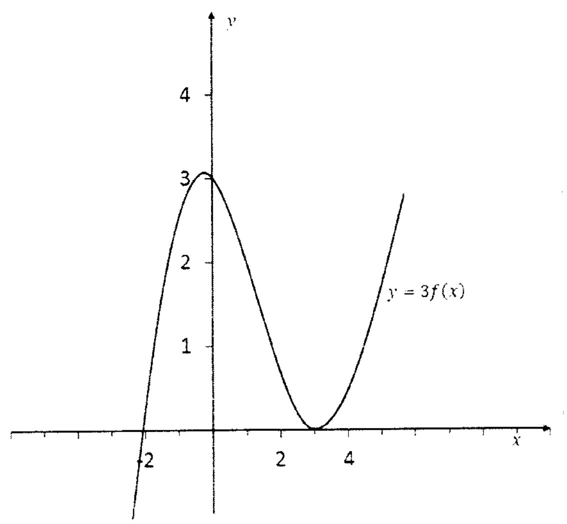
Solution

(i).



An upward vertical shift by 2 units

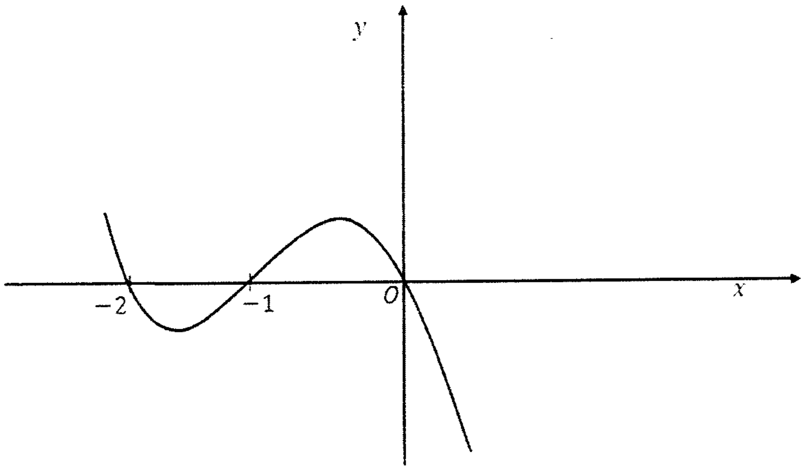
(ii).



A stretch by scale factor 3 parallel to the y-axis. All points sitting on the x axis retain their positions.

Question (Cambridge, June 1996 qp.1)

2.



The graph of $y = f(x)$ is shown in the diagram. On separate diagrams sketch the graph of

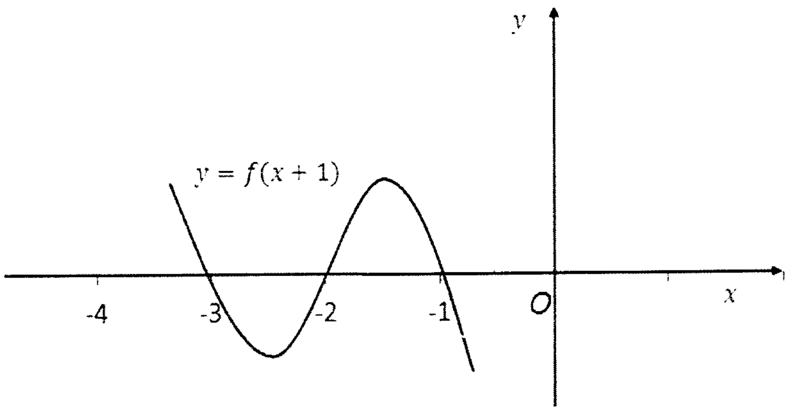
(i). $y = f(x+1)$,

(ii). $y = |f(x+1)|$,

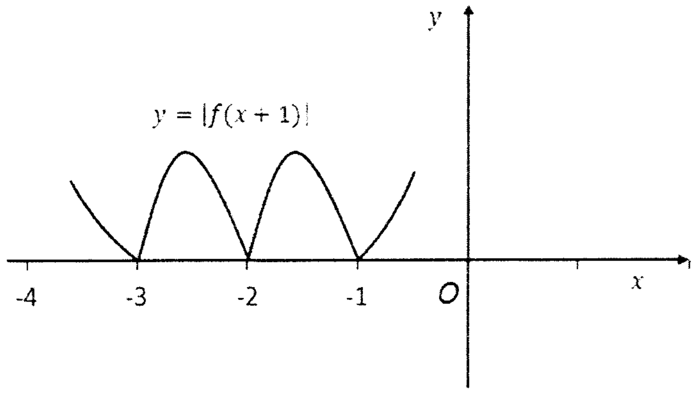
Showing the coordinates of the points where the graphs meet the x-axis. [3]

Solution

(i).



Reverse translation by 1 unit parallel to the x-axis (ii).



Reverse translation by 1 unit parallel to the x-axis followed by a reflection in the mirror line $y = 0$ (that is the x-axis).

Question (Zimsec, June 2010 qp.1)

15. The function f is defined as

$$f: x \longrightarrow \frac{2-x}{x+1}, \qquad x \neq -1$$

a) (i). Express $f(x)$ in form $+\frac{b}{x+1}$, where a and b are constants . [2]

(ii). Hence, give a sequence of three transformations which take the graph of $y = \frac{1}{x}$ onto the graph of $y = f(x)$. [3]

(iii). State the range of f. [1]

Solution

(i) $f(x) = \frac{2-x}{x+1}$

by long division,

$$\Rightarrow f(x) = \begin{array}{r|l} & \quad -1 \\ \hline (x+1) & (-x+2) \\ & -(-x-1) \\ \hline & \quad 3 \end{array}$$

$$\therefore \boldsymbol{f(x) = -1 + \frac{3}{(x+1)}}$$

(ii) $y = \frac{1}{x}$ *is the original graph*

$$\frac{1}{x} \to \frac{1}{(x+1)} \text{ is a reverse translation by one unit parallel to the } x-\text{axis}$$

$$\frac{1}{(x+1)} \to \frac{3}{(x+1)} \text{ is a stretch parallel to the } y-\text{axis by scale factor } 3$$

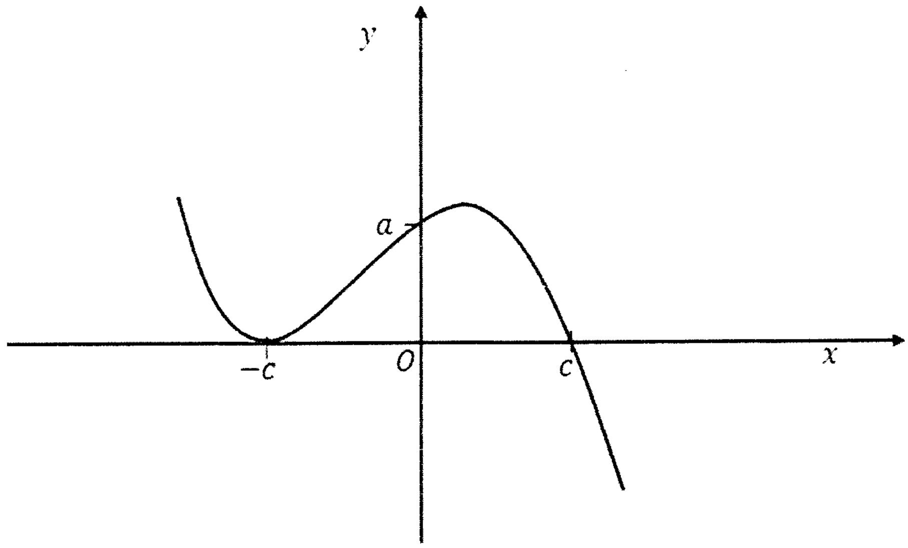
$$\frac{3}{(x+1)} \to -1 + \frac{3}{(x+1)} \quad \text{is a translation by one unit in the negative } y-\text{direction}$$

(iii) *Range*: $f(x) > -1$ *and* $f(x) < -1$.

Revision Questions on Curve Sketching

November 1995 qp.1 (Cambridge)

3.



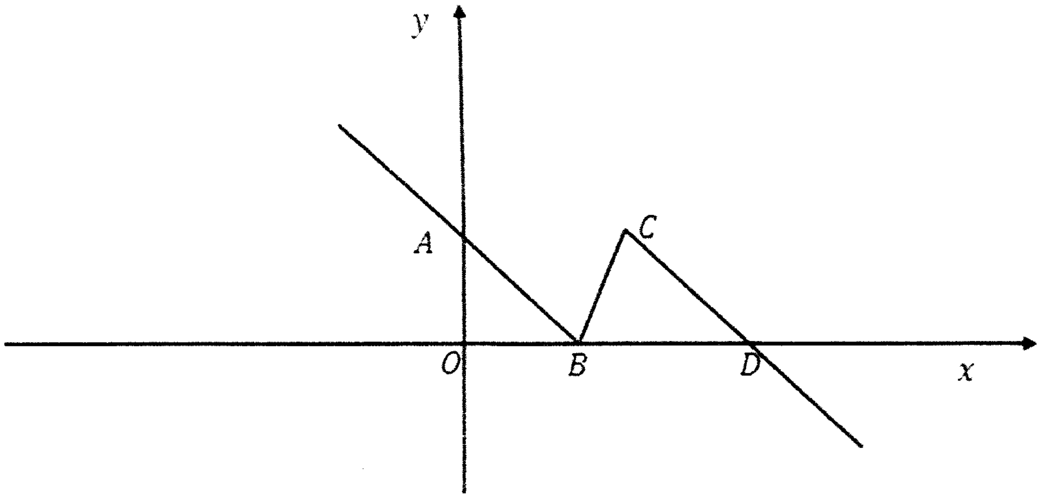
The graph of $y = f(x)$ is shown in the digram. It is given that a, b and c are positive constants. On separate diagrams, sketch the graphs of

i. $y = f(x) + a$, showing the coordinates of the intersection with the y-axis, [2]

ii. $y = -bf(x)$, showing the coordinates of the intersections with the axes. [2]

November 1990 qp.1 (Cambridge)

3.



The graph of $y = f(x)$ is shown above. The points A, B, C and D have coordinates (0, 1), (1, 0), (2, 1) and (3, 0) respectively. Sketch, separately, the graphs of

(i). $y = f(2x)$,

(ii). $y = f(x + 3)$,

stating, in each case, the coordinates of the points corresponding to A, B, C and D. [4]

June 1997 qp.1 (Cambridge)

6. It is given that

$$f(x) \equiv (x - \alpha)(x + \beta), \qquad x \in \mathbb{R}$$

where α and β are positive constants. Sketch on separate diagrams the curves with the following equations, giving in each case the coordinates of the points at which the curve meets the x -axis.

(i). $y = f(x)$, [2]

(ii). $y = |f(x)|$, [2]

(iii). $y = f(x + 2\alpha)$. [3]

November 2001 qp.1 (Zimsec)

13. (a). The functions f and g are defined for all real values of x by

$$f: x \mapsto \sin x \quad and \quad g: x \longmapsto \sin\left(2x - \frac{1}{6}\pi\right)$$

State a sequence of two geometrical transformations under which the graph of $y = f(x)$ is transformed onto the graph of $y = g(x)$. [5]

June 2003 qp.1 (Cambridge)

6 (i) Sketch the graph of the curve $y = 3\sin x$, for $-\pi \leqslant x \leqslant \pi$. [2]

The straight line $y = kx$, where k is a constant, passes through the maximum point of this curve for $-\pi \leqslant x \leqslant \pi$.

(ii) Find the value of k in terms of π. [2]

(iii) State the coordinates of the other point, apart from the origin, where the line and the curve intersect. [1]

November 2004 qp.1 (Cambridge)

4 (i) Sketch and label, on the same diagram, the graphs of $y = 2\sin x$ and $y = \cos 2x$, for the interval $0 \leqslant x \leqslant \pi$. [4]

(ii) Hence state the number of solutions of the equation $2\sin x = \cos 2x$ in the interval $0 \leqslant x \leqslant \pi$. [1]

November 2009 qp.11 (Cambridge)

2 The equation of a curve is $y = 3\cos 2x$. The equation of a line is $x + 2y = \pi$. On the same diagram, sketch the curve and the line for $0 \leqslant x \leqslant \pi$. [4]

November 2010 qp.13 (Cambridge)

4 (i) Sketch the curve $y = 2\sin x$ for $0 \leqslant x \leqslant 2\pi$. [1]

(ii) By adding a suitable straight line to your sketch, determine the number of real roots of the equation

$$2\pi \sin x = \pi - x.$$

State the equation of the straight line. [3]

November 2011 qp.11 (Cambridge)

3 (i) Sketch, on a single diagram, the graphs of $y = \cos 2\theta$ and $y = \frac{1}{2}$ for $0 \leqslant \theta \leqslant 2\pi$. [3]

(ii) Write down the number of roots of the equation $2\cos 2\theta - 1 = 0$ in the interval $0 \leqslant \theta \leqslant 2\pi$. [1]

(iii) Deduce the number of roots of the equation $2\cos 2\theta - 1 = 0$ in the interval $10\pi \leqslant \theta \leqslant 20\pi$. [1]

Chapter Twelve: Differential Equations

"Big jobs go to the men who prove their ability to outgrow small ones."

– Ralph Waldo Emerson

Differential equations are mathematical statements that resemble a state of equilibrium where one of the terms is a differential coefficient. This topic marries three concepts; - proportionality, differentiation and integration.

- Proportionality is used in the formulation of a differential equation where the change in one variable directly or inversely influences the change in the other variable. This is a mere application of 'O' Level proportionality or variation.
- Differentiation pops up in passing in questions that are inclined to **rate of change**. Rate is used to describe the change in one variable *with respect to time*. As such, rate of change in volume, rate of change in area, rate of change in radius and rate of change in x may be expressed as:

$$\frac{dV}{dt} \; ; \; \frac{dA}{dt} \; ; \; \frac{dr}{dt} \; and \; \frac{dx}{dt} \; respectively$$

- Integration plays a pivotal role in problem solving because it is used in the prescription of a general solution.

The process approach to differential equations considers the following steps:

- Formulation of a differential equation using proportion, rate of change or a combination of some sort.
- Solve for k, the constant of variation, using initial conditions.
- Separation of variables; this is the collection of variables of the same family to one side and the variables of the other family to the other side. This is merely rearranging the equation. For example, if a differential equation is in terms of x and t, all the variables in x are to be collected to one side and the variables in t to other side.
- Integration of both the LHS and RHS using relevant integration techniques for each side. It is important that one gains a full appreciation of integration in order to choose the most appropriate integration technique. This process gives birth to the general solution where the constant of integration, c, is unknown.

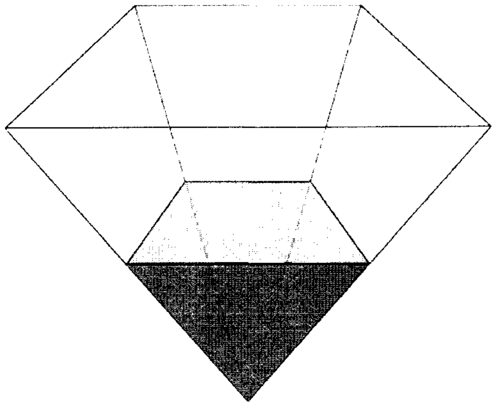
- Solve for c, the constant of integration, using the initial results and substitute c in the general solution.
- Solve other problems, if any exist, using the general solution.

NB: This topic is best explained using practical examples. The forthcoming section unravels some past exam questions on differential equations.

Worked Examination Questions on Differential Equations

Question (Cambridge, November 2008 qp.3)

8



An underground storage tank is being filled with liquid as shown in the diagram. Initially the tank is empty. At time t hours after filling begins, the volume of liquid is V m^3 and the depth of liquid is h m. It is given that $V = \frac{4}{3}h^3$.

The liquid is poured in at a rate of 20 m^3 per hour, but owing to leakage, liquid is lost at a rate proportional to h^2. When $h = 1$, $\frac{dh}{dt} = 4.95$.

(i) Show that h satisfies the differential equation

$$\frac{dh}{dt} = \frac{5}{h^2} - \frac{1}{20}.$$ [4]

(ii) Verify that $\frac{20h^2}{100 - h^2} \equiv -20 + \frac{2000}{(10-h)(10+h)}$. [1]

(iii) Hence solve the differential equation in part **(i)**, obtaining an expression for t in terms of h. [5]

Solution

(i) $The\ rate\ of\ change\ in\ volume,\ \frac{dv}{dt},\ can\ be\ broken\ down\ into\ two:$

- $filling\ rate - accounting\ for\ the\ incease\ in\ volume,$
- $leakage\ rate - accounting\ for\ the\ decrease\ in\ volume.$

$where\ filling\ rate: \frac{dv}{dt} = 20,$

$and\ leakage\ rate: \frac{dv}{dt} \propto h^2$

$\Rightarrow \frac{dv}{dt} = kh^2$

$As\ such, overall\ change\ in\ volume,\ \frac{dv}{dt} = 20 - kh^2,$

$given\ that\ v = \frac{4}{3}h^3$

$\Rightarrow \frac{dv}{dh} = 4h^2,$

$using\ the\ chain\ rule,$

$$\frac{dh}{dt} = \frac{dv}{dt} \div \frac{dv}{dh}$$

$$\Rightarrow \frac{dh}{dt} = \frac{(20 - kh^2)}{4h^2}$$

$by\ substituting\ initial\ results\ to\ find\ k,$

$$4.95 = \frac{20 - k(1)^2}{4(1)^2}$$

$$\Rightarrow 19.8 = 20 - k$$

$$\Rightarrow k = 0.2$$

$$\Rightarrow \frac{dh}{dt} = \frac{20 - 0.2h^2}{4h^2}$$

$$\Rightarrow \frac{dh}{dt} = \frac{20}{4h^2} - \frac{0.2h^2}{4h^2}$$

$$\therefore \boldsymbol{\frac{dh}{dt} = \frac{5}{h^2} - \frac{1}{20}\ (shown)}$$

(ii) $\dfrac{20h^2}{100-h^2} = -20 + \dfrac{2000}{(10-h)(10+h)}$

$let\ h = 0,$

$\Rightarrow \dfrac{20(0)^2}{100-(0)^2} = -20 + \dfrac{2000}{(10-0)(10+0)}$

$\Rightarrow \dfrac{0}{100} = -20 + \dfrac{2000}{100}$

$\therefore \mathbf{0 = 0\ (\mathit{shown})}$

$\boldsymbol{NB}: This\ is\ a\ simple\ process\ confirming\ that\ the\ LHS \equiv RHS$

(iii) $\dfrac{dh}{dt} = \dfrac{5}{h^2} - \dfrac{1}{20}$

$\Rightarrow \dfrac{dh}{dt} = \dfrac{100-h^2}{20h^2}$

$\Rightarrow 20h^2dh = (100-h^2)dt$

$\Rightarrow \displaystyle\int \frac{20h^2}{100-h^2}dh = \int dt$

$where\ \dfrac{20h^2}{100-h^2} \equiv -20 + \dfrac{2000}{(10-h)(10+h)}$

$\Rightarrow \displaystyle\int -20 + \frac{2000}{(10-h)(10+h)}dh = \int dt$

$using\ partial\ fractions,$

$\dfrac{2000}{(10-h)(10+h)} = \dfrac{A}{(10-h)} + \dfrac{B}{(10+h)}$

$\Rightarrow 2000 = A(10+h) + B(10-h)$

$let\ h = 10$ | $let\ h = -10$

$\Rightarrow 2000 = 20A$ | $\Rightarrow 2000 = 20B$

$\Rightarrow a = 100$ | $\Rightarrow B = 100$

$\Rightarrow \dfrac{2000}{(10-h)(10+h)} = \dfrac{100}{(10-h)} + \dfrac{100}{(10+h)},$

$Now,\ \displaystyle\int -20 + \frac{100}{(10-h)} + \frac{100}{(10+h)}dh = \int dt$

$$\Rightarrow \int -20\,dh + 100\int \frac{1}{(10-h)}\,dh + 100\int \frac{1}{(10+h)}\,dh = \int dt$$

$$\Rightarrow -20h - 100\ln(10-h) + 100\ln(10+h) = t + c$$

$$\Rightarrow -20h + 100\ln(10+h) - 100\ln(10-h) = t + c$$

$$\Rightarrow -20h + 100\ln\left(\frac{10+h}{10-h}\right) = t + c$$

$\boldsymbol{NB}$: *Initially the tank is empty, implying that,* $h = 0$ *when* $t = 0$

$$\Rightarrow -20(0) + 100\ln\left(\frac{10+0}{10-0}\right) = 0 + c$$

$$\Rightarrow c = 100\ln(1)$$

$$\Rightarrow c = 0$$

$$\therefore \boldsymbol{t = -20h + 100\ln\left(\frac{10+h}{10-h}\right)}$$

Question (Cambridge, November 2011 qp.31)

4 The variables x and θ are related by the differential equation

$$\sin 2\theta \frac{dx}{d\theta} = (x+1)\cos 2\theta,$$

where $0 < \theta < \frac{1}{2}\pi$. When $\theta = \frac{1}{12}\pi$, $x = 0$. Solve the differential equation, obtaining an expression for x in terms of θ, and simplifying your answer as far as possible. [7]

Solution

$$\sin 2\theta \frac{dx}{d\theta} = (x+1)\cos 2\theta$$

$$\Rightarrow \sin 2\theta\, dx = (x+1)\cos 2\theta\, d\theta$$

$$\Rightarrow \frac{1}{(x+1)}dx = \frac{\cos 2\theta}{\sin 2\theta}d\theta$$

$$\Rightarrow \int \frac{1}{(x+1)}dx = \frac{1}{2}\int \frac{2\cos 2\theta}{\sin 2\theta}d\theta$$

$$\Rightarrow \ln(x+1) = \frac{1}{2}\ln(\sin 2\theta) + c$$

using initial results,

$$\Rightarrow \ln(0+1) = \frac{1}{2}\ln\left[\sin\left(2\times\frac{\pi}{12}\right)\right] + c$$

$$\Rightarrow c = -\frac{1}{2}\ln\left[\sin\left(\frac{\pi}{6}\right)\right]$$

$$\Rightarrow c = -\frac{1}{2}\ln\left(\frac{1}{2}\right)$$

$$\Rightarrow c = -\frac{1}{2}\ln 2^{-1}$$

$$\Rightarrow c = \frac{1}{2}\ln 2$$

Now, $\ln(x+1) = \frac{1}{2}\ln(\sin 2\theta) + \frac{1}{2}\ln 2$

$$\Rightarrow \ln(x+1) = \frac{1}{2}\ln(2\sin 2\theta)$$

$$\Rightarrow \ln(x+1) = \ln(2\sin 2\theta)^{\frac{1}{2}}$$

by introducing e to both sides,

$$\Rightarrow (x+1) = (2\sin 2\theta)^{\frac{1}{2}}$$

$$\Rightarrow x + 1 = \sqrt{2\sin 2\theta}$$

$$\therefore \boldsymbol{x = \sqrt{2\sin 2\theta} - 1}$$

Question (Cambridge, June 2012 qp.31)

7 The variables x and y are related by the differential equation

$$\frac{dy}{dx} = \frac{6xe^{3x}}{y^2}.$$

It is given that $y = 2$ when $x = 0$. Solve the differential equation and hence find the value of y when $x = 0.5$, giving your answer correct to 2 decimal places. [8]

Solution

$$\frac{dy}{dx} = \frac{6xe^{3x}}{y^2}$$

$$\Rightarrow \int y^2\, dy = \int 6xe^{3x}\, dx$$

$$\Rightarrow \frac{y^3}{3} = \int 6xe^{3x}\, dx$$

$$where\ the\ RHS = \int 6xe^{3x}\, dx$$

using integration by parts,

$let\ u = 6x$ *and* $\frac{dv}{dx} = e^{3x}$

$\frac{du}{dx} = 6$ $v = \frac{1}{3}e^{3x}$

$$\Rightarrow RHS = (6x)\left(\frac{1}{3}e^{3x}\right) - \int (6)\left(\frac{1}{3}e^{3x}\right) dx$$

$$\Rightarrow RHS = 2xe^{3x} - \int 2e^{3x}\, dx$$

$$\Rightarrow RHS = 2xe^{3x} - \frac{2}{3}e^{3x} + c$$

$$\Rightarrow RHS = 2e^{3x}\left[x - \frac{1}{3}\right] + c$$

$$Now,\ \frac{y^3}{3} = 2e^{3x}\left(x - \frac{1}{3}\right) + c$$

using initial results,

$$\Rightarrow \frac{(2)^3}{3} = 2e^{(3)(0)}\left(0 - \frac{1}{3}\right) + c$$

$$\Rightarrow \frac{8}{3} = -\frac{2}{3} + c$$

$$\Rightarrow c = \frac{10}{3}$$

$$\Rightarrow \frac{y^3}{3} = 2e^{3x}\left(x - \frac{1}{3}\right) + \frac{10}{3}$$

$$\Rightarrow y^3 = 6e^{3x}\left(x - \frac{1}{3}\right) + 10$$

$$\Rightarrow y = \sqrt[3]{6e^{3x}\left(x - \frac{1}{3}\right) + 10}$$

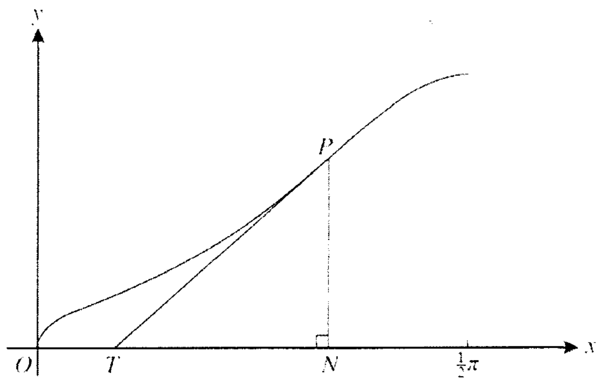
$when\ x = 0.5,$

$$\Rightarrow y = \sqrt[3]{6e^{3(0.5)}\left(0.5 - \frac{1}{3}\right) + 10}$$

$\therefore \boldsymbol{y = 2.44}$

Question (Cambridge, June 2008 qp.3)

8



In the diagram the tangent to a curve at a general point P with coordinates (x, y) meets the x-axis at T. The point N on the x-axis is such that PN is perpendicular to the x-axis. The curve is such that, for all values of x in the interval $0 < x < \frac{1}{2}\pi$, the area of triangle PTN is equal to $\tan x$, where x is in radians.

(i) Using the fact that the gradient of the curve at P is $\frac{PN}{TN}$, show that

$$\frac{dy}{dx} = \frac{1}{2}y^2 \cot x. \qquad [3]$$

(ii) Given that $y = 2$ when $x = \frac{1}{6}\pi$, solve this differential equation to find the equation of the curve, expressing y in terms of x. [6]

Solution

(i) *Area of a triangle is given by,*

$$\frac{1}{2} \times base \times \perp height$$

$$In\ this\ case, \tan x = \frac{1}{2}(TN)(PN) \longrightarrow 1$$

$$and\ gradient,\ \frac{dy}{dx} = \frac{PN}{TN} \longrightarrow 2$$

$where\ PN = y$

$$using\ (1),\quad \tan x = \frac{1}{2}(TN)(y)$$

by expressing TN in terms of y,

$$\Rightarrow 2\tan x = y(TN)$$

$$\Rightarrow TN = \left(\frac{2\tan x}{y}\right)$$

by substituting TN and PN in (2),

$$\Rightarrow \frac{dy}{dx} = \frac{y}{\left(\frac{2\tan x}{y}\right)}$$

$$\Rightarrow \frac{dy}{dx} = \frac{y^2}{2\tan x}$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{2}(y^2)\left(\frac{1}{\tan x}\right)$$

$$\therefore \boldsymbol{\frac{dy}{dx} = \frac{1}{2}y^2 \cot x\ (shown)}$$

(ii) $$\frac{dy}{dx} = \frac{1}{2}y^2 \cot x$$

$$\Rightarrow dy = \frac{1}{2}y^2 \cot x\ dx$$

$$\Rightarrow \int \frac{1}{y^2}dy = \int \frac{1}{2}\cot x\ dx$$

$$\Rightarrow \int y^{-2}dy = \frac{1}{2}\int \cot x\ dx$$

$$\Rightarrow \frac{y^{-1}}{-1} = \frac{1}{2}\ln(\sin x) + c$$

using initial results,

$$\Rightarrow \frac{2^{-1}}{-1} = \frac{1}{2}\ln\left(\sin\frac{\pi}{6}\right) + c$$

$$\Rightarrow -\frac{1}{2} = \frac{1}{2}\ln\left(\frac{1}{2}\right) + c$$

$$\Rightarrow c = -\frac{1}{2} - \frac{1}{2}\ln 2^{-1}$$

$$\Rightarrow c = \frac{1}{2}\ln 2 - \frac{1}{2}$$

$$Now, \quad -\frac{1}{y} = \frac{1}{2}\ln(\sin x) + \frac{1}{2}\ln 2 - \frac{1}{2}$$

$$\Rightarrow -\frac{1}{y} = \frac{1}{2}\ln(2\sin x) - \frac{1}{2}$$

$$\Rightarrow -\frac{1}{y} = \frac{\ln(2\sin x) - 1}{2}$$

$$\Rightarrow -2 = y[\ln(2\sin x) - 1]$$

$$\therefore \mathbf{y = -\frac{2}{[\ln(2\sin x) - 1]}}$$

Revision Questions on Differential Equations

November 2003 qp.1 (Zimsec)

13. A water tank with a uniform cross-section has a tap at its base. When the tap is opened water flows out at a rate proportional to the square root of the depth of water in the tank. Given that the cross-sectional area of the tank is 5cm^2 and the depth of water t minutes after opening is h metres, show that

$$\frac{dh}{dt} = -\frac{k}{5}\sqrt{h}\,. \qquad [6]$$

Given that the tap is opened when the depth of water is 2 metres, find an expression in terms of k for the time taken for the depth to reach 1 metre. [5]

November 2010 qp.1 (Zimsec)

16. (a) Some juice is tapped into a cylindrical container at a rate of 100cm^3 per minute. It is sieved out through a hole at the bottom of the cylinder at a rate of $2.5h\ \text{cm}^3$ per minute, where h is the height of the juice in the cylinder at time t minutes. The radius of the cylinder is 5cm.

(i). Show that $\frac{dh}{dt} = \frac{(40-h)}{10\pi}$. [4]

(ii). Solve the differential equation to find h in terms of t, given that at time $t = 0, h = 0$.

Hence or otherwise state the maximum value of h and explain why this height cannot be exceeded. [8]

(b). The delivering tap in part (a) was closed off when juice was at maximum height, and the juice was allowed to drain out. The differential equation satisfied by the drainage process only is $\frac{dh}{dt} = -\frac{1}{10\pi}h$.

Solve this differential equation to find t in terms of h and show that the time taken to make the juice go down to a height of 0.88 cm is nearly 2 hours. [4]

June 1997 qp.1 (Cambridge)

14. At time $t = 0$ there are 8000 fish in a lake. At time t days the birth rate of fish is equal to one fiftieth of the number N of fish present. Fish are taken from the lake at a rate of 100 per day. Modelling N as a continuous variable, show that

$$50\frac{dN}{dt} = N - 5000$$ [2]

Solve the differential equation to find N in terms of t. [6]

Find the time taken for the population of fish in the lake to increase to 11000. [3]

When the population of fish has reached 11000, it is decided to increase the number of fish taken from the lake from 100 per day to F per day. Write down, in terms of F, the new differential equation satisfied by N. [1]

Show that if $F > 220$, then $\frac{dN}{dt} < 0$ when $N = 11000$ [1]

For this range of values of F, give a reason why the population of fish in the lake continues to decrease. [1]

November 2003 qp.2 (Zimsec Specimen Paper)

5. A race called the Matrices live on an isolated island called Geometry. Demographical studies have shown that the number of births per unit time is proportional to the population, x, at any time t. The number of deaths per unit time is proportional to the square of the population.
   (i). Show that the above information can be modelled by the differential equation $\frac{dx}{dt} = kx - hx^2$, where k and h are positive constants. [2]
   (ii). Solve the differential equation for x in terms of t, given that $x = \frac{k}{3h}$ when $t = 0$. [7]
   (iii). Show that the limit to the size of the population is $\frac{k}{h}$ as t approaches infinity. [1]

November 2001 qp.17 (Zimsec)

17. An infectious disease is spreading in an isolated village of 400 people. The number of people with the disease at time t days is N.

    a) In one model of the spread of the disease, it is assumed that the rate of increase of the number of people with the disease is proportional to the number with the disease at the time. When $t = 0$, 40 people have the disease and it is spreading at a rate of 20 people per day. Taking N to be a continuous variable, show that it satisfies the differential equation $\frac{dN}{dt} = \frac{1}{2}N$. [2]
       (i). Solve the differential equation, obtaining an expression for N in terms of t, and sketch the solution curve of N against t. [4]
       (ii). Show that the time predicted for half of the people in the village to have the disease is 3.2 to 2 significant figures. [2]

    b) In an alternative model, it is assumed that the rate of increase of N varies as the product of the number with the disease at the time and the number of those not yet infected. You are given that with the initial conditions stated in part (a), N satisfies the differential equation $\frac{dN}{dt} = \frac{N(400-N)}{720}$.
       Solve this differential equation and show that the time predicted for half for the people in the village to have the disease is 4.0 days to 2 significant figures. [6]

June 2007 qp.3 (Cambridge)

10 A model for the height, h metres, of a certain type of tree at time t years after being planted assumes that, while the tree is growing, the rate of increase in height is proportional to $(9-h)^{\frac{1}{3}}$. It is given that, when $t=0$, $h=1$ and $\frac{dh}{dt}=0.2$.

(i) Show that h and t satisfy the differential equation

$$\frac{dh}{dt}=0.1(9-h)^{\frac{1}{3}}.$$ [2]

(ii) Solve this differential equation, and obtain an expression for h in terms of t. [7]

(iii) Find the maximum height of the tree and the time taken to reach this height after planting. [2]

(iv) Calculate the time taken to reach half the maximum height. [1]

November 2007 qp.3 (Cambridge)

7 The number of insects in a population t days after the start of observations is denoted by N. The variation in the number of insects is modelled by a differential equation of the form

$$\frac{dN}{dt}=kN\cos(0.02t),$$

where k is a constant and N is taken to be a continuous variable. It is given that $N=125$ when $t=0$.

(i) Solve the differential equation, obtaining a relation between N, k and t. [5]

(ii) Given also that $N=166$ when $t=30$, find the value of k. [2]

(iii) Obtain an expression for N in terms of t, and find the least value of N predicted by this model. [3]

June 2009 qp.3 (Cambridge)

8 **(i)** Express $\frac{100}{x^2(10-x)}$ in partial fractions. [4]

(ii) Given that $x=1$ when $t=0$, solve the differential equation

$$\frac{dx}{dt}=\frac{1}{100}x^2(10-x),$$

obtaining an expression for t in terms of x. [6]

November 2009 qp.31 (Cambridge)

10 In a model of the expansion of a sphere of radius r cm, it is assumed that, at time t seconds after the start, the rate of increase of the surface area of the sphere is proportional to its volume. When $t = 0$, $r = 5$ and $\frac{dr}{dt} = 2$.

(i) Show that r satisfies the differential equation

$$\frac{dr}{dt} = 0.08r^2.$$ [4]

[The surface area A and volume V of a sphere of radius r are given by the formulae $A = 4\pi r^2$, $V = \frac{4}{3}\pi r^3$.]

(ii) Solve this differential equation, obtaining an expression for r in terms of t. [5]

(iii) Deduce from your answer to part **(ii)** the set of values that t can take, according to this model. [1]

November 2009 qp.32 (Cambridge)

9 The temperature of a quantity of liquid at time t is θ. The liquid is cooling in an atmosphere whose temperature is constant and equal to A. The rate of decrease of θ is proportional to the temperature difference $(\theta - A)$. Thus θ and t satisfy the differential equation

$$\frac{d\theta}{dt} = -k(\theta - A),$$

where k is a positive constant.

(i) Find, in any form, the solution of this differential equation, given that $\theta = 4A$ when $t = 0$. [5]

(ii) Given also that $\theta = 3A$ when $t = 1$, show that $k = \ln\frac{3}{2}$. [1]

(iii) Find θ in terms of A when $t = 2$, expressing your answer in its simplest form. [3]

June 2010 qp.31 (Cambridge)

5 Given that $y = 0$ when $x = 1$, solve the differential equation

$$xy\frac{dy}{dx} = y^2 + 4,$$

obtaining an expression for y^2 in terms of x. [6]

June 2010 qp.32 (Cambridge)

7 The variables x and t are related by the differential equation

$$e^{2t}\frac{dx}{dt} = \cos^2 x,$$

where $t \geqslant 0$. When $t = 0$, $x = 0$.

(i) Solve the differential equation, obtaining an expression for x in terms of t. [6]

(ii) State what happens to the value of x when t becomes very large. [1]

(iii) Explain why x increases as t increases. [1]

June 2010 qp.33 (Cambridge)

4 Given that $x = 1$ when $t = 0$, solve the differential equation

$$\frac{dx}{dt} = \frac{1}{x} - \frac{x}{4},$$

obtaining an expression for x^2 in terms of t. [7]

November 2010 qp.31 (Cambridge)

10 A certain substance is formed in a chemical reaction. The mass of substance formed t seconds after the start of the reaction is x grams. At any time the rate of formation of the substance is proportional to $(20 - x)$. When $t = 0$, $x = 0$ and $\frac{dx}{dt} = 1$.

(i) Show that x and t satisfy the differential equation

$$\frac{dx}{dt} = 0.05(20 - x).$$ [2]

(ii) Find, in any form, the solution of this differential equation. [5]

(iii) Find x when $t = 10$, giving your answer correct to 1 decimal place. [2]

(iv) State what happens to the value of x as t becomes very large. [1]

November 2010 qp.33 (Cambridge)

9 A biologist is investigating the spread of a weed in a particular region. At time t weeks after the start of the investigation, the area covered by the weed is A m^2. The biologist claims that the rate of increase of A is proportional to $\sqrt{(2A - 5)}$.

(i) Write down a differential equation representing the biologist's claim. [1]

(ii) At the start of the investigation, the area covered by the weed was 7 m^2 and, 10 weeks later, the area covered was 27 m^2. Assuming that the biologist's claim is correct, find the area covered 20 weeks after the start of the investigation. [9]

June 2011 qp.31 (Cambridge)

10 The number of birds of a certain species in a forested region is recorded over several years. At time t years, the number of birds is N, where N is treated as a continuous variable. The variation in the number of birds is modelled by

$$\frac{dN}{dt} = \frac{N(1800 - N)}{3600}.$$

It is given that $N = 300$ when $t = 0$.

(i) Find an expression for N in terms of t. [9]

(ii) According to the model, how many birds will there be after a long time? [1]

June 2011 qp.32 (Cambridge)

6 A certain curve is such that its gradient at a point (x, y) is proportional to xy. At the point $(1, 2)$ the gradient is 4.

(i) By setting up and solving a differential equation, show that the equation of the curve is $y = 2e^{x^2-1}$. [7]

(ii) State the gradient of the curve at the point $(-1, 2)$ and sketch the curve. [2]

June 2011 qp.33 (Cambridge)

9 In a chemical reaction, a compound X is formed from two compounds Y and Z. The masses in grams of X, Y and Z present at time t seconds after the start of the reaction are x, $10 - x$ and $20 - x$ respectively. At any time the rate of formation of X is proportional to the product of the masses of Y and Z present at the time. When $t = 0$, $x = 0$ and $\frac{dx}{dt} = 2$.

(i) Show that x and t satisfy the differential equation

$$\frac{dx}{dt} = 0.01(10 - x)(20 - x).$$ [1]

(ii) Solve this differential equation and obtain an expression for x in terms of t. [9]

(iii) State what happens to the value of x when t becomes large. [1]

November 2011 qp.33 (Cambridge)

4 During an experiment, the number of organisms present at time t days is denoted by N, where N is treated as a continuous variable. It is given that

$$\frac{dN}{dt} = 1.2e^{-0.02t}N^{0.5}.$$

When $t = 0$, the number of organisms present is 100.

(i) Find an expression for N in terms of t. [6]

(ii) State what happens to the number of organisms present after a long time. [1]

June 2012 qp.32 (Cambridge)

5 The variables x and y satisfy the differential equation

$$\frac{dy}{dx} = e^{2x+y},$$

and $y = 0$ when $x = 0$. Solve the differential equation, obtaining an expression for y in terms of x. [6]

June 2012 qp.33 (Cambridge)

5 In a certain chemical process a substance A reacts with another substance B. The masses in grams of A and B present at time t seconds after the start of the process are x and y respectively. It is given that $\frac{dy}{dt} = -0.6xy$ and $x = 5e^{-3t}$. When $t = 0$, $y = 70$.

(i) Form a differential equation in y and t. Solve this differential equation and obtain an expression for y in terms of t. [6]

(ii) The percentage of the initial mass of B remaining at time t is denoted by p. Find the exact value approached by p as t becomes large. [2]

November 2012 qp.31 (Cambridge)

6 The variables x and y are related by the differential equation

$$x\frac{dy}{dx} = 1 - y^2.$$

When $x = 2$, $y = 0$. Solve the differential equation, obtaining an expression for y in terms of x. [8]

November 2012 qp.33 (Cambridge)

4 The variables x and y are related by the differential equation

$$(x^2 + 4)\frac{dy}{dx} = 6xy.$$

It is given that $y = 32$ when $x = 0$. Find an expression for y in terms of x. [6]

Chapter Thirteen: Functions

"My imagination functions much better when I don't have to speak to people."
– Patricia Highsmith

A function is a mapping in which **one input** value gives **one output** value. For example,

$$x \longrightarrow 3x + 1$$

$$2 \longrightarrow 3(2) + 1 = 7$$

$$1 \longrightarrow 3(1) + 1 = 4$$

If $x \longrightarrow 3x + 1$ {this statement is read as: x maps to $3x + 1$}

Then $f(x) = 3x + 1$ {this statement is real as: function of x is $3x + 1$}

Types of Function

i. **One to One Function**
This is function in which one input value gives exactly one output value. For example:
$f(x) = 2x - 3$
$f(1) = 2(1) - 3 = -1$
$f(-1) = 2(-1) - 3 = -5$

ii. **Many to One Function**
This is a function in which several input values give the same output value. For example:
$f(x) = x^2 - 1$
$f(2) = (2)^2 - 1 = 3$
$f(-2) = (-2)^2 - 1 = 3$

The **One to Many** mapping is not by definition a function (a function gives only one output for one input). It is instead just a mapping in which one input may produce several different outputs.

For example:

$x \longrightarrow \sqrt{x} + 1$

$9 \longrightarrow \sqrt{9} + 1$

$9 \longrightarrow \pm 3 + 1 = 4 \text{ or } -2$

Functions take different forms depending on the nature of expression at hand. Some of the prominent forms include, but not limited to:

- Algebraic functions (linear and quadratic expression, for example)
- Exponential functions
- Logarithmic functions
- Trigonometrical functions

Functional analysis revolves around the determination of:

- Composite functions
- Inverse functions
- Domain and/or range
- Sketch graphs.

Composite Functions

This is used to describe a scenario where two or more functions are combined to come up with one function. For example, given that:

$f(x) = 3\ln(3x - 2)$ *and* $g(x) = x + 1$

$fg(x) = f(x + 1)$

$fg(x) = 3\ln[3(3(x + 1) - 2)]$

$\Rightarrow fg(x) = 3\ln(3x + 3 - 2)$

$\therefore \boldsymbol{fg(x) = 3\ln(3x + 1)}$

$gf(x) = g[3\ln(3x - 2)]$

$gf(x) = [3\ln(3x - 2)] + 1$

$\therefore \boldsymbol{gf(x) = 3\ln(3x - 2) + 1}$

Inverse of a Function

Inverse is given by considering the following steps:

- Transformation of the function into an equation by letting $y = f(x)$;
- Making x the subject of the formula;
- Interchanging the positions of x and y; and
- Writing the final answer in functional notation.

For example, the inverses of $f(x)$ and $g(x)$ where $f(x) = 3\ln(3x - 2)$ and $g(x) = x + 1$ are given by:

let $y = f(x)$

$\Rightarrow y = 3\ln(3x - 2)$

$\Rightarrow \frac{y}{3} = \ln(3x - 2)$

$\Rightarrow e^{\frac{y}{3}} = 3x - 2$

$e^{\frac{y}{3}} + 2 = 3x$

$\Rightarrow x = \frac{e^{\frac{y}{3}} + 2}{3}$

$\Rightarrow y = \frac{e^{\frac{x}{3}} + 2}{3}$

$\therefore \quad \boldsymbol{f^{-1}(x) = \frac{e^{\frac{x}{3}} + 2}{3}}$

and

let $y = g(x)$

$\Rightarrow y = x + 1$

$\Rightarrow y - 1 = x$

$\Rightarrow x - 1 = y$

$\therefore \boldsymbol{g^{-1}(x) = x - 1}$

NB: $f^{-1}(x)$ *and* $g^{-1}(x)$ *is used to describe the notation for inverses of* $f(x)$ *and* $g(x)$ *respectively.*

Domain and Range

- Domain is used to describe the *set of input values* of a function. As such, domain is described in terms of x. For example, if $f(x) = 3\sin x$; $x \geq 30°$.
  Then $x \geq 30°$ is the domain, meaning that this function is only valid for values of x greater than or equal to 30°.

- Range is used to describe the *set of output values* of a function. For example,
  $If\ f(x) = x + 1 \quad ; \quad 0 \leq x \leq 3$
  $\Rightarrow f(0) = 1\ ;\ f(1) = 2\ ;\ f(2) = 3\ ;\ f(3) = 4$
  $\Rightarrow$ the output ranges between 1 and 4 inclusive.
  $\therefore 1 \leq f(x) \leq 4$ is the range of the function. Since the range describes output, it is given in terms of the function.

- Domain of a function is used to describe the range of its inverse function and Range of a function is used to describe the domain of its inverse function. This is summarised by the table below:

Table 13.1

Function	Inverse Function
Domain	Range
Range	Domain

For example, using $f(x) = x + 1 \quad ; \quad 0 \leq x \leq 3$

The **domain of $f(x)$**: $0 \leq x \leq 3$

Which implies that **range of $f^{-1}(x)$**: $0 \leq f^{-1}(x) \leq 3$

And the **range of $f(x)$**: $1 \leq f(x) \leq 4$

Which implies that **domain of $f^{-1}(x)$**: $1 \leq x \leq 4$.

NB:

- The idea here is to just change the notation, but retaining the result as it is.
- Diagrammatically, the graph of an inverse is a reflection of the graph of the original function in the mirror line $y = x$.

Worked Examination Questions on Functions

Question (Cambridge, November 1998 qp.1)

13. The function f is defined by

$$\mathrm{f}: x \longmapsto \ln(x+1), \quad x > -1$$

Find an expression for $\mathrm{f}^{-1}(x)$ and state the domain and range of the inverse function f^{-1}. [4]

The function g is defined by

$$\mathrm{g}: x \longmapsto x - 1, \qquad x \in \mathbb{R}$$

Describe the geometrical relationship between the graphs of $y = \mathrm{fg}(x)$ and $y = \mathrm{gf}(x)$. [4]

Solution

Given that $\mathrm{f}(x) = \ln(x+1), \quad x > -1$

let $y = \mathrm{f}(x)$

$\Rightarrow y = \ln(x+1)$

$\Rightarrow e^y = x + 1$

$\Rightarrow x = e^y - 1$

$\Rightarrow y = e^x - 1$

$\therefore\ \mathbf{f^{-1}(x) = e^x - 1}$

where range of $\mathbf{f^{-1}(x) > -1}$

and domain of $\mathbf{f^{-1}(x)\!: \ x > 0}$

Given also that $\mathrm{g}(x) = x - 1$

$\mathrm{fg}(x) = \mathrm{f}(x-1)$ *and* $\mathrm{gf}(x) = \mathrm{g}[\ln(x+1)]$

$\Rightarrow \mathrm{fg}(x) = \ln([x-1]+1)$

$\Rightarrow \mathrm{gf}(x) = \ln(x+1) - 1$

$\Rightarrow \mathrm{fg}(x) = \ln x$

So $\mathrm{fg}(x) \to \mathrm{gf}(x)$

$\Rightarrow \ln x \to \ln(x+1) - 1$

$y = \text{gf}(x)$ *is the image of* $y = \text{fg}(x)$ *after* $y = \text{fg}(x)$ *has undergone a series of two transformations with the following description:*

- *Translation of one unit in the negative* $x -$ *direction*
- *Translation of one unit in the negative* $y -$ *direction.*

Question (Zimsec, November 2007 qp.1)

5. A function is defined by

$$f : x \longmapsto x^2 + 4x + 1, \text{ for } x \geq -2$$

Find

(i) the range of the function [3]

(ii) an expression for $f^{-1}(x)$, stating its domain. [4]

Solution

Given that, $f(x) = x^2 + 4x + 1$

by completing the square,

$\Rightarrow f(x) = x^2 + 4x + (2)^2 - (2)^2 + 1$

$\Rightarrow f(x) = (x + 2)^2 - 4 + 1$

$\Rightarrow f(x) = (x + 2)^2 - 3$

(i). *The pair of coodinates at the turning point is* $(-2, -3)$ *using a diagram, the snapshot of* $y = f(x)$*is shown below:*

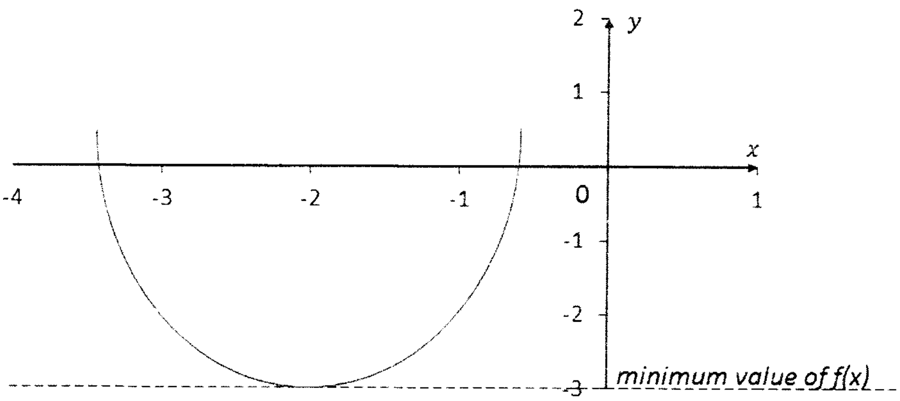


Fig. 13.1

$\therefore$ ***range of*** $\boldsymbol{f(x)}$: $\boldsymbol{f(x) \geq -3}$

(ii). $let \quad y = f(x)$

$\Rightarrow y = (x+2)^2 - 3$

$\Rightarrow y + 3 = (x+2)^2$

$\Rightarrow \pm\sqrt{y+3} = x + 2$

$\Rightarrow x = \pm\sqrt{y+3} - 2$

$\Rightarrow y = \pm\sqrt{x+3} - 2$

$\therefore \boldsymbol{f^{-1}(x) = \pm\sqrt{x+3} - 2}$

$\boldsymbol{Domain\ of\ f^{-1}(x): \quad x \geq -3}$

Question (Cambridge, November 1992 qp.1)

8. The functions f and g are defined by

$$\mathrm{f} : x \longmapsto e^{2x}, \qquad x \in \mathbb{R},$$

$$\mathrm{g} : x \longmapsto \sqrt{x} \quad x \geq 0.$$

Find and simplify

(i). $\mathrm{gf}(x)$,

(ii). $\mathrm{f}^{-1}(x)$,

(iii). $\mathrm{fg}^{-1}(x)$. [3]

Solution

$Given\ that \ \ \mathrm{f}(x) = e^{2x} \quad and \quad \mathrm{g}(x) = \sqrt{x}$

(i). $\mathrm{gf}(x) = \mathrm{g}(e^{2x})$

$\therefore \boldsymbol{\mathrm{gf}(x) = \sqrt{e^{2x}}}$

(ii). $let\ y = \mathrm{f}(x).$

$\Rightarrow y = e^{2x}$

$\Rightarrow \ln y = 2x$

$\Rightarrow x = \frac{\ln y}{2}$

$\Rightarrow y = \frac{\ln x}{2}$

$\therefore \boldsymbol{\mathrm{f}^{-1}(x) = \frac{\ln x}{2}}$

(iii). $let\ y = g(x)$

$\Rightarrow y = \sqrt{x}$

$\Rightarrow x = y^2$

$\Rightarrow y = x^2$

$\Rightarrow g^{-1}(x) = x^2$

$Now,\ fg^{-1}(x) = f(x^2)$

$\therefore\ \mathbf{fg^{-1}(x) = e^{2x^2}}$

<u>Revision Questions on Functions</u>

November 1996 qp.1 (Cambridge)

14. The functions f and g are defined for $x \in \mathbb{R}$ by

$$f : x \longmapsto (x-2)(x-4)$$
$$g : x \longmapsto x^2 - 2$$

(i). Find $fg(x)$ and state the exact values of x for which $fg(x) = 0$. [3]

(ii). Find $gf(x)$. [1]

(iii). Show that $x = 2$ is a root of the equation $fg(x) - gf(x) = 2$, and find all the other roots. [5]

(iv). Solve the inequality $fg(x) - gf(x) < 2$. [3]

June 1997 qp.1 (Cambridge)

4. Functions f and g are defined by

$$f : x \longmapsto \frac{3}{x+3}, \qquad x \in \mathbb{R}, \qquad x \geq 0,$$

$$g : x \longmapsto x + 1, \qquad x \in \mathbb{R}, \qquad x \geq 0.$$

Show that

$$gf : x \longmapsto \frac{x+6}{x+3} \qquad x \in \mathbb{R}, \qquad x \geq 0.$$ [1]

Express fg in a similar form. [1]

Find $(gf)^{-1}(x)$. [3]

November 2007 qp.1 (Zimsec, O Level Additional Mathematics)

14. Functions f and g are defined by

$$f : x \longmapsto \frac{3x-1}{x-2}, \qquad x \neq 2$$

$$g : x \longmapsto \frac{2x-1}{x-3}, \qquad x \neq 3$$

a) Show that fg: $x \longmapsto x$ [3]

b) Evaluate

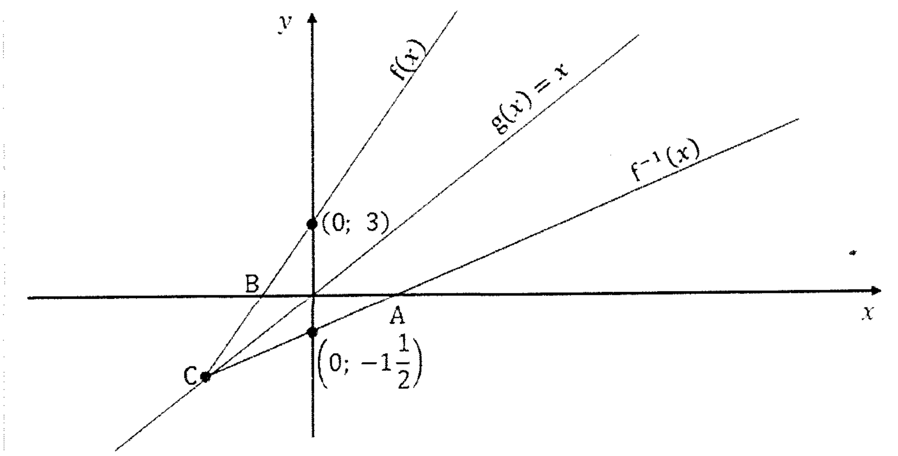
(i). $f^{-1}(5)$, [3]

(ii). $g^{-1}(4)$, [3]

(iii). $ffg(7)$. [3]

November 2010 qp.1 (Zimsec)

7. The diagram below shows the linear graphs of $f(x)$, $g(x) = x$ and $f^{-1}(x)$ which intersect at C. The graph $f^{-1}(x)$ intersects the x-axis at point A and $f(x)$ intersects x-axis at B.



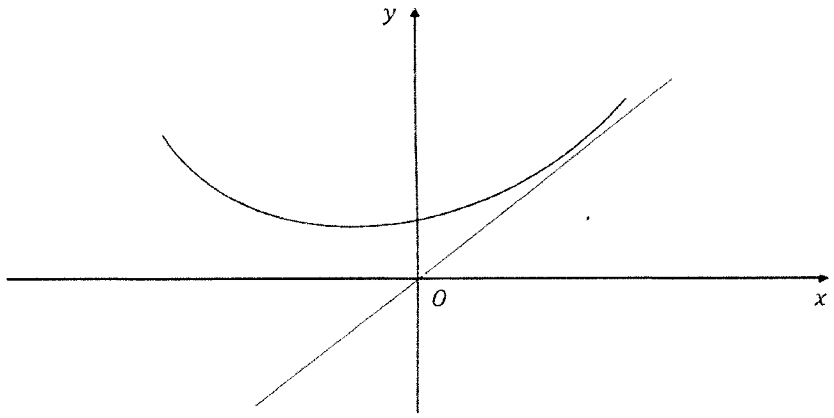
a) State the name given to the function $g(x)$ in relation to the functions $f(x)$ and $f^{-1}(x)$. [1]

b) Write down the coordinates of the points A and B. [3]

c) Calculate the coordinates of the point C. [3]

November 1997 qp.1 (Cambridge)

11.



The diagram shows the curve $= x + e^{-\frac{1}{2}x}$, together with the line $y = x$.

(i). Explain how you can tell from the equation of the curve that the curve approaches the line as x becomes large and positive. [1]

(ii). The minimum point on the curve has coordinates $(h,\ k)$. Use differentiation to find h and k, expressing your answers in terms of logarithms. [4]

(iii). The function f is defined by $\mathrm{f}: x \longrightarrow x + e^{-\frac{1}{2}x}$, $x \in \mathbb{R}$, $x \geq h$

where h has the value found in (ii). State the domain and range of the inverse function f^{-1}, and sketch the graph of $y = \mathrm{f}^{-1}(x)$. [3]

June 1994 qp.1 (Cambridge)

13. (a). Functions g and h are defined by $\mathrm{g}: x \longmapsto \ln x, \quad x \in \mathbb{R}, \quad x > 0$
$\mathrm{h}: x \longrightarrow 1 + x, \quad x \in \mathbb{R}$

The function f is defined by $\mathrm{f}: x \longrightarrow \mathrm{gh}(x), \quad x \in \mathbb{R}, \quad x > -1$

(i). Sketch the graph of $y = \mathrm{f}(x)$. [2]

(ii). Write down expression for $\mathrm{g}^{-1}(x)$ and $\mathrm{h}^{-1}(x)$. [2]

(iii). Write down an expression for $g^{-1}h^{-1}(x)$. [1]

(iv). Sketch the graph of $g^{-1}h^{-1}(x)$. [2]

(b) The function q is defined by $q: x \longrightarrow x^2 - 4x,\ x \in \mathbb{R},\ |x| \le 1$. Show, by means of a graphical argument or otherwise, that q is one-one, and find an expression for $q^{-1}(x)$. [5]

November 1995 qp.1 (Cambridge)

4. The functions f and g are defined for $x \in \mathbb{R}$ by

$$f: x \longmapsto x^3, \qquad g: x \longmapsto 2 - 3x$$

Find

(i). $fg(x)$, [1]

(ii). $(fg)^{-1}(x)$. [2]

November 1999 qp.1 (Cambridge)

10. The functions f and g are defined, for all real values of x, as follows:

$$f: x \longmapsto e^x$$

$$g: x \longmapsto e^{x+a}$$

where a is a positive constant.

(i). State two different geometrical transformations which transform the graph of $y = f(x)$ onto that of $y = g(x)$. [4]

(ii). The inverse function of g can be written as

$$g^{-1}: x \longmapsto \ln(bx), \qquad x > 0,$$

where b is a positive constant. Express b in terms of a. [4]

Chapter Fourteen: Numerical Methods

"Life is a field of unlimited possibilities."

– Deepak Chopra

Some problems in mathematics cannot be determined with substantial accuracy by conventional methods. In such cases, numerical applications are employed to solve the problem. Numerical methods are, therefore, educated approximation techniques used to prescribe a solution to a problem where no particular standard answering method is applicable.

This is a versatile topic that sees its application in different topical areas including, but not limited to, circular measure, trigonometry, differentiation, integration, polynomials and curve sketching.

Errors and Uncertainties

Since they are merely approximations, the solutions obtained by using numeral methods deviate from the exact value of the root of an equation. The greatest deviation (whether positive or negative) of an approximation from the exact (or true) value is known as the **absolute error**. It is given by

$$\boldsymbol{absolute\ error = |actual\ value - estimate\ value|}$$

Often in measurement, the phrase 'to the nearest' refers to the absolute value. For example, a length of 10 cm measured to the nearest 0.1 cm has an absolute value of 0.1 cm. The minimum and maximum values can be calculated as follows:

$$mimimum\ value = exact\ value - absolute\ error = 10 - 0.1 = 9.9 \text{ cm}$$

$$maximum\ value = exact\ value + absolute\ error = 10 + 0.1 = 10.1 \text{ cm}$$

This means that the value of 10 cm is really an approximation that lies between two extreme values of 9.9 cm and 10.1 cm to the nearest 0.1 cm. This can be written as 10 ± 0.1 cm.

An approximation with a very small absolute error is very **accurate**. This means that the approximation is nearly equal to the true/exact value.

The **relative error** of an approximation is a ratio defined by the equation

$$\boldsymbol{relative\ error = \frac{absolute\ error}{true\ value}}$$

NB: The relative error is sometimes also referred to as the ***fractional*** error.

The more accurate an approximation is, the smaller the relative error. If a root obtained from an iteration or any other numerical method is exactly equal (this is very rare) to the exact value, both the absolute and relative errors are equal to zero.

The percentage error is given by

$$\boldsymbol{percentage\ error = \frac{absolute\ error}{true\ value} \times 100 \equiv relative\ error \times 100}$$

Determining Absolute Error

1. **Addition and Subtraction**
   When two quantities are added or subtracted, for example, when determining length, thickness, duration and so on, the absolute error (represented here by Δ) is given as follows:

Addition		Subtraction
$if\ y = A + B$	and	$y = A - B$
$\Delta y = \Delta A + \Delta B$		$\Delta y = \Delta A + \Delta B$
$y \pm \Delta y = (A + B) \pm (\Delta A + \Delta B)$		$y \pm \Delta y = (A - B) \pm (\Delta A + \Delta B)$

NB:

- We always **add** errors
- The notation $\pm$ shows that the true or exact value is the midpoint of two extreme values.

2. **Multiplication and Division**
   We cannot directly add the absolute values of two quantities to obtain the absolute value of when they are multiplied or divided. This is because the absolute value in one quantity has an overall *fractional* effect on the whole product or quotient. Instead, relative (or fractional) errors are employed such that:

Multiplication

$if\ y = AB$ and

$$\frac{\Delta y}{y} = \frac{\Delta A}{A} + \frac{\Delta B}{B}$$

$$\Delta y = \left(\frac{\Delta A}{A} + \frac{\Delta B}{B}\right) y$$

$$\Delta y = \left(\frac{\Delta A}{A} + \frac{\Delta B}{B}\right) AB$$

Division

$$y = \frac{A}{B}$$

$$\frac{\Delta y}{y} = \frac{\Delta A}{A} + \frac{\Delta B}{B}$$

$$\Delta y = \left(\frac{\Delta A}{A} + \frac{\Delta B}{B}\right) y$$

$$\Delta y = \left(\frac{\Delta A}{A} + \frac{\Delta B}{B}\right) \frac{A}{B}$$

In the case that a quantity is squared,

$$y = A^2 = A \times A$$

$$\frac{\Delta y}{y} = \frac{\Delta A}{A} + \frac{\Delta A}{A} = \frac{2\Delta A}{A}$$

The general formula states that if $y = A^m B^n$

$$\frac{\Delta y}{y} = \frac{m\Delta A}{A} + \frac{n\Delta B}{B}$$

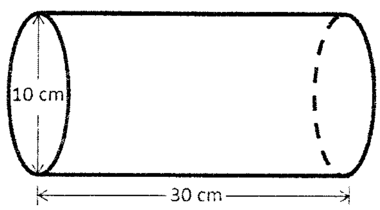
$$\Delta y = \left(\frac{m\Delta A}{A} + \frac{n\Delta B}{B}\right) y$$

$$\Delta y = \left(\frac{m\Delta A}{A} + \frac{n\Delta B}{B}\right) A^m B^n$$

This concept is best explained by a practical example.

Question (Zimsec, June 2003 qp.1)

4. A cylindrical hole is drilled through a block of metal of width 30 cm. The diameter is intended to be 10 cm as shown in the diagram below.



The drilling process produces an error of 0.2 cm in the diameter. Estimate the error which arises in the calculated volume, giving your answer in terms of π.

Solution

$V = \pi r^2 h$

$\Rightarrow V = \dfrac{\pi d^2 h}{4}$ $\qquad \left\{ \begin{array}{l} where\ r = \dfrac{d}{2} \\ \Rightarrow r^2 = \dfrac{d^2}{4} \end{array} \right\}$

$V_{exact} = \dfrac{\pi(10)^2 \times 30}{4}$

$\Rightarrow V_{exact} = 750\pi$

$where \quad \dfrac{\Delta V}{V} = \dfrac{2\Delta d}{d}$

$\Rightarrow \Delta V = \dfrac{2\Delta d}{d} \times V$

$\Rightarrow \Delta V = \dfrac{2(0.2)}{10} \times 750\pi$

$\therefore \Delta V = \mathbf{30\pi\ cm^3}$

The Trapezium Rule

This technique was derived from the formula of calculating the area of a trapezium. In mathematics, integration is a summation process that may be used to find area under a graph or area bound by graphs. In cases where the problem cannot be easily integrated, the trapezium rule provides a good alternative.

- It is assumed that any given region can be divided into trapezia with x –values (absica) and corresponding y –values (ordinates) as shown in Fig. 14.1.

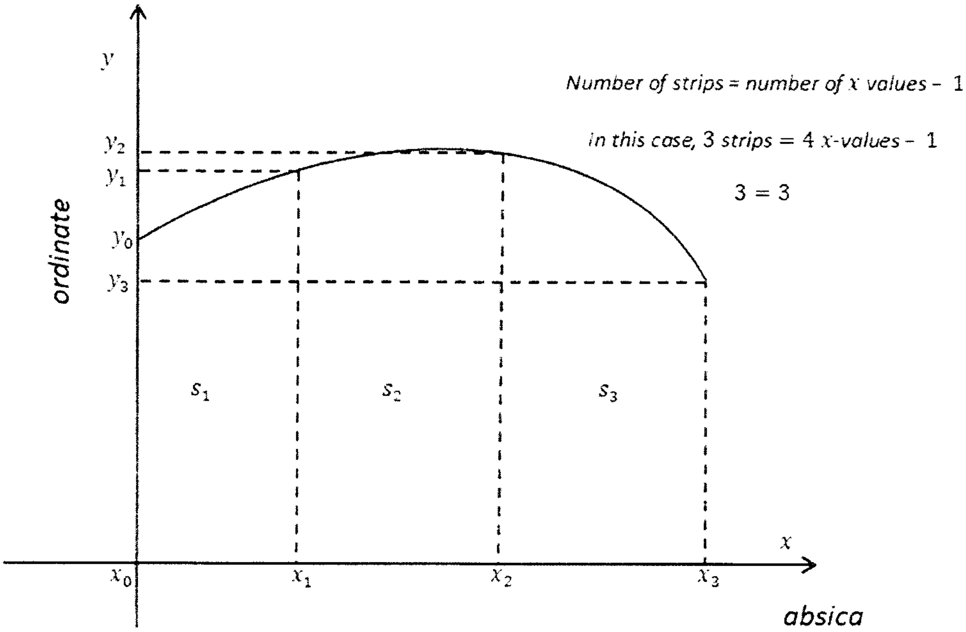


Fig.14.1

- The trapezium rule states that:

$$A = \frac{h}{2}\left[y_o + y_l + 2\left(\sum \text{other } y \text{ values}\right)\right]$$

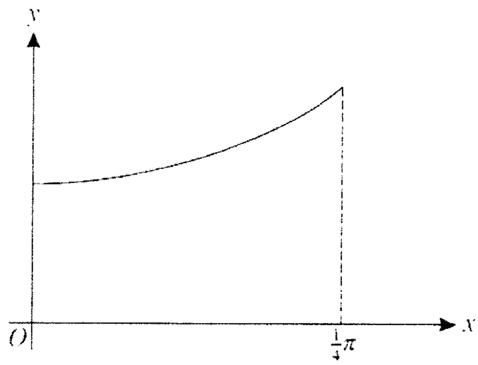
$$\text{where } h = \frac{\text{upper limit} - \text{lower limit}}{\text{number of strips or intervals}}$$

and y represents the values correspnding to x values. y is given by substituting the x − values in the original equation.

y_o is the first y value and y_l is the last.

Example: Question (Cambridge, June 2009 qp.3)

2



The diagram shows the curve $y = \sqrt{(1 + 2\tan^2 x)}$ for $0 \leqslant x \leqslant \frac{1}{4}\pi$.

(i) Use the trapezium rule with three intervals to estimate the value of

$$\int_0^{\frac{1}{4}\pi} \sqrt{(1 + 2\tan^2 x)}\,dx,$$

giving your answer correct to 2 decimal places. [3]

(ii) The estimate found in part **(i)** is denoted by E. Explain, without further calculation, whether another estimate found using the trapezium rule with six intervals would be greater than E or less than E. [1]

Solution

(i) $A = \frac{h}{2}\left[y_o + y_l + 2\left(\sum other\ y\ values\right)\right]$

$where\ h = \dfrac{\frac{\pi}{4} - 0}{3}$ $\quad and \quad$ $\#\ of\ x - values\ = number\ of\ strips + 1$

$\Rightarrow h = \dfrac{\pi}{12}$ $\quad$ $\Rightarrow \#\ of\ x - values = 3 + 1 = 4$

Table 14.1

$x - values\ (absica)$	$y - values\ (ordinate)$
0	1
$\frac{\pi}{12}$	1.0694
$\frac{\pi}{6}$	1.2910
$\frac{\pi}{4}$	1.7321

$Now,\ A = \dfrac{\left(\frac{\pi}{12}\right)}{2}[1 + 1.7321 + 2(1.0694 + 1.2910)]$

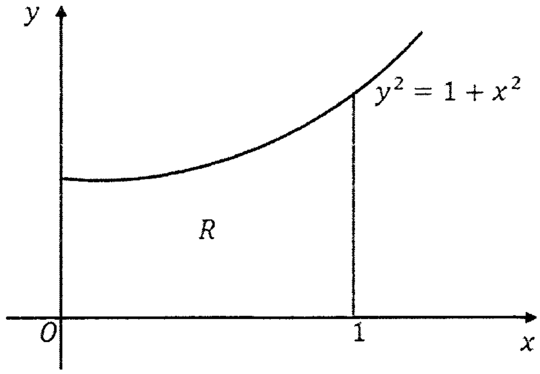
$\therefore \boldsymbol{A = 0.98\ units^2}$

(ii) The general rule states that the greater the number of intervals, the greater the level of accuracy. This is because increasing the number of strips reduces overestimation. As such, the area determined using six intervals gives an answer less than E.

Revision Questions on Trapezium Rule

November 1997 qp.1 (Cambridge)

9.



The diagram shows part of the curve $y^2 = 1 + x^2$. The region R is bounded by the curve, the axes and the line $x = 1$.

i. Show that the volume of the solid formed when R is rotated completely about the x-axis is $\frac{4}{3}\pi\text{units}^3$. [3]

ii. Use the trapezium rule, with ordinates at $x = 0$, $x = \frac{1}{4}, x = \frac{1}{2}, x = \frac{3}{4}$ and $x = 1$, to estimate the area of R, giving your answer correct to 3 significant figures. [3]

iii. State with a reason whether the estimate for the area calculated in (ii) is greater or less than the true value. [1]

June 2010 qp.1 (Zimsec)

18. (a) Use the Trapezium rule with four equally spaced ordinates, to estimate the value of

$$\int_0^{\frac{\pi}{6}} \sin^4 x \, dx,$$

giving your answer to 3 significant figures. [3]

June 1991 qp.1 (Cambridge)

12. Use the trapezium rule, with ordinates at $x = -1$, $x = -\frac{1}{2}$, $x = 0$, $x = \frac{1}{2}$ and $x = 1$, to estimate the value of

$$\int_{-1}^{1} \sqrt{\ln(2+x)}\, dx,$$

giving 2 significant figures in your answer. [4]

November 1999 qp.1 (Cambridge)

4. Use the trapezium rule with 2 intervals to estimate the value of

$$\int_{0}^{\frac{1}{2}\pi} \sqrt{(1+\sin x)}\, dx,$$

giving your answer correct to 3 significant figures. [3]

You are given that the exact value of this integral is 2. Calculate the relative error in your trapezium rule approximation. [2]

Iteration

Iteration refers to a series of trials carried out to locate a root where the actual root cannot be determined with substantial accuracy conventionally. Questions on iteration revolve around five main concepts:

- Curve sketching
- Verification of whether a root lies between two points
- Transformation of an iterative formula into an equation
- Transformation of an equation into an iterative formula
- Determination of the location of a root using a given iterative formula

(i) **Curve sketching**

Refer to Advert 2 (page 238).

(ii) **Verification of whether or not a root lies between two points**

This verification process is made possible by way of transferring all the terms to one side of the equation and equating this to $f(x)$. If a root lies between a and b, this leads to a sign change between $f(a)$ and $f(b)$. If $f(a)$ and $f(b)$ have the same sign (both positive or both negative), then a root does not lie between a and b.

(iii) **Transformation of an iterative formula into an equation**

Omission of all the subscripts automatically transforms an iterative formula into an equation. In some cases, the question will ask for one to re-arrange the resulting equation into a particular form.

(iv) **Transformation of an equation into an iterative formula**

Given an equation in x where x is scattered, target one of the terms in x which has a bias towards satisfying the desired end. The targeted x −value must be expressed as the subject of the formula. It is from this re-arranged equation that subscripts can be introduced to both the LHS and RHS. To the LHS the subscript is introduced as $(n + 1)$ to give x_{n+1} and to the RHS, the subscript is introduced as n to give x_n.

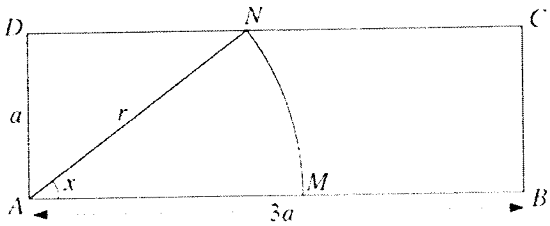
(v) **Determination of the location of a root**

This process makes use of the initial result to estimate the subsequent result by way of substituting the initial result in the iterative formula. If the initial result is not given, use the average of the two boundary limits to the root as the initial approximation. That is, if the root lies between a and b, $x_1 = \left(\frac{a+b}{2}\right)$. The iteration process has to be repeated up to a certain point where all the results are leading to the same conclusion to a specified degree of accuracy.

Worked Examination Questions on Circular Measure and Trigonometry

Question (Cambridge, June 2008 qp.3)

3



In the diagram, $ABCD$ is a rectangle with $AB = 3a$ and $AD = a$. A circular arc, with centre A and radius r, joins points M and N on AB and CD respectively. The angle MAN is x radians. The perimeter of the sector AMN is equal to half the perimeter of the rectangle.

(i) Show that x satisfies the equation

$$\sin x = \tfrac{1}{4}(2 + x).$$ [3]

(ii) This equation has only one root in the interval $0 < x < \frac{1}{2}\pi$. Use the iterative formula

$$x_{n+1} = \sin^{-1}\left(\frac{2 + x_n}{4}\right),$$

with initial value $x_1 = 0.8$, to determine the root correct to 2 decimal places. Give the result of each iteration to 4 decimal places. [3]

Solution

(i) $P_{rectangle} = 2(3a + a)$

$\Rightarrow P_{rectangle} = 8a$

$P_{AMN} = r + r + rx$

$\Rightarrow P_{AMN} = r(2 + x)$

where r:

using trig ratios,

$\sin x = \dfrac{opp}{r}$

$\Rightarrow opp = r \sin x$

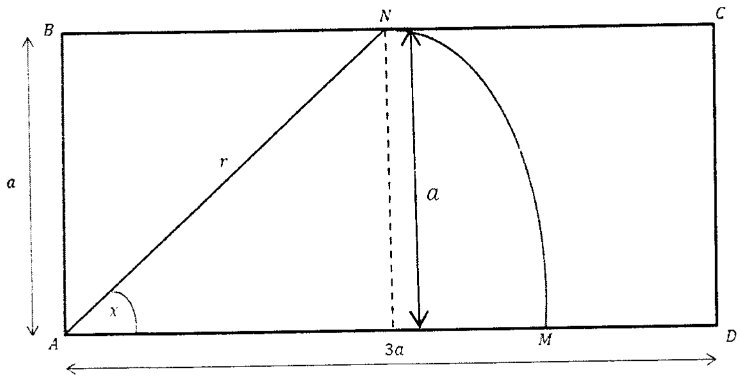


Fig. 14.2

$$but\ opp\ is\ \parallel\ and\ equal\ to\ AD = a$$

$$\Rightarrow a = r\sin x$$

$$\Rightarrow r = \frac{a}{\sin x}$$

$$Given\ that\ P_{AMN} = \frac{1}{2}P_{rectangle}$$

$$\Rightarrow r(2+x) = \frac{1}{2}(8a)$$

$$\Rightarrow \frac{a(2+x)}{\sin x} = 4a$$

$$\Rightarrow \frac{2+x}{\sin x} = 4$$

$$\Rightarrow 2 + x = 4\sin x$$

$$\therefore \mathbf{\sin x = \frac{1}{4}(2+x)(\mathit{shown})}$$

(ii) $Given\ that\ x_{n+1} = \sin^{-1}\left(\frac{2+x_n}{4}\right)$

Table 14.3

n	x_n	x_{n+1}
1	$x_1 = 0.8$	$x_2 = 0.7754$
2	$x_2 = 0.7754$	$x_3 = 0.7668$
3	$x_3 = 0.7668$	$x_4 = 0.7638$
4	$x_4 = 0.7638$	$x_5 = 0.7628$

$$\therefore \mathbf{x = 0.76\ \mathit{to}\ 2\ \mathit{d.p}}$$

Question (Cambridge, June 2012 qp.31)

10 **(i)** It is given that $2\tan 2x + 5\tan^2 x = 0$. Denoting $\tan x$ by t, form an equation in t and hence show that either $t = 0$ or $t = \sqrt[3]{(t + 0.8)}$. [4]

(ii) It is given that there is exactly one real value of t satisfying the equation $t = \sqrt[3]{(t + 0.8)}$. Verify by calculation that this value lies between 1.2 and 1.3. [2]

(iii) Use the iterative formula $t_{n+1} = \sqrt[3]{(t_n + 0.8)}$ to find the value of t correct to 3 decimal places. Give the result of each iteration to 5 decimal places. [3]

(iv) Using the values of t found in previous parts of the question, solve the equation

$$2\tan 2x + 5\tan^2 x = 0$$

for $-\pi \leqslant x \leqslant \pi$. [3]

Solution

(i) $\quad$ *Given that* $2\tan 2x + 5\tan^2 x = 0$

where $\tan 2x = \dfrac{2\tan x}{1 - \tan^2 x}$

$$\Rightarrow 2\left(\frac{2t}{1-t^2}\right) + 5t^2 = 0$$

$$\Rightarrow \frac{4t}{1-t^2} + 5t^2 = 0$$

$$\Rightarrow 4t + 5t^2 - 5t^4 = 0$$

$$\Rightarrow 5t^4 - 5t^2 - 4t = 0$$

$$\Rightarrow t(5t^3 - 5t - 4) = 0$$

$$\Rightarrow either\ t = 0$$

$$or\ 5t^3 - 5t - 4 = 0$$

$$\Rightarrow 5t^3 = 5t + 4$$

$$\Rightarrow t^3 = t + 0.8$$

$$\Rightarrow t = \sqrt[3]{t + 0.8}$$

$$\therefore \boldsymbol{t = 0\ or\ \sqrt[3]{t + 0.8}}\ \ (\textbf{\textit{shown}})$$

(ii) $t = \sqrt[3]{t + 0.8}$

by collecting all the terms to one side,

$\Rightarrow t - \sqrt[3]{t + 0.8} = 0$

$let\ f(t) = t - \sqrt[3]{t + 0.8}$

$where\ f(1.2) = 1.2 - \sqrt[3]{1.2 + 0.8}$

$\Rightarrow f(1.2) = -0.0599$

$and \Rightarrow f\ (1.3) = 1.3 - \sqrt[3]{1.3 + 0.8}$

$\Rightarrow f(1.3) = 0.0194$

$\therefore$ ***there is a root between 1.2 and 1.3 because of sign change.***

(iii) $t_{n+1} = \sqrt[3]{t_n + 0.8}$

$where\ t_1 = \dfrac{1.2 + 1.3}{2}$

$\Rightarrow t_1 = 1.25$

Table 14.4

n	t_n	t_{n+1}
1	$t_1 = 1.25$	$t_2 = 1.27033$
2	$t_2 = 1.27033$	$t_3 = 1.27452$
3	$t_3 = 1.27452$	$t_4 = 1.27538$
4	$t_4 = 1.27538$	$t_5 = 1.27555$
5	$t_5 = 1.27555$	$t_6 = 1.27559$

$\therefore \boldsymbol{t = 1.276\ to\ 3\ d.p}$

(iv) $2\tan 2x + 5\tan^2 x = 0$

$where\ t = 0$	and	$t = 1.276$
$\Rightarrow \tan x = 0$	and	$\tan x = 1.276$
$\Rightarrow x = \tan^{-1}(0)$	and	$x = \tan^{-1}(1.276)$
$\Rightarrow x = 0\ (PV)$		$\Rightarrow x = 0.906\ (PV)$

$using\ the\ general\ solution\ for \tan x,$
$x = 0 + \pi n$

$when\ n = 0;\ x = 0$

$when\ n = 1;\ x = \pi$

$when\ n = -1;\ x = -\pi$

$x = 0.906 + \pi n$

$when\ n = 0;\ x = 0.906$

$when\ n = 1;\ x = out\ of\ range$

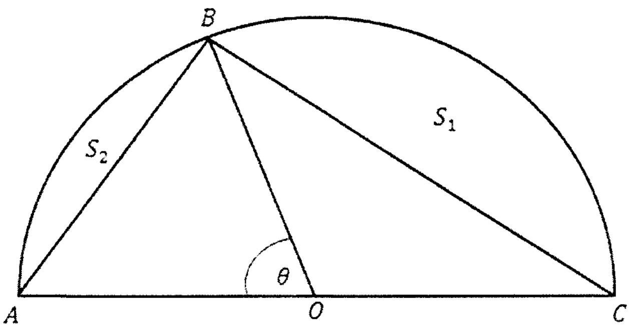
$when\ n = -1;\ x = -2.24$

$\therefore \boldsymbol{x = (-\pi;\ \ -2.24\ ;\ 0\ ;\ \ 0.906\ and\ \pi)}$

<u>Revision Questions on Circular Measure and Trigonometry</u>

November 1998 qp.1 (Cambridge)

10.



The diagram shows a semicircle ABC on AC as diameter. The mid-point of AC is O, and angle $AOB = \theta$ radians, where $0 < \theta < \frac{1}{2}\pi$. The area of the segment S_1 bounded by the chord BC is twice the area of the segment S_2 bounded by the chord AB. Show that

$$3\theta = \pi + \sin\theta. \qquad [3]$$

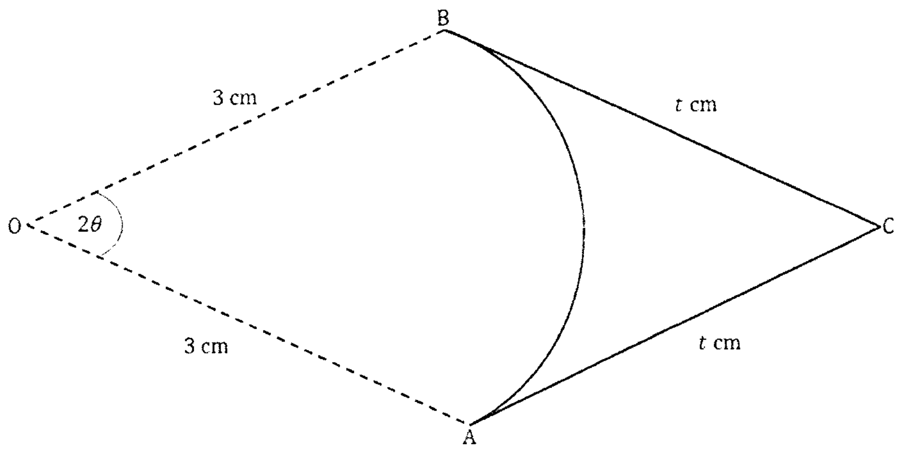
Use the iterative formula

$$\theta_{n+1} = \frac{1}{3}(\pi + \sin\theta_n),$$

together with a suitable starting value, to find θ correct to 3 significant figures. You should show the value of each approximation that you calculate. [3]

November2006 qp.1 (Zimsec)

6.



The diagram shows the shape ABC formed from a piece of wire 10cm in length. AB is the arc of circle centre O and radius 3cm. CA and CB are tangents to the circle, each of length t cm.

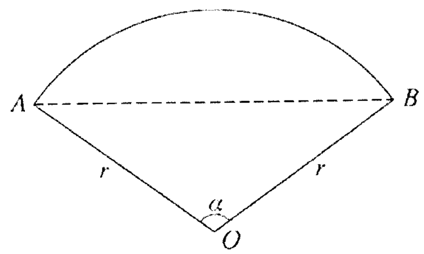
Given that the angle subtended by the arc at the centre is 2θ radians, show that $t = 5 - 3\theta$. [2]

Hence show that $t = 5 - 3\tan^{-1}\left(\frac{t}{3}\right)$. [2]

Use the iterative formula $t_{n+1} = 5 - 3\tan^{-1}\left(\frac{t_n}{3}\right)$ with $t_1 = 3$ to find t_2, t_3 and t_4. [3]

June 2007 qp.3 (Cambridge)

6



The diagram shows a sector AOB of a circle with centre O and radius r. The angle AOB is α radians, where $0 < \alpha < \pi$. The area of triangle AOB is half the area of the sector.

(i) Show that α satisfies the equation

$$x = 2\sin x.$$ [2]

(ii) Verify by calculation that α lies between $\frac{1}{2}\pi$ and $\frac{2}{3}\pi$. [2]

(iii) Show that, if a sequence of values given by the iterative formula

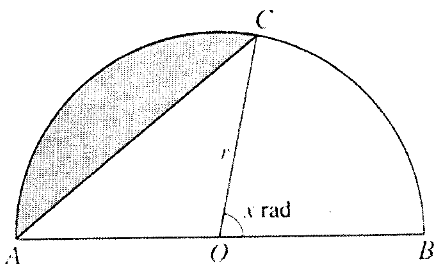
$$x_{n+1} = \tfrac{1}{3}(x_n + 4\sin x_n)$$

converges, then it converges to a root of the equation in part **(i)**. [2]

(iv) Use this iterative formula, with initial value $x_1 = 1.8$, to find α correct to 2 decimal places. Give the result of each iteration to 4 decimal places. [3]

June 2010 qp.31 (Cambridge)

6



The diagram shows a semicircle ACB with centre O and radius r. The angle BOC is x radians. The area of the shaded segment is a quarter of the area of the semicircle.

(i) Show that x satisfies the equation

$$x = \frac{3}{4}\pi - \sin x. \qquad [3]$$

(ii) This equation has one root. Verify by calculation that the root lies between 1.3 and 1.5. [2]

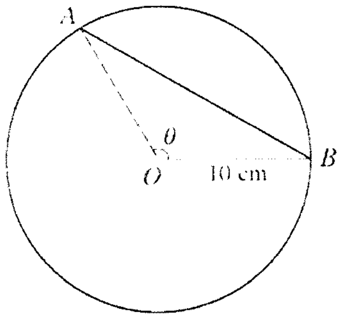
(iii) Use the iterative formula

$$x_{n+1} = \frac{3}{4}\pi - \sin x_n$$

to determine the root correct to 2 decimal places. Give the result of each iteration to 4 decimal places. [3]

June 2011 qp.31 (Cambridge)

6



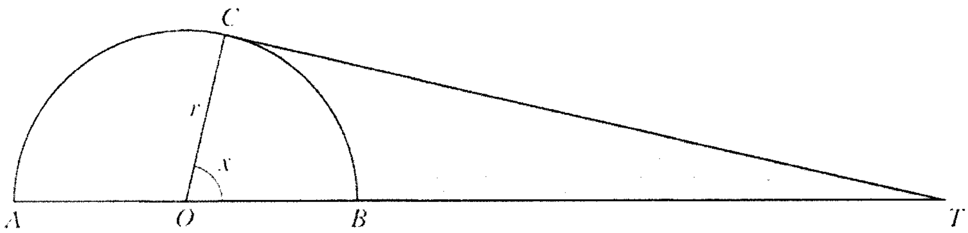
The diagram shows a circle with centre O and radius 10 cm. The chord AB divides the circle into two regions whose areas are in the ratio 1 : 4 and it is required to find the length of AB. The angle AOB is θ radians.

(i) Show that $\theta = \frac{2}{5}\pi + \sin\theta$. [3]

(ii) Showing all your working, use an iterative formula, based on the equation in part **(i)**, with an initial value of 2.1, to find θ correct to 2 decimal places. Hence find the length of AB in centimetres correct to 1 decimal place. [5]

June 2011 qp.32 (Cambridge)

4



The diagram shows a semicircle ACB with centre O and radius r. The tangent at C meets AB produced at T. The angle BOC is x radians. The area of the shaded region is equal to the area of the semicircle.

(i) Show that x satisfies the equation

$$\tan x = x + \pi.$$ [3]

(ii) Use the iterative formula $x_{n+1} = \tan^{-1}(x_n + \pi)$ to determine x correct to 2 decimal places. Give the result of each iteration to 4 decimal places. [3]

November 2010 qp.31 (Cambridge)

4 **(i)** By sketching suitable graphs, show that the equation

$$4x^2 - 1 = \cot x$$

has only one root in the interval $0 < x < \frac{1}{2}\pi$. [2]

(ii) Verify by calculation that this root lies between 0.6 and 1. [2]

(iii) Use the iterative formula

$$x_{n+1} = \tfrac{1}{2}\sqrt{(1 + \cot x_n)}$$

to determine the root correct to 2 decimal places. Give the result of each iteration to 4 decimal places. [3]

June 2011 qp.33 (Cambridge)

6 **(i)** By sketching a suitable pair of graphs, show that the equation

$$\cot x = 1 + x^2,$$

where x is in radians, has only one root in the interval $0 < x < \frac{1}{2}\pi$. [2]

(ii) Verify by calculation that this root lies between 0.5 and 0.8. [2]

(iii) Use the iterative formula

$$x_{n+1} = \tan^{-1}\left(\frac{1}{1 + x_n^2}\right)$$

to determine this root correct to 2 decimal places. Give the result of each iteration to 4 decimal places. [3]

Worked Examination Question on Integration

Question (Cambridge, November 2008 qp.3)

9 The constant a is such that $\int_0^a xe^{\frac{1}{2}x}\,dx = 6$.

(i) Show that a satisfies the equation

$$x = 2 + e^{-\frac{1}{2}x}.$$ [5]

(ii) By sketching a suitable pair of graphs, show that this equation has only one root. [2]

(iii) Verify by calculation that this root lies between 2 and 2.5. [2]

(iv) Use an iterative formula based on the equation in part **(i)** to calculate the value of a correct to 2 decimal places. Give the result of each iteration to 4 decimal places. [3]

Solution

(i) *Given that,*

$$\int_0^a xe^{\frac{1}{2}x}\,dx = 6$$

using integration by parts,

$let\ u = x$ $\quad$ *and* $\quad \frac{dv}{dx} = e^{\frac{1}{2}x}$

$\frac{du}{dx} = 1$ $\quad\quad v = 2e^{\frac{1}{2}x}$

$$\Rightarrow 2xe^{\frac{1}{2}x} - \int_0^a 2e^{\frac{1}{2}x} = 6$$

$$\Rightarrow \left[2xe^{\frac{1}{2}x} - 4e^{\frac{1}{2}x}\right]_0^a = 6$$

$$\Rightarrow \left(2ae^{\frac{1}{2}a} - 4e^{\frac{1}{2}a}\right) - \left(2(0)e^{\frac{1}{2}(0)} - 4e^{\frac{1}{2}(0)}\right) = 6$$

$$\Rightarrow 2ae^{\frac{1}{2}a} - 4e^{\frac{1}{2}a} + 4 = 6$$

$\Rightarrow 2e^{\frac{1}{2}a}(a-2) = 2$

$\Rightarrow a - 2 = \dfrac{2}{2e^{\frac{1}{2}a}}$

$\Rightarrow a = e^{-\frac{1}{2}a} + 2$

since a is an x − value,

$\Rightarrow x = e^{-\frac{1}{2}x} + 2$

$\therefore \boldsymbol{x = 2 + e^{-\frac{1}{2}x}}$ (***shown***)

(ii)

$let\ y = LHS$ $\quad$ *and* $\quad$ $y = RHS$

$\Rightarrow y = x$ $\quad$ *and* $\quad$ $y = 2 + e^{-\frac{1}{2}x}$

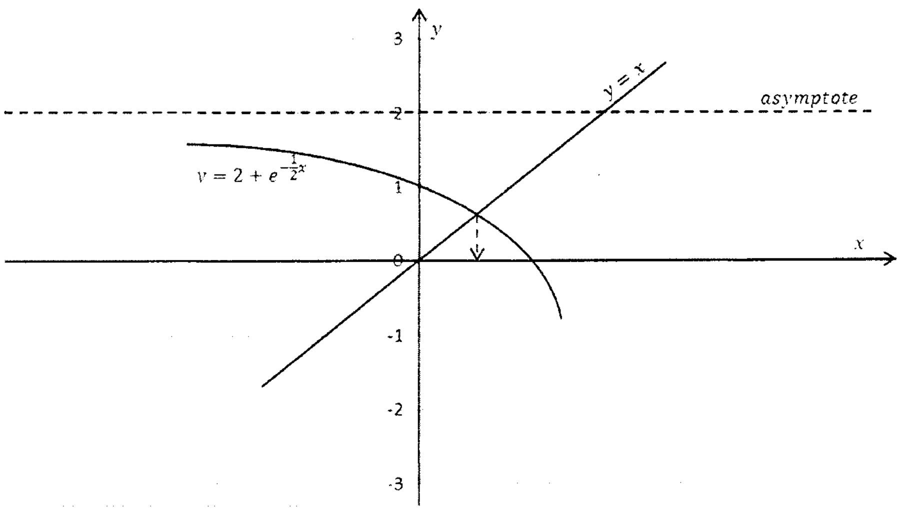


Fig.14.3

(iii) $Since\ \ x = 2 + e^{-\frac{1}{2}x}$

by collecting all the terms to one side,

$\Rightarrow x - 2 - e^{-\frac{1}{2}x} = 0$

$let\ f(x) = x - 2 - e^{-\frac{1}{2}x}$

$where\ f(2) = 2 - 2 - e^{-\frac{1}{2}(2)}$

$\Rightarrow f(2) = -0.368$

$and\ f(2.5) = 2.5 - 2 - e^{-\frac{1}{2}(2.5)}$

$\Rightarrow f(2.5) = 0.213$

$\therefore$ ***there is a root between* 2 *and* 2.5 *because there is a sign change*.**

(iv) $x = 2 + e^{-\frac{1}{2}x}$

by introducing subscripts,

$x_{n+1} = 2 + e^{-\frac{1}{2}x_n}$

$where\ x_1 = \frac{2 + 2.5}{2}$

$\Rightarrow x_1 = 2.25$

Table 14.5

n	x_n	x_{n+1}
1	$x_1 = 2.25$	$x_2 = 2.3247$
2	$x_2 = 2.3247$	$x_3 = 2.3128$
3	$x_3 = 2.3128$	$x_4 = 2.3146$
4	$x_4 = 2.3146$	$x_5 = 2.3143$

$\Rightarrow x = 2.31$

$\therefore \boldsymbol{a = 2.31}$

Revision Questions on Integration

November 2010 qp.33 (Cambridge)

7 (i) Given that $\int_1^a \frac{\ln x}{x^2}\,dx = \frac{2}{5}$, show that $a = \frac{5}{3}(1 + \ln a)$. [5]

(ii) Use an iteration formula based on the equation $a = \frac{5}{3}(1 + \ln a)$ to find the value of a correct to 2 decimal places. Use an initial value of 4 and give the result of each iteration to 4 decimal places. [3]

November 2011 qp.33 (Cambridge)

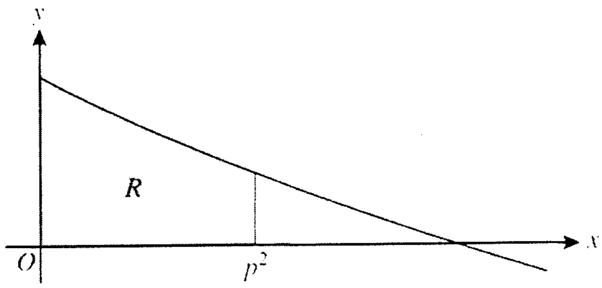
5 It is given that $\int_1^a x \ln x\,dx = 22$, where a is a constant greater than 1.

(i) Show that $a = \sqrt{\left(\frac{87}{2\ln a - 1}\right)}$. [5]

(ii) Use an iterative formula based on the equation in part **(i)** to find the value of a correct to 2 decimal places. Use an initial value of 6 and give the result of each iteration to 4 decimal places. [3]

June 2012 qp.33 (Cambridge)

7



The diagram shows part of the curve $y = \cos(\sqrt{x})$ for $x \geqslant 0$, where x is in radians. The shaded region between the curve, the axes and the line $x = p^2$, where $p > 0$, is denoted by R. The area of R is equal to 1.

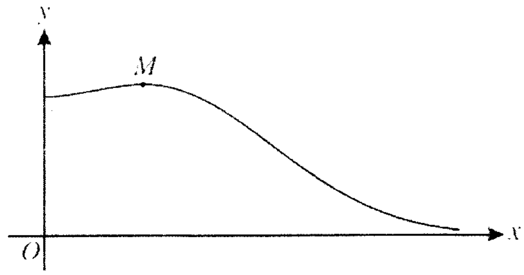
(i) Use the substitution $x = u^2$ to find $\int_0^{p^2} \cos(\sqrt{x})\,dx$. Hence show that $\sin p = \frac{3 - 2\cos p}{2p}$. [6]

(ii) Use the iterative formula $p_{n+1} = \sin^{-1}\left(\frac{3 - 2\cos p_n}{2p_n}\right)$, with initial value $p_1 = 1$, to find the value of p correct to 2 decimal places. Give the result of each iteration to 4 decimal places. [3]

Worked Examination Question on Differentiation

Question (Cambridge, November 2012 qp.31)

8



The diagram shows the curve $y = e^{-\frac{1}{2}x^2}\sqrt{(1 + 2x^2)}$ for $x \geqslant 0$, and its maximum point M.

(i) Find the exact value of the x-coordinate of M. [4]

(ii) The sequence of values given by the iterative formula

$$x_{n+1} = \sqrt{(\ln(4 + 8x_n^2))},$$

with initial value $x_1 = 2$, converges to a certain value α. State an equation satisfied by α and hence show that α is the x-coordinate of a point on the curve where $y = 0.5$. [3]

(iii) Use the iterative formula to determine α correct to 2 decimal places. Give the result of each iteration to 4 decimal places. [3]

Solution

(i) $M \text{ is a turning point which implies that } \frac{dy}{dx} = 0.$

$$y = e^{-\frac{1}{2}x^2}\sqrt{(1 + 2x^2)}$$

$$\Rightarrow y = e^{-\frac{1}{2}x^2}(1 + 2x^2)^{\frac{1}{2}}$$

using the product rule,

$$\frac{dy}{dx} = (1 + 2x^2)^{\frac{1}{2}}\left(-xe^{-\frac{1}{2}x^2}\right) + \left(e^{-\frac{1}{2}x^2}\right)\left(\frac{1}{2}\right)(1 + 2x^2)^{-\frac{1}{2}}(4x)$$

$$\Rightarrow \frac{dy}{dx} = -xe^{-\frac{1}{2}x^2}(1 + 2x^2)^{\frac{1}{2}} + 2xe^{-\frac{1}{2}x^2}(1 + 2x^2)^{-\frac{1}{2}}$$

$$\Rightarrow 0 = -xe^{-\frac{1}{2}x^2}(1 + 2x^2)^{\frac{1}{2}} + \frac{2xe^{-\frac{1}{2}x^2}}{(1 + 2x^2)^{\frac{1}{2}}}$$

$$\Rightarrow xe^{-\frac{1}{2}x^2}(1+2x^2)^{\frac{1}{2}} = \frac{2xe^{-\frac{1}{2}x^2}}{(1+2x^2)^{\frac{1}{2}}}$$

$$\Rightarrow xe^{-\frac{1}{2}x^2}(1+2x^2) = 2xe^{-\frac{1}{2}x^2}$$

$$\Rightarrow 1 + 2x^2 = 2$$

$$\Rightarrow 2x^2 = 1$$

$$\Rightarrow x^2 = \frac{1}{2}$$

$$\therefore \boldsymbol{x} = \sqrt{\frac{\mathbf{1}}{\mathbf{2}}}$$

(ii) *Given the iterative formula,*

$$x_{n+1} = \sqrt{[\ln(4 + 8x_n{}^2)]}$$

by ignoring the subscripts,

$$\therefore \boldsymbol{x} = \sqrt{[\mathbf{ln}(\mathbf{4} + \mathbf{8}\boldsymbol{x}^{\mathbf{2}})]}$$

when $y = 0.5$,

$$\Rightarrow (0.5)^2 = \left[e^{-\frac{1}{2}x^2}\sqrt{(1+2x^2)}\right]^2$$

$$\Rightarrow \frac{1}{4} = e^{-x^2}(1+2x^2)$$

$$\Rightarrow \frac{1}{4} = \frac{(1+2x^2)}{e^{x^2}}$$

$$\Rightarrow e^{x^2} = 4(1+2x^2)$$

$$\Rightarrow x^2 = \ln(4+8x^2)$$

$$\therefore \boldsymbol{x} = \sqrt{\mathbf{ln}(\mathbf{4} + \mathbf{8}\boldsymbol{x}^{\mathbf{2}})}\ (\boldsymbol{shown})$$

(iii) $x_{n+1} = \sqrt{[\ln(4 + 8x_n{}^2)]}$

$where\ x_1 = 2$

Table 14.6

n	x_n	x_{n+1}
1	$x_1 = 2$	$x_2 = 1.8930$
2	$x_2 = 1.8930$	$x_3 = 1.8672$
3	$x_3 = 1.8672$	$x_4 = 1.8607$
4	$x_4 = 1.8607$	$x_5 = 1.8591$
5	$x_5 = 1.8591$	$x_6 = 1.8587$

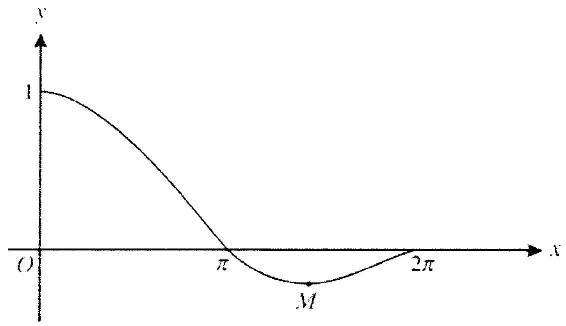
$\Rightarrow x = 1.86$

$\therefore \alpha = \mathbf{1.86}$

Revision Questions on Differentiation

June 2010 qp.32 (Cambridge)

4



The diagram shows the curve $y = \dfrac{\sin x}{x}$ for $0 < x \leqslant 2\pi$, and its minimum point M.

(i) Show that the x-coordinate of M satisfies the equation

$$x = \tan x.$$ [4]

(ii) The iterative formula

$$x_{n+1} = \tan^{-1}(x_n) + \pi$$

can be used to determine the x-coordinate of M. Use this formula to determine the x-coordinate of M correct to 2 decimal places. Give the result of each iteration to 4 decimal places. [3]

June 2010 qp.33 (Cambridge)

6 The curve $y = \dfrac{\ln x}{x+1}$ has one stationary point.

(i) Show that the x-coordinate of this point satisfies the equation

$$x = \frac{x+1}{\ln x},$$

and that this x-coordinate lies between 3 and 4. [5]

(ii) Use the iterative formula

$$x_{n+1} = \frac{x_n + 1}{\ln x_n}$$

to determine the x-coordinate correct to 2 decimal places. Give the result of each iteration to 4 decimal places. [3]

Worked Examination Questions on Algebraic Expressions

Question (Cambridge, November 2009 qp.31)

3 The sequence of values given by the iterative formula

$$x_{n+1} = \frac{3x_n}{4} + \frac{15}{x_n^3},$$

with initial value $x_1 = 3$, converges to α.

(i) Use this iterative formula to find α correct to 2 decimal places, giving the result of each iteration to 4 decimal places. [3]

(ii) State an equation satisfied by α and hence find the exact value of α. [2]

Solution

(i) $x_{n+1} = \frac{3x_n}{4} + \frac{15}{{x_n}^3}$

$where\ x_1 = 3$

Table 14.7

n	x_n	x_{n+1}
1	$x_1 = 3$	$x_2 = 2.8056$
2	$x_2 = 2.8056$	$x_3 = 2.7834$
3	$x_3 = 2.7834$	$x_4 = 2.7831$
4	$x_4 = 2.7831$	$x_5 = 2.7832$

$\Rightarrow x = 2.78$

$\therefore \boldsymbol{\alpha = 2.78}$

(ii) $using\ the\ iterative\ formula,$

$x_{n+1} = \frac{3x_n}{4} + \frac{15}{{x_n}^3}$

$by\ ignoring\ subscripts,$

$x = \frac{3x}{4} + \frac{15}{x^3}$

$\Rightarrow 4x^3(x) = \frac{3x}{4}(4x^3) + \frac{15}{x^3}(4x^3)$

$\Rightarrow 4x^4 = 3x^4 + 60$

$\Rightarrow x^4 = 60$

$\Rightarrow x = \sqrt[4]{60}$

$\therefore \boldsymbol{\alpha = \sqrt[4]{60}}$

Revision Questions on Algebraic Expressions

June 2001 qp.1 (Cambridge)

13. Use the iterative formula

$$x_{n+1} = \frac{2(x_n^3 + 1)}{3x_n^2},$$

where $x_0 = 1$, to calculate x_1, x_2 and x_3. Give your answers correct to 6 decimal places. [3]

Given that the iteration converges, find and simplify an equation in x whose real root is approximated by this iteration. [2]

Hence write down the exact value of this root. Find the absolute errors e_1, e_2, e_3 in the first three approximations x_1, x_2, x_3, giving your answers to 6 decimal places. [3]

June 2009 qp.3 (Cambridge)

4 The equation $x^3 - 2x - 2 = 0$ has one real root.

(i) Show by calculation that this root lies between $x = 1$ and $x = 2$. [2]

(ii) Prove that, if a sequence of values given by the iterative formula

$$x_{n+1} = \frac{2x_n^3 + 2}{3x_n^2 - 2}$$

converges, then it converges to this root. [2]

(iii) Use this iterative formula to calculate the root correct to 2 decimal places. Give the result of each iteration to 4 decimal places. [3]

November 2009 qp.32 (Cambridge)

2 The equation $x^3 - 8x - 13 = 0$ has one real root.

(i) Find the two consecutive integers between which this root lies. [2]

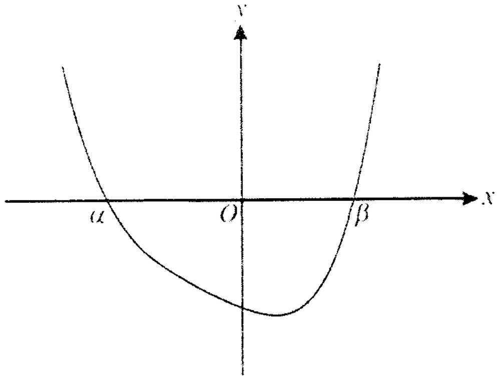
(ii) Use the iterative formula

$$x_{n+1} = (8x_n + 13)^{\frac{1}{3}}$$

to determine this root correct to 2 decimal places. Give the result of each iteration to 4 decimal places. [3]

November 2012 qp.33 (Cambridge)

6



The diagram shows the curve $y = x^4 + 2x^3 + 2x^2 - 4x - 16$, which crosses the x-axis at the points $(\alpha, 0)$ and $(\beta, 0)$ where $\alpha < \beta$. It is given that α is an integer.

(i) Find the value of α. [2]

(ii) Show that β satisfies the equation $x = \sqrt[3]{(8 - 2x)}$. [3]

(iii) Use an iteration process based on the equation in part **(ii)** to find the value of β correct to 2 decimal places. Show the result of each iteration to 4 decimal places. [3]

<u>Worked Examination Questions involving curve sketching</u>

Question (Cambridge, November 2007 qp.3)

6 **(i)** By sketching a suitable pair of graphs, show that the equation

$$2 - x = \ln x$$

has only one root. [2]

(ii) Verify by calculation that this root lies between 1.4 and 1.7. [2]

(iii) Show that this root also satisfies the equation

$$x = \tfrac{1}{3}(4 + x - 2\ln x).$$ [1]

(iv) Use the iterative formula

$$x_{n+1} = \tfrac{1}{3}(4 + x_n - 2\ln x_n),$$

with initial value $x_1 = 1.5$, to determine this root correct to 2 decimal places. Give the result of each iteration to 4 decimal places. [3]

Solution

(i) $let\ y = LHS$ and $y = RHS$

$\Rightarrow y = 2 - x$ and $y = \ln x$

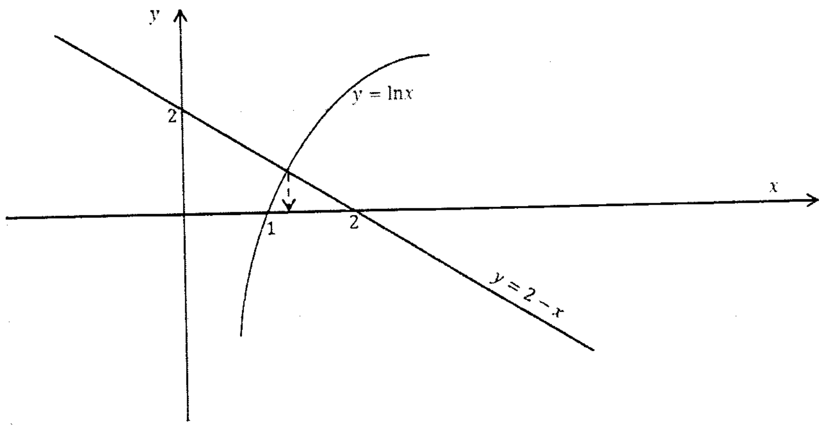


Fig.14.4

(ii) $2 - x = \ln x$

$\Rightarrow 2 - x - \ln x = 0$

$let\ f(x) = 2 - x - \ln x$

$where\ f(1.4) = 2 - 1.4 - \ln 1.4$

$\Rightarrow f(1.4) = 0.264$

$and\ f(1.7) = 2 - 1.7 - \ln 1.7$

$\Rightarrow f(1.7) = -0.231$

$\therefore$ ***there is a root between 1.4 and 1.7 because there is a sign change***

(iii) $x = \frac{1}{3}(4 + x - 2\ln x)$

$by\ re-arranging\ the\ equation,$
$3x = 4 + x - 2\ln x$

$\Rightarrow 2\ln x = 4 - 2x$

$\Rightarrow \ln x = 2 - x$

$\therefore \mathbf{2 - x = \ln x}$ ***(shown)***

(iv) $x_{n+1} = \frac{1}{3}(4 + x_n - 2\ln x_n)$

$where\ x_1 = 1.5$

Table 14.8

n	x_n	x_{n+1}
1	$x_1 = 1.5$	$x_2 = 1.5630$
2	$x_2 = 1.5630$	$x_3 = 1.5566$
3	$x_3 = 1.5566$	$x_4 = 1.5572$
4	$x_4 = 1.5572$	$x_5 = 1.5571$

$\therefore \boldsymbol{x = 1.56}$

Newton-Raphson Method

This method is based on finding the linear approximation for a function. As such, the method makes use of the gradient function, $f'(x)$, to estimate the probable location of a solution. In this module, we will overlook the derivation process of the Newton-Raphson Method because it is beyond the scope of the ZIMSEC syllabus. The method states that:

$$\boldsymbol{x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}}$$

Where:

- x_n is the first approximation, which is given in most cases.
- $f(x_n)$ is the result obtained by substituting the first approximation in the function.
- $f'(x_n)$ is the result obtained by substituting the first approximation in the first derivative of the function.
- x_{n+1} is the result obtained by the Newton-Raphson method.

All iterative methods share the same principles. Once the first approximation (commonly known as an iteration) is found, it is used as input in the general formula to give a better approximation. This is fed in the general formula again to give an even better approximation. This process is repeated until the required degree of accuracy has been satisfied.

Reviewing a practical example helps in improving on the understanding of the Newton-Raphson Method. Using the example below:

Question (Zimsec, November 2011 qp.1)

12. (c) Starting with $x_0 = 1$ use Newton-Raphson method to find the smallest positive root of $2x - \tan x = 0$, giving your answer correct to 3 decimal places. [4]

Proposed Solution

$let\ f(x) = 2x - \tan x$

$\Rightarrow f'(x) = 2 - \sec^2 x$

$\Rightarrow f'(x) = 2 - \frac{1}{\cos^2 x}$

$using\ x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$

Table 14.2

n	x_n	$f(x_n)$	$f'(x_n)$	x_{n+1}
0	$x_0 = 1$	0.44259	−1.142552	1.31048
1	$x_1 = 1.31048$	−1.13335	−13.09487	1.22393
2	$x_2 = 1.22393$	−0.31853	−6.65294	1.17605
3	$x_3 = 1.17605$	−0.04820	−4.76145	1.16593
4	$x_4 = 1.16593$	−0.001643	−4.44521	1.16556

$\therefore \boldsymbol{x = 1.166\ to\ 3\ d.p}$

Worked Examination Question on Newton-Raphson Method

Question (Zimsec, November 2002 qp.1)

13. Sketch on the same axes, the graphs of

$$y = e^{3x} \text{ and } y = 2x + 10$$ [2]

(i). Show that the equation

$$e^{3x} - 2x - 10 = 0$$

has a root between -5 and -4. [2]

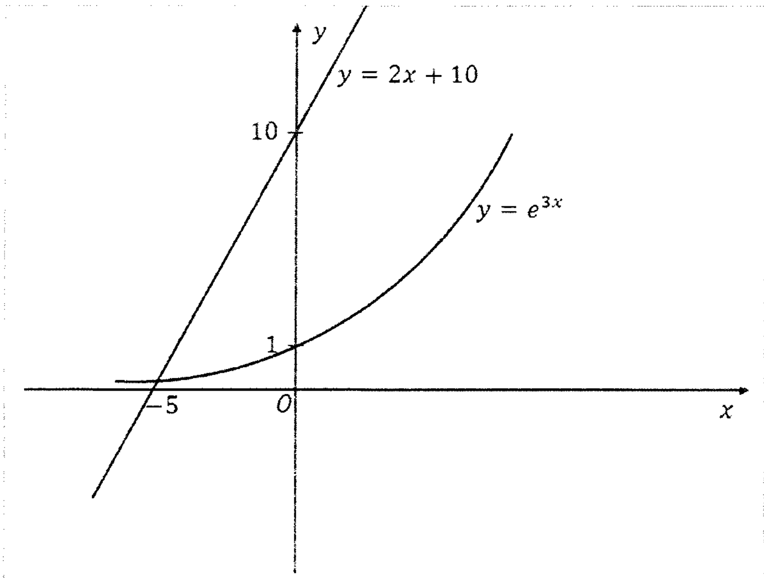
(ii). State a pair of consecutive integers between which another root lies. [1]

(iii). Taking $x_1 = -4$, use the Newton-Raphson method once to find a second approximation x_2, giving your answer correct to 5 decimal places. [3]

Solution

Given that $y = e^{3x}$ *and* $y = 2x + 10$,

Diagrammatically the relationship connecting the two is as shown below:



i. *Given the equation,* $e^{3x} - 2x - 10 = 0$

let $f(x) = e^{3x} - 2x - 10$

where $f(-5) = e^{3(-5)} - 2(-5) - 10$

$\Rightarrow f(-5) = 3.06 \times 10^{-7} \quad (positive)$

$and \ f(-4) = e^{3(-4)} - 2(-4) - 10$

$\Rightarrow f(-4) = -1{,}999994 \ (negative)$

$\therefore$ **there is a root between -4 and -5 because $f(-5) > 0$ and $f(-4) < 0$.**

ii. $0 \ and \ 1$

iii. $If \ f(x) = e^{3x} - 2x - 10$

$then \ f'(x) = 3e^{3x} - 2$

$$using \ x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$\Rightarrow x_2 = -4 - \frac{[e^{3(-4)} - 2(-4) - 10]}{[3e^{3(-4)} - 2]}$$

$\therefore \mathbf{x_2 = -5{,}00001 \ to \ 5 \ d.p}$

Question (Zimsec, November 2010 qp.1)

9. (a). Show, by sketching two appropriate graphs, that the equation $x^3 + 3x - 3 = 0$ has only one real root. [2]

(b). Show, by calculation, that the root of the equation in (a) lies between $x = 0.8$ and $x = 1$. [1]

(c) Obtain approximations to the root of the equation in (a), to 3 significant figures

(i). by performing one application of the Newton-Raphson procedure using $x = 0.8$ as a first approximation, [2]

(ii). by performing two iterations, using $x_{n+1} = \frac{3 - x_n^3}{3}$ and starting with $x = 0.8$. [2]

Solution

9. (a). *Given that* $x^3 + 3x - 3 = 0$,

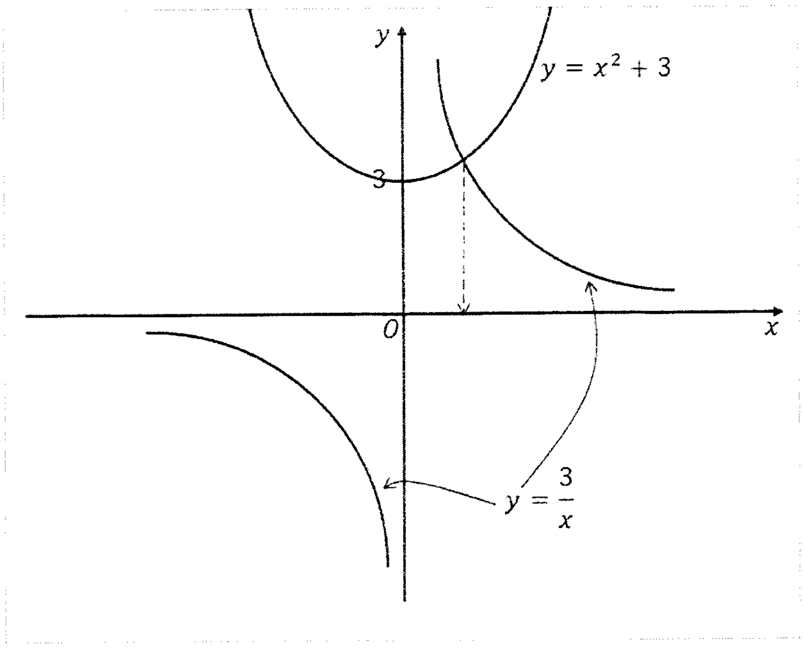
by dividing throughout by x,

$$\Rightarrow x^2 + 3 - \frac{3}{x} = 0$$

$$\Rightarrow x^2 + 3 = \frac{3}{x}$$

Let $y = x^2 + 3$ *and* $y = \frac{3}{x}$,

The diagram below shows the relationship between the two graphs:



$\therefore$ ***there one real root because there is one point of intersection.***

(b). *Since* $x^3 + 3x - 3 = 0$,

let $f(x) = x^3 + 3x - 3$

where $f(0.8) = -0.088$ *(negative)*

and $f(1) = 1$ *(positive)*

$\therefore$ ***there is a root between*** $\mathbf{0.8}$ ***and*** $\mathbf{1}$ ***because*** $f(0.8) < 0$ ***and*** $f(1) > 0$ ***(shown)*** (c). (i) $Since\ f(x) = x^3 + 3x - 3$

$then\ f'(x) = 3x^2 + 3$

$$using\ \ x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$\Rightarrow x_{n+1} = 0.8 - \frac{[0.8^3 + 3(0.8) - 3]}{[3(0.8)^2 + 3]}$$

$$\therefore \boldsymbol{x_{n+1} = 0.818\ \ to\ \ 3\ s.f}$$

(ii). $Given\ that,\ x_{n+1} = \frac{3-x_n^3}{3}$

$where\ \ x_1 = 0.8$

$$\Rightarrow x_2 = \frac{3 - 0.8^3}{3}$$

$$\Rightarrow x_2 = \frac{311}{375}$$

$$and\ \ x_3 = \frac{3 - \left(\frac{311}{375}\right)^3}{3}$$

$$\therefore\ \boldsymbol{x_3 = 0.810\ \ to\ \ 3\ s.f}$$

Revision Questions on Newton-Raphson Method

November 1997 qp.1 (Zimsec)

16. Show, by means of a graphical argument, that the equation $x = 10e^{-\frac{1}{4}x}$ has exactly one real root (denoted by α), and determine the pair of consecutive integers between which α lies. [3]

The iterative formula $x_{n+1} = 10e^{-\frac{1}{4}x_n}$ can be used to find α. Starting with the nearest integer above α, carry out four applications of this iteration, and state the number of significant figures to which α has been determined as a result of these calculations. [3]

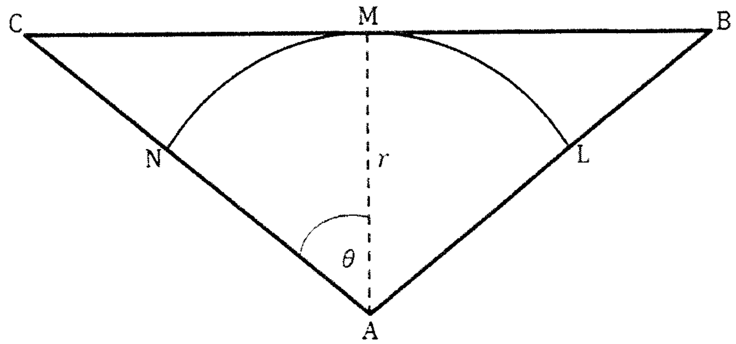
Show that the equation $x = 10e^{-\frac{1}{4}x}$ may be written in the form

$$x + 4\ln x - 4\ln 10 = 0. \quad [2]$$

Using the same initial value as previously, carry out two iterations of the Newton-Raphson method, applied to the equation $x + 4\ln x - 4\ln 10 = 0$, and state the number of significant figures to which α has been determined as a result. [5]

November 2001 qp.1 (Zimsec)

14.



The diagram shows an isosceles triangle ABC in which $AB = AC$. The mid-point of BC is M, and $AM = r$. A circular arc with centre A and radius r is drawn and meets the sides AB and AC at the points L and N respectively. Angle $CAM = \theta$ radians. Given that the area of the sector $ALMN$ is one quarter of the area of triangle ABC, show that $\theta = \frac{1}{4}\tan\theta$. [3]

Show, by sketching graphs, that this equation has exactly one root between 0 and $\frac{1}{2}\pi$ and verify by calculating that this root lies between 1.3 and 1.4. [3]

Use the Newton-Raphson method with 1^{st} approximation 1.4 to determine this root correct to 3 decimal places, showing the value of each approximation that you calculate. [4]

June 2003 qp.1 (Zimsec)

10. Show graphically that the equation $\ln x - 2 + x = 0$ has only one real root. [2]

State two consecutive integers between which this root lies. Using the smaller of these two integers as initial approximation apply the Newton-Raphson method to obtain the root correct to 3 significant figures. [5]

Chapter Fifteen: Complex Numbers

"I tell you, with complex numbers you can do anything."

– John Derbyshire

Mathematics is centred on the analysis and use of number systems. Complex numbers $(\boldsymbol{z})$ is one of the number systems that form the basis of 'A' Level mathematics. A complex number is a hybrid made up of a **real number** and an **imaginary number**.

It is important to highlight on the concept of imaginary numbers before reflecting on complex numbers.

Imaginary Numbers

In real terms, the square-root of a negative number is undefined. Imaginary numbers elucidate the mystification that the square-root of a negative number is undefined. It is believed that,

$$\sqrt{-1} = i$$

$For\ example,\ \sqrt{-25} = \sqrt{25} \times \sqrt{-1} = 5i$

$Since\ i = \sqrt{-1}$

$\Rightarrow i^2 = -1$

This proposition is the backbone, and is therefore central, to the study of complex numbers.

Complex Numbers

Since z is a hybrid of real and imaginary numbers, it implies that z is the sum of the two. As such,

$$\boldsymbol{z = real\ number + imaginary\ number}$$

With x denoting the real part and y denoting the imaginary part, the general rule therefore states that:

$$\boldsymbol{z = x + yi}$$

NB: x and y are real numbers.

Addition and Subtraction of z

This concept follows the same principle as the addition and subtraction of any number system. For example, given that:

$z_1 = 3 - 2i \quad and \quad z_2 = -2 - i$

$\Rightarrow z_1 + z_2 = (3 - 2i) + (-2 - i)$

$\Rightarrow z_1 + z_2 = (3 - 2) + (-2i - i)$

$\therefore \boldsymbol{z_1 + z_2 = 1 - 3i}$

$and \ z_1 - z_2 = (3 - 2i) - (-2 - i)$

$\Rightarrow z_1 - z_2 = 3 - (-2) - 2i - (-i)$

$\therefore \boldsymbol{z_1 - z_2 = 5 - i}$

NB: the idea here is to collect like terms and simplify them.

Multiplication of z

Multiplication of complex numbers uses the normal expansion of brackets. For example, using z_1 and z_2 from the example above,

$z_1 \times z_2 = (3 - 2i)(-2 - i)$

$\Rightarrow z_1 \times z_2 = 3(-2 - i) - 2i(-2 - i)$

$\Rightarrow z_1 \times z_2 = -6 - 3i + 4i + 2i^2$

$\Rightarrow z_1 \times z_2 = -6 + i + 2(-1)$

$\therefore \boldsymbol{z_1 \times z_2 = -8 + i}$

Division of z

It is an error in principle in the imaginary world to have a complex number in the denominator. It is therefore imperative that the complex number being used as the divisor be eliminated. This process of eliminating the complex number in the denominator draws its inspiration from the 'difference of two squares'.

$Given\ that\ x^2 - y^2 = (x + y)(x - y),$

$(x + y)(x - y)\ are\ a\ conjugate\ pair\ of\ factors.$

$In\ this\ case,\ \ if\ z_1 = 3 - 2i\ \ its\ conjugate\ is\ (3 + 2i)\ and\ if\ \ z_2 = -2 - i\ \ its\ conjugate$

$is\ (-2 + i).$

$A\ conjugate\ is\ given\ by\ changing\ the\ sign\ of\ the\ imaginary\ part.$

So when dividing complex numbers, multiply both the numerator and the denominator by the conjugate of the denominator.

$$For\ example,\ \ \frac{z_1}{z_2} = \frac{3 - 2i}{-2 - i}$$

$$\Rightarrow \frac{z_1}{z_2} = \frac{(3 - 2i)(-2 + i)}{(-2 - i)(-2 + i)}$$

$$\Rightarrow \frac{z_1}{z_2} = \frac{-6 + 3i + 4i - 2i^2}{4 - 2i + 2i - i^2}$$

$$\Rightarrow \frac{z_1}{z_2} = \frac{-6 + 7i - 2(-1)}{4 - (-1)}$$

$$\Rightarrow \frac{z_1}{z_2} = \frac{-4 + 7i}{5}$$

$$\therefore \frac{\boldsymbol{z_1}}{\boldsymbol{z_2}} = -\frac{\mathbf{4}}{\mathbf{5}} + \frac{\mathbf{7}}{\mathbf{5}}\boldsymbol{i}$$

The Argand Diagram

This is an instrument used to locate complex numbers in space. It is really a disguised Cartesian plane. The x –axis is used to represent the real part and the y –axis represents the imaginary part. For example,

$If\ \ z_1 = 3 - 2i \qquad and \qquad z_2 = -2 - i$

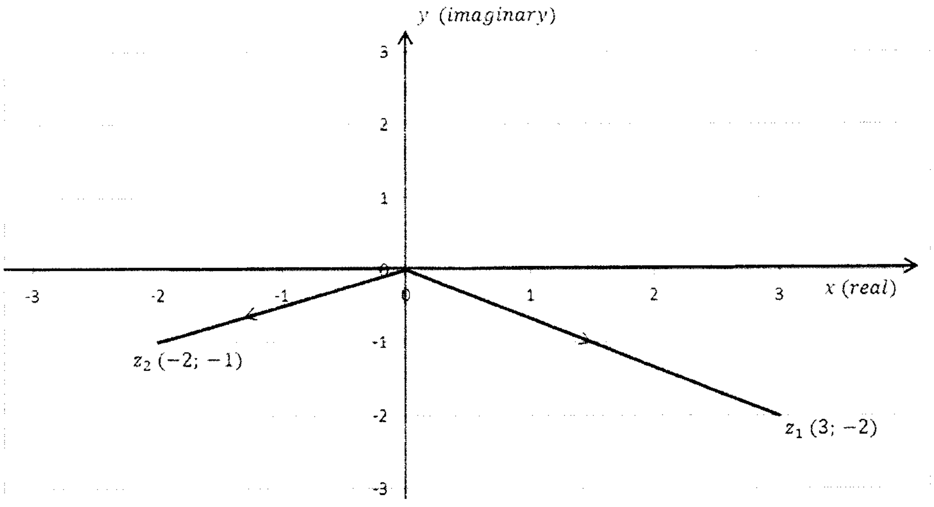


Fig.15.1

NB: complex numbers are analysed using the same set of principles as the one used for vector analysis. An arrow is incorporated to show the direction of travel.

Magnitude of a Complex Number

This refers to the length or size of a complex number. It is given by the Pythagoras theorem, that is $z^2 = x^2 + y^2$,

$\Rightarrow |z| = \sqrt{x^2 + y^2}$

For example,

$if\ z_1 = 3 - 2i$	and	$z_2 = -2 - i$
$\Rightarrow |z_1| = \sqrt{(3)^2 + (-2)^2}$		$\Rightarrow |z_2| = \sqrt{(-2)^2 + (-1)^2}$
$\Rightarrow |z_1| = \sqrt{9 + 4}$		$\Rightarrow |z_1| = \sqrt{4 + 1}$
$\Rightarrow |z_1| = \sqrt{13}\ units$		$\Rightarrow |z_1| = \sqrt{5}\ units$

Argument of a Complex Number

Argument refers to the size of the angle measured from the positive x –axis to the complex number. Argument is greater than $-\pi$ radians, but less than or equal to π radians i.e.

$$-\pi < \arg z \leq \pi$$

The Argand diagram is used in the determination of both the ***size*** and ***sign*** of the argument in relation to its location in space.

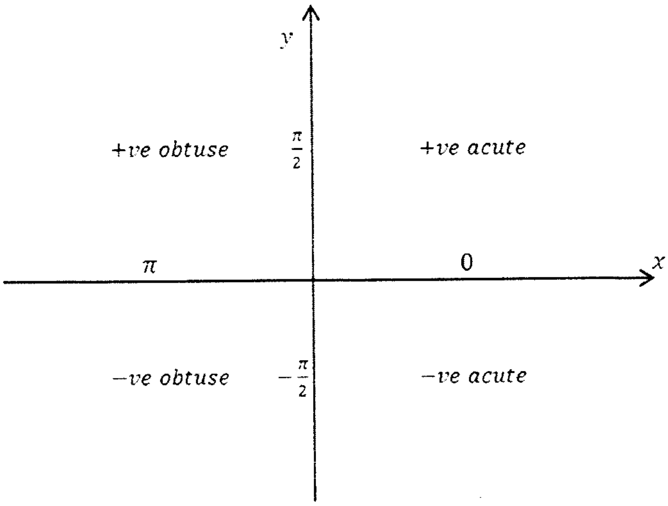


Fig 15.2

Argument is given by,

$$\mathbf{arg(z) = tan^{-1}\left(\frac{y}{x}\right),}$$

but one has to ignore the signs of x and y when calculating the argument. The actual value of the $\arg(z)$ is given by making relevant adjustments to the answer using the sketch of an Argand diagram. For example,

using $z_1 = 3 - 2i$ *and* $z_2 = -2 - i$

$\Rightarrow \arg(z_1) = \tan^{-1}\left(\frac{2}{3}\right)$

$\Rightarrow \arg(z_1) = 0.588\ rad$

$\therefore \mathbf{\arg(z_1) = -0.588\ rad}$

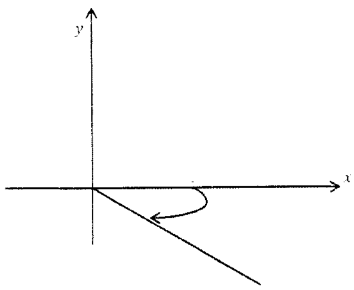


Fig. 15.3

$and \ \arg(z_2) = \left(\frac{1}{2}\right)$

$\Rightarrow \arg(z_2) = \tan^{-1}\left(\frac{1}{2}\right)$

$\Rightarrow \arg(z_2) = 0.464\ rad$

$\Rightarrow \arg(z_2) = -(\pi - 0.464)$

$\therefore \mathbf{arg(z_2) = \ -2.68\ rad}$

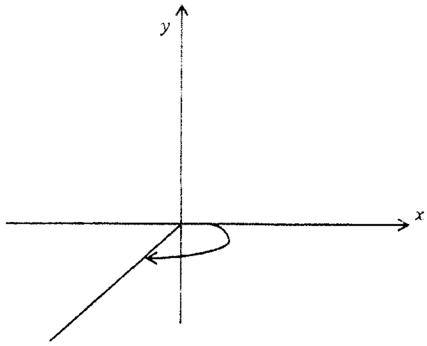


Fig. 15.4

NB: Given two complex numbers u and v,

- $\arg\left(\frac{u}{v}\right) = \arg u - \arg v$
- $\arg(uv) = \arg u + \arg v$

Geometrical Effects of Complex Operations

1. **Conjugating a Complex Number**

A conjugate of a complex number is, geometrically, a reflection of a complex number in the mirror line $y = 0$, that is the x −axis. Given that,

$$z = 3 - 2i, \text{ its conjugate, } \bar{z} = 3 + 2i.$$

Below is a diagram illustrating the relationship between z and $\bar{z}$.

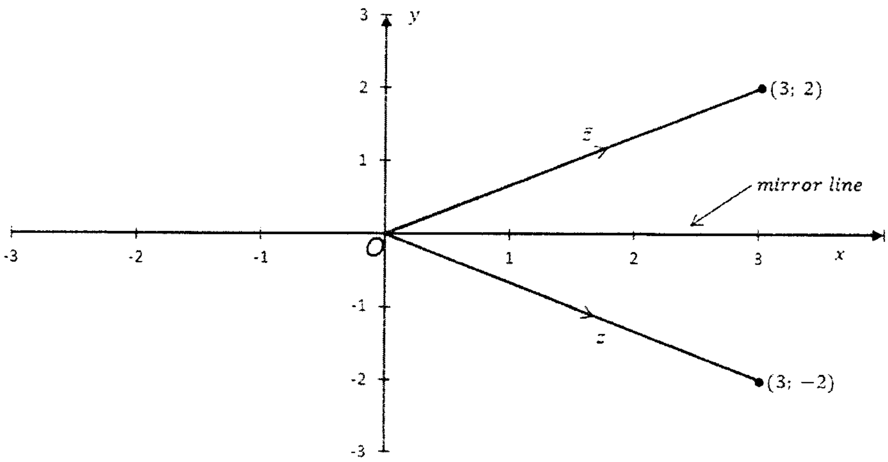


Fig. 15.5

2. Addition and Subtraction

Adding or subtracting two complex numbers leads to the formation of a parallelogram. Using the examples on page 309:

$z_1 = 3 - 2i$ *and* $z_2 = -2 - i$

Where $z_1 + z_2 = 1 - 3i$ *and* $z_1 - z_2 = 5 - i$

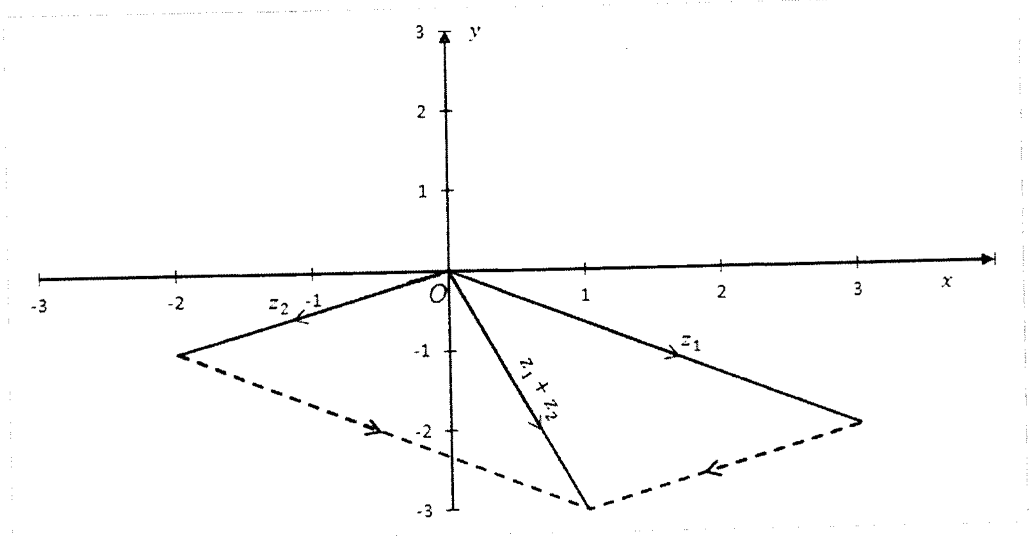


Fig. 15.6

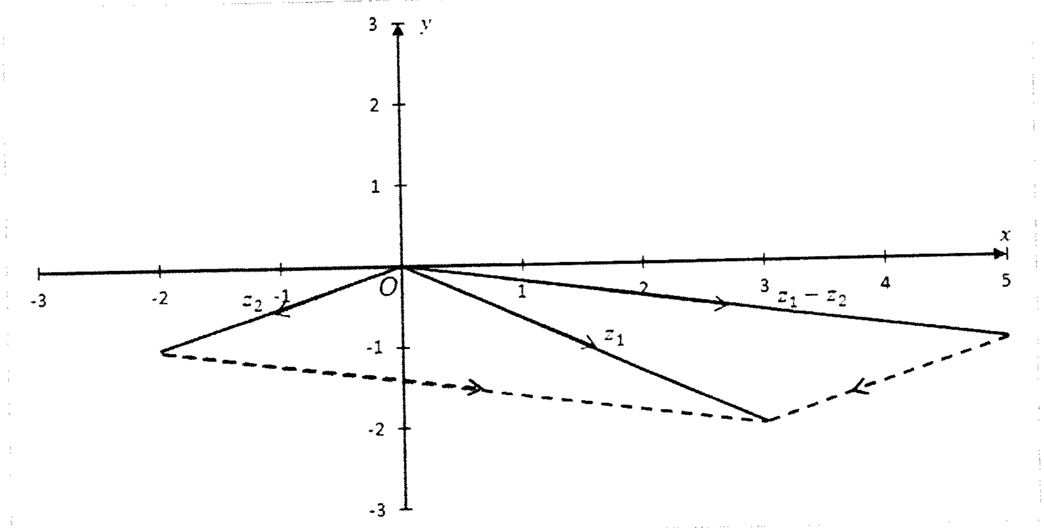


Fig. 15.7

Worked Examination Questions on Complex Numbers

Question (Cambridge, June 2012 qp.32)

7 **Throughout this question the use of a calculator is not permitted.**

The complex number u is defined by

$$u = \frac{1+2\text{i}}{1-3\text{i}}.$$

(i) Express u in the form $x + \text{i}y$, where x and y are real. [3]

(ii) Show on a sketch of an Argand diagram the points A, B and C representing the complex numbers u, $1 + 2\text{i}$ and $1 - 3\text{i}$ respectively. [2]

(iii) By considering the arguments of $1 + 2\text{i}$ and $1 - 3\text{i}$, show that

$$\tan^{-1} 2 + \tan^{-1} 3 = \tfrac{3}{4}\pi.$$ [3]

Solution

(i) $u = \dfrac{1+2i}{1-3i}$

$$\Rightarrow u = \frac{(1+2i)}{(1-3i)}\frac{(1+3i)}{(1+3i)}$$

$$\Rightarrow u = \frac{1+3i+2i+6i^2}{1-9i^2}$$

$$\Rightarrow u = \frac{1-6+5i}{1-9(-1)}$$

$$\Rightarrow u = \frac{-5+5i}{10}$$

$$\therefore \boldsymbol{u = -\frac{1}{2} + \frac{1}{2}i}$$

(ii)

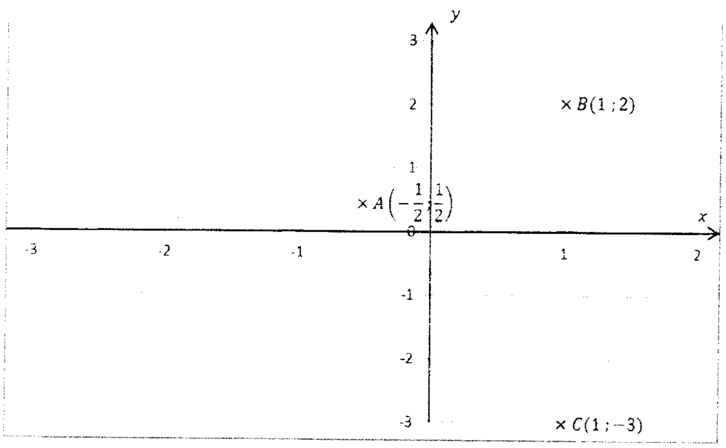


Fig. 15.8

(iii) *The general rule states that:*

$$\arg\left(\frac{B}{C}\right) = \arg(B) - \arg(C)$$

In this case, $\arg\left(\frac{1+2i}{1-3i}\right) = \arg(1+2i) - \arg(1-3i)$

where $\arg\left(\frac{1+2i}{1-3i}\right) = \arg\left(-\frac{1}{2}+\frac{1}{2}i\right)$

$\Rightarrow \arg\left(\frac{1+2i}{1-3i}\right) = \tan^{-1}\left(\frac{1/2}{1/2}\right)$

$\Rightarrow \arg\left(\frac{1+2i}{1-3i}\right) = \frac{\pi}{4}$

but $\arg\left(\frac{1+2i}{1-3i}\right) = \pi - \frac{\pi}{4}$

$\Rightarrow \arg\left(\frac{1+2i}{1-3i}\right) = \frac{3\pi}{4}$ *because of its location,*

and $\arg(1+2i) = \tan^{-1}\left(\frac{2}{1}\right)$

$\Rightarrow \arg(1+2i) = \tan^{-1}(2)$

and $\arg(1-3i) = \tan^{-1}\left(\frac{3}{1}\right)$

$\Rightarrow \arg(1-3i) = -\tan^{-1}(3)$ *because of its location*

$\frac{3\pi}{4} = \tan^{-1}(2) - -\tan^{-1}(3)$

$\therefore \mathbf{\tan^{-1}(2) + \tan^{-1}(3) = \frac{3\pi}{4}}$

Question (Cambridge, November 2007 qp.3)

8 **(a)** The complex number z is given by $z = \frac{4-3i}{1-2i}$.

(i) Express z in the form $x + iy$, where x and y are real. [2]

(ii) Find the modulus and argument of z. [2]

Solution

(a) (i) $z = \dfrac{4-3i}{1-2i}$

$z = \dfrac{(4-3i)}{(1-2i)}\dfrac{(1+2i)}{(1+2i)}$

$\Rightarrow z = \dfrac{4+8i-3i-6i^2}{1-4i^2}$

$\Rightarrow z = \dfrac{4+6+5i}{1-4(-1)}$

$\Rightarrow z = \dfrac{10+5i}{5}$

$\therefore \boldsymbol{z = 2+i}$

(ii) $|z| = \sqrt{(2)^2+(1)^2}$

$\therefore |\boldsymbol{z}| = \sqrt{\mathbf{5}}\ \boldsymbol{units}$

$and\ \arg(z) = \tan^{-1}\left(\dfrac{1}{2}\right)$

$\therefore \mathbf{arg}\,\boldsymbol{z} = \mathbf{0.464}\ \boldsymbol{rad\ or}\ \mathbf{26.6}^{\circ}$

Revision Questions on Complex Numbers

November 2006 qp.1 (Zimsec)

9. (a) The complex number $z = x+iy$ satisfies the equation $\frac{z}{z+2} = 2-i$
Find the value of x and the value of y. [4]

November 2007 qp.1 (Zimsec)

4. The complex number $\frac{3+2i}{2+ai}$ can be expressed in the form $x+iy$, where x and y are real.
Find the value of a given that $x = y$. [5]

June 2010 qp.1 (Zimsec)

10. The complex number $z_1 = 1 - 2i$ and the complex number z_2 is such that

$$z_1 z_2 = -10i$$

Find z_2 in the form $a + ib$ and sketch it on an Argand diagram. [5]

November 2004 qp.1 (Zimsec)

2. Given that $z = 4 - 2i$, find

(i). $|z|$ and arg z, [2]

(ii). $\frac{z}{\bar{z}}$ in the form $a + bi$, where $\bar{z}$ represents the conjugate of z, and a and b are real numbers. [2]

November 2010 qp.1 (Zimsec)

3. Express $z = \frac{1+i}{3+4i}$ in form $a + bi$, where a and b are real. [3]

Hence or otherwise find $|z|$ in the form $c\sqrt{d}$ where d is a prime number. [2]

June 2007 qp.3 (Cambridge)

8 The complex number $\dfrac{2}{-1+i}$ is denoted by u.

(i) Find the modulus and argument of u and u^2. [6]

November 2009 qp.32 (Cambridge)

7 The complex numbers $-2 + i$ and $3 + i$ are denoted by u and v respectively.

(i) Find, in the form $x + iy$, the complex numbers

(a) $u + v$, [1]

(b) $\frac{u}{v}$, showing all your working. [3]

(ii) State the argument of $\frac{u}{v}$. [1]

In an Argand diagram with origin O, the points A, B and C represent the complex numbers u, v and $u + v$ respectively.

(iii) Prove that angle $AOB = \frac{3}{4}\pi$. [2]

(iv) State fully the geometrical relationship between the line segments OA and BC. [2]

June 2010 qp.31 (Cambridge)

7 The complex number $2 + 2i$ is denoted by u.

(i) Find the modulus and argument of u. [2]

November 2010 qp.31 (Cambridge)

6 The complex number z is given by

$$z = (\sqrt{3}) + i.$$

(i) Find the modulus and argument of z. [2]

(ii) The complex conjugate of z is denoted by z^*. Showing your working, express in the form $x + iy$, where x and y are real.

(a) $2z + z^*$.

(b) $\dfrac{iz^*}{z}$. [4]

(iii) On a sketch of an Argand diagram with origin O, show the points A and B representing the complex numbers z and iz^* respectively. Prove that angle $AOB = \frac{1}{6}\pi$. [3]

November 2010 qp.33 (Cambridge)

3 The complex number w is defined by $w = 2 + i$.

(i) Showing your working, express w^2 in the form $x + iy$, where x and y are real. Find the modulus of w^2. [3]

June 2011 qp.31 (Cambridge)

8 The complex number u is defined by $u = \dfrac{6 - 3i}{1 + 2i}$.

(i) Showing all your working, find the modulus of u and show that the argument of u is $-\frac{1}{2}\pi$. [4]

June 2011 qp.32 (Cambridge)

7 **(a)** The complex number u is defined by $u = \dfrac{5}{a + 2i}$, where the constant a is real.

(i) Express u in the form $x + iy$, where x and y are real. [2]

(ii) Find the value of a for which $\arg(u^*) = \frac{3}{4}\pi$, where u^* denotes the complex conjugate of u. [3]

June 2012 qp.31 (Cambridge)

4 The complex number u is defined by $u = \dfrac{(1 + 2i)^2}{2 + i}$.

(i) Without using a calculator and showing your working, express u in the form $x + iy$, where x and y are real. [4]

Answers to Revision Questions

Chapter One: Quadratics

Revision Exercise on completing the square

1. (i) $16-(x-4)^2$

   (ii) $(4,\ 16)$

2. $2(x-2)^2+3$
3. $2(x-1)^2-1;\ A(1,\ -1)$
4. $2(x+2)^2-18$
5. $(x-2)^2-4+k$

Revision Questions on Quadratics

- ***Unknown Source***
  $k<-4\ and\ k>4$
- ***N02/P1/Q1(Z)***
  (a) $k\le\frac{3}{2}\ and\ k\ge\frac{3}{2}$
- ***N00/P3/Q1(C)***
  (i) $x=-\frac{2}{5}\ and\ -2$
  (ii) $k>-\frac{1}{3}$
- ***N02/P1/Q8(ZOAM)***
  (a) $k\le\frac{1}{2}\ and\ k\ge\frac{3}{2}$
  (b) $c>-4$
- ***N07/P1/Q1(C)***
  $k<-4$
- ***N09/P12/Q10(C)***
  (ii) $k<\frac{31}{8}$
- ***J11/P13/Q2(C)***
  $m>2\ and\ m<-10$
- ***J12/P13/Q10(C)***
  (ii) $k<-\sqrt{48}\ and\ k>\sqrt{48}$

Chapter Two: Polynomials

- ***N03/P2/Q1(ZOAM)***
  (a) $a=-32$
  (b) $x=1\ or\ 2\ or\ -\frac{1}{2}$
  (c) $p=-1\ and\ q=-9$
- ***N06/P1/Q1(Z)***
  $q=-57\ and\ p=25$
- ***J07/P3/Q2(C)***
  (i) $a=4$
  (ii) x^2-2x+2
- ***N07/P1/Q13(ZOAM)***
  (a) (ii) x^2+2x+2
  (b) $x<-3\ and\ x>2$
- ***N07/P3/Q2(C)***
  $a=4;\ \ x^2-x+2$
- ***N07/P1/Q1(Z)***
  $a=2;\ b=0;\ c=5$
- ***J11/P31/Q4(C)***
  (i) $(x+2)(4x+3)(3x-2)$
  (ii) $3^y=\frac{2}{3};\ -0.369$
- ***J11/P33/Q5(C)***
  (i) $a=2\ and\ b=-3$
  (ii) x^2-x+2
- ***N11/P31/Q3(C)***
  (i) $a=1$
  (ii) $x=-3\ or\ -1$
- ***N11/P33/Q7(C)***
  (i) $a=2;\ (2x-1)(x^2+2)$
  (ii) $\frac{-4}{(2x-1)}+\frac{2x+5}{(x^2+2)}$
- ***J12/P31/Q3***
  (i) $a=4$ (ii) (a) $(x-2)^2(x+4)$

Chapter Three: Analytical Geometry

- ***N01/P1/Q1(Z)***
  $y = -2x + 8$
- ***N01/P1/Q2(Z)***
  $x^2 + y^2 + 2x - 4y - 31 = 0$
- ***N02/P1/Q9(C)***
  (i) $y = \frac{1}{2}x + \frac{11}{2}$
  (ii) $(13,\ 12)$
  (iii) $16\sqrt{5}$
- ***N02/P1/Q1(ZOAM)***
  $x = \frac{1}{2};\ y = \frac{3}{2}$
  $x = -2;\ \ y = -1$
- ***N02/P1/Q3(ZOAM)***
  $h = 15$
- ***J03/P1/Q7(C)***
  (i) $y = \frac{1}{2}x + \frac{1}{2}$
  (ii) $\sqrt{20}$
- ***N03/P1/Q14(Z)***
  (i) $y = 2x + 15$
  (ii) $x^2 + y^2 + 4x - 22y + 25 = 0$
- ***N03/P1/Q5(C)***
  (I) BC: $y = \frac{1}{2}x + 4$
  CD: $y = -2x + 29$
  (ii) $(10,\ 9)$
- ***N04/P1/Q12(Z)***
  $\left(\frac{3\sqrt{2}}{2};\ \frac{4 + 3\sqrt{2}}{2}\right)$ *and* $\left(\frac{-3\sqrt{2}}{2};\ \frac{4 - 3\sqrt{2}}{2}\right)$
- ***J05/P1/Q5(C)***
  $M\ (4,\ 6);\ A\ (-8,\ 0);\ C(16,\ 12)$
- ***N07/P1/Q2(ZOAM)***
  $x = 1;\ \ y = 4$
  $x = -4;\ \ y = -1$
- ***J08/P1/Q11(C)***
  (i) $(4,\ 6)$
  (ii) $(6,\ 10)$
  (iii) 40.9
- ***N09/P12/Q9(C)***
  (i) *correct explanation*
  (ii) $AD = \frac{8}{h}$ *or* $\frac{h - 12}{8}$
  $CD = \frac{8}{12 - h}$ *or* $\frac{-h}{8}$
  (iii) $x_D = 16;\ \ x_B = -4$
  (iv) 160
- ***N10/P1/Q10(Z)***
  (i) $(x - 4)^2 + (y + 3)^2 = 25$
  or $x^2 + y^2 - 8x + 6y = 0$
  (ii) *The distance from* $(x,\ y)$
  to any point on the circumference is less than or equal to 5
  (iii) *see diagram*
  (iv) $1 < x < \frac{27}{5}$
- ***N11/P12/Q9(C)***
  (i) $M\ (5,\ 2);\ \ D(7,\ \ -2)$
  (ii) $3:2$
- ***J12/P11/Q5(C)***
  (i) $\frac{4}{9}$ *and* $\frac{1}{4}$
  (ii) $k = \frac{49}{24}$
- ***N12/P13/Q10(C)***
  (ii) $k = 2$ *or* 10
  (iii) $(2,\ \ -2);\ \ (4,\ 2);$
  $y = 2x - 6$

Chapter Four: Logarithmic and Exponential Functions

Questions on Exponential Functions

- ***J03/P1/Q1(Z)***
  $x = \left(\frac{-3 + \ln 2}{2}\right)$
- ***J04/P3/Q4(C)***
  (i) $y^2 - y - 1 = 0$
  (ii) $x = 0.694$
- ***J06/P3/Q1(C)***
  $y = \frac{\ln 4 - \ln x}{\ln 3}$
- ***N11/P31/Q1(C)***
  1.10
- ***N07/P1/Q1 (ZOAM)***
  (b) $x = \ln\sqrt{3}$
- ***J07/P3/Q4(C)***
  0.802
- ***J10/P32/Q1(C)***
  0.585
- ***N09/P31/Q2(C)***
  0.107
- ***N12/P31/Q2(C)***
  1.14

Questions on Natural Logarithms

- ***N08/P3/Q1(C)***
  0.313
- ***N09/P32/Q1(C)***
  1.38 *and* 3.62
- ***J12/P32/Q1(C)***
  2.30
- ***J12/P33/Q2(C)***
  1.39
- ***N12/P33/Q1(C)***
  $\frac{5}{e-1}$

Questions on Linear Law

- ***N03/P2/Q3(ZOAM)***
  (a) *graph see diagram*
  $p = \frac{4}{5}$ *and* $q = -\frac{12}{25}$
  (b) $y = e^{\frac{1}{2}x}$
  (c) $a = -6$ *and* $b = 3$
- ***N05/P3/Q2(C)***
  $1.97 \leq A \leq 2.03$; $n = 0.25$
- ***J10/P31/Q3***
  (i) $n = 1.50$;
  $C = 6.00$
  (ii) $\ln y = -n \ln x + \ln C$
  $(linear; y = mx + c)$
- ***J10/P1/Q5(Z)***
  (i) $x - axis : x$
  $y - axis : \ln y$
  (ii) $a = e^{1.1}$ *and* $b = e^{1.6}$

Questions on Common Logarithms

- ***N02/P3/Q3(C)***
  (ii) $\frac{\sqrt{425} - 5}{2}$
- ***N07/P1/Q1(ZOAM)***
  $\log 3$
- ***N10/P1/Q1(Z)***
  $b = a^2$

Chapter Five: Modulus Functions and Inequalities

Question on Simple Interpretation

- ***N06/P3/Q1(C)***
  $1.83 < x < 1.95$

Question on Graphical method

- ***J06/P3/Q2(C)***
  $x > \frac{1}{3}$
- ***N07/P1/Q3(Z)***
  graph see diagrams
  $x > 2/3$

Questions on Quadratic Inequality Approach

- ***J08/P3/Q1(C)***
  $-1 < x < -\frac{1}{7}$
- ***J10/P1/Q3(Z)***
  $x \leq -5 \;\; and \;\; x \geq \frac{3}{5}$
- ***J10/P33/Q1(C)***
  $-5 < x < \frac{1}{3}$
- ***J11/P32/Q1(C)***
  $x < -5; \; x > -\frac{5}{3}$
- ***N10/P31/Q1(C)***
  $-7 < x < 1$

Chapter Six: Vectors

- ***N12/P12/Q7(C)***
  (i) $24.1°$

  (ii) $1 \; or \; \frac{2}{3}$
- ***N12/P13/Q9(C)***
  (i) $p = 2; \frac{1}{\sqrt{6}}\begin{pmatrix}2\\1\\1\end{pmatrix}$

  (ii) $p = 0 \;\; or \;\; 5$

  (iii) $\begin{pmatrix}1\\1\\2\end{pmatrix}$
- ***N11/P11/Q8(C)***
  (i) $25 + p^2$

  (iii) $p = \pm\sqrt{15}$
- ***N10/P11/Q5(C)***
  $48.0°$
- ***N06/P1/Q12(Z)***
  (i) $\overrightarrow{AB} = -4i + j - 5k;$

  $\overrightarrow{AC} = 8i - 2j + 10k$

  $|\overrightarrow{AB}| = \sqrt{42}$

  (ii) $\overrightarrow{AB}$ *and* $\overrightarrow{AC}$ *are opposite vectors and* $\overrightarrow{AC}$ *is twice* $\overrightarrow{AB}$
  sketch see diagrams
- ***J03/P1/Q12(Z)***
  (i) $\overrightarrow{OM} = \begin{pmatrix}3/2\\2\\3/2\end{pmatrix}; \; \overrightarrow{CM} = \begin{pmatrix}-5/2\\2\\3/2\end{pmatrix}$

  (ii) $O\widehat{M}C = 76.0°; \;\; Area = 5$

- ***N10/P1/Q5(Z)***

$\overrightarrow{OP} = \begin{pmatrix} -8/5 \\ 2 \\ 6/5 \end{pmatrix}$

- ***N07/P1/Q8(Z)***

(a)(i) $\sqrt{33}$

(ii) $\frac{1}{2}i + k$

(b) 0.77

- ***J01/P1/Q7(C)***

$\overrightarrow{OD} = i - j$

(i) $\overrightarrow{AV} = -i - j + (\sqrt{6})k$

$\overrightarrow{CV} = i + j + (\sqrt{6})k$

Advert One: The Concept of Partial Fractions

Linear-Factor Approach

1. $-\frac{1}{(x-1)} + \frac{4}{(x-2)} - \frac{2}{(x+1)}$

2. $\frac{1}{(x+1)} + \frac{2}{(x-2)}$

3. $\frac{1}{4y} + \frac{1}{4(4-y)}$

Quadratic-Factor Approach

4. $\frac{2}{(2-x)} + \frac{2x+4}{(1+x^2)}$

5. $\frac{4}{(2-x)} + \frac{4x+1}{(1+x^2)}$

6. $-\frac{16}{19(x+4)} + \frac{16x+12}{19(x^2+3)}$

7. $\frac{2}{(x+2)} + \frac{x-1}{(x^2+1)}$

Repeated-Factor Approach

8. $-\frac{3}{(3x+1)} + \frac{1}{(x+1)} + \frac{2}{(x+1)^2}$

9. $\frac{1}{(2x+1)} + \frac{4}{(x-2)} + \frac{8}{(x-2)^2}$

10. $\frac{2}{(2x+1)} - \frac{1}{(x+1)} + \frac{3}{(x+1)^2}$

Improper-Fractions Approach

11. $1 - \frac{1}{(x-1)} + \frac{2x}{(x^2+1)}$

12. $x - 3 + \frac{4}{(x+2)} - \frac{3}{(x-1)}$

13. $2 + \frac{1}{2x+1} - \frac{3}{x+2}$

Chapter Seven: Sequences and Series

Questions on Arithmetic and Geometric Progressions

- ***N03/P2/Q2(ZOAM)***
  (a) (i) $a = 178;\ d = -4$
  (ii) $n = 90$
  (b) (i) $a = 100$
  (ii) 1.44
  (iii) $n = 28$
- ***N03/P1/Q12(Z)***
  (a) $U_1 = 3\ or\ 9$

  $oscillating\ for\ both\ values\ of U_1$
- ***N02/P1/Q8(Z)***
  (i) $(n-3);\ (n-6);\ (n-9)$
  (ii) $-\frac{7}{2}n(n+1)$
- ***N01/P1/Q16(Z)***
  (a)(i) $oscillating$
  (ii) $converging$
  (iii) $diverging$
  (b)(i) 2; 5; 8; 11
  (ii) $Arithmetic\ progression$
  $U_n = 3n - 1$
  (c) $r = \frac{1}{3}$ and $S_\infty = 27$
- ***J08/P1/Q7(C)***
  (i) $\frac{2}{3}$
  (ii) 243
  (iii) 270
- ***J11/P13/Q6(C)***
  (a) 45
- ***N05/P1/Q6(C)***
  (i) 369 000
  (ii) 3 140 000
  (iii) 14 300
- ***N07/P1/Q4(C)***
  (i) $a + 4d;\ a + 14d$
  (ii) 2.5

Questions on Series Expansion

- ***N10/P33/Q1(C)***
  $1 - 6x + 24x^2$
- ***J11/P31/Q1(C)***
  $1 - 2x - 4x^2 - \frac{40}{3}x^3$
- ***J12 /P31/Q2(C)***
  (i) $1 + 2x + 6x^2$
  (ii) 5
- ***J12/P33/Q1(C)***
  $\frac{1}{2} - \frac{3}{16}x + \frac{27}{256}x^2$
- ***N12/P31/Q4(C)***
  (i) $\frac{1}{\sqrt{2}}$
  (ii) $1 - x\sqrt{2} + \frac{3}{2}x^2$
- ***N09/P32/Q8(C)***
  (i) $\frac{\left(\frac{2}{3}\right)}{1-x} + \frac{\frac{2}{3}x + \frac{1}{3}}{2 + x^2}$
  (ii) $\frac{1}{2} + x + \frac{3}{4}x^2$
- ***J10/P33/Q9(C)***
  (i) $\frac{1}{1-2x} + \frac{1}{2+x} - \frac{2}{(2+x)^2}$
  (ii) $1 + \frac{9}{4}x + \frac{15}{4}x^2$
- ***N10/P31/Q8(C)***
  (i) $\frac{-1}{1+x} + \frac{2x+1}{1+2x^2}$
  (ii) $3x - 3x^2 - 3x^3$

- ***J11/P32/Q8(C)***
  (i) $\frac{-2}{1+x}+\frac{x+4}{2+x^2}$
  (ii) $\frac{5}{2}x-3x^2+\frac{7}{4}x^3$
- ***N07/P1/Q3(C)***
  (i) $32+80u+80u^2$
  (ii) 160
- ***J12/P12/Q3(C)***
  5

Chapter Eight: Trigonometry

Equations of the form $R\ \cos(\theta \pm \alpha)$ / $R \sin(\theta \pm \alpha)$

- ***N99/P1/Q6 (C)***
  $\tan\alpha = \frac{1}{2}$ *and* $R = \sqrt{20}$
  $\sqrt{20}\ \sin\left[\theta + \tan^{-1}\left(\frac{1}{2}\right)\right]$
  $\theta = 111.3°;\ 15.6°$
- ***J94/P1/Q6(C)***
  $\sqrt{34}\cos(\theta + 59.0°)$;
  $\theta = \pm 69.9° + 360°n - 59.0°$
- ***J97/P1/Q10(C)***
  $5\cos(x + 53.13°)$
  $\theta = 6.6°$; $120.2°$; $186.6°$; $300.2°$
- ***N03/P2/Q4(ZOAM)***
  (a) $\alpha = 18.43°$ *and* $R = \sqrt{10}$
  $\theta = 69.2°;\ 327.7°$
  (b)(i) $-\frac{56}{25}$
  (ii) $-{}^{7}/_{25}$
  (iii) $-{}^{120}/_{169}$
  (iv) $-\frac{24}{7}$
- ***N90/P1/Q5(C)***
  $13\cos(\theta + 67.4°)$
- ***N10/P33/Q8(C)***
  (i) $4\cos(\theta - 52.24°)$
  (ii) (a) 232.2
  (b) 21.7
- ***N11/P31/Q6(C)***
  (i) $\sqrt{10}\cos(x - 71.57°)$
  (ii) $61.2°;\ 10.4°$
- ***N11/P33/Q3(C)***
  (i) $17\cos(\theta - 61.93°)$
  (ii) $107.0°;\ 16.8°$
- ***N12/P33/Q2(C)***
  (i) $25\sin(\theta - 16.26°)$
  (ii) $59.1°$

More Trig Equations

- ***J03/P1/Q14(Z)***
  (ii) $x = 1.79;\ 2.92;\ 4.94;\ 6.06$
- ***J10/P1/Q1(Z)***
  $-\frac{\sqrt{6}}{12}$
- ***J97/P1/Q1(C)***
  $x = 29.0°;\ 331.0°$
- ***N02/P1/Q14(ZOAM)***
  (a)(i) $x = 63.4°;\ 153.4°;\ 243.4°;\ 333.4°$
  (Ii) $x = 60°;\ 109.5°;\ 250.5°;\ 300°$
  (b) $\sin x = -\frac{1}{2}$
  $\cos x = -\frac{\sqrt{3}}{2}$

- ***J91/P1/Q6(C)***
  $\theta = 26.4°; \ 333.6°$
- ***Unknown Source/Q7***
  (ii) $\theta = 0.902$
- ***N06/P1/Q3(Z)***
  $2 \operatorname{cosec} 2\theta; \quad \theta = 30°; \ 60°$
- ***N07/P3/Q5(C)***
  (ii) 22.5°; 112.5°
- ***N09/P32/Q4(C)***
  $\alpha = 45°, \ \beta = 26.6°;$
  $\alpha = 116.6°, \ \beta = 135°$
- ***J10/P31/Q2(C)***
  48.6°; 131.4°; 270°
- ***J10/P33/Q3(C)***
  15.7°; 119.3°
- ***N10/P31/Q3(C)***
  9.9°; 189.9°
- ***J11/P32/Q3(C)***
  29.0°; 180°
- ***J11/P33/Q4(C)***
  (ii) 16.8°; 163.2°
- ***J12/P32/Q4(C)***
  201.5°; 338.5°
- ***J12/P33/Q6(C)***
  (ii) 16.8°; 163.2°
- ***N12/P31/Q3(C)***
  105.9°

Chapter Nine: Circular Measure

- ***N02/P1/Q12(ZOAM)***
  (a) $\theta = \frac{1}{6}\, rad$
  (b) 12.6 m
  (c) $1.97a^2$
- ***J03/P1/Q9(Z)***
  $x = 8$
- ***J91/P1/Q5(C)***
  $\frac{2\pi}{3}\, rad \ or \ 120°$
  $\left(\frac{\pi}{3} - \frac{\sqrt{3}}{4}\right)$
- ***N04/P1/Q13(Z)***
  (a)(i) $12\theta - 4\sin 3\theta$
  (ii) $4\sin 3\theta$
- ***J10/P1/Q13(Z)***
  (ii) 6.29
  (iii) 85.6
- ***N96/P1/Q6(C)***
  6.92
- ***N95/P1/Q2(C)***
  (ii) $\frac{2\sqrt{3}\pi}{9}$

Chapter Ten: Differentiation

Question on Differentiation and Mensuration

- ***N05/P1/Q5(C)***
  (i) $h = 12 - 2r$
  (ii) 64π
- ***J10/P12/Q8***
  (i) $h = \frac{24}{x} - \frac{x}{2}$
  (ii) 64
  (iii) *maximum*

- ***N10/P11/Q8(C)***
  (i) $y = 30 - x - \frac{\pi x}{4}$
  (iii) 15
  (iv) *maximum*
- ***N10/P12/Q10***
  (i) $h = \frac{8}{x^2}$
  (ii) 2
- ***N11/P11/Q7(C)***
  (i) $y = \frac{1}{6(48 - 8x)}$
  (iii) 72
- ***N12/P12/Q3(C)***
  (ii) 975

Question on Implicit Differentiation

- ***N96/P1/Q10(C)***
  2
- ***J01/P1/Q8(C)***
  (ii) $(1,\ -2);\ (-1,\ 2)$
- ***J08/P3/Q6(C)***
  $(a;\ -2a)$
- ***N09/P32/Q3(C)***
  (i) $\frac{3x^2 - 2xy}{x^2 + 3y^2}$
  (ii) $8x - 7y - 9 = 0$
- ***J10/P32/Q6(C)***
  (ii) $4x + y + 1 = 0$
- ***J11/P31/Q5(C)***
  (ii) $k = 5;\ c = 68$
- ***J12/P31/Q6(C)***
  (i) $\frac{12}{7}$

Questions on Parametric Differentiation

- ***J10/P1/Q9(Z)***
  -2
  straight line
- ***N01/P1/Q8(Z)***
  (ii)(a) $(4,\ 0)$
  (b) $(0,\ 4)$
- ***J94/P1/Q7(C)***
  $\frac{1 - e^{-t}}{1 + e^{t}}$
  $(1,\ 1)$
- ***J09/P3/Q6(C)***
  (i) $\frac{dy}{dx} = -\tan t$
- ***J11/P32/Q5(C)***
  (i) $\frac{dy}{dx} = 2\sin^2 t \cos^2 t\ oe$
  (ii) $y = \frac{1}{2}x + \frac{1}{2}$
- ***N11/P31/Q2(C)***
  $\frac{dy}{dx} = -\cos t$
- ***N11/P33/Q8(C)***
  (ii) $\frac{3}{2}$
  (iii) $1.9\ and\ 2.8$
- ***N12/P33/Q3(C)***
  (i) $\frac{dy}{dx} = \frac{1}{3}(2t + 3)$
  (ii) 2

Question on Differentiation of Products

- ***J10/P1/Q2(Z)***
  (i) $e^{-2t}(\cos t - 2\sin t)$
  (ii) $6\sec^2(3t - 100)\tan(3t - 100)$
- ***N90/P1/Q9(C)***
  (i) $3u(2\sin 5u + 5u\cos 5u)$
- ***J07/P3/Q3(C)***
  $y = x$
- ***N07/P3/Q4(C)***
  (i) $\frac{1}{4}\pi\ or\ 0.785$
  (ii) *maximum*

Questions on Differentiation of Fractions

- ***N90/P1/Q9(C)***

  (ii) $\dfrac{3-3t^2}{(t^2+1)^2}$
- ***N95/P1/Q6(C)***

  (c) $x = 0 \; or \; x = 1$

  $when \; x = 0, \; maximum \; point$

  $when \; x = 1, \; minimum \; point$
- ***N08/P3/Q3(C)***

  $-\frac{1}{4}\pi \; or -0.785$
- ***J11/P31/Q2(C)***

  (i) $\dfrac{2\cos 2x}{1+\sin 2x}$

  (ii) $\dfrac{x\sec^2 x - \tan x}{x^2}$
- ***J11/P33/Q2(C)***

  $e^{\left(\frac{1}{3}\right)} \; or \; 1.40$

Questions on Maclaurin Series

- ***J10/P1/Q8(Z)***

  $y = e^{\left(x^2/2\right)}$

  $1 + \dfrac{x^2}{2}$
- ***J01/P1/Q15(Z)***

  (i) $e^x[\cos(x\sqrt{3}) - \sqrt{3}\sin(x\sqrt{3})]$

  (iv) $1 + x - x^2 - \frac{4}{3}x^3$
- ***J03/P1/Q18(Z)***

  (a) $2x - 6x^2 + \frac{23}{3}x^3$
- ***N01/P1/Q11(Z)***

  $\dfrac{dy}{dx} = e^{-x}(\cos x - \sin x)$

  $x - x^2 + \frac{1}{3}x^3$
- ***N96/P1/Q19(Z)***

  (b) $1 + 6x + 18x^2 + 72x^3$

Questions on Simple and Complex Algebraic Expressions

- ***N92/P1/Q9(C)***

  $(2, \; -3); \; miniumum$

  $\left(-\frac{2}{3}, \; \frac{175}{27}\right); \; maximum$
- ***N02/P1/Q18(Z)***

  (i) $3x^2 - 2x - 5$

  (ii) $y = 3x - 7$

  (iii) $\left(\frac{5}{3}, \; -\frac{40}{27}\right); \; minimum$

  $(-1, \; 8); \; maximum$
- ***N06/P1/Q8(Z)***

  (i) $tangent: y = -8x - 10$

  $normal: y = \frac{1}{8}x + \frac{25}{4}$
- ***N02/P1/Q2(ZOAM)***

  (i) $36x^2 - 60$

  (ii) $y = -\frac{1}{84}x + \frac{85}{42}$
- ***J06/P1/Q1(C)***

  12
- ***J08/P1/Q4(C)***

  (i) $2 \; or \; 1\frac{1}{2}$
- ***J10/P12/Q10(C)***

  (i) $(2x-3)^2 - 4$

  (ii) $y = 5x - \frac{9}{2}$

  (iii) $x > 2\frac{1}{2} \; and \; x < \frac{1}{2}$
- ***N08/P1/Q8(C)***

  (ii) $(-8, \; 6)$

  (iii) $5\sqrt{5}$
- ***N10/P13/Q5(C)***

  (i) $\dfrac{dy}{dx} = \dfrac{-1}{(x-3)^2} + 1;$

  $\dfrac{d^2y}{dx^2} = \dfrac{2}{(x-3)^3}$

  (ii) $A(2, \; 1); \; B = (4, \; 5)$

- ***N12/P11/Q5(C)***
  minimum
- ***N12/P13/Q11(C)***
  (i) $a = 2$
  (ii) $b = \frac{2}{3}$
  (iii) $\frac{4}{3}$
  (iv) $-\frac{4}{3}$

Questions on Rate of Change

- ***N06/P1/Q10(Z)***
  $h = \frac{200}{\pi r^2}$
  $\frac{dr}{dt} = -0.2011\ cm/minute$
  (i) $\frac{dS}{dr} = 4\pi r - \frac{400}{r^2}$
  (ii) $-5.08cm^2/minute$
  The surface area is decreasing
- ***J12/P12/Q2(C)***
  (i) $\frac{dy}{dx} = \frac{2}{x^{\frac{1}{2}}} - \frac{1}{x^{\frac{3}{2}}}$
  (ii) 0.105
- ***J12/P11/Q4(C)***
  $k = 0.0032$
  0.096
- ***J11/P11/Q2(C)***
  0.0398

Chapter Eleven: Integration

Questions on the Method of Substitution

- ***N01/P1/Q12(Z)***
  $2 + 4\ln\left(\frac{2}{3}\right)$
- ***N10/P1/Q8(Z)***
  (a) $\frac{2}{3}$
  (b) 45°; 75°; 165°; 195°
- ***J03/P1/Q7(Z)***
  $\frac{1}{3}(x^2-1)^{\frac{3}{2}} + (x^2-1)^{\frac{1}{2}} + c$
- ***J10/P1/Q12(Z)***
  $\frac{a^2\pi}{4}$
- ***N95/P1/Q7(C)***
  $\frac{-1}{4(2x+3)} + \frac{3}{8(2x+3)^2} + c$
- ***J02/P3/Q10(C)***
  (i) $x_A = 1$
  (iv) $e - 2$
- ***N09/P32/Q6(C)***
  (ii) $\frac{1}{8}(\pi + 2)$
- ***N10/P31/Q5(C)***
  (ii) $\frac{1}{3}\pi - \frac{\sqrt{3}}{2}$
- ***J11/P31/Q7(C)***
  (ii) $15\ln 5 - 4$
- ***J11/P33/8(C)***
  (i) 0.886 *radians*
  (ii) $\frac{2}{3}$
- ***N12/P33/Q7(C)***
  (i) $\frac{1}{24}$
  (ii) 10

Questions on Integration by Parts

- ***N92/P1/Q11(C)***

  (ii) $\frac{1}{4}(5e^4 - 1)$

- ***N95/P1/Q8(C)***

  $\frac{1}{4}(\sin 2x - 2x\cos 2x) + c$

- ***N10/P31/Q9(C)***

  (i) $\left(e^{\left(-\frac{1}{3}\right)}, \ -\frac{1}{3e}\right)$

  (ii) $4\ln 2 - \frac{15}{16}$

- ***J11/P32/Q10(C)***

  (ii) 2

  (iii) 1

- ***J12/P32/Q9(C)***

  (i) $y = x - 1$

  (ii) $\frac{1}{4}\pi(e^2 - 1)$

- ***N12/P33/Q5(C)***

  (i) $6e$

  (ii) $-\frac{3}{4}$

Questions on Partial Fractions

- ***N10/P1/Q14(Z)***

  (ii) $\frac{5}{(2x-1)} + \frac{-2x}{(x^2+4)}$

  (iii) 3.07

- ***N99/P1/Q13(C)***

  (i) $\frac{1}{x} - \frac{1}{(x+1)}$

- ***J02/P3/Q6(C)***

  (i) $\frac{-3}{(3x+1)} + \frac{1}{(x+1)} + \frac{2}{(x+1)^2}$

- ***J94/P1/Q9(C)***

  (i) $\frac{-1}{x} - \frac{1}{x^2} + \frac{1}{(x-1)}$

  $\ln\left(\frac{x-1}{x}\right) + \frac{1}{x} + c$

- ***J08/P3/Q7(C)***

  (i) $1 + \frac{\left(\frac{1}{2}\right)}{x+1} + \frac{\left(-\frac{3}{2}\right)}{x+3}$

- ***J10/P31/Q8(C)***

  (i) $\frac{1}{x+1} - \frac{1}{x+3}$

- ***J10/P32/10(C)***

  (i) $A = 1;\ B = 2;\ C = 1;$

  $D = -3$

- ***N11/P31/Q8(C)***

  (i) $\frac{3}{2-x} + \frac{4x}{4+x^2}$

Questions on Standard Integrals

- ***J04/P1/Q7(C)***

  (i) $y = 2x - 9$

- ***J01/P1/Q12(C)***

  $\frac{\pi}{64}\left[\frac{h^3}{3} + 16h^2 + 256h\right]$

- ***N04/P1/Q15(Z)***

  (i) $A(1,\ -1)$ *and* $B(4,\ -{}^1/_4)$

  (iii) $\frac{9\pi}{4}$

- ***J08/P1/Q9(C)***

  (ii) 7.2

- ***J07/P3/Q5(C)***

  (i) $2\cos\left(\theta - \frac{\pi}{3}\right)$

- ***N07/P3/Q1(C)***

  $\frac{1}{2}(e^2 + 1)$

- ***J10/P33/Q5(C)***

  (i) $\ln 2$

- ***J10/P33/Q7(C)***

  (ii) $\frac{2}{3} - \frac{3}{8}\sqrt{3}$

- ***J11/P31/Q9(C)***
  (ii) (a) $0.572; \quad -0.572$
  (b) $\frac{3}{32}\pi + \frac{1}{4}$
- ***J12/P31/Q5(C)***
  (ii) $8 + 2\ln\frac{1}{2}$

<u>Advert Two: Proportionality and Curve Sketching:</u>

<u>Questions on Proportionality</u>

- ***N97/P1/Q4(C)***
  10 700
- ***J03/P1/Q3(Z)***
  $x = \frac{kc^2}{z^2}$
  $where\ k\ and\ c\ are\ constants$
- ***N03/P1/Q10(ZSP)***
  260%
- ***J01/P1/Q3(C)***
  (i) $z = k\left(\frac{x^3}{y^2}\right)$
  (ii) 76.8%; $decrease$

<u>Questions on Curve Sketching</u>

- ***N95/P1/Q3***
  (i) $see\ diagram$
  (ii) $see\ diagram$
- ***N90/P1/Q3***
  (i) $see\ diagram$
  (ii) $see\ diagram$
- ***J97/P1/Q6***
  (i) $see\ diagram$
  (ii) $see\ diagram$
  (iii) $see\ diagram$
- ***N01/P1/Q13(Z)***

$A\ reduction\ by\ shrink\ factor\ 2\ parallel\ to\ the\ x\ axis$

$or\ a\ stretch\ by\ scale\ factor\ \frac{1}{2}\ parallel\ to\ the\ x\ axis$

$A\ translation\ by\ \frac{\pi}{6}\ units\ in\ the\ positive\ x\ direction$

- ***J03/P1/Q13(C)***
  (i) $see\ diagram$
  (ii) $k = \frac{6}{\pi}$
  (iii) $\left(-\frac{\pi}{2}; -3\right)$
- ***N04/P1/Q4(C)***
  (i) $see\ diagram$
  (ii) 2
- ***N09/P11/Q2(C)***
  $see\ diagram$
- ***N10/P13/Q4(C)***
  (i) $see\ diagram$
  (ii) $see\ diagram$; 3 $roots$;
  $y = \frac{\pi - x}{\pi}$
- ***N11/P11/Q3(C)***
  (i) $see\ diagram$
  (ii) 4 $roots$
  (iii) 20 $roots$

Chapter Twelve: Differential Equations

- ***N03/P1/Q13(Z)***

  $t = \frac{10}{k}(\sqrt{2} - 1)$

- ***N10/P1/Q16(Z)***

  (a) (ii) $h = 40 - e^{\left[\frac{10\pi \ln 40 - t}{10\pi}\right]}$

  $h_{max} = 40cm$

  approaches 40 as t increases

  (b) $t = 10\pi \ln\left(\frac{40}{h}\right)$

- ***J97/P1/Q14(C)***

  $N = e^{\left(\frac{t}{50} + \ln 3000\right)} + 5000$

  $t = 35\ days$

  $50\frac{dN}{dt} = N - 50F$

  $fishing\ rate\ is\ greater\ than\ birth\ rate$

- ***N03/P2/Q5(ZSP)***

  (ii) $x = \frac{ke^{kt}}{h(2 + e^{kt})}$

- ***N01/P1/Q17(Z)***

  (a)(i) $N = 40e^{\frac{t}{2}}$

  $see\ diagram$

  (b) $t = \frac{9}{5}\ln\left(\frac{9N}{400 - N}\right)$

- ***J07/P3/Q10(C)***

  (ii) $h = 9 - \left(4 - \frac{1}{15}t\right)^{\frac{3}{2}}$

  (iii) $h = 9m;\quad t = 60\ years$

  (iv) 19.1

- ***N07/P3/Q7(C)***

  (i) $\ln N = 50k\sin(0.02t) + \ln 125$

  (ii) 0.0100479

  (iii) $N = 125e^{0.502\sin 0.02t}$; 75.6

- ***J09/P3/Q8(C)***

  (i) $\frac{1}{x} + \frac{10}{x^2} + \frac{1}{(10 - x)}$

  $or\ \frac{x + 10}{x^2} + \frac{1}{(10 - x)}$

  (ii) $t = \ln\left(\frac{9x}{10 - x}\right) - \frac{10}{x} + 10$

- ***N09/P31/Q10(C)***

  (ii) $r = \frac{5}{(1 - 0.4t)}$

  (iii) $0 \le t < 2.5$

- ***N09/P32/Q9(C)***

  (i) $\ln\left(\frac{\theta - A}{3A}\right) = -kt$

  (iii) $\theta = \frac{7}{3}A$

- ***J10/P31/Q5(C)***

  $y^2 = 4(x^2 - 1)$

- ***J10/P32/Q7(C)***

  (i) $x = \tan^{-1}\left(\frac{1}{2} - \frac{1}{2}e^{-2t}\right)$

  (ii) $x\ approaches\ \tan^{-1}\left(\frac{1}{2}\right)$

  (iii) $correct\ explanation$

- ***J10/P33/Q4(C)***

  $x^2 = 4 - 3e^{-\frac{1}{2}t}$

- ***N10/P31/Q10(C)***

  (ii) $\ln\left(\frac{20}{20 - x}\right) = \frac{t}{20}$

  (iii) $x = 7.9$

  (iv) $x\ approaches\ 20$

- ***N10/P33/Q9(C)***

  (i) $\frac{dA}{dt} = k\sqrt{2A - 5}$

  (ii) $A = 63$

- ***J11/P31/Q10(C)***

  (i) $N = \frac{1800e^{\frac{1}{2}t}}{5 + e^{\frac{1}{2}t}}$

  (ii) $N\ approaches\ 1800$

- ***J11/P32/Q6(C)***

  (ii) $gradient$

  $= -4;\ sketch\ see\ diagram$

- ***J11/P33/Q9(C)***

  (ii) $x = \frac{20(e^{0.1t} - 1)}{2e^{0.1t} - 1}$

  (iii) $x\ approaches\ 10$

- ***N11/P33/Q4(C)***

  (i) $N = (40 - 30e^{-0.02t})^2$

  (ii) $N\ approaches\ 1600$

- ***J12/P32/Q5(C)***

  $y = \ln\left(\frac{2}{3 - e^{2x}}\right)$

- ***J12/P33/Q5(C)***

  (i) $y = 70e^{(e^{-3t}-1)}$

  (ii) $p\ approaches \frac{100}{e}$

- ***N12/P31/Q6(C)***

  $y = \frac{x^2 - 4}{4 + x^2}$

- ***N12/P33/Q4(C)***

  $y = \frac{1}{2}(x^2 + 4)$

Chapter Thirteen: Functions

- ***N96/P1/Q14(C)***

  (i) $fg(x) = (x^2 - 4)(x^2 - 6)$

  $x \pm 2\ \ or\ \ x = \pm\sqrt{6}$

  (ii) $gf(x)$

  $= [(x-2)(x-4)]^2$
  $- 2$

  (iii) $x = \frac{2}{3}\ or \frac{5}{2}$

  (iv) $x < \frac{2}{3}\ and\ 2 < x < 2.5$

- ***J97/P1/Q4(C)***

  $fg(x) = \frac{3}{x+4}$

  $(gf)^{-1}(x) = \frac{6-3x}{x-1}$

- ***N07/P1/Q14(ZOAM)***

  (b)(i) $f^{-1}(5) = 4\frac{1}{2}$

  (ii) $g^{-1}(4) = 5\frac{1}{2}$

  (iii) 4

- ***N10/P1/Q7(Z)***

  (a) *mirror line*

  (b) $A(3;0);\ \ B(-1\frac{1}{2};0)$

  (c) $C = (-3;\ \ -3)$

- ***N97/P1/Q11(C)***

  (i) $e^{-\frac{1}{2}x}\ approaches\ 0$

  (ii) $h = \ln\frac{1}{4}$

  $k = \ln\frac{1}{4} + 2$

  (iii) $range: f^{-1}(x) \geq \ln\frac{1}{4}$

  $domain: x \geq \ln\frac{1}{4} + 2$

  see diagram

- ***J94/P1/Q13(C)***

  (a)(i) *see diagram*

  (ii) $g^{-1}(x) = e^x$

  $h^{-1}(x) = x - 1$

  (iii) $g^{-1}h^{-1}(x) = e^{x-1}$

  (iv) *see diagram*

  (b) *see diagram*

  $q^{-1}(x) = 2 - \sqrt{x+4}$

- ***N95/P1/Q4(C)***

  (i) $(2 - 3x)^3$

  (ii) $\frac{2 - \sqrt[3]{x}}{3}$

- ***N99/P1/Q10(C)***

  (i) *strech in y − axis by scale factor e^a or translation in the negative x direction by **a** units*

  (ii) $b = e^{-a}$

Chapter Fourteen: Numerical Solutions of Equations

Questions on Trapezium Rule

- ***N97/P1/Q9(C)***
  (ii) 1.15
  (iii) *The estimate is greater than the true value because it includes the region outside R.*
- ***J10/P1/Q18(Z)***
  0.00800
- ***J91/P1/Q12(C)***
  1.3
- ***N99/P1/Q4(C)***
  1.97; 0.015

Questions Involving Circular Measure and Trigonometry

- ***N98/P1/Q10(C)***
  1.37
- ***N06/P1/Q6(Z)***
  $t_2 = 2.64$
  $t_3 = 2.83$
  $t_4 = 2.73$
- ***J07/P3/Q6(C)***
  (iv) 1.90
- ***J10/P31/Q6(C)***
  (iii) 1.38
- ***J11/P31/Q6(C)***
  (ii) $\theta = 2.11$; $AB = 17.4$
- ***J11/P32/Q4(C)***
  (ii) 1.35
- ***N10/P31/Q4(C)***
  (i) *see diagram*
  (iii) 0.73
- ***J11/P33/Q6(C)***
  (i) *see diagram*
  (iii) 0.62

Questions involving Integration

- ***N10/P33/Q7(C)***
  (ii) 3.6
- ***N11/P33/Q5(C)***
  (ii) 5.86
- ***J12/P33/Q7(C)***
  (ii) 1.25

Questions involving Differentiation

- ***J10/P32/Q4(C)***
  (ii) 4.49
- ***J10/P33/Q6(C)***
  (ii) 3.59

Questions involving Newton-Raphson Method

- ***N97/P1/Q16(Z)***
  see diagram; 3 *and* 4
  3.7 *to* 2 *s.f*
  3.83 *to* 3 *s.f*
- ***N01/P1/Q14(Z)***
  $x = 1.393$
- ***J03/P1/Q10(Z)***
  see diagram
  root is between 1 *and* 2
  $x = 1.56$

Questions on Algebraic Expressions

- ***J01/P1/Q13(C)***
  $x_1 = 1.333333$
  $x_2 = 1.263889$
  $x_3 = 1.259933$

  $x^3 = 2$
  $x = \sqrt[3]{2}$
  $e_1 = 0.073412$
  $e_2 = 0.003968$
  $e_3 = 0.000012$

- ***J09/P3/Q4(C)***
  (iii) 1.77
- ***N09/P32/Q2(C)***
  (i) $x = 3\ and\ x = 4$
  (ii) 3.43
- ***N12/P33/Q6(C)***
  (i) -2
  (iii) 1.67

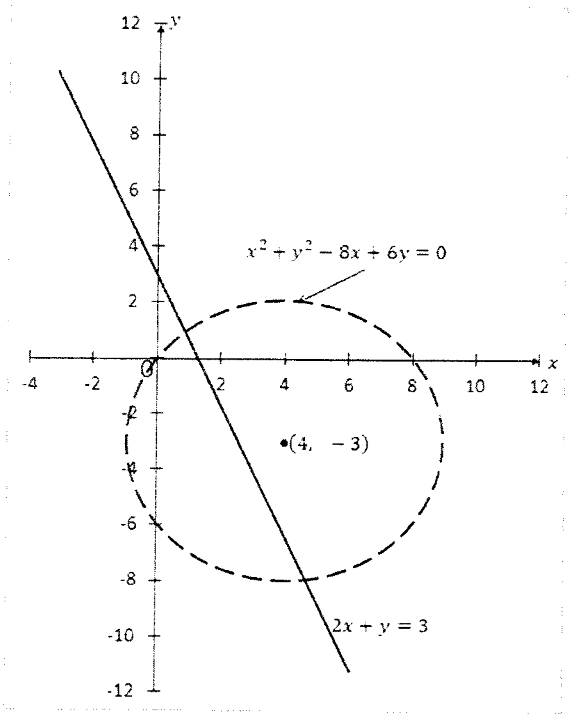
Chapter Fifteen: Complex Numbers

- ***N06/P1/Q9(Z)***
  (i) $x = -3;\ y = -1$
- ***N07/P1/Q4(Z)***
  $a = -\frac{2}{5}$
- ***J10/P1/Q10(Z)***
  $z_2 = 4 - 2i$
- ***N04/P1/Q2(Z)***
  (i) $\sqrt{20}$
  $-26.6°/-0.464rad$
  (ii) $\frac{3}{5} - \frac{4}{5}i$
- ***N10/P1/Q3(Z)***
  $\frac{7}{25} - \frac{1}{25}i$
  $\frac{1}{5}\sqrt{2}$
- ***J07/P3/Q8(C)***
  (i) $|u| = \sqrt{2}$,
  $\arg u = -\frac{3}{4}\pi\ or - 135°$
  $|u^2| = 2,\ \ \arg u^2 = \frac{1}{2}\pi\ or\ 90°$
- ***N09/P32/Q7(C)***
  (i) (a) $1 + 2i$
  (b) $-\frac{1}{2} + \frac{1}{2}i$
  (ii) $\frac{3}{4}\pi\ rad\ or\ 135°$
  (iv) $OA = BC,\ \ OA \parallel BC$
- ***J10/P31/Q7(C)***
  (i) $|u| = \sqrt{8}$, $\arg u = \frac{1}{4}\pi\ or\ 45°$
- ***N10/P31/Q6(C)***
  (i) $|z| = 2$, $\arg z = \frac{1}{6}\pi\ or\ 30°$
  (ii) (a) $3\sqrt{3} + i$
  (b) $\frac{1}{2}\sqrt{3} + \frac{1}{2}i$
  (iii) *see diagram*
- ***N10/P33/Q3(C)***
  (i) $3 + 4i;\ 5$
- ***J11/P31/Q8(C)***
  (i) 3
- ***J11/P32/Q7(C)***
  (a) (i) $\frac{5a}{a^2 + 4} - \left(\frac{10}{a^2 + 4}\right)i$
  (ii) -2
- ***J12/P31/Q4(C)***
  (i) $-\frac{2}{5} + \frac{11}{5}i$

DIAGRAM ANSWERS TO REVISION QUESTIONS

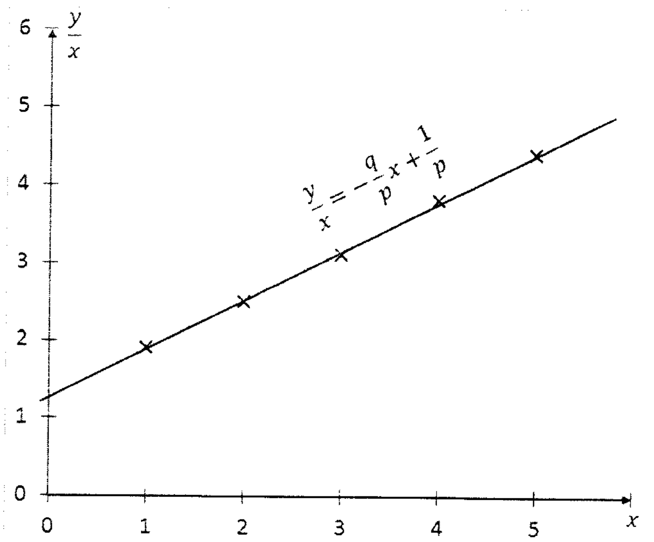
Chapter Three: Analytical Geometry

N10/P1/Q10 (iii) (C)



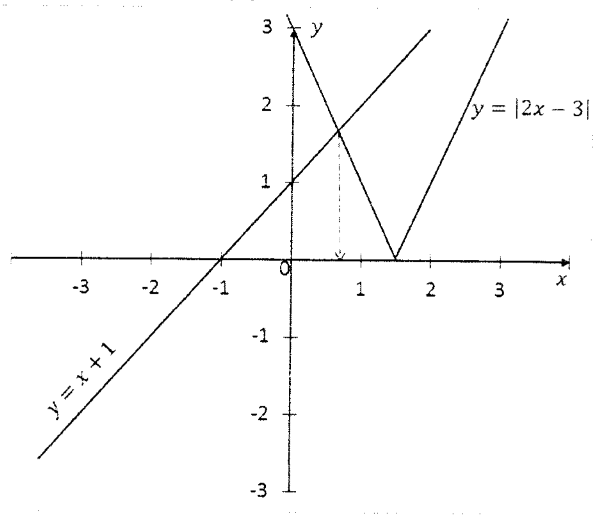
Chapter Four: Logarithmic and Exponential Functions

- ***N03/P2/Q3 (a)(ZOAM)***



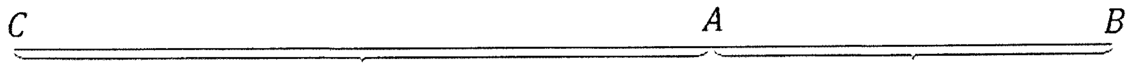
Chapter Five: Modulus Functions and Inequalities

- ***N07/P1/Q3 (Z)***



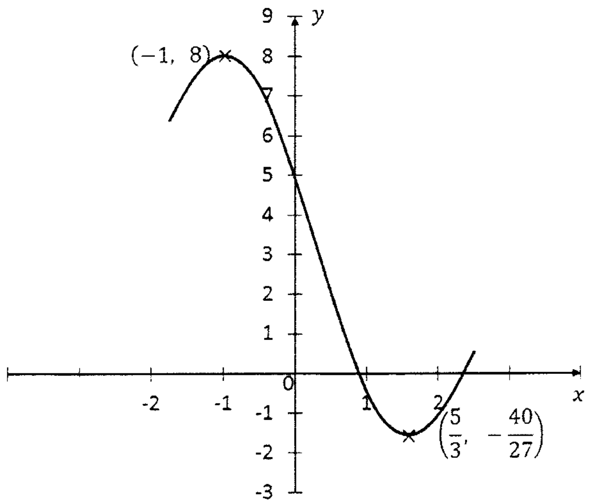
Chapter Six: Vectors

- ***N06/P1/Q12(Z) (ii)***



Chapter ten: Differentiation

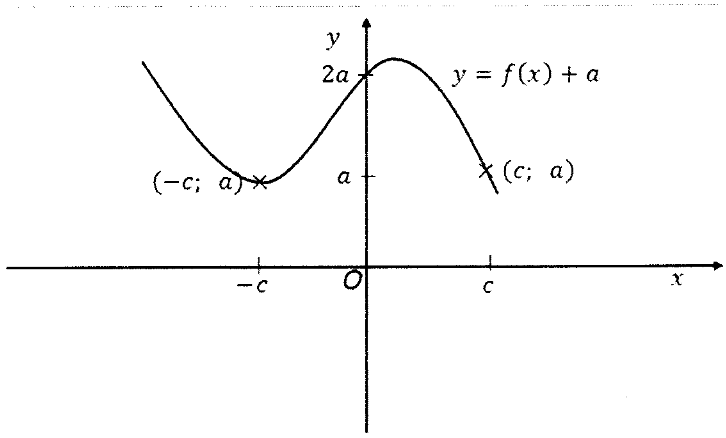
- ***N02/P1/Q18(Z)(iii)***



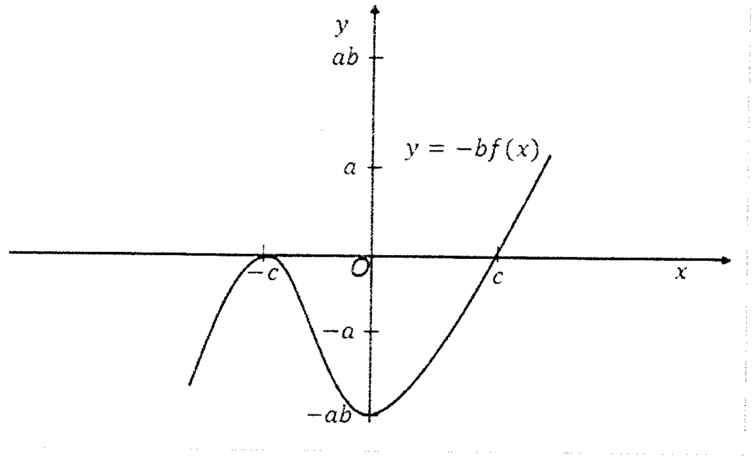
Advert Two: Proportionality and Curve Sketching

- ***N95/P1/Q3(C)***

  (i).

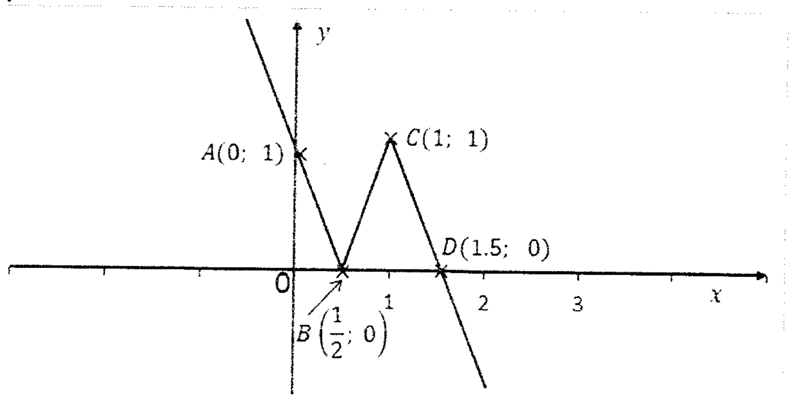


(ii).

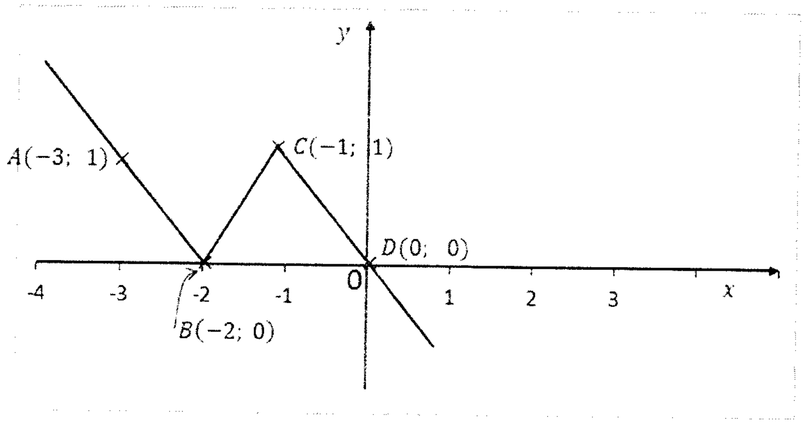


- ***N90/P1/Q3(C)***

(i).

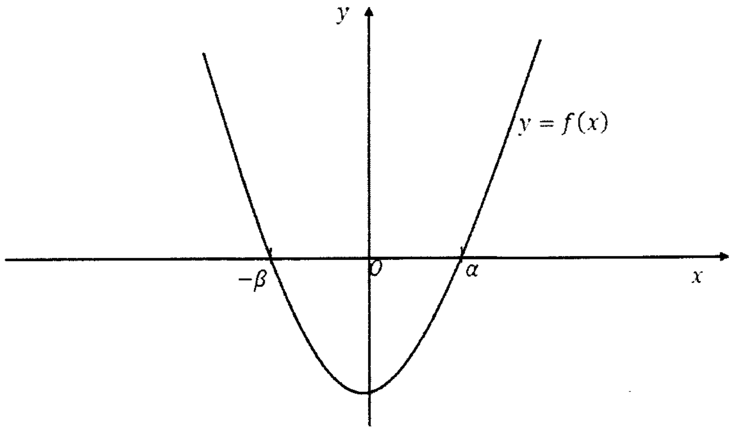


(ii).

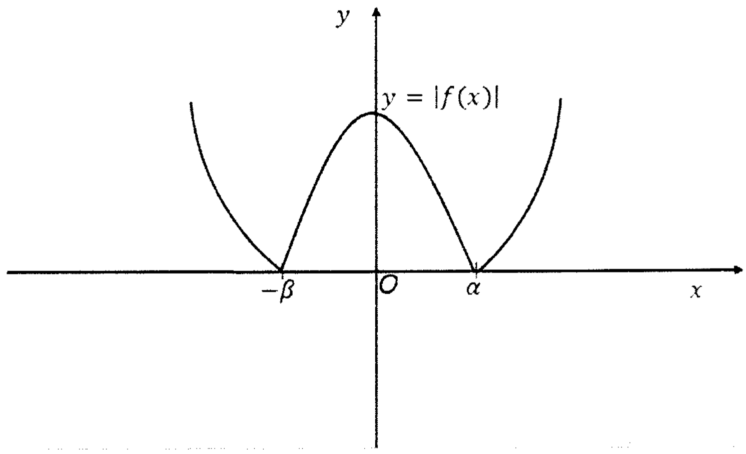


- ***J97/P1/Q6(C)***

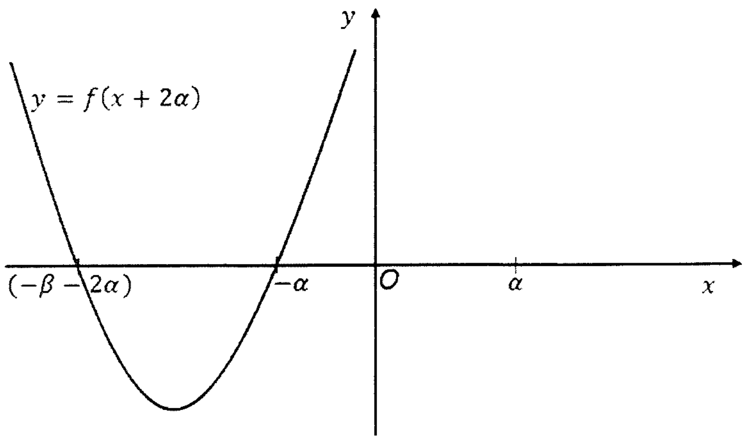
(i).



(ii).

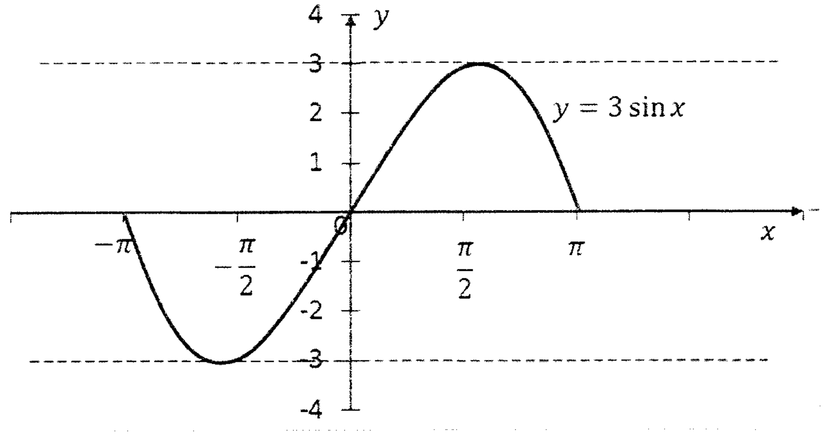


(iii).



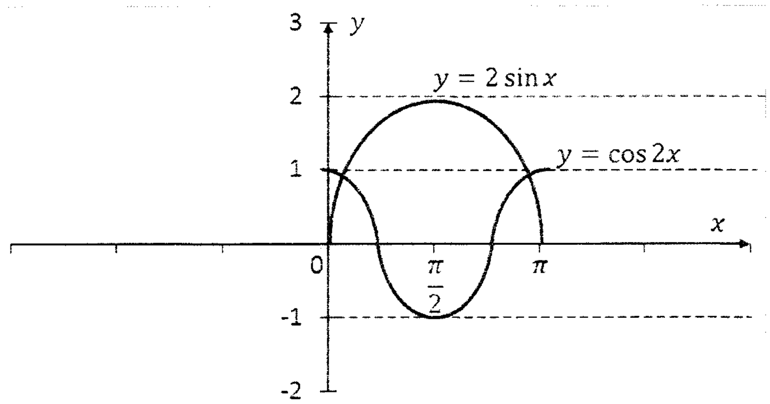
- ***J03/P1/Q6(C)***

  (i).

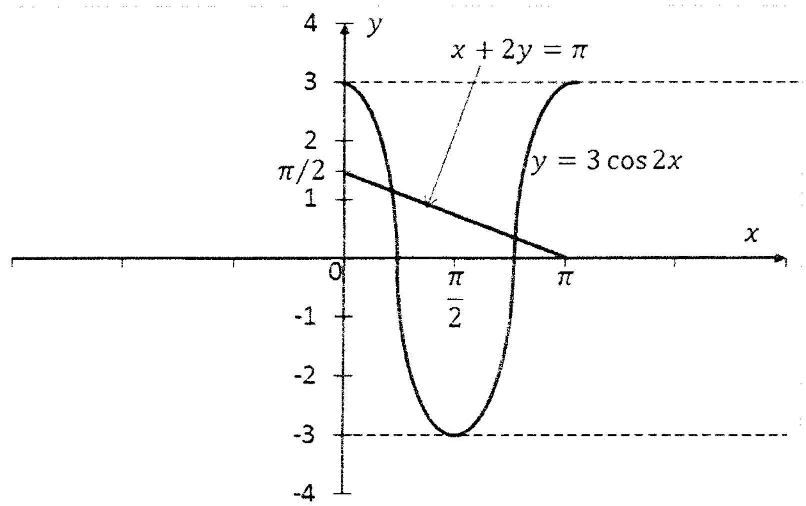


- ***N04/P1/Q4(C)***

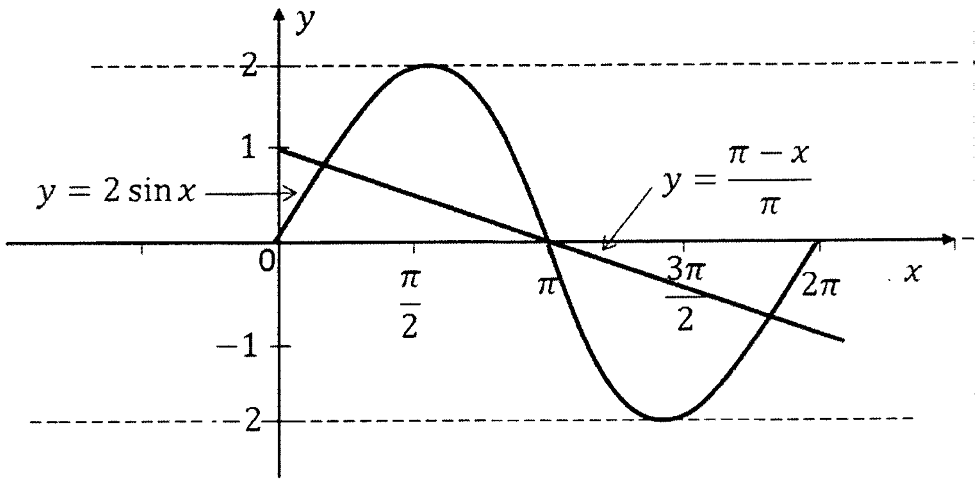
  (i).



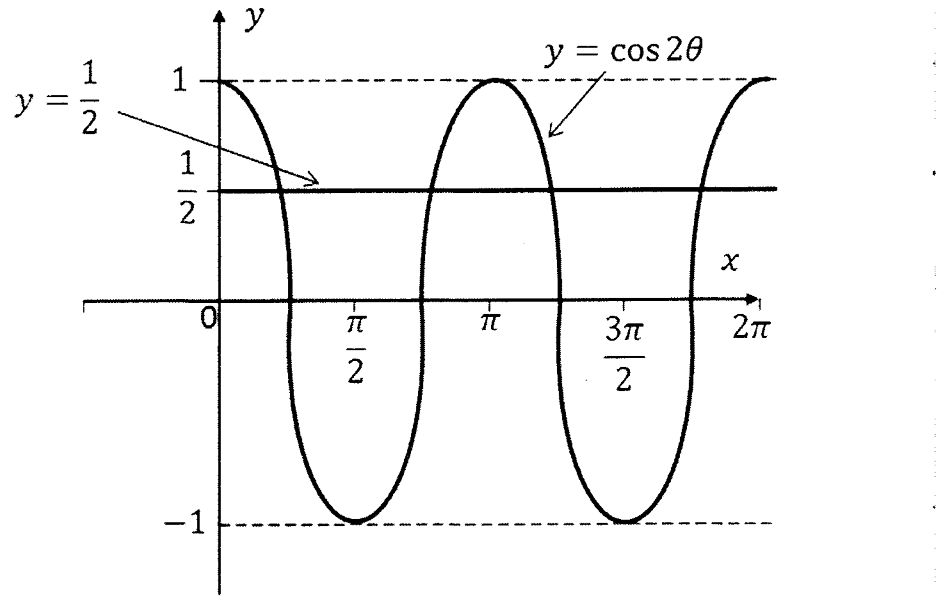
- ***N09/P11/Q2(C)***



- ***N10/P13/Q4(C)***



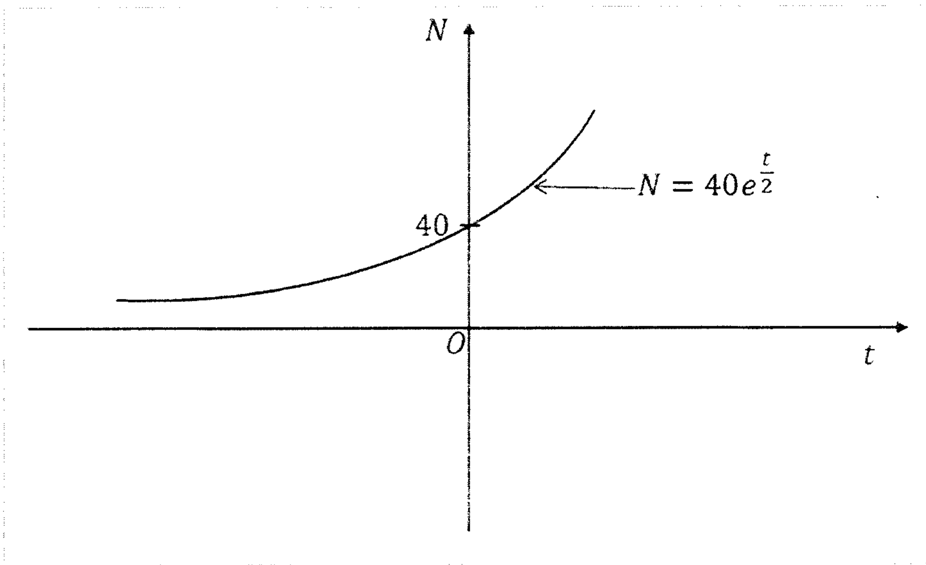
- ***N11/P11/Q3(C)***
  (i).



Chapter Twelve: Differential Equations

- ***N01/P1/Q17(Z)***

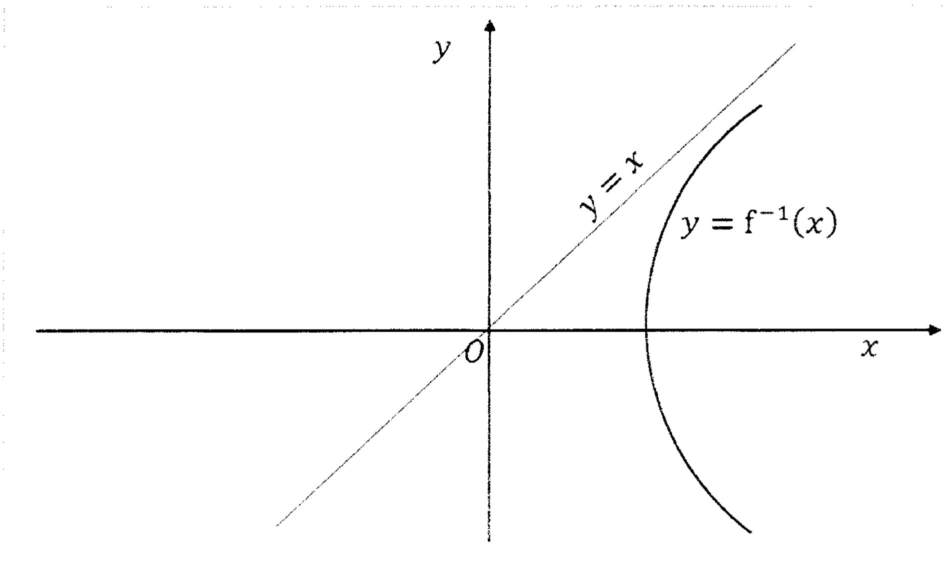
  (i).



Chapter Thirteen: Functions

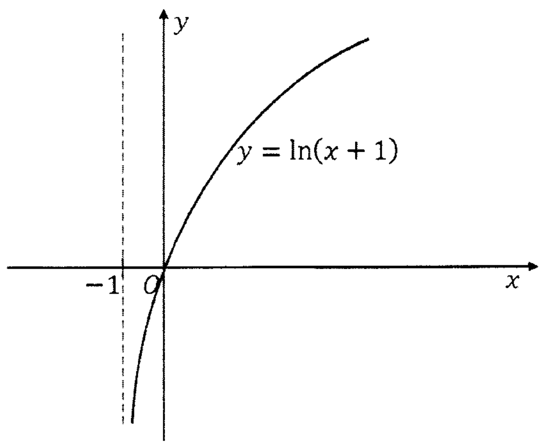
- ***N97/P1/Q11(C)***

  (iii).

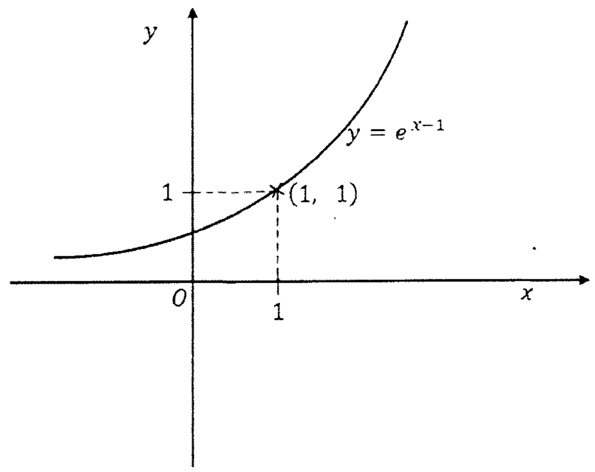


- ***J94/P1/Q13(C)***

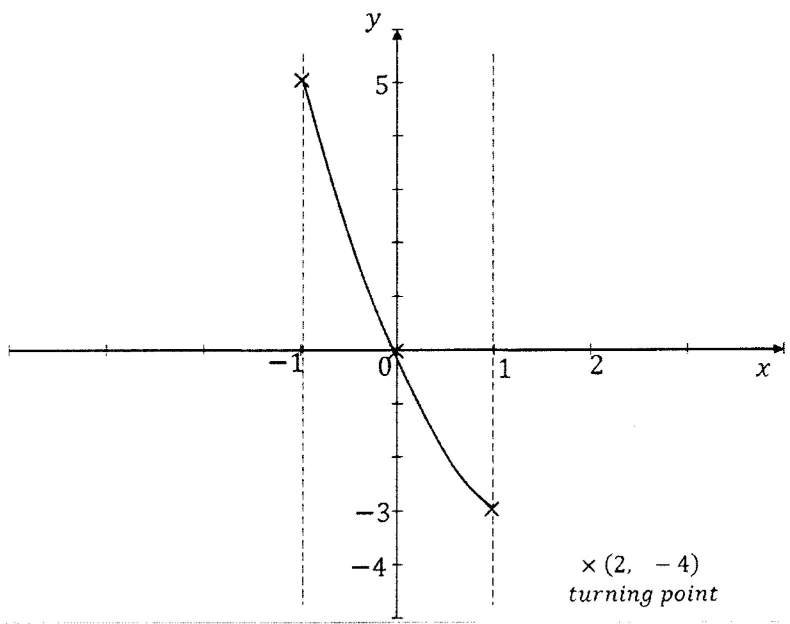
(a)(i).



(iv).

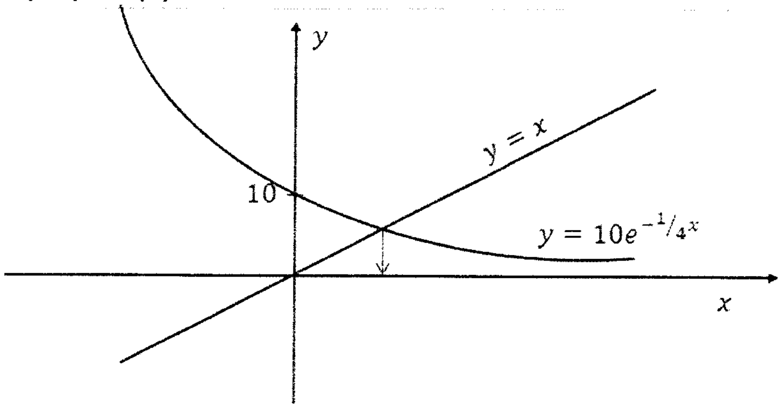


(b).

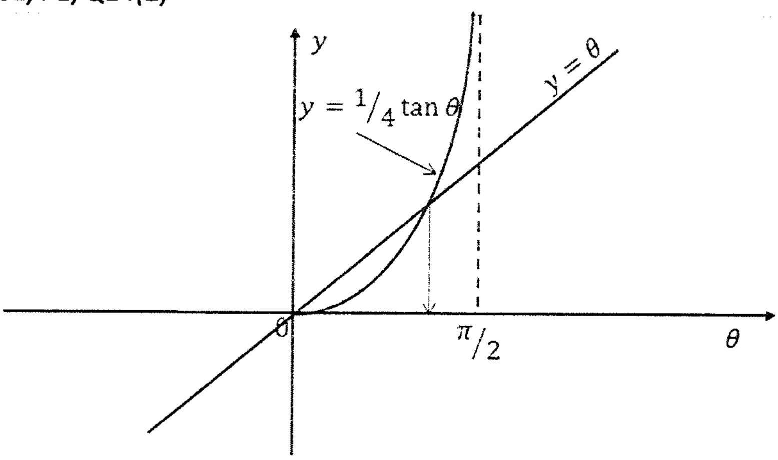


Chapter Fourteen: Numerical Methods

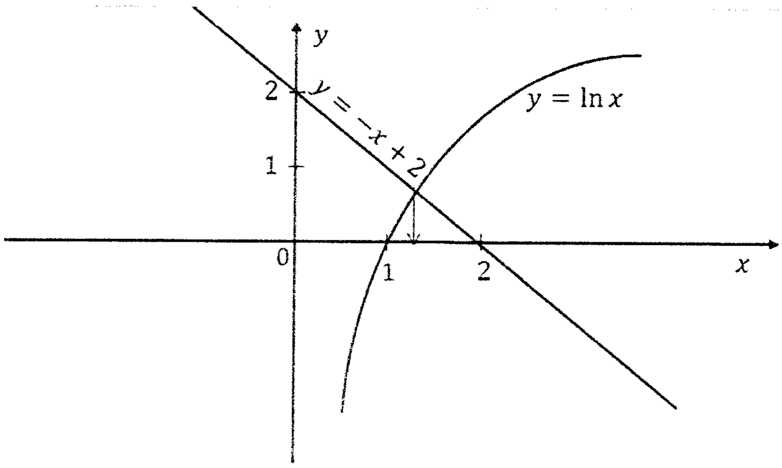
- *N97/P1/Q16(C)*



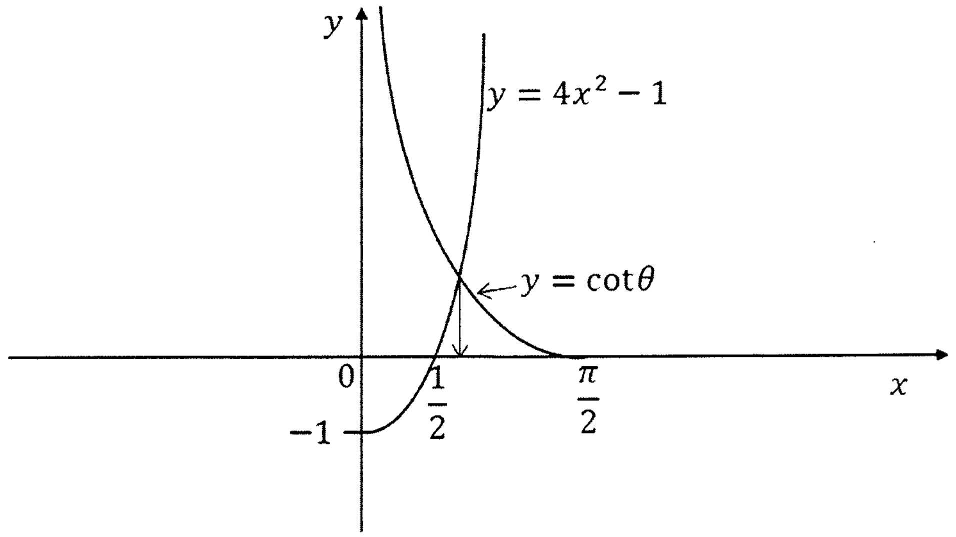
- *N01/P1/Q14(Z)*



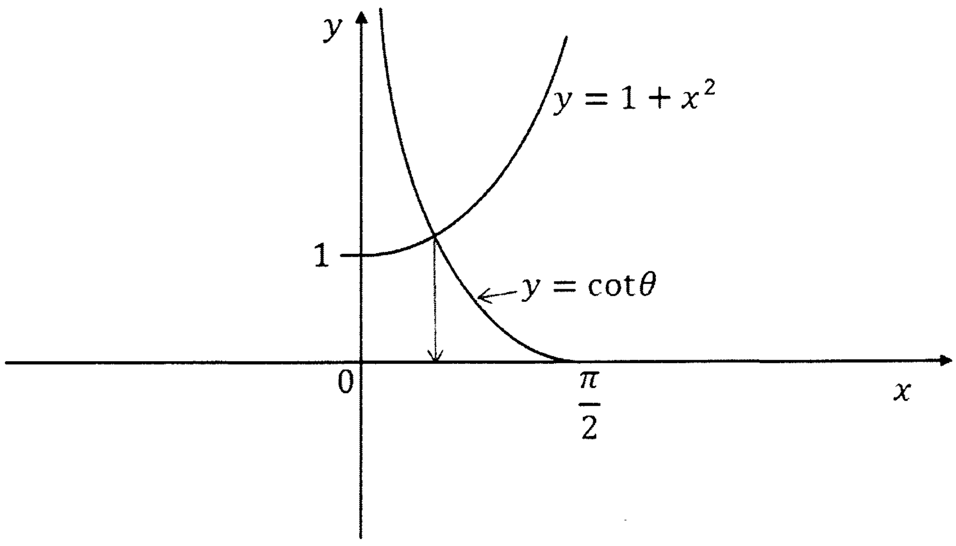
- *J03/P1/Q10(Z)*



- ***N10/P31/Q4(C)***
  (i).



- ***J11/P33/Q6(C)***
  (i).



<u>***Chapter Fifteen: Complex Numbers***</u>

- ***N10/P31/Q6(C)***

  (iii).