'A' LEVEL

PURE MATHS

STUDY PACK

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CHAPTER 1

INDICES AND PROPORTIONALITY

OBJECTIVES

By the end of the chapter the student should be able to :

- Simplify indices
- Simplify surds
- Solve exponential equations
- Solve problems involving propotionality

Laws of indices

The student will recall from earlier studies the following laws of indices.

(i) $b^p x b^q = b^{p+q}$

(ii) $b^p \div b^q = b^{p-q}$

(iii) $b^0 = 1$

(iv) $b^{-p} = \frac{1}{b^p}$

(v) $(b^p)^q = b^{pq}$

(vi) $b^{\frac{1}{p}} = \sqrt[p]{b}$

(vii) $(b^{\frac{1}{q}})^p = b^{\frac{p}{q}} = (\sqrt[q]{b})^p$

Examples

Simplify each of the following:

a) $x^5 \times x^4$ b) $(4d)^2 \div (2d)^3$ c) $4^{1/2}$ d) $4^{-1/2}$

e) $(\frac{343}{512})^{\frac{-2}{3}}$ f) $(\frac{3}{11})^{-2}$

Solution

a) $x^5 . x^4 = x^{5+4} = x^{+9}$ b) $(4d)^2 \div (2d)^3 = (2d)^4 \div (2d)^3 = (2d)^{4-3} = 2d$

c) $4^{\frac{1}{2}} = \sqrt{4} = 2$ d) $4^{\frac{-1}{2}} = (\frac{1}{4})^{\frac{1}{2}} = \frac{1}{2}$

e) $(\frac{343}{512})^{\frac{-2}{3}} = (\frac{512}{343})^{\frac{2}{3}} = (\frac{8}{7})^2 = \frac{1}{2}$

f) $(\frac{3}{11})^{-2} = (\frac{11}{3})^2 = \frac{121}{9}$

Practice Questions

1.Evaluate the following

i) $\frac{1}{(16)^{-\frac{1}{4}}}$ ii) $(\frac{100}{9})^0$ iii) $(\frac{1}{9})^{\frac{-3}{2}}$

iv) $(\frac{125}{27})^{\frac{-1}{3}}$ v) $\frac{9^{\frac{1}{3}} \times 27^{\frac{-1}{2}}}{3^{\frac{-1}{6}} \times 3^{\frac{-2}{3}}}$

Surds

Square roots of irrational numbers when left in exact form are called surds
Examples of surds are abound.

(a) $\sqrt{50}$ (b) $\frac{6}{\sqrt{3}}$ (c) $\frac{4}{\sqrt{6+2}}$

Simplifying surds.

Surds can be simplified by leaving them in either of the two forms.

(i) $p\sqrt{q}$, where, q is a number that cannot be expressed in the form r/s i.e. a rational number

(ii) $p + \sqrt{q}$

Example

Simplify: $\sqrt{500}$

Solution

$\sqrt{500} = \sqrt{100 \times 5}$

$= \sqrt{100} \times \sqrt{5}$

$= 10 \times \sqrt{5}$

$\mathbf{= 10\sqrt{5}}$

Express 500 as a product of a perfect square and another number, which is not a prefect square.

Example:
Simplify: $(\sqrt{5} – 1)(\sqrt{5} – 1)$

Solution

$(\sqrt{5} – 1)(\sqrt{5} – 1)$
$=\sqrt{5}.\sqrt{5} - \sqrt{5} - \sqrt{5} + 1$
$=5 - 2\sqrt{5} + 1$
$=6 – 2\sqrt{5}$

Example

Simplify: (i) $\frac{3}{\sqrt{2}}$ (ii) $\frac{1}{\sqrt{2-1}}$ (iii) $\frac{1-3\sqrt{2}}{3\sqrt{2+2}}$

Solution

(i) $3/\sqrt{2}$
we multiply the denominator and the numerator by $\sqrt{2}$, the process is called **rationalizing.**

Hence:

$\frac{3}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}}$

$= \frac{3\sqrt{2}}{2}$

$= \frac{3\sqrt{2}}{2}$

We multiply the denominator by the conjugate of $\sqrt{2} – 1$; $\sqrt{2} +1$

(ii) $\frac{1}{\sqrt{2}-1}$

$=\frac{\sqrt{2}+1}{(\sqrt{2}-1)(\sqrt{2}+1)}$

$=\frac{1+\sqrt{2}}{2-1}$

$=1+\sqrt{2}$

(iii) $\frac{(1-3\sqrt{2})}{(3\sqrt{2}+2)}$

$=\frac{(1-3\sqrt{2})(3\sqrt{2}-2)}{(3\sqrt{2}+2)(3\sqrt{2}-2)}$

$=\frac{3\sqrt{2}-2-9\times 2+6\sqrt{2}}{9\times 2-6\sqrt{2}+6\sqrt{2}-4}$

$=\frac{9\sqrt{2}-20}{18-4}$

$=\frac{-20+9\sqrt{2}}{14}$

$=\frac{1}{14}(-20+9\sqrt{2})$

Exponential Equations

These are equations where the variable is found in the exponent. In this case we, deal with two types. The third type is dealt with in chapter 6.

(a) Type 1

These are solved by introducing a common base.

Example
Solve for x

$9^{3x+4} = 27^{x+1}$

Solution

$$9^{3x+4} = 27^{x+1}$$
$$3^{2(x+4)} = 3^{3(x+1)}$$

$\therefore \quad 3^{6x+8} = 3^{3x+3}$

$\therefore \quad 6x + 8 = 3x + 3$

$\therefore \quad 3x = -5$

$x = \dfrac{-5}{3}$

Example

Solve for x: $\dfrac{1}{4^{-x}} = 16^{x}$

Solution $\dfrac{1}{4^{-x}} = 4^{x}$

Hence:

$4^x = 16^x$
$(2^2)^x = (2^4)^x$
$2^{2x} = 2^{4x}$
$2x = 4x^2$
$4x^2 - 2x = 0$
$2x(x - 2) = 0$
$2x = 0$ or $x = 2$
$x = 0$ or $x = 2$

For equal bases, equate exponents.

(b) **Type 2**

These are solved by introducing a new variable.

Example :

$y^{\frac{2}{3}} - 9y^{\frac{1}{3}} + 20 = 0$

$(y^{\frac{1}{3}})^2 - 9y^{\frac{1}{3}} + 20 = 0$

Let $u = y^{\frac{1}{3}}$

$u^2 - 9u + 20 = 0$
$(u - 4)(u - 5) = 0$
$u = 4$ or 5

$y^{\frac{1}{3}} = 4$ or $\quad y^{\frac{1}{3}} = 5$

$(y^{\frac{1}{3}})^3 = 4^3$ or $\quad (y^{\frac{1}{3}})^3 = 5^3$

$y = 64$ or $\quad 125$

Example

$2x^{\frac{1}{4}} = 9 - 4x^{\frac{-1}{4}}$

Let $y = x^{\frac{1}{4}}$

Hence, $2y = 9 - 4/y$

$2y^2 - 9y + 4 = 0$

$y = \frac{9 \pm \sqrt{49}}{4}$

$y = 4$ or $y = \frac{1}{2}$

$x^{1/4} = 4$ or $x^{\frac{1}{4}} = \frac{1}{2}$

$x = 256$ or $x = \frac{1}{16}$

Practice Questions

1. Solve the following equations in x.

   (i) $2^{2x+1} - 9.\ 2^x + 4 = 0$
   (ii) $3^{3x} - 13.\ 3^{2x} + 13.\ 3^{x+1} - 27 = 0$
   (iii) $4^x - 5\ .2^{x+1} + 16 = 0$

2. Solve for x
   i) $x - 10x^{1/2} + 24 = 0$
   ii) $4x^2 = 64$
   iii) $2x^{-\frac{1}{3}} = 16$

3. i) $3^{2x+1} = 9^{-(3x+4)}$
   ii) $(\frac{1}{2})^{-3x} - (\frac{1}{16})^{-x+4} = 0$

   iii) $64 = (\frac{1}{16})^{-4x}$

Proportionality

This topic is a requisite to the study of differential equations. It has been covered to a certain extent at "O" level. However we are going to cover it in detail in this study pack. Variation comes in the following ways:

a) Direct proportion
b) Inverse proportion
c) Joint variation

Direct Variation

Two variables vary directly if they increase or decrease together or if one increases and the other decreases i.e. if the ratio of y to x is always constant y is said to vary directly as x , $y \propto x$. The equation connecting the two quantities is $y = kx$.

Example

The weight (w) of a pile of this study pack varies directly as the number of packs in the pile.

This is written as:

$W \alpha N$ which gives the general formula.
$W = kN$ where N is the number of packs.

Where k is called the constant of variation or the constant of proportionality

Given the 10 packs that weigh 20kgs, we can calculate the value of K:

$$20 = K(10)$$
$$K = 2$$

Giving us the particular formula.

$$W = 2N$$

From this we can calculate W given N or N given W.

Inverse Variation

This is where one variable increases as the other decreases.

Example

The number of slices of a loaf of bread each person gets (N) decreases as the number of people sharing it.

We say : $N \ \alpha \ \frac{1}{p}$

$N = \frac{k}{p}$

Joint Variation

This is when a variable is related to two or more variables.

Example

The volume of a conical sand heap is directly proportional to its height (*h*) and to the square of the radius of its base.

That is:
$$V \ \alpha \ h$$
$$V \ \alpha \ r^2$$
$$V \ \alpha \ hr^2$$
$$V = khr^2$$

Given that the volume is $200m^2$ when the radius is 2m and the height is 5m, find the volume of the heap when the radius is 3m and the height is 8m.

$$V = khr^2$$

Substituting: $200 = k(5)(2^2)$

$$k = 10$$

$$V = 10\,hr^2$$

When h = 8 and r = 3

$$V = 10\,(8)(3^2)$$
$$V = 720m^3$$

Example

If $x \ \alpha \ y$ and $y \ \alpha \ z^2$. How does x vary with z?

Solution

$x \propto y$

$\therefore\ x = k_1y \quad (1)$

$y \propto z^2$

$\therefore\ y = k_2z^2 \quad (2)$

from (1) $y = \dfrac{x}{k_1}$ - (3), $k_1 \neq 0$

sub: (3) in (2) to obtain

$$\frac{x}{k_1} = k_2z^2$$

$\therefore x = k_1.k_2z^2$

$\therefore x = kz^2$; where $k = k_1.k_2$

$\therefore \mathbf{x \propto z^2}$

i.e x varies directly as the square of x

Examination Type Questions

1. Write down the formulae to express the following.
   i) The volume of a cylinder varies as the square of the radius and directly as the height
   ii) The pressure of a given mass of gas varies directly as the temperature and inversely as the volume.
   iii) The electrical resistance of a wire varies directly as the length and inversely as the square of the radius.

2. The positive variables x, y and z are related as follows:
   x is proportional to the square root of y
   y is inversely proportional to z.
   Express x in the form Az^n, where A is a constant of proportionality and where the value of n is to be stated. Find the factor by which x is multiplied as a result of z being multiplied by 100.

3. The volume of a cone varies jointly as the height and square of the base. Calculate the percentage change in volume if the base radius is increased by 70% and the height decreased by 10%.

4. Solve the equation $2x^2 + 3 – 7\ (2x+1) – 15 = 0$

CHAPTER 2

QUADRATIC AND CUBIC FUNCTIONS

OBJECTIVES

By the end of the chapter the student should be able to :

- Identify nature of roots
- Complete square of a quadratic expression
- Sketch graph of quadratic functions
- Solve cubic equations
- Sketch graphs of cubic functions

Quadratic Functions

These are equations of the form $y = ax^2 + bx + c$, where $a \neq 0$
The expression $ax^2 + bx + c$, $a \neq 0$ is called the quadratic expression:
If $y = 0$ we have $ax^2 + bx + c = 0$, which is a quadratic equation.

1.1.1 Solving quadratic equations

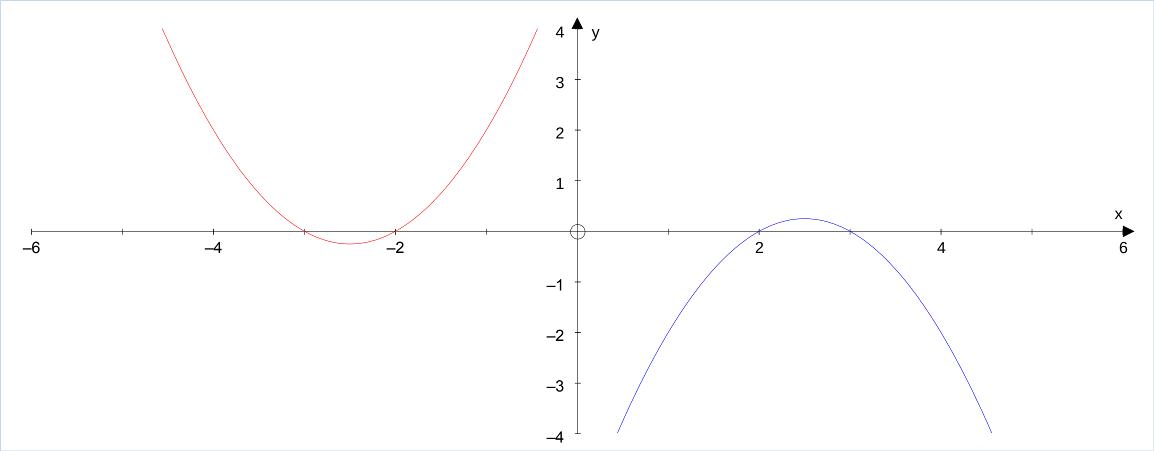
Quadratic equations can be solved either by

i. graphical method or
ii. Using the quadratic formula.
iii. Completing the square

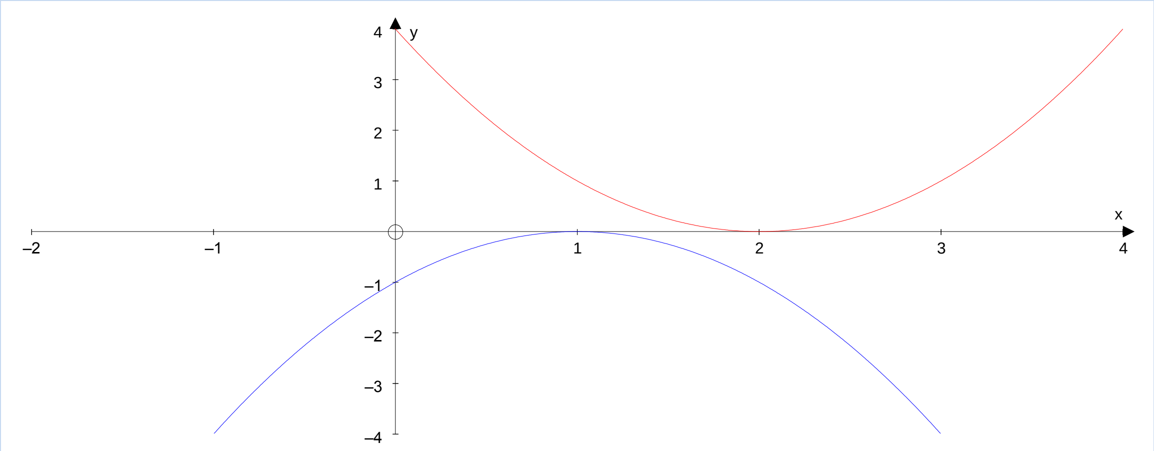
Recall from 'O' level the quadratic formula: $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, the expression

Let $\Delta = b^2 - 4ac$ is called the discriminant.

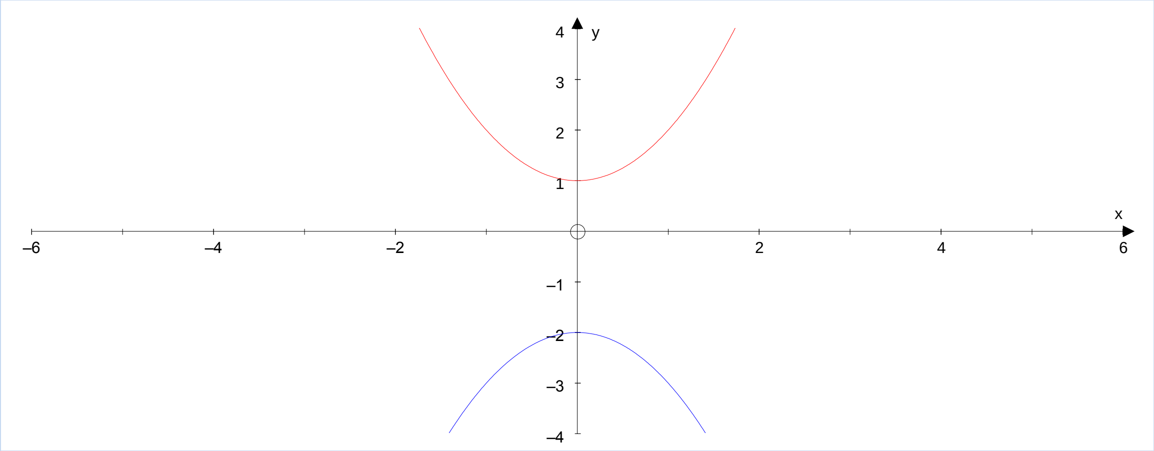
a) If $\Delta > 0$, there are 2 distinct roots as shown below



b) If $\Delta = 0$, there is a repeated root



c) If $\Delta < 0$,there are no real roots



Example

Determine the number of roots in each of the following equations

i. $2x^2 + 2x - 6 = 0$
ii. $-3x^2 - 3x - 8 = 0$
iii. $x^2 + 2x + 1 = 0$

<u>Solution</u>

i. $\Delta = 2^2 - 4(2)(-6)$
$= 52 > 0$
there are two distinct roots

ii. $\Delta = (-3)^2 \ 4(-3)\ (-8)$
$= -87 < 0$
there are no real roots

iii. $\Delta = 2^2 - 4\,(1)(1)$
$= 0$
there is a repeated root

<u>Completing the square.</u>

If the quadratic expression is expressed/ written in the form $a\,(x + A)^2 + B$ where a, A and B are real values, the process is called completing the square.

Example
Method 1.
Complete the square $x^2 - 4x + 5$

<u>Solution</u>

We express $x^2 - 4x + 5$
In the form $a(x + A)^2 + B$
i.e. $x^2 - 4x + 5 = a(x + A)^2 + B$
$x^2 - 4x + 5 = a\,(x^2 + 2Ax + A^2) + B$, expanding $(x + A)^2$
$x^2 - 4x + 5 = ax^2 + 2aAx + aA^2 + B$, multiplying by a.
Equating coefficients of x^2

$1 = a$

Equating coefficients of x
$-4 = 2aA$
$-4 = 2A$
$-2 = A$

Equating independent terms of x.

$5 = aA^2 + B$
$5 = (1)(-2)^2 + B, \quad a = 1 , \; A = -2$
$5 = 4 + B$
$1 = B.$

Hence: $x^2 - 4x + 5 = 1\,(x - 2)^2 + 1$
$\therefore \qquad x^2 - 4x + 5 = (x-2)^2 + 1$

Method 2.

Find half of – 4, the coefficient of x; which is – 2: square (-2) to obtain 4. Add and subtract 4 to the quadratic expression.

i.e. $x^2 - 4x + 5 \quad = x^2 - 4x + 4 + 5 - 4$
$\qquad\qquad\qquad = x^2 - 4x + 4 + 1$
$\qquad\qquad\qquad = (x - 2)^2 + 1$

Example

Complete square.
$-3x^2 + 6x - 5$

<u>Solution</u>

Method 1

$-3x^2 + 6x - 5 \equiv a(x + A)^2 + B$
$\qquad\qquad\quad = ax^2 + 2aAx + aA^2 + B$

Equating coefficients

$a = -3 \;\; ; aA^2 + B = -5$
$6 = 2aA \qquad (-3)(-1)^2 + B = -5$
$A = -1 \qquad B - 3 = -5$
$\qquad\qquad\; B = -2$
Hence: $-3x^2 + 6x - 5 = -3\,(x-1)^2 - 2$

Method 2.

In this case the coefficient of x^2 is –3 different from 1, hence factor out –3 to obtain.

$-3\,(x^2 - 2x + 5/3)$; since ½ of –2 is –1 and $(-1)^2 = 1$
$-3\,(x^2 - 2x + 1 + 5/3 - 1)$
$-3\,[\,(x-1)^2 + 2/3]$; multiplying by -3.
$-3\,(\,x-1)^2 - 2.$

Sketching graphs of quadratic functions

To sketch the graph of a quadratic function you need to find:

- **The roots.**
- **Determine the form of the graph using the value of a.**
- **Find the turning point.**
- **Find the y – intercept.**

Solution.

1. $y = 0$

$x^2 - 5x + 4 = 0$

$$x = \frac{-5 \pm \sqrt{(-5)^2 - 4(1)(4)}}{2(1)}$$

$$x = \frac{5 \pm \sqrt{25 - 16}}{2}$$

$$x = \frac{5 \pm \sqrt{9}}{2}$$

$$x = \frac{5 \pm 3}{2}$$

$$x = \frac{2}{2} or \frac{8}{2}$$

$x = 1$ or $x = 4$ $\therefore$ the graph crosses the line x –axis at $x = 1$ and at $x = 4$

Since a $= 1>0$, the graph open, upwards.

The turning point:
There are two methods:

Method 1

- Add the roots: $1 + 4 = 5$
- Divide by 2: $5/2 = 2.5$
- Find the value of y:

$$y = (2.5)^2 - 5(5/2) + 4$$

$$= -\frac{25}{4} - \frac{25}{2} + 4$$

If $ax^2 + bx + c = a(x + A)^2 + B$

The turning point is $(-A ; B)$

$$= \frac{25-50+16}{4}$$

$$= \frac{-9}{4}$$

Hence: the turning point is (2,5 ; -9/4)

Method 2

By completing the square.

$x^2 - 5x + 4 = x^2 - 5x + \frac{25}{4} + 4 - \frac{25}{4}$

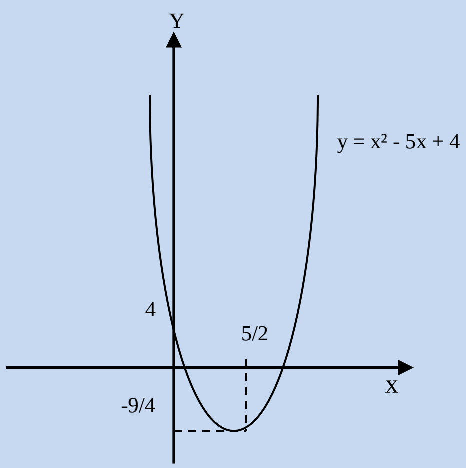
$$= (x - 5/2)^2 - 9/4.$$

Hence the turning point is (5/2 ; - 9/4)

The y - intercept

Set x = 0 ; hence $y = 0^2 - 5(0) + 4$

$y = 4$



Example

Sketch the graph of $y = -3x^2 + 7x + 4$.

<u>**Solution**</u>.

1. $y = 0$

$-3x^2 + 7x + 4 = 0$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-7 \pm \sqrt{7^2 - 4(-3)(4)}}{2(-3)}$$

$$x = \frac{-7 \pm \sqrt{49 + 48}}{-6}$$

$$x = \frac{-7 \pm \sqrt{97}}{-6}$$

$$x = \frac{7 - \sqrt{97}}{6} \; or \; \frac{7 + \sqrt{97}}{6}$$

x=-0,474 or 2,8

2. Since a = -3<0, the graph opens downwards.

4. The turning point.

$-3x^2 + 7x + 4 = a(x + A)^2 + B$
$\qquad\qquad\quad = ax^2 + 2aAx + aA^2 + B$
$a = -3: \quad 2aA = 7$
$\qquad\qquad 2(-3) = 7$

$$A = \frac{-7}{6}$$

$aA^2 + B = 4$

$(-3)\ (-7/6)^2 + B = 4$

$-3\ (49/36) + B = 4$

$-49/12 + B = 4$

$B = 4 + 49/12$

$\mathbf{B} = \frac{97}{12}$ **hence: the turning point is** $\left(\frac{7}{6}; \frac{97}{12}\right)$

<u>Solving cubic equations</u>

A cubic function is a polynomial where the highest power of the variable is 3 i.e. of the form $y = ax^3 + bx^2 + cx + d$ where $a \neq 0$

In solving the cubic equations, we expect to find at most three roots and at least one root

i. 1 root : This occurs when we have three (3) coincident points. That is to say when we have a cubed factor.

Example $(x + 2)^3 = 0$

$x = -2$ thrice

ii. 2 roots : This occurs when we have 2 coincident points, that is, when we have a squared factor.

Example **:** $(x - 3)^2(x + 1) = 0$

$\Rightarrow$ $x = 3$ twice or -1

iii. 3 different roots: This occurs when we have three distinct factors

Example : $(x + 1)(x - 4)(x + 5) = 0$

$\Rightarrow$ $x = -1, -5$ or 4

Example : Solve $x^3 + 2x^2 - 5x - 6 = 0$

To solve this equation, we will have to factorise it first. In order to do this we must first find one of its factors using the factor theorem.

If $x - b$ is a factor of $x^3 + 2x^2 - 5x - 6$ then the possible values of b are found by considering all the factors of the constant term, -6. These are $\pm 1, \pm 2, \pm 3, \pm 6$

Trying $f(-1) = (-1)^3 + 2(-1)^2 \;\; -5(-1) - 6$

$f(-1) = 0$

Hence: $x + 1$ is a factor since $f(-1) = 0$: **this method of factorization will be explored further in the next chapter.**

To find the other factors, we divide $f(x)$ by $x + 1$

$$
\begin{array}{r|l}
 & x^2 + x - 6 \\
\hline
x + 1 & x^3 + 2x^2 - 5x - 6 \\
 & \underline{x^3 + x^2} \\
 & \quad x^2 - 5x
\end{array}
$$

$\underline{x^2 + x}$

$- 6x - 6$

$\underline{- 6x - 6}$

$$f(x) = (x + 1)(x^2 + x - 6) = 0$$

$$(x + 1)(x - 2)(x + 3) = 0$$

$$x = -1, 2, -3$$

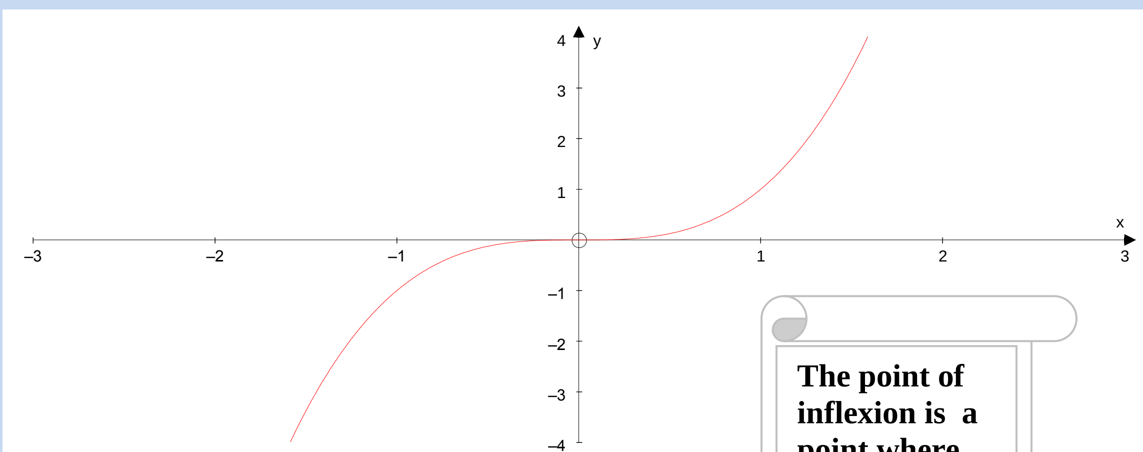
Cubic functions

There are three basic forms of graphs of cubic functions which depend on the number of roots of the cubic equation.

CASE 1 :

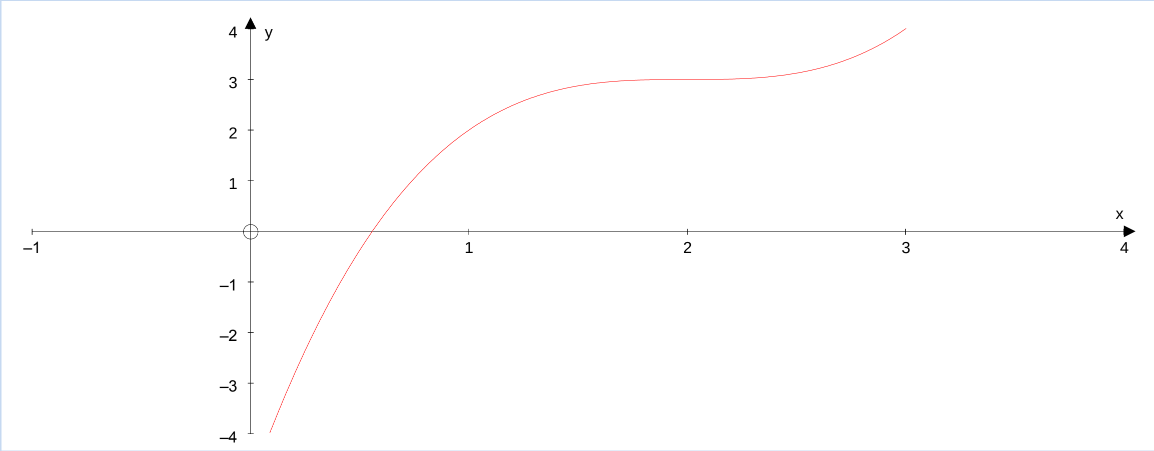
If the equation is $y = x^3$ then the graph will have a point of inflexion at x = 0

Example : **Sketch the graph of $y = x^3$**



The point of inflexion is a point where the graph changes concavity e.g. x = 0 for the function y = x^3

Example : $y = (x - 2)^3 + 3$



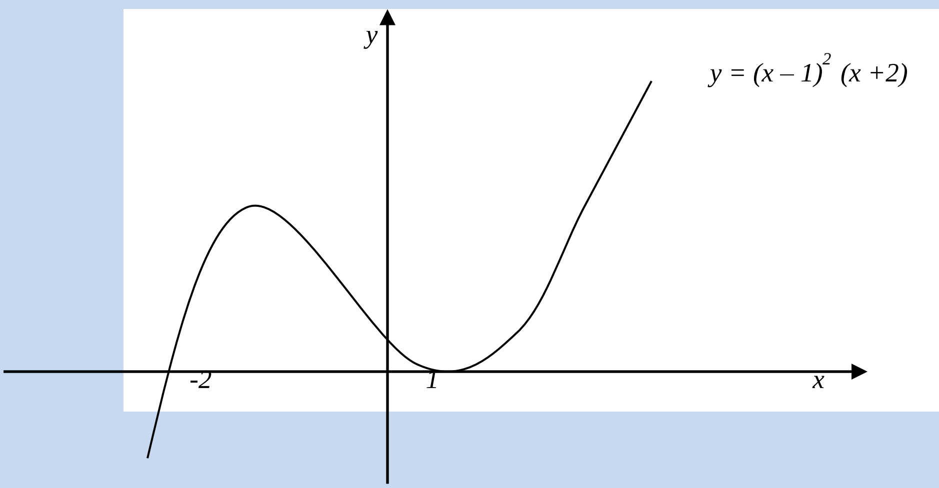
CASE 2:

With a repeated root in the equation, the graph both touches and crosses the axis.

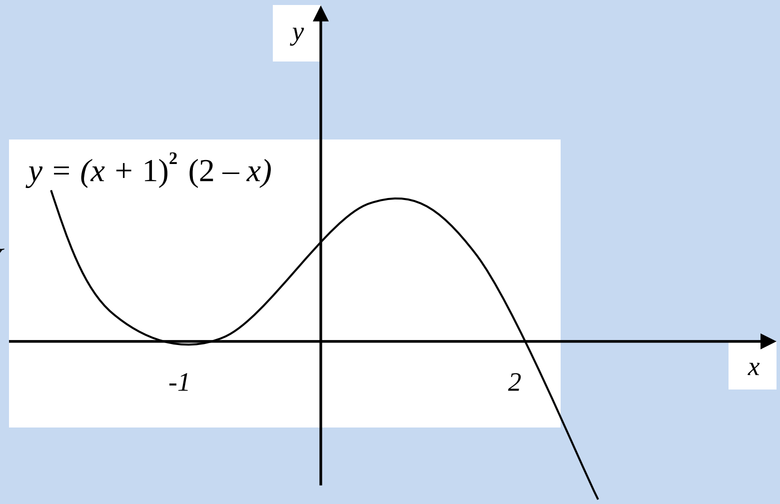
Example : Sketch the graph of y = $(x - 1)^2(x + 2)$

Putting $(x - 1)^2(x + 2) = 0$

$x = 1$ twice or -2



Example : *y*

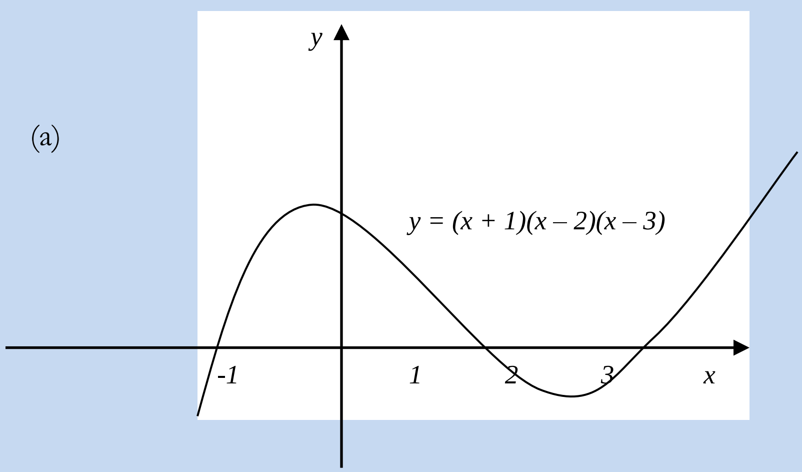


CASE 3
Three Distinct Roots

Example
Sketch the graph of $y = (x + 1)(x - 2)(x - 3)$

$x = -1$, 2 or 3

(a)



Examination Type Questions

1 (a) Solve $y^2 - 7y + 10 = 0$

(b) Hence find the solution to $(x^2 + 1)^2 - 7(x^2 + 1) + 10 = 0$

(c) Solve for x:: $x^3 + 7 = \frac{8}{x^3}$

(d) (i) By using the substitution $p = x + \frac{1}{x}$, show that the equation

x

$2x^4 + x^3 - 6x^2 + x - 2 = 0$ reduce to $2p^2 + p - 10 = 0$.

(ii) Hence solve $2x^4 + x^3 - 6x^2 + x + 2 = 0$.

2 (a) Write $x^2 + 6x + 14$ in the form $(x + a)^2 + b$, where a and b are constants to be found.

(b) Find the value of x for which this minimum occurs.

(c) Write down the maximum value of the function $\frac{1}{x^2 + 6x + 14}$.

3 Find the range of values of k for which the quadratic equation

$$(3 + k)x^2 + 4x + k = 0$$

has real roots

4 Show that the elimination of x from the simultaneous equations

$$x - 2y = 1$$
$$3xy - y^2 = 8$$

produces the equation

$$5y^2 + 3y - 8 = 0$$

Solve this quadratic equation and hence find the pairs (x, y) for which the simultaneous equations are satisfied

CHAPTER 3

TRANSFORMATIONS

OBJECTIVES

By the end of the chapter the student should be able to :

- Sketch grafphs using transformations

Transformations of graphs

To transform a graph is to change either its shape or its location or both. These techniques come in handy when we wish to sketch the graphs of given functions.

We start with either a standard graph or a given curve.

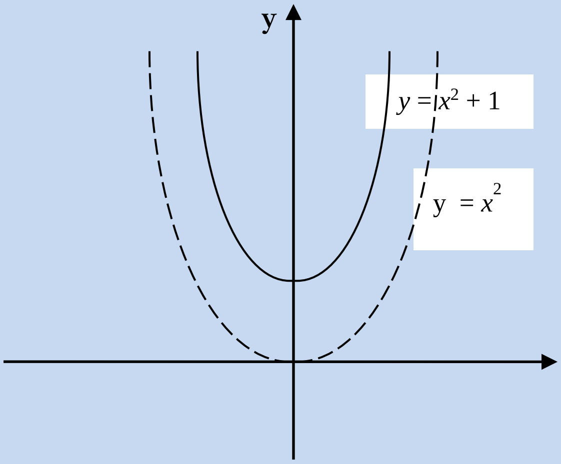
There are several transformations of interest to you.

i. $y = f(x) + k$

This represents a translation of the graph of $f(x)$ by k units in the direction of y

Example: **Sketch the graph of $y = x^2 + 1$**

We start by sketching the graph of $y = x^2$ and then move it upwards by 1 unit.

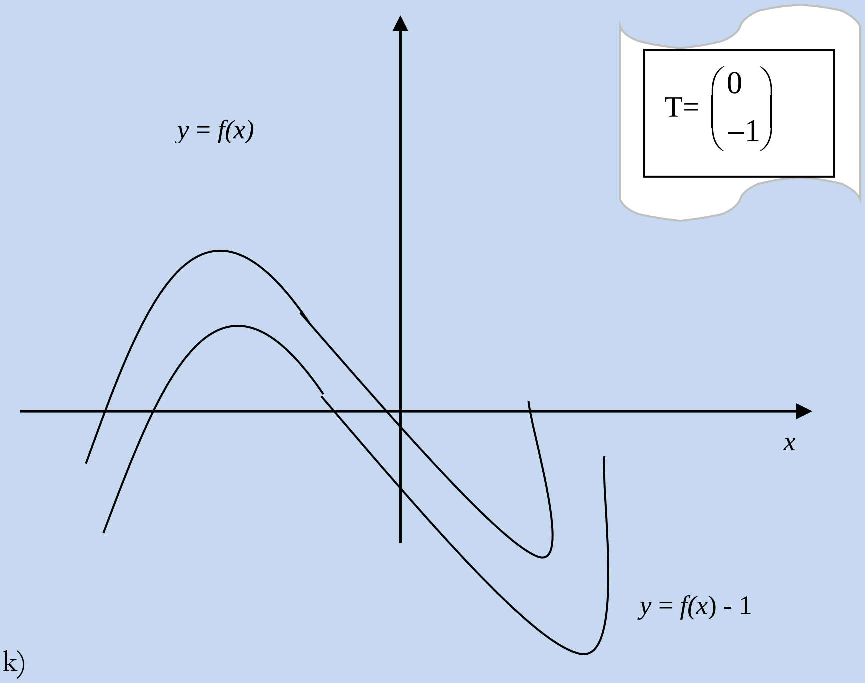


The graph has been translated by 1 unit upwards hence

T = $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$

1

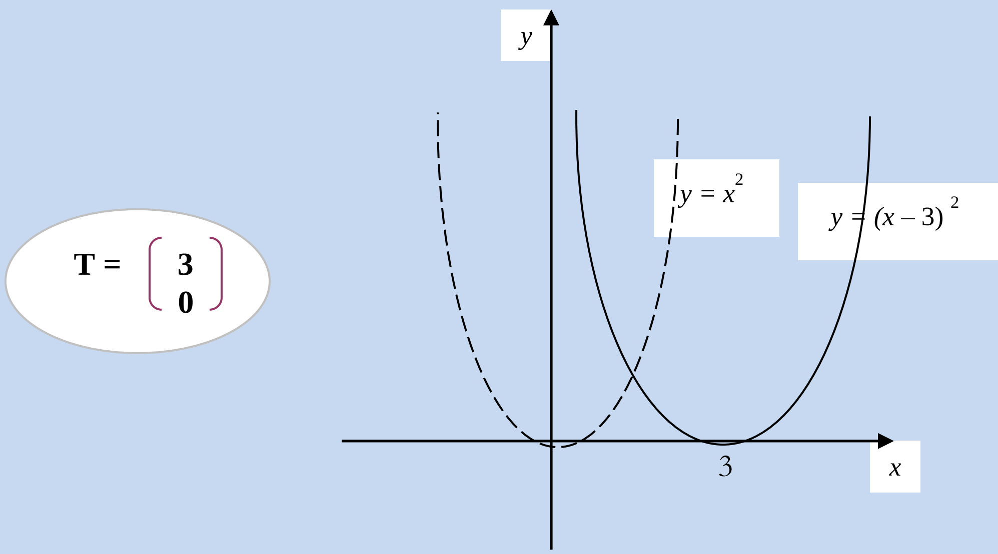
Example Given the graph of y = f(x) below sketch the graph of y = f(x) -1



ii. $y = f(x - k)$

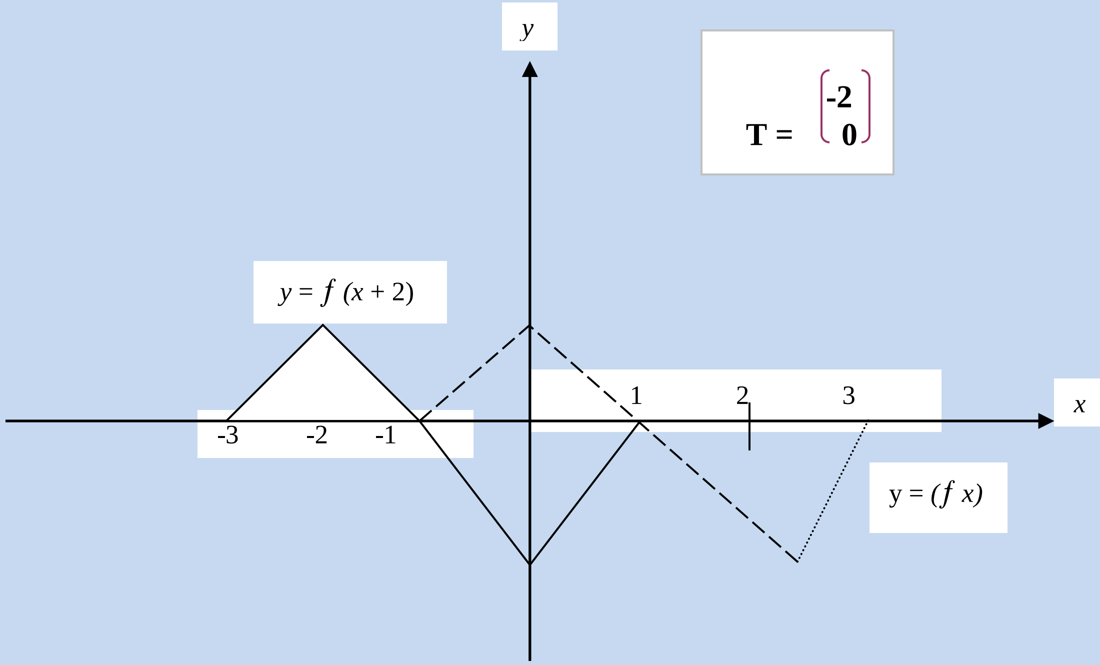
This represents a translation by k units in the positive direction of x.

Example: Sketch the graph of $\boldsymbol{y = (x-3)^2}$



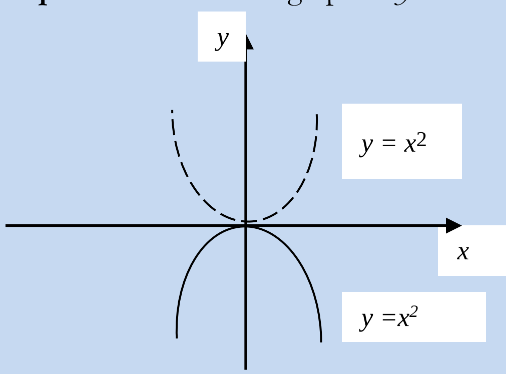
Example

Given the graph of y = $f(x)$ below sketch the graph of y = $f(x + 2)$



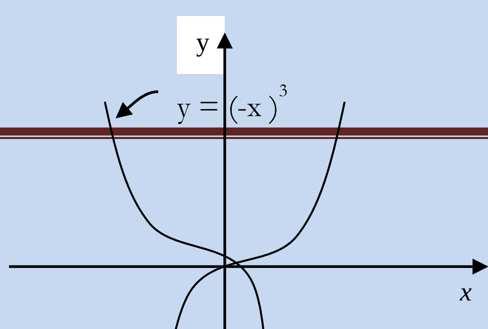
iii. $y = -f(x)$: This represents a reflection about the line $y = 0$

Example Sketch the graph of $y = -x^2$



iv. $y = f(-x)$

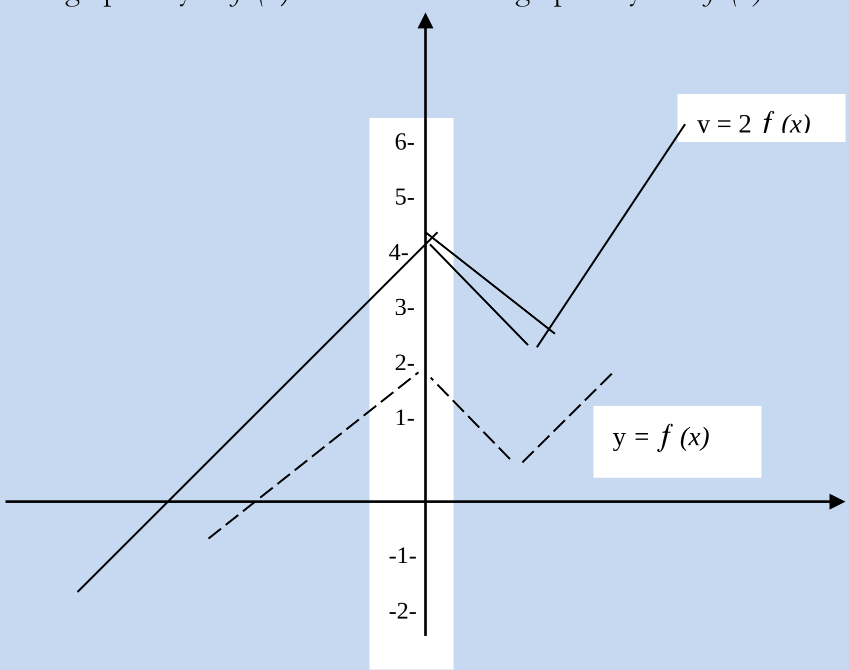
Example : Sketch the graph of $y = (-x)^3$



v. $y = \mathrm{k}\, f\,(x)$ This represents a stretch by a factor k in the direction of y.

Example

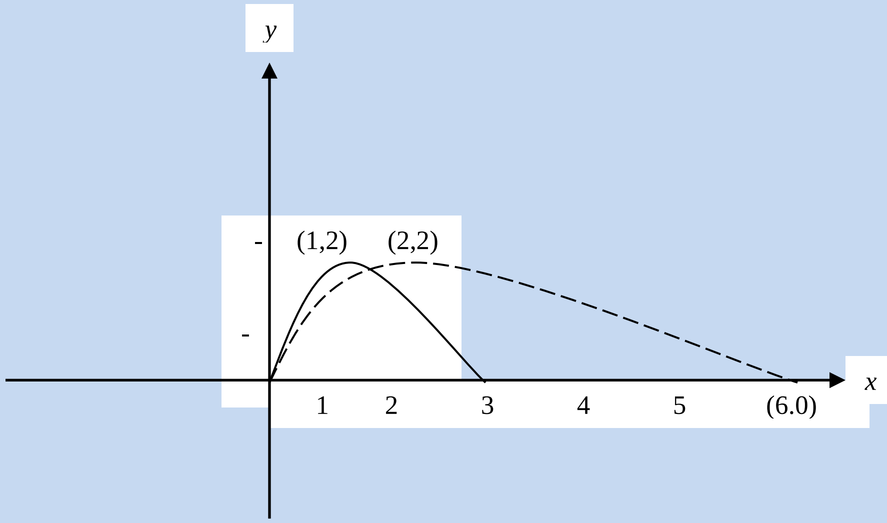
Given the graph of $y = f\,(x)$ below sketch the graph of $y = 2\,f\,(x)$



vi. $y = f\,(ax)$: This represents a shear by a factor of 1/a in the direction of x.

Example

Given the graph of $f\,(x)$ below sketch the graph of $y = f\,(2x)$



Example

a) A curve whose equation is $y = x^3$ undergoes the following successive transformations

1. A translation of 2 units in the direction of x
2. scaling parallel to the y-axis by a factor 5
3. A translation of 3 units in the direction of y

Give the equation of the resulting curve

b) Another curve undergoes the same transformations as above and the resulting function is $y = \dfrac{3x-1}{x-2}$

What is the original equation?

a) <u>Solution</u>

i. Translating by 2 in the direction of the x-axis gives us $y = (x-2)^3$
ii. The second transformation leads to $y = 5(x-2)^3$
iii. The final transformation gives us $y = 5(x-2)^3 + 3$

b) <u>Solution</u>

Method 1

3

$$\begin{array}{r|l} x-2 & 3x-1 \\ & \underline{3x-6} \\ & \quad\; 5 \end{array}$$

$$\frac{3x-1}{x-2} = 3 + \frac{5}{x-2}$$

From this, we deduce that the original function was y = $\frac{1}{x}$

Method 2 : In this case we reverse the transformations starting with the last one

$$y = \frac{3x-1}{x-2}$$

1. Translating by -3 units in the direction of y, we get

$$y = \frac{3x-1}{x-2}$$

$$= \frac{3x-1-3(x-2)}{x-2}$$

$$= \frac{5}{x-2}$$

2. Scaling parallel to the y-axis by a factor $\frac{1}{5}$

$$y = \frac{5}{x-2} \times \frac{1}{5}$$

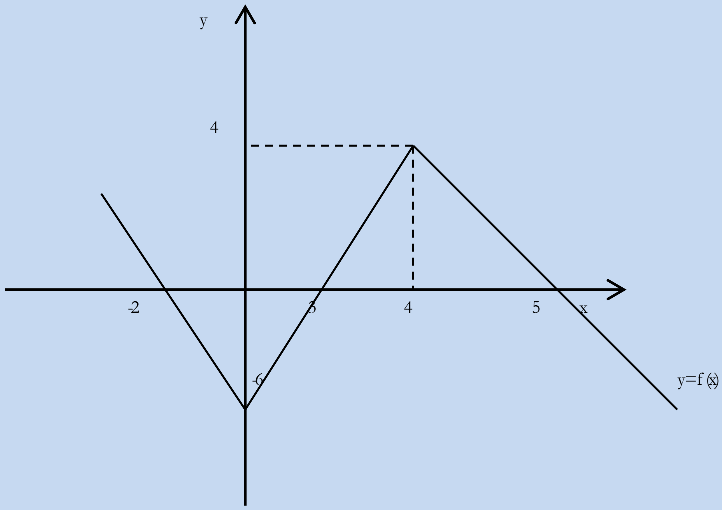
$$= \frac{1}{x-2}$$

3. Translating by -2 in the direction of x:

$$y = \frac{1}{x-2+2}$$

$$= \frac{1}{x}$$

Example



Given the above graph of $y = f(x)$ sketch the following graphs of

(a) $y = f(x-2)$
(b) $y = f(x) + 3$
(c) $y = 4f(x)$
(d) $y = f(-2x)$, showing clearly points of intersections.

Solution

1. $y = f(x-2)$ this is a translation $T = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$ i.e. only the x – value s are affected.

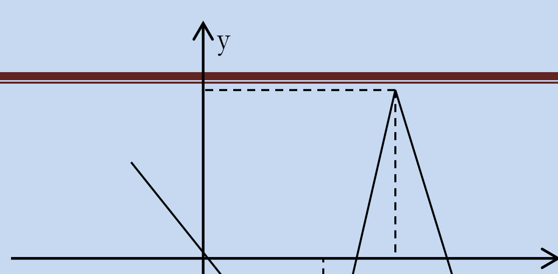
$x - 2 = -2$
$x = 0$ i.e. $-2 \rightarrow 0$

$x - 2 = 0$
$x = 2$ i.e. $0 \rightarrow 2$

$x - 2 = 3$
$x = 5$ i.e. $3 \rightarrow 5$

$x - 2 = 4$
$x = 6$ i.e. $4 \rightarrow 6$

$x - 2 = 5$
$x = 7$ i.e. $5 \rightarrow 7$



4

$y = f(x-2)$

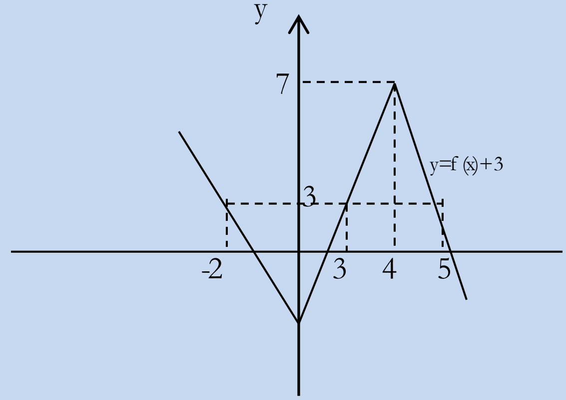
2

0 5 6 7 x

-6

The shape does not change.

b) $y = f(x) + 3$ this is a translation $T\begin{pmatrix}0\\3\end{pmatrix}$ i.e. only y–values change.



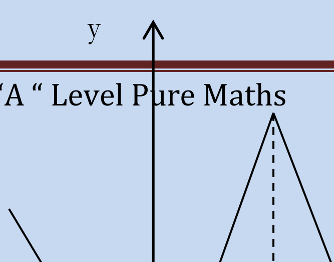
$y = 0 + 3 = 3$; $y = -6 + 3 = -3$
$0 \to 3$. ; $-6 \to -3$

$y = 4 + 3 = 7$
$4 \to 7$

The shape does not change..

c) $y = 4\,f(x)$. A stretch S.F in y–direction.
$y = 4\,f(-2) = 0$
$0 \to 0$
$y = 4f(0) = 4x - 6 = -24$
i.e $-6 \to -24$

$y = 4f(4) = 4 \times 4 = 16$
hence $4 \to 16$



16 - - - - - - - - -

$y = 4f(x)$

-2 3 4 5 x

-24

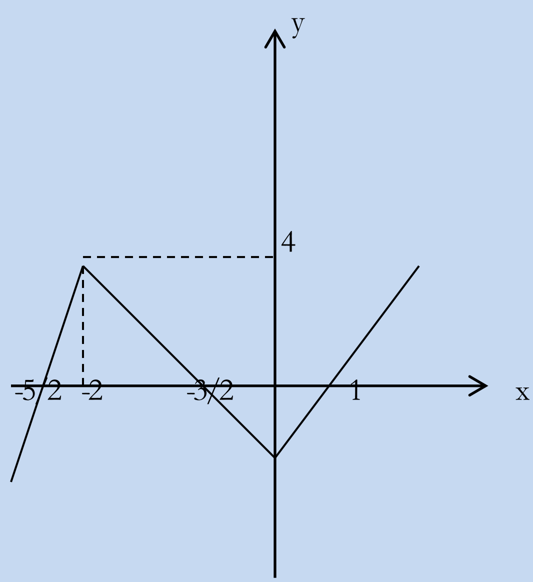
d) $y = f(-2x)$
$-2x = -2$
$x = 1$
i.e. $-2 \rightarrow 1$

$-2x = 0$
$x = 0$
i.e. $0 \rightarrow 0$

$-2x = 3$
$x = -3/2$
i.e. $3 \rightarrow -3/2$

$-2x = 4$
$x = -2$
i.e. $4 \rightarrow -2$

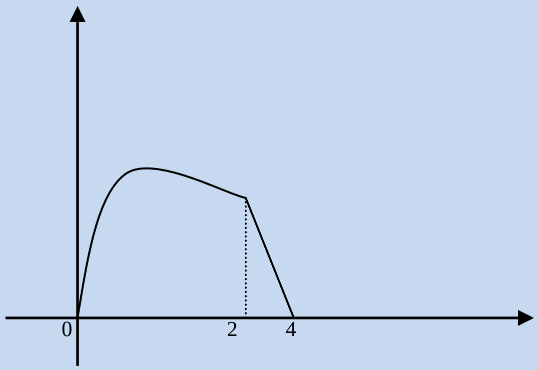
$-2x = 5$
$x = -5/2$
i.e. $5 \rightarrow -5/2$



Examination Type Questions

1. The function f is defined by f (x) = 3/x, x $\in \Re$, x > 0. The graph of g (x) is obtained by translating the graph of f (x) by – 4 units parallel to the x – axis, and the graph of h (x) is obtained by reflecting the graph of g (x) in the axis.
   (a) Sketch the graphs of f (x) and h (x) on the same set of axis.
   (b) Find the equations of g (x) and h (x).
   (c) Find an expression for the image of the function f (x) under a translation $\begin{pmatrix} p \\ q \end{pmatrix}$

2. The graph of y = f(x) is given below



Sketch the graphs of

a) y = f(x – 2) b) y = f (4 – x) c) y = f(2 – x)

CHAPTER 4

FUNCTIONS

OBJECTIVES

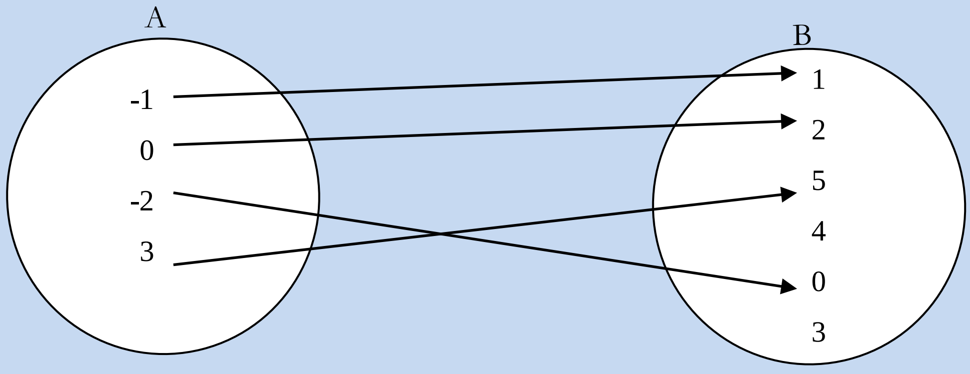
By the end of the chapter the student should be able to :

- State the domain and range of functions
- Compute universese of functions
- Identify the graphical relationship between the function and its universe
- Compute compound functions

Definition

A function is a law or instruction for producing a single number from another number. We say a function maps one number into another number.
Consider two non – empty sets A and B. A mapping from A to B is a rule which associates with each element of A an element of B.

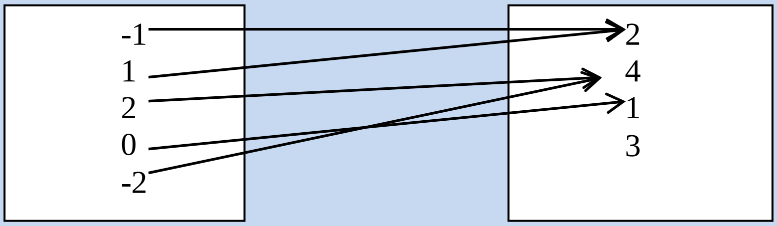
Case 1.



In this case each element of A maps to one and only one element of B. This is called a one to one mapping.

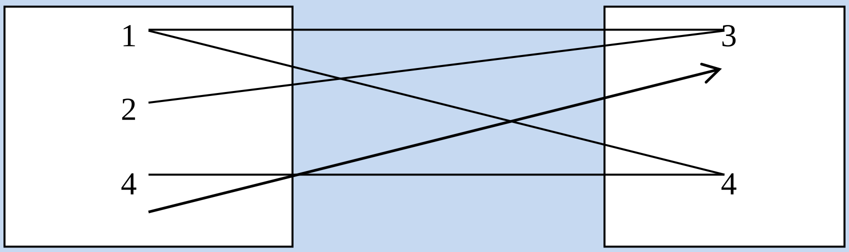
Case 2.

In this case two elements of A map to one element of B. This is a two to one mapping or simple many to one mapping.



Case. 3.

In this case one element of A maps to two elements of b. This is a many to many mapping.



A one to one or a many to one mapping is called a function. Hence cases 1 and 2 represent functions, whilst case 3 does not represent a function.

We denote rules, which associates each element of A to elements of B by small letter e.g. f, g, h etc.

Example: $f(x) = x + 2$ or $f: x \to x + 2$

Domain of a function

The set A is called the domain of the function. These are input values of x, such that f(x) is defined – "meaningful ".

Example

$f(x) = \sqrt{(x-2)}$ is "meaningful" for $x - 2 \geq 0$, hence all the real values of x for which $x - 2 \geq 0$ constitute the domain of the function $f(x)$

Example $f(x) = 1/(x-1)$, $f(x)$ is "meaningful" for

$x - 1 \neq 0$, hence $x \neq 1$, $x \in \Re$

i.e.all real values for which $x \neq 1$ constitute the domain.

The range of a function

The points of set B, where x has corresponding values, is called the range of f (x), i.e the range of f (x) is the set of all images of the pre images of the function.

For example;

Example: add 4 to a given number

Example: find the square root of a given number

These are written as:

(i) $f : x \rightarrow x + 4$: read as f is the function such that x is mapped onto x + 4

(ii) $f : x \rightarrow \sqrt{x}$: read as f is the function such that x is mapped onto $\sqrt{x}$

The output values could be written as:

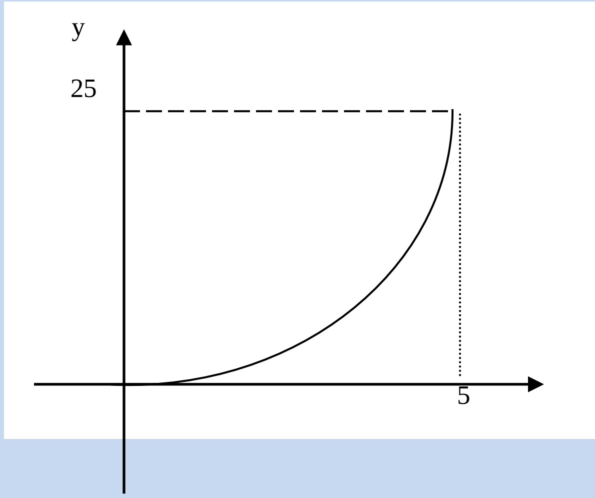
$f(x) = x + 4$
$f(x) = \sqrt{x}$

Example
Find the domain $f : x \rightarrow 1/(x + 2)$
Solution
$x + 2 \neq 0$
$x \neq -2$

Example $f : x \rightarrow x^2$, $1 < x < 10$
For each domain there is a corresponding set of output values. This set is called the image set on the range.

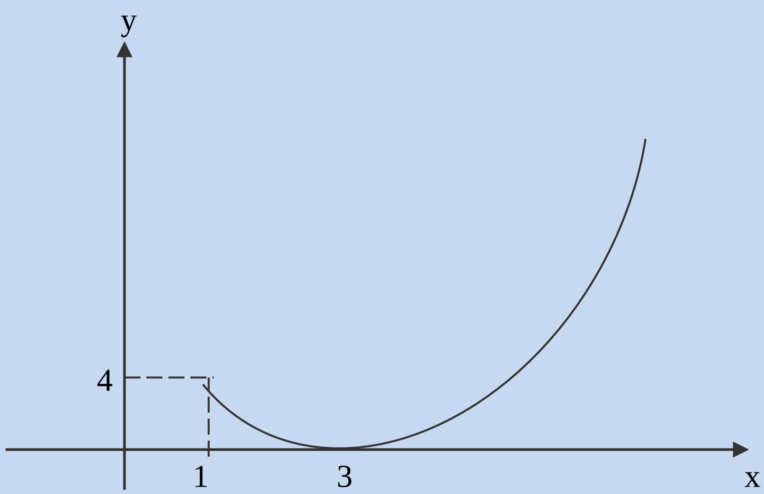
Example
Sketch the following functions stating their range

(i) $f: x \rightarrow x^2$ $0 \leq x \leq 5$



The range is $0 \leq f \leq 25$

(ii) $f : x \rightarrow (x - 3)^2$, $x \in \text{IR}$, $x \geq 1$



The range is $f \geq 0$

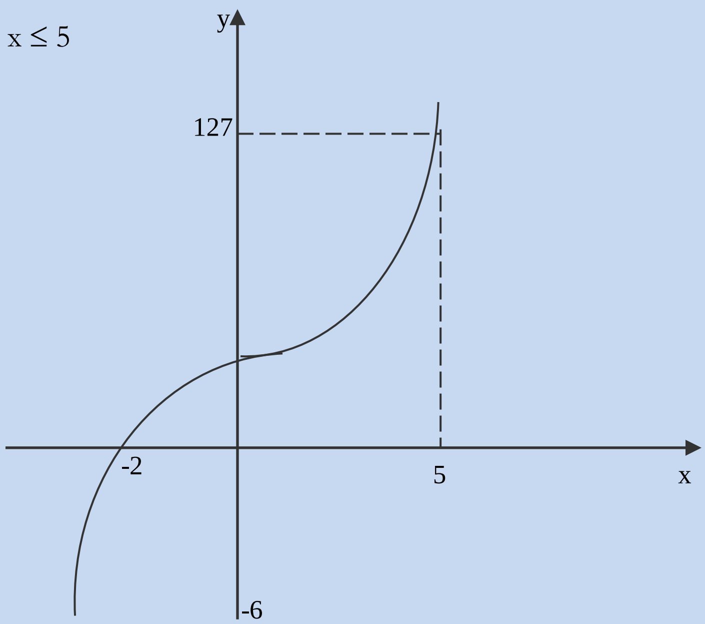
Example

(iii) $f : x \rightarrow 1/(x-3)$, $x \neq 3$, $x \in \Re$

Range: $y \neq 0$, $y \in \Re$

The range is $f \neq 0$

(iv) $f : x \to x^3 + 2 \quad , -2 \leq x \leq 5$



The range is $-6 \leq f \leq 127$

Inverse Functions

The inverse of a function (written as f^{-1}) is a function that reverses or undoes the effect or mapping of a given function. It has the effect of mapping the range of a given function back onto its domain.

If $f : x \to 2x + 5$, find the inverse

$$\text{Let } y = 2x + 5$$

$$x = \frac{y-5}{2}$$

Interchanging x for y we get

$$y = \frac{x-5}{2} \quad \text{, i.e. } f^{-1} : x \to \frac{x-5}{2}$$

That is $f^- : x \longrightarrow (x + 3)^3$ find f^{-1}

Solution

$$y = (x + 3)^3$$

$$x = y^{1/3} - 3$$

$$f^{-1} \quad x \to x^{1/3} - 3$$

Not every function has an inverse. The inverse function exists if and only if the function is a one to one function.

Example: Does $f : x \to x^2$ have an inverse?

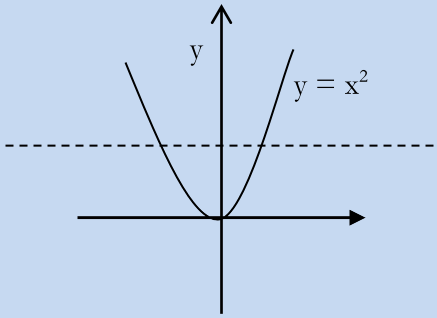
Solution

$y = x^2$

$x = \pm y$

$y = \pm x$

Testing for one to one function: draw a line parallel to the x – axis. If it crosses the graph more than once, the function is not one to one.



The function is one to one if for $f(x_1) = f(x_2)$ then $x_1 = x_2$.

The line crosses at two different points

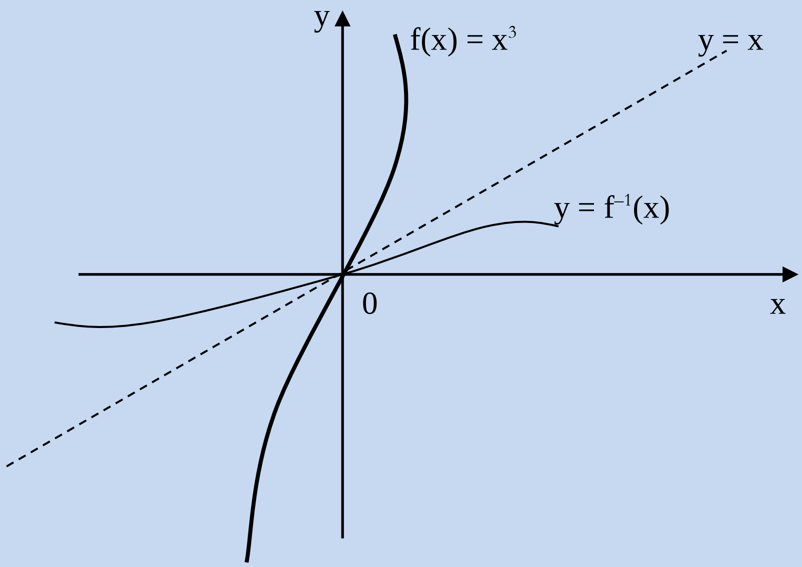
$f : x \to \pm x$ is not a function. Hence $f : x \to x^2$ has no inverse

$f(x_1) = f(x_2)$
$x_1^2 = x_2^2$
$x_1 = \pm\sqrt{x_2^2}$ hence: x_1 equals to the different points.

The Graphs of Functions and their Inverses

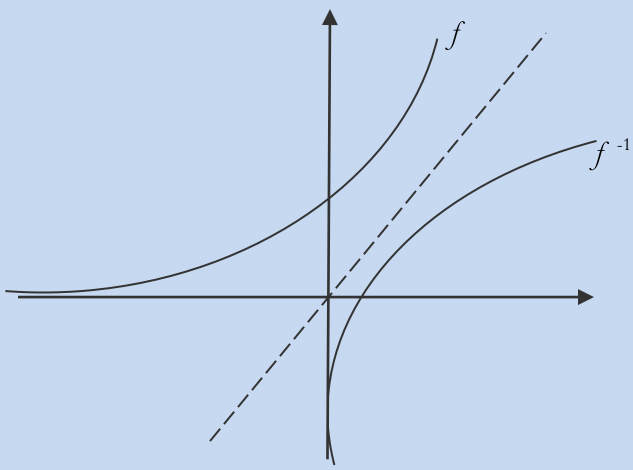
The graph of f^{-1} is found by reflecting the graph of f about the line $y = x$ and vice versa

Example
If $f : \to x^3$ sketch the graph of f^{-1}



Example

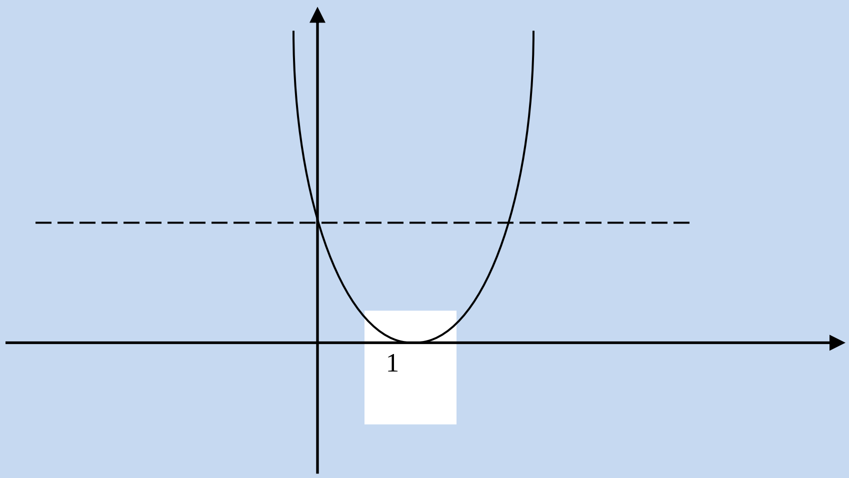
If $f : x \to e^x$, , Sketch the graph of f^{-1}



Example

Is $f : x \to (x - 1)^2$ a one to one function?

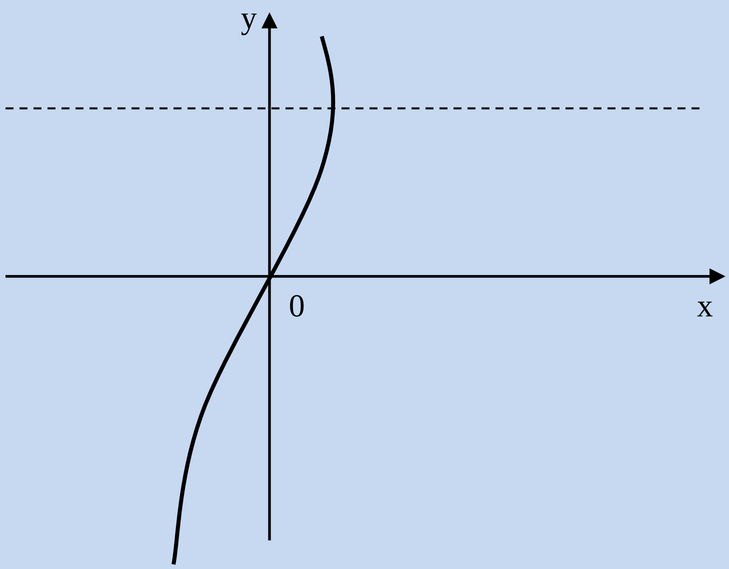
Solution



$f : x \to (x-1)^2$ is not a one to one function

Example

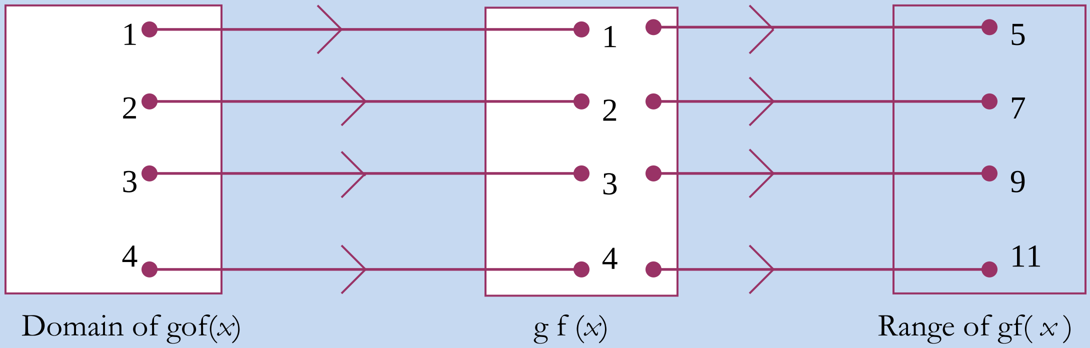
Is $f : x \to x^3$



f is a one to one function

Compound / Composite Functions

Consider the two functions f (x) and g (x) = 2x + 3 where the domain of f is { 1,2,3,4} and the domain of g is the range of f.



The function of g of (x) whose domain is that of f and range is that of g, is called a composite function. It is denoted by either g f(x), g of (x), or g [f(x)].

Hence: g f(x) = g f(x) = g[f(x)]
g f(x) = 2x + 3.

g.f(x) exist if Domain g $\cap$ Range f $\neq \phi$

Now: fog (x) = f [g (x)]
f.f(x) = f^2(x) = f of (x) = f[f(x)]

Example

Functions f(x) and g(x) are designed by f: x $\rightarrow \frac{3}{x}$; g: x$\rightarrow$ x + 5

a) Write down an expression for f g (x) and hence solve the equation f g (x) = 2.
b) Write down an expression for g f(x) and hence solve the equation of g f(x) = 7.

Solution

a) f:g(x) = f [g(x)] = $\dfrac{x}{x+3}$

Hence $\dfrac{3}{x+5} = 2$

3= 2x + 10

2x = -7

x = -3.5

b) g f (x) = g[f(x)]

$= \dfrac{x}{3} + 5$

hence $\dfrac{x}{3}$ +5=7

i.e. x + 15 =21

$\therefore$ x=6

Example

Functions p(x) and q(x) are defined such that q(x) = x^2 + 7, and qp(x) = $9x^2 + 6x + 8$. find possible expressions for p(x)

Solution.

qp(x) = q[p(x)] = $9x^2 + 6x + 8$, from the composite function $9x^2 + 6x + 8$. p(x) = a x+ b. We find the value of a and b.

Now: $a^2x^2 + 2abx + b^2 + 7 = 9x^2 + 6x + 8$, equating coefficients.

$a^2 = 9$ i.e $a = \pm 3$

$b^2 + 7 = 8$

$b^2 = 1$, hence p(x) $\pm$ (3x + 1)

b = ± 1.

Example

The function f is designed by f (x) , $-\dfrac{x}{x-1}$, $x \neq 1$

(i) Find and simplify an expression for f.f(x).

(ii) Hence or otherwise, find an expression for f^{-1} (x).

(iii) State the range of f.

(iv) Suggest the rule for f^{n}(x).

Solution.

(i) f.f (x) = f (f (x)) = x

(ii) Since f.f(x) = x, hence $f^{-1}(x) = \dfrac{x}{x-1}$

(iii) The range of f is the domain of $f^{-1}(x) = \dfrac{x}{x-1}$ hence, range f: $y \neq 1$, $y \in \Re$

(iv) f·f(x) = x

$f_0 \cdot f \cdot (x) = \dfrac{x}{x-1}$

$$f \circ f \circ f \circ f\,(x) = x$$

$$f^5(x) = \frac{x}{x-1}$$

Hence. $$f^n(x) = \begin{cases} x, & n \text{ even} \\ \dfrac{x}{x-1}, & n \text{ odd} \end{cases}$$

Examination Type Questions

1. The functions f and g are defined with their respective domains by:

   f: $x \to \dfrac{4}{3x-1}$ $x \varepsilon \Re$, $x \neq 1/3$ and g: $x \to 2x^2 + 3$, $x \varepsilon \Re$,

   (a) Find the values of x for which f(x) = 3x
   (b) Find the range of g.
   (c) The domain of fg is 1$\Re$. Find fg(x) and state its range.
   (d) Find $f^{-1}(x)$, the inverse of f(x)

2. The functions f and g are defined by f: $x \to x^2 - 9$, $x \varepsilon \Re$,

   (a) Find ff(x). Find all the values of x for which ff(x) = 40.
   (b) Find gf(x). Sketch the graph of gf(x). Hence or other wise, solve the equation gf(x) = 5x

3. Given that f (x) = $x^2 + 3$, g(x) = 2x + a and fg(x) = $4x^2 - 8x + 7$, calculate the value of the constant a

4. i) Show that $x^2 + 4x + 7 = (x + 2)^2 + a$, where a is to be determined.
   ii) Sketch the graph of $y = x^2 + 4x + 7$,giving the equation of the axis of symmetry and the coordinates of its vertex

   The function f is defined by f : $x \to x^2 + 4x + 7$ and has its domain the set of all real numbers.
   iii) Find the range of f.
   iv) Explain, with reference to your sketch, why f has no inverse with its given domain. Suggest a domain for f for which it has an inverse.

5. The function f with domain $\{x: x \geq 0\}$ is defined by f(x) = 8 / (x + 2)
   (a) Sketch the graph of f and state the range of f
   (b) Find $f^{-1}(x)$, where f^{-1} denotes the inverse of f
   (c) Calculate the value of x for which $f(x) = f^{-1}(x)$.

6. The function f is given by

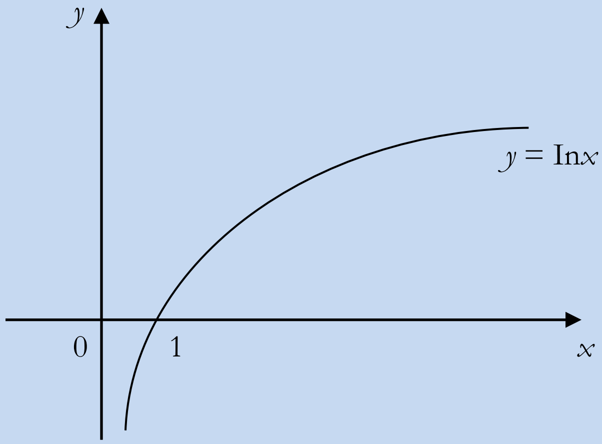
   $f{:}x \longrightarrow x^2 - 8x$, $x \in \Re$, $x \leq 4$

(i) Sketch the graph of f(x) .

(ii) Determine the range of f(x)

(iii) Find the value of x for which f(x) =20

(iv) Find $f^{-1}(x)$



7. The diagram shows the graph of y =1nx, sketch the graph of y=1n(x +a) , where a is a constant such that a >1, and state the co –ordinates of the points of intersection of the graph with the axis.

8. Functions f and g are defined by

f: x $\longrightarrow \dfrac{3}{x+3}$, $x \in \Re$, $x \geq 0$

g : x x +1 $x \in \Re$, ; $x \geq 0$

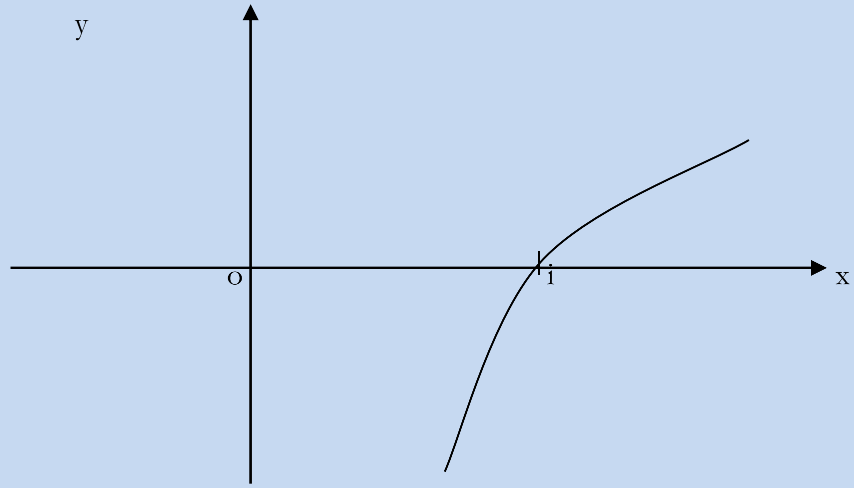
Show that.

gf : x:$\rightarrow \dfrac{x+6}{x+3}$, $x \in \Re$, ; $x \geq 0$

Express fg in a similar form.

Find $(fg)^{-1}(x)$

4.



The diagram shows the graph of $y = 1nx$, Sketch the graph of $y = 1n\ (x+a)$, where a is a constant such that $a > 1$, and state the co-ordinates of the points of intersection of the graph with the axes.

CHAPTER 5

INEQUALITIES, IDENTITIES, EQUATIONS AND POLYNOMIALS

OBJECTIVES
By the end of the chapter the student should be able to :

- Solve disguised quadratic equations
- Solve simultaneous equations where one is at least quadratic
- Solve quadratic inequalities
- Solve rational inequalities
- Factorise polynomials using the remainder theorem

Equations

Disguised quadratic equations
We will now look at equations which do not appear to be quadratic, but in fact are.

Example
Solve for x :$x^4 - 13x^2 + 36 = 0$

Solution

Let $y = x^2$, then $y^2 - 13y + 36 = 0$
$(y - 4)(y - 9) = 0$
$y = 4$ or $y = 9$
hence $x^2 = 4$ or $x^2 = 9$
$x = \pm 2$ or $x = \pm 3$

Example

Solve for x :$x^6 = 8 + 2x^3$

Solution
Letting $y = x^3$, then $y^2 = 8 + 2y$
i.e $y^2 - 2y - 8 = 0$
$(y - 4)(y + 2) = 0$
$y = 4$ or $y - 2$
hence $x^2 = 4$ or $x^2 = - 2$
We see that $x^2 = - 2$ gives no real solutions.
However, $x^2 = 4$ gives $x = \pm 2$

Practice Questions

In each case solve for x

1. $x^4 - 2x^2 - 3 = 0$
2. $x - 6\sqrt{x} + 5 = 0$
3. $x^8 + 16 = 17x^4$

Simultaneous Equations

We now look at simultaneous equations where one is at most quadratic.

Example

Solve the simultaneous equations $y = 3x - 4$ and $y = x^2 - 4x + 6$

<u>Solution</u>

We eliminate y by substituting in the second equation $3x - 4$ for y. to obtain

$3x - 4 = x^2 - 4x + 6$

$x^2 - 7x + 10 = 0$

$(x - 5)(x - 2) = 0$

$x = 5$ or $x = 2$

When $x = 2$: $y = 3(2) - 4 = 2$

When $x = 5$: $y = 3(5) - 4 = 11$

The solutions are $x = 2$, $y = 2$ and $x = 5$, $y = 11$

Example

Solve the simultaneous equations $x + y = 2$, $x^2 + 2y^2 = 11$

<u>Solution</u>

$x + y = 2 \Rightarrow y = 2 - x$ …(1)

$x^2 + 2y^2 = 11$……(2)

Substituting equation (1) in (2) gives

$x^2 + 2(2 - x)^2 = 11$

$x^2 + 2(4 - 4x + x^2) = 11$

$x^2 + 8 - 8x + 2x^2 - 11 = 0$

$3x^2 - 8x - 3 = 0$

$(3x + 1)(x - 3) = 0$

$x = -1/3$ or $x = 3$

When, $x = -\frac{1}{3}$, $y = 2 - (-\frac{1}{3}) = 7/3$

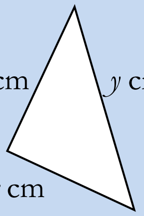
When, $x = 3$, $y = 2 - 3 = -1$

$S = \{(-\frac{1}{3}, \frac{7}{3}), (3, -1)\}$

Practice Questions

1. Solve each of the following pairs of simultaneous equations
   a) $y + 2x = 3$ $\quad y^2 + xy = 13 - 16x$
   b) $x + y = 5$ $\quad xy = 6$
2. A right- angled triangle has sides of length x cm, y cm and $(y - 2)$cm, as shown in the diagram. Given that the perimeter of the triangles is 60cm and its area is 120cm^2, derive the equations.

   $x + 2y = 62$ $\quad$ [1]
   $xy - 2x = 240$ $\quad$ [2]



   Find the pairs of values of x and y which satisfy these equations. Which of the answers works in practice.

Inequalities

1.1.2 Quadratic Inequalities

Definition: Every inequality, which patterns in either of the forms:
$ax^2 + bx + c \geq 0$ or $ax^2 + bx + c \leq 0$, where $a \neq 0$ is called a quadratic inequality.
The following properties help to solve quadratic inequalities

1. $a.b \geq 0 \Leftrightarrow a \geq 0$ and $b \geq 0$ or $a \leq 0$ and $b \leq 0$
2. $a.b \leq 0 \Leftrightarrow a \geq 0$ and $b \leq 0$ or $a \leq 0$ and $b \geq 0$

Example

Solve for x: $x^2 + 2x - 8 \geq 0$

Solution

and b^2

$(x + 4)(x - 2) \geq 0$

hence, $x + 4 \geq 0$ and $x - 2 \geq 0$ or $x + 4 \leq 0$ and $x - 2 \leq 0$

$x \geq -4$ and $x \geq 2$ or $x \leq -4$ and $x \leq 2$

$x \geq 2$ or $x \leq -4$

Example

Solve for x: $x^2 - 3x - 10 < 0$

Solution

$x^2 - 3x - 10 < 0$

$x + 2 > 0$ and $x - 5 < 0$ or $x + 2 < 0$ and $x - 5 > 0$

$x > -2$ and $x < 5$ or $x < -2$ and $x > 5$

$x > -2$ and $x < 5$

Given $(x - a)(x - b)$ with $a < b$, the solution : $(x - a)(x - b) > 0$ is $x < a$ or $x > b$ and the solution of $(x - a)(x - b) < 0$ is $a < x < b$.

Example
Solve $2x^2 + 5x > 12$

Solution

$2x^2 + 5x - 12 > 0$

$(2x - 3)(x - 4) > 0$

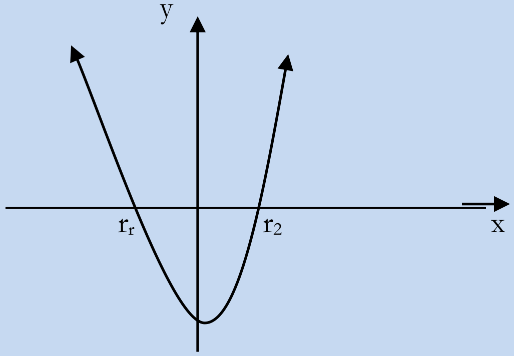
$2(x - 3/2)(x - (-4)) > 0$

$x < -4$ or $x > 3/2$

However, a more elegant method is abound. From the notion of the graphs of quadratic functions, we know that:

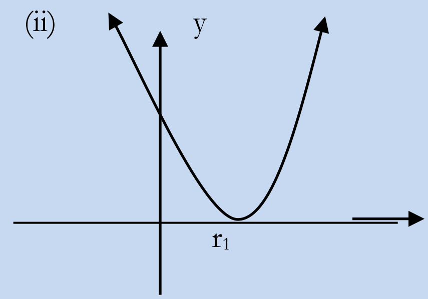
(a) if $a > 0$ the graph of the quadratic functions takes the following three forms

(i)



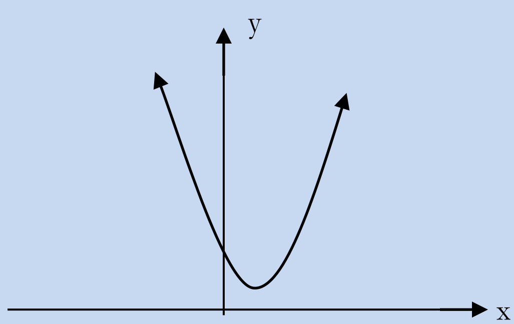
$b^2 - 4ac > 0$

(ii)



$b^2 - 4ac \geq 0$

(iii)



$b^2 - 4ac < 0$

For part (i) $f(x) > 0$ for $x < r_1 \cup x > r_2$
$f(x) < 0$ for $r_1 < x < r_2$

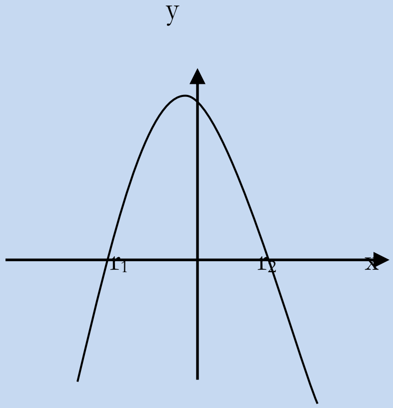
For part (ii) $f(x) \geq 0, x \in \Re / \{r_1\}$

$f(x) < 0$ for no real value of x
$f(x) = 0$ for $x = r_1$

For part (iii) $f(x) > 0$ for $x \in \Re$
$f(x) < 0$ for no value of x.

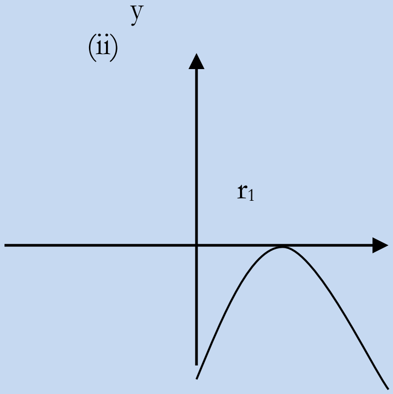
(b) If a < 0 the graph of the quadratic functions takes the following three forms.

(i)



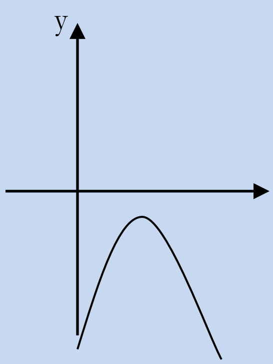
$b^2 - 4ac \geq 0$

(ii)



$b^2 - 4ac = 0$

(iii)



For part (i) $f(x) > 0$ for $r_1 < x < r_2$
$f(x) < 0$ for $x < r_1$ or $x > r_2$

For part (ii) $f(x) < 0, x \in \Re / \{r_1\}$
$f(x) > 0$ for no real value of x
$f(x) = 0$ for $x = r_1$

For part (iii) $f(x) < 0$ for $x \in \Re$

$f(x) > 0$ for no value of x.

Summary

1. If $a > 0$ and $b^2 - 4ac > 0$ the sign of the quadratic expression, $ax^2 + bx + c$ changes over the roots of the quadratic equation, $ax^2 + bx + c = 0$ as shown in the number line below

+(plus) - (minus) +(plus)

2. . If $a > 0$ and $b^2 - 4ac = 0$ the sign of the quadratic expression, $ax^2 + bx + c$ changes over the roots of the quadratic equation, $ax^2 + bx + c = 0$ as shown in the number line below

+(plus) = (equals) +(plus)

3. . If $a > 0$ and $b^2 - 4ac > 0$ the sign of the quadratic expression, $ax^2 + bx + c$ changes over the roots of the quadratic equation, $ax^2 + bx + c = 0$ as shown in the number line below

+(plus)

4. . If $a < 0$ and $b^2 - 4ac > 0$ the sign of the quadratic expression, $ax^2 + bx + c$ changes over the roots of the quadratic equation, $ax^2 + bx + c = 0$ as shown in the number line below

- (minus) +(plus) - (minus)

5. If $a < 0$ and $b^2 - 4ac = 0$ the sign of the quadratic expression, $ax^2 + bx + c$ changes over the roots of the quadratic equation, $ax^2 + bx + c = 0$ as shown in the number line below

- (minus) =(equals) - (minus)

6. If $a < 0$ and $b^2 - 4ac < 0$ the sign of the quadratic expression, $ax^2 + bx + c$ changes over the roots of the quadratic equation, $ax^2 + bx + c = 0$ as shown in the number line below

- (minus)

Example

Solve for x

1. $(x + 4)(x - 3) < 0$
2. $(3 + x)(6 - x) \leq 0$
3. $x + 1 > 6x^2$
4. $5x^2 + 11x + 6 \geq 0$

1. $(x + 4)(x - 3) < 0$

$a = 1 > 0$ and $b^2 - 4ac = 49 > 0$,

hence,

$-4 < x < 3$

+ - +

- 4 3

2. $(3 + x)(6 - x) \le 0$

$a = -1 < 0$ and $b^2 - 4ac = 81 > 0$, hence,

$x \le -3$ or $x \ge 6$

3. $x + 1 > 6x^2$

$0 > 6x^2 - x - 1$

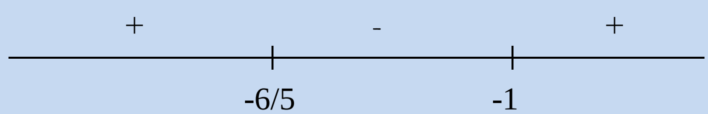
$6x^2 - x - 1 < 0$

$a = 6 > 0$ and $b^2 - 4ac = 25 > 0$, hence,

$-1/3 < x < 1/2$

4. $5x^2 + 11x + 6 \ge 0$

$a = 5 > 0$ and $b^2 - 4ac = 1 > 0$, hence,



$x \le -6/5$ or $x \ge -1$

5 $5x^2 + 5x + 6 < 0$

$a = 5 > 0$ and $b^2 - 4ac = -95 < 0$, hence, $S = \{\ \}$

<u>Rational Inequalities</u>

An inequality of the form $f(x) / g(x) < 0$ (or $>$), where $g(x) \neq 0$ is called a rational inequality, e.g. $\dfrac{3x+3}{2x+1} > 3$

Let $f(x) / g(x) < 0$, hence multiplication by $(g(x))^2$ does not affect the inequality sign.

Example

Solve for x

$\dfrac{x-3}{x+1} > 2$

Solution

$$\frac{x-3}{x+1} > 2 \Rightarrow \frac{x-3}{x+1} - 2 > 0 \Rightarrow_{-} \frac{x-3-2(x+1) \succ 2}{x+1}$$

$$\Rightarrow \frac{x-5}{x+1} > 0 \Rightarrow (-x - 5)(x + 1) > 0$$

The critical points are - 5 and – 1

- + -

-5 -1

-5 < x < -1

Polynomials

Definition

A polynomial is an algebraic expression of the form $a_0 + a_1x + a_2x^2 + a_3x^3 + \ldots + a_nx^n$, where , a_i are constants.

Polynomials can be added, multiplied or subtracted using the methods learnt at "O" level. In this chapter we will only be concerned with the division of the polynomials

1.1.3 Division of Polynomials

One way of writing if 17 divided by 3 is $\frac{17}{3} = 5$ remainder 2

OR

$$19 = 3 \underbrace{\times 5}_{\text{called the quotient}} + \underbrace{2}_{\text{called the remainder}}$$

We apply the same method to polynomial division.

Example

Find the quotient and the remainder when the polynomial $x^3 + 4x^2 - 3x + 2$ is divided by $x - 1$.

Solution

Writing $x^3 + 4x^2 - 3x + 2$ in terms of a quotient and a remainder gives an expression of the form.

$$x^2 + 4x^2 - 3x + 2 \equiv (x - 1)\underbrace{(\quad)}_{\textbf{Quotient}} + \underbrace{(\qquad)}_{\textbf{Remainder}}.$$

We can see, that the quotient, q (x) is of the form $q(x) = ax^2 + bx + c$. Therefore,

$x^3 + 4x^2 - 3x + 2 \equiv (x-1)(ax^2 + bx + c) + r$

Expanding and collecting like terms.

$$x^3 + 4x^2 - 3x + 2 \equiv ax^3 + bx^2 + cx - ax^2 - bx - c + r$$
$$\equiv ax^3 + (b-a)x^2 + (c-b)x + r - c$$

Comparing the coefficients of the x^3 terms gives: $a = 1$
Comparing the coefficients of the x^2 terms gives:

$$5 = b - a \ldots(1)$$

Comparing the coefficients of the x terms gives:

$$-3 = c - b. \ldots(2)$$

Comparing the constant terms gives

$$1 = r - c \ldots(3)$$

$$\begin{aligned} b - a &= 5 \\ c - b &= -3 \\ r - c &= 1 \\ \hline b &= 4 + a \\ b &= 4 + 1 \\ \underline{b} &\underline{= 5} \end{aligned} \qquad\qquad \begin{aligned} b - c &= 3 \\ -c &= 3 - b \\ \underline{c} &\underline{= 2} \\ \\ r - c &= 2 \\ r &= 2 + c \\ \underline{r} &\underline{= 4} \end{aligned}$$

Therefore $q(x) = x^2 + 5x + 2$ and the remainder is 4. Now,
let $f(x) = x^3 + 4x^2 - 3x + 2$,substituting $x = 1$ in f (x), we obtain: $f(1) = 4$.
It is not a coincidence that the remainder is equals f (1).

Alternatively, we can use long division.

$$\begin{array}{r|l} & \quad x^2 + 5x + 2 \\ \hline x - 1 & x^3 + 4x^2 - 3x + 2 \\ & \underline{x^3 - x^2} \text{ (subtract)} \\ & \qquad 5x^2 - 3x + 2 \text{ (subtract)} \\ & \qquad \underline{5x^2 - 5x} \\ & \qquad\qquad 2x + 2 \\ & \qquad\qquad \underline{2x - 2} \text{ (subtract)} \\ & \qquad\qquad\quad \underline{4} \end{array}$$

hence, $q(x) = x^2 + 5x + 2$ and the remainder: $r = 4$
Let f (x) be a polynomial of degree n, and $g(x) = ax + b$ be the divisor and q (x) a polynomial of degree $(n-1)$, be the quotient and r, the remainder, hence:

$f(x) = g(x) . q(x) = r.$

If $g(x) = 0$ i.e. $ax + b = 0$, then $x = -b/a$, $a \neq 0$
Hence,$f(-b/a) = 0 . q(-b/a) + r$

Then:

$$\boxed{f(-b/a) = r}$$

Note:
1.f (- b/a) is the remainder, if the polynomial f (x) is divided by ax +b
2.If f (-b/a) = 0, then ax + b is the factor of the polynomial f (x)

The Remainder Theorem

If the polynomial f (x) is divided by ax + b, the remainder is f (- b/a)

Example

The expression $2x^3 - 3x^2 + ax - 5$ gives a remainder of 17 when divided by $x - 2$. Find the value of the constant a.

Solution

$f(x) = 2x^3 - 3x^2 + ax - 5$
$x - 2 = 0$
$x = 2$
$f(2) = 17$
$2(2)^3 - 3(2)^2 + a(2) - 5 = 17$
$16 - 12 - 2a - 5 = 17$
$-2a = 18$
$\underline{a = -9}$

Example

The expression $x^3 + px^2 + qx - 10$ is divisible by $x - 2$, and leaves a remainder of 5 when divided by $x + 3$. Find the values of the constants p and q

Solution

$f(x) = x^3 + px^2 + qx - 10$
$f(2) = 0$
$(2)^3 + p(2)^2 + q(2) - 10 = 0$
$8 + 4p + 2q - 10 = 0$
$2p + q = 1$ …..**(1)**
$f(-3) = 5$
$(-3)^3 + p(-3)^2 + q(-3) - 10 = 5$
$-27 + 9p - 3q - 10 = 5$
$9p - 3q = 42$
$3p - q = 14$ …..**(2)**

$2p + q = 1$
$\underline{3p - q = 14}$
$5p = 15$
$\underline{p = 3}$ $2(3) + q = 1$, i.e. $\underline{q = -5}$

Example

Show that the equation $x^3 - 5x^2 + 2x - 10 = 0$ has only one real solution, and state its value.

Solution

Let $f(x) = x^3 - 5x^2 + 2x - 10$

Try, factors of -10, i.e. $\pm 1, \pm 2, \pm 5, \pm 10$

$$
\begin{array}{r|l}
 & x^2 + 2 \\ \hline
x - 5 & x^3 - 5x^2 + 2x - 10 \\
 & \underline{x^3 - 5x^2} \\
 & \qquad\quad 2x - 10 \\
 & \qquad\quad 2x - 10 \\
 & \qquad\quad \overline{\overline{}}
\end{array}
$$

Hence $x^3 - 5x^2 + 2x - 10 \equiv (x - 5)(x^2 + 2)$
i.e. $(x - 5)(x^2 + 2) = 0$
$x - 5 = 0$ or $x^2 + 2 \neq 0$
$\underline{x = 5}$ $\quad$ <u>is the only root</u>

Example

$f(x) \equiv 2x^3 - 9x^2 + 7x + 6$

(a) Factorise $f(x)$ into its linear factors
(b) Solve the equation $f(x) = 0$
(c) Solve the in equation $f(x) \leq 0$

Solution

(a) $f(x) \equiv 2x^3 - 9x^2 + 7x + 6$
Try the factors of 6, $\pm 1, \pm 2, \pm 3, \pm 6$
$f(2) = 2^4 - 9 \times 4 + 7 \times 2 + 6$
$f(2) = 0$

hence : $x - 2$ is a factor.
$f(3) = 2(3)^3 - 9(3^2) + 7(3) + 6$
$f(3) = 0$: $x - 3$ is a factor. Therefore $(x - 2)(x - 3)$ is also a factor i.e $x^2 - 5x + 6$

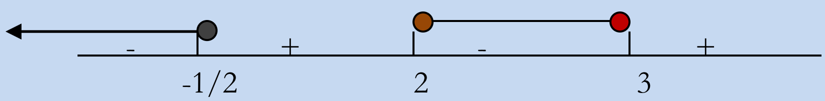
$$
\begin{array}{r|l}
 & \quad 2x + 1 \\ \hline
5x^2 - 5x + 6 & 2x^3 - 9x^2 + 7x + 6 \\
 & \underline{2x^3 - 10x^2 + 12x} \\
 & \qquad\quad x^2 - 5x + 6 \\
 & \qquad\quad \underline{\underline{x^2 - 5x + 6}}
\end{array}
$$

b) $f(x) = 0$ i.e. $(2x+1)(x-2)(x-3) = 0$

$2x+1 = 0$ or $x-2 = 0$ or $x-3 = 0$

$x - \frac{1}{2}$ or $x = 2$ or $x = 3$

c) $f(x) \leq 0$: . $(2x+1)(x-2)(x-3) = 0$



$S = \{ x: x \leq -\frac{1}{2} \text{ or } 2 \leq x \leq 3\}$

Examination Type Questions

1. Show that $(x-2)$ is a factor of $x^3 - 9x^2 + 26x - 24$. Find the set of values of x for which $x^3 - 9x^2 + 26x - 24 < 0$

2. The cubic polynomial $x^3 + Ax - 12$ is exactly divisible by $x + 3$. Find the constant A, and solve the equation $x^3 + Ax - 12 = 0$ for this value of A.

3. Given that $f(x) = mx^3 + (m+n)x^2 + (m+n)x + 2$ is exactly divisible by $(x+1)$, express n in terms of m. Show that,if in this case, the equation $f(x) = 0$ has only one real root then $m^2 - 6m + 4 < 0$

4. Show that $x + 1$ is a factor of $x^3 - 3x^2 + 4$. Hence solve the equation $x^3 - 3x^2 + 4 = 0$.

Given that $x = 3^y$, show that $x^3 - 3x^2 + 4 = 3^{3y} - 3^{2y+1} + 4$

Hence, solve the equation $3^{3y} - 3^{2y+1} + 4 = 0$

Giving your answer to 3d.p.

5. A function is defined as $f(x) = 2x^4 - 3x^3 - 12x^2 + 7x + 6$

(i) Find $f(1)$ and $f(-2)$, hence state a factor of $f(x)$.

(ii) Express $f(x)$ in the form $(x+a)(x+b)(cx^2 + dx + e)$ where a, b, c, d, and e are constants to be determined.

(iii) Solve the equation $f(x) = 0$

(iv) Solve the inequality $f(x) \leq 0$

(v) Without further calculations, sketch the graph of $y = f(x)$.

6. The cubic polynomial $x^3 + Bx - 9$ is exactly divisible by $x + 2$. Find the constant B, and solve the equation $x^3 + Bx - 9$ for this value of B.

7. Solve the inequality

$$\frac{3x+1}{x-1} > \frac{3}{x-2}$$

8. Solve $y^2 - 7y + 10 = 0$

Hence find the solutions to $(x^2+1)^2 - 7(x^2+1) + 10 = 0$

9. In order to make a new type of beer, a brewer decides to mix *x* kg of malt with *y* kg of hops, in such a way that *x* and *y* satisfy the following equations:

$x + y = 8$ $\qquad\qquad$ $x - y = 24/x$

Find the pairs of values of x and y which satisfy these equations. Which of these answers can the brewer use in practice?

10. If $x + 2$ and $2x - 1$ are factors of $fc(x) = 2x^3 + ax^2 + bx + 6$, find *a* and *b* determine the third factor.

11. If $x + 2$ and $2x - 1$ are factors of $f(x) = 2x^3 + ax^2 + bx + 6$, find a and b and determine the third factor.

12. Show that both $(x - \sqrt{3})$ and $(x + \sqrt{3})$ are factors of $x^4 + x^3 - x^2 - 3x - 6$. Hence write down one quadratic factor of $x^4 + x^3 - x^2 - 3x - 6$, and find a second quadratic factor of this polynomial.

13. Express $2x^2 + 5x + 4$ in the form $a(x + b)^2 + c$, stating the numerical values of a, b and c. Hence, or otherwise, write down the co – ordinates of the minimum point on the graph of $y = 2x^2 + 5x + 4$

14. Express in partial fractions

$$\frac{x^2 + 3x + 1}{(x + 1)(x + 2)}$$

15. Solve the simultaneous equations

$2x + 3y = 5$

$y^2 - yx = 5$

CHAPTER 6

MODULUS FUNCTIONS.

OBJECTIVES

By the end of the chapter the student should be able to :

- Define the basic modullus functions $|x|$
- Sketch graphs of modulus functions
- Solve modulus equations
- Solve modululs inequalities

Modulus functions

The term modulus means the size or magnitude or absolute value

So $\quad |-8| = 8$

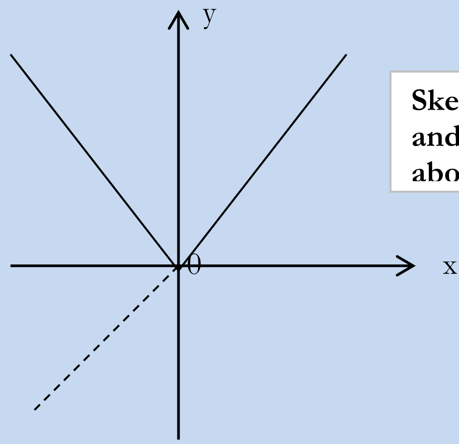
and $\quad |-8| = 8$

The graph of y= $|\,f(x)\,|$

To sketch the graph of y = $|\,f(x)\,|$, we should first sketch the graph of y = $f(x)$ and then reflect the negative part about the x-axis all the parts of this graph which lie below the x-axis.

The modulus function is given by the equation y = | x |

The graph.



Sketch the graph of y = x and reflect its negative point about the x – axis.

Def: $|x| = \begin{cases} x & \text{for } x \geq 0 \\ -x & \text{for } x < 0 \end{cases}$

Properties.

Domain: $x \in \Re^*$

Range: $y \geq 0$

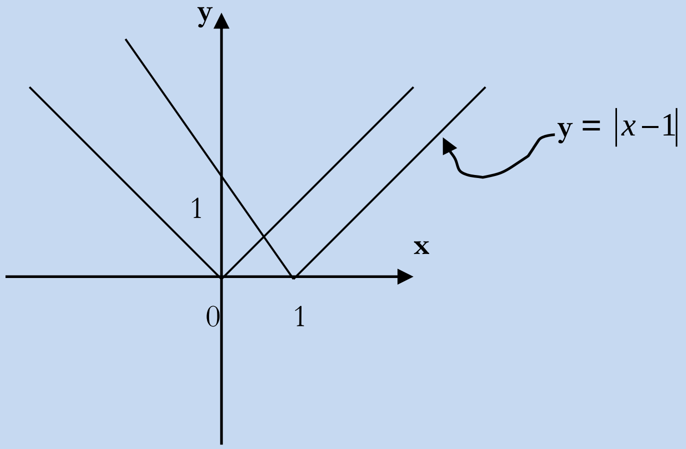
Min point (0,0)

Not one to one, hence does not posses an inverse.

Example Sketch the graph of $y = |x-1|$

Solution

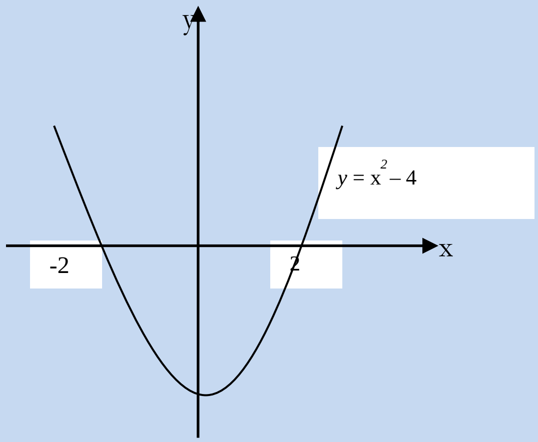
Sketch the graph of $y = |x|$ and translate it through 1 unit in the positive x – direction, the resultant graph is the graph of $y = |x-1|$



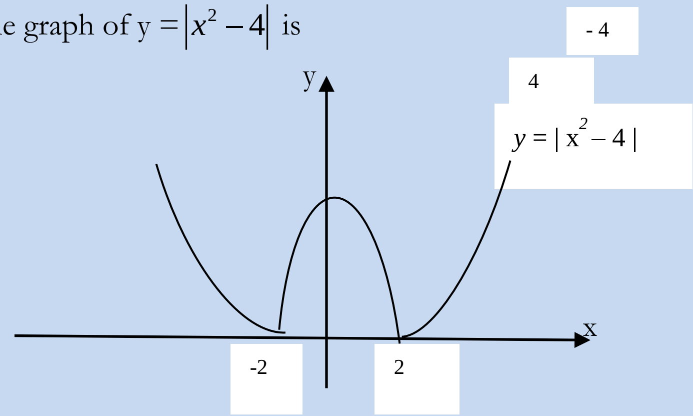
Example Sketch the graph of $y = |x^2 - 4|$

Solution

The graph of $y = x^2 - 4$ is



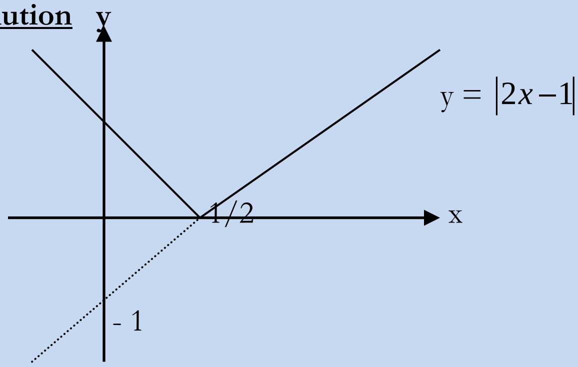
The graph of y = $\left|x^2-4\right|$ is



Example

Sketch the graph of y = $\left|2x-1\right|$

Solution

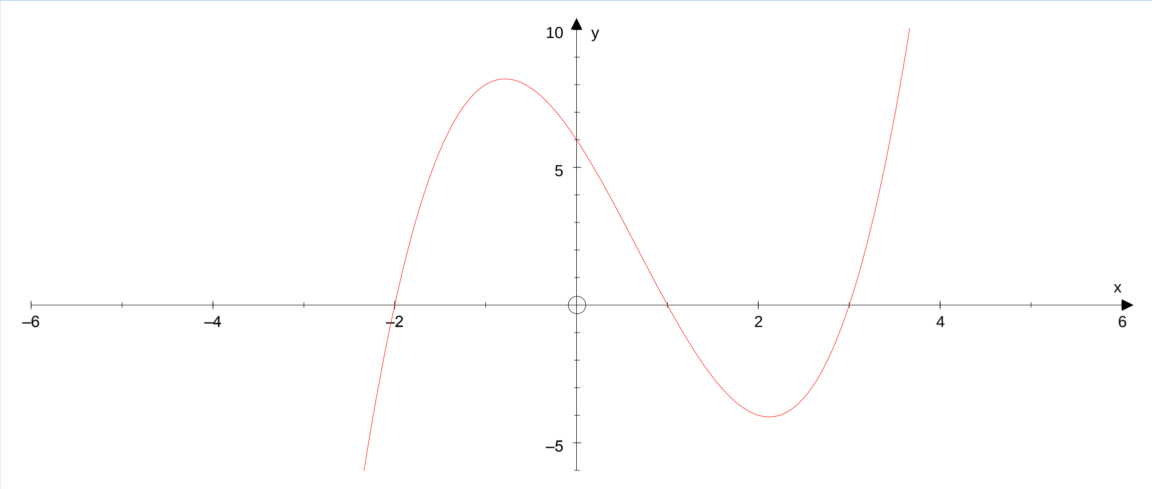


Example

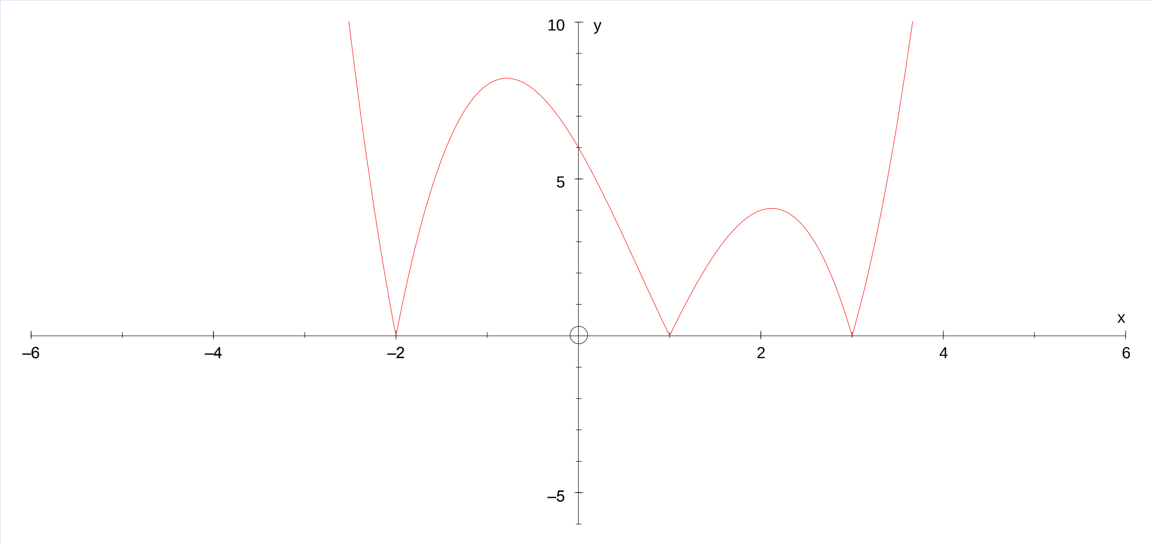
Sketch the graph of $y = \left|x^3-2x^2-5x+6\right|$

Hence $y = \left|(x-1)(x+2)(x-3)\right|$

The graph of y = (x – 1)(x +2)(x – 3) is



The graph of $y = |\ x^3 - 2x^2 - 5x + 6\ |$ is



Example

Sketch the graph y = $|x-2| + |x+1|$

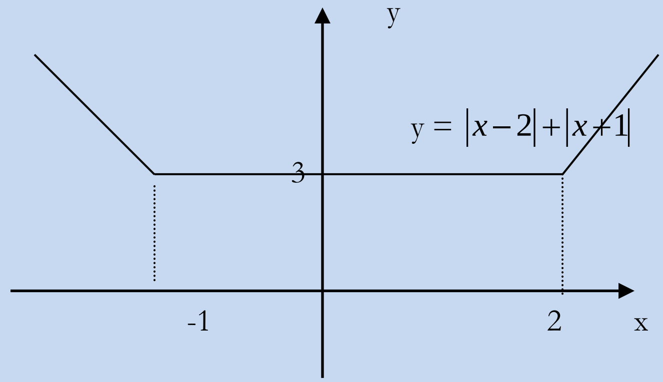
x<-1

y = -(x-2) – (x+1)
= -x + 2 – x – 1
= -2 x + 1

-1 ≤ x < 2

y = -x(x – 2) + (x +1)
= -x + 2 = x + 1
=3
x ≥ 2

y = x – 2 + x + 1
=2x - 1



Modulus Equations.

CASE 1: Modulus on both sides

(a) Algebraic Method
Method 1: Squaring both sides

Example
: Solve $|2x+3| = |x-1|$

$(2x + 3)^2 – (x – 1)^2$
$4x^2 + 12x + 9x^2 – 2x + 1$
$3x^2 + 14x + 8 = 0$
$(3x + 2)(x + 4) = 0$

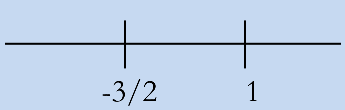
x = - 2/3 and - 4

The method uses the definition of the modulus functions.

Method 2: The analytic method

Solve $|2x+3| = |x-1|$

Solution:
The critical points are: 2x + 3 = 0 or x – 1 = 0
i.e. x = -3/2 or x = 1



Case 1. $\underline{x \leq -3/2}$

$-(2x + 3) = 3 - (x - 1)$
$-2x - 3 = -x + 1$
$2x - x = -3-1$
$x = -4$

Hence: $x = -4$ is admissible since $\mathbf{-4 < -3/2}$

Case 2.

$\underline{-3/2 \leq x \leq 1}$
$2x + 3 = -(x - 1)$
$2x + 3 = -x + 1$
$3x = -2$
$\mathbf{x = -2/3}$

$x = -2/3$ is admissible since $-2/3 \in [-3/2;1]$

Note: $|2x+3| = \begin{cases} 2x + 3, & x \geq -3/2 \\ -2x - 3, & x < -3/2 \end{cases}$

And. $|x-1| = \begin{cases} x - 1, & x \geq 1 \\ -x + 1, & x < 1 \end{cases}$

$2x + 3 = -x + 1$

$3x = -2$

$x = \dfrac{-2}{3}$

S = { -2/3 , -4 }

Case 3.

$x \geq 1$
$2x + 3 = x - 1$
$2x - x = -1-3$
$x = -4$
$x = -4$ is admissible since $x = | - 4 \ngeq 1$

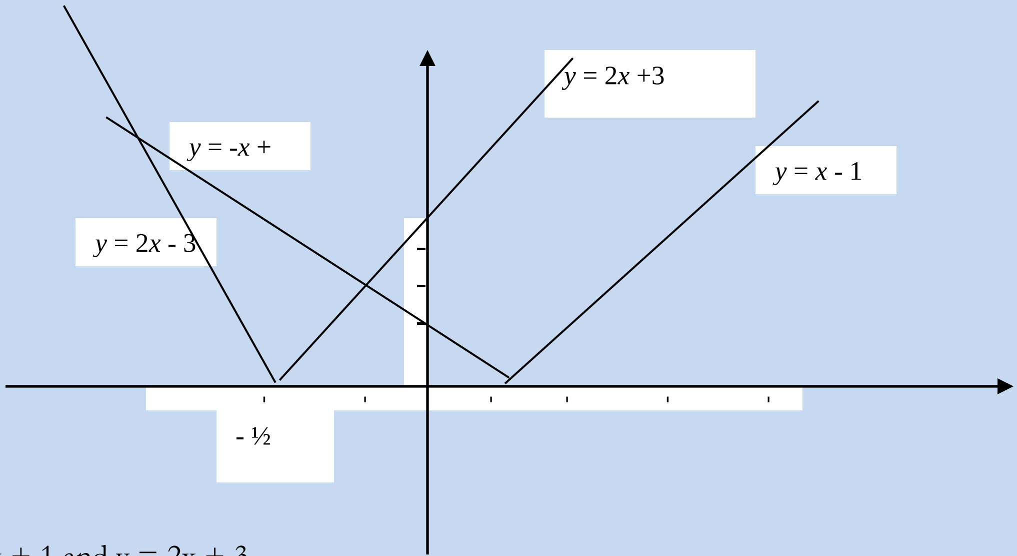
Hence: **S = {-4; -2/3 }**

(b) The Graphical Method

Sketch the graphs of $y = |2x+3|$ and $y = |x-1|$ on the same axis. It is important to make them realistic.

We note that the line segments which intersect are
(i) $y = x + 1$ and $y = 2x – 3$

at the point of intersection: $-x + 1 = - 2x – 3$
$x = - 4$



(ii) y = -x + 1 and y = 2x + 3
- x + 1 = 2x + 3
x = - 2/3

Example

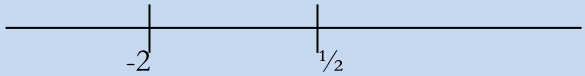
: Solve the equation $\left|\frac{1-2x}{3x+6}\right| = 1$

(a) $(1 - 2x)^2 = (3x + 6)^2$
$1 – 4x + 4x^2 = 9x^2 + 36x + 36$
$5x^2 + 40x + 35 = 0$
$x^2 + 8x + 7 = 0$
$(x + 1)(x + 7) = 0$

***x* = -1 and -7**

Alternatively.

$|1\text{-}2x| = 1 – 2x$; $x < ½$ and $2x – 1$; $\geq ½$
$|3x + 6| = 3x + 6$; $x \geq -2$ and $-3x – 6$; < -2



<u>x< -2</u>

-3x –6 = 1 –2x
-3x + 2x = +7
-x = +7
x = -7 is admissible.

<u>-2≤ x < ½</u>
3x + 6 = 1 – 2x
5x = -5
x = -1 admissible
<u>x ≥ ½</u>
3x + 6 = 2x – 1
x = -7 is admissible.
Hence: s = { -7 ; -1 }

Method 3.

$$\left|\frac{1-2x}{3x+6}\right|=1$$

$$|1-2x|=|3x+6|$$

1 – 2x = ± (3x +6)

1 – 2x = 3x + 6 or 1- 2x = -(3x + 6)

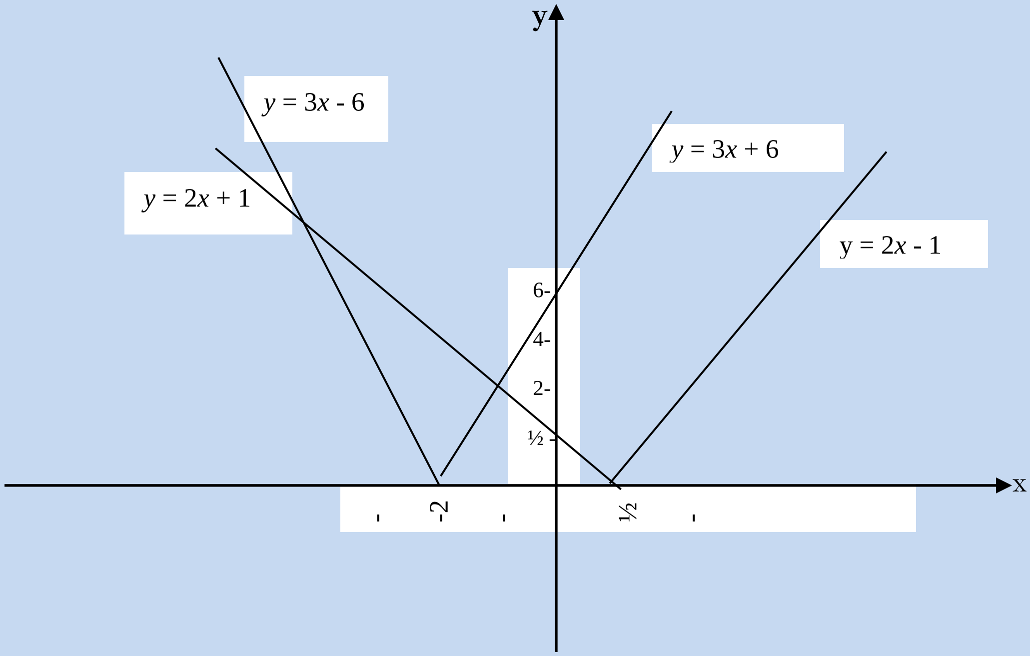
-2x –3x = 6 – 1 -2x + 3 = -6 –1

-5x = 5 **x = -7**

x = -1

S = {-7 ; -1}

The graphical method



(i) $1 - 2x = 3x + 6$

$5x = -5$

$x = -1$

(ii) $1 - 2x = 3x - 6$

$x = -7$

$\boldsymbol{x}$ **= -1 and - 7**

CASE 3 : Modulus on one side only

Example

Solve $|3x-3| = 2x+1$

Method 1: squaring both sides.

Since $|3x-3| = 2x+1$

Hence: $2x + 1 \geq 0$

i.e. **x ≥ -½, this is the domain of the equation** then $|3x-3|=2x+1$ for values x ≥ -1 hence square both sides.

$|3x - 3|^2 = (2x+1)^2$
$(3x - 3)^2 = (2x+1)^2$
$9x^2 - 18x + 9 = 4x^2 + 4x + 1$
$9x^2 - 4x^2 - 18x - 4x + 8 = 0$
$5x^2 - 22x + 8 = 0$

Square both sides only if both sides are positive or both sides are negative.

$$(3x-3)(3x-3)=9x^2-9x-9x+9$$
$$=9x^2-18x+9$$

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$
$$x=\frac{22\pm\sqrt{22^2-4\times5\times8}}{2(5)}$$
$$x=\frac{22\pm\sqrt{484-16-}}{10}$$
$$x=\frac{22\pm\sqrt{324}}{10}$$
$$x=\frac{22\pm18}{10}$$
$$x=\frac{4}{10} or x=\frac{40}{10}$$
$$x=\frac{2}{5} or 4$$
$$Since \frac{2}{5},4\geq-\frac{1}{2}$$
$$S(0,4,4)$$

Method 2

$$|3x-3|=2x+1$$

$3x - 3 = 0$; $2x + 1 = 0$
$x = 1$ $\quad x = -½$

We only analyse for x ≥ -½

Case 1:

<u>$-\frac{1}{2} \leq x < 1$</u>

$2x + 1 = -3x + 3$

$5x = 2$

$x = 2/5 > -\frac{1}{2}$ admissible.

Case 2.

<u>$x \geq 1$</u>

$3x - 3 = 2x + 1$

$x = 4 > 1$ admissible

Hence: $S = \{ 2/5 \ ; 4\}$

Method 3.

<u>$x \geq -\frac{1}{2}$</u>

$|3x - 3| = 2x + 1$

$3x - 3 = \pm (2x + 1)$

$3x - 3 = 2x + 1$

$x = 4$

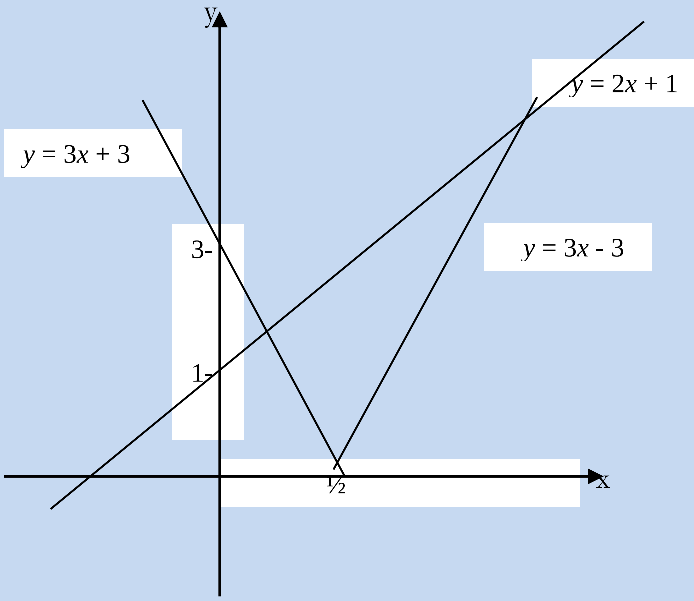
$3x - 3 = 2x - 1$

$5x = 2$

$x = 2/5$

$\mathbf{S = \{ 2/5 \ ; 4\}}$

The graphical Method



(i) $2x + 1 = 3x - 3$

$2x - 3x = -4$

x =-4

(ii) $2x + 1 = -3x + 3$

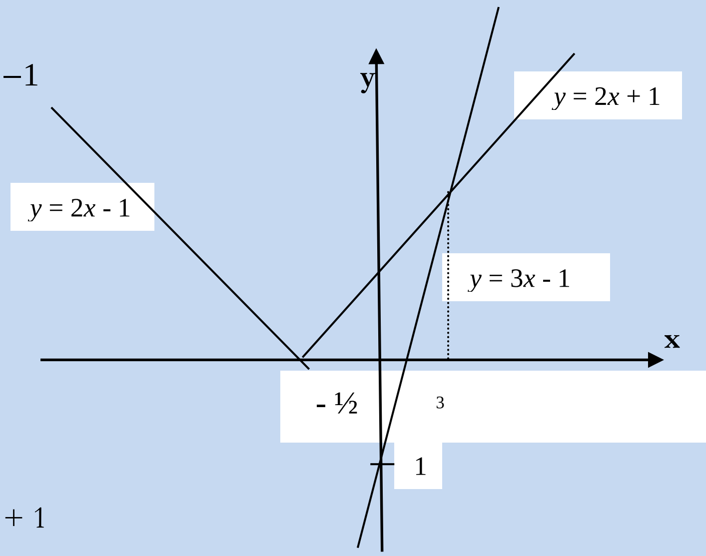
$5x = 2$

$x = 2/5$

$x = 2/5$ or $x = 4$

Example

Solve $|2x+1| = 3x-1$



$3x - 1 = 2x + 1$

$x = 2$

Squaring both sides

$3x - 1 \geq 0$

$x \geq 1/3$

Hence: $|2x+1|^2 = (3x-1)^2$

$(2x + 1)^2 = 9x^2 - 6x + 1$

$4x^2 + 4x + 1 = 9x^2 - 6x + 1$

$9x^2 - 4x^2 - 6x - 4x = 0$

$5x^2 - 10x = 0$

$5x(x-2) = 0$

$x = 0$ or $x = 2$

S = {2}

Alternatively.

$|2x+1|^2 = 3x-1$

$2x + 1 = \pm (3x - 1)$

$2x + 1 = 3x - 1$

$-x = -2$

x = 2

2x + 1 = -3x + 1

$5x = 0$

x = 0< 1/3

Since 0<1/3, inadmissible hence S = {2}

Example

Solve $\left|x^2 - 5x + 6\right| = \left|x + 1\right|$

Solution

You may square both sides, but algebraic calculations are tedious we apply the second method, that of considering cases.

$$\left|x^2 - 5x + 6\right| = \left|x + 1\right|$$

$$\left|(x-2)(x-3)\right| = \left|x + 1\right|$$

Method 3.

$$\left|x^2 - 5x + 6\right| = \left|x + 1\right|$$

$x^2 - 5x + 6 = \pm(x + 1)$

$x^2 - 5x + 6 = x + 1$

$x^2 - 5x - x + 6x - 1 = 0$

$x^2 - 6x + 5 = 0$

x = 1 or 5

s = {1;5}

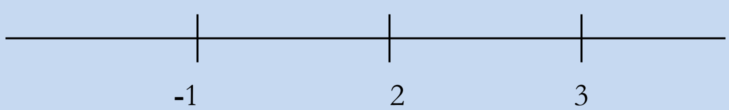
OR

$x^2 - 5x + 6 = -x - 1$

$x^2 - 4x + 7 = 0$

$x = \dfrac{4 \pm \sqrt{16 - 28}}{2}$

no real solutions.



Case 1

x < -1

$x^2 - 5x + 6 = -(x + 1)$

$x^2 - 5x + x + 6 + 1 = 0$

$x^2 - 4x + 7 = 0$

$$x = \frac{-b \pm \sqrt{16 - 28}}{2}$$

16 – 28 <0

No real solutions

$S_1 = \phi$

Case 2.

-1 ≤ x < 2

$x^2 - 5x + 6 = x + 1$

$x^2 - 5x - x + 6 - 1 = 0$

$x^2 - 6x + 5 = 0$

$(x – 1)(x – 5) = 0$

$x = 1$ or $x = 5$ **inadmissible.**

$S_2 = \{1\}$

Case 3.

$2 \leq x < 3$

$-(x^2 – 5x + 6) = x + 1$

$-x^2 + 5x – 6 = x + 1$

$-x^2 + 5x – x – 6 – 1 = 0$

$-x^2 + 4x – 7 = 0$

$= 16 – 28 < 0$

No real solutions

$S_3 = \phi$

Case 4.

$x \geq 3$

$x^2 – 5x + 6 = x + 1$

$x^2 - 5x + 6 – x – 1 = 0$

$x^2 - 6x + 5 = 0$

$(x-1)(x – 5) = 0$

$x = 5$ or $x = 5$

$x = 5$ admissible

$S_4 = \{5\}$

Hence: $S = S_1 \cup S_2 \cup S_3 \cup S_4$

$S = \{1;5\}$

Modulus Inequalities

These can be solved using similar methods used in modulus equations The algebraic method is preferable.

1.1.4 Properties of the modulus inequalities

> Used for solving modulus inequalities.

(i) $| x | \leq a \Leftrightarrow - a \leq x \leq a$

(ii) $| x | \geq a \Leftrightarrow x \leq - a$ or $x \geq a$

Example: $|x| < 3$

$- 3 < x < 3$

Example: $|4x - 3| < 5$

$- 5 < 4x - 3 < 5$

$-2 < 4x < 8$

$- ½ < x < 2$

Example $|\ x|\ > 7$

$\therefore\ \boldsymbol{x} > 7$ **or** $\boldsymbol{x} < -7$

Example $|2x + 1| > 9$

$2x + 1 < -9$ or $2\mathbf{x} + 1 > 9$

$2x < -10$ or $2x > 8$

$x < -5$ or $x > 4$

Example

Solve $|x-1| \succ |x+2|$

Squaring both sides : $(x - 1)^2 > (x + 2)^2$

$x^2 - 2x + x^2 + 4x + 4$

$6x < 3$

$\boldsymbol{x < -\frac{1}{2}}$

Example

Solve $|2x-1| \succ |3x+2|$

(a) Squaring both sides $(2x - 1)^2 > (3x - 2)^2$

$4x^2 - 4x + 1 > 9x^2 - 12x + 4$

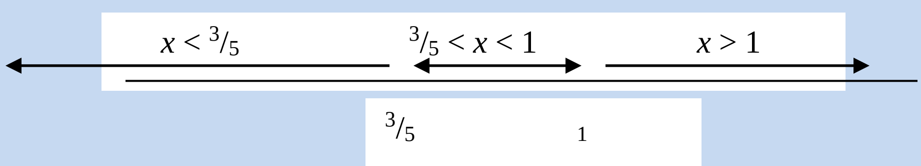
$5x^2 - 8x + 3 < 0$

$(5x - 3)(x - 1) < 0$

Changing this to an equation $(5\ x - 3)(\ x - 1) = 0$

$x = {}^3/_5$ and x $= 1$

These are the critical values of x.. These divide the number line into three regions as shown below.



S ={ x: x real: $^3/_5 < x < 1$}

Alternatively.

We make the following table determining the sign of each factor in each region by substitution.

	$x < {}^3/_5$	$^3/_5 < x\ 1$	$x > 1$
$5x - 3$	-	+	+
$x - 1$	-	-	+
$(5x - 3)(x-1)$	+	-	+

From the last row $(5x - 3)(x-1) < 0$ (that is to say is negative when $^3/_5 < x < 1$ which is our solution.

If the question was changed to:

Solve $|2x-1| \succ |3x+2|$

The solution set would be where
$(5x-3)(x-1) > 0$ and this would be
$x < {}^{3}/_{5}$ or $x > 1$

Example

Solve $|2x+5| \succ |x-4|$

$4x^2 + 20x + 25 > x^2 - 8x + 16$

$3x^2 + 28x + 9 > 0$

$(3x + 1)(x + 9) > 0$

The critical values are $x = \dfrac{-1}{3}$ and $x = -9$

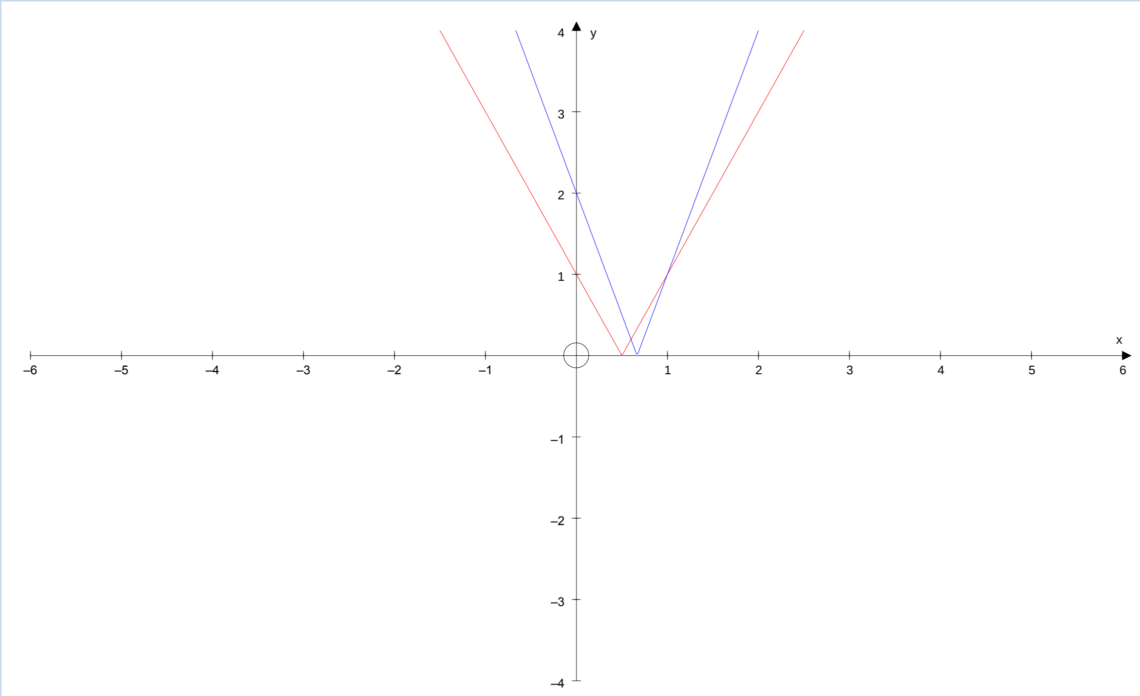
	$x < -9$	$-9 < x < -⅓$	$x > -⅓$
$3x + 1$	-	-	+
$x + 9$	-	+	+
$f(x)$	+	-	+

$|2x+5| \succ |x-4|$ if $x < -9$ or $x > -⅓$

Example

Solve $|2x-1| \prec |3x-2|$

We will start by sketching the two graphs: $y = |2x-1|$ and $y = |2x-1|$



Solving for x at P : $2x - 1 = -3x + 2$

$5x = 3$

$x = {}^{3}/_{5}$

Solving for x at Q : $2x - 1 = 3x - 2$

$x = 1$

From the question we want to find the range of values of x for which the graph of $y = |\ 2\ x - 1|$ lies below the graph of $y\ = |3x - 2|$. This is ${}^{3}/_{5} < x < 1$ as we found earlier.

The Analytic Method

Solve for $|2x-1| \prec |3x-2|$

<u>Solution</u>

The critical points are 1/2 and 2/3

<u>x < ½</u>

$-(2x - 1) < -(3x - 2)$

$x < 1$

$S_1 = \{ x:\ ½ < x < 1\}$

<u>½ < x< 2/3</u>

$2x - 1 < -(3x - 2)$

$5x < 3$

$x < 3/5$

$S_2 = \{x: ½ < x < 3/5\}$

<u>x > 2/3</u>

$2x - 1 < 3x - 2$

$-x < -1$

$x > 1$

S_3 **= { x:**x >1}

$S = S_1 \cap S_2 \cap S_3$

S = { x: = ½ < x < 3/5}

Example

Solve $2x + 2 \succ |3x - 6|$

<u>Solution</u>

Squaring both sides: since 2 x + 2 > 0

i.e. **x ≥ -1.**

$(2x + 2)^2 > (3x - 6)^2$

$4x^2 + 8x + 4 > 9x^2 - 36x + 36$

$-5x^2 + 44x - 32 > 0$

$5x^2 - 44x + 32 < 0$

$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

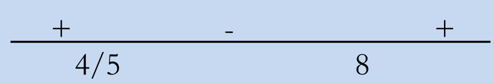
$x = \frac{44 \pm \sqrt{(44^2 - 4 \times 5 \times 32)}}{2(5)}$

$x = \frac{44 \pm \sqrt{(1936 - 640)}}{10}$

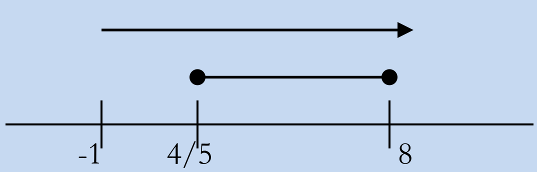
$x = \frac{44 \pm \sqrt{1296}}{10}$

$x = \frac{44 \pm 36}{10}$

x = 8 or 4/5

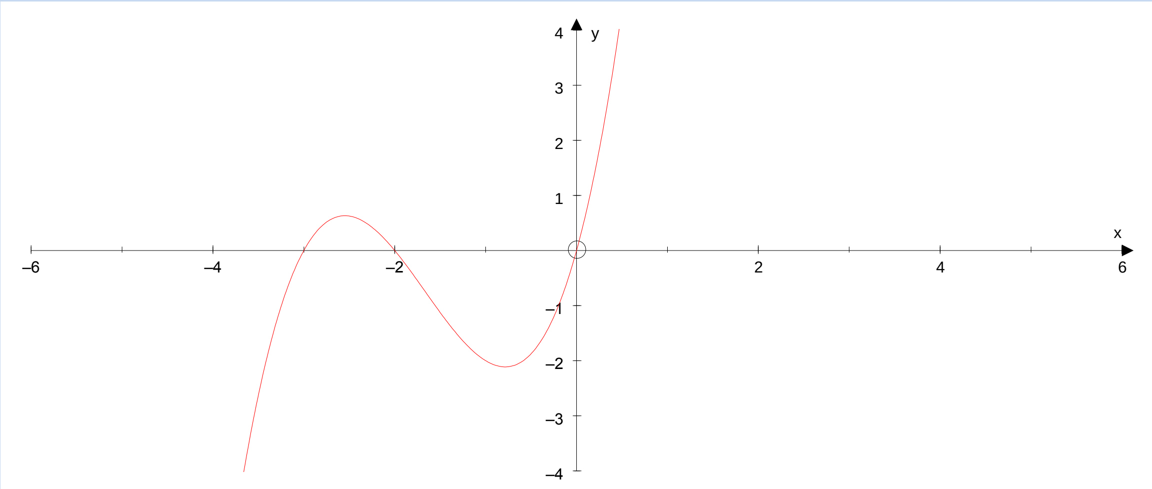


4/5 < x < 8



Examination Type Questions

1. The diagram shows the graph of y = f(x)



y = f(x)

On separate diagram sketch the graphs of: (i) $y = |f(x)|$
(ii) $y = f(|x|)$
(iii) $y = |\ f(-1 \text{ x } 1)|$

2. (a) Solve the inequality $|x| \prec |x-2|$

(a) The function f is defined by f(x) $= |x-3|$, x∈$\Re$, sketch the graph of f. Solve the inequality $|x-3| <$ ½x.

3. (a) On the same diagram. Sketch the graphs of $y = \frac{1}{x-2}$ and $y = 4|x-2|$

show clearly the coordinates of any points of intersection with the coordinate axes.

(c) Hence, or otherwise, find the set of values of x for which $\frac{1}{x-2} < 4|\,|x-2|$

4. The figure shows the graphs of y = f(x) with f(x) = 0 for $|\ x\ | \geq 2$.

On separate diagrams, sketch the graphs of:

(a) y = f(x + 2) (b) y = f(2x) (c) y = 2 f(½x)

(d) $y = |\ f(x\ |$ (f) y = f(1 x 1) (g) y = f(x) – 3

5. Sketch the graph of y = $|x + 2|$ and hence, or otherwise, solve the inequality.

$|x+2|$ > 2x +1 , $X \in \Re$.

6. Find the set of values of x for which

$|2x-1| > |x+1|$

7. Sketch the graph of y = 1x + 2l and hence, or otherwise, solve the inequality

1x + 2l > 2x +1, $X \in \Re$.

8. The curve $y = \frac{1}{4}x^2 - 1$ and the line 2y = x+10 intersect at the points A and B, O is the origin. Calculate the co-ordinates of A and B, and hence show that OA and OB are perpendicular.

11. The position vectors of three points A, B and C with respect to a fixed origin O are 2i – 2j + k, 4i + 2j + k and i + j + 3k respectively. Find the unit vectors in the directors of CA and CB. Calculate angle ACB in degrees, correct to 1 decimal place.

CHAPTER 7

LOGARITHMIC AND EXPONENTIAL FUNCTIONS

OBJECTIVES

By the end of the chapter the student should be able to :

- Sketch graphs of exponential and logarithmic functions
- Solve equations of type $a^x = b$
- Solve inequalities of type $a^x \succ b$
- Solve problems involving exponential growth and decay

Theorem of logarithms

The student should recall that:

(i) If $a^x = n$ $\log_a n = x$

(ii) $\log (a.b) = \log a + \log b$

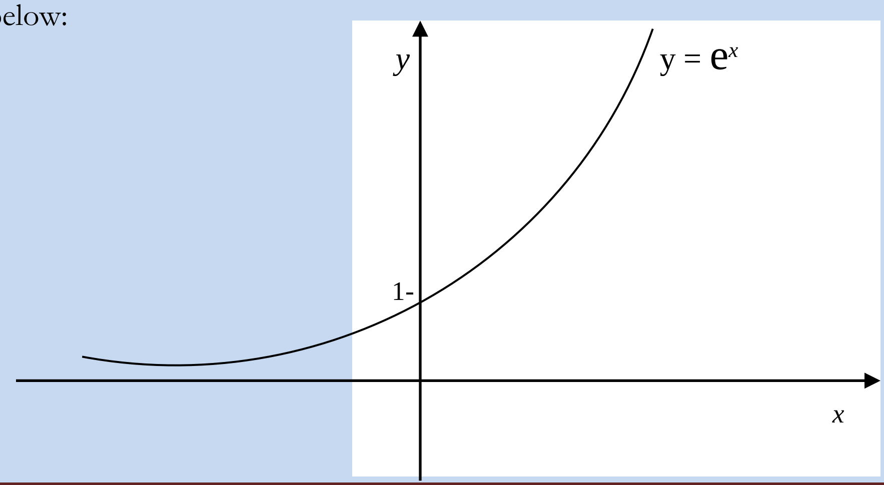
(iii) $\log (a \div b) = \log a - \log b$

(iv) $\log a^m = m \log a$

(v) $\log a^{-1} = -\log a$

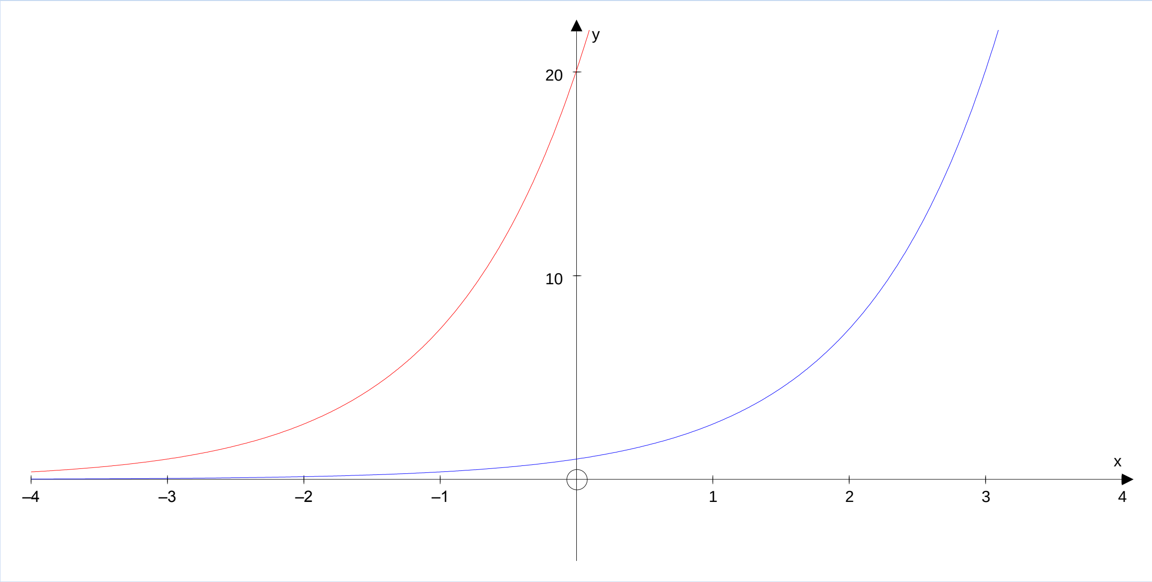
The Graph of the Exponential Function

An expression of the form $y = a^x$ is called an exponential function. When the base is the number e, that is when $y = e^x$, this is called the exponential function. Its graph is as shown below:

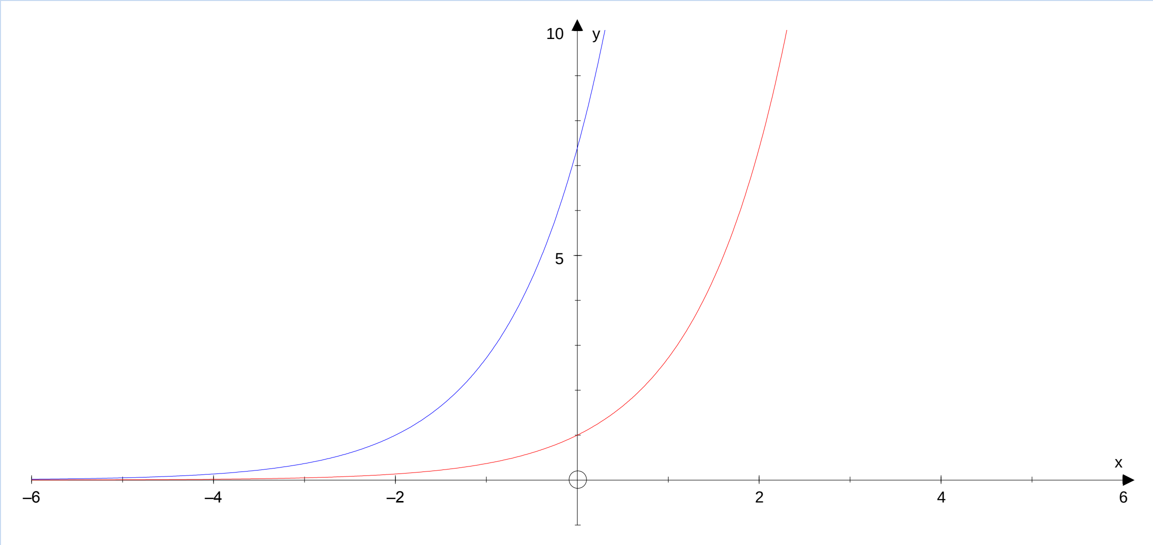


By transforming this standard graph, we can sketch the graphs of related functions

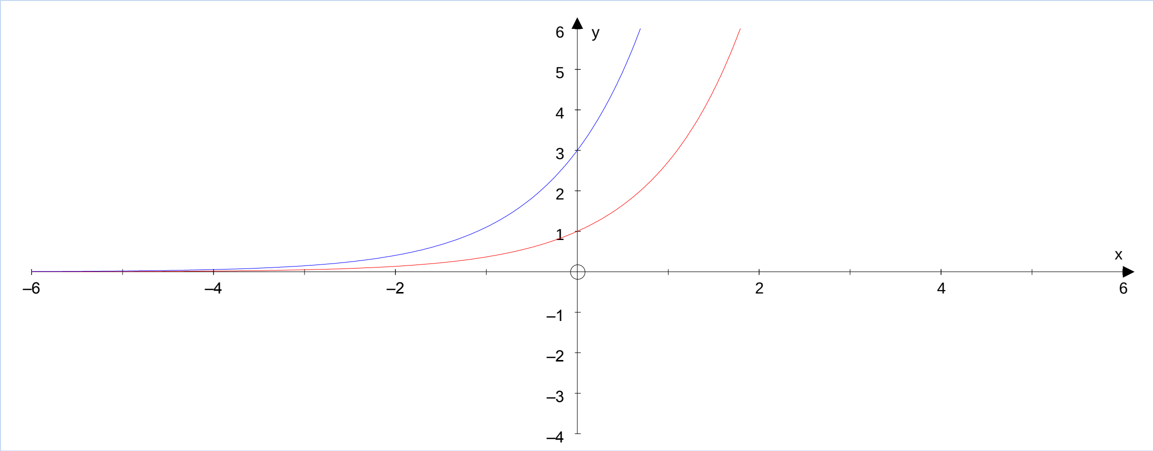
Example: Sketch the graph of $y = e^{x} + 3$



Example: $y = e^{x + 2}$

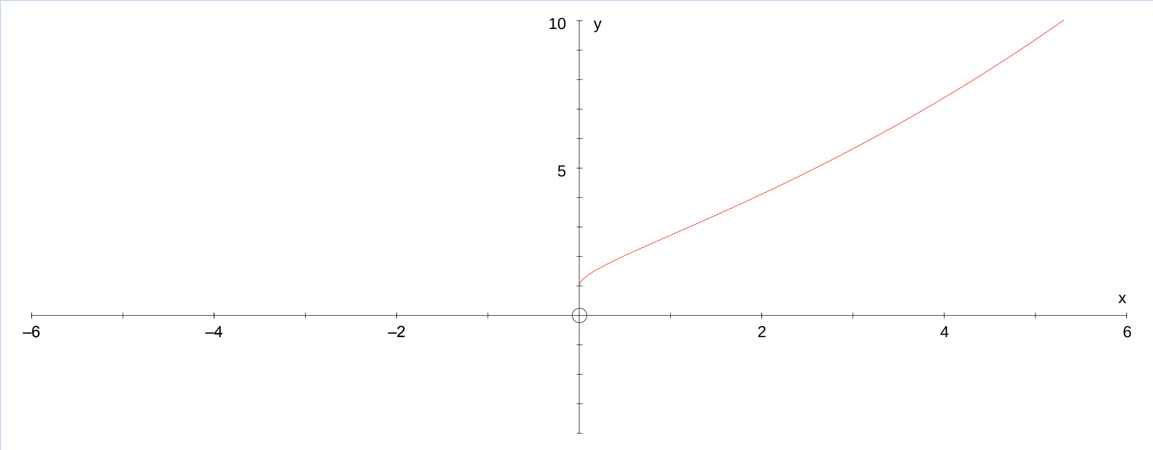


Example: $y = 3e^{x}$



Example

$y = e^{\sqrt{x}}$



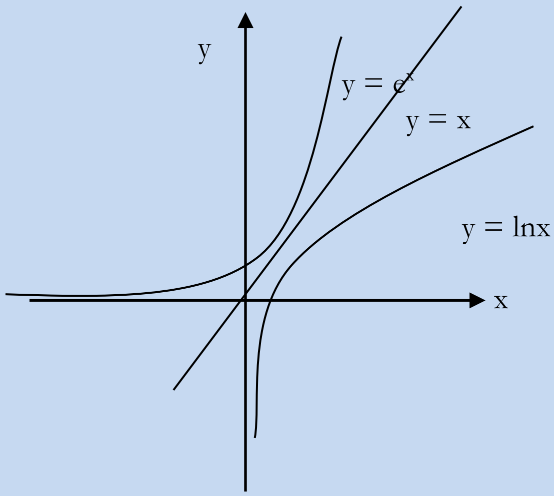
The Graphs of the Logarithm Functions

Logarithms to base e are called Natural or Naperian logarithms while those to base 10 are called decimal logarithms. Earlier we stated that:

If $a^x = n \Leftrightarrow \log_a n = x$

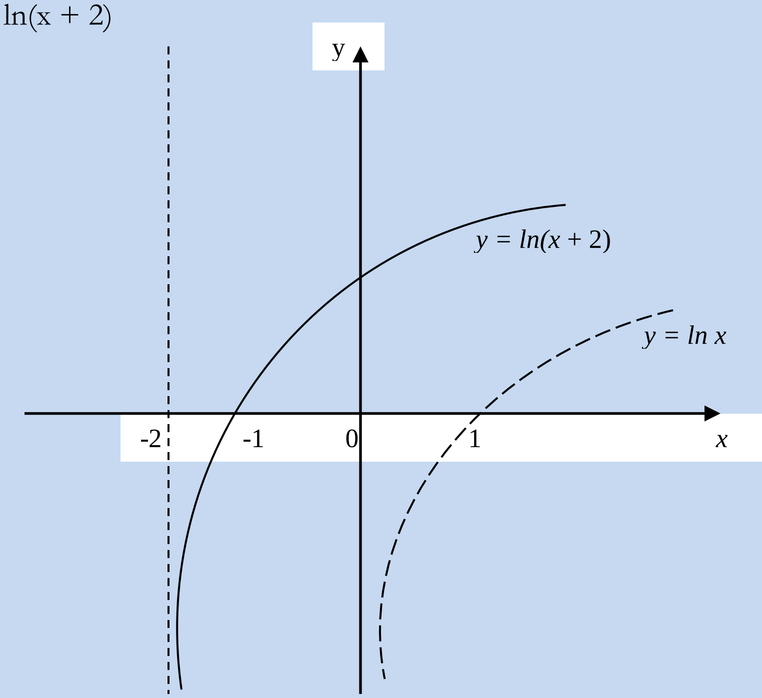
Hence , $e^x = n \Leftrightarrow \ln n = x$

$y = e^x$ and $y = \ln x$ are universe functions. The graph of $y = \ln x$ is therefore a reflection of the graph of $y = e^x$ about the line $y = x$

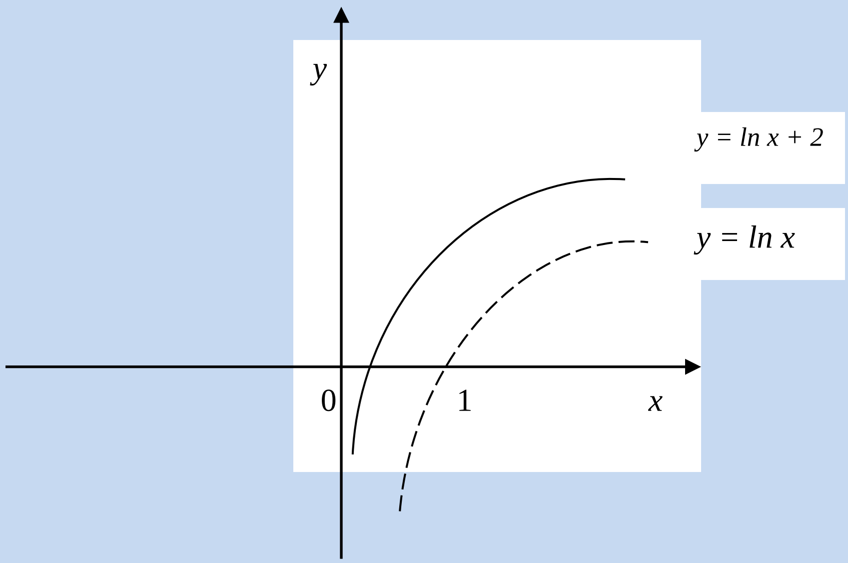


NB: $y = \ln x$ exists if and only if $x > 0$
From the standard graph of $y = \ln x$ we can obtain the graphs of other logarithms functions by transformations.

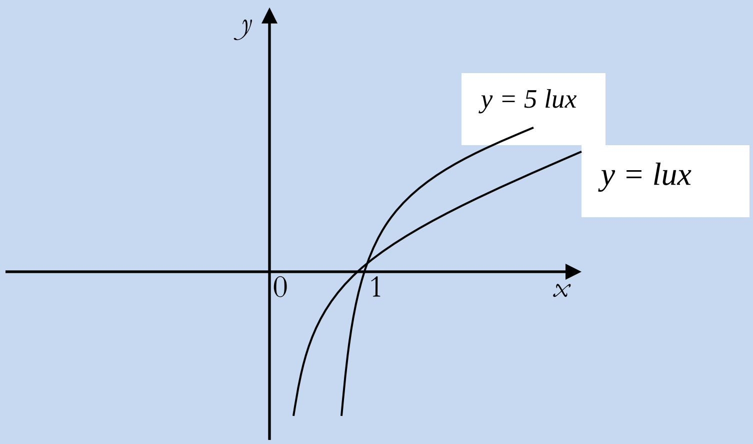
Example : $y = \ln(x + 2)$



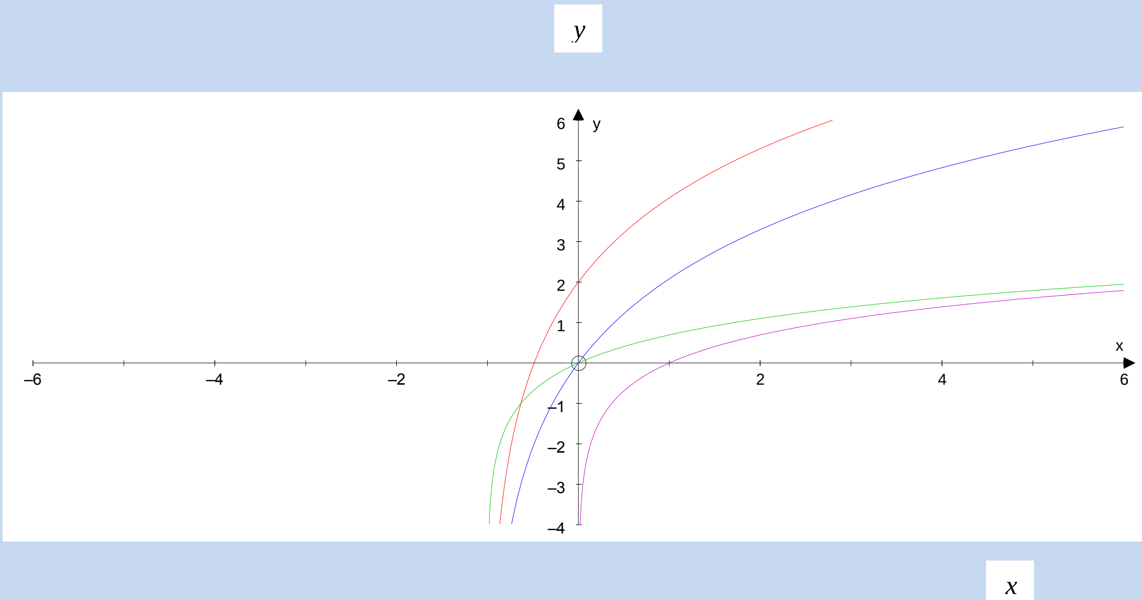
Example
$y = ln\ x + 2$



Example
y = 5 lnx



Example
$y = 3 \ln(x + 1) + 2$



<u>Equations of the Type $a^x = b$</u>

These are solved by taking logarithms on both sides.

Example: Solve : $3^x = 10$

Taking logs : $\ln 3^x = \ln 10$

$$x \ln 3 = \ln 10$$

$$x = \frac{\ln 10}{\ln 3}$$

$$= \underline{2.1}$$

Example Solve : $3^{2x+1} = 10^{3x}$

Solution : $\ln(3^{2x+1}) = \ln 10$

$$(2x + 1) \ln 3 = 3x \ln 10$$

$$2x \ln 3 + \ln 3 = 3x \ln 10$$

$$\ln 3 = 3x \ln 10 - 2x \ln 3$$

$$x = \frac{\ln 3}{3\ln 10 - 2 \ln 3}$$

$$= \underline{\mathbf{0.23}}$$

Inequalities of the type $a^x > b$

NB: It is important to remember that when you divide or multiply an inequality by a negative number, the sign of the inequality changes.

Example : Solve $2^{x-3} > 43$ where x is an integer and find its least value.

Solution :

$(x-3)\ ln\ 2 > ln\ 43$

$x\ ln\ 2 > \ln 43 + 3\ lu\ 2$

$x > ln\ 43 + 3\ ln\ 2$

$> \mathbf{8.4}$

The least value of x is 9.

Example : Solve $(0.34)^{n-10} > 12$

Solution :

$(n-10)\ ln\ 0.34 > ln\ 12$

$n\ ln\ 0.34 > ln\ 12 + 10 \ln 0.34$

$n < \dfrac{\ln 12 + 10\ ln\ 0.34}{ln\ 0.34}$

$n < \mathbf{7.69}$

Exponential growth and decay

An exponential growth or decay function is a special case of a geometric progression. It represents a function, which increases or decreases by a constant factor (the growth factor) over fixed line intervals. It is of the form, $x = x_0e^{-kt}$ for a decay function and $x = x_0e^{kt}$ for a growth function.

Example

The number of people (N) who visited a New Start Center t months after it opened is given by $48e^{0,4\ (t-2)}$

(a) State the value of N when t = 3
(b) Determine the value of t to the nearest whole number when N has risen to 95.

Solution

(a) $N = 48e^{0,4\ (t-2)}$

when t = 3, $N = 48e^{0,4}$

$= 71.6$

$= 72$ people

(b) $N = 48\ e^{0,4\ (t-2)}$

when N = 95 => $95 = 48e^{0.4\ (t-2)}$

$$\frac{95}{48} = e^{0.4\,(t-2)}$$

$$\ln 95 - \ln 48 = 0.4\,(t-2)$$

$$t - 2 = \frac{1}{0.4}(\ln\ 95 - \ln 48)$$

$$t = 2 + \frac{1}{0.4}(\ln\ 95 - \ln 48)$$

$$= 3.71 \text{ months}$$

$$= 4 \text{ months (to the nearest whole number)}$$

Example

At time t days, the number of radio active atoms present when a particular element decays satisfies the equation $N = 10^{16} e^{-4t}$

(a) State the number of radio-active atoms present initially.

(b) Determine the time for the number of atoms to be reduced to half their initial number (that is the half-size of the element)

Solution

(a) when $t = 0$, $N = 10^{16}$

(b) at the half-life $N = \frac{1}{2}(10^{16})$

$$\Rightarrow \tfrac{1}{2} = \quad - 4t$$

$$\ln(\tfrac{1}{2}) = \quad - 4t$$

$$t = -\tfrac{1}{4} \ln 0.5$$

$$\mathbf{= 0.17 \text{ days}}$$

Examination Type Questions

1. The function $f(x) = 2 - e^x$ has an inverse function f^{-1}. Find

   a) $f^{-1}(x)$ b) the domain of f^{-1}

2. The amount of radioactive element decreases spontaneously over time because of the process of radioactive decay. The increase is known to be exponential, so that the amount y, in grams of a particular element present after time t, in years of decay can be described by the equation

$$y = Ae^{-kt}$$

where A and k are positive constants.

a) The amount (y grams) present at time t (years) in a sample of the radio active element radium is given approximately by

$$y = 1000e^{-0.0005t}$$

(i) Calculate the initial amount of radium in the sample

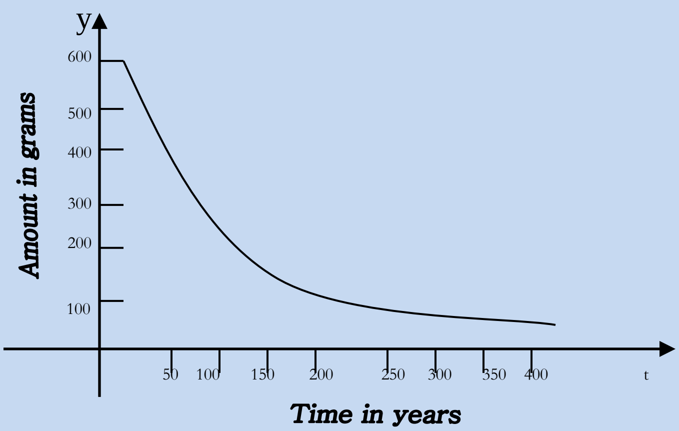
ii) Calculate the amount of radium in the sample after 15 years.

iii) What is the average rate of decay (in grams per year) of the radium sample over the first 500 years?

iv) Write down an equation for the instantaneous rate of change at any time t of the amount of the given sample of radium.

v) Sketch the graph of the rate of change of the amount of radium versus time in the sample as it decays radioactively.

(b) A sketch of the equation for the amount of a sample of the Radioactive element uranium is shown below.



It is known that the half-life of this element is 50 years. that is it takes 150 years for a given sample of uranium to decay to half its initial amount. The equation which models the amount of uranium (y grams) present in a given sample at any time (t years) is given by

$$y = ae^{-bt}$$

From the information given above, find a and b for the element uranium. Give the value b to four decimal places.

3. Solve the inequality $(0,96)^{n-1} < 56$ giving the least or largest value.

CHAPTER 8

COORDINATE GEOMETRY

OBJECTIVES
By the end of the chapter the student should be able to :

- Calculate distance and gradient between two points
- Identgify relationships between lines using gradients
- State te equation of a circle
- State the centre and radius of a circle
- Convert parametric equatons to Cartesian equations and vice versa

Distance and gradient

The student should recall that given two points A $(x_1; y_1)$ and B $(x_2; y_2)$ on a line as shown below:

(i) The distance AB = |AB| = $\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$

(ii) The gradient of AB (m) = $\dfrac{y_3 - y_1}{x_2 - x_1}$

(iii) The midpoint (M) of x

AB = $\left(\dfrac{x_1 + x_2}{2}; \dfrac{y_1 + y_2}{2}\right)$

(i) **perpendicular lines**: If two lines with gradients m_1 and m_2 are perpendicular, then $m_1 . m_2 = -1$

(ii) **parallel lines**: If two lines are parallel then $m_1 = m_2$

(iii) the equation of a line is: $y = mx + c$

Example:
Given the points A (2,2); B (3,4) and C (1,5). Find (i) the equation of the line joining C to the midpoint (m) of AB. (ii) its length

Solution:

$$M \quad \left[\frac{2+3}{2};\frac{2+4}{2}\right]$$

$$= \qquad M \left[\frac{5}{3};3\right]$$

Gradient of CM $= \dfrac{5-3}{1-\frac{5}{2}}$

$$G = \frac{-4}{3}$$

Finding the equation: $\dfrac{y-5}{x-1}=\dfrac{-4}{3}$

$$3\ (y-5) \qquad = \qquad -4(x-1)$$

$$3y \qquad = \qquad -4x + 19$$

$$|CM| \quad = \quad \sqrt{(\frac{5}{2}-1)+(5-3)^2}$$

$$= \qquad 2{,}5$$

Example Show that $5y = -2x + 5$ and $2y = 5x + 6$ are perpendicular**.**

<u>**Solution**</u>

$5y = 2x + 5$

$y = \dfrac{2x}{5} + 1$

$m_1 = \dfrac{-2}{5}$

$2y = 5x + 6$

$y = \dfrac{5x}{2} + 3$

$m_2 = \dfrac{5}{2}$

$m_1 \times m_2 \quad = \quad \frac{-2}{5} \times \frac{5}{2}$

$\quad = \quad -1$

$5y = -2x + 5$ and $2y = 5x + 6$ are perpendicular

Example:

(i) Find the points of intersection of the line $2x + y = 5$ and the circle $x^2 + y^2 = 50$

(ii) How far apart are they ?

Solution

$2x + y = 5$

$y = -2x + 5$

Substituting for y in $x^2 + y^2 = 50$
$x^2 + (-2x + 5)^2 = 50$
$x^2 + 4x^2 – 20x + 25 = 50$
$5x^2 – 20x – 25 = 0$
$x^2 – 4x – 5 = 0$
$(x + 1)(x - 5) = 0$
$x = -1; 5$

when $\quad x = -1, y = -2 (-1) + 5 = 7$
$\quad y = 5, \; y = -2 (5) + 5 = -5$

The points of intersection are $= (-1. 7), (5. -5)$

(ii) The distance apart $= \sqrt{\{(5 - - 1)^2 + (-5 -7)^2\}}$

$= \sqrt{\{6^2 + 12^2\}}$

$= 13.4$ units

The Equation of A Circle

This is of the form $(x - a)^2 + (y - b)^2 = r^2$ where the coordinates of the centre are (a, b) and the radius is r.

Example. Find the centre and radius of the circle

$$x^2 + y^2 + 2x + 4y - 4 = 0$$

rearranging: $x^2 + 2x + y^2 + 4y - 4 = 0$

completing the square: $(x + 1)^2 - 1^2 + (y + 2)^2 - 2^2 - 4 = 0$

$$(x + 1)^2 + (y + 2)^2 = 3^2$$

$$C(-1, -2),\ r = 3$$

The student should recall that in an equation of a circle, the coefficients of x^2 and y^2 must be equal.

Example:

Given that $x^2 + y^2 + 2x + 2y - 2 = 0$ and $x^2 + y^2 + 6x + 8y = 0$. Find the length of the line joining the two centres.

Solution:

$x^2 + y^2 + 2x + 2y - 2 = 0$

$x^2 + 2x + y^2 + 2y - 2 = 0$

$(x + 1)^2 - 1^2 + (y + 1)^2 - 1 - 2 = 0$

$(x + 1)^2 + (y + 1)^2 = 4$

$C(-1, -1)$, $\quad r = 2$

$x^2 + y^2 + 6x + 8y = 0$

$x^2 + 6x + y^2 + 8y = 0$

$(x + 3)^2 + (y + 4)^2 = 42 = 0$

$(x + 3)^2 + (y + 4)^2 = 25$

$C(-3, -4)$, $\quad r = 5$

The distance between the two centres

$$= \sqrt{(-3 - -1)^2 + (-4 - -1)^2}$$

$$= \sqrt{(-2)^2 + (-3)^2}$$

$$= 3.606$$

The general equation of a circle is given by $x^2 + y^2 + 2fx + 2gy + c = 0$, where the coordinates of the center are given by $(-f, -g)$ and $r = \sqrt{f^2 + g^2 - c}$

Example

Show that the following equations are equations of a circle.

i) $2x^2 + 2y^2 - 10x + 6y - 15 = 0$

ii) $36x^2 + 36y^2 + 48x - 108y + 97 = 0$

iii) $x^2 + y^2 - 8x + 6y + 29 = 0$

Solution

i) $2x^2 + 2y^2 - 10x + 6y - 15 = x^2 + y^2 - 5x + 3y - 7.5$

$f = 5/2$, $g = -3/2$ and $r = \sqrt{\{2.5^2 + (-1.5)^2 + 7.5\}} = \sqrt{16} = 4 > 0$. It is an equation of a circle centre $(5/2, -3/2)$ and radius $r = 4$

ii) $36x^2 + 36y^2 + 48x - 108y + 97 = x^2 + y^2 + 4/3x - 3y + 97/36 = 0$

$f = -2/3$, $g = 3/2$ and $r = \sqrt{\{-2/3)^2 + (1.5)^2 - 97/36\}} = 0$.It is not an equation of a circle. It is a point $(-2/3, 3/2)$

iii) $x^2 + y^2 - 8x + 6y + 29 = 0$

$f = 4$, $g = -3$ and $r = \sqrt{\{4^2 + (-3)^2 - 29\}} = \sqrt{\{16 + 9 - 29\}} = -4 < o$

It is not an equation of a circle

Parametric equations

A Cartesian equation gives us a direct relationship between two variables x and y. sometimes this relationship is given indirectly through a third variable called a parameter

Example

Given two parametric equations $x = 3t + 4$ and $y = t^2$, find the Cartesian relationship of x and y

Solution

$$x = 3t + 4$$

$$t = \frac{x-4}{3}$$

Substituting for t in

$$y = t^2$$

$$y = \left[\frac{x-4}{3}\right]$$

$$= \frac{x^2 - 8x + 16}{9}$$

$$9y = x^2 - 8x + 16$$

Example $x = 3\cos\theta$
$y = 5\sin\theta$

Solution

$\cos\theta = x/3$

$\sin\theta = y/5$

using $\sin^2\theta + \cos^2\theta = 1$

$$\left(\frac{x}{3}\right)^2 + \left(\frac{y}{5}\right)^2 = 1$$

$$\frac{x^2}{9} + \frac{y^2}{5} = 1$$

$25x^2 + 9y^2 = 225$

Please note that the identity $1 + \tan^2\theta = \sec^2\theta$ may also prove useful.

Examination Type Questions

1. Given the coordinates of the points A (6; 2), B (2; 4), C (-6; - 2) and D (-2, - 4).
   (i) Calculate the gradients of the lines AB, CB, DC, and DA. Hence describe the shape of the figure ABCD.
   (ii) Show that the equation of the line DA is $4y - 3x = -10$ and find the length DA.
   (iii) Calculate the gradient of a line, which is perpendicular to DA, and hence find the equation of the line ℓ through B, which is perpendicular to DA.
   (iv) Calculate the coordinates of the point P where ℓ meets DA
   Calculate the area of the figure ABCD

2. A circle touches the line $y = \frac{3}{4}x$ at the point (4,3) and passes through the point (-12, 11).
   Find (i) the equation of the perpendicular bisector of the line passing through the points (4,3) and (- 12, 11)
   (ii) the equation of the circle

3. In each of the following parametric equations, sketch the curve and obtain the Cartesian equation represented by the parametric equation
   i. $x = 2t,\ y = 4t$
   ii. $x = t + 2,\ y = 2t + 3$
   iii. $x = 2t^2,\ y = 3/t^2$
   iv. $x = 2\sin\theta - 3,\ y = 4\cos\theta - 4$

4. You are given the coordinates of the four points A(6,2), B(2,4), C(- 6, - 2) and D(- 2, - 4).
   Calculate the gradients of the lines AB, CB, DC and DA. Hence describe the shape of the figure ABCD.
   Show that the equation of the line DA is $4y - 3x = -10$, and the length DA
   iii) Calculate the gradient of a line which is perpendicular to DA, and find the equation of the line ℓ through B which is perpendicular to DA.
   iv) Calculate the coordinates of the point P where ℓ meets DA.
   v) Calculate the area of the figure ABCD

5. Find the equation of the circle which passes through the points A(1,2), B(2,5) and C(- 3, 4)

6. The points P, Q and R have coordinates (2,4), (8, - 2) and (6,2) respectively.
   a) Find the equation of the straight line ℓ whish is perpendicular to the line PQ and which passes through the mid- point of PR.
   b)The line ℓ cuts PQ at S. Find the ratio PS:SQ.
   c) The circle passing through P<q and R has center C. Find the coordinates of C and the radius of the circle.
   d) Given that angle PCQ = θ radians, show that $\tan\theta = 24/7$
   Prove that the smaller segment of the circle cut off by the chord PQ has area $25\theta - 24$

7 A circle has center at the point with co-ordinates (-1, 2) and has radius 6. Find the equation of the circle, giving your answer in the form $x^2 + y^2 + ax + by + c =$

CHAPTER 9

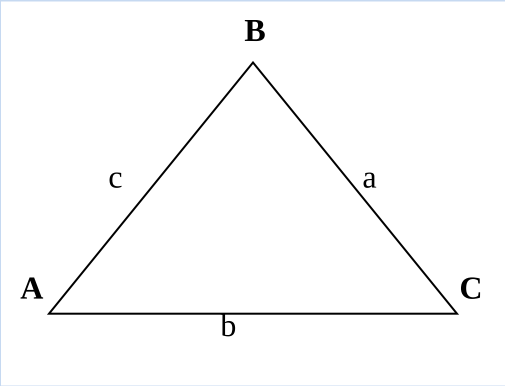
PLANE TRIGONOMETRY

OBJECTIVES
By the end of the chapter the student should be able to:

- State the sine rfule
- State the cosine rule
- Calculate lengrus of sides of triangles using the cosine and sine rfule
- Calculate angles of triangles using the sine and lor cosine rule
- Convert radius to degrees and vice versa
- Compute areas of sectors
- Compute lengths of segments

The Sine and the Cosine rule.

(i) The Sine Rule



$$\frac{a}{SinA} = \frac{b}{SinB} = \frac{c}{SinC}$$

Example :
In triangle EFG, e = 6 , g = 4 and G = 35^o

From the sketch, two triangles can be drawn using the same rule;

$$\frac{SinE}{6} = \frac{Sin35^\circ}{4}$$

Sin E = 0.8604

$E \quad = \quad \mathbf{59.36^o}$ **or** $\mathbf{120.64^o}$

In this case, we have two perfectly acceptable answers because $\sin\theta = \sin(180 - \theta)$. The student should also remember that $\cos(180 - \theta) = -\cos\theta$ and $\tan(180 - \theta) = -\tan\theta$

(ii) The Cosine Rule

$a^2 = b^2 + c^2 - 2bc \cos A$
$b^2 = a^2 + c^2 - 2ac \cos B$
$c^2 = a^2 + b^2 - 2ab \cos C$

(iii) Area of a triangle

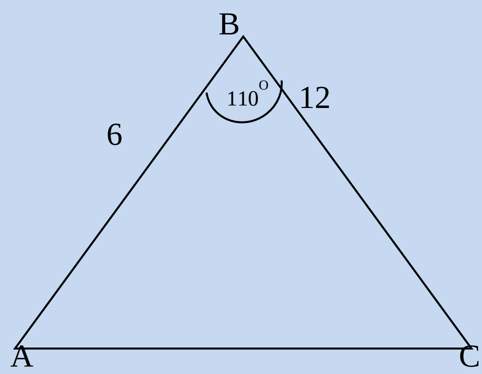
$\triangle = \frac{1}{2} ab \sin C$

$= \frac{1}{2} bc \sin A$
$= \frac{1}{2} ac \sin B$

Example

A, B and C are points on level ground such that $\angle ABC = 110$,
AB =2km and BC = 6km. Find (i) AC and (ii) the area of triangle ABC.

<u>Solution</u>

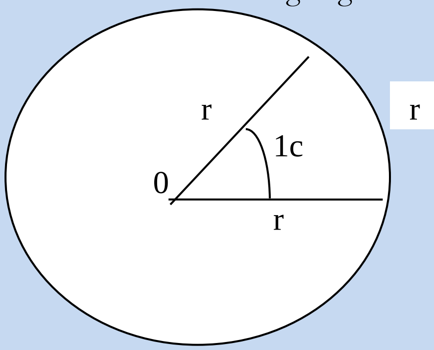


(i) $AC^2 = a^2 + c^2 - 2ac \cos 110^O$
$= 6^2 + 12^2 - 2(6)(12) \cos 110^O$
$= 229,25$
$= 15.km$

(ii) $\triangle = \frac{1}{2}\ ac \sin 110^0$
$= \frac{1}{2}(6)(12)(0.93969262)$
$= 33.83km^2$

Radians

A radian, like a degree is a unit for measuring angles.



One radian (1c) is the angle subtended at the centre of a circle by an arc equal in length to the radius.

$$180^\circ = \pi \text{ rad}$$

$$360^\circ = 2\pi \text{ rad}$$

Angle in radians $= \dfrac{\pi \text{ (angle in degrees)}}{180^\circ}$

Example:
Convert the following angles to degrees

(a) 3πrad (b) $\dfrac{2\pi}{3}$

<u>**Solution**</u>

(a) 3π rad $= 3\pi \times \dfrac{180^\circ}{\pi}$

$= 540^\circ$

(b) $\dfrac{2\pi rad}{3} = \dfrac{2\pi \times 180^\circ}{3\pi}$

$= 120^\circ$

Example
Convert the following angles to radians

(a) 45° (b) 300°

Solution:

(a) $45^{o} = \dfrac{45^{\circ} \times \pi rad}{180^{\circ}}$

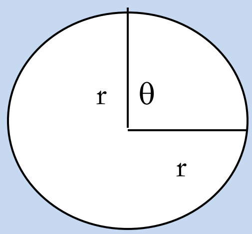
$= \dfrac{\pi rad}{4}$

(b) $300^{o} = \dfrac{300^{\circ} \times \pi rad}{180^{\circ}}$

$= \dfrac{5\pi rad}{3}$

Sectors and segments

With reference to the diagrams below the essential formulae are:

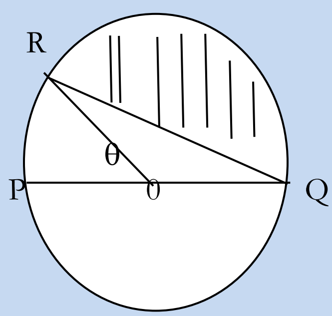


(i) Arc length (s) = $r\,\theta$

(ii) Area of a sector = $\frac{1}{2} r^2 \theta$

(iii) Area of a segment = $\frac{1}{2} r^2 (\theta - \sin\theta)$, where θ is in radians.

Example



Given the semi-circle above and that the area of the shaded part is a quarter of the area of the sector ROP, show that $\theta = 4\,(\pi - \theta - \sin\theta)$

Solution:

Area of shaded segment = $\frac{1}{2} r^2 (\theta - \sin\theta)$

In this case Area $= \frac{1}{2} r^2 \{ (\pi - \theta - \sin \theta)$

Area of sector ROP $= \frac{1}{2} r^2 \theta$

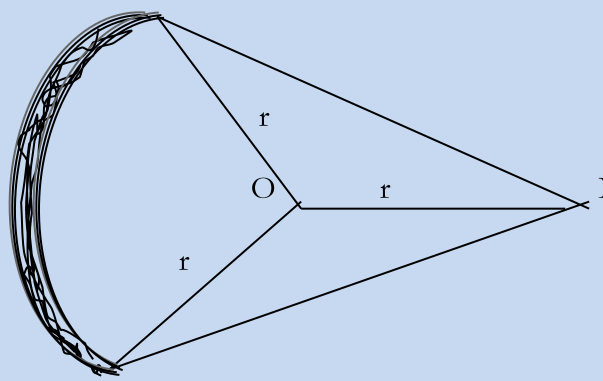
Area of sector ROP = 4 (Area of shaded segment)

$= \frac{1}{2} r^2 \theta = \frac{1}{2} 4r^2 (\pi - \theta - \sin \theta)$, hence,

$\theta = 4(\pi - \theta - \sin \theta)$

Examination Type Questions

1. A



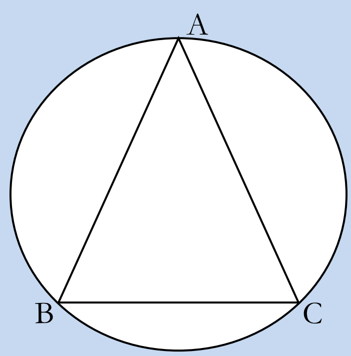
The left edge of the shaded crescent- shaped region, shown in figure, consists of an arc of a circle of radius r cm with centre O the angle AOB = $2/3\pi$ radians. The right edge of the shaded region is a circular arc with centre X, where OX = rcm.

a) show that angle AXB = $1/3\pi$ radians.

b) show that AX = $r\sqrt{3}$cm

c)calculate, in terms of r, π and $\sqrt{3}$, the area of the shaded region.

2.



The diagram shows a shape ABC, which has a triangular hole in the middle. The hole is an equilateral triangle ABC of side 8cm. AB, AB and CA are circular arcs with centers at C, A and B respectively,Calculate

a) the area of the triangle ABC

b) the area of the sector ABC

c)the area of the area outside the triangle

3. a) Express the length ℓ of a chord of a circle with radius r cm as a function of the central angle θ

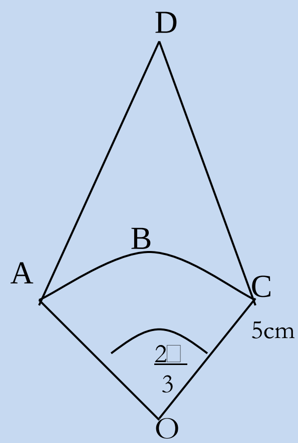
b) If $\ell = 6$cm when $\theta = \pi/3$ radian, find the value of r. Hence show that $\ell = 12 \sin \frac{\theta}{2}$

c) Sketch the graph of $\ell = 12 \sin \frac{\theta}{2}$ where $0 < \theta < \pi$. Hence state the range of ℓ.

d) Find:

i. $d\ell/d\theta$

ii. the maximum value of ℓ



In the diagram, ABC is an arc of a circle with center O and radius 5cm. The lines Ad and CD are tangents to the circle at A and C respectively. Angle AOC = $\frac{2□}{3}$ radius.

(i) Show that the exact length of AD is $(5\sqrt{3})$ cm

(ii) Find the area of the sector AOC, giving your answer in terms of □.

Calculate the area of the region enclosed by AD, DC and arc ABC, giving your answer correct to 2 significant figures

CHAPTER 10

SEQUENCES AND SERIES

OBJECTIVES
By the end of the chapter the student should be able to :

- Write down terms of a sequence
- Determine whether a sequence is convergent or divergent
- State whether the sequence is oscillating or periodic
- Write firsyt few terms of an A-P
- Compute the general term and sum to terms of an A-P
- Write first few terms of G.P
- Compute the general term and sum to terms and sum to infinity of A-G-P
- Solve problems involving A.Ps and G.Ps

Definition

A sequence is a set of numbers in a particular order with each number in the sequence being derived from a particular rule. For example, consider the sequence 1,2,4,8,16,
The first term is 1, the second term is 2, the n^{th} term is 2^n
We write the n^{th} term as U_n. In this example, $U_n = 2^n, n \geq 1$.

Sequences can be given / defined in the following three ways:
a) By listing the terms of a sequence. For example 2, 4, 6, 8,. . ..
b) By giving an n^{th} term, for example $U_n = 3n - 1$, $n \geq 1$
c) By using a recurrence formula, f (n) = f (n –1): e.g. $U_n = U_{n-1}/(3 - U_{n-1})$, $U_1 = 2$.

Example:

Write down the first 4 terms of the following sequences.

a. $U_{n+1} = (U_{n-1} - 1) / U_n$, $U_1 = 7$ and $U_2 = 3$

b. $U_n = 3 + \dfrac{1}{n(n+1)}$, $n \geq 1$

c. $U_{n+1} = U_n + \dfrac{1}{U_n}$ $U_1 = 2$

Solution

a. $U_{n+1} = (U_{n-1} - 1) / U_n$
$U_1 = 7$
$U_2 = 3$
$U_3 = (U_1 - 1)/ U_2 = (7 -1)/3 = 2$
$U_4 = (U_2 - 1)/ U_3 = (3 - 1)/ 2 = 1$
Hence the sequence is : 7, 3, 2, 1, ….

b $U_n = 3 + I / (n + 1)$
$U_1 = 3 + ½ = 7/2$
$U_2 = 3 + 1/6 = 19/6$
$U_3 = 3 + 1/12 = 37/12$
$U_4 = 3 + 1/20 = 61/20$
The sequence is : 7/2, 19/6, 37/12, 61/20 ….

c. $U_{n+1} = U_n + \dfrac{1}{U_n}$
$U_1 = 1$
$U_2 = U_1 + 1/U_1 = 1 + 1 = 2$
$U_3 = U_2 + 1/U_2 = 2 + ½ = 5/2$
$U_4 = U_3 + 1/ U_3 = 5/2 + 2/5 = 29/10$
The sequence is : 1, 2, 2.5, 2.9, ……

Properties of Sequences

We now investigate the behaviour of sequences as n gets bigger and bigger i.e. as n tends to infinity. In mathematical symbols:

$$\lim_{n \to \infty} U_n$$

If $\lim_{n\to\infty} U_n = a$, the sequence U_n is said to be convergent, other wise , U_n is divergent.

Note:

- a is called the limit of the sequence U_n as n tends to infinity
- $a \neq \pm \infty$
- a is unique i.e the limit , if it exists, it is only one.

Example

Write down the first six terms of the following sequences and determine which of the sequences are convergent and which are divergent.

(a) $U_n = n / (n + 1)$
(b) $U_n = \dfrac{(-1)^{n+1}}{n^2}$
(c) $U_n = (-1)^n n$

(d) $U_{n+1} = 1 + 2U_n$, , $U_1 = 3$

(e) $U_{n+1} = 2/ U^2_n$, $U_1 = 1$

(f) $U_{n+1} = U_n (U_{n-1} - 2)$, $U_1 = 3$, $U_2 = 1$

(g) $U_{n+1} = U_n / U_{n-1}$, $U_1 = 4$, $U_2 = 2$

Solution

(a) $U_n = n / (n + 1)$

$U_1 = 1/2$

$U_2 = 2/3$

$U_4 = 4/5$

$U_5 = 5/6$

$U_6 = 6/7$

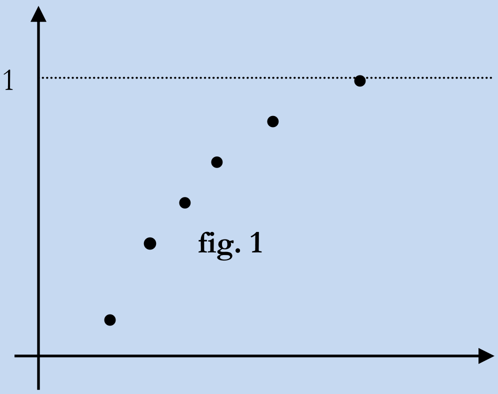


fig. 1

The sequence is convergent and it approaches 1 as n tends to infinity, hence

$\text{Lim}_{n\to\infty}\ n / (n + 1) = 1$

b) $U_n = \dfrac{(-1)^{n+1}}{n^2}$

$U_1 = 1$

$U_2 = - 0.25$

$U_3 = 1/9$

$U_4 = - 1/16$

$U_5 = 1/25$

$U_6 = - 1/36$

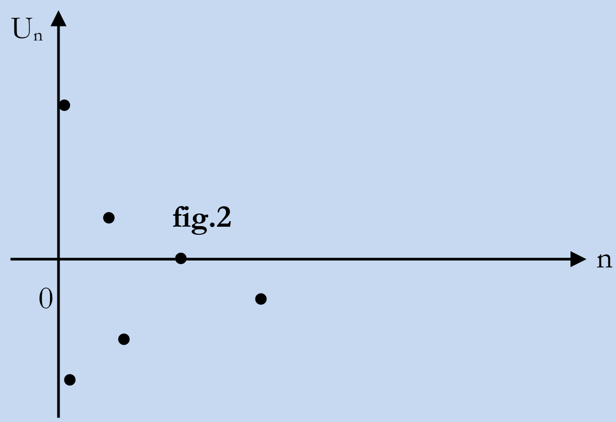


fig.2

The sequence is convergent and $\text{Lim}_{n\to\infty} \dfrac{(-1)^{n+1}}{n^2} = 0$

(c) $U_n = (-1)^n n$

$U_1 = - 1$

$U_2 = 2$

$U_3 = - 3$

$U_4 = 4$

$U_5 = - 5$

$U_6 = 6$

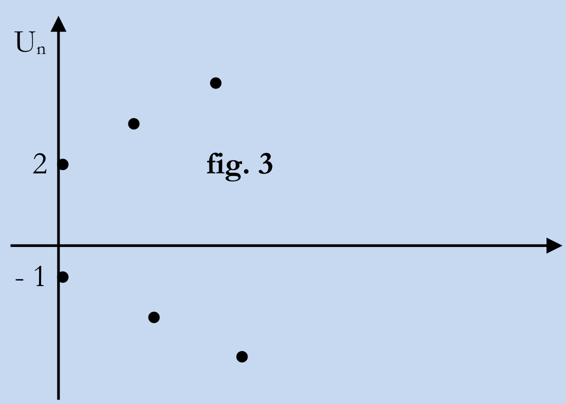


fig. 3

The sequence is divergent

$$\lim_{n \to \infty} (-1)^n n = \begin{cases} +\infty, & \text{for n even} \\ -\infty, & \text{for n odd} \end{cases}$$

(d) $U_{n+1} = 1 + 2U_n$

$U_1 = 3$

$U_2 = 7$

$U_3 = 15$

$U_4 = 31$

$U_5 = 63$

$U_6 = 127$

The sequence is increasing indefinitely hence it is divergent.

(e) $U_{n+1} = 2/U^2_n$

$U_1 = 1$

$U_2 = 2$

$U_3 = ½$

$U_4 = 8$

$U_5 = 1/32$

$U_6 = 2048$

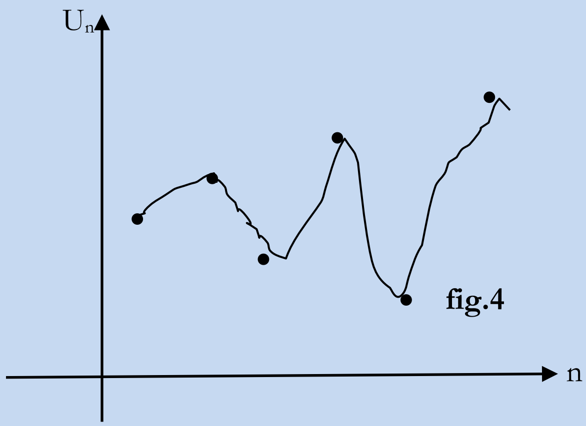


fig.4

$$\lim_{n \to \infty} 2/U^2_n = \begin{cases} 0, & \text{for n odd} \\ \infty, & \text{for n even} \end{cases}$$

The sequence is divergent

(f) $U_{n+1} = U_n(U_{n-1} - 2)$

$U_1 = 3$

$U_2 = 1$

$U_3 = 1$

$U_4 = -1$

$U_5 = 1$

$U_6 = -3$

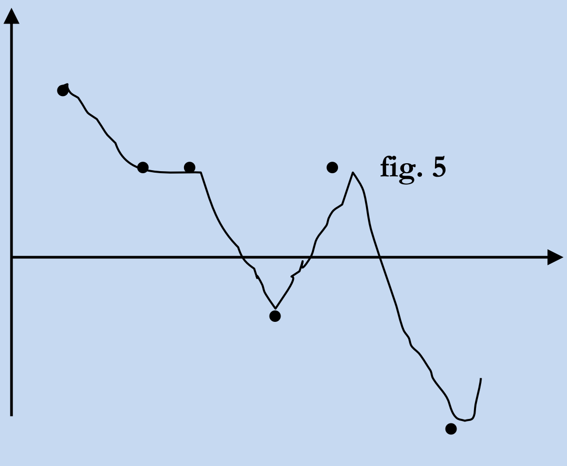


fig. 5

The sequence is divergent

(g) $U_{n+1} = U_n / U_{n-1}$

$U_1 = 4$

$U_2 = 2$

$U_3 = 0.5$

$U_4 = 0.25$

$U_5 = 0.5$

$U_6 = 2$

$U_7 = 4$

The sequence is divergent.

Oscillating and Periodic Sequences

10.3.1 Oscillating Sequences

Consider the sequences

(a) $U_n = \frac{(-1)^{n+1}}{n^2}$

(b) $U_n = (-1)^n n$

(c) $U_n = 2/U^2_n, U_1 = 1$

- For sequence (a),as n tends to infinty the sequence oscillates about the value 0, see **fig. 6** below

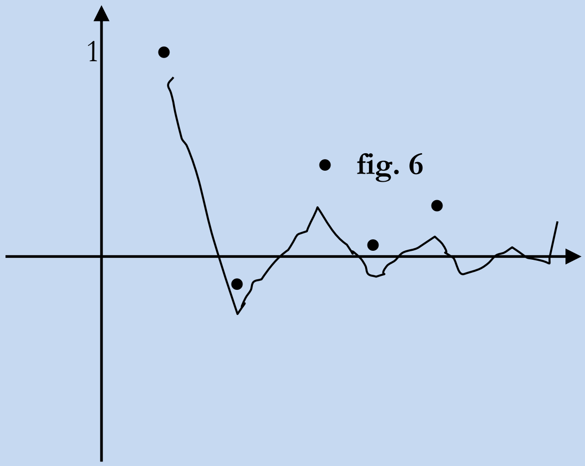
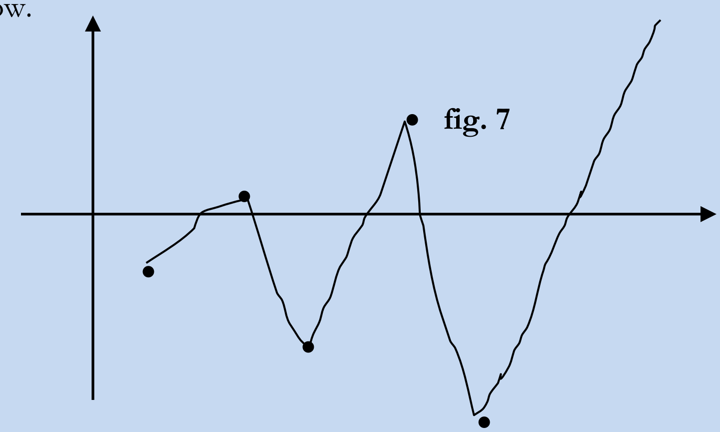


fig. 6

In this case the sequence is convergent to a limit 0

The sequence can oscillate and converge

- For the sequence (b),as n tends to infinity the sequence oscillates about the value 0, see **fig. 7** below.

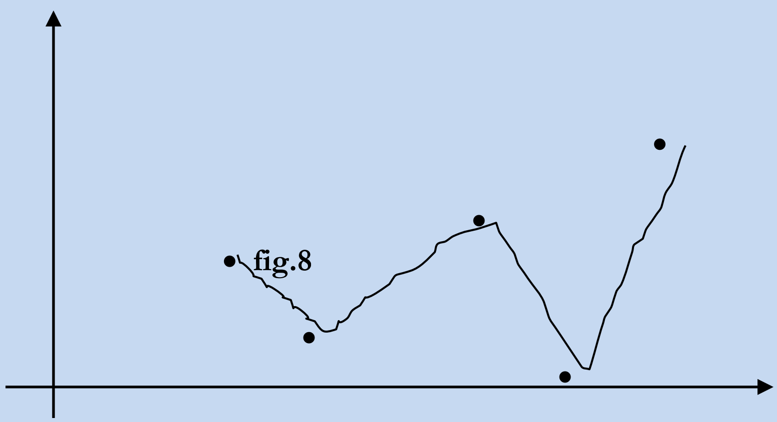


In this case the sequence is divergent

The sequence can oscillate and diverge

For the sequence (c),as n tends to infinity the sequence oscillates. See **fig.8** below

$U_n = 2/U^2_n$
$U_1 = 1$
$U_2 = 2$
$U_3 = ½$
$U_4 = 8$
$U_5 = 1/32$
$U_6 = 2048$



The sequence is also oscillating , but it is divergent.

Periodic Sequences

Consider the sequences:

(a) $U_{n+1} = \dfrac{U_n}{U_{n+1}}$, $U_1 = 4, U_2 = 2$

(b) $U_{n+1} = \dfrac{1}{U_1}$, $U_1 = 7$

For the sequence (a) $U_{n+1} = \dfrac{U_n}{U_{n+1}}$,

$U_1 = 4$
$U_2 = 2$
$U_3 = 1/2$
$U_4 = 1/4$
$U_5 = 1/2$
$U_6 = 2$
$U_7 = 4$

The sequence is periodic i.e. it repeats itself after a period of six, hence the sequence is divergent

For the sequence (b) $U_{n+1} = \frac{1}{U_n}$

$U_1 = 7$
$U_2 = 1/7$
$U_3 = 7$
$U_4 = 1/7$

The sequence is also periodic, hence it is divergent.

A sequence can be convergent or divergent. It can be also oscillatory divergent or oscillatory convergent. But all periodic sequences are divergent. A constant sequence is always convergent.

Example

The sequence {Un} is defined by the recurrence relation $U_{n+1} = U_n + U_{n-1}$

(a) Given that $U_{11} = 683$ and $U_8 = 85$, deduce the equations;
$U_{10} + 2U_9 = 683$ and $U_{10} – U_9 = 170$

(b) Hence find the value of U_9

Solution

(a) $U_{n+1} = U_n + 2U_{n-1}$

$U_{10} = U_9 + 2U_8$
$U_{10} = U_9 + 2 \text{x}\ 85$
$U_{10} - U_9 = 170$ (shown)
$U_{11} = 683$
$U_{11} = U_{10} + 2U_9$
$U_{10} + 2U_9 = 683$ (shown)

(b) $U_{10} + 2U_9 = 683$
$\underline{U_{10} - U_9 = 170}$
$3U_9 = 513$
$U_9 = 171$

Example

The sequence $\{V_n\}$ is defined by $V_n = 3V_{n-1} - 2V_{n-2}$, where $V_1 = 1$ and $V_2 = 2$. Find an expression for V_n in terms of n

Solution

$V_n = 3V_{n-1} - 2V_{n-2}$

$V_1 = 1$

$V_2 = 2$
$V_3 = 3 \times 2 - 2 \times 1 = 4$
$V_4 = 4 \text{ x } 4 - 2 \text{ x } 2 = 8$
$\therefore \{ 1, 2, 4, 8, \ldots\}$

$V_n = 2^{n-1}, n \geq 1$

Example

Show that $U_n = 3 + 2^n$ satisfies the recurrence relation $U_{n+1} = 3U_n - 2U_{n-1}$, where $U_1 = 5$ and $U_2 = 7$. Hence find the value of U_{16}

Solution

$U_n = 3 + 2^n$
$U_1 = 5$
$U_2 = 7$
$3U_n - 2U_{n-1} = 3(3 + 2^n) - 2(3 + 2^{n-1})$
$= 9 + 3 \text{ x } 2^n - 6 - 2^n$
$= 3 + 2^{n+1}$
$= U_{n+1}$
hence $U_{n+1} = 3U_n - 2U_{n-1}$, $U_1 = 5$, $U_2 = 7$

$U_{16} = 3 + 2^{16}$
$U_{16} = 65539$
Alternatively, $U_{16} = 3U_{15} - 2U_{14}$
$= 3(3 + 2^{15}) - 2(3 + 2^{14})$
$= 9 + 3 \text{ x } 2^{15} - 6 - 2^{15}$
$= 65539$

Example

The sequence $U_1, U_2, U_3, \ldots$ is defined by $U_{n+1} = U^2_n - 1$.

(a) Describe the behaviour of the sequence for each of the following cases.
(i) $U_1 = 0$
(ii) $U_2 = 2$

(b) Given that $U_2 = U_1$, find the two possible values of U_1 in exact form.

(c) Given also that $U_3 = U_1$, show that $U^4_1 - 2U^2_1 - U_1 = 0$

Solution

(a) (i) $U_{n+1} = U^2_n - 1$
$U_1 = 0$
$U_2 = -1$
$U_3 = 0$
$U_4 = -1$
The sequence is periodic, hence it is divergent.

(ii) $U_{n+1} = U^2_n - 1$
$U_1 = 2$

$U_2 = 3$
$U_3 = 8$
$U_4 = 63$
The sequence is increasing indefinitely ,hence it is divergent

(b) $U_2 = U_1$
$U_2 = U^2_1 - 1$
Hence $U^2_1 - 1 = U_1$
$U^2_1 - U_1 - 1 = 0$
$U_1 = \frac{1 \pm \sqrt{5}}{2}$

(c) $U_3 = U_1$
$U_3 = U^2_2 - 1$
$U_2 = U^2_1 - 1$
$U_3 = (U^2_1 - 1)^2 - 1$
$U_3 = U^4_1 - 2U2_1 + 1 - 1$
$U_3 = U^4_1 - 2U^2_1$
$\therefore \; U^4_1 - 2U^2_1 = U_1$
$U^4_1 - 2U^2_1 - U_1 = 0$

Practice Questions

1. Write down the first six terms of each of the following sequences, and determine their behavior as n tends to infinity
   (i) $U_{n+1} = 2 + U_n$. $U_1 = 5$
   (ii) $U_{n+1} = 1 - 1/U_n$, $U_1 = 2$
   (iii) $U_{n+1} = U_n / U_{n-1}$, $U_1 = 4$, $U_2 = 2$
   (iv) $U_n = 1/ (n^2 + 1)$

2. For the sequence $U_1, U_2, U_3, \ldots$ the terms are related by $U_{n+1} = U_n - U_{n-1}$, where $U_1 = 1$ and $U_2 = 3$
   (a) Show that the sequence is periodic.
   (b) Find the values of U_{13}, U_{63} and U_{89}

Series and Sigma Notation

A series is the sum of the terms of a sequence. We write the sum of the first n terms of a sequence as S_n, where :
$S_n = U_1 + U_2 + U_3 + U_4 + \ldots.. + U_n$

Hence $U_! + U_2 + U_3 + \ldots. + U_n = \sum_{i=1}^{n} u_i$, Σ is the summation sign and is called sigma

$$S_n = \sum_{i=1}^{n} u_1$$

Example

$$\sum_{i=1}^{4} (6i - 1) = 5 + 11 + 17 + 23$$

$$\sum_{i=1}^{\infty} Ui = U_1 + U_2 + U_3 + \ldots\ldots\ldots\ldots$$

Example

Write each of the following series in sigma ($\sum$) notation

(a) $\frac{2}{3}+\frac{5}{9}+\frac{8}{27}+\frac{11}{81}+\frac{14}{243}+\frac{17}{729}$

(b) $\frac{1}{2\times 3}+\frac{2}{3\times 4}+\frac{3}{4\times 5}+\ldots\ldots\ldots\ldots+\frac{n}{(n+1)(n+2)}$

(c) 4 – 8 +16 - 32 + +64 – 128 + 256 – 512 + 1024

<u>Solution</u>

(a) $\frac{2}{3}+\frac{5}{9}+\frac{8}{27}+\frac{11}{81}+\frac{14}{243}+\frac{17}{729}=\sum_{n=1}^{6}(3n-1)/n^2$

(b) $\frac{1}{2\times 3}+\frac{2}{3\times 4}+\frac{3}{4\times 5}+\ldots\ldots\ldots\ldots+\frac{n}{(n+1)(n+2)}=\sum_{r=1}^{n}r/(r+1)(r+2)$

(c) 4 – 8 +16 - 32 + +64 – 128 + 256 – 512 + 1024 = $\sum_{r=1}^{9}(-1)^{r+1}2^{r+1}$

Practice Questions

1. Write down all the terms in each of these series

(a) $\sum_{r=3}^{6}r^3$ (b) $\sum_{r=1}^{7}(-1)^{r-1/r}$ c) $\sum_{r=1}^{10}\lfloor 1-(-1)^r r^2 \rfloor$

2. Write each of these series in $\sum$ notation

(a) 1 + 2 + 3 + 4 + 5

(b) 7 + 10 + 13 + 16 + 19 + 22 + 25

(c) $3^4+4^4+5^4+\ldots\ldots\ldots+n^4$

(d) $\frac{2}{3}+\frac{5}{9}+\frac{8}{27}+\frac{11}{81}+\frac{14}{243}+\frac{17}{729}$

Arithmetic Progressions (A.P.)

An arithmetic progression (A.P) is a sequence of numbers in which any term can be obtained from the previous term by adding a certain number called the common difference

- The fisrt term of an A.P. is denoted by **a**
- Its common difference is denoted by d

The terms of an A.P. are given by

$a + (a + d) + (a + 2d) + (a + 3d) + \dots\dots\dots + a + (n - 1)d + \dots,$

where $T_n = a + (n - 1)d$.

T_n is also called the last term and is denoted by ℓ, i.e. $\ell = a + (n - 1)d$

Sum of the first n nterms of an A.P.

$S_n = a + (a + d) + (a + 2d) + (a + 3d) + \dots\dots\dots + a + (n - 1)d.$ **1**

$S_n = a + (n - 1)d + a + (n - 2)d + \dots\dots\dots\dots\dots\dots + a + d + a$ **2**

Adding **1** and **2** term to term we obtain:

$2S_n = 2a + (n - 1)d + 2a + (n-1)d + \dots\dots 2a + (n - 1)d$

$2S_n = n(2a + (n - 1)d)$

$S_n = \frac{1}{2} n (2a + (n - 1)d)$ --- the formula of the sum to n terms

$S_n = \frac{1}{2} n (a + \ell)$

If $\sum_{n=1}^{\infty} T_n$ is an A.P. hence $\sum_{n=1}^{\infty} T_n$ coverages if and only if $S_n = \sum_{r=1}^{n} T_r$

Where lim $\lim_{x \to \infty} S_n = \sum_{n=1}^{\infty} T_n$ conversely if S_n diverges, $\sum_{n=1}^{\infty} T_n$ also diverges

Example

(a) Write down the term indicated

$1.25x + 1.5x + 1.75x + \dots\dots$ 10^{th} term

(b) Find the sum as far as the indicated term

$14c + 4c - 6c - \dots..$ 30^{th} term

(c) Find the number of terms

$7 + 9 + \dots\dots\dots \dots + (2n + 1)$

(d) Find the sum of the following A. P.

$1 + 2 + 3 + 4 + \dots\dots\dots.. + 2n$

Solution

(a) Write down the term indicated

$1.25x + 1.5x + 1.75x + \ldots\ldots..$

$d = 1.5x - 1.25x = 0.25x$

$a = 1.25x$

$T_n = a + (n - 1)d$

$T_{10} = 1.25x + 9(0.25x)$

$T_{10} = 3.5x$

(b) $14c + 4c - 6c - \ldots..$

$a = 14c$

$d = 4c - 14c = -10c$

$T_n = a + (n - 1)d$

$T_n = 14c + (n - 1)(-10c)$

$T_{30} = -276c$

$S_n = \frac{1}{2} n (a + \ell)$

$S_{30} = 15(14c - 276c)$

$S_{30} = -3930c$

(c) $7 + 9 + \ldots\ldots\ldots.. + (2n + 1)$

$a = 7$

$d = 2$

$N = ?$

$T_N = a + (N - 1)d$

$a + (N - 1)d = 2n + 1$

$7 + (N - 1)2 = 2n + 1$

$N - 1 = \frac{1}{2}(2n + 1 - 7)$

$N = n - 2$

(d) $1 + 2 + 3 + + \ldots\ldots\ldots + 2n$

$a = 1$

$d = 1$

$T_N = 2n$

$1 + (N - 1) = 2n$

$N = 2n$

$S_N = S_{2n} = \frac{1}{2}(2n)(1 + 2n)$

$S_N = 2n^2 + n$

Example

Given that $\sum_{r=n+3}^{2n} = 312$. Find the value n

Solution

$N = 2n - n - 3 + 1 = n - 2$; there are $n - 2$ terms

$a = n + 3$

$d = 1$
$S_N = ½(n – 2) [2n + 6 + (n – 2 – 1)]$
$S_{N+}= ½(n – 2)(3n + 3)$
$312 = ½(n – 2)(3n + 3)$
$624 = (n – 2)(3n + 3)$
$624 = 3n^2 – 6n + 3n – 6$
$630 = 3n^2 – 3n$
$3n^2 –3n – 630= 0$
$n^2 – n – 210= 0$
$n = ½(1 \pm\sqrt{841})$
$n = - 14$ **(reject)** or 15
$n = 15$

Example
A child is collecting bottle tops. He collects six bottle tops on the first day of the month and stores them in a box. On the second day of the month he collects another 10 bottle tops, and adds them to his box. He continues in this way, each day collecting four more bottle tops more than he collected on the previous day. Find the day of the month on which the number of bottle tops in his box will first exceeds 1000

Solution
$a = 6$
$T_2 = 10$
$T_3 = 14$
It is an A.P.
$T_n = a + (n – 1)d$
$T_n = 6 + (n – 1)4$
$T_n = 4n + 2$

$S_n = 1/2n (a + \ell)$
$S_n = 1/2n (6 + 4n + 2)$
$S_n = 1/2n (8 + 4n)$
$S_n = 1000$
$n (4 + 2n) > 1000$
$2n^2 + 4n – 1000 > 0$
$n^2 + n – 500 > 0$
$n = ½(1 \pm\sqrt{2004})$
$n > 2138$
$n = 22$
i.e on the 22nd day of the month

Example
The 8th term of an arithmetic series is 5 and the sum of the first 16 terms is 84. Calculate the sum of the first ten terms

Solution
$T_n = a + (n – 1)d$

$T_8 = a + 7d$
$a + 7d = 5$ ….**(1)**
$S_{16} = 8\,(2a + 15d)$……$S_n = \frac{1}{2}(n)(2a + (n-1)d)$
$8\,(2a + 15d) = 84$
$16a + 120d = 84$
$8a + 60d = 42$
$4a + 30d = 21$…..**(2)**
$a + 7d = 5$
$\underline{4a + 30d = 21}$
$4a + 28d = 20$
$\underline{4a + 30d = 21}$
$2d = 1$
$\underline{d = \frac{1}{2}}$

$a + 3.5 = 5$
$\underline{a = 1.5}$

$S_n = \frac{1}{2}(n)(2a + (n-1)d)$
$S_{10} = 5\,(2 \times 1.5 + 0.5 \times 9)$
$S_{10} = 15 + 22.5$
$S_{10} = 37.5$

Geometric Progression (G. P.)

A geometric progression (G. P) is a sequence of numbers in which any term can be obtained from the previous term by multiplying by a certain number called the common ratio

- the first term is denoted by ***a***
- the common ratio by ***r***

hence
$a + ar + ar^2 + ar^3 + \ldots\ldots + ar^n + \ldots\ldots$
is a geometric progression, where

$T_n = ar^{n-1}$, $n \geq 1$ or
$T_n = ar^n$, $n \geq 0$

Sum of the first n terms of a G.P.

Let $S_n = a + ar + ar^2 + ar^3 + \ldots\ldots + ar^n$
and $rS_n = ar + ar^2 + ar^3 + ar^4 + \ldots\ldots + ar^{n-1}$, hence
$S_n - rS_n = a - ar^n$
$S_n(1 - r) = a - ar^n$

$$S_n = \frac{a(1-r^n)}{1-r}, \quad r \neq 1 \text{ or } S_n = \frac{a(r^n-1)}{r-1}$$

Example

In a G. P. the second term is - 12 and the fifth term is 768. Find the common ratio and the first term

<u>Solution</u>

$T_2 = ar$, hence $ar = -12$ … **(1)**

$T_5 = ar^4$, hence $ar^4 = 768$ … **(2)**

Divide (2) by (1) to obtain

$ar^4 / ar = -768 / 12$

$r^3 = -64$

$r = -4$, hence

$ar = -12$

$a(-4) = -12$

$a = 3$

Example

A child tries to negotiate a new deal for her pocket money for the 30 days of the month of May. She wants to be paid \$ 10000. 0n the first of the month, \$ 20 000 on the second of the month and in general \$ 2^{n-1} 0000 on the nth day of the month. Calculate how much she would get, in total , if this were accepted.

<u>Solution</u>

$a = 10000$

$T_2 = 20000$

$T_3 = 40000$

$r = 2$

$$S_n = a\frac{(r^n-1)}{r-1}$$

$$S_{30} = 10000\frac{(2^{30}-1)}{2-1}$$

$S_{30} = \$1.073741823 \times 10^{13}$

Example

Given that $x - 5$, $x - 2$ and $3x$ are the first, second and fourth terms of a G.P. Find the possible values of x

Solution

$a = x - 5$, $ar = x - 2$ and $ar^3 = 3x$, hence

$(x - 5)\, r = x - 2$

$r = (x - 2) / (x - 5)$

$ar^3 = 3x$

$(x - 5)(x - 2)^3 / (x - 5)^3 = 3x$

$(x - 2)^3 = 3x\,(x - 5)^2$

$x^3 - 6x^2 + 12x - 8 = 3x^2 - 30x^2 + 75x$

$2x^3 - 24x^2 + 63x + 8 = 0$

Let $f(x) = 2x^3 - 24x^2 + 63x + 8$

$f(8) = 2(8)^3 - 24(8)^2 + 63 \times 8 + 8$

$f(8) = 0$, hence $x - 8$ is a factor

$2x^2 - 8x - 1$

$x - 8 \;\big)\; 2x^3 - 24x^2 + 63x + 8$

$2x^3 - 16x^2$

$- \;\; 8x^2 + 63x + 8$

$- \;\; 8x^2 + 64x$

$- \;\; x + 8$

$- x + 8$

$2x^2 - 8x - 1 = 0$

$x = \frac{1}{2}(8 \pm \sqrt{72})$, hence the values of x are 8, $2 \pm \sqrt{4.5}$

Infinite Geometric Series

$$S_\infty = \lim_{n \to \infty} a(1 - r^n) / (1 - r) = a/(1 - r), \text{ for } -1 < r < 1$$

Example

∞

Given that $\sum_{r=0} 5 \times a^r = 15$, find the value of a

Solution

$T_1 = 5$
$T_2 = 5a$
$T_3 = 5a^2$
$r = a$
$S_\infty = a/(1 - r)$
$15 = 5/(1 - a)$
$15 - 15a - 5 = 0$
$15a = 10$
$a = 2/3$

Example

(a) Find the third term of the geometric series whose first two terms are 3 and 4

(b) Given that x, 4, and x + 6 are consecutive terms of the geometric series, find

i) The possible values of x

ii) The corresponding values of the common ratio of the geometric series

Given that x, 4, and x + 6 are the sixth, seventh and eighth terms of a geometric series and that the sum to infinity of the series exists, find

iii) The first term

iv) The sum to infinity

Solution

(a) $a = 3$
$T_2 = 4$
$r = 4/3$
$T_3 = 3(4/3)^2$
$T_3 = 16/3$

(b)

(i) $4/x = (x + 6)/4$
$16 = x^2 + 6x$
$x^2 + 6x - 16 = 0$
$x = \frac{1}{2}(-6 \pm \sqrt{100})$
$x = -8$ or $x = 2$

(ii) $T_6 = -8$ or 2, hence $r = -0.5$ or 2

(iii) $T_6 = ar^5$ $r = -0.5$ for sum to infinity to exist, hence
$a(-0.5)^5 = -8$
$a = 256$

$S_\infty = 256/(1 - (-0.5))$
$S_\infty = 512/3$

Example

A savings scheme pays 5 % per annum compound interest . A deposit of $ 100 is invested in this scheme at the start of each year.

(a) Show that at the start of the third year, after the annual deposit has been made, the amount in the scheme is $ 315.25

(b) Find the amount in the scheme at the start of the fortieth year, after the annual deposit has been made.

<u>**Solution**</u>

(a) $a = 100$

$T_2 = 100 \times 1.05 + 100 = 205$

$T_2 = 205 \times 1.05 + 100 = 315.25$

(b) $a = 100$

$T_2 = 100 \times 1.05 + 100$

$T_3 = (100 \times 1.05^2 + 100)\,1.05 + 100$, hence

$T_n = 100 \times 1.05^n + 100 \times 1.05^{n-1} + 100 \times 1.05^{n-2} + \ldots + 100$

$T_n = 100(r^n + r^{n-1} + \ldots + 1)$

$T_n = 100(1 + r + r^2 + \ldots + r^n)$

$$T_{40} = \frac{100(1.05^{40} - 1)}{0.05}$$

$\underline{\underline{T_{40} = \$\ 12079.98}}$

Example

Express the recurring decimal $0.\dot{1}\dot{2}$ in the form a/b, where a and b are integers

<u>**Solution**</u>

$0.\dot{1}\dot{2} = 0.\dot{1}\dot{2}\dot{1}\dot{2} = a + ar + ar^2 + ar^3 + \ldots$

$0.\dot{1}\dot{2} = 12/100 + 12/1000 + 12/10000 + \ldots$

$a = 12/100$

$a = 3/25$

$r = 0.01$

$S_\infty = a/(1 - r)$

$S_\infty = 3/25 \div (1 - 0.01)$

$\underline{\underline{S_\infty = 4/33}}$

Summary

To differentiate between an A.P and a G.P , find the first three terms of the series, if there is a common difference , the series is an A.P, if there is a common ratio the series is a G.P. This comes hand when dealing with word problems

Examination Type Questions

1. The nth terms of two sequences are defined as follows:
   a) $u_n = 1 - 1/n$ b) $u_n = 1 - 1/u_{n-1}$ where $u_1 = 2$
   Decide in each case whether the sequence is convergent, divergent, oscillating or periodic, giving reasons for your answers

2. Determine the behaviour of the following sequences
   a) $U_{n+1} = U_n(U_n - 2)$, $U_1 = 1$
   b) $U_{n+1} = U^2_n - U_{n-1}$, $U_1 = 8$, $U_2 = -3$
   c) $X_{n+1} = X_n + X_{n-1}$, $X_1 = X_2 = 1$ (the famous fibonacci series).

3. A woman saves $120 during the first year, $150 in the second year and $180 in the third year. If she continues her savings according to this scheme, in which year will she save $1020?

4. The price of a loaf of bread is $65000. If the price of a loaf of bread increases by $25000 every month , find how long it will take for the to be $5750000.

5. The 1st term of an arithmetic series is 38 and the tenth term is 2. Given that the sum of
   the first n terms of the series is 72, calculate the possible values of n.

6. The first term of a geometric series is 5 and the common ratio is 1.2. Find for this series:
   a) the 16th term, giving your answer to the nearest integer.
   b) the sum of the first 30 terms, giving your answer to the nearest integer.
   c) Give a reason why this series has no sum to infinity.

7. A proprietor of a certain industry makes a list of her five favourite charities, donating $6million to the first on the list. If each remaining charity receives half of the amount donated to the preceding charity on the list, how much money does she donate? altogether?

8. Express 0.3 4 5 as a rational fraction.

9. In 1996, a certain school had a population of 950. Students had been leaving the school at a rate of 25 per year. Also, because of a false rumour that the school was closing down there was a decrease in the population of the school at the of 5% of the previous year's population:
   a) Set up a recurrence equation representing the given data. Hence create a mathematical model for projecting the population of the school
   b) Determine when the population of the school will be less than 500

10. In a potato race, the first potato is 5m from the finishing line, the second is a further 10m away, the third a further 10mm away, and soon. There are 6 potatoes

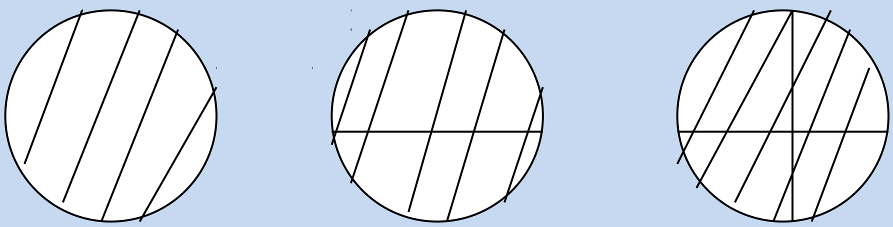
per competitor. If a competitor starts at the finishing line, runs to the first potato, brings it back to the finishing line, runs to the second, and brings it back, etc. how much distance will the competitor cover?

11. A car manufacturing company is planning to manufacture a new car with expected sales of 3000 in the first year. The plan for market penetration of this car is to increase the sales by 650 cars per year. Assuming that such a program for expansion of sales is feasible, determine:

    (a) the first year in which the number of cars sold will exceed 5550.
    (b) how long it will take to sell 42200 cars.

12. You and five other students each bring a cake to a birth day party. These cakes are cut into pieces as follows:

    - the first cake is served whole (i.e. in one piece).
    - The second cake is divided (by one single cut) into two pieces.
    - The third cake is divided (by two intersecting cuts) into 4 pieces.



This process continues until the sixth cake, which receives five cuts.

(a) Calculate the total number of pieces produced using all the cakes.
(b) How many pieces would be produced if 51 students attended the birth day party?
(c) Create a general formula which will enable you to compute the total number of pieces produced if n students attended the party.
(d) How many students attended the party if the total number of pieces generated is 9901.

13. Given that $\sum_{r=n+3}^{2n}$ = 312, find the value of n

14. In an A. P. the nth term is 11, the sum of the first n terms is 72, and the first term is 1. Find the value of n.

15. The nth term of two sequences are defined as follows:

(b) $t_n = 1 - \dfrac{1}{n}$ (b) $u_n = 1 - \dfrac{1}{u_{n-1}}$

where $u_1 = 2$

Determine in each case whether the sequence is convergent, divergent oscillating or periodic.

16. If the sum of the infinite geometric series $x^2 + -\frac{x^2}{1-x} + \frac{x^2}{(1-x)^2} + \ldots\ldots\ldots\ldots is 380$

what are the two possible values of x?

(i) Evaluate $\sum_{r=1}^{500}(3r+2)$

(ii) Find the consonants *a* and *b*.

A sequence U_r is defined by $U_r = (n-3r)$

(i) Write down the first 3 terms of the sequence.
(ii) Find in terms of *n* a formula for

$$\sum_{r=n}^{2n} ur$$

Three sequences are defined below, for

(a) n = 1, 2, 3…. Describe the behaviour of each sequence as *n* tends to infinity.

(i) $a_n = (-1)^n$
(ii) $b_n = 2 - n$
(iii) $C_n = (-1)^n + 3n$

(b) A sequence $U_1, U_2, U_3, \ldots$ is defined by $U_1 = 2$ and $U_{n+1} = U_n + 3$ for $n \geq 1$

(i) Write down the first four terms of the sequence.
(ii) State what type of a sequence it is and express U_n in terms of *n*.

(c) Geometric progression of possible terms is such that the sum of its first two terms is 24 and the third term is 2. Find the common ratio and the sum to infinity of this *P*.

17. In the sequence 1.0, 1.1, 1.2, …., 99.9, 100.0 each number after the first is 0.1 greater than the preceding number. Find

(i) How many numbers are there in the sequence
(ii) The sum of all the numbers in the sequence

CHAPTER 11

VECTORS (I)

OBJECTIVES
By the end of the chapter the student should be able to :

- Add and subtract vectors
- Calculate unit vectors
- Calculate a scalar product
- Calculate an angle between two line segments
- Solve problems involving vectors in space

Conventions

There are two types of quantities that the student meets in life. Vector quantities and scalar quantities. Vector quantities are those that have both magnitude and direction whereas scalar quantities have magnitude only.

There are a number of ways of denoting a vector.

(i) **AB**

(ii) $\underline{a}$

(iii) $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$

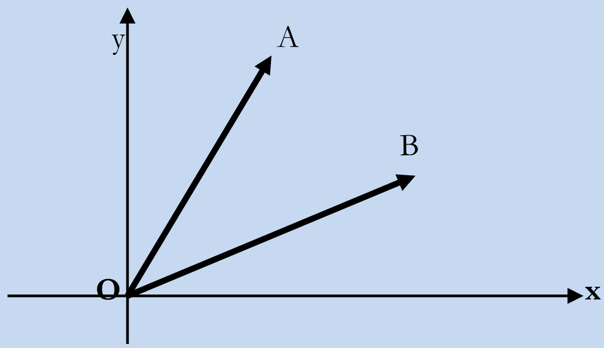
(iv) x**i** + y**j** + z**k**

Position and displacement vectors

On the Cartesian plane vectors are represented by lines

(i) Position or localized vectors are those that are fixed in space. They are usually written as **OA, OB** e.t.c where **O** is the origin

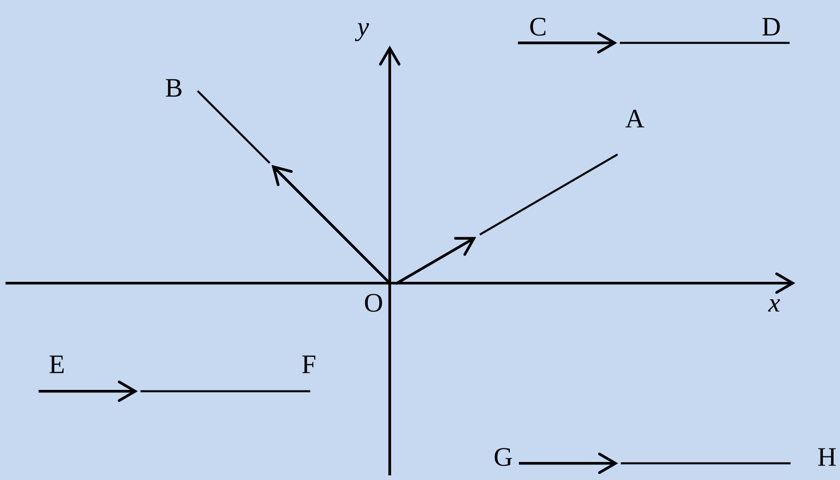
They are drawn with one end at the origin (see the diagram below)



Free or displacement vectors are not fixed in space. In the diagram below **CD** and **EF** are equivalent vectors.

CD and **GH** are opposite vectors as they have the same magnitude but are opposite in direction.

Addition and Subtraction



Vectors can be added and subtracted.

Example. If **AB** = 2i + 3j + 4k
CD = 5i + 2j + k

AB + CD = 2i + 3j + 4k + (5i + 2j + k)

= 7i + 5j + 5k

AB – CD = 2i + 3j + 4k - (5i + 2j + k)

= - 3i+ 2j + 3k

Zero vectors

$$O = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Magnitude of a vector

I f **a** = a_1**i** + a_2**j** + a_3**k** , then $|\mathbf{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$

Example

If

$$\mathbf{AB} = \begin{pmatrix} 2 \\ 4 \\ 5 \end{pmatrix}$$

then $|\mathbf{AB}| = \sqrt{(2^2 + 4^2 + 5^2)}$

$= 3\sqrt{5}$

The unit vector

A unit vector is a vector with magnitude of 1. A unit vector in the direction of vector **a** is given by

$$\hat{e} = \frac{a_1 i + a_2 j + a_3 k}{\sqrt{a_1^2 + a_2^2 + a_3^2}}$$

Example

Given that

$$\mathbf{CD} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

The unit vector in the direction of **CD**

$$\mathbf{CD} = \frac{i + 2j + 3k}{\sqrt{1^2 + 2^2 + 3^2}}$$

$$\mathbf{CD} = \frac{i + 2j + 3k}{\sqrt{14}}$$

Multiplication by A Scalar

If

$$\mathbf{a} = \begin{pmatrix} 2 \\ 3 \\ 5 \end{pmatrix}$$

then

$$3\mathbf{a} = 3\begin{pmatrix} 2 \\ 3 \\ 5 \end{pmatrix} = \begin{pmatrix} 6 \\ 9 \\ 15 \end{pmatrix}$$

The Dot or Scalar Product.

Given two vectors **a** and **b**, we define multiplication of vectors.

a . b = $|a||b| \cos\theta$, where θ = angle between vectors **a** and **b**

Example

If **a** = 2**i** +3**j** – **k** and **b** = 4**i** + 5**j** – 3**k** and $\angle(a,b) = 30^0$, find the scalar product of **a** and **b.**

Solution

a.b = $|a|b| \cos\theta$

A scalar product is a number

$|\mathbf{a}| = \sqrt{2^2 + 3^2 + (-1)^2}$

$= \sqrt{4+9+1}$

$= \sqrt{14}$

$|\mathbf{b}| = \sqrt{4^2 + 5^2 + (-1)^2}$

$= \sqrt{16 + 25 + 1}$

$= \sqrt{42}$

Hence: $\mathbf{a.b} = \sqrt{14} \,.\, \sqrt{42} \cos 30^\circ$

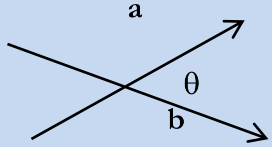
$\mathbf{a.b} = \sqrt{14} \,.\, \sqrt{42}\, \dfrac{\sqrt{3}}{2}$

$\mathbf{a.b} = \dfrac{14.3}{2}$

$\mathbf{a.b} = 21$

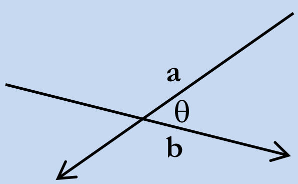
1.1.5 Angle between two vectors a and b.

a)



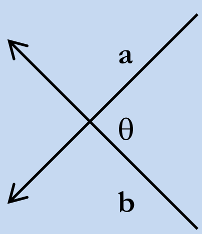
$\theta = \angle(a, b)$

b)



$180 - \theta = \angle(a, b)$

c)



$\theta = \angle(a,b)$

1.1.6 Properties of the scalar product.

$\mathbf{a} \perp \mathbf{b}$ if and only if $\mathbf{a} . \mathbf{b} = 0$

a) $\mathbf{a.b} = 0$ if $\theta = 90^0$
b) $\mathbf{a.b} = |a|.|b|$ if $\theta = 0^0$

c) **a.b** = - |a| |b| if $\theta = 180^0$.

i ⊥ j, i ⊥ k, j ⊥ k. hence i.j = i.k = j.k = 0 and i.i = j.j = k.k = 1

Given that vectors **a** = $x_1\mathbf{i} + y_1\mathbf{j} + z_1\mathbf{k}$

and **b** = $x_2\mathbf{i} + y_2\mathbf{j} + z_2\mathbf{k}$

a·b = $(x_1\mathbf{i} + y_1\mathbf{j} + z_1\mathbf{k})(x_2\mathbf{i} + y_2\mathbf{j} + z_2\mathbf{k}$

$= x_1 \cdot x_2\, \mathbf{i.i} + x_1y_2\, \mathbf{i.j} + x_1z_2\mathbf{i.k} + y_1x_2\, \mathbf{i.j} + y_1y_2\, \mathbf{j.j} + y_1z_2\, \mathbf{j.k} + z_1x_2\, \mathbf{i.k} + z_1y_2\, \mathbf{j.k} + z_1z_2\, \mathbf{k.k}$

$= x_1 \cdot x_2 + y_1y_2 + z_1 \cdot z_2$

Hence.

$$\boxed{\mathbf{a \cdot b = x_1 \cdot x_2 + y_1y_2 + z_1 \cdot z_2.}}$$

Example Find **a · b**, if **a** = 2i + 3j – 3k and **b** = -4i + 5j - k

Solution
A (2,3, - 3) ; B = (-4,5,- 1)
Hence: **a·b** = 2(-4) + 3(5) + (-3)(-1)
a·b = -8 + 15 + 3
a·b = 10

Example
Let **AB** = 4i + j – k; **CD** = 3i – j – 2k find the angle between AB and CD.

Solution
a·b = |a| |b| cos < (a ,b)

$$\boxed{< (a,b) = \cos^{-1} \frac{a.b}{|a||b|}}$$

AB·CD = 4(3) + 1(-1)+ (-1)(-2)
=12 – 1 + 2
= 13

$|\mathbf{AB}| = \sqrt{4^2 + 1^2 + (-1)^2}$

$= \sqrt{18}$

$|\mathbf{CD}| = \sqrt{3^2 + (-1)^2 + (-2)^2}$

$= \sqrt{9+1+4}$

$= \sqrt{14}$

hence: $\theta = \cos^{-1} \dfrac{AB.CD}{|AB||CD|}$

$= \cos^{-1} \dfrac{13}{\sqrt{18\sqrt{14}}}$

$= 35°$

Example

If **a** = 4i + λj – k and **b** = 3i – j + k, find the values of λ, if **a** is perpendicular to **b**.

Solution

Given **a** ⊥**b** then **a·b** = 0

a·b = 4(3) + λ(-1) + (-1)(1)

= 12 - λ- 1

=11 - λ

Hence 11 - λ = 0

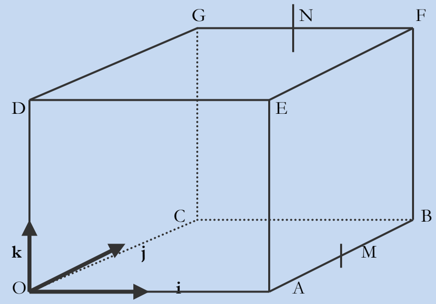
i.e. λ = 11

Examination Type Questions

1. Given that O (3,4,1), Q (2,- 3, 1) and R (3, - 1, 2), find the scalar product of **OQ** and **OR** and show that cos $Q\hat{O}R = \dfrac{7\sqrt{13}}{26}$

   Hence find the exact area of triangle OQR

2.



In the diagram OABCDEFG is a cube in which the length of each edge is 4 units. Unit vectors i, **j**, and k are parallel to **OA, OC, OD** respectively. The mid points of AB and FG are M and N, respectively

i) Express each of the vectors **ON** and **MG** in terms of **i, j** and **k**

ii) Find the acute angle between the directions of **ON** and **MG**, correct to the nearest 0.1^0

3. The points A and B have positions vectors i + 2**j** + 2**k** and 4**i** + 3**j**, respectively relative to an origin O

a) Find the lengths of OA

b) Find the scalar product of **OA** and **OB** and hence find angle AOB

c) Find the area of the triangle AOB, giving your answer correct to 2 d.p.

d) The point C divides AB in the ratio $\lambda: 1 - \lambda$.

(iii) Find an expression for **OC**

(iv) Show that $OC^2 = 14\lambda^2 + 2\lambda + 9$

(v) Find the position vectors of the two points on AB whose distance from O is $\sqrt{21}$

(vi) Show that the perpendicular distance of O from AB is approximately 2.99.

4. The points A, B, C have position vectors

a = 2**i** + **j** – **k**   **b** = 3**i** + 4**j** – 2**k**   **c** = 5**i** – **j** + 2**k**

respectively, relative to the origin O.

a) Evaluate the scalar product (a – **b**) **.** (c – **b**).

Hence calculate the size of angle ABC, giving your answer to the nearest 0.1^0

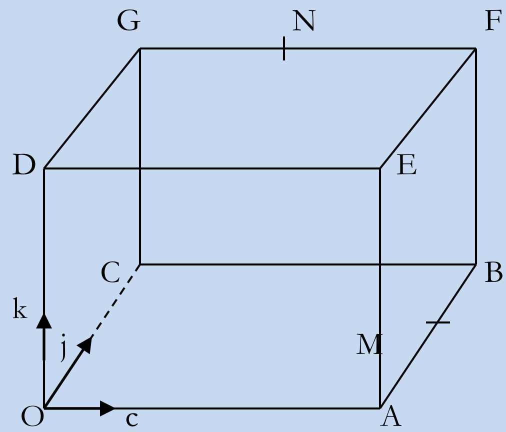
b) Given that ABCD is a parallelogram:

i) determine the position vector of D

ii) calculate the area of ABCD

c) The point E lies on BA produced so that **BE** = 3**BA**. Write down the position vector of E. The line CE cuts the line AD at X. Find the position vector of X.

5. The position vectors of three points A, B and C with respect to a fixed origin O are 2i – j + k, 4i + j + k and i + j + 3k respectively. Find the unit vector in the directions of **CA** and **CB**. Calculate angle $A\hat{C}B$ in degrees, correct to 1 decimal place.



6.. In the diagram, OABCDEFG is a cube in which the length of each edge is 2 units. Unit vectors i, j, k are parallel to OA, OC, OD respectively. The mid points of AB and FG are M and N respectively.

(i) Express each of the vectors ON and MG in terms of i, j and k.

(ii) Find the angle between the directions of ON and MG, correct to the nearest 0, 1°.

CHAPTER 12

TRIGONOMETRIC FUNCTIONS

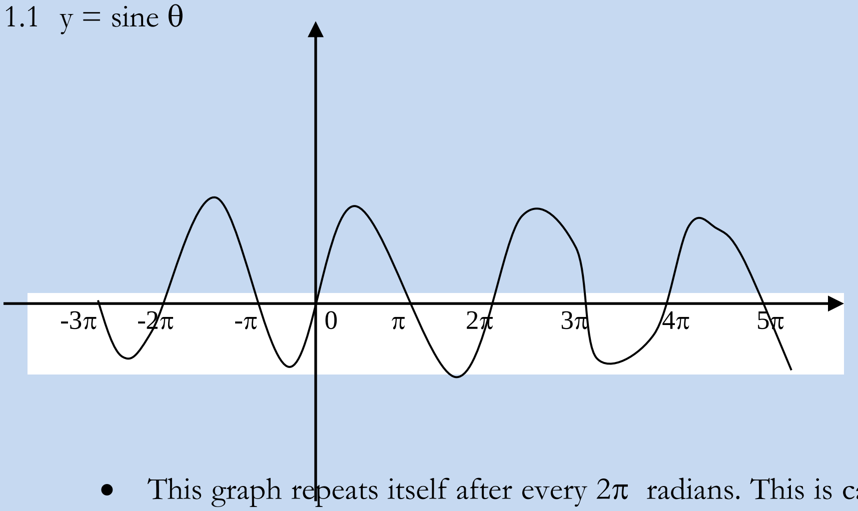
OBJECTIVES
By the end of the chapter the student should be able to :

- Sketch graphs of trig functions
- Prove trig identities
- Solve trig equations

The graphs of the Trigonometric Functions

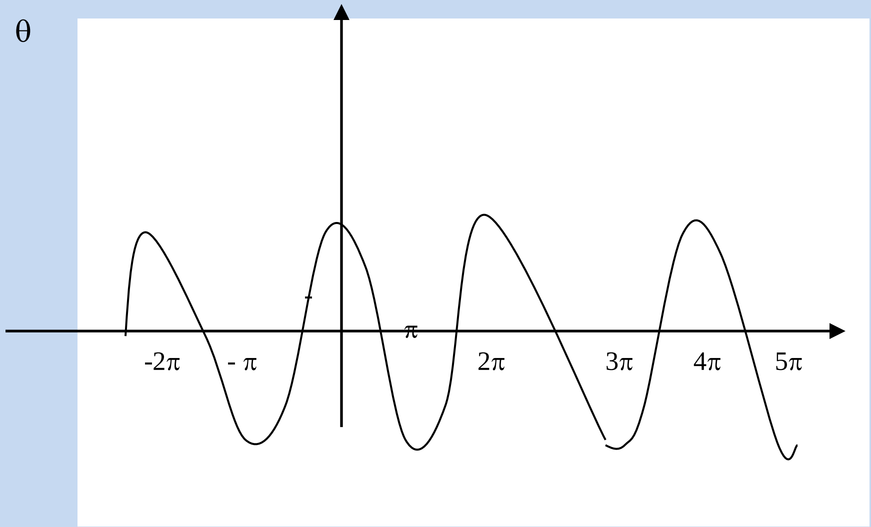
We are interested in the functions Sine, Cosine and tangent, secant , cosecant and cotangent.

1.1 y = sine θ



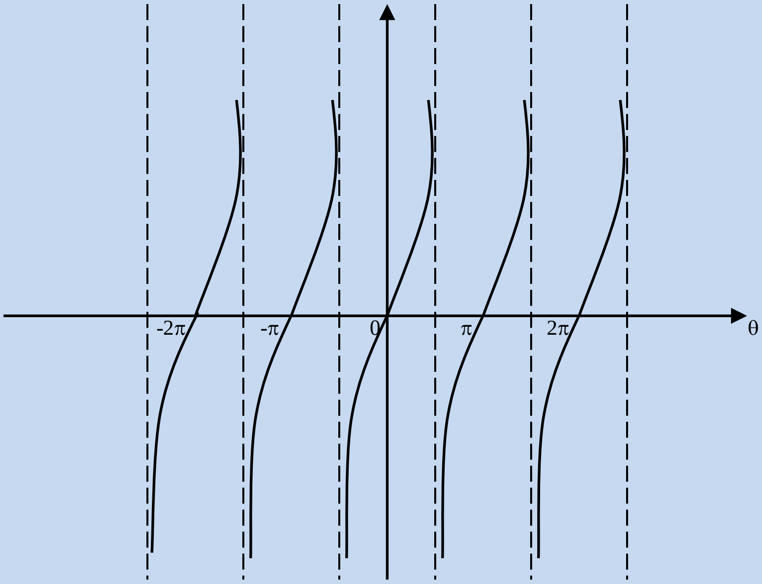
- This graph repeats itself after every 2π radians. This is called the period of the graph.
- $-1 \le \sin\theta \le 1$
- $\sin\theta = 0$ for $\theta = n\pi$, n is an integer
- the function is not one to one, however, if the domain is restricted to $-\pi \le \theta \le \pi$, the function becomes one to one and posses an inverse.

1.2 y = cosine θ



- The cosine function has a period of 2π
- $-1 \le \cos\theta \le 1$
- $\cos\theta = 0$ for $\theta = \frac{1}{2}\pi \pm 2n\pi$
- the function is not one to one, but if the domain is restricted to $0 \le \theta \le \pi$, the function becomes one to one.

1.3 y = tanθ



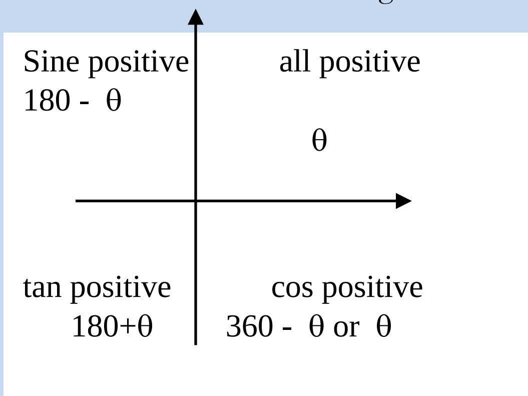
(i) The period is π

(ii) $-\infty < \tan\theta < \infty$

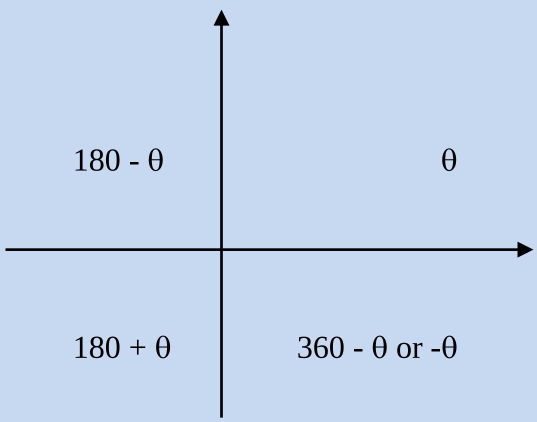
Surds

The student should recall the following

(a)

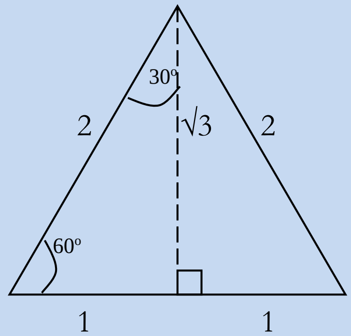


(b)

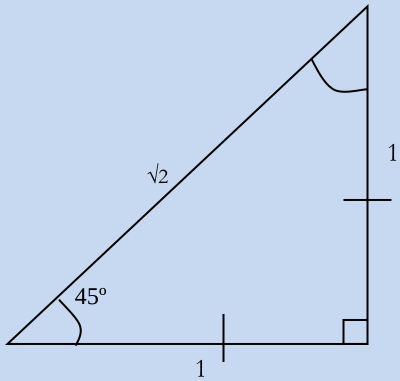


Where θis the principal angle to be calculated?

(c)



(d)



Example

$\sin 60^{o} = \frac{\sqrt{3}}{2}$

$\cos 45^{o} = \sqrt{2}/2$

Addition formulae

These are:

(i) $\cos (A \pm B) = \cos A \cos B \pm \sin A \sin B$

(ii) $\sin (A \pm B) = \sin A \cos B \pm \cos A \sin B$

(iii) $\tan (A \pm B) = \frac{\tan A + \tan B}{1 + \tan A . \tan B}$

The double angle formulae

(By putting B = A in the formulae above)

(iv) $\sin 2A = 2\sin A \cos A$

(v) $\cos 2A = \cos^2 A - \sin^2 A$

$= 1 - 2\sin^2 A$

$2\cos^2 A - 1$ using $\sin^2 A + \cos^2 A = 1$

(vi) $\quad \tan 2A \quad = \dfrac{2\tan A}{1-\tan^2 A}$ -

Example Given that $\sin A = \dfrac{12}{13}$ $\quad 0 < A < 90°$ and

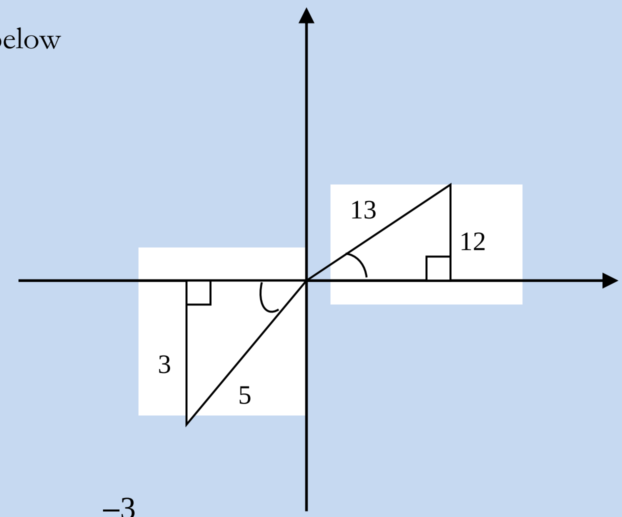
$\cos B = -4/5$, $180° < B < 270°$ find, without using a calculator, $\sin(A - B)$

<u>Solution</u>

It is important to draw appropriate sketches in the correct quadrants.

$\sin(A - B) = \sin A \cos B - \cos A \sin B$

From the diagram below



$\cos A = \dfrac{5}{13}$, $\quad \sin B = \dfrac{-3}{5}$

$$\sin(A - B) = \left(\frac{12}{13}\right)\left(\frac{-4}{5}\right) - \left(\frac{5}{13}\right)\left(\frac{-3}{5}\right)$$

$$= \frac{-33}{65}$$

<u>Identities</u>

In proving identities the student should recall that

(i) $\tan x \quad = \dfrac{Sinx}{Cosx}$

(ii) cot x $= \dfrac{1}{\tan x} = \dfrac{\cos x}{\sin x}$

(iii) sec x $= \dfrac{1}{\cos x}$

(iv) cosec x $= \dfrac{1}{\sin x}$

(v) $\sin^2 x + \cos^2 x = 1$

(vi) $\tan^2 x + 1 = \sec^2 x$

(vii) $1 + \cot^2 x = \text{cosec}^2 x$

We prove them by following three basic steps.

Example

Prove the identity $\cot\theta + \tan\theta = \sec 2 \ \text{cosec}\,\theta$

<u>Solution</u>:

LHS = $\cot\theta + \tan\theta$

Step 1 : Introduce $\sin\theta$ and/or $\cos\theta$

$$\text{LHS} = \frac{\cos\theta}{\sin\theta} + \frac{\sin\theta}{\cos\theta}$$

Step 2 : Find the LCM

$$\text{LHS} = \frac{\cos^2\theta + \sin^2\theta}{\sin\theta\cos\theta}$$

Step 3 : since $\cos^2\theta + \sin^2\theta = 1$

$$\text{Then LHS} = \frac{1}{Sin\theta\cos\theta}$$

$$= \text{cosec}\,\theta \sec\theta = \text{RHS}$$

Example

Prove that $\frac{1-\cos 2A}{\sin 2A} = \tan A$

Proof

$$\text{LHS} = \frac{1-\cos 2A}{\sin 2A}$$

$$= \frac{1-(\cos^2 A - \sin^2 A)}{2\sin A\cos A}$$

$$= \frac{1-\cos^2 A + \sin^2 A}{2\sin A\cos A}$$

$$= \frac{1-(1-\sin^2 A)+\sin^2 A}{2\sin A\cos A}$$

$$= \frac{2\sin^2 A}{2\sin A\cos A}$$

$$= \frac{\sin A}{\cos A}$$

$= \tan A$

$=$ RHS

Solving trigonometric equations

There are various techniques

Example Solve $\sin x = \frac{\sqrt{3}}{2}$

Solution: Always think about the quadrants in which x lies. In this case, sin is positive therefore x lies in the first and second quadrants.

$x = 60°, 120°$

Example.

Solve $\text{Cos}3x = \frac{1}{2}$ $0 < x < 360°$

Solution

The range of x is 360° therefore the range of 3x is $0 \le 3x \le 1080°$

$3x = 60°, 330°, 420°, 690°, 780°, 1050°$

$x = 20°, 110°, 140°, 230°, 260°, 350°$

Example.

$\tan(\theta + 10°) = \frac{1}{\sqrt{3}}.\ 0 \le \theta \le 360°$

Solution $\tan(\theta + 10) = -\frac{1}{\sqrt{3}}$

$\theta + 10° = 30°, 210°$

$\theta = 20°, 200°$

Example

$5\cos\theta - 4\sin^2\theta = 2$

using $\cos^2\theta + \sin^2\theta = 1$

$5\cos\theta - 4(1 - \cos^2\theta) = 2$

$5\cos\theta - 4 + (4\cos^2\theta) = 2$

$4\cos^2\theta + 5\cos\theta - 6 = 0$

$(4\cos\theta - 3)(\cos\theta + 2) = 0$

$\cos\theta = \frac{3}{4}$ $\cos\theta = -2$

$= 41.4°, 318.6°$

$= 41.4°, 131.8°$

Example Solve $\sin^2 x - 3\sin x\cos x = 0$

$\sin x(\sin x - 3\cos x) = 0$

$\sin x = 0$ or $\sin x - 3\cos x = 0$

$\sin x = 3\cos x$

$\frac{\sin x}{\cos x} = 3$

$\tan x = 3$

$x = 0°, 180°, 360°, 71.6°, 251.6°$

$x = 0°, 71.6°, 180°, 251.6°, 360°$

Expressions of the form acosx + bsinx

We will look at how to express the function f(x) = acosx + bsinx in the form

Rcos(x ± α) or Rsin(x ± α)

Where R >0 is a constant and α is acute.

This alternative form will enable us to solve equations of the form

acosx + bsinx

and to find the maximum and minimum values of such functions.

Results

$$\textbf{acosx + bsinx} \equiv \begin{cases} \textbf{Rcos(x - }\alpha\textbf{)} \\ \textbf{Rsin(x +}\alpha\textbf{)} \end{cases}$$

$$\textbf{acosx – bsinx} \equiv \begin{cases} \textbf{Rcos(x +}\alpha\textbf{)} \\ \textbf{Rsin(x - }\alpha\textbf{)} \end{cases}$$

-acosx + bsinx ≡ Rsin(x - α)

Practice questions

a. Express 7sinx – 4cosx in the form Rsin(x -α)
b. Solve the equation 7sinx – 4cosx = 3, for $0< x < 360^0$
c. Find
   i) the maximum and the minimum values of 7sinx – 4cosx
   ii) the maximum and the minimum values of $(7\sin x - 4\cos x)^{-1}$

Examination type questions

1. Express 35 cosx + 12 sinx in the form R cos (x – α) where R > 0 and α is an acute angle. Find the solutions of 35 cos x + 125 sin x = 20 in the range $0^o \leq x \leq 360^o$.

2. Prove the following identities

(i) $$\frac{Sinx + \sin 2x + \sin 3x}{Cosx + \cos 2x + \cos 3x}$$

(ii) $$\frac{\sqrt{(1-\sin x)}}{1+\sin x} \equiv \text{Sec x} - \tan x$$

(iii) $\dfrac{1-\sin x}{\cos x} \equiv 1/(\sec x + \tan x)$

4. Solve the following equations for $-180 \le x \le 180°$

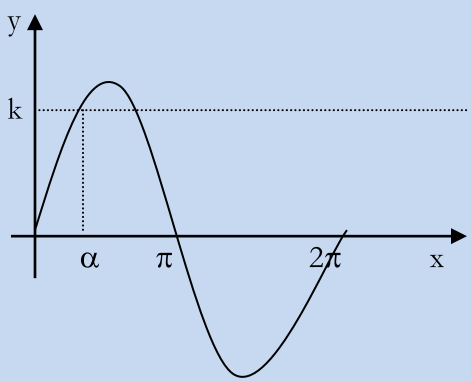
(i) $3\sin^3 x + 10\cos^2 x + 9\sin x = 12$

(ii) $25\cos x = 16\sin x \tan x$.

5. The diagram shows part of the Graph of $y = \sin x$, where x is measured in radians, and values α on the x – axis and k on the y – axis such that $\sin\alpha = k$. Write down, in terms of α

i) a value of x between $\frac{1}{2}\pi$ and π such that $\sin x = k$.

ii) two values of x between 3π and 4π such that $\sin x = -k$



6. The function f is defined for real values of x by

$f(x) = (\cos x - \sin x)(17\text{cox} - 7\sin x)$

a) By first multiplying out the brackets, show that f(x) may be expressed in the form $5\cos 2x - 12\sin 2x + k$, where k is a constant, and state the value of k.

b) Given that $5\cos 2x - 12\sin 2x \equiv R\cos(2x + \alpha)$, where $R > 0$ and $0 < \alpha < \frac{1}{2}\pi$, state the value of R and find the value of α in radians to three d.p.

c) Determine the greatest and least values of $39/(f(x) + 14)$ and state a value of a x at which the greatest value occurs.

7. Assuming the identities $\sin 3\theta \equiv 3\sin\theta - 4\sin^3\theta$ and $\cos 3\theta \equiv 4\cos^3\theta - 3\cos\theta$, prove that

$\cos 5\theta \equiv 5\cos\theta - 20\cos^3\theta + 16\cos^5\theta$

Find the set of all values of θ in the interval $0 \le \theta \le \pi$ for which $\cos 5\theta > 16\cos^5\theta$

8. (i) Prove that $\tan\theta + \text{Cot}\,\theta \equiv 2\,\text{Cosec}\,2\theta$

(i) Hence solve the equation; $\tan\theta + \text{Cot}\,\theta = 4$ giving all values of θ such that $0° < \theta < 90°$.

9. Given that $\text{Sin}(\theta - 30°) + \text{Cos}(\theta + 45°) = 0$

Show that $\tan\theta = \dfrac{\sqrt{2}-2}{\sqrt{6}-2}$

and hence solve the given equation for $0 \le \theta \le 180°$.

10. Given $\text{Cos}\,\propto = 3/5$, $\text{Cos}(\propto + \beta) = 5/13$ and that $\propto$ and β are acute, evaluate $\text{Sin}\,\beta$ without using a calculator.

11. Find all values of θ , where $0^0 < \theta < 360^0$, satisfy $\cos 2\theta = 2 \cos\theta$ giving your answer correct to the nearest $0,1^0$.

12. Solve the equation

$$3 \cos x - 4 \sin x = 2 \text{ ; where } 0^0 < x < 360^0,$$ giving your answer correct to the nearest $0,1^0$.

CHAPTER 13

DIFFERENTIATION

OBJECTIVES
By the end of the chapter the student should be able to :

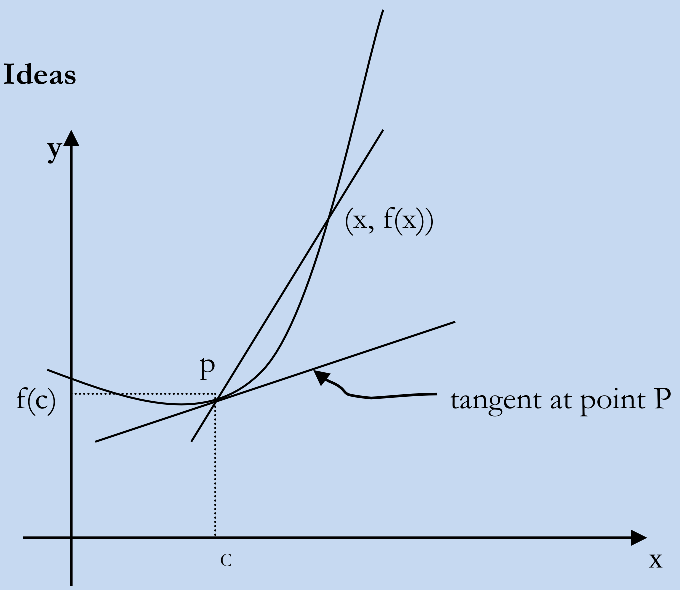
- Differentiate polynomials
- Differentiate comoposite functions
- Differentiate log,trig and exponential functions
- Differentiate parametric and imoplicit functions
- Find the equation of tangent and nominals
- Find stationary points and investigate their nature
- Investigate whether the functionbs in increasing or decreasing
- Compute the Maclaurin series of given functions.

Introduction

In this unit, we are going to examine the process of differentiation. Beginning by considering how the gradient to a curve at a given point is obtained, we define precisely what is meant by saying a function has a derivative or is differentiable at a given point. You will see that not every function has a derivative at each point in R. Indeed, there are functions that have no derivatives at every point in R.

The Derivative

Intuitive Ideas



Suppose we are given a curve whose function is y = *f* (*x*) and we want to find the gradient or slope of the curve at a point P (c, *f*(*c*). We can do it by finding the slope or gradient of the tangent at P. Now to get the tangent at P, we consider what happens to the chord PQ where Q is some point (x, *f* (*x*) on the curve seen below. We see that as Q moves towards P on the curve and eventually approaches P, the chord PQ gradually becomes a tangent to the curve at P.

Thus the tangent at P is the limit of the chord PQ as Q tends to P on the curve. Note that::

Gradient of PQ = $\lim\limits_{x \to c} f\dfrac{(x) - f(c)}{x - c}$

Grad at P = lim $\lim\limits_{x \to c} f\dfrac{(x) - f(c)}{x - c}$

Definition: The process of finding a general expression for the gradient of a curve at any point is known as differentiation

Note:

1. The gradient of a function at a point exist if and only if *lim* $\lim\limits_{x \to c} f\dfrac{(x) - f(c)}{x - c}$

   exists and is unique

2. Findingthe gradient using this method is known as differentiation from first principles.

Notation

$f^1(c)$ = lim $\lim\limits_{x \to c} f\dfrac{(x) - f(c)}{x - c}$

Other notations for $f^1(c)$ are **df/dx, dy/dx, f^1 and y^1**

Now: $f^1(c)$ = lim $\lim\limits_{x \to c} f\dfrac{(x) - f(c)}{x - c}$

Let h = x – c, hence if x → c, then h → 0, i.e.

$f^1(c) =$ *lim* $\lim\limits_{b \to 0} \dfrac{f(c+h) - f(c)}{h}$

But c is an arbitrary real number, for convenience, we substitute c for x and use the expression:

$$f^1(x) = lim \quad \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$f^1(x)$ is called the gradient function, or simple the derived function or the first derivative of f (x)

Example

If f (x) = 2x, find df/dx

Solution

$f^1(x) = \lim \quad \lim_{b\to 0}\frac{f(x+h)-f(c)}{h}$

$\lim_{h\to 0} = 1/h(\ 2(x + h\ ) - 2x)$

$f^1(x) = \lim \quad \lim_{h\to 0}\frac{2x+2h-2x}{h}$

$f^1(x) = \lim \quad \lim_{h\to o} \quad \frac{2h}{h}$

$= 2$

hence $f^1(x) = 2$

Example

If y = 1/x, find $f^1(x)$

Solution

$f^1(x) = \lim \quad \lim_{h\to 0}\frac{f(x+h)-f(c)}{h}$

$f^1(x) = \lim \quad \lim_{h\to 0}\frac{1(x+h)-1(x)}{h}$

$f^1(x) = \quad \lim \quad \lim_{h\to o} \quad \frac{1}{h} \quad \left(\frac{x-x-h}{x(x+h)}\right)$

$f^1(x) = \lim_{h\to o} \quad - \frac{-1}{x^2+xh}$

$^1(x) = \frac{-1}{x^2}$

Derivation from first principles involves too many complicated calculations. However, through its use, many easy techniques have been developed. We will explore these techniques in the subsequent sections

Differentiation of a constant

If y = c ~ a constant, hence dy/dx = $f^1(x) = 0$

Example

If $f(x) = e^2$, find $f^1(x)$

Solution

$f^1(x) = 0$, since e^2 is a constant

Differentiation of $y = ax^n$

If $y = ax^n$,then $y^1 = anx^{n-1}$

Example:

Differentiate w.r.t. x

(a) $y = 4x^3$
(b) $y = 1/x$
(c) $y = x^{-1/3}$
(d) $y = 3x^{2.5}$

Solution

(a) $y = 4x^3$: $y^1 = 4.3x^{3-1} = 12x^2$
(b) $y = 1/x$: $y = x^{-1}$: $y^1 = -1.\ x^{-1-1} = -x^{-2}$
(c) $y = x^{-1/3}$: $y^1 = -1/3\ x^{-4/3}$
(d) $y = 3x^{2.5}$: $y^1 = 3 \bullet 5/2x^{5/2-1} = 7.5x^{3/2}$

Differentiation of polynomials

Let $f(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \ldots + a_nx^n$, hence
$f^1(x) = a_1 + 2a_2x + 3a_3x^2 + \ldots + na_nx^{n-1}$

Example

Find dy/dx
a $y = x^3 + x^4 + 3$

b $y = \dfrac{x^5 + x^2}{x}$

c $y = (2x + 5)^2$

Solution

(a) $y = x^4 + x^3 + 3$: $dy/dx = 4x^3 + 3x^2$

(b) $y = \dfrac{x^5 + x^2}{x} = x^{5-1} + x^{2-1} = x^4 + x : y^1 = 4x^3 + 1$

(c) $y = (2x + 5)^2 = 4x^2 + 20x + 25$: $y^1 = 8x + 20$

Practice Exercise

Differentiate with respect to x

1. $y = x^5 + x^8 + 1$
2. $y = 3 – 2x^4$
3. $y = (5x^2 – 2)^3$
5. $y = (x^2 – 1)^{4}$

Result

If $y = f_1(x) + f_2(x) + \ldots\ldots\ldots\ldots\ldots + f_n(x)$, then
$y^1 = f_1^1(x) + f_2^1(x) + \ \ldots. + f_n^1(x)$

Derivative of composite functions

Let $y = f[g(x)]$, hence $y^1 = f^{\,1}[g(x)] \bullet g^1(x)$ i.e. $dy/dx = dy/du \bullet du/dx$

This result is achieved by the change of variable technique. In this case we let $u = g(x)$

Example

If $y = (2x + 5)^4$, find $f^1(x)$

Solution

Let $u = 2x + 5$, hence $y = u^4$, then $du/dx = 2$, $dy/du = 4u^3$

$y^1 = 4u^3 \text{ x } 2 = 8(2x + 5)^3$

Example

If $y = 4(3x^2 – 2x + 1)^3$, find y^1

Solution

Let $u = 3x^2 – 2x + 1$, $du/dx = 6x – 2$, $y = 4u^2$, $dy/du = 8u$

$dy/dx = dy/du \bullet du/dx = 8u \,.\, (6x – 2) = 8(3x^2 – 2x + 1)(6x – 2)$

Derivative of Product of Two Functions

If y = u(x) . v(x), then dy/dx = vdu/dx + udv/dx

Example

$$\frac{dy}{dx} = vu^1 + uv^1$$

If $y = x^2(2x^2 - 3)^4$,find y^1

Solution

$y = x^2(2x^2 - 3)^4$

$u = x^2$ $\quad\quad$ $v = (2x^2 - 3)^4$

$du/dx = 2x$ $\quad\quad$ $dv/dx = 16x(2x^2 - 3)^3$

$y^1 = (2x^2 - 3)^4(2x) + x^2(2x^2 - 3)^3(16x)$

$y^1 = 2(2x^2 - 3)^3(10x^3 - 3x)$

Practice Questions

Find dy/dx if;

1. $y = (x^2 + 2)(x^3 + 7)$
2. $y = x^{-5} + x^{-\frac{3}{2}}$
3. $y = (3x^5 + 4x + 2)^6$
4. $y = \dfrac{(x-1)^3}{x}$
5. Derivative of a quotient of two functions

Let y = u/v, v ≠ 0, then dy/dx = $\dfrac{xdu/dx - udv/dx}{V^2}$

These formulas are derived using differentiation from first principles

Example

If $y = -\dfrac{2x}{2x^2 + 4}$

Solution

y = $-\dfrac{2x}{2x^2+4}$

u = 2x　　　　　　　　v = $2x^2 + 4$

du/dx =2　　　　　　　dv/dx = 4x

$y^1 = \dfrac{2(2x^2+4)-4x(2x)}{(2x^2+4)^2} = \dfrac{2-x^2}{(2x^2+4)^2}$

Derivative of trigonometric functions

y = sin *x* ,then y^1 = cos x

y =cos *x* , then y^1 = - sin x

y = tan x, then y^1 = $\sec^2$ x

Example

Find dy/dx in each of the following cases:

(a) y = cos 3x　　　(b) y =sin ^{5}2x　　　(c) y = $\tan^3$ (2x + 1)

Solution

(a) y = cos 3x :dy/dx =- sin 3x d/dx (3x)
=-3 sin 3x

(b) y = sin ^{5}2x

To differentiate this function, we must follow the P.T.A rule, where P = Power, T = Trigonometric function, A = Algebraic function.

Now: dy/dx =5 sin ^{4} 2x . d/dx (sin 2x)
=5 sin ^{4} 2x . cos 2x . d/dx (2x)
=10 sin 4 2x cos 2x

(c) y = tan ^{3} (2x + 1)

dy/dx = 3 $\tan^2$ (2x + 1) d/dx$_x$ [tan (2x + 1) }

=3 $\tan^2$ (2x + 1) $\sec^2$ (2x + 1) . d/dx (2x + 1)

=6 $\tan^2$ (2x + 1) $\sec^2$ (2x + 1)

Practice Questions

Differentiate with respect to x

1. $y = \cos 2x$
2. $y = \tan (5x + 1)$
3. $y = \cos^2 (3x + 2)$
4. $y = \sin^7 x$
5. $y = \sin^{10} (3x)$
6. $\cos (2x - \pi/3)$

Derivative of Exponential Functions

$$\boxed{\text{If } f(x) = e^x, \quad f^1(x) = e^x.}$$

Example

Differentiate (a) e^{3x} (b) $\exp(4x^2)$

Solution

(a) Let $f(x) = e^{3x}$: $f^1(x) = 3e^{3x}$

(b) Let $y = \exp(4x^2)$: $f^1(x) = 8x \exp(4x^2)$

Result

$$\boxed{\text{If } y = e^{g(x)}, \text{ then } y = e^{g(x)}.g'(x) =}$$

Practice Questions

Differentiate with respect to x.

1. e^{2x}
2. $e^{3x + 1}$
3. $e^{1 - 4x}$
4. $e^{2x^2 + 5}$
5. $e^{\sin x}$
6. $e^{\sin 2x}$
7. $e^{\cos 2x}$
8. $e^{8x^2 + 4}$
9. $e^{\cos 3x}$
10. $e^{1 - 3x}$

Derivative of Logarithmic Functions

If y = lnx, then $y' = \frac{1}{x}$

Result

If f(x) = ln{g(x)}, then $f'(x) = \frac{g'(x)}{g(x)}$

Example
Find dy/dx in each of the following cases

(a) y = In 3x

(b) y = In $(x^2 + 1)$

Solution
(a) y = In 3x: then dy/dx = 1/3x . d/dx (3x) $\Rightarrow$ 1/3x . 3 = 1/x

(b) y = In $(x^2 + 1)$then: dy/dx = $\frac{1}{x^2+1}\frac{d}{dx}(x^2+1)$

$= \frac{2x}{x^2+1}$

x

Practice Questions

Differentiate with respect to x

1. In (1 + x)
2. In $(x^2 + 1)$
3. In (2 – 3x)
4. In cos x
5. In sin ^{2}x
6. In $(x^2 + 2x)$
7. In tan x
8. In $(x^2 + 5x)$

Example
Find dy/dx in each of the following cases

(a) y = x^2 (3x + 2)

(b) y = 2x sin 3x

Solution

(a) y = x^2 (3x + 2)

dy/dx = x^2 d/dx (3x + 2) 3x + 2) d/dx (x^2)

$= 3x^2 + 2x\ (3x + 2)$

Hence, $dy/dx = 9x^2 + 4x$

(b) $y = \text{Sin}3xe^{2x}$

$dy/dx \quad = \quad e^{2x}\ d/dx\ (\sin 3x) + \sin 3x\ \ d/dx\ (e^{2x})$

$= \quad e^{2x}\ .\ \cos 3x\ .\ 3 + \sin 3x\ .\ \ e^{2x}\ \ .\ 2$

$= \quad 3\ \ (e^{2x}\ \cos 3x\ \ + 2\ \ e^{2x}\ \sin 3x$

Hence $dy/dx \quad = \quad e^{2x}\ (\ 3\cos 3x + 2\sin 3x)$

Practice Questions

Differentiate with respect to x.

1. $y = x\ (x^3 + 1)$
2. $y = x^2\ (3x + 4)$
3. $y = (2x^3 + 1)\ (x^2 + 4)$
4. $y = (x + 1)^2\ (x + 2)^3$
5. $y = x\ e^{x}$
6. $y = \ 3x \sin 3x$
7. $y = x^3\ e^{2x}$
8. $y = \cos 2x \ln x$
9. $y = \sin x \ln x$
10. $y = \ 5x \cos 3x$
11. $y = \ 2x\ a^x$
12. $y = x\ (4 – x)3$

Example

Find dy/dx in each of the following cases

(a) $y = \frac{x^2}{2x+1}$ (b) $y = \frac{\sin x}{x}$

<u>**Solution**</u>

(a) $y = \frac{x^2}{2x+1}$

$\therefore\ dy/dx \quad = \quad \frac{(2x+1)d/dx(x^2) - dy/dx(2x+1)}{(2x+1)^2}$

$$= \frac{2x(2x+1)-2x^2}{(2x+1)^2}$$

$$= \frac{4x^2+2x-2x^2}{(2x+1)^2}$$

$$= \frac{2x^2+2x}{(2x+1)^2}$$

(b) $y = \frac{\sin x}{x}$

$$\text{dy/dx} = \frac{xd/dx(\sin x)-\sin xd/dx(x)}{x^2}$$

$$\text{Hence dy/dx} = \frac{x\cos x-\sin x}{x^2}$$

Practice questions

Differentiate with respect to x

1. $\frac{2x+5}{3x+1}$

2. $\frac{\ln x}{\ln(x+1)}$

3. $\frac{2x}{Cosx}$

4. $\frac{\sin x}{\cos x}$

5. $\frac{3x+2}{2x}$

6. $\frac{3x^2}{x+1}$

7. $\frac{Sinx}{2-\cos x}$

8. $\frac{1-x^2}{x^2}$

Implicit Differentiation

The functions we've differentiated so far have given one variable usually y, explicitly in terms of other variables, usually x, in the form of $y = f(x)$. However, when y is mixed with the x's, given implicitly, we have to expand technique of differentiation so that we can still find the gradient function dy/dx at any point.

Consider the function $x^2 + y^2 = 2$. we differentiate each term w.r t. x

Hence: $$\frac{d(x^2)}{dx} + \frac{d(y^2)}{dx} = \frac{d(2)}{dx} \qquad \textbf{(1)}$$

By the chain rule.

$$-\frac{d(x^2)}{dx} + \frac{d(y^2)}{dx} \cdot \frac{dy}{dx} = 0$$

Hence: (1) becomes $2x + 2y\dfrac{dy}{dx} = 0$

Rearranging for $\dfrac{dy}{dx}$ gives $\dfrac{dy}{dx} = \dfrac{-x}{y}$

Alternatively:

Let $F = x^2 + y^2 - 2$

Hence: $\dfrac{df}{dx} = 2x$, (y^2, 2 is a constant)

$\dfrac{df}{dx} = 2y$, holding x^2-2 as a constant.

Hence: $$\frac{dy}{dx} = \frac{-df/dx}{df/dy}$$

$$= \frac{-2x}{2y}$$

$$= \frac{-x}{y}$$

Example

Given that $x^2 + xy + y^2 - 3x - y = 3$

Show that $\frac{dy}{dx} = \frac{3-2x-y}{x+2y-1}$

Set $F = x^2 + xy + y^2 - 3x - y - 3$

$$\frac{df}{dx} = 2x + y - 3$$

$$\frac{df}{dy} = x + 2y - 1$$

Hence: $\frac{dy}{dx} = \frac{df/dx}{df/dy}$

i.e. $\frac{dy}{dx} = \frac{(2x+y-3)}{x+2y-1}$

$$\frac{dy}{dx} = \frac{3-2x-y}{x+2y-1}$$

Practice Questions

Find dy/dx in terms of x and y

1. $x^2 + 2y^2 + 6x = 1$
2. $x^2 + 3xy - 2x = 1$
3. $2x^3 + 3y^3 + 5xy - 3 = 0$
4. $x^2 + y^2 + 4y = 21$
5. $x^2 + y^2 - 8x + 4y + 2 = 0$
6. $x^2 + xy + y^2 + 3y = 0$
7. $y^2 = 2y + 8x - 17$
8. $x^2y + y^2 = 10$

Parametric Differentiation

In this section we are going to see that *x* and *y* are given separately as a function of a third variable, usually *t* or θ, and we have to find the gradient function at a given parameter.

In this case $$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$$

Example

If $x = 2 + t^2$ and $y = 3 + t^3$, find dy/dx

Solution

We differentiate *x* and *y* separately with respect to *t* and replace in the above formula.

Now $x = t^2$ and $y = 3 + t^3$

$$\frac{dx}{dt} = 2t \qquad \frac{dy}{dt} = 3t^2$$

We have $$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx} = 3t^2 \times \frac{1}{2t} = \frac{3t}{2}$$

Example

If x = a cos θ and y = b sin θ 2 , find dy/dx

Solution

x= a cos θ and y = b sin θ

$$\frac{dx}{d\theta} = a\sin\theta \qquad \frac{dy}{d\theta} = -b\cos\theta$$

$$\frac{dx}{dx} = \frac{-b\cos\theta}{a\sin\theta} = \frac{-b\cot\theta}{a}$$

Practice Questions

Find dy/dx

1. $x = t^3 - t$, $y = t^2 + t$
2. x = a cos θ , y = a sin θ
3. x = a (1 + cos θ) , y = a (1 – Sin θ)

4. $x = -\frac{1}{1+t}$, $y = \frac{3t}{1+t}$

5. $x = t^2 - 2t$, $y = 2t - 1$

6. $x = 3 \sin \theta$, $y = \cos t$

7. $x = 3 - 2 \cos \theta$, $y = 4 \sin \theta - 1$

8. $x = 2 \sin \theta$, $y = 3 \cos \theta$

Application of Differentiation.

Example Find the gradient of the curve $f(x) = \frac{4-x^3}{x^2}$, at (-2,3)

$$= 4x^{-2} - x$$

Hence: $f^1(x) = 4(-2)\, x^{-2+1} - 1$

$$= \frac{-8}{x^3} \quad -1$$

Hence: $m = f^1(-2)$

$$= \frac{-8}{(-2)^3} - 1$$

$$\mathbf{= \ 0}$$

The gradient is 0

Example

A curve whose equation is y = a /x+ c passes through the point (3,9) with gradients 5. Find the values of the constraints a and c.

Solution

$y = \frac{a}{x} + c$

$y^1 = \frac{-a}{x^2}$

$m = 5$

Hence: $\frac{-a}{x^2} = 5$

$\frac{-a}{5} = x^2$

-a $= 5(3^2)$

a = -45.

Now $y = \frac{-45}{x} + c$

i.e. $a = \frac{-45}{3} + c$

a + 15 = c

c = 24

Equation of A Tangent and A Normal

The equation of a straight line is
y = mx + c
where m = gradient and c

Example

Find the equation of the tangent and the normal to the curve $y = 3\sqrt{x} + \frac{1}{\sqrt{x}}$

where $x = \frac{1}{4}$

Since the tangent and normal are perpendicular, if the gradient is $m = f^1(x_0)$ **then the gradient of a normal is** $\frac{1}{m} = -\frac{1}{f^1(x_0)}$

$y = 3\sqrt{x} + \frac{1}{\sqrt{x}}$ **(1)**

$y = 3x^{\frac{1}{2}} + x^{\frac{-1}{2}}$

$y^1 = \frac{1}{2} x^3 x^{\frac{1}{2}\,1} + \frac{-1}{2} x^{\frac{-1}{2}\,-1}$

$y^1 = 3/2\, x^{\frac{-1}{2}} - \underline{x}^{\frac{-3}{2}}$

$$2$$

$$y^{1}\tfrac{1}{4} = \frac{3}{2}\left(\tfrac{1}{4}\right)^{\frac{-1}{2}} \left(\tfrac{1}{4}\right.^{\frac{-3}{2}}$$

$y^1(\tfrac{1}{4}) = 3/2\ 4^{\frac{1}{2}} - \tfrac{1}{2}\ 4^{3/2}$

$y^1\ (\tfrac{1}{4}) = 3/2^{x2} - \tfrac{1}{2}\cdot 2^3$

$y^1\ (\tfrac{1}{4}) = 3\text{-}4$

$\mathbf{y^1(\tfrac{1}{4}) = -1}$

Now $y(\tfrac{1}{4}) = 3(\tfrac{1}{4})^{y2} + (\tfrac{1}{4})^{-\frac{1}{2}}$

$= 3/2 + 2$

$= \dfrac{7}{2}$

Hence. $y = mx + c$

$y = -x + c$

$\dfrac{7}{2} = -\tfrac{1}{4} = \dfrac{15}{4}$

i.e. $y = -x + \dfrac{15}{4}$

$4y + 4x = 15$. equation of the tangent.

The gradient of the normal is 1.

Hence. $y = x+c$

$\dfrac{7}{2} = \tfrac{1}{4} + c$

$C = \dfrac{7}{2}\ \tfrac{1}{4} = \dfrac{13}{4}$

$y = x + \dfrac{13}{4}$

$4y - 4x = 13$ equation of the normal.

Example

A curve has equation $y = Ax^3 + Bx^2 + Cx + d$. where A,B,C and D are constants. Given that the curve has a gradient -4 at the point (1,2) and gradient 8 at the point (-1,6) find A,B,C and D.

Solution.

$y = Ax^3 + Bx^2 + Cx + D$
$y^1 = 3Ax^2 + 2Bx + c$
$y(1) = A + B + C + D$
Hence. $A + B + C + D = 2$ **(1)**
$y(-1) = -A + B - C + D$
Hence. $-A + B - C + D = 6$ **(2)**
$y^1(1) = 3A + 2B + C$
Hence. $3A + 2B + C = -4$ **(3)**
$y^1(-1) = 3A - 2B + C$
Hence. $3A - 2B + C = 8$ **(4)**

From (1) $D = 2 - A - B - C$ **(5)**

Sub (5) in (2)

$-A + B - C + 2 - A - B - C = 6$
$-2A - 2C = 4$ **(6)**

From 6
$A = -C - 2$ **(7)**
Sub (**7**) in (**4**)

$3(-c-2) - 2B + C = 8$
$-3 - 6 - 2B + C = 8$
$-2B - 2c = 14$ **#**

Sub **(7) in (3)**

$3(-c-2) + 2B + C = -4$
$-3c - 6 + 2B + c = -4$
$-2c + 2B = 2$ **@**

Taking # and @ and solving simultaneous equation.

$-2B - 2C = 14$
$\underline{2B - 2C = 2}$
$-4C = 16$
C = -4

$-2B = 14 + 2C$
$-\underline{2B = 14 - 8}$
$-2B = 6$
B = -3

$A = -c - 2$
$A = -(-4) - 2$
A = 2

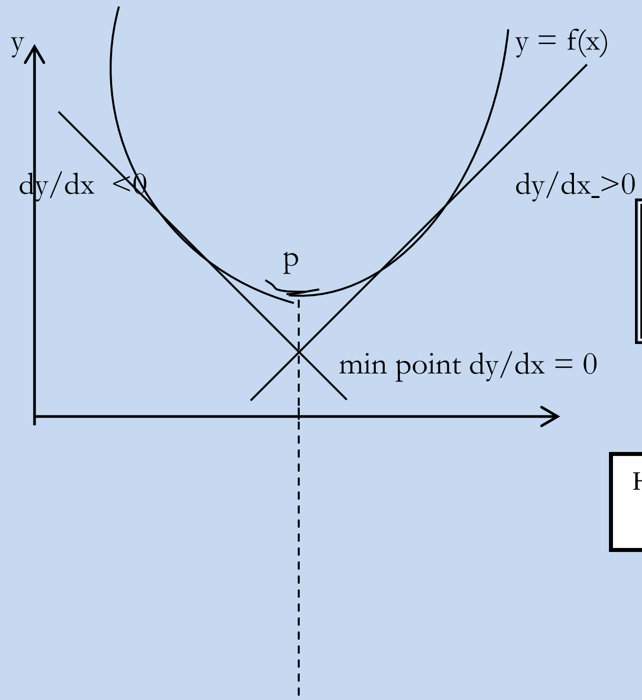
$D = 2 - A - B - C$
$D = 2 - 2 + 3 + 4$
D = 7

Maximum, Minimum and Point of Inflexion.

A point on a curve at which the gradient is zero i.e. $dy/dx = 0$ is called a stationary point.

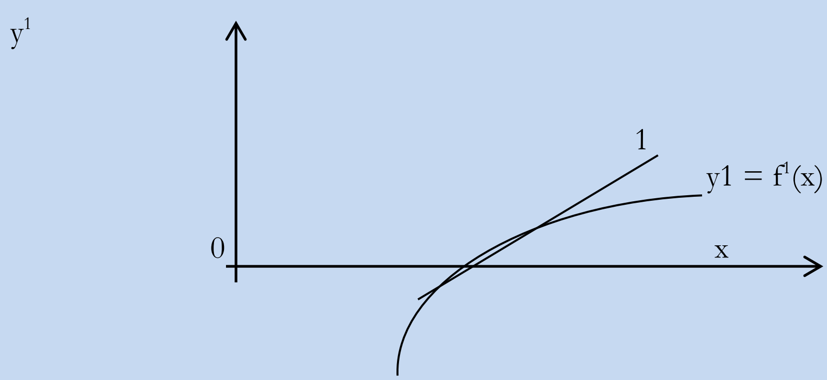
There are three types of stationary points, which are maximum, minimum and point of inflexion. We will not deal with conditions of obtaining stationary points in this course.

Minimum point.

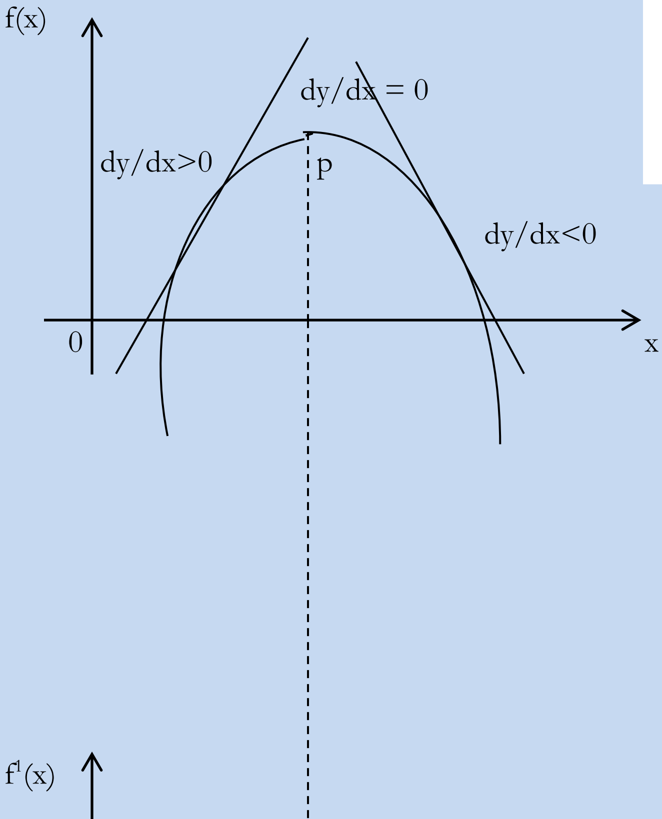


If P is a minimum point then dy/dx changes from negative to positive.

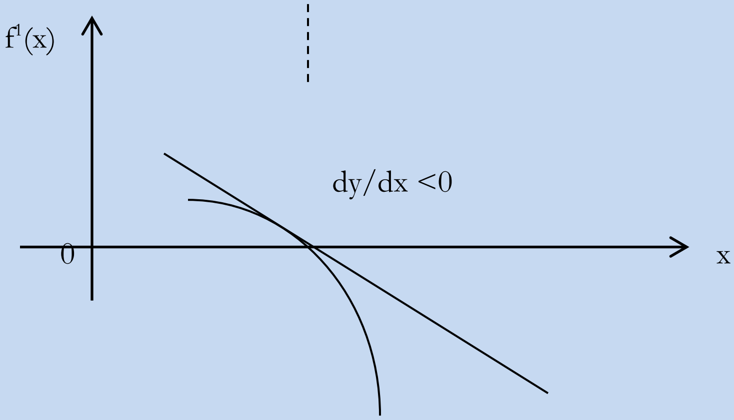
Hence $y^{11} = d^2y/dx^2 > 0$



Maximum point.



If P is a maximum point dy/dx changes from positive to negative.



Summary of results.

	Maximum	Minimum
Sign of $\frac{dy}{dx}$	+ 0 -	- 0 +
Sign of $\frac{d^2y}{dx^2}$	Negative (or zero)	Positive (zero)

Example

Find the co-ordinates of the stationary points and determine their nature.

$$y = x + \frac{1}{x}$$

<u>Solution</u>.

$y = x + \frac{1}{x}$

$y^1 = 1 - \frac{1}{x^2}$

$\frac{dy}{dx} = 0$

$1 - \frac{1}{x^2} = 0$

$\frac{x^2 - 1}{x^2} = 0$

$x^2 - 1 = 0$

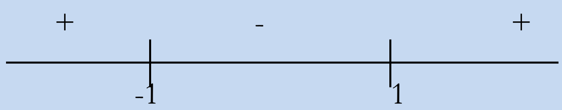
$\mathbf{x = \pm 1}$

For $x = 1$, $y = 1 + \frac{1}{1} = 2$

For $x = -1$, $y = -1 + \frac{1}{1} = -2$

Hence the stationary points are (1;2) and (-1;-2)

$f^1(x) = \dfrac{x^2 - 1}{x^2}$ the sign of the numerator $x^{2\,-1}$



Hence. At x = -1, the point (-1,2) is a maximum point at x = 1, the point (1,2) is a minimum point.

Example

A large tank in the shape of a cuboid is to be made from $54m^2$ of sheet metal. The tank has a horizontal rectangular base and no top. The height of the tank is x meters. Two of the opposite vertical faces are squares.

a) Show that the volume, vm^3, of the tank is given by v = 18x – 2/3.

b) Given that x can vary, find the maximum value of v.

c) Justify that the value of v you have found is a maximum.

<u>Solution</u>.

V = length x breath x height.

Let the height of the tank be x meters surface area of the tank = 2bx + 2ℓx + bℓ

2bx + 2ℓx + bℓ = $54m^2$.

Since two of the opposite vertical faces are squares we assume b = x , hence

$$2x^2 + 2\ell x + \ell x = 54$$

$$\ell(2x + x) = 54 - 2x^2$$

$$\ell = \frac{54 - 2x^2}{3x}$$

$$\ell = \frac{18}{x} - \frac{2x}{3}$$

Hence. $V = (18/x - 2^x/3)x^2$

$\mathbf{V = 18x - \tfrac{2}{3}x^3}$

b) $V = 18x - \dfrac{2x^3}{3}$

$V^1 = 18 - 2x^2$

$V^1 = 0$ i.e. $18 - 2x^2 = 0$

$2x^2 = 18$

Hence. $x = 3$

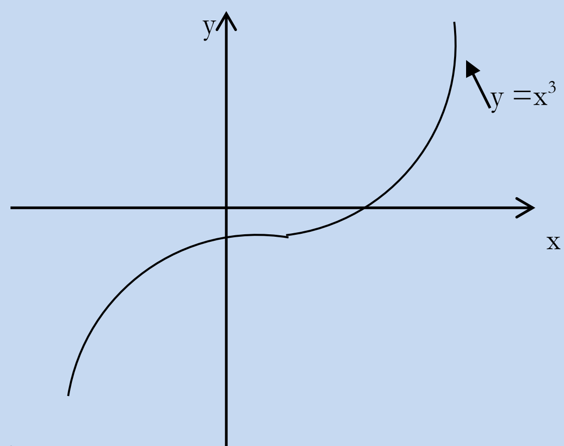
Hence. $V_{max} = 18 \text{ x } 3 - \dfrac{2 \times 3^3}{3}$

$$= \mathbf{36m^3}$$

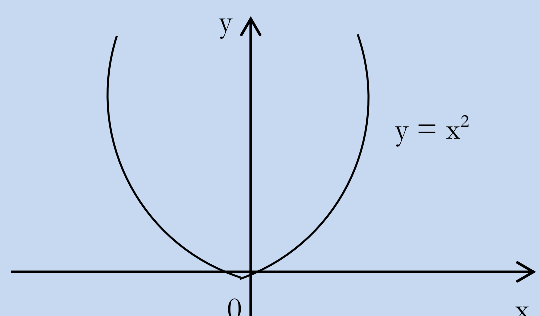
c) Since $v^{11} = -12 < 0$ hence at $x = 3$ there is a maximum.

<u>Increasing and Decreasing Functions.</u>

Consider the graph of the function $y = x^3$ drawn below.



The graph of $y = x^3$, is always increasing for all values of x. Check $\frac{dy}{dx} = 3x^2 \geq 0$
For all values f x.
Now consider the graph of $y = x^2$.



The graph above show a decrease when $x < 0$, and increases when $x > 0$.

Check: $\frac{dy}{dx} < 0$ for $x < 0$ and $\frac{dy}{dx} > 0$ for $x > 0$

$\frac{dy}{dx} = 2x$, depends on the sign of x.

Example Determine the intervals of increase and decrease given that
$y = x^4 + 4x^3 - 8x^2 - 48x^2 - 48x + 20$

<u>Solution</u>.

$y = x^4 = 4x^3 - 8x^2 - 48x + 20$

$y^1 = 4x^3 + 12x^2 - 16x - 48$

Let $g(x) = 4x^3 + 12x^2 - 16x - 48$

We factorise g(x)

$g(2) = 4(2)^3 + 12(2)^2 - 16x^2 - 48$

$\quad = 32 + 48 - 32 - 48$

$\quad \mathbf{= 0}$

Hence. $x - 2$ is a factor of $g(x)$. Since $g(2) = 0$

$$
\begin{array}{r|l}
 & \underline{4x^2 + 20x + 24} \\
x - 2 & 4x^3 + 12x^2 - 16x - 48 \\
 & \underline{4x^3 - 8x^2} \\
 & \quad 20x^2 - 16x - 48 \\
 & \quad \underline{20x^2 - 40x} \\
 & \qquad 24x - 48 \\
 & \qquad 24x - 48
\end{array}
$$

Start with a negative since a = 4>0 for cubic functions.

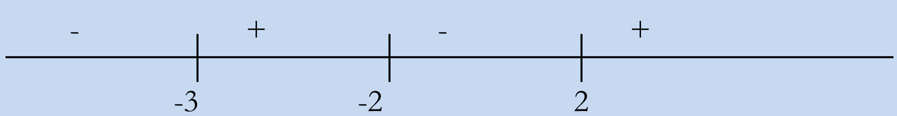
$(x-2)(4x^2 + 20x + 24) = 4(x - 2)(x^2 + 5x + 6)$

$\qquad = 4(x-2)(x + 2)(x + 3)$

For: $g(x) = 0$

$4(x - 2)(x + 2)(x + 3) = 0$

$x = -3,\ x = -2,\ x = 2.$



Hence. $\frac{dy}{dx} < 0$ for $x < -3 \cup -2 < x < 2$

$\frac{dy}{dx} > 0$ for $-3 < x < -2 \cup x > 2$

The function increases for $-3 < x < -2 \cup x > 2$ and decreases for $x < -3 \cup -2 < x < 2$.

Example

given that $x^2 + xy + y^2 - 3x - y = 3$

a) Show that $\dfrac{dy}{dx} = \dfrac{3-2x-y}{x+2y-1}$

b) Find, and classify, the maximum and minimum values of y.

c) Determine the co –ordinates of the points on the curve where the tangents to the curve are parallel to the y – axis.

<u>Solution</u>

(a)

Now: $\dfrac{dy}{dx} = \dfrac{3-2x-y}{x+2y-1}$

$$\frac{d^2y}{d^2x} = \frac{(x=2y-1)d/dx(3-2x-y)-(3-2x-y)d/dx(x+2y-1)}{(x+2y-1)2}$$

Hence. $\dfrac{d^2y}{d^2x} = \dfrac{(x+2y-1)(-2-dy/dx)-(3-2x-y)(1+2dy/dx)}{(x+2y-1)^2}$

<u>Solution.</u>

a) $x^2 + xy + y^2 - 3x - y = 3$

$$\frac{d(x^2)}{dx} + \frac{d(x-y)}{dx} + \frac{d(y^2)}{dx} - \frac{-3d(x)}{dx} - \frac{-d(y)}{dx} = \frac{d(3)}{dx}$$

$$2x + y + \frac{dy}{dx} + 2y\frac{dy}{dx} - 3 - \frac{dy}{dx} = 0$$

$$\frac{dy}{dx}(x+2y-1) = 3-2x-y$$

$$\frac{dy}{dx} = \frac{3-2x-y}{x+2y-1}$$

Hence. We evaluate d^2y/dx^2 at stationary points.

i.e. $= \dfrac{(3+2x(3)-1)(-2-0)-(3-2x3-(-3)(1+0)}{(3+2(-3)-1)^2}$

$= \dfrac{(3-6-1)(-3)-(3-6+3)}{(3-6-1)^2}$

$= \dfrac{-8}{16}$

$= \frac{-1}{2} \quad < 0$

Hence. (1/3, 7/3) is a maximum point?

b) $\dfrac{dy}{dx} = 0$ i.e. $\dfrac{3-2x-y}{x+2y-1} = 0$

3 – 2x – y = 0
y = 3 – 2x

sub: y = 3 – 2x onto x
$x^2 + xy + y^2 - 3x - y = 3$

$x^2 + x(3 - 2x) + (3 - 2x)^2 - 3x - (3 - 2x) = 3$
$x^2 + 3x - 2x^2 + 9 - 12x + 4x^2 - 3x - 3 + 2x = 3$
$3x^2 - 10x + 3 = 0$

$x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$x = \dfrac{-(10) \pm \sqrt{(-10)^2 - 4\times 3\times 3)}}{2\times 3}$

$x = \dfrac{10 \pm \sqrt{(100-36)}}{6}$

$x = \dfrac{10 \pm 8}{6}$

x = 3 and x = $\dfrac{1}{3}$

Hence. for x = 3; y = 3 – 2x3 = -3

for $x = \dfrac{1}{3}$; $y = 3 - 2x\ \dfrac{1}{3} = 7/3$

Hence the stationary points are (3;3) and (1/3; 7/3)

Alternatively.

Let $F = x^2 + xy + y^2 - 3x - y - 3$

$\frac{df}{dx} = 2x + y - 3$, freeze y

Hence: $\frac{dy}{dx} = -\frac{df/dx}{df/dy}$

i.e $\frac{dy}{dx} = \frac{(2x + y - 3)}{x + 2y - 1}$

$\frac{dy}{dx} = \frac{3 - 2x - y}{x + 2y - 1}$

If the tangents are parallel to the y – axis , their gradients are equal to infinity.

Hence: $\frac{dy}{dx} = \infty$

i.e $\frac{3 - 2x - y}{x + 2y - 1} = \infty$

? $x + 2y - 1 = 0$

$2y = 1 - x$

$y = \frac{1}{2} - \frac{x}{2}$

sub $y = \frac{1}{2} - \frac{x}{2}$ onto $x^2 + xy + y^2 - 3x - y = 3$

i.e $x^2 + x(\frac{1}{2} - \frac{x}{2}) + (\frac{1}{2} - \frac{x}{2})^2 - 3x - (\frac{1}{2} - \frac{x}{2}) = 3$

$x^2 + \frac{x}{2} - \frac{x^2}{2} + ¼ - x/2 + x^2/4 - 3x - ½ + x/2 = 3$

$4x^2 + 2x - 2x^2 + 1 - 2x + x^2 - 12x - 2 + 2x = 12$

$3x^2 - 10x - 13 =$

x = $\dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

x = $\dfrac{-(-10) \pm \sqrt{(-10)^2 - 4(3)(-13)}}{2 \times 3}$

x = $\dfrac{10 \pm \sqrt{256}}{6}$

x = $\dfrac{26}{6}\, or\, \dfrac{-6}{6}$

x = $\dfrac{13}{2}\, or\, -1$

for x = -1; y = $\dfrac{1}{2} - (-\dfrac{1}{2}) = 1$

x = $\dfrac{13}{2}$; y =

hence, the points are (13/5; -5/3) and (-1;1)

Example

a) For the curve x = $\dfrac{3t-1}{t}$; y = $\dfrac{t^2+4}{t}$; show

that (i) $\dfrac{dy}{dx}$ = $t^2 - 4$

(ii) $\dfrac{d^2y}{dx^2}$ = $2t^3$

b) Hence find and classify any stationary values on the curve.

<u>**Solution**</u>

a)(i) x = $\dfrac{3t-1}{t}$

$$\frac{dx}{dt} = \frac{t(3) - (3t-1)(1)}{t^2}$$

$$\frac{dx}{dt} = \frac{3t - 3t + 1}{t^2} = \frac{1}{t^2}$$

$$y = \frac{t^2+4}{t}; \frac{dy}{dt} = \frac{2t(t) - 1(t^2+4)}{t^2}$$

$$\frac{dy}{dt} = \frac{2t^2 - t^2 - 4}{t^2} = \frac{t^2 - 4}{t^2}$$

hence, $\frac{dy}{dt} = \frac{dy}{dt} . \frac{dt}{dx} = \frac{t^2 - 4}{t^2}$

= t^2 –4

(ii) $\frac{d^2y}{dx^2} = \frac{d(dy/dx)}{dx} = \frac{d(t^2 - 4)}{dx}$

but $\frac{d(t^2 - 4)}{dx} = \frac{d(t^2 - 4)}{dt} . \frac{dt}{dx}$

hence , $\frac{d^2y}{dx^2} = 2t.\frac{dt}{dx}; \frac{dt}{dx} = \frac{1/dx}{dt} = t^2$

hence, $d^2y = 2t \cdot dt$; $dt = 1 = t^2$

hence, $\frac{d^2y}{dx^2} = 2t(t^2)$

= $2t^3$

$\frac{dy}{dx} = 0$

i.e $t^2 - 4 = 0$ i.e $t = \pm 2$

for t = 2.

$$\frac{d^2y}{dx^2} = 2(2)^3 = 16 \succ 0$$

Hence, at t = 2, there is a minimum i.e for t = 2; x = 3(2) –1 = $\frac{5}{2}$ and y = $\frac{2^2 + 4}{2}$ = 4

hence, $(\frac{5}{2}; 4)$ is minimum point.

For **t = -2**

$\frac{d^2y}{dx^2} = 2(-2)^3 = -16 < 0$, there is a maximum point

$x = \frac{3(-2)-1}{2} = \frac{-7}{2}$; y = -4

hence, $(-\frac{-7}{2}; -4)$ is a maximum point

The Maclaurin Series.

Let f(x) be a defined function, such that it is possible to express:

$f(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 +$ ---------------

Where a_1, a_1, a_2--- are unknown constants.

Hence $f(0) = a_0$

$f'(x) = a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3 + 5a_5x^4 +$ -----

$f'(0)) = a_1$

$f^{11}(x) = 2a_2 + 2\cdot3a_3x + 3\cdot4a_4x^2 + 4.5a_5x^3 +$ ----

$f^{11}(0) = 2a_2$

i.e $a_2 = \frac{f^{11}(0)}{2}$

$f^{111}(x) = 1\cdot2\cdot3a_3 + 2\cdot3\cdot4a_4x + 3\cdot4\cdot5a_5x^2 +$ -----

$f^{111}(0) = 1\cdot2\cdot3\ a3$ i.e $a_3 = \frac{f^{111}(0)}{1\cdot2\cdot3} = \frac{f^{111}(0)}{3!}$

a. y = 1n (1 + x) ; y (0) = 1n1 = 0

$y^1 = \frac{1}{1+x}$; $y^1(0) = 1$

$y^{11} = -\frac{1}{(1+x)^2}$; $y^{11}(0) = -1$

$y^{111} = \frac{2(1+x)}{(1+x)^4}$; $y^{111}(0) = 2$

hence, $1n(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} +$-----

$= x - \frac{1}{2}x^2 + 1/3\, x^3 - \frac{1}{4}x^4 +$-------

(b) y = 1n(4 + 5x) , we use the expression for : y = 1n(1 + x)

hence y = 1n(4 + 5x) = 1n[4(1 + 5x/4)]

= 1n4 + 1n(1 + 5x/4)

Addition law of logs.

$1n(1+5x/4) = 5/4x - \frac{1}{2}(5/4x)^2 + 1/3(5x/4)^3 +$-----

$= \underline{5x} - \underline{25x^2} + \underline{125x^3} +$ -----

4 32 192

hence $\ln(4+5x) = \ln 4 + \frac{5x}{4} - \frac{25x^2}{32} + \frac{125x^3}{192} + \text{----}$

(c) $y = \sin x$; $y(0) = 0$; $y^1 = \cos x$; $y^1(0) = 1$

$y^{11} = -\sin x$; $y^{11}(0) = 0$; $y^{111} = -\cos x$; $y^{11}(0) = -1$

hence: $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \text{-----}$

hence, $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \text{----}$

Differentiate sin x and its expansion.

(d) hence,

$\sin 2x = 2x - \frac{(2x)^3}{3!} + \frac{(2x)^5}{5!} + \text{--------}$

$= 2x - \frac{8x^3}{6} + \frac{32x^5}{120} + \text{----}$

$\mathbf{= 2x - \frac{4x^3}{3} + \frac{4x^5}{15} + \text{-----}}$

Rate of Changes.

We know that $\frac{dy}{dx} \approx \frac{\delta y}{\delta x}$

Example

If $\frac{dy}{dx} = 4$, then y is increased 4 times as fast as x..

Example

Air is pumped onto a ball at the rate of 250cm3 per second. When the radius of the ball is 15cm, calculate:

1) The rate at which its radius is increasing?
2) The rate at which its surface area is increasing?

Solution

(i) Assume the ball is perfectly spherical, hence $v = \frac{4}{3}\pi r^3$ and $s = 4\pi r^2$.

$\frac{dr}{dt} = 250$, $v = \frac{4}{3}\pi r^3$

$\frac{dr}{dt} = 4\pi r^2$

And $\frac{dr}{dt} = \frac{dr}{dr} \cdot \frac{dr}{dt}$

$= \frac{1}{4\pi r^2} \cdot 250$

$= \frac{250}{4\pi r^2}$

hence, $\frac{dr}{dt} = \frac{250}{4x\pi x15^2} = \frac{5 \text{ cm/s}}{18\pi}$

(ii) $S = 4\pi r^2$

$\frac{ds}{dr} = 8\pi r$

$\frac{ds}{dt} = \frac{ds}{dr} \cdot \frac{dr}{dt}$

$= 8\pi rr \text{ x } \frac{5}{18\pi} \text{cm}^2/\text{s}$

$= \frac{8x15 \text{ x } 5}{18} \text{ cm}^2/\text{s}$

$= \frac{8x25}{6} \text{cm}^2/\text{s}$

$= \frac{100}{3} \text{ cm}^2/\text{s}$

Examination Type Questions

1. Use differentiation to find the coordinates of the stationary points on the curve $y = x + \frac{9}{x}$ and determine whether each stationary point is maximum point or a minimum point. Find the set of values of x for which y increases as x increases.

2. Express $\frac{14x + 2}{(x - 1)(2x + 3)}$ in partial fractions. Hence find the value of

$$\frac{d}{dx}\left(\frac{4x+2}{(x-1)(2x+3)}\right)$$

when $x = 3$.

3. (i) Show that $\frac{d}{dx}(\ln(\sec x + \tan x)) = \sec x$

(ii) Show that d $(\ln(\tan \frac{1}{2}x)) \equiv \operatorname{cosec} x$

(iii) Find and classify all stationary values on the curve $y = e^x \cos x$, in the range $0 \leq x \leq 2$□ .

4. (i) Show that $\frac{d}{dx}(\ln(\operatorname{Sec} x + \tan x)) = \operatorname{Sec} x$.

(iv) Show that $\frac{d}{dx}(\ln(\tan \frac{1}{2}x)) \equiv \operatorname{cosec} x$

(v) Find and classify all stationary values on the curve $y = e^x \cos x$, in the range $0 \leq x \leq 2$□

5. The inside of a glass is cylindrical in shape, the height of the glass is 8cm and radius 4cm. Wine is poured into the glass at the rate of $5cm^3/s$. Find the rate at which the depth of the wine in the glass is increasing.

6. It is given that $y = \frac{1}{1 + \operatorname{Sin} x}$. Show that when $x = 0$, $\frac{d^2y}{dx^2} = 2$

Find the first three terms in the Madaurin series for y.

8. The parametric equations of a curve are $x = t - e^{2t}$, $y = t + e^{2t}$, where

(i) Express $\frac{dy}{dx}$ in terns of t.

(ii) Hence find the value of t for which the gradient of the curve is 3, giving your answer in logarithmic for m

(iii) Show that , for points on the curve, the greatest value of x is ½ (1n½ - 1)

9. A curve C is given by the equation

$$y^3 + y^2 + y = x^2 - 2x$$

(i) Show that the point (3,1) is the only point of intersection of the line $x = 3$ and the curve.

(ii) Show that the tangent to C at the point (-1, 1) has equation $2x + 3y - 1 = 0$.

10. Wheat is dropped from an elevator shaft on to the ground at a steady rate of $20m^3/\ min$. It forms a conical pile whose height remains equal to the radius of its base. At what rate is the height increasing?

    (a) When the height is 10 meters

    (b) After 10 minutes

11. The parametric equations of a curve are

$$x = t - \frac{1}{t} \qquad y = t - \frac{2}{t} \qquad \text{where } t > 0$$

(i) Find $\frac{dy}{dx}$ in terms of *t* and hence find the value of $\frac{dy}{dx}$ when $x = 0$

(ii) Write down the first two terms of the Maclaurin's series for *y* in terms of *x*.

12. A rectangular block has a base which measurers 2x cm by 3x cm. Given that its volume is 1 800cm³, prove that the total surface area, A cm² is given by

$$A = 12x^2 + \frac{3\,000}{x}$$

Calculate the value of *x* for which *A* has a stationary value. Determine whether this value of *x* makes *A* a maximum or minimum and find optimal value of *A*.

CHAPTER 14

COMPLEX NUMBERS (I)

OBJECTIVES
By the end of the chapter the student should be able to :

- Write a complex number in the form a+yi
- Add , subtract , divide and multiply complex numbers
- Represent complex numbers in an Argand diagram

Introduction

We know that the square of any real number is non-negative, for example $(-4)^2 = 16 > 0$, $(\frac{3}{4})^2 = \frac{9}{16} > 0$, $(-\frac{2}{5})^2 = \frac{4}{9} > 0$, $(-1)^2 = 1 > 0$. As a result, the equation $x^2 + 1 = 0$ has no solution, since $x^2 = -1$. we cannot find the square roots of a negative number, in the real number system

Let us now consider the quadratic equation

$$x^2 - 4x + 13 = 0$$

Using the quadratic equation formula, we have

$$x = \frac{4 \pm \sqrt{(16-52)}}{2}$$

$$x = \frac{4 \pm \sqrt{-36}}{2}$$

$$x = \frac{4 \pm \sqrt{-1}\sqrt{.36}}{2}$$

$$x = \frac{4 \pm 6i}{2}$$

$$x = 2 \pm 3i$$

Then let $i = \sqrt{-1}$. Clearly i cannot be a real number.

Thus $x = \frac{4+6i}{2}$ *or* $\frac{4-6i}{2}$

$= 2 + 3i$ or $2 - 3i$

Also the equation $x^2 + 1 = 0$ has solutions $x = \pm i$. The new numbers i, -i , 2 + 3i, 2 – 3i are examples of numbers known as complex numbers.

Complex numbers

Complex numbers are usually denoted by the letter z, where

$$z = x + iy$$

x and y are real numbers and $i^2 = -1$. We can represent i and –i in the form

$$i = 0 + i, \text{ and } \quad -i = 0 - i$$

Thus a complex number has two parts: the real part and the imaginary part.
Re (z) and lm (z) can denote these respectively

Therefore for the complex number $Z = x + iy$

$$\text{Re}(z) = x \text{ and } \text{lm}(z) = y$$

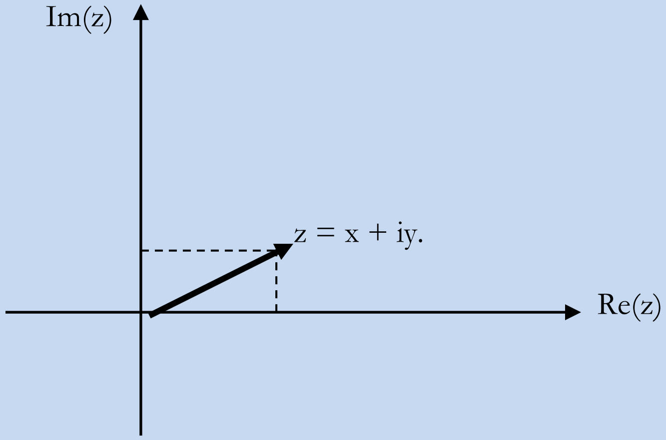
Example
Find the real and the imaginary parts of the complex number 6 – 2i

Solution
Re (6 – 2i) = 6 and lm (6 – 2i) = -2

Representation of Complex Numbers in the Plane

The x-axis is called the real axis and the y-axis is called the imaginary axis. The *xy*-plane in which the complex numbers are represented is called the complex plane or the Argand diagram. We usually use the latter.

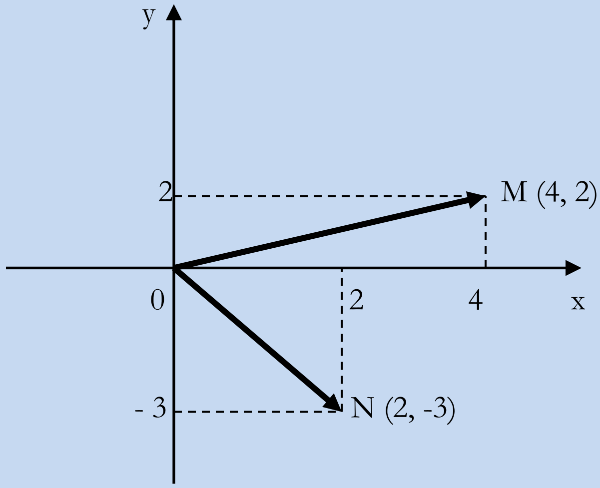
The Argand Diagram



The complex number z = x + iy is represented as a vector in the Argand diagram

Example
Represent the following complex numbers in an Argand Diagram

a) z = 4 + 2i b) z = 2 - 3i



A complex number x + iy is represented on an Argand diagram by the point P (x, y). Thus the complex numbers 4 + 2i and 2 – 3i are represented on the Argand diagram by the points M (4,2) and N (2, -3)

Practice Questions
Represent the following complex numbers on an Argand diagram.

(a) - 2 + 4i
(b) 2 + 3i
(c) 3 – 2i
(d) $2[\cos 45^0 + i\sin 45^0]$

Operations Involving Complex Numbers

(a) Equality of complex numbers
Two complex numbers $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ are equal if and only if the real parts are equal and the imaginary parts are equal, that is

$z_1 = z_2$ if and only if $x_1 = x_2$ and $y_1 = y_2$.

Note that if $x_1 = x_2$ but $y1 \neq y_2$ then $Z_1 \neq Z_2$ and also if $y_1 = y_2$ but $x_1 \neq x_2$ then $Z_1 \neq Z_2$.

Example
If 2 + iv = u + 4i, find value of u and v

Solution

Since the complex numbers are equal, the real parts are equal and the imaginary parts are equal also. Thus u = 2 and v = 4

Practice Questions

For each of the following pairs of complex numbers, find the unknowns in the real and imaginary parts, if all unknowns are real numbers.

(a) $-3 + 4i = x + iy$
(b) $u + 3i = -2 + iv$
(c) $10 + (3u + v)i = 6u - v + 8i$

Addition of complex numbers

The sum $z_1 + z_2$ of two complex numbers z_1 and z_2 is obtained by adding the real parts and the imaginary parts of z_1 and z_2 separately. i.e. if $z_1 = x_1 + iy$ and $z_2 = x_2 + iy_2$ then

$$z_1 + z_2 = (x_1 + x_2) + i(y_1 + y_2)$$

Subtraction of complex numbers

We also subtract the real parts and the imaginary parts separately as with addition

$$z_1 - z_2 = (x_1 - x_2) + i(y_1 - y_2)$$

Geometrical, this is accomplished by taking z_1 and z_2 as addition of vectors hence the parallelogram rule applies

Practice Questions

Evaluate the following:

(a) $(2 + 3i) + (1 - 5i)$
(b) $(3 + 2i) - (2 + i)$
(c) $(5 + 1\tfrac{1}{2} i) - (2\tfrac{1}{2} - 2\tfrac{1}{2} i)$
(d) $(-3 + 5i) - (-5 - 2i)$

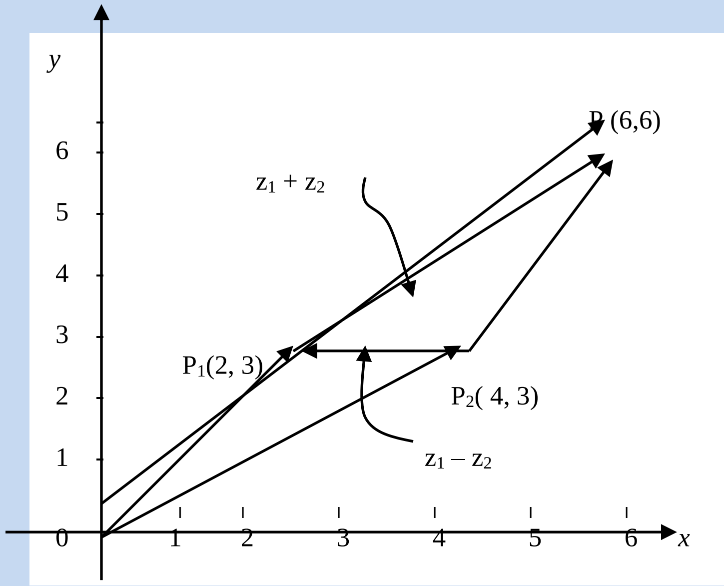
A complex number $z = x + iy$ can be regarded as a vector **OP** whose initial point is O, the origin and whose terminal point is the point P (x, y)

Example

Find graphically

(i) (2 + 3i) + (4 + 3i)

(ii) (2 + 3i) – (4 + 3i)



Exercise 4

Find graphically
 (a) (2 + 4i) + (3 + 2i)
 (b) (2 + 4i) – (3 + 2i)

Multiplication of complex numbers

Example

Find (3 + 5i) (2 – 2i)

Solution

$(3 + 5i)(2 - 2i) = 6 + i(10 - 6) - i^2$
$=16 + 4i$

Conjugation

Let z= x + iy be any complex number, then x – iy is called the conjugate of z and it is denoted by . Thus, if z = x + iy then = x - iy

For example, the conjugate of 5 + 2i is 5 – 2i and that of 2 – 7i is 2 + 7i

Now z + = (x + iy) + (x – iy) = 2x

While z = (x + iy)(x – iy) = x^2 + i(xy – xy) – i2y2

= x^2 – i2y2

= x^2 – (-1) y^2

= $x^2 + y^2$

Division of complex numbers

The quotient z_1/z_2 is obtained by noting that

z_2 $_2$ = $x^2 + y^2$ which is real

z_1 = $x_1 + iy_1$

z_2 = $x_2 + iy_2$

Example 5

Simplify $\frac{3+2i}{2-i}$

<u>Solution</u>

$$\frac{3+2i}{2-i} = \frac{(3+2i)(2+i)}{(2-i)(2+i)}$$

$$= \frac{6-2+i(4+3)}{4+1}$$

$$= \frac{4+7i}{5}$$

Exercise 5

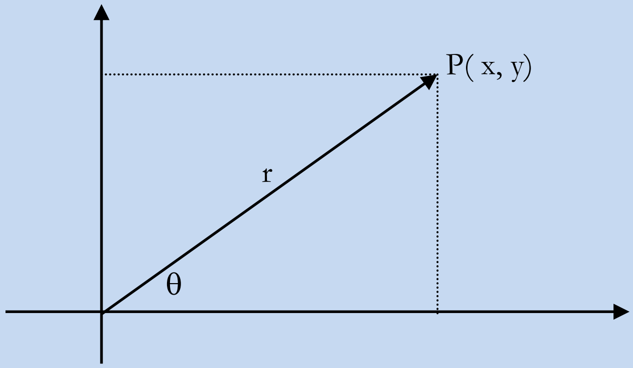
1. Expand (2 + 3i)(1 + 2i)
2. Show that z – = 2iy and hence show that lm (z) = ½ i (z –)
3. Simplify $\frac{2+3i}{5+2i}$

<u>Modulus</u>

The modulus or absolute value of the complex number z denoted by | z | is defined as

| z | = $\sqrt{(x^2 + y^2)}$, where z = x + iy

The Polar Form of a Complex Number



Let z= x + iy be a complex number represented in the diagram above, let

$r = |z|$, $x = r\cos\theta$ and $y = r\sin\theta$, hence

$z = x + iy = r\cos\theta + ir\sin\theta$

$\therefore$ $z = r[\cos\theta + i\sin\theta]$, where $\theta = \tan^{-1}(y/x)$ and $\theta = \arg(z)$, $-\pi \leq \theta \leq \pi$.

There are many complex numbers that can be represented in this case:

$z = r[\cos\theta + i\sin\theta] = r[\cos(\theta + 2n\pi) + i\sin(\theta + 2n\pi)]$, $n \in$

$\theta + 2n\pi = \text{Arg}(z)$

Example

Express the following complex numbers in polar form.

(a) $2\sqrt{3} + 2i$
(b) $-1 = -1 + 0i$
c) $1 + 2i$
d) $-2 - 2i$
e) $3 - 2i$
f) $-4 + i$

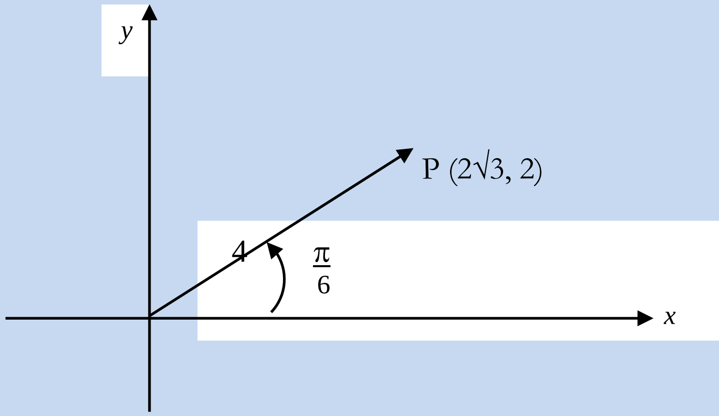
Solutions

(a) $z = 2\sqrt{3} + 2i$

$r = \sqrt{(2\sqrt{3})^2 + 2^2} = \sqrt{(12 + 4)} = \sqrt{16} = 4$

$\theta = \tan^{-1}(1/\sqrt{3}) = \frac{\pi}{6}$ since the vector is in the fisrt quadrant

. Therefore $2\sqrt{3} + 2i = 4(\cos \pi/6 + i \sin \pi/6)$ as shown below on the diagram



(b) $z = -1 = -1 + 0i$

$r = 1$

$\theta = \pi$

so $-1 = \cos \pi + i \sin \pi$

(c) $z = 1 + 2i$

$r = \sqrt{(1 + 2^2)} = \sqrt{5}$ and $\theta = \tan^{-1}(2) = 63.4^0$ since the vector is in the first quadrant

$1 + 2i = \sqrt{5}(\cos 63.4^0 + i \sin 63.4^0)$

(d) $z = -2 - 2i$

$r = \sqrt{((-2)^2 + (-2)^2)} = \sqrt{8} = 2\sqrt{2}$ and $\theta = \tan^{-1}(1) = 5/4\pi$, since the vector is in the third quadrant

(e) $z = 3 - 2i$

$r = \sqrt{(3^2 + (-2)^2)}$ and $\theta = \tan^{-1}(-2/3) = 326.3^0$, since the vector is in the fourth quadrant

$3 - 2i = \sqrt{5}(\cos 326.3^0 + i \sin 326.3^0)$

(f) $z = -4 + i$

$r = \sqrt{(-4)^2 + 1^2)}$ and $\theta = \tan^{-1}(-2/3) = 326.3^0$, since the vector is in the fourth quadrant

Practice Questions

Express each of the following numbers in polar form and represent each number on an Argand diagram

1. $1 + i$
2. $-3 + 4i$
3. $3 - 3\sqrt{3}i$
4. -1 – I

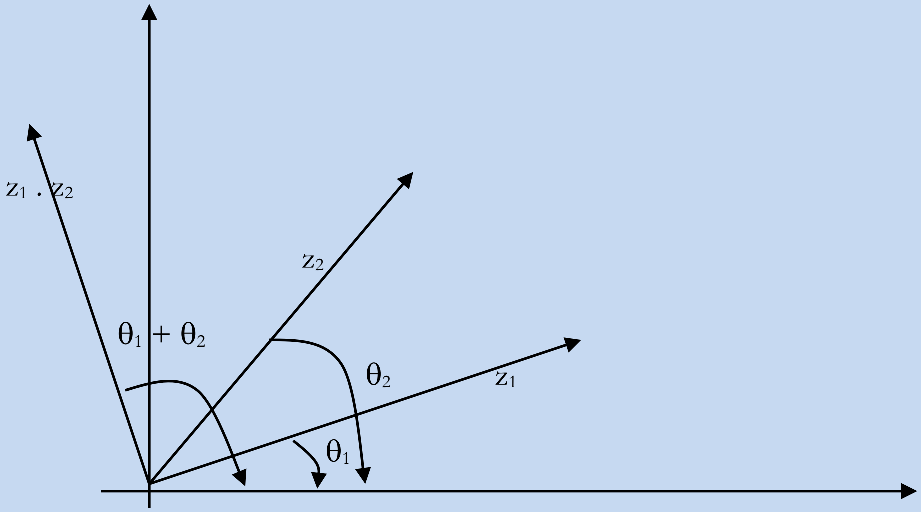
Multiplication Complex Numbers in Polar Form

Let $z_1 = r_1 (\cos \theta_1 + i\sin \theta_1)$ and $z_2 = r_2 (\cos \theta_2 + i\sin \theta_2)$, then

$$
\begin{aligned}
z_1 . z_2 &= r_1 (\cos \theta_1 + i\sin \theta_1). r_2 (\cos \theta_2 + i\sin \theta_2) \\
&= r_1.r_2 [\cos \theta_1 \cos \theta_2 + i\cos\theta_1\sin\theta_2 + i\sin\theta_1\cos\theta_2 + i.i \sin\theta_1\sin\theta_2] \\
&= r_1.r_2 [\cos \theta_1 \cos \theta_2 + i(\cos\theta_1\sin\theta_2 + \sin\theta_1\cos\theta_2) - \sin\theta_1\sin\theta_2] \\
&= r_1.r_2 [\cos (\theta_1 + \theta_2)+ i\sin(\theta_1 + \theta_2)], \text{ hence}
\end{aligned}
$$

$|z_1 z_2| = |z_1| . |z_2| = r_1 . r_2$ and $\arg (z_1 .z_2) = \theta_1 + \theta_2$

Geometrical, multiplication of the complex numbers is shown in the diagram below



Example

Given that $z = 4(\cos 2/3 \pi + i \sin 2/3 \pi)$ and $w = 2 (\cos 2\pi + i \sin 2\pi)$

Find $z. w$

Solution

$$
\begin{aligned}
z . w &= 8 (\cos(2/3 \pi +2\pi) + i \sin (2/3 \pi + 2\pi) \\
&= 8 (\cos(8/3\pi) + i \sin (8/3\pi)
\end{aligned}
$$

Division of Complex Numbers in Polar Form

The following are important results:

1. $\dfrac{1}{\cos \theta + i\sin \theta} = \cos\theta - i\sin\theta$

2 $\dfrac{1}{\cos \theta + i\sin \theta} = \cos\theta + i\sin\theta$

3. $\cos\theta - i\sin\theta = \cos(-\theta) + i\sin(-\theta)$

hence $\dfrac{z_1}{z_2} = r_1/r_2\,[\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)]$

Example
If $z_1 = 1 + i$, $z_2 = 1 - i$, display and label clearly on an Argand diagram,
a) $z_1 \cdot z_2$ b) z_1 / z_2

Solution

$z_1 = \sqrt{2}(\cos \pi/4 + i \sin \pi/4)$
$z_2 = \sqrt{2}[\cos(-\pi/4) + i \sin(-\pi/4)]$
a) $z_1 \cdot z_2 = 2[\cos(\pi/4 - \pi/4) + i \sin(\pi/4 - \pi/4)]$
$= 2[\cos 0 + i \sin 0]$
$= 2$
b) $z_1 / z_2 = 2[\cos(\pi/4 + \pi/4) + i \sin(\pi/4 + \pi/4)]$
$= 2[\cos \pi/2 + i \sin\pi/2]$
$= i$

Examination Type Questions

1. Solve for x and y
a) $x - 2y = i$
$(1 + i)x - 2iy = 3 + i$
b) $(1 - i)x - (1 + i)y = -1 + i$
$(-2 + 2i)x - 2y = -4$

2. Solve the following equations.
a) $|z| - 2z = -1 - 8i$ b) $2|z| - 3z = -1 - 12i$

3. Calculate
a) i^n b) $i^n + i^{n+1} + i^{n+2} + i^{n+3}$

4. Find the least value of n for which $(1 - i)^n$ is a positive number.

5. Find n if $(1 + i)^n = (1 - i)^n$

6. Find the summation,
a) $\sin x + \sin 2x + \sin 3x + \ldots + \sin(nx)$.
b) $\cos x + \cos 2x + \cos 3x \ldots + \cos nx$.

7. Given that $z_1 = 1 + 3i$ and $z_2 = -3 + 2i$, find
i) $|z_1|$
ii) $\arg z_2$

iii) $z_1 . z_2$

iv) z_1/z_2

Show the complex numbers z_1 and z_2 on the same Argand diagram, clearly labeling $|z_1|$ and arg z_2

8. The complex numbers *u*, *v*, and *w* are such that $\frac{1}{u} = \frac{1}{v} = \frac{1}{w}$.. If u = + 3i and v = 3 + 2i. Find *w* in the form a + bi.

9. If $(a+bi)^2 = x + yi$, show that $a^2 - b^2 = x$ and $2ab = y$. Hence evaluate $\sqrt{5+12i}$.

10 The complex numbers z and w are given by z = -3+2i and w=5+4i.
Find

(i) $|z|$

(ii) Argz

(iii) $\frac{z}{w}$ in the form a+bi where a and b are exact. Hence represent $\frac{z}{w}$ in an Argand diagram *zimsec 2009*

CHAPTER 15

SERIES EXPANSION

OBJECTIVES

By the end of the chapter the student should be able to :

- Expand a biamonials using the biamonia series
- Estimate values e.g $(x+ 0{,}2)^2$ using the binomial expansion
- Find coefficient from a bimomial expansion

The Binomial Expansion

In the following, we shall be concerned with the expansion of the expression $(x + y)^n$, where n is a non- negative integer. It can be shown by ordinary multiplication that

$(x + y)^0 \quad = \quad 1$

$(x + y)^1 \quad = \quad x + y$

$(x + y)^2 \quad = \quad x^2 + 2xy + y^2$

$(x + y)^3 \quad = \quad x^3 + 3x^2y + 3xy^2 + y^3$

$(x + y)^4 \quad = \quad x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$

Notice that the coefficients of the terms in the above expansions form what is known as a Pascal's triangle.

n = 0					1				
n = 1				1		1			
n = 2			1		2		1		
n = 3		1		3		3		1	
n = 4	1		4		6		4		1

Each row of Pascal's triangle starts and ends with a 1; others can be obtained by adding the two terms on either side of it in the preceding row.

In general, if n is a positive integer, the expansion of $(x + y)^n$ is given by

$$(x + y)^n = x^n + \overset{n}{\underset{1}{C}} x^{n-1} y + \overset{n}{\underset{2}{C}} x^{n-2} y^2 + \ldots + \overset{n}{\underset{n-1}{C}} xy^{n-1} + y^n$$

This is called the Binomial Theorem for any positive integral value of n, where

$$\binom{n}{r} = {}^{n}C_r = \frac{n!}{r!(n-r)!} = 1 \text{ x } 2 \text{ x } 3 \text{ x } 4 \ldots\ldots\ldots \text{x } n$$

Example

Expand $(2x + y)^5$ and simplify each term

Solution

$(2x + y)^5 \quad = (2x)^5 + 5(2x)^4y + 10\,(2x)^3\,y^2 + 10\,(2x)^2y^3 + 5\,(2x)(y^4) + y^5$

$\quad = 32x^5 + 80x^4y + 80x^3y^2 + 40x^2y^3 + 10xy^4 + y^5$

Example

Expand $(2x + {}^1/_x)^4$ and simplify each term.

Solution

$(2x + {}^1/_x)^4 = (2x)^4 + 4(2x)^3\ {}^1/_x + 6(2x)^2\ ({}^1/_x)^2 + 4\,(2x)\,({}^1/_x)^3 + ({}^1/_x)^4$

$\quad = 16x^4 + 32x^3 + 24 + 8/_{x}{}^2 + ({}^1/_x)^4$

Example

Evaluate $(1.02)^4$ using the binomial expansion giving your answer to 6 decimal places

Solution

We write $(1.02)^4 = (1 + 0.02)^4$

Therefore

$(1.02)^4 \quad = \quad 1^4 + 4.1^3\ (0.02) + 6.1^2\ (0.02) + 4.^1\ (0.02)^3 + (0.02)^4$

$\quad = \quad 1 + 0.08 + 0.00024 + 0.000032 + 0.00000016$

$\quad = \quad 1.082432$

The Binomial Series

If n is a positive integer, the binomial expansion

$(1 + x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \ldots\ x^n$ is a

finite series consisting of n + 1 terms. Consider the infinite geometric series.

$1 + x + x^2 + x^3 + \ldots$

which has a finite sum of $1/(1 - x)$ provided $|x| < 1$.

Thus, we write

$$1/(1 - x) = (1 - x)^{-1} = 1 + x + x^2 + x^3 + \ldots.$$

Which can also be stated as the expansion of $(1 - x)^{-1}$. This result can be obtained by putting n = -1 in the following binomial expansion for positive index n.

For instance:

$$(1 + x)^{-1} = 1 + (-1)x + \frac{(-1.\ -2)}{1.2}x^2 + \frac{(-1.\ -2\ .\ -3)}{1.2.3}x^3 + \ldots$$

$$= 1 - x + x^2 - x^3 + \ldots$$

Also we have,

$$(1 - x)^{-1} = 1 + x + x^2 + x^3 + \ldots$$

Example

a) Given that $\dfrac{2 - 6x + 10x^2}{(1 - 3x)(1 + x^2)} = \dfrac{A}{1 - 3x} + \dfrac{Bx + c}{1 + x^2}$

find the value of the constants A,B and C.

b) Write down the series expansion, up to and including a term in x^3 of

$1 / (1 - 3x)$ and $1/ (1 + x^2)$

c) Deduce that, if x is small enough for terms in x^4 and higher powers of x to be ignored, then: $\dfrac{2 - 6x + 10x^2}{(1 - 3x)(1 + x^2)} = 2 + ax^2 + bx^3$, where a and b are constants to be determined.

Solution

$$\frac{2 - 6x + 10x^2}{(1 - 3x)(1 + x^2)} = \frac{A}{1 - 3x} + \frac{Bx + C}{1 + x^2}$$

i.e.

$$2 - 6x + 10x^2 = A(1 + x^2) + (1 - 3x)(Bx + c)$$

$$2 - 6x + 10x^2 = Ax^2 + A + Bx + C - 3Bx^2 - 3cx$$

Equating coefficients.

$A + C = 2$, $A - 3B = 10$, $B - 3c = -6$

$A = 2 - c$

Hence:

$B - 3c = -6 \times 3$

$\underline{-3B - c = 8 \times 1}$

$3B - 9c = -18$

$\underline{-3B - c = 8}$

$-\quad 10c = -10$

C = 1

$A = 2 - 1$

A = 1

B – 3c = -6
B = -6 + 3c
B = -6 + 3(1)
B = -3

Hence: $\frac{2 - 6x + 10x^2}{(1 - 3x)(1 + x^2)} = \frac{1}{1 - 3x} + \frac{1 - 3x}{1 + x^2}$

$(1 - 3x)^{-1} = 1 + (-1)(-3x) + \frac{(-1)(-2)}{2!}(-3x)^2 + \frac{(-1)(-2)(-3)}{3!}(-3x)^3 \ldots.$
$= 1 + 3x + 9x^2 + 27x^3$

$(1 + x^2)^{-1} = 1 + (-1)x^2$
$= 1 - x^2$

Hence: $\frac{1}{1 - 3x} + \frac{1 - 3x}{1 + x^2} =$
$= 1 + 3x + 9x^2 + 27x^3 + (1 - 3x)(1 - x^2)^{-1}$
$= 1 + 3x + 9x^2 + 27x^3 + 1 - x^2 + 3x^3$
$= 2 + 8x^2 + 30x^3$

Hence: **a = 8 and b = 30.**

Examination Type Questions

1. i) Show that $(4 - x)^{-½} = 1/2\,(1 - x/4)^{-1/2}$
ii) Write down the first three terms in the binomial expansion of $1/2\,(1 - x/4)^{-½}$ in ascending powers of x, stating the range of values of x for which this expansion is valid.
iii) Find the first three terms in the expansion of $2(1 + x)(4 - x)^{-½}$ in ascending powers of x, for small values of x

2. When $(1 - 1.5x)^p$ is expanded in ascending powers of x, the coefficient of x is 24.
a) find the value of p
b) find the coefficient of x^2 in the expansion
c) find the coefficient of x^3 in the expansion

3. a)Obtain the first four non-zero terms of the binomial expansion in ascending powers of x of $(1 - x^2)^{-½}$ given that $|x| < 1$
b) Show that ,when x = 1/3, $(1 - x^2)^{-½} = 0.75\sqrt{2}$
c) Substitute x = 1/3 into your expansion and hence obtain an approximation to $\sqrt{2}$,giving your answer to five decimal places.

Expand fully $(a + 2b)^6$, simplify the co-efficients.

Hence or otherwise write down the term independent of *x* in the expansion of

$(x^2 + 2/x)^6$

CHAPTER 16

INTEGRATION – AREAS AND VOLUMES OF REVOLUTION.

OBJECTIVES
By the end of the chapter the student should be able to:

- Intergrate polynomials
- Intergrate using the change of variable technique
- Intergrate trig. Functions
- Intergrate exponential functions
- Calculate areas and volumes of revolutions

Indefinite Integration
This process is the reverse of differentiation. In other words, we wish to find the original function given its derivative.

We know from differentiation that if $y = 5x^2 + 3$, then $y^1 = 10x$. Now suppose that we are given $y^1 = 10x$ and asked to find y in terms of x. This process is the reverse of differentiation and is called **integration**

In this particular case, we know that $y = 5x^2 + 3$ will satisfy $y^1 = 10x$, but so will $y = 5x^2$ and $y = 5x^2 + 3$. In general , $y = 5x^2 + k$, where k is a constant will satisfy $y^1 = 10x$, hence the **general solution** of $y^1 = 10x$ is $y = 5x^2 + k$ and the latter is called the **integral** of 10x.

This is written as

$$\int 10x \quad \underbrace{dx}_{\text{Indicating that the integration is with respect to the variable x}} \quad = \quad 5x^2 + k$$

Integral sign

k is called the constant of integration and we would need further information to find its value.

Algebraic functions

Result

If dy/dx = ax^n, then $y = \frac{ax^{n+1}}{n+1} + k$, $n \neq -1$.

One way of remembering this is " add one to the power and divide by the new power".

Example

Find a) $\int x^5 dx$ b) $\int 4x^3 dx$ c) $\int 5/x^3\, dx$

Solution

(a) $\int x^5 dx = \frac{x^{5+1}}{5+1} + k = x^6 + 6$

(5 + 1) 6

(b) $\int 4x^3 dx = \frac{4x^{3+1}}{3+1} = x^4 + k$

(c) $\int \frac{5}{x^3} dx = \frac{5x^{-3+1}}{-3+1} + k = -\frac{5}{2}x^{-2} + k$

Example

Find $\int (x^4 – x^2 – 3x - 2)\, dx$

Solution

$\int (x^4 – x^2 – 3x - 2)\, dx = 1/4x^4 – 1/2x^2 – 3/x + k$

Result

If $y^1 = (ax + b)^n$, then $\int (ax + b)^n dx = \frac{(ax + b)^{n+1}}{a(n + 1)} + k$

Example

Find $\int (2x + 3)^4 dx$

Solution

$\int (2x + 3)^4 dx = 1/10 (2x + 3)^5 + c$

Example

Find $\int \frac{2}{(4x-7)^3} dx$

<u>Solution</u>

$$\int \frac{2}{(4x-7)^3}dx = \int 2(4x-7)^{-3}dx$$

$= \quad 2\,(4x - 7)^{-2}\ (\tfrac{1}{2})\ (\tfrac{1}{4}) + c$

$= \quad -\tfrac{1}{4}\,(4x - 7)^{-2} + c$

$= \quad - \dfrac{-1}{4(4x-7)^2} + c$

Example

Find $\int (6x + 1)^{1/3}\ dx$

<u>Solution</u>

$\int (6x + 1)^{1/3}\ dx = 1/8(6x + 1)^{4/3} + k$

Practice Questions

1. Integrate with respect to x

a) $\dfrac{2}{3x^4}$

b) $\sqrt{x^5}$

c) $5x^{-3/2} - 2x^{-2/3}$

d) $\dfrac{5x^2-4}{\sqrt{x}}$

e) $2x^{1/3} - 3x^{4/3} - 5x^{7/3}$

Find each of the following integrals

a) $\int (3\sqrt{x} - 4)\ dx$

b) $\int (x^{1/4} - x^{-1/4})\ dx$

(c) $\int \sqrt{x}(x - 3)\ dx$

<u>Change of Variable Technique</u>

Consider the integral $\int x(x^2 + 1)^3\ dx$, one way of solving this integral is by expansion i.e. writing $x(x^2 + 1)^3$ as a polynomial , the other way is to use the change of variable technique. In this section we will navigate through the latter.

Let $u = x^2 + 1$, then $du = 2xdx$ i.e. $\tfrac{1}{2}du = xdx$, hence

$\int x(x^2 + 1)^3\ dx = \int \tfrac{1}{2}\,(u + 1)^3\ du$

$= 1/8\,(u + 1)^4 + k$

$= 1/8(x^2 + 1)^4 + k$

Result

If $y^1 = f^1(x)f(x)$, then $\int f^1(x)f(x)dx = ½ (f(x))^2 + k$

Example

Find each of the following integrals using the suggested substitution

(a) $\int x(x-3)^2\,dx$, $\quad u = x - 3$

(b) $\int x(x+4)\,dx$, $\quad u = x + 4$

(c) $\int (x-4)(x-1)^3\,dx$, $\quad u = x - 1$

<u>Solution</u>

(a) $\int x(x-3)^2\,dx$, $\quad u = x - 3$,then $du = dx$, hence

$\int x(x-3)^2\,dx = \int (u+3)u^2$

$= \int (u^3 + u^2)du$

$= ¼\, u^4 + \downarrow u^3 + c$

$= ¼\,(x-3)^4 + (x-3)^3 + c$

(b) $\int x(x+4)\,dx$, $\quad u = x + 4$, then $du = dx$, hence

$\int x(x+4)\,dx = \int u(u-4)du$

$= \int (u^2 - 4u)du$

$= 1/3(x+4)^3 - (x+4)^2 + c$

(c) $\int (x-4)(x-1)^3\,dx$, $\quad u = x - 1$, $x = u + 1$, $dx = du$

$\int (x-4)(x-1)^3\,dx = \int (u + 1 - 4)u^3\,du = \int (u^4 - u^3)du$

$= 1/5u^5 - 1/3u^3 + k$

$= 1/5(x-1)^5 - 1/3(x-1)^3 + k$

<u>Integration of Trigonometric Functions</u>

$\int \sin x\,dx = -\cos x + c$

$\int \cos x\,dx = \sin x + c$

$\int \sec^2 x \, dx = \tan x + c$

Example: $\int \cos 3x \, dx = {}^1/_3 \sin 3x + c$

Example: $\int \text{Sin } (3x + 5) \, dx = -{}^1/_3 \cos (3x + 5) + c$

Example: $\int \{2 \sin x + 3 \sec^2 x\} dx = -2 \cos x + 3 \tan x + c$

Integration of type $\int \sin^2 x \, dx$ and $\int \cos^2 x \, dx$

Recall that $\cos^2\theta = \frac{1}{2}(1 + \cos 2\theta)$

$\sin^2\theta = \frac{1}{2}(1 - \cos 2\theta)$

Example

Find $\int \cos^2 5\theta \, d\theta$

Solution

$$\int \cos^2 5\theta \, d\theta = \frac{1}{2}\int \{1 + \cos 10\theta\} \, d\theta$$

$$= \frac{1}{2}(\theta + 1/10 \text{ Sin } 10\theta) + c$$

$$= \frac{1}{2}(\theta + 1/10 \text{ Sin } 10\theta) + c$$

Example

Find $\int \sin^2\theta \, d\theta$

Solution

$$\int \sin^2\theta \, d\theta = \int \frac{1}{2}(1 - \cos 2\theta) \, d\theta$$

$$= \frac{1}{2}\theta - \frac{1}{4}\sin 2\theta + k$$

Result

$$\int \textbf{Sin}^n x \, dx = \frac{\sin^{n+1} x + c}{n+1}$$

$\int$ Cos ^{n}x Sin x = $\dfrac{\cos^{n+1} x + c}{n+1}$

Example: $\int \text{Sin}^2 x \cos x \, dx = \dfrac{\sin^3 x + c}{3}$

Example : $\int \cos^5 2x \text{ Sin } 2x \, dx = \dfrac{\cos^5 2x + c}{10}$

Integration of Exponential Functions

$\int e^x \, dx = e^x + c$

Example : $\int e^{3x} \, dx = \dfrac{1}{3} e^{3x} + c$

Example : $\int \{e^{5x} + e^{3x}\} dx = \dfrac{1}{5} e^{5x} + \dfrac{1}{3} e^{3x} + c$

Result

If y = 1/x, then $\int$ 1/x dx = ln│x│ + k

Example

Evaluate $\int$ tan x dx

Solution

$\int$ tan x dx = $\int$ sinx/cosx dx, let u = cosx, then du = - sinxdx

$\int$ tan x dx =$\int$ (- 1/u)du= - ln cosx + c

Example

Find $\int 2x/(1 + x^2)$ dx, using $u = 1 + x^2$

Solution

$\int 2x/(1 + x^2)$ dx, $u = 1 + x^2$, du = 2xdx,then ½ du = xdx

$\int 2x/(1 + x^2)$ dx = ½ $\int$ 1/udu

= ½ lnu + c

= ln $(1 + x^2)$ + c

Example:

Find $\int x^2/(2x^3 + 3)\,dx$, using $u = 2x^3 + 3$

Solution

$\int x^2/(2x^3 + 3)\,dx$, using $u = 2x^3 + 3, du = 6x^2dx$

$$\int x^2/(2x^3 + 3)\,dx = \frac{1}{6}\int 1/u\,du$$

$$= \frac{1}{6}\ln u + c$$

$$= \frac{1}{6}\ln(2x^3 + 3) + c$$

Use of Partial Fractions in Integration

Example

Find $\int 1/(x^2 - 1)dx$

Solution

$1/(x^2 - 1) \equiv A/(x - 1) + B/(x + 1)$

$1 = A(x + 1) + B(x - 1)$

set $x = 1$: $1 = A(2) \Rightarrow A = ½$

set $x = -1$: $1 = B(-2) \Rightarrow B = -½$

$$\frac{1}{x^2 - 1} = \frac{1}{2}\left(\frac{1}{x-1} - \frac{1}{x+1}\right)$$

$$\int 1/(x^2 - 1)dx = \frac{1}{2}\int \{1/(x - 1) - 1/(x -+1)\}dx$$

$$= \frac{1}{2}\ln|x - 1| - ½\ln|x + 1| + c$$

Example

Find $\int (x + 2)/(x + 5)dx$

Solution

$$\frac{x+2}{x+5} \equiv A + \frac{B}{x+5}$$

You may use long division as well

$x + 2 = A(x + 5) + B$

set $x = -3$: $-3 = B$

set x = 1: 3 = 6A + B

3 = 6A – 3

6 = 6A

A = 1

$\int (x + 2)/(x + 5)dx = \int \{1 - 3/(x + 5)\}dx$
$= x - 3 \ln (x + 5) + c$

Example
Find $\int(x^2 + x - 11)/(x + 4)dx$

You may use long division

Solution
$(x^2 + x - 11)/(x + 4) \equiv Ax + B + C/(x + 4)$

$x^2 + x - 11 = Ax(x + 4) + B(x + 4) + C$

set x = - 4 : $(-4)^2 - 4 - 11 = C$
$16 - 15 = C$
$C = 1$
set x = 0 : $-11 = 4B + C$
$-11 = 4B + 1$
$-12 = 4B$
$B = -3$
Set x = 1 : $1 + 1 - 11 = 5A + 5B + C$
$-9 = 5A + 5(-3) + 1$
$-9 + 15 - 1 = 5A$
$5 = 5A$
$A = 1$
$\int(x^2 + x - 11)/(x + 4)dx = \int\{x - 3 + 1/(x + 4)\}dx$

$$= \frac{1}{2}x^2 - 3x + \ln|x + 4| + k$$

Practice Questions
Integrate with respect to x

1. $\dfrac{x+2}{x(x+4)}$ 2. $\dfrac{x^2}{(x+1)(x-3)}$ 3. $\dfrac{x^2+2}{x-2}$

Integration by Parts

The student will recall that

$$\int \frac{d(dx)}{dx} = \int \frac{dx}{dx} + \int \frac{vdudx}{dx}$$

Integrating both sides w.r. t. x

$\int d/dx(\ uv)dx = \int udv/dx(dx) + \int vdu/dx(dx)$
$uv = \int udv + \int vdu$
$\int udv = uv - \int vdu$ … this is the formula used in integration by parts

Example
Find $\int x\ e^{4x}\ dx$

Solution
Let $u = x$, $du = dx$ and $dv = e^{4x}$, $\int dv = \int e^{4x}dx \Rightarrow v = \frac{1}{4} e^{4x}$
$\int x\ e^{4x}\ dx = \frac{1}{4} xe^{4x} - \frac{1}{4} \int e^{4x}dx$
$= \frac{1}{4} xe^{4x} - 1/16\ e^{4x} + k$

Example

Find $\int \ln x dx$

Solution
Let $u = \ln x$, $du = \frac{dx}{x}$

$dv = dx \Rightarrow \int dv = \int dx \Rightarrow v = x$
$\int \ln x dx = x\ln x - \int x(1/x)dx$
$= x\ln x - \int dx$
$= x\ln x - x + k$

Example
Find $\int x\ln x\ dx$

Solution
Let $u = \ln x$, $du = \frac{dx}{x}$

$dv = xdx \Rightarrow v = \frac{1}{2} x^2$
$\int x\ln x\ dx = \frac{1}{2} x^2\ln x - \int 1/x(\frac{1}{2} x^2)dx$
$= \frac{1}{2} x^2\ln x - \int \frac{1}{2} xdx$
$= \frac{1}{2} x^2\ln x - \frac{1}{4} x^2 + k$

Example

Find $\int e^{x} \sin x\ dx$

Solution

Let $u = e^x$, $du = e^x dx$,

You may use $u = \sin x$, but this is not always the case. Care should be taken in choosing u that will simplify calculations

$dv = \sin x dx \Rightarrow \int dv = \int \sin x dx \Rightarrow v = -\cos x$
$\int e^x \sin x\, dx = -e^x \cos x + \int e^x \cos x\, dx$

Applying the formula again

Let $u = e^x$, $du = e^x dx, dv = \sin x dx \Rightarrow \int dv = \int \cos x dx \Rightarrow v = \sin x$
$\int e^x \sin x\, dx = -e^x \cos x + \{ e^x \sin x - \int e^x \sin x\, dx \}$
$\int e^x \sin x\, dx = -e^x \cos x + e^x \sin x - \int e^x \sin x\, dx$
$2\int e^x \sin x\, dx = -e^x \cos x + e^x \sin x$
$\int e^x \sin x\, dx = ½(-e^x \cos x + e^x \sin x) + k$

Practice Questions

Find

1. $\int x\cos x dx$
2. $\int x^3 \ln x dx$
3. $\int \sqrt{x} \ln x dx$
4. $\int (x + 1)e^x dx$
5. $\int x^2(x - 2)^{-1/2} dx$
6. $\int e^{-x} \sin x\, dx$

Definite Integration

We define definite integration as:

$$\int_a^b f(x)dx = F(b) - F(a)$$

- The dx indicates that the limits **a** and **b** are x limits
- The constant a is called the lower limit of the integral
- The constant b is called the upper limit of the integral

Example

Evaluate $\int_0^1 x^2 dx$

Solution

$$\int_0^1 x^2 dx = [1/3\, x^3 + c]_0^1 = 1/3(1^3) + c - (1/3(0^3) + c) = 1/3$$

The constant of integration c disappears through subtraction. It is always left out when integrating

Example

Evaluate $I = \int_1^{16} (\sqrt{x-4})/\sqrt{x}dx$

Solution

$$\frac{\sqrt{x-4}}{\sqrt{x}} = \frac{\sqrt{x}}{\sqrt{x}} - \frac{4}{\sqrt{x}} = 1 - 4x^{-\frac{1}{2}}$$

$$I = \int_1^{16}(1 - 4x^{-1/2})dx$$

$$= (x - 8\sqrt{x}) \Big|_1^{16} = 16 - 8\sqrt{16} - (1 - 8\sqrt{1}) = 16 - 32 - 1 + 8 = -9$$

Example

Evaluate $I = \int_3^4 3x(x-2)^{-½} dx$, using the substitution $u = (x-2)^{½}$

Solution

Given $u = (x-2)^{½}$, then $du = ½ (x-2)^{-1/2} dx = \frac{dx}{2(x-2)^{\frac{1}{2}}} = \frac{dx}{2u}$

i.e. $2udu = dx$

The limits must be changed from x limits to u limits. This is done by calculating the value of u when x =3 and x =4. You may also find the indefinite integral and then evaluate the x limits

x	u
3	1
4	√2

for $x = 3$, $u = (3-1)^{1/2} = 1$

for $x = 4$, $u = \sqrt{(4-2)} = \sqrt{2}$

Also we have, $u^2 = x - 2 \Rightarrow x = u^2 + 2$, hence

$$I = \int_1^{\sqrt{2}} 3/u(u^2 + 2)2udu = \int_1^{\sqrt{2}}(6u^2 + 12)du$$

$$= [2u^3 + 12u]_1^{\sqrt{2}}$$

$$= 16\sqrt{2} - 14$$

Example

Evaluate $I = \int_0^1 (1 + x^2)^{-1} dx$, using the substitution $x = \tan\theta$

Solution

Given that, $x = \tan\theta$, $dx = \sec^2\theta d\theta$ and $x^2 = \tan^2\theta$

x	θ
0	0
1	¼ π

Therefore, $I = \int_0^{\frac{1}{4}\pi} \sec^2\theta/(1 + \tan^2\theta)\, d\theta = \int_0^{\frac{1}{4}\pi} d\theta = \left.\theta\right]_0^{\frac{1}{4}\pi} = \frac{1}{4}\pi$

Example

By using the substitution $x = 2\cos\theta$,show that:

$I = \int_1^2 x^{-2}(4 - x^2)^{-\frac{1}{2}}\, dx = \frac{1}{4}\sqrt{3}$

Solution

Given that $x = 2\cos\theta$, $dx = -2\sin\theta d\theta$ and $x^2 = 4\cos^2\theta$

For $x = 1$, $\theta = \downarrow\pi$

For $x = 2$, $\theta = 0$,then

$$\boxed{\int_a^b f(x)\, dx = -\int_b^a f(x)dx}$$

$I = \int_\pi^0 -2\sin\theta(4\sec^2\theta)(4 - 4\cos^2\theta)^{-1/2}\, d\theta = -\int_0^\pi(-2\sin\theta(4\sec^2\theta)(4 - 4\cos^2\theta)^{-1/2}\, d\theta$

$= \frac{1}{4}\int_0^\pi \sec^2\theta\, d\theta = \frac{1}{4}\tan x\Big]_0 = \frac{1}{4}\sqrt{3}$

Example

Evaluate $\int_0^3 (x + 2)(x + 3)^{-2}(x + 1)^{-1}\, dx$

Solution

$$\frac{x+2}{(x+3)^2(x+1)} = \frac{A}{x+3} + \frac{B}{(x+3)^2} + \frac{C}{x+1}$$

$x + 2 = A(x + 3)(x + 1) + B(x + 1) + C(x + 3)^2$

set $x = -1$: $1 = 4C \Rightarrow C = \frac{1}{4}$

set $x = -3$: $-1 = -2B \Rightarrow B = \frac{1}{2}$

Equating independent terms: $2 = 3A + B + 9C$

$2 = 3A + \frac{1}{2} + 9(\frac{1}{4})$

$2 = 3A + 11/4$

$-\frac{3}{4} = 3A$

$A = -\frac{1}{4}$

3

Therefore, $I = \int_{0} \{ -\tfrac{1}{4}/(x+3) + \tfrac{1}{2}/(x+3)^2 + \tfrac{1}{4}/(x+1)\}dx$

$I = [-\tfrac{1}{4}\ln(x+3) - \tfrac{1}{2}(x+3)^{-1} + \tfrac{1}{4}\ln(x+1)]_{0}^{3}$

$I = -\tfrac{1}{4}\ln 6 - \tfrac{1}{2}(1/6) + \tfrac{1}{4}\ln 4 + 0\ln 3 + \tfrac{1}{2}(1/3)$

$I = \tfrac{1}{4}\ln(\,^\circ\,) + \tfrac{1}{4}\ln 3 - 1/12 + \tfrac{1}{6}$

$I = \tfrac{1}{4}\ln 2 + 1/12$

Example

Evaluate $I = \int_{0}^{1} x^2 e^{2x}dx$

$$\boxed{\int_a^b u\,dv = uv\Big|_a^b - \int_a^b v\,du}$$

Solution

Let $u = x^2$, $du = 2xdx$

$v = \tfrac{1}{2}e^{2x}$

$I = \tfrac{1}{2}x^2e^{2x}\Big|_0^1 - \int_0^1 xe^{2x}dx$

$I = \tfrac{1}{2}e^2 - \int_0^b xe^{2x}dx$, let $u = x$, $du = dx$ and $v = \tfrac{1}{2}e^{2x}$

$I = \tfrac{1}{2}e^2 - (\tfrac{1}{2}xe^{2x}\Big|_0^1 - \tfrac{1}{2}\int_0^1 e^{2x}dx)$

$I = \tfrac{1}{2}e^2 - \tfrac{1}{2}e^2 + \tfrac{1}{4}e^{2x}\Big|_0^1$

$I = \tfrac{1}{4}(e^2 - 1)$

Example

Evaluate $I = \int_{\pi/6}^{\pi/3} x\cos 3x\, dx$

Solution

Let $u = x$, $du = dx$

$v = \tfrac{1}{3}\sin 3x$

$$\boxed{\textbf{In this case you cannot take } u = \cos 3x}$$

$I = \tfrac{1}{3}x\sin 3x\Big|_{\pi/6}^{\pi/3} - \tfrac{1}{3}\int_{\pi/6}^{\pi/3}\sin 3x\,dx$

$I = \tfrac{1}{3}x\sin 3x + 1/9\cos 3x\Big|^{\pi/3}$

$$I = -\frac{(\square + 2)}{18}^{\square/6}$$

Practice Questions

1. Evaluate $I = \int_3^6 x^2 (x-2)^{-\frac{1}{2}}\, dx$, using $u = (x-2)^{\frac{1}{2}}$

2. $I = \int^{\square} x^2 \sin x\, dx$, find I

3. a) Show that $\int_0^a f(x)\, dx = \int_0^a f(a-t)\, dt$: Hint: take $x = a - t$

   b) Deduce that $\int_0^{\pi} x \sin x\, dx = \frac{1}{2}\pi \int_0^{\pi} \sin x\, dx$

   c) Hence evaluate $\int_0^{\pi} x \sin x\, dx$

3. a) By using the substitution $u = x^3$, show that $\int 3x^5/(x^3 - 1)\, dx = \int u/(u-1)\, du$
   b) Deduce that $\int 3x^5/(x^3 - 1)\, dx = x^3 + \ln(x^3 - 1) + c$
   c) Hence evaluate

   $\int_2^3 3x^5/(x^3 - 1) dx$

4. Express as the sum of partial fractions $2/x(x+1)(x+2)$

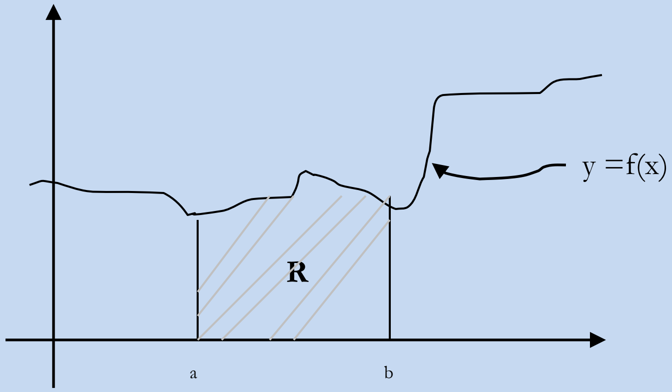
   Hence show that $\int_2^4 2/x(x+1)(x+2)\, dx = 3\ln 3 - 2\ln 5$

Application of Integration

Definite integration can be used to calculate areas and volumes of revolutions.

1.1.7 Area under a curve

Consider the area of region R bounded by a curve $y = f(x)$, the x – axis, the lines $x = a$ and $x = b$



Hence $A_R = \int_a^b f(x)\,dx$, where, A_R is the area of region R

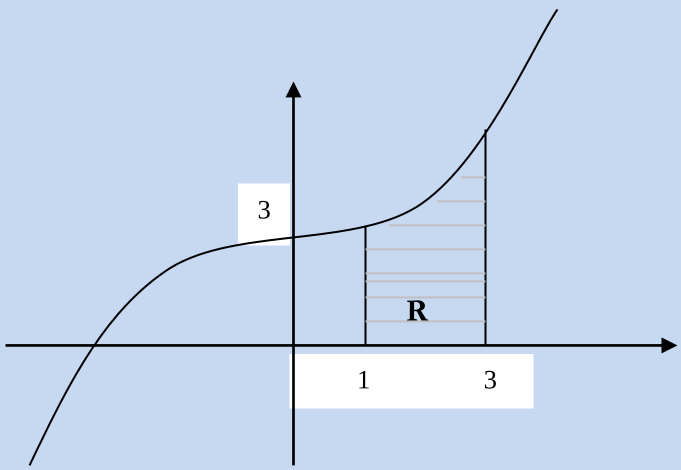
The essential relationship is:

$$\textbf{Area} = \int_a^b y\,dx$$

Example

Find the area of the region bounded by the graph $y = x^3 + 3$, the x- axis and the lines $x = 1$ and $x = 3$

Solution



$A_R = \int_1^3 y\,dx$

$= \int_1^3 (x^3+3)\,dx$

$= \frac{1}{4}x^4 + 3x \Big|_1^3$

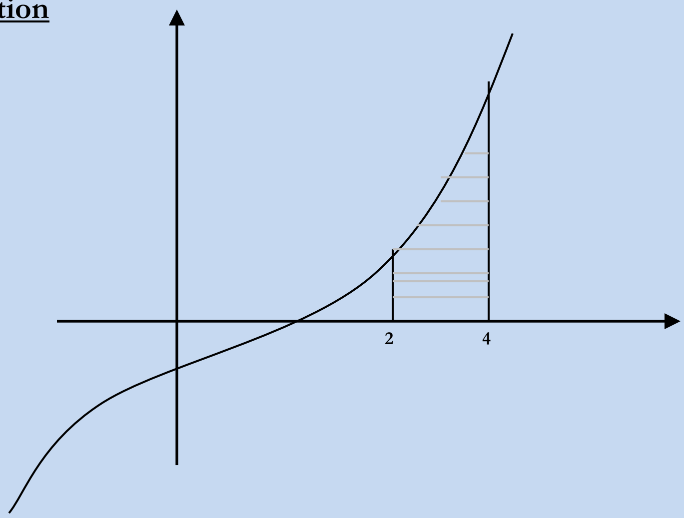
$= \frac{1}{4}(81) + 9 - (\frac{1}{4} + 3)$

= 26 units squared

Example

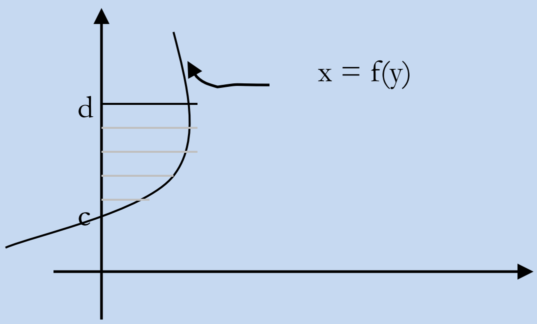
Sketch the graph of the region bounded by the curve $y = x^3 - 5$, the lines $x = 2$ and $x = 4$ and the x – axis . Find the area enclosed.

<u>Solution</u>



$A_R = \int_2^4 (x^3 - 5)\,dx = \frac{1}{4}x^4 - x \Big|_2^4 = \frac{1}{4}4^4 - 4 - (\frac{1}{4}(2) - 2)$ = 61.5 units squared

Area of a region bounded by the curve x = f(y), they-axis, the lines y =c and y =d

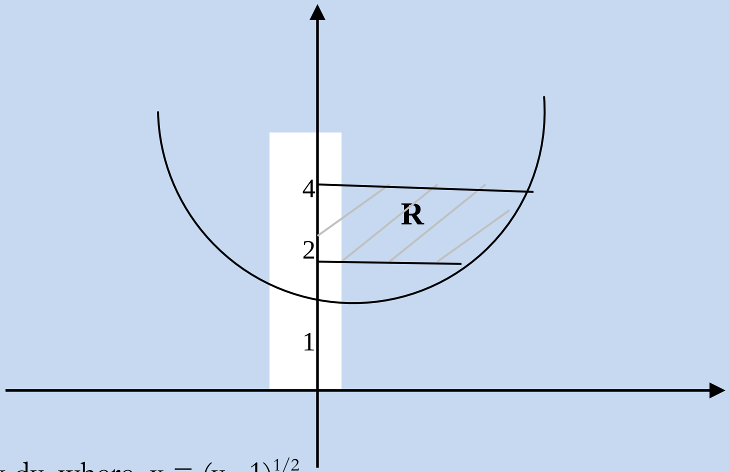


Hence, the area is given by the following formula **:**

$$\mathbf{A_R = \int^{d} x\,dy}$$

Example

Find the area between $y = x^2 + 1$, the y – axis and the lines $y = 2$ and $y = 4$



$A_R \quad = \int_a^b x\, dy$, where $x = (y-1)^{1/2}$

$$= \int_2^4 \sqrt{(y-1)}\, dy$$

$$= \int_2^4 (y-1)^{1/2}\, dy$$

$$= \frac{2}{3}(y-1)^{3/2}\Big|_2^4$$

$$= \frac{2}{3}\{(4-1)^{3/2} - (2-1)^{3/2}\}$$

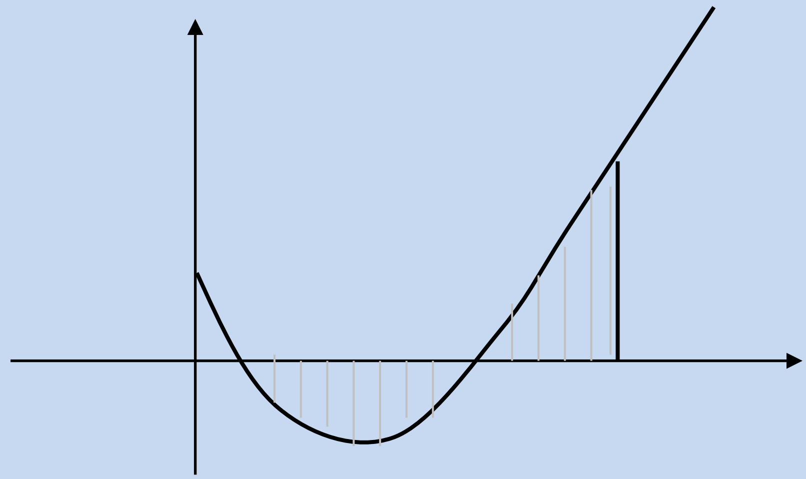
$$= \frac{2}{3}(\sqrt{27} - 1)\ u^2$$

1.1.8 Regions below the x- axis

Remember: area is always positive

If the region is under the x-axis, then AR = $-\int_a^b f(x)\,dx = \left|\int_a^b f(x)\,dx\right|$

Example Find the area under the curve $y = (x-1)(x-3)$ from $x = 1$ to $x = 5$



A_2

1

3 5

A_1

$A_R = A_1 + A_2$

$A_1 = -\int_1^3 (x^2 - 4x + 3)\,dx$

$= -\left(1/3\,x^3 - 2x^2 + 3x\right)\Big|_1^3$

$= -\frac{3^3}{3} + 2(3^2) - 3(3) + \frac{1^3}{3} + 2(1^2) - 3(1)$

$= 4/3\,u^2$

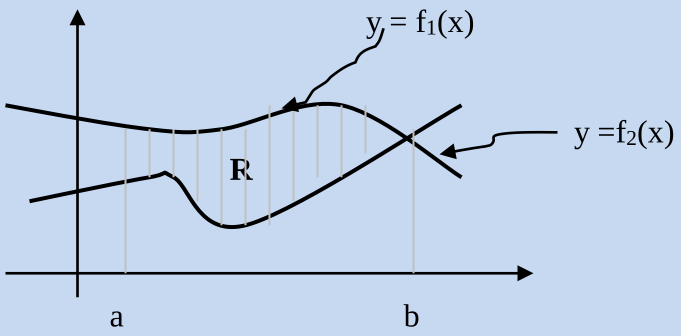
$A_2 = \int_3^5 (x^2 - 4x + 3)\,dx$

$= \frac{5^3}{3} - 2(5^2) + 5(3) - \frac{3^3}{3} + 2(3^2) - 3\,(3)$

$= 20/3$

The required is $A_R = A_1 + A_2 = 4/3 + 20/3 = 8u^2$

1.1.9 Area of the region under two curves

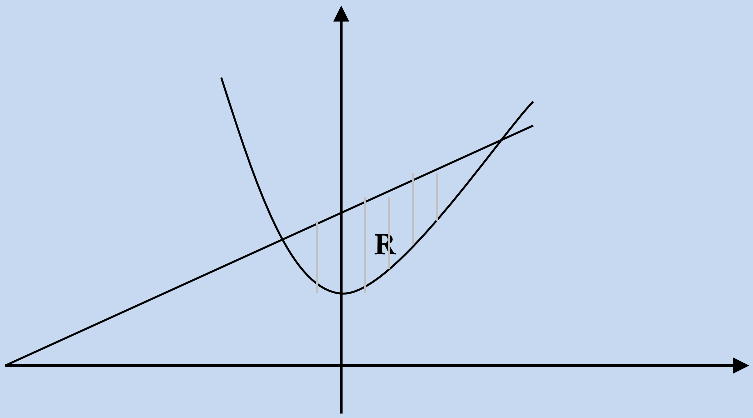


$$A_R = \int_a^b \{f_1(x) - f_2(x)\}dx$$

Example

Find the area bounded by the graphs $y = x^2 + 1$, $y = 2x + 4$, the y- axis and the x-axis

<u>Solution</u>



We first find the points of intersection of the two graphs.

$x^2 + 1 = 2x + 4$

$x^2 - 2x - 3 = 0$

$(x + 1)(x - 3) = 0$

$x = -1$ or $x = 3$

Find the x – intercept of $y = 2x + 4$

When $y = 0$: $2x + 4 = 0$

$x = -2$

$f_1(x) = 2x + 4$ and $f_2(x) = x^2 + 1$, hence

$$A_R = \int^{3} \{2x + 4 - x^2 - 1\}dx$$

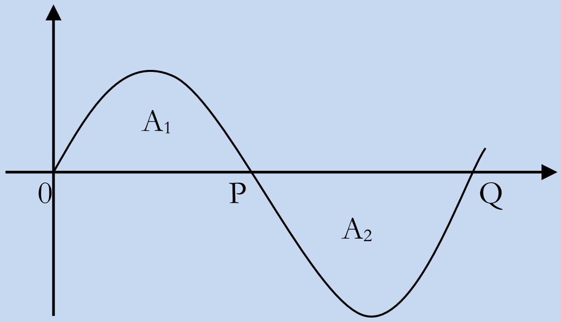
-1

$$= x^2 + 4x - \frac{x^3}{3} - x \Big|_{-1}^{3}$$

$$= \frac{16}{3}u^2$$

Practice Questions

1.

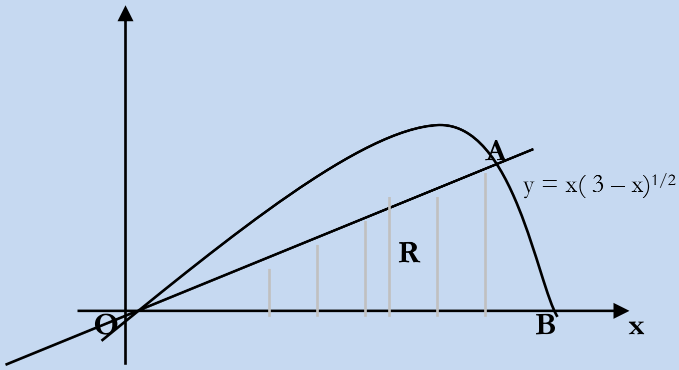


The diagram shows the curve with equation $y = x\sin x$ for values of x between 0 and 2π. The curve cuts the x- axis at O, P and Q.

a) Find the coordinates of P and Q

b) Calculate the area bounded by the curve $y = x\sin x$ and the x -axis

2.



The diagram shows part of the curve with eq**uation** $y = x(3 - x)^{1/2}$, together with the line segment, OA.. The curve has a maximum at A, and crosses the x – axis at B.

a) Find the coordinates of the points A and B.

b) Find the area of the region R bounded by the line segment OA, the arc of the curve AB, and the x –axis.

Volumes of Revolution

The objective is to find the volume of the solid generated when the region R is rotated through one revolution, i.e. through 360^0 , about the x-axis or the y-axis.

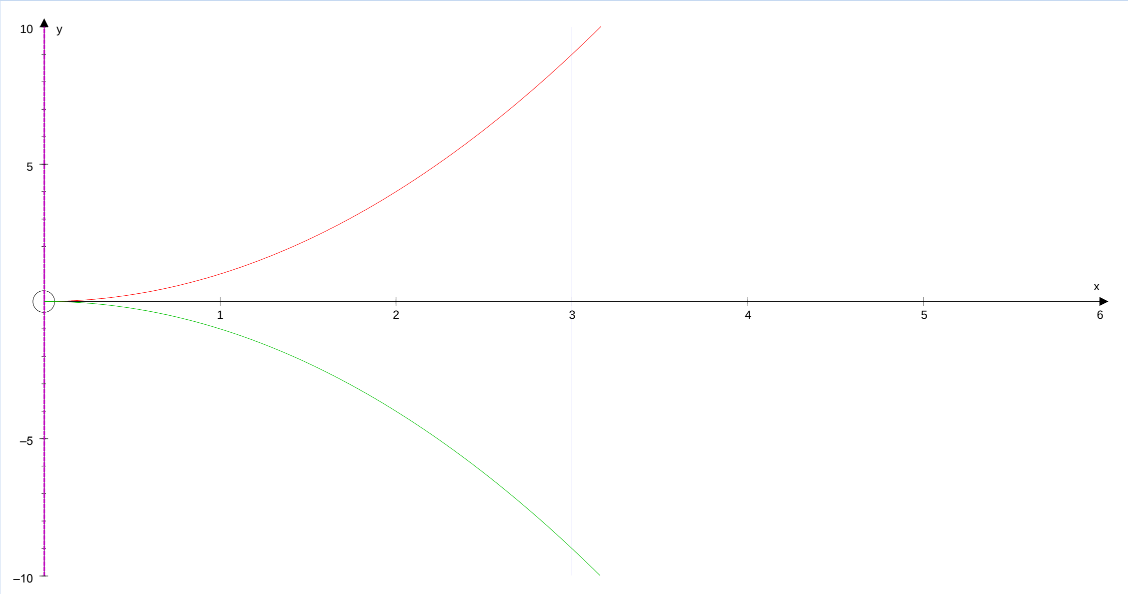
$$V_x = \pi \int_a^b y^2\, dx$$

$$V_y = \pi \int_a^b x^2\, dy$$

Example

Find the volume of the solid generated when the region under the graph $y = x^2$ from $x = 0$ to $x = 3$ is rotated about the x – axis through 360^0

Solution



$$V_x = \pi \int_0^3 y^2 dx$$

$$= \pi \int_0^3 x^4\, dx$$

$$= \frac{\pi}{5} \left. x^5 \right|_0^3$$

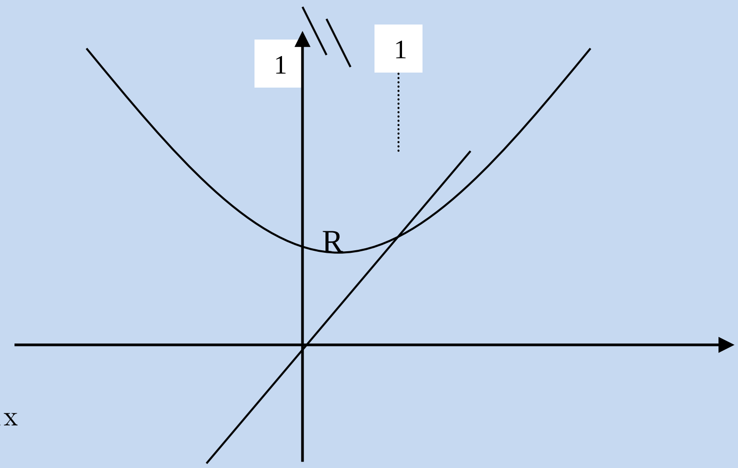
$$= \frac{\pi}{5} (3^5 - 0^5)$$

$$= \frac{243}{5} \pi u^3$$

Example

Let R be the region bounded by the curve $y = x^2 + 1$, $y = 2x$ and the y - axis. Find the volume of the solid generated when R is rotated through four right angles about the x-axis.

Finding the point of intersection



$x^2 + 1 = 2x$
$x^2 - 2x + 1 = 0$
$(x - 1)(x - 1) = 0$
$x = 1$

$$V_x = \pi \int_0^1 \{y_1^2 - y_2^2\}\,dx$$

$$= \pi \int_0^1 \{(x^2 + 1)^2 - (2x)^2\}\,dx$$

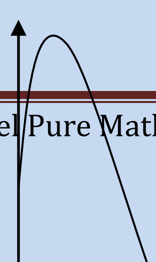
$$= \pi \int_0^1 \{x^4 - 2x^2 + 1\}\,dx$$

$$= \pi \left(\frac{x^5}{5} - 2x^3 + x\right)\Bigg|_0^1$$

$$= \frac{8\pi}{15}\,u^3$$

Examination Type Questions

1.



0 $\pi/2$ π

The figure shows a sketch of the curve with equation

$$y = \frac{3\cos x}{2-\sin x} \text{ for } 0 \le x \le \pi$$

a. Find the values of x, 8 in the given interval for which $dy/dx = 0$ giving tour answers in radians.

b. Determine the range of values taken by y

c. Determine the equation of the normal to the curve at the point $A(\pi/2,0)$

d. Calculate the area of the finite region bounded by the curve, the y- axis and the normal at A

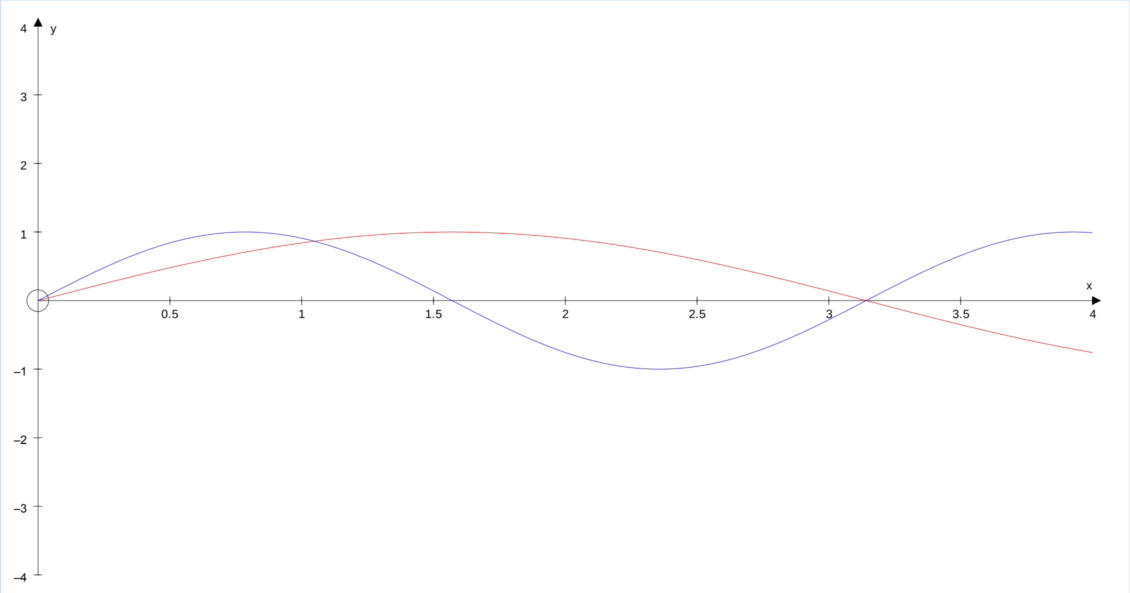
2. The region R in the first quadrant is bounded by the curve $y = e^{-x}$, the x – axis and the line x = 2.Show that the volume of the solid generated when R is completely rotated about the x- axis is:

$$\pi\int_0^2 e^{-2x}dx$$

and evaluate this volume, giving your answer in terms of e and π

3. Given that $y = \ln(x^2 - 4x + 5)$, find an expression for dy/dx

Hence find $\int_3^4 (x-2)/(x^2-4x+5)dx$



y =sin2x

The diagram shows the two curves with equations y = sinx and y = sin2x for values of x between 0 and π.

The curves meet at the origin, and at the points P and Q.

a) Find P and Q.

b) Find the area $A = A_1 + A_2$

5. Let f(cx) = $\dfrac{4x^2 3x+2}{(x-1)(x+2)}$

(i) Express f(cx) in partial fractions

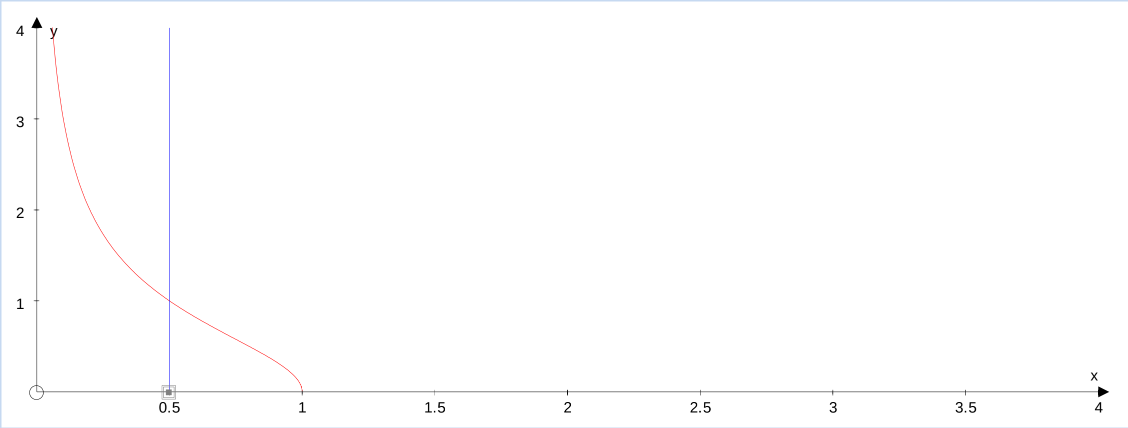
(ii) Hence show that: $\int f(cx)\,dx = 8 + 1n3 + 81n\left(\frac{2}{3}\right)$

6. (a) Find ---- $\int \text{Cos}^2 X\,dx$

(b) By use of the substitution u = Sin x, or otherwise, find ----- $\int \text{Cos}^3 X\,dx$

7. The diagram shows the region bounded by the x – axis, the line x = ½ and the curve

$$y^2 = \frac{1-x}{x}$$



(i) Find the volume of the solid formed when R is rotated through 360^0 about the x axis.

(ii) Use the substitution x = sin2 θ to show that the area of R may be expressed as:

$2\cos^2\theta$ do and hence find this area.

7 (a) Show that $\frac{d}{dx}\left(\frac{x}{\sqrt{4-x^2}}\right) = \frac{4}{(4-x^2)^{\frac{3}{2}}}$

Hence evaluate $\int_0^1 \frac{8}{(4-x^2)^{\frac{3}{2}}}dx$

(b) The region enclosed by the curve $y = \frac{x}{\sqrt{(4-x^2)}}$,the x-axis and the lines x= -1,and x=1,is rotated about the x-axis through four right angles.

Sho that the volume geratted is given by $2\pi(ln3\text{-}1)$ *zimsec 2009*

CHAPTER 17

VECTORS (II)

OBJECTIVES
By the end of the chapter the student should be able to :

- Write down the equation of a line passing through a point and given direction vector
- Find the Cartesian equation of a line
- State the relative positions between lines
- Calculate the dist from a perpendicular to a line
- Find the vector and Cartesian equation of a plane
- Calculate the angle between a plane and a vector

The Equation of a Line

The equation of a line in vector form is given by:

$$\mathbf{r} = \mathbf{a} + t\mathbf{b}$$

Where, r is a general point i.e. $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$

a is a point through which the line passes
t is a scalar or constant
b is a vector in the same direction as the line called the direction vector.

Example

Write down the vector equation of the line passing through the point (1,2,3) and in the same direction as the vector 4i + 5j + 6k

Solution

$$\mathbf{r} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + t\begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix}$$

Cartesian form: it can be shown that if

$$r = \begin{pmatrix} x \\ y \\ z \end{pmatrix} + t\begin{pmatrix} u \\ v \\ w \end{pmatrix}$$

then its Cartesian

Equation is: $\dfrac{x-x_1}{u} = \dfrac{y-y_1}{v} = \dfrac{z-z_1}{w}$

Example

If $\mathbf{r} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + t\begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix}$

What is the Cartesian equation of the line?

Solution

$$\frac{x-1}{4} = \frac{y-2}{5} = \frac{z-3}{6} = t$$

Example

Find (i) the vector equation and the Cartesian equation of the line passing through the points A and B having position vectors – **i** +2**j** +4**k** and 3**i** – 5**j** – **k**, respectively

Solution

AB = OB - OA = $\begin{pmatrix} 3 \\ -5 \\ -1 \end{pmatrix} - \begin{pmatrix} -1 \\ -2 \\ 4 \end{pmatrix} = \begin{pmatrix} 4 \\ -7 \\ -5 \end{pmatrix}$

(i) r = $\begin{pmatrix} -1 \\ 2 \\ 4 \end{pmatrix} + \mathrm{t}\begin{pmatrix} 4 \\ -7 \\ -5 \end{pmatrix}$ using point A or

$\mathbf{r} = \begin{pmatrix} 3 \\ -5 \\ -1 \end{pmatrix} + t\begin{pmatrix} 4 \\ -7 \\ -5 \end{pmatrix}$

The Cartesian equation is:

$$\frac{x+1}{4} = \frac{y-2}{-7} = \frac{z-4}{-5} = t \text{ or}$$

$\frac{x-3}{4}=\frac{y+5}{-7}=\frac{z+1}{-5}=t$ respectively.

Example

Show that the lines with direction vectors 5**i** + 2**j** – 4**k** and 2**i** + 3 **j** + 4**k** are perpendicular.

Solution

We compute the scalar product of their direction vectors i.e. :

(5**i** + 2**j** – 4**k**)**.** (2**i** + 3 **j** + 4**k**) = 5(2) + 2(3) – 4(4) = 0; hence the two lines are perpendicular

The equation of the plane π.

The equation of a plane in vector form is given by:

(r – a) . n = 0,

where **n** is a vector normal to the plane and A is apoint on the plane.

Rearranging we obtain

r .n = a .n

r . n = d, where d = **a .n**

In Cartesian coordinates ,we have ux + vy + wz + d = 0,where **n =** u**i +** v**j +** w**k**

The parametric equation form of a plane is given by:

r =**a** + r**b** + s**c,**

where r and s are parameters and **b** and **c** are vectors on the plane

Example

Find the vector equation for the plane π passing through the point with position vector 5i + 3 + k and perpendicular to the vector 2i + 9i +6k.

Solution

$$\mathbf{r}\,.\begin{pmatrix}2\\9\\6\end{pmatrix}=\begin{pmatrix}5\\3\\1\end{pmatrix}.\begin{pmatrix}2\\9\\6\end{pmatrix}$$

$$=\mathbf{r}.\begin{pmatrix}2\\9\\6\end{pmatrix}=43$$

Example
What is the Cartesian form of the equation of the plane?

$$\mathbf{r}.\begin{pmatrix}1\\2\\3\end{pmatrix} = 4$$

Solution
$x + 2y + 3z = 4$

Determining the Normal Vector to a Plane

The cross product
Let $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ and $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$, the cross product of vectors **a** and **b** is given by

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\end{vmatrix} = \mathbf{i}\begin{vmatrix}a_2 & a_3\\ b_2 & b_3\end{vmatrix} - \mathbf{j}\begin{vmatrix}a_1 & a_3\\ b_1 & b_3\end{vmatrix} + \mathbf{k}\begin{vmatrix}a_1 & a_2\\ b_1 & b_2\end{vmatrix}$$

The vector $\mathbf{v} = \mathbf{a} \times \mathbf{b}$ is perpendicular to both vectors **a** and **b**

Example

Given vectors $3\mathbf{i} + 2\mathbf{j} + \mathbf{k}$ and $5\mathbf{i} + 6\mathbf{j} + 7\mathbf{k}$ find their cross product.

Solution
The cross product of two vectors is a normal vector to the plane containing the two vectors.

$$\mathbf{n} = \begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 3 & 2 & 1\\ 5 & 6 & 7\end{vmatrix} = \mathbf{i}\begin{vmatrix}2 & 1\\ 6 & 7\end{vmatrix} - \mathbf{j}\begin{vmatrix}3 & 1\\ 5 & 7\end{vmatrix} + \mathbf{k}\begin{vmatrix}3 & 2\\ 5 & 6\end{vmatrix}$$

$$= 8\mathbf{i} - 16\mathbf{j} + 8\mathbf{k}$$

Example
Find the vector and Cartesian equation of the plane containing the points P, Q and R with position vectors $4\mathbf{i} + 2\mathbf{j} + 6\mathbf{k}$, $5\mathbf{i} + \mathbf{j} + 3\mathbf{k}$ and $2\mathbf{i} + 3\mathbf{j} + 4\mathbf{k}$ respectively

Solution

$$\begin{pmatrix}5\\1\\3\end{pmatrix}-\begin{pmatrix}4\\2\\6\end{pmatrix}=\begin{pmatrix}1\\-1\\-3\end{pmatrix}$$

$$\mathbf{PR} = \begin{pmatrix}2\\3\\4\end{pmatrix}-\begin{pmatrix}4\\2\\6\end{pmatrix}=\begin{pmatrix}-2\\1\\-2\end{pmatrix}$$

$$\mathbf{n} = \begin{vmatrix} i & j & k \\ -1 & 1 & 3 \\ -2 & 1 & -2 \end{vmatrix} = i\begin{vmatrix} 1 & 3 \\ 1 & -2 \end{vmatrix} - j\begin{vmatrix} -1 & 3 \\ -2 & -2 \end{vmatrix} + k\begin{vmatrix} -1 & 1 \\ -2 & 1 \end{vmatrix}$$

$$= -5i - 6j + k$$

Using point P

$$\mathbf{r.} = \begin{pmatrix}-5\\-6\\1\end{pmatrix}=\begin{pmatrix}-5\\-6\\1\end{pmatrix}\begin{pmatrix}4\\2\\6\end{pmatrix}$$

$$\mathbf{r.}\begin{pmatrix}-5\\-6\\1\end{pmatrix} = -26$$

And the Cartesian equation is $-5x - 6y + z = -26$

Alternatively:

$\mathbf{r} = \mathbf{a} + r\mathbf{b} + s\mathbf{c}$, let $\mathbf{b} = \mathbf{PQ} = \mathbf{i} - \mathbf{j} + 3\mathbf{k}$, $\mathbf{a} = \mathbf{OP} = 4\mathbf{i} + 2\mathbf{j} + 6\mathbf{k}$ and $\mathbf{c} = \mathbf{PR} = -2\mathbf{i} + \mathbf{j} - 2\mathbf{k}$, **hence**

$\mathbf{r} = 4\mathbf{i} + 2\mathbf{j} + 6\mathbf{k} + r(\mathbf{i} - \mathbf{j} + 3\mathbf{k}) + s(-2\mathbf{i} + \mathbf{j} - 2\mathbf{k})$

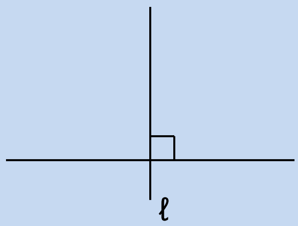
The Perpendicular Distance from a Point P to a Line ℓ

Example

Find the perpendicular shortest distance from the point P (3, -1, 5) to the line ℓ given by

P

$$\mathbf{r} \;.\; \begin{pmatrix} 8 \\ 0 \\ -1 \end{pmatrix} + t\begin{pmatrix} -6 \\ 1 \\ 4 \end{pmatrix}$$



Rearranging

$$\mathbf{r} = \begin{pmatrix} 8-6t \\ 0+t \\ -1+4t \end{pmatrix}$$

$$\mathbf{QP = OP - OQ =} \begin{pmatrix} 8-6t \\ 0+t \\ -1+4t \end{pmatrix} - \begin{pmatrix} 3 \\ -1 \\ 5 \end{pmatrix}$$

$$= \begin{pmatrix} 5-6t \\ 1+t \\ -6+4t \end{pmatrix}$$

since QP is perpendicular to C, then

$$\begin{pmatrix} 5-6t \\ 1+t \\ -6+4t \end{pmatrix}\begin{pmatrix} -6 \\ 1 \\ 4 \end{pmatrix} = 0$$

$$-30 + 36t + 1 + t - 24 + 16t = 0$$
$$t = 1$$

$$\begin{pmatrix} 8 - 6\,(1) \\ 0 \;+\; 1 \\ -1 \;+4\,(1) \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}$$

and QP = $\sqrt{2-3^2} + 1-1^2 + 3-5^2$
$= 3$

The Perpendicular Distance from a Point P(x_1,y_1,z_1) to a Plane π

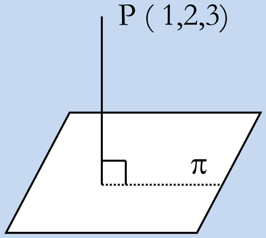
The distance d from the point P(x_1,y_1,z_1) to the plane ux + vy + wz + d = 0 is given by:

$$\mathbf{d} = \frac{ux_{2+}vy_1 + wz_1 + d}{\sqrt{u^2 + v^2 + w^2}}$$

Example

Find the shortest distance from the point P (1,2, 3) to the plane with the equation **r.** (5i + 3j + k) = 2

<u>Solution</u>



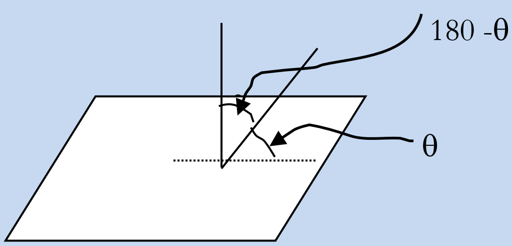
The formula for the distance d is: $d = \left|\frac{ux_1 + vy_1 + wz_1 + d}{\sqrt{u^2 + v^2 + w^2}}\right|$

Where $\mathbf{n} = \begin{pmatrix} u \\ v \\ w \end{pmatrix}$ is the normal vector to the plane

$$d = \frac{5(1) + 3(2) + 1(3) - 2}{\sqrt{5^2 + 3^2 + 1^2}}$$

$$= \frac{12}{\sqrt{35}}$$

The Angle Between the Line ℓ and a Plane π



We want to calculate the angle θ

We know that cos (180 - θ) = sin θ = $\dfrac{n.a}{|n||a|}$

Example
Find the angle between the between the line ℓ having the equation

$$\mathbf{r} = \begin{pmatrix}1\\2\\3\end{pmatrix} + t\begin{pmatrix}5\\-4\\3\end{pmatrix}$$ and the equation of the plane $\mathbf{r}\,.\begin{pmatrix}3\\1\\1\end{pmatrix} = 5$

<u>**Solution**</u>

$$\text{Sin }\theta = \frac{(3i+j+k)(5i-4j+3k)}{\sqrt{11}\sqrt{50}}$$

$$= \frac{14}{\sqrt{11}\sqrt{50}}$$

$$\therefore \sin^{-1}\frac{14}{\sqrt{11}\sqrt{50}} = \theta$$

$$\therefore \theta = 40.7^0$$

<u>The Angle Between Two Planes</u>

Here we find the angle between the normals to the planes.
Example

Find the angle between the planes.

$$\mathbf{r.}\begin{pmatrix}1\\2\\1\end{pmatrix} = 4 \quad \text{and} \quad \mathbf{r.}\begin{pmatrix}4\\3\\3\end{pmatrix} = 9$$

<u>**Solution**</u>

$$\text{Cos }\theta = \frac{n_1.n_2}{|n_1|.|n_2|}$$

$$= \frac{(i+2j+k).(4i+3j+3k)}{\sqrt{6}\sqrt{29}}$$

$$= \frac{12}{\sqrt{174}}$$

$$\cos^{-1} \frac{12}{\sqrt{174}} = \theta$$

$\theta = 24.5^\circ$

Parallel lines and planes

Two lines are parallel if they have the same direction vectors or if the direction vectors are multiples of each other.

Example

Show that the lines **r** = $\begin{pmatrix}1\\2\\3\end{pmatrix} + t\begin{pmatrix}4\\5\\6\end{pmatrix}$ and $\mathbf{r} = \begin{pmatrix}3\\1\\4\end{pmatrix} + m\begin{pmatrix}20\\25\\30\end{pmatrix}$ are parallel

Solution

Since $\begin{pmatrix}20\\25\\30\end{pmatrix} = 5\begin{pmatrix}4\\5\\6\end{pmatrix}$

hence the two lines are parallel

Example

Show that the planes

$\mathbf{r}\,.\begin{pmatrix}1\\2\\3\end{pmatrix} = 5$ **and** $\mathbf{r}.\begin{pmatrix}3\\6\\9\end{pmatrix} = 1$ are parallel

Solution

Since $\begin{pmatrix}3\\6\\9\end{pmatrix}=3\begin{pmatrix}1\\2\\3\end{pmatrix}$

Hence the two lines are parallel

Example

Show that the line

$$\mathbf{r}=\begin{pmatrix}1\\2\\3\end{pmatrix}+t\begin{pmatrix}2\\1\\2\end{pmatrix} \quad \text{is parallel to the plane } \mathbf{r}.\begin{pmatrix}-1\\0\\-1\end{pmatrix}=-3.$$

<u>Solution</u>

If a line and a plane are parallel then their direction vectors must be perpendicular. i.e. **a . n** $=0$ hence

$$\begin{pmatrix}0\\1\\1\end{pmatrix}.\begin{pmatrix}-1\\0\\1\end{pmatrix}=0 \text{ there fore the lines are parallel}$$

<u>Intersecting Lines</u>

Example

Show that the lines $\mathbf{r}_1$=2i+ 2 +λ (- i + 2)

$\mathbf{r}_2$ = i – 2 + s(i + 3)

Intersect and find the co-ordinates of their point of intersection.

<u>Solution</u>

At the point of intersection

$$\begin{pmatrix}2-\lambda\\1+\lambda\\0+0(\lambda)\end{pmatrix}=\begin{pmatrix}-1+3\\-2-3s\\0+0\end{pmatrix}$$

= 2 -λ = 1+s

1 + λ = -2 – 3s

5 + λ = 1

$3s + \lambda = -3$

$-\quad 2s = 4$

$s = -2$

$\underline{\lambda = 4}$

Therefore we have a solution to the simultaneous equations the two lines intersect.

(ii) The point of intersection is found by substituting for either S or λ.

For S = -2

$$\mathbf{r}_1 = \begin{pmatrix} 2-3 \\ 1+4 \\ 0+0 \end{pmatrix} = \begin{pmatrix} -1 \\ 4 \\ 0 \end{pmatrix}$$

for $\lambda = 4$

$$\mathbf{r}_2 = \begin{pmatrix} 1-2 \\ -2+6 \\ 0+0 \end{pmatrix} = \begin{pmatrix} -1 \\ 4 \\ 0 \end{pmatrix}$$

Example

Determine whether the two lines

$$\mathbf{r}_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \textbf{ and } \mathbf{r}_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

<u>Solution</u>

rearranging

$$r_1 \begin{pmatrix} 1+\lambda \\ 0+\lambda \\ 1+0\lambda \end{pmatrix} \mathbf{r}_2 \begin{pmatrix} 1+ot \\ 1+t \\ 0+t \end{pmatrix}$$

$$\mathbf{r}_1 = \mathbf{r}_2 = \begin{pmatrix} 1+\lambda \\ 0+\lambda \\ 1+0\lambda \end{pmatrix} = \begin{pmatrix} 1+0t \\ 1+t \\ 0+t \end{pmatrix}$$

$= 1 + \lambda = 0$

therefore: $\lambda = 0$

$\lambda = 1 + t$

$t = -1$

but 1 = t (from the third row) which is inconsistent therefore the two lines do not intersect. They are said to be skew

Examination Type Questions

1. **i, j** and **k** are unit vectors parallel to x, y and z axes of the Cartesian plane Oxyz, with origin O. A line ℓ_1, passes through the point (3, 6, 1) and is parallel to the vector

2i + 3j – k. A line ℓ_2 passes through the point (3, - 1, 4) and is parallel to the vector 4i – 2j +k

(a) Using the form r = a + bt, write down the vector equations of the lines ℓ_1 and ℓ_2. Show that the lines intersect and find the coordinates of the point of intersection.

(b) What is the acute angle between the lines?

(c) The point A(5, a, 0) lies on ℓ_1, whilst the point B(5, a, b) lies on ℓ_2. Find a and b and hence find the point = C which lies on the line AB such that AC = 1:2.

What is the unit vector parallel to $\overrightarrow{OC}$?

2. The points A and B have coordinates (2, 3, - 1) and (5, -2, 2) respectively. Calculate the acute angle between AB and the line with the equation.

$$\overrightarrow{V} = \begin{Bmatrix} 2 \\ 3 \\ -1 \end{Bmatrix} + t \begin{Bmatrix} 1 \\ -2 \\ -2 \end{Bmatrix}$$ giving your answer to the nearest degree.

3. Show that the two lines given by the equations
x = 3 + t, y = 4 + 2t, z = - 2t
x =8 – 2s, y = 5 – s, z = - 4 + 2s
intersect. Find the coordinates of their common point. Find also the acute angle between these lines

4. The equation vector of the plane $\prod_1$ is x + z = 0 and the equation of the line l is:

$$\mathbf{r} = \begin{pmatrix} 7 \\ 3 \\ 2 \end{pmatrix} +t \begin{pmatrix} -2 \\ 1 \\ 3 \end{pmatrix}$$ where t is a parameter

Find

(i) The position vector of the point of intersection of $\prod_1$ and l.

(ii) The length of the perpendicular from the origin to l giving your answer to 3 significant figures.

(iii) Given that place $\prod_2$ has the equation $6x - 5y - 2z = 0$, find the acute angle between $\prod_1$ and $\prod_2$, giving your answer correct to the nearest 0.1.

CHAPTER 18

FIRST ORDER DIFFERENTIAL EQUATIONS

OBJECTIVES
By the end of the chapter the student should be able to :

- Formulate and solve differential equations
- Represent solutions graphical

Formulation of Odes

An ODE is an equation containing the derivative function dy / dx

The student should recall that the differential operator is also called the gradient function and that the gradient is the rate of change of a given variable with respect to another – usually time (t). and also that an increasing function has a positive gradient while a decreasing one has a negative gradient.

Example
Formulate the following differential equations.

(i) The rate of increase of a population is constant.
(ii) The rate of a chemical reaction increases with time
(iii) The rate at which water flows out of a tank decreases with the water level.

Solution

(i) $-\dfrac{dP}{dt} = k$

(ii) $\dfrac{dM}{dt} \infty .t$

$$\frac{dM}{dt} = kt$$

(iii) $\dfrac{-dV}{dt} \infty h$

$$\frac{-dV}{dt} = kh$$

Solving Odes

An ODE is an indirect way of expressing the relationship between given variables. In solving ODES, we will be aiming at expressing this relationship directly. This is essentially an integration problem.

Odes with Separable Variables

This is when similar terms can be grouped on one side of the equation.

Example Solve the following ode

$$\frac{dH}{dt} = kH(1+t)$$

Separating the variables and integrating:

$$\int \frac{dH}{H} = \int k\,(1 + t)\,dt$$

$$\text{LnH} = k\,(t + t^2/2) + c \qquad (1)$$

This is the general solution.

If we are further told that H = 1, when t = 1 and H = e^2 when t = 4 , we can then find the values of k and c leading to the particular solution.

Substituting in (1) above

$$\ln 1 = k\,(1 + ½) + c$$

$$-c = 1.5\,k \qquad (2)$$

$$\ln e^2 = k\,(4 + 8) + c$$

$$2 = 12k + c \qquad (3)$$

$$2 = 12k - 1.5k$$

$$k = 4/21$$

$$c = \frac{-3}{2} \quad \frac{4}{21}$$

$$= \frac{-2}{7}$$

The particular solution is $\ln H = \frac{21}{2}\left(t+\frac{t^2}{2}\right)-\frac{2}{7}$

Graphical Representation of Solutions

The general solution can be represented on a family 0f curves and a particular solution by one number of this family.

Example Sketch the two solution curves passing through the given co-ordinates.

$\frac{dy}{dx}$ = x : (0, 1) , (1, 2)

Solution

$$\frac{dy}{y} = xdx$$

$$\frac{x^2}{2}+c$$

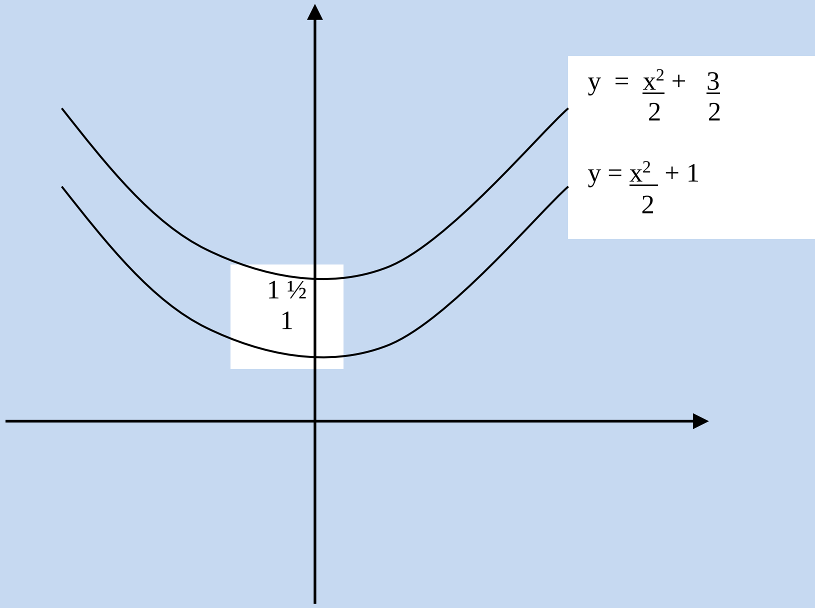
when x = 0 and y = 1 c = 1

$$y = \frac{x^2}{2}+1$$

when x = 1 and y = 2 $2 = \frac{1}{2} + c$

$c = \frac{3}{2}$

$$y = \frac{x^2}{2}+\frac{3}{2}$$



Example

The growth rate of a colony of mice increases at a rate proportional to its size. In 30 days, the size of the colony rises from 2000 to 4000. (i) Derive and solve a differential equation for the population size at time t days after the size was 2000 (ii) What size is it when t = 90 days? (iii) How long did it take for the population to increase from 3000 to 4000.

<u>Solution</u>

(i) $\frac{dp}{dt} \propto P$

$\frac{dp}{dt} = kP$

$\frac{dp}{P} = kdt$

Ln p $= kt + c$

When $t = 0$, $p = 2000$

$\ln 2000 = k(0) + c$

$c = \ln 2000$

when $t = 30$, $P = 4000$

$\ln 4000 = k(30) + c$

$= 30k + \ln 2000$

k $= \frac{In2}{30}$

ln P $= \frac{tIn2}{30} + \ln 2000$

When t = 90 , $\ln P = \frac{90In2}{30} + \ln 2000$

hence P = 2000 x 2^3 = 16 000

ln P $= \frac{tIn2}{30} + \ln 2000$

when P = 3000 , $\ln 3000 = \frac{tIn2}{30} + \ln 2000$

t $= \frac{30In1.5}{In2}$

= 17.549

The time to increase from 3000 to 4000

= 30 – 17.549 days

= 12.45 days

Examination Type Questions

1. A bottle is shaped so that when the depth of water is xcm, the volume of water in the bottle is $(x^2 + 4x)$ cm^3, $x \geq 0$. Water is poured into the bottle so that at time t_s after pouring commences, the depth of water is xcm and the rate of increase of the volume of the water is $(x^2 + 25)$ cm^3/s.

(a) Show that $\frac{dx}{dt} = \frac{x^2+25}{2x+4}$ - Given that the bottle was empty at t = 0

(b) Solve this differential equation to obtain t in terms of x.

(c) Use integration by parts to find $\int x \sec^2 x \, dx$. By separating the variables, solve the differential equation $\cos^2 x \frac{dy}{dx} = xy^2$ $(0 < x < \frac{1}{2})$ given that y = 1 when x= 0

2. In an industrial process, a control mechanism supplies heating or cooling in such a way that the temperature H and the time t (each measured in suitable units are related by the differential equation:

$$\frac{dH}{dt} = (100 - H^3)$$

   (i) Solve this differential equation, giving t in terms of H and an arbitrary constant.

   (ii) Given that H = 95 when $t = 0$, find the value of t when H = 99

   (iii) Given instead that H = 101 when $t = 0$, express H in terms of t.

3. A rain water Butt has a height of 100cm and a uniform cross-sectional area of 2000cm^2. At a time when the butt is full of water it begins to leak from a small hole in the base. The depth of the water which remains t minutes after the leak begins is $x$cm. Given that the water leaks out at the rate of $100\sqrt{x}\ \text{cm}^3$ per minute and that no water enters the butt, show that:

$$\frac{dx}{dt} = \frac{-1\sqrt{x}}{20}$$

   When the leak is first noticed, the butt is found to be only half full. Find, to the nearest minute, the time for which the butt has been leaking.

CHAPTER 19

COMPLEX NUMBERS (II)

OBJECTIVES
By the end of the chapter the student should be able to :

- Prove trig, identities using Moirve formulae
- Solve polynomial equations
- Calculate units
- Roots of complex numbers
- Describe and sketch loci of complex numbers

Further Properties of the Conjugate

The following are some properties of the conjugate of a complex number.

(a) Z = **Z** if and only if z is real

(b) $\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$

(c) $\overline{z_1 - z_2} = \overline{z_1} - \overline{z_2}$

(d) $\overline{z_1 . z_2} = \overline{z_1} . \overline{z_2}$

(e) $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\overline{z_1}}{\overline{z_2}}$

Example
Given that $z = x + iy$, express $y + ix$ and $6ix - 2y$ in terms of Z and or ⇓

Solution
y + ix = i (x + 1 / i y)
= i(x + i/ i2 y)
= i(x + i/ -1 y)
= i(x – iy)
= i

6ix – 2y = 6i (Z +) – 2 (Z –) = 3i (Z +) + i(Z –)
= 4iZ + 2 I = 2 i(2Z +)

Practice Questions

1. Given that z = x + iy, express each of the following in terms of z and or z

(a) – x + iy
(b) – y + ix
(c) – x – iy
(d) 2x – iy
(e) 2ix –y
(f) 5x – 2iy

Example

Find Re $\frac{1}{z}$

Solution

$$\frac{1}{z^2} = \frac{1}{(x+iy)^2}$$

$$\frac{x^2+y^2-2iy}{(x^2+y^2+2iy)(x^2+y^2-2iy)}$$

$$= \frac{(x^2-y^2)-2ixy}{(x^2-y^2)+4x^2y^2} = \frac{(x^2-y^2)-2ixy}{x^4-2x^2y^2+y^4+4x^2y^2}$$

$$= \frac{(x^2-y^2)+2ixy}{x^4-2x^2y^2+y^4} = \frac{(x^2-y^2)-2ixy}{(x^2+y^2)^2}$$

$$= \frac{x^2-y^2-2ixy}{(x^2+y^2)^2}$$

Therefore Re $(\frac{1}{2}z) = \frac{x^2-y^2}{(x^2+y^2)^2}$

Practice Questions

1. Find (i) lm $\frac{1}{1+i}$ (ii) Re $\frac{(1-i)^2}{4+2i}$

2. Show that $|\cos\theta + i\text{Sin }\theta| = 1$

3. Prove result (2) by using $|Z|^2 = Z^2$.

De Moivre's Formulae

Recall that $r_1 (\cos\theta_1 + i \sin\theta_1) r_2 (\cos\theta_2 + i\sin\theta_2) =$
$= r_1 \; r_2 (\cos(\theta_1 + \theta_2) + i \sin(\theta_1 + \theta_2)$

We extend this result to the case where we have n complex numbers. We can prove by induction that:

$r_1 (\cos\theta_1 + i\sin\theta_2) \ldots. r_n (\cos\theta_n + i\sin\theta)$
$= r_1 . r_2 \ldots\ldots r_n \{\cos (\theta_1 + \theta_2 + \ldots +\theta n) + i \sin(\theta_1 + \theta_2 + \ldots + \theta_n)\}$

Now consider the case where $r_1 = r_2 = \ldots r_n = r$ and $\theta_1 = \theta_2 = \ldots = \theta_n = \theta$

Hence $Z^n = r^n (\cos\theta + i\sin\theta)^n = = r^n (\cos n\theta + i\sin n\theta)$

For r =1

$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$

This is De Moivre's first formula. In fact, this result is true for any real number .

De Moivre's formula can be used to find the expansion of multiple angles.

Example
By De Moivre's formula $\cos 5\theta + i\sin 5\theta = (\cos\theta + i\sin\theta)^5$

$(\cos\theta + i\sin\theta)^5 =$
$= \cos^5\theta + 5 \text{Cos}^4\theta (i\text{Sin }\theta) + 10 \cos^3\theta (i\text{Sin}\theta)^2 + 10 \cos^2\theta (i\sin\theta)^3 + 5 \cos\theta (i\text{Sin }\theta)^4 + (i\text{Sin }\theta)^5$
$= \cos^5\theta + 5 i\sin\theta \cos^4\theta + 10 i^2 \cos^3\theta \sin^2\theta + 10 i^5 \cos^2\theta \sin^5\theta + 5 i^4 \cos^2\theta\sin^4\theta + i^5 \sin^5\theta$
$= \cos^5\theta + 5 i\text{Sin }\theta \cos^4\theta - 10 \cos^3\theta \sin^2\theta - 10 i\cos^2\theta \sin^3\theta + 5 \cos^2\theta\sin^4\theta + i\text{Sin}^5\theta$

$= \cos^5\theta - 10 \cos^3\theta \sin^2\theta + 5 \cos^2 \sin^4 2\theta + i(\cos^4\theta \sin 2 - 10 \cos^2\theta \sin^3\theta + \sin 520$

Equating the real parts, we have

$\cos 5\theta$ = $\cos^5\theta - 10 \text{ Cos }^3\theta \sin^2\theta + 5 \cos^2 \sin^4\theta$
= $\cos^5\theta - 10 \cos^3\theta \ (1 - \cos^2\theta) + 5 \cos^2 (1 - \cos^2\theta)^2$
= $\cos^5\theta - 1 \cos^3\theta \ (1 - \cos^3\theta\ ) + 5 \cos^2 (1 - 2 \cos^2\theta \ \_ \cos^4\theta\ )$
= $\cos^5\theta - 10 \cos^3\theta + 10 \cos^5\theta + 5 \cos\theta - 10 \cos^3\theta + 5 \cos^5\theta$
= $16 \cos^5\theta - 20 \cos^3\theta + 5 \cos \theta$

If we had equated the imaginary parts , we would obtain the expansion of sin 5θ..

Practice Questions

1. Prove by induction that $r_1 (\cos \theta_1 + i\sin \theta_1) \ldots\ldots r_n (\cos \theta_n + i\sin \theta_n) =$
   $= r_1 . r_2 \ldots r_n (\cos (\theta_1 + \theta_2 + \ldots + \theta_n) + i\sin (\theta_1 + \theta_2 + \ldots. + \theta_n)\ )$

2. Prove that $\sin 3\theta = \frac{3}{4} \sin \theta - \frac{1}{4} \sin^3\theta$
3. Simplify without using the calculator

$$\frac{(\cos\frac{\pi}{7} - i\sin\frac{\pi}{7})^3}{(\cos\frac{\pi}{7} + i\sin\frac{\pi}{7})^4}$$

4. Prove using De Moivre's Theorem
   a) $\cos 4\theta = 8 \cos^4\theta - 8 \cos^2\theta + 1$
   b) $\sin 5\theta = 5\sin\theta - 20\sin^3\theta + 16\sin^5 \theta$
   c) $\tan 4\theta = \dfrac{4\tan\theta - 4\tan^3\theta}{1 - 6\tan^2\theta + \tan^4\theta}$

Important Results

If $z = \cos \theta + i\sin \theta$
$\frac{1}{z} = \cos \theta - i\sin \theta$

hence $z + \frac{1}{z} = \cos \theta + i\sin\theta + \cos \theta - i\sin \theta = 2\cos \theta$ **and**

$z - \frac{1}{z} = \cos \theta + i\sin\theta + (\cos \theta - i\sin \theta) = 2i\sin \theta$

If $Z^n = (\cos n\theta + i\sin n\theta)$ for $|z| = 1$, then $Z^{-n} = (\cos n\theta - i\sin n\theta)$

Hence

$$\mathbf{Z^n + \frac{1}{z^n} = 2\cos n\theta \text{ and } Z^n - \frac{1}{z^n} = 2i\sin n\theta.}$$ This is another version of De Moivre's theorem

Example

Prove that $\cos^5\theta = \frac{1}{16}(\cos 5\theta + 5\cos 3\theta + 10\cos\theta)$

Solution

Let $z = \cos\theta + i\sin\theta$, hence $z + \frac{1}{z} = 2\cos\theta$, then

$(z + 1/z)^5 = 32\cos^5\theta$. On expanding the left hand side we obtain

$z^5 + 1/z^5 + 5(z^3 + \frac{1}{z^3}) + 10(z + \frac{1}{z}) = 2\cos 5\theta + 10\cos 3\theta + 20\cos\theta$, hence

$32\cos^5\theta = 2\cos 5\theta + 10\cos 3\theta + 20\cos\theta$

$\therefore\ \cos^5\theta = \frac{1}{16}(\cos 5\theta + 5\cos 3\theta + 10\cos\theta)$

Practice Questions

1. Prove $\cos^4\theta + \sin^4\theta = \frac{1}{4}(\cos 4\theta + 3)$

2. Prove that $\sin^5\theta = \frac{1}{16}(\sin 5\theta - 5\sin 3\theta + 10\sin\theta)$, hence find $\int (10\sin\theta - 16\sin^5\theta)d\theta$

3. Using De Moivre's Theorem, evaluate $\int_0^{\pi/4} 8\cos^4\theta d\theta$

4. Find expressions for $\cos^3\theta$ and $\sin^3\theta$, hence evaluate $\int \cos^3\theta d\theta$ and $\int \sin^3\theta d\theta$

5. Find expressions for $\cos^{3/2}\theta$, hence evaluate $\int \cos^{\frac{3}{2}}\theta d\theta$ θ

Polynomial Equations

Let us first show that we can now always find the solution of every quadratic equation with real coefficients. Suppose we are given the quadratic equation.

$$ax^2 + bx + c = 0$$

where $a \neq 0$, b and c are real numbers, then

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

If $b^2 \geq 4ac$, the equation has two real roots

If $b^2 < 4ac$, the equation has complex roots.

For in this case we have
$$x = \frac{-b \pm \sqrt{(-1)}\sqrt{4ac - b^2}}{2a}$$

$$= \frac{-b \pm \sqrt{(-1)}\,\sqrt{(4ac - b^2)}}{2a}$$

$$= \frac{-b \pm i\sqrt{4ac - b^2}}{2a}$$

Example

Solve the equation $x^2 - 2x + 10 = 0$

Solution

$$x = \frac{2 \pm \sqrt{(4 - 4 \times 10)}}{2}$$

$$= \frac{2 \pm \sqrt{-36}}{2}$$

$$= \frac{2 \pm 6i}{2} = 2 + 3i \quad \text{or} \quad 2 - 3i$$

Note that the roots are conjugate complex numbers. If we look carefully at the general solution of quadratic equation with real coefficients, we see that if a quadratic equation has complex roots, then the roots are conjugate.

This result hold for all polynomial equations with real coefficients.

Example

Solve $x^3 - 4x^2 + 9x - 10 = 0$

Solution

Let f (x) = $x^3 - 4x^2 + 9x - 10$. Try factors of 10 i.e. +1 , +2 , +5, +10

f (2) = 8 - 16 + 18 – 10 = 0. Therefore x = 2 is a root of f (x) = 0 and so x – 2 is a factor of f (x). On dividing, we obtain the factor, $x^2 - 2x + 5$

Now $x^2 - 2x + 5 = 0$ i.e.x $= \dfrac{2 \pm \sqrt{4 - 4\times 10}}{2}$

$= \dfrac{2 \pm \sqrt{-36}}{2}$

$= \dfrac{2 \pm 6i}{2}$

$= 1 \pm 2i$

Therefore x = 1 + 2i or 1 – 2i,hence the roots are 2 , 1 + 2i or 1 – 2i

Practice Questions

Solve the following equations:

1. $5x^2 + 2x + 2 = 0$
2. $x^3 - 7x^2 - 17x - 15 = 0$
3. $x^6 + x^5 + 11x^4 + 13x^3 + 10x2 + 36x - 72 = 0$ given that 2i and 3i are roots of the equation
4. $x^4 + x^3 + 3x^2 + 7x + 20 = 0$, given that 1 – 2i is a root of the equation.
5. $x^4 - x^3 + x^2 - 1 = 0$. (Hint: Let $y = x^2$)
6. Given that one root of the equation $z^4 - 6z^3 + 23z^2 - 34z + 26 = 0$ is $1 + i$, find the other roots.

Roots of Unity and of Complex Numbers

The n^{th} roots of unity, where n is a positive integer

Consider the equation $Z^n = 1$,we find the n^{th} roots of 1. Clearly

$1 = \cos(0 + 2k\pi) + \sin(0 + 2k\pi)$, where k is an integer, then

$1^{1/n} = \cos(0 + 2k\pi) + \sin(0 + 2k\pi)^{1/n}$, using De Moivre's theorem, we have

$Z_k = \cos\left(\dfrac{0+2k\pi}{n}\right) + \sin\left(\dfrac{0+2k\pi}{n}\right)$ **,where k = 0, 1, 2,…, n - 1**

It is clear that we get distinct values of z for k = 0 , 1 , 2 …, n – 1 and starting with k = n, we start repeating again the cycle of n values up to 2n – 1 and so on. As a result, 1 has exactly n distinct nth roots.

These roots can be represented as vertices of an n-sided regular polygon inscribed in the unit circle with point z = 1 as one of its vertices.

Example

Find the sixth roots of unity

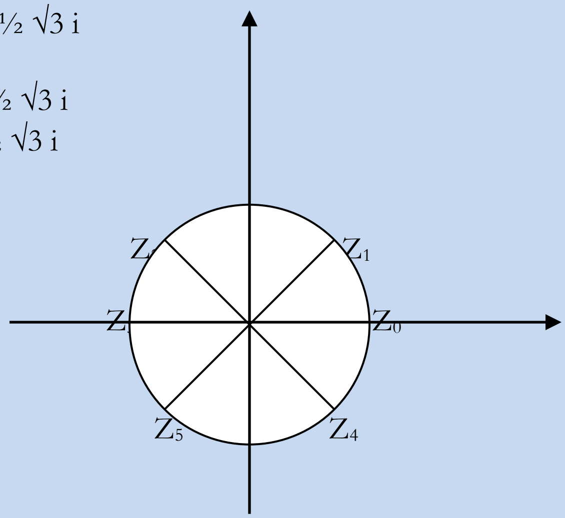
<u>Solution</u>

The sixth roots of unity are given by

$$Z_k = \cos\left(\frac{0+2k\pi}{6}\right) + \sin\left(\frac{0+2k\pi}{6}\right) \quad \textbf{,where k = 0, 1, 2,…, 5}$$

$$Z_k = \cos(1/3k\pi) + i\sin(1/3\pi)$$

For k = 0, $Z_0 = 1$
For k = 1, $Z_1 = ½ + ½ \sqrt{3}\, i$
For k = 2, $Z_2 = -½ + ½ \sqrt{3}\, i$
For k = 3, $Z_3 = - 1$
For k = 4, $Z_4 = -½ - ½ \sqrt{3}\, i$
For k = 5, $Z_5 = ½ - ½ \sqrt{3}\, i$
See figure below:



Example

Find $(-1 + i)^{1/4}$

<u>Solution</u>

Expressing -1 + i in polar form, we have

$$-1 + i = 2 (\cos (3\pi /4 + 2k\pi) + i\text{Sin} (3\pi/4 + 2k\pi)$$

Therefore, $(-1 + i)^{1/4} = 2^{1/8}(\cos(3\pi/16 + 2k\pi/4) + i\sin(3\pi/16 + 2k\pi/4))$

If $k = 0$, $z_1 = 2^{1/8}(\cos 3\pi/16 + i\sin 3\pi/16)$

If $k = 1$, $z_2 = 2^{1/8}(\cos 11\pi/16 + i\sin 11\pi/16)$

If $k = 2$, $z_3 = 2^{1/8}(\cos 19\pi/16 + i\sin 19\pi/16)$

If $k = 3$, $z_4 = 2^{1/8}(\cos 27\pi/16 + i\sin 27\pi/16)$

The four answers above are the fourth roots of $(-1 + i)$

Example

Find the square roots of $12 + 5i$

<u>Solution</u>

Let $Z = p + iq$ where p and q are real numbers be the required square root.

Then $(p + iq)^2 = p^2 - q^2 + 2pqi = 12 + 5i$

$$p^2 - q^2 = 12 \quad (1)$$

$$2pq = 5 \quad (2)$$

From (2) $q = 5/2p$ and substituting this into (1), we have $p^2 - 25/4p^2 = 12$ and so $p^4 - 48p^2 - 25 = 0$

Thus $(2p^2 + 1)(2p^2 - 25) = 0$

$p^2 = -\frac{1}{2}$(reject) or $p^2 = \frac{25}{2}$

$p^2 = \frac{25}{2}$ and so $p \pm \frac{5}{2}$ and consequently, $q = \pm 1$

thus the square roots of 12 + 5i are: $\frac{5\sqrt{2}}{2} + 1\frac{\sqrt{2}}{2}i$ or $-\frac{25}{2} - \frac{\sqrt{2}}{2}\sqrt{2}\,i$

Practice Questions

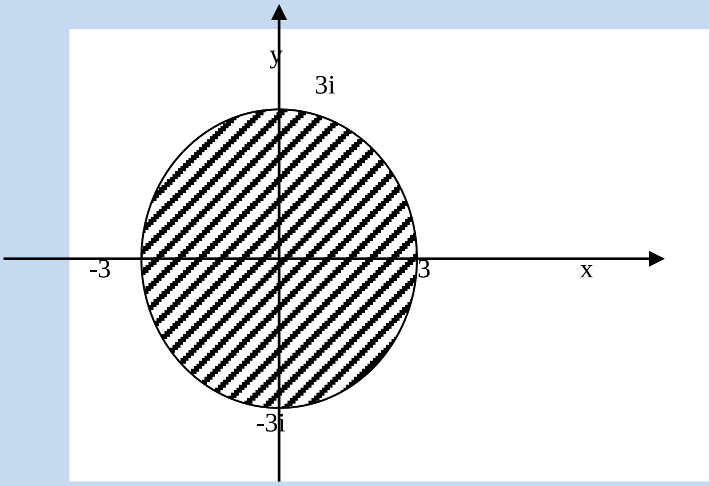
1. Find the cube root of $3 + 4i$
2. find the square root of $8 - 6i$
3. find the root indicated (a) $(8 + 3i), \frac{1}{5}$ (b) $(3 + 6i), \frac{1}{8}$
4. solve the equation $z^3 + 8 = 0$

Loci

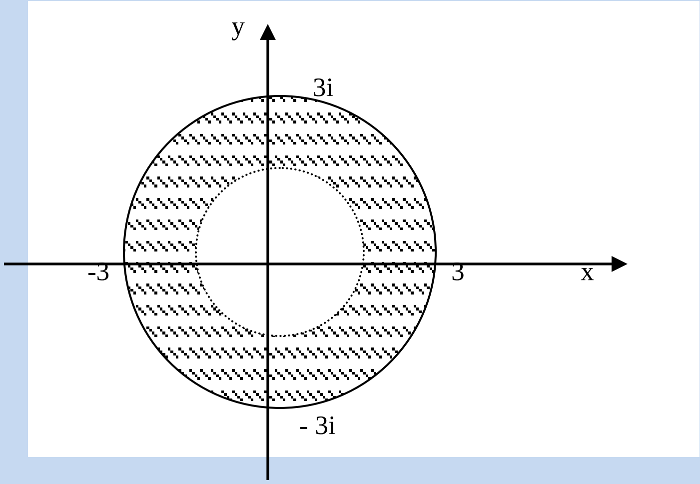
Consider the following examples

1. Find the locus of points z such that Re (z) ≥ 0. In other words we wish to find the locus of all points such that the real part is positive. This is the half plane consisting of all points to the right of the origin.

2. Find the locus of points z:$|z| \leq 3$. Recall that $|z|$ can be viewed as the distance of z from the origin. Thus $|z| \leq 3$ gives the locus of all points whose distance from the locus is less than or equal to 3 i.e. a circle of radius 3 including the boundary as shown in figure below:



Similarly $1 < |z| \leq 3$ gives all points whose distance from the origin is greater than 1 but less than or equal to 3 , i.e. all points in an annulus or ring of inner radius 1 and outer radius 3, excluding the inner boundary but including the outer boundary as shown on diagram.



In general $|z - z_0|$, where z_0 is a complex number gives the distance of z from the point z_0

Let z = x + iy and z_0 = a + bi , then

$$|z - z_0| = |(x + iy) - (a + bi)| = |(x - a) + i(y - b)|$$

$$= \sqrt{(x-a)^2 + (y-b)^2}$$

Which is the expression for the distance between the points (x , y) and (a , b)

Example

Represent geometrically the set of values of z for which

(a) $\left|\frac{z-i}{z+2}\right| = 1$

(b) $\left|\frac{z-i}{z+2}\right| = 2$

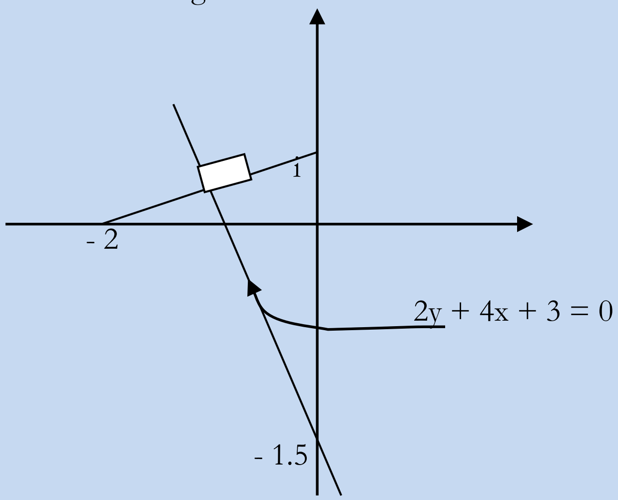
(c) $\left|\frac{z-i}{z+}\right| < 2$

(d) $\left|\frac{z-1}{z+2}\right| > 2$

<u>Solution</u>

(a) $\left|\frac{z-i}{z+2}\right| = 1 \Rightarrow |z - i| = |z + 2|$.

Consequently the points z are such that the distance from the point z to the point z = i equals the distance from the point z to the point z = 2. Note that $|z + 2| = |z - (-2)|$. Thus they are points that are equidistant from the points z = i and z = -2. we know that these are points are on the perpendicular bisector of the straight line joining the points z = i and z = -2 as shown on the figure below.



$|z - i| = |z + 2| \Rightarrow \sqrt{[x^2 + (y-1)^2]} = \sqrt{[(x + 2)^2 + y^2]}$

$\Rightarrow x^2 + (y-1)^2 = (x + 2)^2 + y^2$

$\Rightarrow x^2 - 2y + 1 = x^2 + 4x + 4 + y^2$

$\Rightarrow - 2y + 1 = 4x + 4$, hence the Cartesian equation of the perpendicular bisector is $2y + 2x + 3 = 0$

b. $\left|\frac{z-i}{z+2}\right| = 2$

$\Rightarrow \; |z-1| = 2 \; |z+2|$

$\Rightarrow |x+iy-i| = 2 \; |x+iy-2|$ squaring both sides

$\Rightarrow x^2 + (y-1)^2 = 2[(x+2)^2 + y^2]$

$\Rightarrow x^2 + y^2 - 2y + 1 = 4[x^2 + 4x + 4 + y^2]$

$\Rightarrow 3x^2 + 3y^2 + 16x + 2y + 15 = 0$

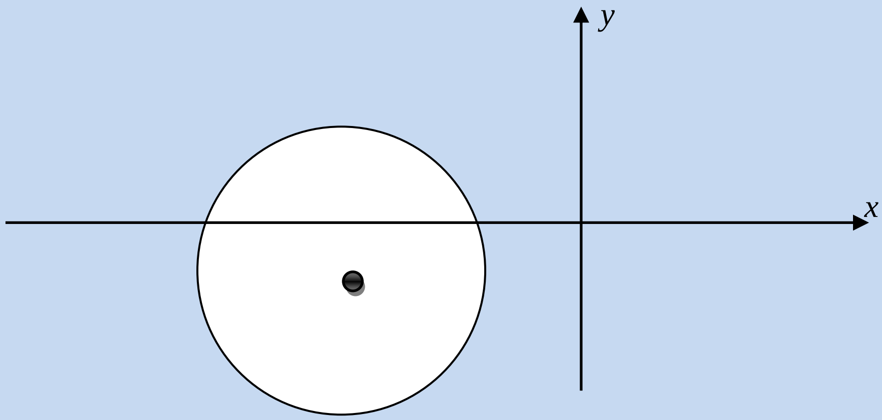
$\Rightarrow 3x^2 + 16x + 3y^2 + 2y = -15$

$$\Rightarrow 3(x^2 + \frac{16}{3x} + \frac{64}{9} - \frac{64}{9}) + 3(y^2 + \frac{2}{3y}\frac{1}{9} - \frac{1}{9}) = -15$$

$$\Rightarrow 3(x+\frac{8}{3})^2 + 3(y+\frac{2}{3})^2 = -15 + \frac{64}{3} - \frac{1}{3}$$

$$\Rightarrow (x+\frac{8}{3})^2 + (y+\frac{2}{3})^2 = \frac{20}{9}$$

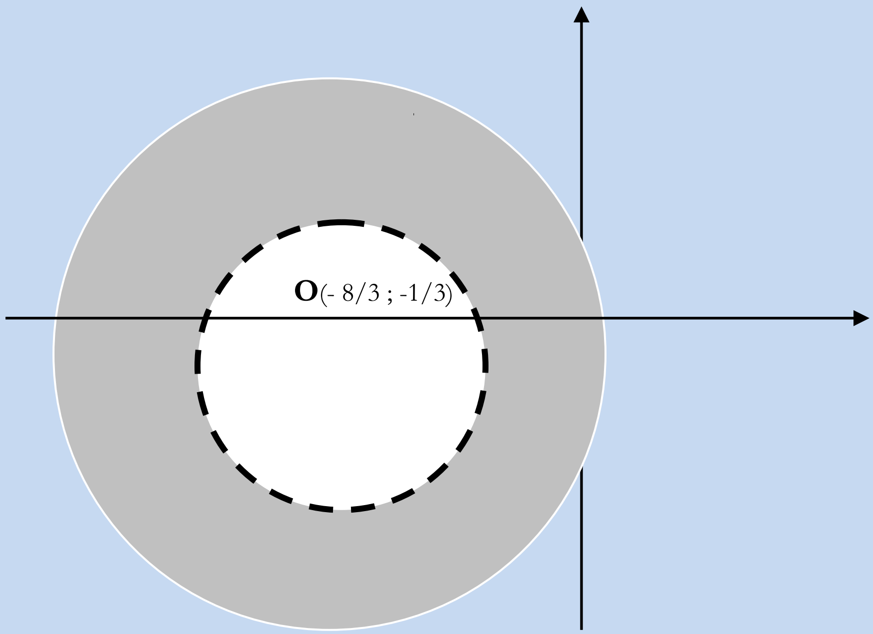
This is a circle , center $\frac{-8}{3};\frac{2}{3}$ and radius $r = \frac{2}{3\sqrt{5}}$



c. $\left|\frac{z-i}{z+2}\right| < 2$

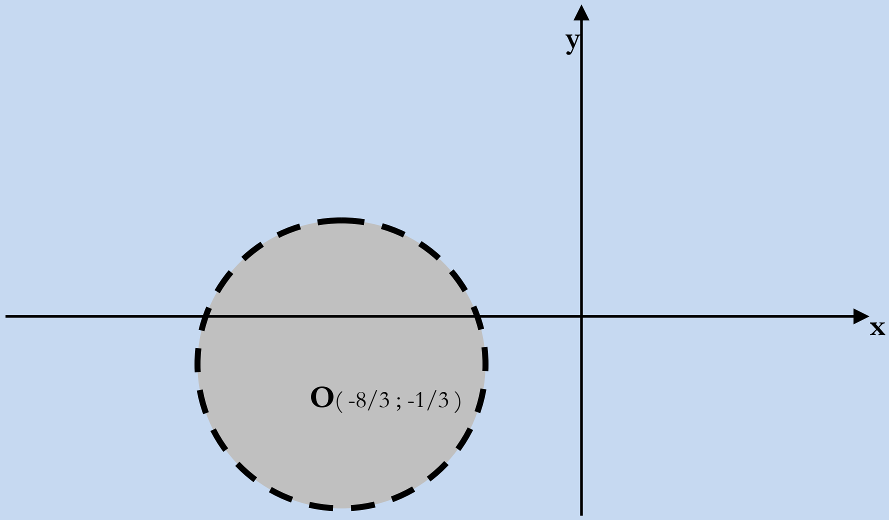
$$\Rightarrow ((x+\frac{8}{3})^2 + (y+\frac{2}{3})^2 = \frac{20}{9} x + \text{; Out side the circle center}$$

(-8/3 ; - 1/3), r = 2/3√5



(d) $\left|\frac{z-i}{z+2}\right| > 2$

$\Rightarrow (x+\frac{8}{3})^2 + (y+\frac{2}{3})^2 = \frac{20}{9}$; Inside the circle center (-8/3 ; - 1/3), r = 2/3√5



Example

Sketch the locus of the point P (x, y) where P represents the complex number z = x + iy on an Argand diagram and write down the Cartesian equation of each locus

a) arg z = π/2

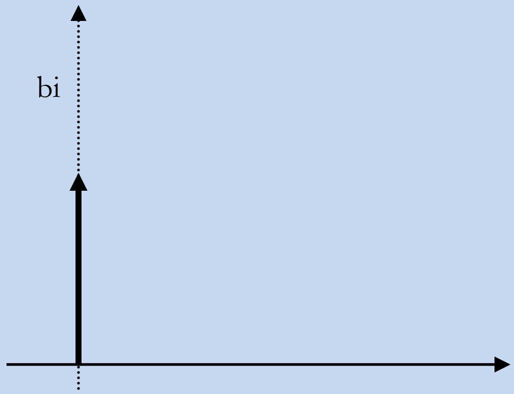
b) arg (z – 3i) = π/3

c) 0 < arg (z + 3) < π/6

d) arg [(z – 1) / (z + 1) = π/2

Solution

a)



Let z = x + iy, then

arg z = π/2

arg (x +iy) = π/2

arg (x + iy) = $\tan^{-1}$(y/x), hence

$\tan^{-1}$(y/x) = π/2

y/x = tan π/2

y/x = ∞

y > 0 and x = 0 is the Cartesian equation i.e. the y – axis and y > 0

b) arg (z – 3i) = π/3

arg (x + iy – 3i) = π/3

arg (x +i (y –3)) = π/3

arg (x +i (y –3)) = $\tan^{-1}$((y –3)/x)

$\tan^{-1}$((y –3)/x) = π/3

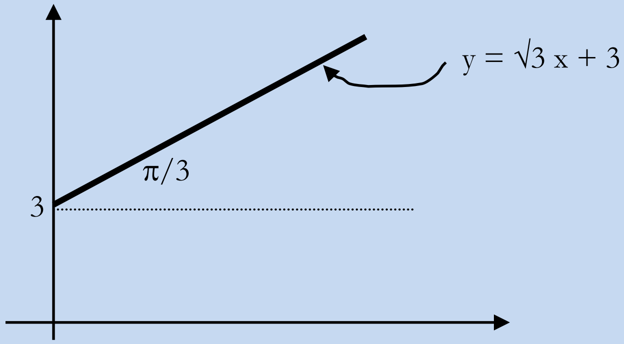
(y – 3) / x = tan π/3

(y – 3) / x = √3

y – 3 = x√3

y = √3 x + 3, y > 3 and x > 0,since π/3 is in the first quadrant

The locus of P is shown below by the equation of the line



c) $0 < \arg(z + 3) < \pi/6$

$0 < \arg(x + iy + 3) < \pi/6$

$0 < \arg(x + 3 + iy) < \pi/6$, hence

$\arg(x + 3 + iy) = \tan^{-1}(y / (x + 3))$, then

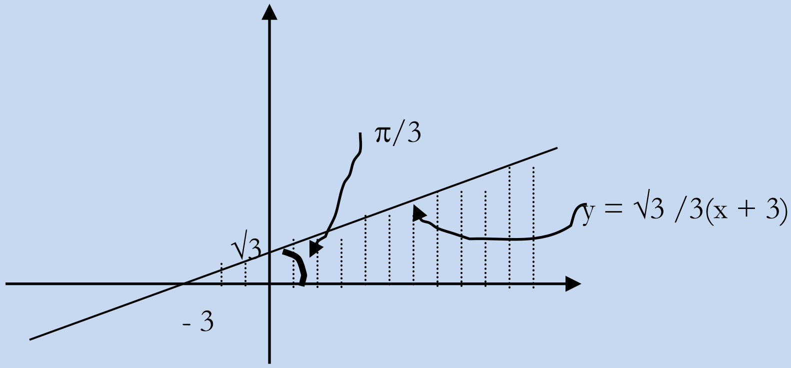
$\tan^{-1}(y / (x + 3)) = \pi/6$

$y / (x + 3) = \tan \pi/6$

$y = \dfrac{\sqrt{3}}{3(x+3)}$, $y > 0$ and $x > -3$, since $\pi/6$ is in the first quadrant.

Hence $0 < y < \dfrac{\sqrt{3}}{3(x+3)}$

The diagram is shown in the figure below



d) $\arg[(z - 1) / (z + 1) = \pi/2$

$$\arg\left[\frac{x+iy-1}{x+iy+1}\right]=\arg\left[\frac{(x-1+iy)(x+1-iy)}{(x+1+iy)(x+1-iy)}\right]$$

$$\arg\left[\frac{x^2-1+y^2+2iy}{x^2+2x+1+y^2}\right]$$

$$=\tan^{-1}\left(\frac{2y}{x^2-1+y^2}\right) hence$$

$$\tan^{-1}\left(\frac{2y}{x^2-1+y^2}\right)=\frac{\pi}{2}$$

$$\frac{2y}{x^2-1+y^2}=\tan\frac{\pi}{2}$$

$$\frac{2y}{x^2-1+y^2}=\infty$$

$$x^2-1+y^2=0, and\ -1\prec\times\prec 1$$

$$x^2+y^2=1$$

The locus of the point P is a semi – circle centre (0,0) and radius 1

$$x^2 + y^2 = 1$$

Examination Type Questions

1. Given that $z^2 + 2i$, express z in the form $r(\cos\theta + i\sin\theta)$, where r is a positive real number and $-\pi < \theta < \pi$. On the same diagram, display and label clearly the numbers z, z^2 and $4/z$
   Find the values of $|z + z^2|$ and $\arg(z + 4/z)$

2. Sketch the locus of the point P (x, y) where P represents the complex number $z = x + iy$ on an Argand diagram and write down the Cartesian equation of each locus
   a) $\arg z = -\pi/4$
   b) $\arg(2z + 4i)$
   c) $0 < \arg(z - 2i) < \pi/8$ 0
   d) $\arg[(z + i)/(z - 3i)]$

3. Use de Moivre's theorem to express Sin 5θ in terms of powers of Sin θ.

4. Show that $2 + 3i$ is a root of the equation $z^3 - 3z^2 + 13 = 0$. Hence find the other two roots.

5. The point P represents the complex number $z = x + iy$ on an argand diagram.
   Describe the locus of P. if

   a) $|z-1| = |z+i|$ b) arg $\left[\dfrac{z+i}{z-1}\right] = \dfrac{\pi}{4}$

6. Find , in the form $r(\cos\theta + i\sin\theta)$ all the complex number $z = x + iy$, such that

   $$z^3 = \frac{5+i}{2+3i}$$

7. A complex number z has modulus 4 and agreement $\dfrac{\pi}{4}$ Another complex number w has modulus ½ and argument $\dfrac{\pi}{8}$.

   a) Write each of the following complex numbers in the form of $a + bi$

   (i) ZW^4 (ii) $\dfrac{z^2}{w^2}$

   b) Find the smallest value of n such that $|w^n| < 0{,}01$

CHAPTER 20

MATRICES

OBJECTIVES
By the end of the chapter the student should be able to :

- Calculate the determinant of a 3x3 matrice
- Find the inverse of a 3x3 matrice
- Solve simultaneous equation of 3 variables using matrices
- Transform points and shapes using matrices

Definition

A matrix is a rectangular array or arrangement of numbers.
The student should recall that more than two matrices can be added together.

Example
If

$$A = \begin{pmatrix} 2 & 3 & 4 \\ 5 & 2 & 1 \\ 2 & 8 & 9 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 2 & 12 \\ 13 & 4 & 3 \end{pmatrix}$$

$$A+B = \begin{pmatrix} 2 & 3 & 4 \\ 5 & 2 & 1 \\ 2 & 8 & 9 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 1 \\ 1 & 2 & 12 \\ 13 & 4 & 3 \end{pmatrix} = \begin{pmatrix} 3 & 3 & 5 \\ 6 & 4 & 13 \\ 15 & 12 & 12 \end{pmatrix}$$

$$A - B = \begin{pmatrix} 2 & 3 & 4 \\ 5 & 2 & 1 \\ 2 & 8 & 9 \end{pmatrix} - \begin{pmatrix} 1 & 0 & 1 \\ 1 & 2 & 12 \\ 13 & 4 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 3 & 3 \\ 4 & 0 & -11 \\ -11 & 4 & 6 \end{pmatrix}$$

$$3A = 3\begin{pmatrix} 2 & 3 & 4 \\ 5 & 2 & 1 \\ 2 & 8 & 9 \end{pmatrix} = \begin{pmatrix} 6 & 9 & 12 \\ 15 & 6 & 3 \\ 6 & 24 & 27 \end{pmatrix}$$

Definition
Identify (unit) matrix. This is a matrix with ones in the leading diagonal and zeros elsewhere

Example $\quad I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ or $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

Definition
Null (zero) matrix: this is a matrix with zeros throughout.

$$O = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \text{ or } O = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

Definition
An **m x n** matrix is a matrix with m rows and n columns.

Example $\quad A_{2x3} =$

$$\begin{pmatrix} 2 & 4 & 51 \\ 9 & 13 & 13 \end{pmatrix}$$

Definition: the transpose of a matrix is the matrix obtained when the rows and columns of a matrix are interchanged.

Example. $\quad$ If $M = \begin{pmatrix} 2 & 3 & 4 \\ 5 & 6 & 7 \end{pmatrix}$ $\quad$ then $M^T = \begin{pmatrix} 2 & 5 \\ 3 & 6 \\ 4 & 7 \end{pmatrix}$

The Adjoint of a 2x2 Matrix

If $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ then , the adjoint $M = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$

The Determinant of a 2x2 Matrix

If M = $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ then $|M|$ = det M = ad – bc

The Inverse of a 2x2 Matrix

If

M = $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ then $M^{-1} = \dfrac{Adjo\operatorname{int} ofM}{\det M}$

Example If A = $\begin{pmatrix} 3 & 1 \\ 5 & 9 \end{pmatrix}$ find the inverse of A

Solution

Adjoint of A = $\begin{pmatrix} 9 & -1 \\ -5 & 3 \end{pmatrix}$

det A = 3(9) -5 (1)
= 22

$A^{-1} = \dfrac{Adjo\operatorname{int} ofA}{\det A}$

$= \dfrac{1}{22}\begin{pmatrix} 9 & -1 \\ -5 & 3 \end{pmatrix}$

We now find the inverse of the 2 x 2 matrix using the **method of reduction.** We will navigate through this method using an example.
Example. Find an inverse of

A = $\begin{pmatrix} 3 & 1 \\ 5 & 9 \end{pmatrix}$

Solution
We write the matrix A with the identity matrix I_2 on the right hand side, i.e.

$\left(\begin{array}{cc|cc} 3 & 1 & 1 & 0 \\ 5 & 9 & 0 & 1 \end{array}\right)$, the objective is to transform this matrix until the identity matrix

$\left(\begin{array}{cc|cc} 1 & 0 & a & b \\ & & & \end{array}\right)$

0 1 c d , hence the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is the inverse matrix.

$$\left(\begin{array}{cc|cc} 3 & 1 & 1 & 0 \\ 5 & 9 & 0 & 1 \end{array}\right) \sim \begin{pmatrix} 1 & \frac{1}{3} & \frac{1}{3} & 0 \\ 5 & 9 & 0 & 1 \end{pmatrix} \text{sub. } r_1 \text{ by } 1/3\, r_1$$

$$\sim \begin{pmatrix} 1 & \frac{1}{3} & \frac{1}{3} & 0 \\ 0 & \frac{22}{3} & \frac{-5}{3} & 1 \end{pmatrix} \text{sub } r_2 \text{ by } r_2 - 5r_1$$

$$\sim \begin{pmatrix} 1 & \frac{1}{3} & \frac{1}{3} & 0 \\ 0 & 1 & \frac{-5}{22} & \frac{3}{22} \end{pmatrix} \text{sub } r_2 \text{ by } 3/22 r_2$$

$$\begin{pmatrix} 1 & 0 & \frac{9}{22} & \frac{-1}{22} \\ 0 & 1 & \frac{-5}{22} & \frac{3}{22} \end{pmatrix}, \text{ hence}$$

$$A^{-1} = \begin{pmatrix} \frac{9}{22} & \frac{-1}{22} \\ \frac{-5}{22} & \frac{3}{22} \end{pmatrix}$$

Definition
A **singular** matrix is one whose determinant is zero. This implies that a singular matrix does not posses an inverse.

Example Find the inverse of $M = \begin{pmatrix} 3 & 6 \\ 2 & 4 \end{pmatrix}$ adjoint of M = $\begin{pmatrix} 4 & -6 \\ -2 & 3 \end{pmatrix}$

$$\begin{aligned} \text{Det } M &= 3(4) - 2\,(6) \\ &= 0 \end{aligned}$$

Since det M is equals zero then M is called a singular matrix. A non-singular matrix is a matrix whose detminant is not zero.

The Determinant of a 3x3 Matrix

Definition
Minor

Let A= $\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$

If we cross out the entries in the row one and column two through entry 2 the resultant matrix is $\begin{pmatrix} 4 & 6 \\ 7 & 8 \end{pmatrix}$

the determinant of this matrix, denoted by M_{22}, is called the minor of entry 2. Each minor has an associated sign given by $(-1)^{i+j}$, where i and j denominate the row and column of the associated entry .

Definition
Cofactor. A minor and its sign is called a cofactor and is denoted by

$$(-1)^{i+j} \ M_{ij} = A_{ij}$$

The signs of minors can be found from the diagram below:

$$\begin{vmatrix} + & - & + \\ - & + & - \\ + & - & + \end{vmatrix}$$

The determinant is the sum of the products of the entries of any row or column and their cofactors.

Det A $= a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13}$
$= a_{21}A_{21} + a_{22}A_{22} + a_{23}A_{23}$
$= a_{31}A_{31} + a_{32}A_{32} + a_{33}A_{33}$

Det A $= a_{11}A_{11} + a_{21}A_{21} + a_{31}A_{31}$
$= a_{12}A_{12} + a_{22}A_{22} + a_{32}A_{32}$
$= a_{13}A_{13} + a_{23}A_{23} + a_{33}A_{33}$

Example Given A= $\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$ find $|A|$

Solution: **Method 1.** $\text{Det } A = 1\begin{vmatrix} 5 & 6 \\ 8 & 9 \end{vmatrix} - 2\begin{vmatrix} 4 & 6 \\ 7 & 9 \end{vmatrix} + 3\begin{vmatrix} 4 & 5 \\ 7 & 8 \end{vmatrix}$ (using entries of row one)

$= 1(-3) - 2(-6) + 3(-3) = 0$, A is a singular matrix.

Method 2

Find the determinant of the matrix A = $\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$

<u>Solution</u>

$$\begin{array}{ccccc} + & + & + \; - & - & - \\ 1 & 2 & 3 & 1 & 2 \\ 4 & 5 & 6 & 4 & 5 \\ 7 & 8 & 9 & 7 & 8 \\ - & - & - & & \end{array}$$

Repeat the first two columns at the end.

Det of A = 1 x 5 x 9 + 2 x 6 x 7 + 3 x 4 x 8 – 3 x 5 x 7 – 1 x 6 x 8– 2 x 4 x 9
= 45 + 84 + 96 - 105 - 48 - 72
= 225 – 225
= 0

Note : You may add the rows at the end.

Method 3.

Find the determinant of the matrix A = $\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$

<u>Solution</u>.

1. Start with the principal diagonal. $1 \times 5 \times 9 = 45$
$2 \times 6 \times 7 = 84$
$4 \times 8 \times 3 = 96$
add: 225
2. Consider the second diagonal $3 \times 5 \times 7 = 105$
$1 \times 8 \times 6 = 48$
$2 \times 4 \times 9 = 72$
subtract: 225

Hence: Det = 225 – 225
= 0

Method 4.
Consider column one, the objective is to transform matrix A so that the entries of row one and that of row two are both zeros

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} \sqcup \begin{pmatrix} 1 & 2 & 3 \\ 0 & 3 & 6 \\ 0 & 6 & 12 \end{pmatrix}$$ obtained by substituting r_2 by $4r_1 - r_2$ and r_3 by $7r_1 - r_3$

$$\text{Det } A = \begin{vmatrix} 3 & 6 \\ 6 & 12 \end{vmatrix}$$

$$= 3 \times 12 - 6 \times 6$$
$$= 0$$

<u>Inverse of a 3 x 3 Matrix</u>

The objective is to find a matrix M^{-1} such that $MM^{-1} = M^{-1} M = I_3$

$$\text{Where } I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Recall that from a 2 x 2 matrix , we have $M^{-1} = \dfrac{\text{adj } M}{\det M}$

Similarly in a 3x 3 matrix

$$M^{-1} = \frac{\text{adj } M}{\det M}$$

Let A_{ij} be the cofactors of the matrix $A = (a_{ij})$, where i= 1, 2, 3 and j = 1,2,3, furthermore let $B = (A_{ij})$ i.e.

$$B = \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix} \qquad \begin{pmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{pmatrix} \text{ then}$$

$$\text{adj } M = B^T$$

$$\text{adj } M = \begin{pmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{pmatrix}$$

$M^{-1} = \dfrac{B^T}{\Delta}$, where Δ = det M

$$M^{-1} = \begin{pmatrix} \frac{A_{11}}{\Delta} & \frac{A_{21}}{\Delta} & \frac{A_{31}}{\Delta} \\ \frac{A_{12}}{\Delta} & \frac{A_{22}}{\Delta} & \frac{A_{32}}{} \\ \frac{A_{13}}{\Delta} & \frac{A_{23}}{\Delta} & \frac{A_{33}}{\Delta} \end{pmatrix}$$

Fix j = 1 and vary i
Fix j = 2 and vary i
Fix j = 3 and vary i

Example

The matrix C = $\begin{pmatrix} 0 & -1 & -1 \\ -2 & 1 & 1 \\ -3 & 0 & 0 \end{pmatrix}$ Calculate the inverse of C

Solution

Method 1 : Using co- factors

$$\Delta = -(-1)\begin{vmatrix} -1 & -1 \\ 0 & 1 \end{vmatrix} + (-3)\begin{vmatrix} -1 & -1 \\ 1 & 1 \end{vmatrix}$$

$$\Delta = -2$$

$A_{11} = \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} = 1$ $\quad A_{21} = -\begin{vmatrix} -1 & -1 \\ 0 & 1 \end{vmatrix} = 1$ $\quad A_{31} = \begin{vmatrix} -1 & -1 \\ 1 & 1 \end{vmatrix} = 0$

$A_{12} = -\begin{vmatrix} -2 & 1 \\ -3 & 1 \end{vmatrix} = -1$ $\quad A_{22} = \begin{vmatrix} 0 & -1 \\ -3 & 1 \end{vmatrix} = -3$ $\quad A_{32} = -\begin{vmatrix} 0 & -1 \\ -2 & 1 \end{vmatrix} = 2$

$A_{13} = \begin{vmatrix} -2 & 1 \\ -3 & 0 \end{vmatrix} = 3$ $\quad A_{23} = -\begin{vmatrix} 0 & -1 \\ -3 & 0 \end{vmatrix} = 3$ $\quad A_{33} = \begin{vmatrix} 0 & -1 \\ -2 & 1 \end{vmatrix} = -2$

$$C^{-1} = \begin{pmatrix} -\frac{1}{2} & -\frac{1}{2} & 0 \\ \frac{1}{2} & \frac{3}{2} & -1 \\ -\frac{3}{2} & -\frac{3}{2} & 1 \end{pmatrix}$$ by a direct substitution in the formula

Method 2 : Using elementary row operations

We write CI_3 side by side. The objective is to carry out elementary row operations until we obtain I_3

$$\left(\begin{array}{ccc|ccc} 0 & -1 & -1 & 1 & 0 & 0 \\ -2 & 1 & 1 & 0 & 1 & 0 \\ -3 & 0 & 1 & 0 & 0 & 1 \end{array}\right)$$

$$\sim\left(\begin{array}{ccc|ccc} 0 & -1 & -1 & 1 & 0 & 0 \\ 1 & -\frac{1}{2} & -\frac{1}{2} & 0 & -\frac{1}{2} & 0 \\ -3 & 0 & 1 & 0 & 0 & 1 \end{array}\right)$$ sub. r_2 by $-r_2/2$

$$\sim\left(\begin{array}{ccc|ccc} 1 & \frac{1}{2} & \frac{1}{2} & -1 & -\frac{1}{2} & 0 \\ 1 & -\frac{1}{2} & -\frac{1}{2} & 0 & -\frac{1}{2} & 0 \\ 0 & -\frac{3}{2} & -\frac{1}{2} & 0 & -\frac{3}{2} & 1 \end{array}\right)$$ sub. r_1 by $r_2 - r_1$

sub. r_3 by $3r_2 + r_3$

$$\sim\left(\begin{array}{ccc|ccc} 1 & \frac{1}{2} & \frac{1}{2} & -1 & -\frac{1}{2} & 0 \\ 0 & 1 & 1 & -1 & 0 & 0 \\ 0 & -\frac{3}{2} & -\frac{1}{2} & 0 & -\frac{3}{2} & 1 \end{array}\right)$$ sub. r_2 by $r_1 - r_2$

$$\sim\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 1 & 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -\frac{3}{2} & -\frac{3}{2} & 1 \end{array}\right)$$ sub. r_1 by $r_1 - ½\, r_2$

sub. r_3 by $3/2r_2 + r_3$

$$\sim\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 1 & 0 & \frac{1}{2} & \frac{3}{2} & -1 \\ 0 & 0 & 1 & -\frac{3}{2} & -\frac{3}{2} & 1 \end{array}\right) \text{ sub. } r_2 \text{ by } r_2 - r_3$$

hence

$$C^{-1} = \begin{pmatrix} -\frac{1}{2} & -\frac{1}{2} & 0 \\ 1/2 & \frac{3}{2} & -1 \\ -\frac{3}{2} & -\frac{3}{2} & 1 \end{pmatrix}$$

Practice Questions

1. The matrix $C = \begin{pmatrix} 2 & 3 & 4 \\ 1 & 2 & 3 \\ 3 & 4 & 1 \end{pmatrix}$ Calculate the inverse of C

2. The matrix $C^{-1} = \begin{pmatrix} 1 & 3 & -1 \\ 2 & 1 & -1 \\ 0 & -2 & 2 \end{pmatrix}$ Calculate C.

Solving Simultaneous Equations.

Given the following equations:

$4x + 3y = 8$

$-3x + 2y = 11$

we can re- write these in matrix form as follows:

$$\begin{pmatrix} 4 & 3 \\ -3 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 8 \\ 11 \end{pmatrix}$$

or in general: **AX = C**

Pre multiplying both sides by A^{-1} we get,

$$A^{-1}(AX) = A^{-1}C$$
$$(A^{-1}A)(X) = A^{-1}C$$
but $A^{-1}A = \mathbf{1}$

Therefore $\mathbf{X = A^{-1}C}$

Using the method above: $A^{-1} = \frac{1}{17}\begin{pmatrix} 4 & 3 \\ -3 & 2 \end{pmatrix}$

$$A = \begin{pmatrix} 4 & 3 \\ -3 & 2 \end{pmatrix}$$

-

$$\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{17}\begin{pmatrix} 2 & -3 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 8 \\ 11 \end{pmatrix}$$

$$= \frac{1}{17}\begin{pmatrix} -17 \\ 68 \end{pmatrix}\begin{pmatrix} -1 \\ 4 \end{pmatrix}$$

hence x = - 1 and y = 4

Example

Solve the following simultaneous equations:

$$x - 2y + 0z = 1$$
$$2x - y + 4z = 2$$
$$x \qquad\qquad = 1$$

In matrix form.

$$\begin{pmatrix} 1 & -2 & 0 \\ 2 & -1 & 4 \\ 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \quad \text{Let } A = \begin{pmatrix} 1 & -2 & 0 \\ 2 & -1 & 4 \\ 1 & 0 & 0 \end{pmatrix}$$

Therefore

$$A^{-1} = -\frac{1}{8}\begin{pmatrix} 0 & 0 & -8 \\ 4 & 0 & -4 \\ 1 & 0 & 4 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = -\frac{1}{8}\begin{pmatrix} 0 & 0 & -8 \\ 4 & 0 & -4 \\ 1 & 0 & 4 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$$

$$= -\frac{1}{8}\begin{pmatrix} -8 \\ 0 \\ 5 \end{pmatrix}$$ therefore x = 1, y = 0 and z = -5/8

Example

If
$$A = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 4 & -3 \end{pmatrix}$$

(i) Show that A satisfies the equation $A^3 = 11A - 14\mathbf{I}_3$

(ii) Show that the equation can be rewritten as $A^{-1} = \frac{1}{14}(11A - A^2)$

and hence find A^{-1}

<u>Solution</u>

(i) LHS = A^3 = AAA =

$$= \begin{pmatrix} 2 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 4 & -3 \end{pmatrix}\begin{pmatrix} 2 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 4 & -3 \end{pmatrix}\begin{pmatrix} 2 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 4 & -3 \end{pmatrix}$$

$$= \begin{pmatrix} 4 & 3 & 1 \\ 0 & 5 & -2 \\ 0 & -8 & 13 \end{pmatrix}\begin{pmatrix} 2 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 4 & -3 \end{pmatrix}$$

$$= \begin{pmatrix} 8 & 11 & 0 \\ 0 & -3 & 11 \\ 0 & 44 & -47 \end{pmatrix}$$

RHS = $11A - 14\mathbf{I}_3$

$$= 11\begin{pmatrix} 2 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 4 & -3 \end{pmatrix} - 14\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 22 & 11 & 0 \\ 0 & 11 & 11 \\ 0 & 44 & -33 \end{pmatrix}\begin{pmatrix} 14 & 0 & 0 \\ 0 & 14 & 0 \\ 0 & 0 & 14 \end{pmatrix}$$

$$= \begin{pmatrix} 8 & 11 & 0 \\ 0 & -3 & 11 \\ 0 & 44 & -47 \end{pmatrix} = \text{LHS}$$

(iii) $A^3 = 11A - 14\mathbf{I_3}$

$A^{-1}A^3 = 11\ A^{-1}A - 14A^{-1}\mathbf{I_3}$

$A^2 = 11\mathbf{I}_3 - 14A^{-1}$, hence re arranging

$A^{-1} = \frac{1}{14}\ (11I - A^2)$

(iv) $A^{-1} = \frac{1}{14}(11\mathbf{I}_3 - A^2)$

$$A^{-1} = \frac{1}{14}\left\{\begin{pmatrix} 11 & 0 & 0 \\ 0 & 11 & 0 \\ 0 & 0 & 11 \end{pmatrix} - \begin{pmatrix} 4 & 3 & 1 \\ 0 & 5 & -2 \\ 0 & -8 & 13 \end{pmatrix}\right\}$$

therefore

$$A^{-1} = \frac{1}{14}\begin{pmatrix} 7 & -3 & -1 \\ 0 & 6 & 2 \\ 0 & 8 & -2 \end{pmatrix}$$

Examination Type Questions

1. Solve for x, y and z

   a) $x + y = 6$

   $y + z = 13$

   $x + z = 3$

   b) $x + 2y + 3z = 32$

   $2x - 3y + 4z = 29$

   $3x + 4y - 5z = -8$

2. Find the two numerical values of λ such that

$$\begin{pmatrix} 4 & 3 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} u \\ 1 \end{pmatrix} = \lambda \begin{pmatrix} u \\ 1 \end{pmatrix}$$

Hence, or otherwise, find the equations of the two lines through the origin which are invariant under the transformation of the plane defined by

$$\begin{pmatrix} x1 \\ y1 \end{pmatrix} = \begin{pmatrix} 4 & 3 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} n \\ y \end{pmatrix}$$

3. Find the equations of the lines that are mapped on to themselves under the transformation.

$$\begin{pmatrix} 2 & 1 \\ 3 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x1 \\ y1 \end{pmatrix}$$

4. Find the determinant of the matrix

$$A = \begin{pmatrix} 2 & 3 & 5 \\ 1 & 0 & 4 \\ 2 & 5 & 6 \end{pmatrix}$$

Hence find A-1

5. It is given that $A = \begin{pmatrix} 1 & 0 & 2 \\ 2 & -1 & 1 \\ 3 & -1 & 0 \end{pmatrix}$ and

Matrix B is such that

$$AB = \begin{pmatrix} -1 & 0 & 2 & 1 \\ 1 & -5 & 1 & 1 \\ 0 & 2 & 3 & 0 \end{pmatrix}$$

(a) State the dimensions of the matrix B
(b) Find (i) the inverse of A
(ii) the matrix B

6. M is the matrix $\begin{pmatrix} 3 & 1 & -3 \\ 1 & 2a & 1 \\ 0 & 2 & a \end{pmatrix}$

(a) Find two values of *a* for which M is singular.

(b) Solve the equation $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -3\frac{1}{2} \\ 5\frac{1}{2} \\ 5 \end{pmatrix}$ in the case $a = 2$ and determine

whether or no solution exist for each of the two values of *a* found in (a).

7. Find the inverse of A^{-1} of the matrix

$A = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 3 & 2 & 1 \end{pmatrix}$ find also $(AB)^{-1}$ where

$B^{-1} = \begin{pmatrix} 1 & -4 & 14 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{pmatrix}$

CHAPTER 21

NUMERICAL METHODS

OBJECTIVES
By the end of the chapter the student should be able to :

- Calculate errors
- Estimate areas using the trapezium rule
- Locate roots graphical and analytical estimate roots using the general iterative formula and the Newton Raphson method

Approximate Solutions

These are methods used to find approximate solutions to problems that cannot be solved analytical

Errors

When measuring continuous variables such as length and height, it is not possible to get an exact value. These quantities can only be measured to some degree of accuracy.

Example

If height (h) of student is recorded as 1.5m (correct to 1 decimal place) this implies that

$$1.45 \leq h < 1.55$$

is the lower bound and 1.55 is the upper bound.
The absolute error $= |\text{estimate} - \text{exact}|$ in this ease since we do not know the true height we will take the measured value as the exact value.

For the same reason we are unable to calculate the absolute error but we can find the maximum absolute error.

Maximum absolute error = upper bound - lower bound = 1.5 – 1.45 = 0.05
Therefore $-0.05 < E < 0.05$

The relative error = Absolute error

Actual value

$$= \frac{0.05}{1.5}$$

= 0.033 (to 3 decimal places)

The percentage error = $\frac{\text{Absolute error}}{\text{Actual value}}$ x 100%

= 3.33% to 2 d. p.

Example

If the quantities m and n measured correct to 1 d. p. are n = 10.3 and m = 16.3, find the maximum and minimum values of:

(i) m + n

(ii) m – n

(iii) $\dfrac{m}{n}$

(iv) $\dfrac{m^2}{m-n}$

<u>Solution</u>

10:25 ≤ n < 10.35

16.25 ≤ m < 16.35

(i) a) max (m + n) = 16. 35 + 10.35
= 26.7

b) min(m + n) = 16 . 25 + 10. 25
= 26.5

(ii) a) max (m-n)= max (m) - min(n)
= 16.35 - 10.25
= 6.1

b) min (m-n) = min(m) – max (n)
= 16.25 - 10.35
= 5.9

(iii) a) max(m/n) =16.35 / 10.25
= 1.60 to 2 d. p.

b) min (m/n) = 16.25/10.35

$$= \frac{16.25}{10.35}$$

= 1.57 to 2 d. p.

(iv) a) max $(m^2/(m-n)) = \dfrac{16,35^2}{(16,25-10,35)}$

$= 45.31$ to 2 d. p.

b) min $(m^2/(m-n)) = \dfrac{16,25^2}{(16,35-10,25)}$

$= 43.29$ to 2 d. p.

Example

The period (T) of a pendulum of length ℓ is given by $T = 2\pi\ell/g$, where g is the acceleration due to gravity.

a) Given that for a particular pendulum g = 9.81 and L = 0.53 correct to 2 decimal places, find the least possible value of T to 2 decimal places.
b) If the above values are taken as exact and the value of g as 10 calculate the percentage error in calculating the value of T to 2 decimal places.

Solution

(a) $9.805 \le g < 9.815$ and $0.525 \le \ell < 0.535$

T (least) $= 2\pi\dfrac{\sqrt{0.525}}{9.815}$

$= 0.46$ to 2 d. p.

(b) T (actual) $= 2\pi\dfrac{\sqrt{0.53}}{9.81}$

$= 0.466282155$

Estimate $= 2\pi\dfrac{\sqrt{0.53}}{10}$

$= 0.457422794$

$\therefore$ Percentage error $= \dfrac{0.466282155 - 0.457422794}{0.4662821550}$

$= 1.94\%$

Small Errors

Here we make use of the relationship

$$\frac{\delta y}{\delta x} \approx \frac{dy}{dx}$$

where δx and δy are small changes in x and y respectively

Example

If there was a 3 % error in measuring the radius of a cone, find the percentage error in the volume..

Solution

V = $\frac{1}{3}\pi r^2 h$, hence, dv/dr = $\frac{2}{3}\pi rh$

and $\frac{\delta y}{\delta x} \approx \frac{dy}{dx}$

$\therefore \quad \delta v = \frac{dy}{dr}.\ \delta r = \frac{2}{3}\pi rh.\frac{3r}{100}$

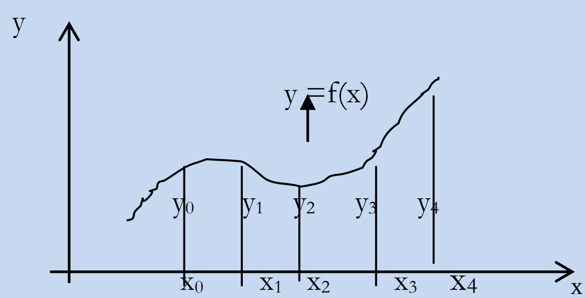
Hence $\frac{\delta y}{v} \times 100 = \frac{2\pi r^2 h}{100}.\frac{3\times 100\%}{\pi r^2 h}$

$= 6\%$

The Trapezium Rule

This is a method for approximating the area under a graph. It is an alternative to integration.

Given the diagram below,



And considering the individual trapezium, it can be shown that,

$$A \approx \frac{h}{2} [y_0 + y_4 + 2 (y_1 + y_2 + y_3)]$$

Where h is the common width of the trapezium in general, for n strips.

$h = \frac{(b-a)}{n}$)/ n

This is easily remembered as:

A = half width × (first + last + twice the rest)

Please note that the greater the number of strips the more accurate is the approximation. For n strips the formula is:

$$\int_a^b f(x)\, dx = \frac{1}{2} h [y_0 + 2(y_1 + y_2 + y_3 + \dots + y_{n-1}) + y_n]$$

Example

Use the trapezium rule with 5 ordinates to find an approximate value of the area between the x – axis, the curve $y = e^{3x}$ and the lines x = -2 and x = 2.

<u>Solution</u>

These are 5 ordinates, therefore the No. of strips n = 5 – 1 = 4

The common width h = (upper limit - lower limit) / No. of strips

$$= \frac{(2-(-2)}{4}$$

$$= 1$$

$x_0 = -2$	$y_0 = e^{-6}$
$x_1 = -1$	$y_1 = e^{-3}$
$x_2 = 0$	$y_2 = 1$
$x_3 = 1$	$y_3 = e^{3}$
$x_4 = 2$	$y_4 = e^{6}$

$$\therefore \quad A = \frac{h}{2} [y_0 + y_4 + 2 (y_1 + y_2 + y_3)]$$

$$= \frac{1}{2} [e^{-6} + e^{6} + 2 (e^{-3} + e^{0} + e^{3})]$$

$$= 222.85 \text{ to 2 d. p.}$$

Example

Find (i) the approximate value of the area between the x – axis, the curve $y = x^3$ and the lines x = 0 and x = 3 with 4 ordinates

(ii) Find the exact value.

(iii) Does the trapezium rule over or under estimate the area ?

(iv) the percentage error in using the trapezium rule to calculate the area.

Solution

$$h = \frac{(3-0)}{3}$$

$x_0 = 0$ $\quad$ $y_0 = 0$

$x_1 = 1$ $\quad$ $y_1 = 1$

$x_2 = 2$ $\quad$ $y_2 = 8$

$x_3 = 3$ $\quad$ $y_3 = 27$

$$A = \frac{1}{2}\,[\;0 + 27 + 2\,(\,1 + 8\,)\,]$$

$= 22.5$

(ii)

$$A = \int_0^3 x^3 dx$$

$$= \left|\frac{x^4}{4}\right.$$

$= 20.25$

(iii) The area is overestimated by the trapezium rule.

(iv) the absolute error $= |22.5 - 20.25|$

$= 2.25$

$\therefore$ the percentage error $= \dfrac{2.25}{20,25} x100\%$

$= 11.11\%$

Approximation of Roots.

The first step is to locate the root and the root is computed using iterative methods. There are two methods used to locate the roots, these are the graphical method and the change of sign method.

1.1.10 The Graphical method

Consider the equation h (x) = 0. Assume it is possible to express h (x) = f (x) – g (x). The roots of h (x) = 0 i.e. f (x) – g (x) = 0, are the values of x where the curves y = f (x) and y = g (x) intersect. It involves making a sketch of the curves

It is important to recall that with one curve the roots are the x – values where the curve cuts the x – axis and with two curves these are the x – values of their points of intersection.

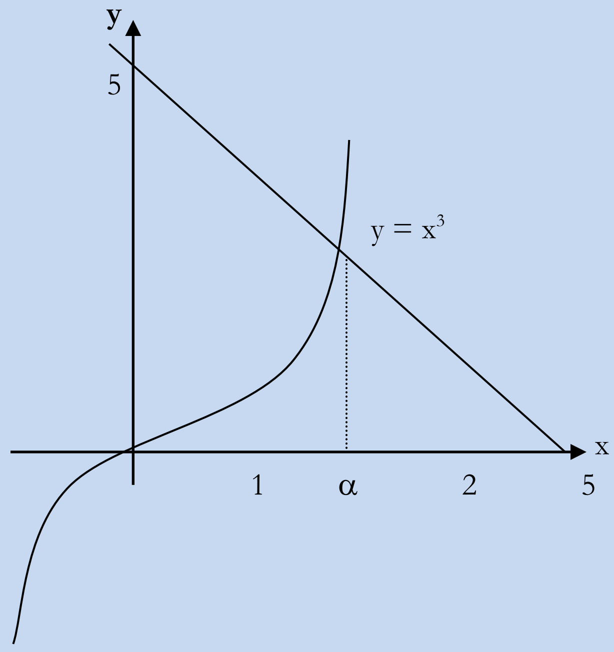
Example

Locate the root of the curve $x^3 + x - 5 = 0$

Solution

$x^3 + x - 5 = 0$

$x^3 = -x + 5$, hence, f (x) = x^3 and g (x) = $-x + 5$. Sketching the graphs of the curves, we obtain the following diagram below, where α is the root of the equation h (x) = 0



∴ α is between 1 and 2

1.1.11 By a change in sign

Example

Show that the function $y = x^3 + x - 5$ has a root between x = 1 and x = 2.

Solution

$f(x) = x^3 + x - 5$

$f(1) = 1^3 + 1 - 5 = -3$ (negative)
$f(2) = 2^3 + 2 - 5 = 5$ (positive)
$\therefore$ $f(1) \times f(2) < 0$, change of sign means there is a root between
$x = 1$ and $x = 2$.
Note: This method does not guarantee the uniqueness of the root , it only guarantees the existence of a root. This means that there could be more than one root in the interval in question.

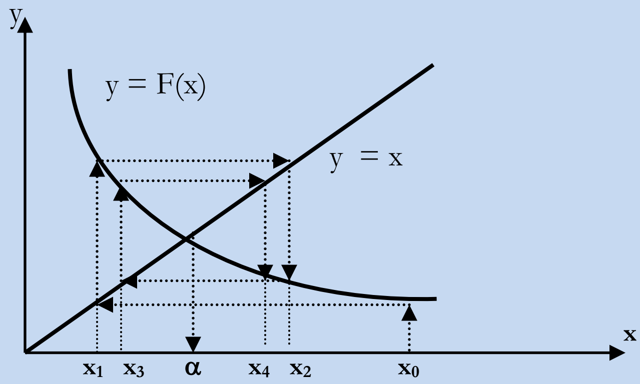
Iterative Methods

1.1.12 General iterative method

An iteration is a successive approximation. If we want to solve the equation $f(x) = 0$ by an iterative method, we need a relationship

$$x_{r+1} = F(x_r),$$

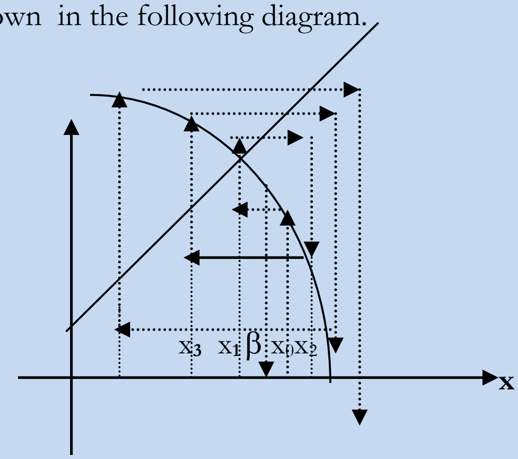
where x_{r+1} is a better approximation to the solution of $f(x) = 0$ than is x_r. To find such a relationship, we need to rearrange $f(x) = 0$ into the form $x = F(x)$.
We now show how to create this iterative formula by geometrical means.
Suppose that the graphs of $y = x$ and $y = F(x)$ are as shown below.



Let x_0 be the fisrt approximation, hence from the graph :
$x_1 = F(x_0)$
$x_2 = F(x_1)$
$x_3 = F(x_2)$
$x_4 = F(x_3)$
…………
…………
…………
$\mathbf{x_{n+1} = F(x_n)}$

If this process is continued for n large the sequence will converge to the root α
Note : Recall from previous studies that such sequences may diverge. The onus is on choosing the function F(x) such that the sequence $x_{n+1} = F(x_n)$ does converge. In this course the iterative formula $x_{n+1} = F(x_n)$ will be suggested, i.e the function F(x) will be suggested

An iteration that fails to lead to a solution in this case is said to be divergent. This can be shown in the following diagram.



From the diagram x_{n+1} is drifting away from the root β,hence, the sequence diverges

Example

Given $x^2 - 5x + 1 = 0$, show that the equation mat be rearranged onto the form

$x = \frac{1}{5}(x^2 + 1)$, hence, suggest the iterative formula.

<u>Solution</u>

Rearranging $5x = x^2 + 1$

$$\Rightarrow \qquad x = \frac{1}{5}(x^2 + 1)$$

$$\Rightarrow \qquad x_{n+1} = \frac{1}{5}(x_n^2 + 1)$$

Example

a) Show that the equation $3x^3 - 2x^2 - 9x + 2 = 0$ has a root between 0 and 1.

b) Show that the equation $3x^3 - 2x^2 - 9x + 2 = 0$ can be rearranged in the form

$$x = \frac{1}{9}(3x^3 - 2x + 2)$$

c) Use an iteration based on this arrangement with an initial value $x_0 = 0.5$ to find this correct to two decimal places.

<u>Solution</u>

Rearranging $9x = 3x^3 - 2x + 2$

$$x = \frac{1}{9}(3x^3 - 2x + 2)$$

$$x_{n+1} = \frac{1}{9}(3x_n^3 - 2x_n^2 + 2)$$

$$x_0 = 0.5$$

$\therefore \quad x_1 = \frac{1}{9}[3(0.5^3) - 2(0.5^2) + 2]$

$= 0.208333333333$

$x_2 = \frac{1}{9}[3(0.2083^3) - 2(0.2083^2) + 2]$

$= 0.215591242$

$x_3 = \frac{1}{9}[3(0.2156^3) - 2(0.2156^2) + 2]$

$= 0.215233622$, **stability has been reached up to 3 d.p.**

$= 0.22$ to 2 d. p.

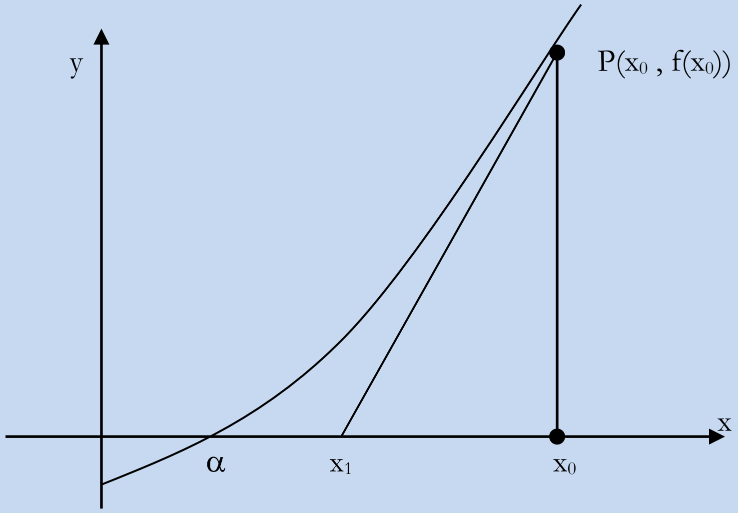
$\therefore \quad \alpha = 0.22$ to 2 d. p.

1.1.13 The Newton – Raphson Method

The iterative formula given by :

$$x_{n+1} = x_n - \frac{f(x_n)}{f^1(x_n)}$$

is refered to as the Newton –Raphson's iterative formula. We deduce this formula using geometrical methods.



Note that α is a root of the equation f(x) = 0. Let x_0 be the fisrt approximation to the root, hence the tangent line to the curve at P(x_0, f(x_0)) cuts the x- axis at x_1, which is a better approximation to α.

The gradient, m , of the tangent is given by

$$m = \frac{f(x_0)}{x_0 - x_1}$$

We also know that m = $f^1(x_0)$. There fore, we have

$$f^1(x_0) = f(x_0) / (x_0 - x_1)$$

Rearranging, we obtain

$$(x_0 - x_1)f^1(x_0) = f(x_0)$$

which gives

$$x_0 - x_1 = \frac{f(x_0)}{f^1(x_0)}$$

$$\therefore x_1 = x_0 - \frac{f(x_0)}{f^1(x_0)}$$

....

....

$$\mathbf{x_{n+1} = x_n -} \frac{f(x_n)}{f^1(x_n)}$$

which is the **Newton- Raphson formula**

Example

Given that $x^3 - 3x^2 - 1 = 0$ has a root lying between 3 and 4. Hence find the root of f(x) = 0 using the Newton – Raphson method correct to 2 decimal places.

Solution

Let $x_0 = 3.5$ and $f^1(x) = 3x^2 - 6x$, hence

$$x_{n+1} = x_0 - \left[\frac{x_n^{\ 3} - 3x_n^{\ 2} - 11}{3x_n^{\ 2} - 6x_n}\right]$$

$$= 3.174603175$$

$$x_2 = 3.1067 - \left[\frac{3.1067^3 - 3(3.1067^2) - 1}{3(3.1067^2) - 6(3.1067)}\right]$$

$= 3.106694909$

$x_3 = 3.103808523$

$x_4 = 3.103803403$, stability has been reached up to 5 d. p.

$\therefore\ \alpha = 3.10$ to 2 d.p.

Examination Type Questions

1. Show that the equation $x^5 - 5x - 6 = 0$ has a root in the interval (1,2). Stating the values of the constants p, q and r, use an iteration of the form $x_{n+1} = (px_n + q)^{1/r}$, the appropriate number of times to calculate this root, to 3 d.p. Show sufficient working to justify your answer.

2. Use the trapezium rule with five ordinates and interval width 0.25 to evaluate approximately the integral

$$\int_1^2 \ln(1 + x^2)\, dx$$

Show your working and give your answer correct to 2 d.p.

3. A chord of circle subtends an angle of θ radians ($\theta < \pi$)at the center. If the chord divides the circle into two segments whose areas are in the ratio 3:1, prove that

$$\sin\theta = \theta - \frac{\pi}{2}$$

Using the iterative formula $\theta_{n+1} = \sin\theta_n + \frac{\pi}{2}$

$\theta_0 = 0.75\pi$, find the root of the equation

$$\sin\theta = \theta - \frac{\pi}{2}, \text{ correct to 3d.p.}$$

4. Given the values in the table below estimate the value of

x	0	0.25	0.5	0.75	1
$\cos\sqrt{x}$	1	0.878	0.76	0.648	0.04

$$\int_0^1 \cos\sqrt{x}\,dx$$

a) by the trapezium rule
b) by means of the Maclaurin series for $\cos\sqrt{x}dx$
c) Evaluate the integral by using the substitution $x = t^2$

d) Find the relative percentage error for using the trapezium rule as the approximation to the integral.

5. Show that the equation $x^3 - x - 2 = 0$ has a root between 1 and 2. Using Newton - Raphson's approximation with starting point 1.5, determine by means of two iterations an approximation to this root, giving your answers to 2 d.p.

6. Show that the equation $x^3 - x^2 - 2 = 0$ has a root *a* which lies between 1 and 2.

   Using 1.5 as a first approximation for α, use the Newton – Raphson method once to obtain a second approximation for α, giving your answer to 3 decimal places.

7. Show that, if $y = \sqrt{x}$, then the error, δy in y, resulting from a small error δx in x is given by

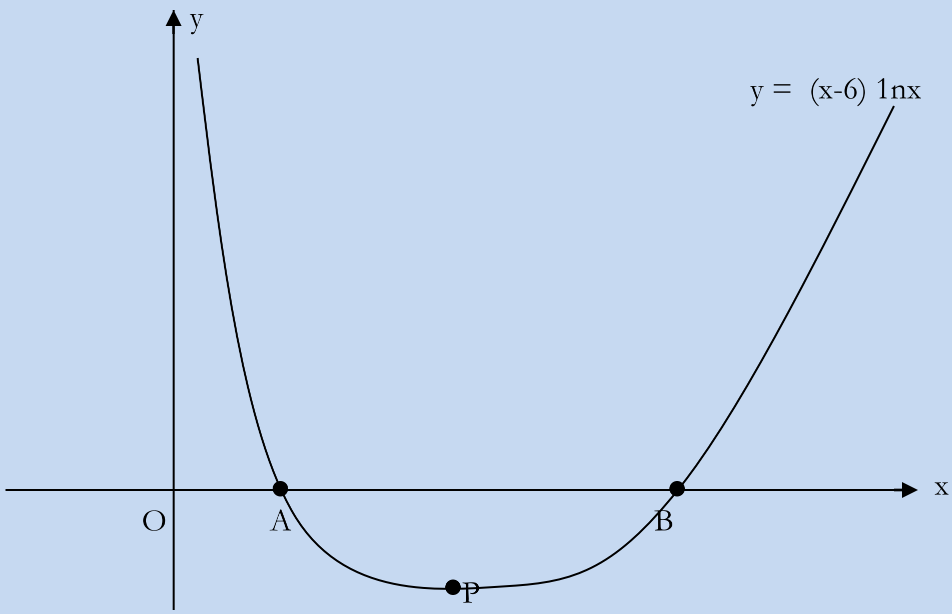
$$\delta y \approx \frac{(1)}{2\sqrt{x}})\ \delta x$$

8. Use the trapezium rule with 6 ordinates to estimate the value of $\frac{1}{1+x}$ dx

   Find the exact value of the integral and hence the percentage error involved.

9. Use the trapezium rule with 4 ordinates to estimate the value of $\int$Inxdx

Usi ng integration by parts, or otherwise, verify that $\int$Inxdx = xInx – x + C. Find the exact value of the integral and hence the percentage error involved.



The diagram shows the curve y = (x-6) 1nx and its minimum point P. The curve cuts the x-axis at the points A and B.

(i) Write down the co-ordinates of A and B.
(ii) Show that the x-co-ordinates of P satisfies the equation

$$x = \frac{6}{1+1nx}$$

(iii) Use the iteration formula

$$X_{n+1} = \frac{6}{1+1nx}$$

to find the x-co-ordinate of P correct to decimal place, showing the result of each iteration that you calculate.

CHAPTER 22

THE PRINCIPLE OF MATHEMATICAL INDUCTION

OBJECTIVES
By the end of the chapter the student should be able to :

- Prove mathematical results using the principle of mathematical induction

The Method of Mathematical Induction

This is a method of proving a mathematical relationship. It has exactly four steps.

Step 1 : Show that the relationship is true for n =1

Step 2 : Assume that its true for n = k.

Step 3 : Show that if it is true for n = k, then it is true for n = k + 1

Step 4 : Conclusion

There is a wide variety of questions that may be posed and we shall deal with the most common.

Proofs of Mathematical Results

Example

Prove by induction that

$$1^3 + 2^3 + \ldots + n^3 = \frac{1}{4}n^2(n+1)^2$$

Let $P_n : 1^3 + 2^3 + \ldots + n^3 = \frac{1}{4}n^2(n+1)^2$

Step 1. For $n = 1$ LHS $= 1^3 = 1$

RHS $= \frac{1}{4}1^3(1+1)^2$

$= 1$

P_1 is true

Step 2 : Assume that it is true for n = k

i.e. P_k : $1^3 + 2^3 + \ldots + k^3 = \sum_{r=1}^{k} r^3 = \frac{1}{4} k^2 (k+1)^2$

Step 3 : If Pk is true we now prove that P_{k+1}

i.e. P_{k+1}: $1^3 + 2^3 + \ldots + k^3 + (k+1)^3 = \sum_{r=1}^{k+1} r^3 = \frac{1}{4}(k+1)^2(k+2)^2$

LHS: $\sum_{r=1}^{k+1} r^3 = \sum r^3 + (k+1)^3$

$= \frac{1}{4}(k+1)^2 k^2 + (k+1)^3$

$= \frac{1}{4}\{(k+1)^2 k^2 + 4(k+1)^3\}$

$= \frac{1}{4}(k+1)^2 [k^2 + 4(k+1)]$

$= \frac{1}{4}(k+1)^2 [k^2 + 4k + 4]$

$= \frac{1}{4}(k+1)^2 (k+2)^2$

= RHS

Step 4 : Since P_1 is true and from P_k, P_{k+1} is true, hence by PMI P_n is true for all positive integral values of n.

Example

Prove by induction that $n^3 - n$ is divisible by 6

<u>Solution</u>

Step 1 : If n = 2, $2^3 - 2 = 6$, 6 is divisible by 6

P_2 is true

Step 2 : Assume that P_n is true for k

$k^3 - k = 6m$, say

Step 3 : Proving that P_{k+1} is true

When $n = k + 1$, $n^3 - n = (k + 1)^3 - (k + 1)$

$$= k^3 + 3k^2 + 3k + 1 - k - 1$$

$$= (k^3 - k) + (3k^2 + 3k)$$

$$= (k^3 - k) + \frac{6(k^2 + k)}{2}$$

$$= (k^3 - k) + \frac{6k}{2}(k + 1)$$

which is divisible by 6 , since by P_k., $k^3 - k$ is divisible by 6

Step 4 : Since P_1 is true and from P_k, P_{k+1} is true, hence by PMI P_n is true for all positive integral values of n.

Example

Prove by the PMI that if

$$A = \begin{pmatrix} 2 & a \\ 0 & 1 \end{pmatrix} \text{ then } A^n = \begin{pmatrix} 2n & (2^n - 1)a \\ 0 & 1 \end{pmatrix}$$

<u>Solution</u>

Let P_n be the statement above

Step 1 : when $n = 1$ LHS = $\begin{pmatrix} 2 & a \\ 0 & 1 \end{pmatrix}$ RHS

$$= \begin{pmatrix} 2 & a \\ 0 & 1 \end{pmatrix}$$

P_1 is true

Step 2 : Assume P_k is true.

If $A = \begin{pmatrix} 2 & a \\ 0 & 1 \end{pmatrix}$ then $A^k = \begin{pmatrix} 2k & (2^k - 1)a \\ 0 & 1 \end{pmatrix}$

Step 3 : Proving that P $_{k+1}$ is true.

$$A^{k+1} = \begin{pmatrix} 2k & (2^k - 1)a \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & a \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 22^k & a^2 k + a(2^k - 1) \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 2^{k+1} & a(2.2^k - 1) \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 2^{k+1} & 2^{k+1} - a \\ 0 & 1 \end{pmatrix}$$

= which is true for n = k + 1

Step 4 :P_1 is true, P_k is true, P_{k+1} is true. Therefore P_n is true for an integer value of n.

Example

The function f is defined by f (x) = x + 1; x $\in \Re$

1. Write expressions for the compound functions f^2 (x) and f^3 (x)
2. Using one result from point (i), write down an identity for f^n (x)

Prove by induction that this is true for all positive integers n.

<u>Solution.</u>

$f^2(x) = f.f(x)$
$= x + 1 + 1$
$= x + 2$

$f^3(x) = f^2.\ f(x)$
$= x + 1 + 2$
$= x + 3$

hence: $f^n(x) = x + n$

p(n): $f^n(x) = x + n$
p(1): $f(x) = x + 1$
true
$p^{(k)}$: $f^k(x) = x + k$
p(k +1): $f^{k+1}(x) = x + k + 1$

Hence: $f^{k+1}(x) = f^k . f(x)$
$= x + 1 + k$
$= x + k + 1$

since p(1) is true and from p(k) , p(k + 1) is true , hence using the point 1. p(n) is true for all n integer.

Example

If f(n) = 3^{2n} +7. Show that f(n+1) – f(n) is divisible by 8 hence , prove by induction that 3^{2n} +7 is divisible by 8

Solution

i) f(n) = $3^{2n} + 7$
f(n+1) –f(n) =
$3^{2n+2} + 7 - 3^{2n-7}$
$3^{2n+2} - 3^{2n}$
$3^{2n}(3^2 - 1)$
8×3^{2n}
which divisible by 8
p(1) is true
assume for n =k: p(k) is true

f(k+1): 3^{2k+2}+7

3^{2k+2} 7-(3^{2k} +7)+3^{2k}

$8.3^{2k} + 3^{2k} + 7$

By 8 by hipothesis

Conclusion

Since p(1) is true and from p(k): p(k+1) is true hence p(n) is true for all values of n

Examination Type Questions

1. If y = sinx, find $y^{11} = d^2y/dx^2$, hence prove by induction that :

   $y^{2n} = d^{2n}y/dx^{2n} = (-1)^n \sin x$

2. If A = $\begin{pmatrix} 2 & -1 \\ 1 & 0 \end{pmatrix}$, prove by induction that

   $A^n = \begin{pmatrix} n+1 & -n \\ n & 1-n \end{pmatrix}$ for all values n

3. Prove by induction that :$4^n \geq 3n^2 + 1$, for all n

4. Show by induction or otherwise that $3^{2n} - 1$ is divisible by8 for all positive integer values of n.

5. Prove by induction that the following results true for all positive integers n.

   Given that y = xe^x, then $\dfrac{d^n y}{dx^n}$ = $(x + n)e^x$

6. Prove by mathematical induction that the following result is true for all positive integers n.

   $$\sum_{r=1}^{n}(r+2) = \frac{n(n+5)}{2}$$

7. Prove by induction that $\sum_{1}^{n}(r+1)(2^{r-1}) = n.2^n$

8. Prove 7^n (3n+1)-1 is always divisible by 9

9. Prove that f(n) = 2^{3n} +6 is always divisible by 7

10. Prove that 3^{2n} -5 is always divisible by 4

11. The function f is defined by f(x) = x+1 $x \in \Re$

12. Write expressions for the compound functions $f^2(x)$ and $f^3(x)$

    Using the results from point (i) write down an identity for $f^n(x)$ prove by induction that this is true for all positive integers

13. If M= $\begin{pmatrix} 2 & -1 \\ 1 & 0 \end{pmatrix}$,deduce that $M^n = \begin{pmatrix} n+1 & -n \\ n & 1-n \end{pmatrix}$ Prove this result by induction

14. Prove by induction that $\begin{pmatrix} 1 & a \\ 0 & a \end{pmatrix}^n = \begin{pmatrix} 1 & \dfrac{a^{n+1}-a}{a-1} \\ 0 & a^n \end{pmatrix}$ hence simplify $\begin{pmatrix} 1 & 2 \\ 0 & 2 \end{pmatrix}^6$

15. If y sinx, find y' and y y'' hence prove by induction that $\dfrac{d^{2n}y}{dx^{2n}} = (-1)^n \sin x$