'A' LEVEL

STATISTICS AND MECHANICS

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1.2 The need for data

The following examples refer to the need for data:

1. In Hwange National Park, as in most game reserves, management decisions have to be made from time to time: how many elephants, if any, should be culled year, whether an additional borehole is needed at a certain locality, whether or not to burn grass in a given area. All these decisions depend on the availability of data and tremendous effort is made to obtain good data by marking and tracking animals, by aerial and ground surveys, etc. A large quantity of data is collected in this way, and statistical analysis essential in order to "cool" it and "distil" the essential information out.
2. In Zimbabwe, a census is carried out after every 10 years, data is obtained about the number of school going children, literacy rate, number of people above a certain specific age, etc. The purpose of this is to plan a heard and allocate appropriate resources by the government.
3. In Zimbabwe pregnant (expecting mothers) are tested for H.I.V AIDS. Data obtained is used to measure the rate of H.I.V. infection among the Zimbabweans. Hence appropriate intervention strategies are implemented to bring down the rate
4. A manufacturing company wants to market a new product but it can be an expensive exercise if the public is not going to buy this new product. Hence, the company needs to do a market survey first, obtaining data about the preferences of at least a part of a public.

1.3 Methods of data collection

1. Face to face questionnaire interviewing

   Advantages

   - There is a high response rate

- The presence of the interviewer ensures that nobody else contributes to the answering of the questionnaire other than the specific subject
- Allows probing for further explanations
- Allows interviewer to observe subject gestures

Disadvantages

- Very expensive to carry out
- It is time consuming
- Possibility of introducing interviewer bias
- Usually people do not open doors to strangers
- Data not reliable

Telephone Interview

Advantages

- Savings in terms of money and time e.g. telemarketing is widely used in Zimbabwe by companies
- Data is generated at source and there is instant data entry at the time of interview
- Can cover a wider geographical area at the shortest time possible

Disadvantages

- Not all people have telephones and not all people are listed in the telephone directory
- It increases the rate of non- response i.e. difficult to establish rapport and trust with the interviewer
- It can be only used in countries with a well developed telephone network.

Self-administered questionnaires

These are posted to the subject to answer without supervision. Magazine houses, newspapers, and big stores like Edgars use this method

Advantages

- It is a relatively a cheaper method
- Cover a wider geographical area
- Greater accessibility to respondents I.e. those not at home will always get their mails
- Respondents feel more anonymous and hence they may respond to more sensitive questions

Disadvantages

- Low response rate usually people do not fill-in questionnaires in magazines and newspapers.
- Bias is introduced; usually the unintended subjects e.g. school children are the ones who are keen to fill in magazine questions.
- Assumes everybody is literate some questions are left unanswered.

Direct observation

It involves taking details as the subject behaves unknowingly e.g. espionage. Another example could be that of observing the number of people coming to a banking hall observing a machine churning out items the advantage of this method is that accurate results may be obtained. The disadvantage of this method is that it offers no time for probing.

Existing Records

Institutions like the Central Statistics Office, Museums, hospitals and other government departments keep records for use by people and other organizations. Statistics like the poverty datum line, wages, consumer price index etc can be obtained from the C. S O and labour unions and political parties usually use these for their own purposes. Statistics on people suffering from TB can be obtained from the hospitals. The advantage of this method is that the data is always available though in some cases the data is not problem specific

1.4 Types of Data

There are two types of data.

Categorical data (qualitative)

It only describes the attributes of an entity. For example shirts may be classified by colour and people may be classified by tribe, race etc.
Categorical data does not permit us to make statements about the ordering of members of the data or the number of this particular member is greater than or less than another. Usually bar graphs and charts are used to represent qualitative data.

Numerical data

It is data that assigns a numerical value on some numerical scale. Such data is collected either by measurement or by counting e.g. weights, mass, ages, etc
There are two types of numerical data

Continuous data

Continuous data
It takes any value in an interval and is obtained by measuring e.g. height: a person, who is 162 cm tall, can take any value between 161.5 to 162.5 cm. There are infinity values in between 161.5 and 162.5

Discrete data

Exact data
It can take only whole (integer) values and are obtained by counting e.g. the number of goals scored by Bosso against De Mbare in 2005 season.
Once data has been collected it has to be arranged, analysed and interpreted. In the following section we will look at different ways of representation of data.

1.5 Stem and leaf Plot

It is a device to group data while displaying most of the original data.
Each score is considered to have two parts i.e., a stem and a leaf. The leading digit(s) is called the stem and the subsequent digit(s) is called a leaf.
A stem and leaf plot should be always accompanied by a key, which enables to interpret it.

Example

The ages of the 50 university students are given below

64 50 23 24 19 72 75 65
61 59 33 34 59 18 16 55
32 31 38 26 29 52 28 66
67 56 58 68 72 17 20 23
55 56 $\bar{x}$ 48 46 45 49 63
52 27 38 35 43 59 68 48
50 51

Prepare a stem and leaf plot of the above data. Comment briefly on what your analysis shows

Solution

Highest score = 75
Lowest score = 16
We take the number of tens as stems and the number of units as leaves

Stem	Leaf
1	6 7 8 9
2	0 1 3 4 6 7 8 9
3	1 2 3 4 5 8
4	3 5 6 8 9
5	0 1 2 5 6 8 9
6	1 3 4 5 6 7 8
7	2 5

Key 6 | 1 means 61

A stem and leaf plot is not unique. There can be more than one plots displaying the same data

Comment

Most university students are aged between 50 and 59 inclusive

Advantages of using a stem and leaf plot

- It retains the original data for further analysis
- It can be used for comparing distributions of two or more data sets
- It is easy to visualize the skewness of the distribution

Disadvantages

- Difficult to construct for continuous data

Back to back stem and leaf plots

Example

The results of 24 students in the subject of mathematics for class A and Class B are as follows:

Class A: 30 47 40 46 43 35 60 55 51 48 63 66 84 82 80 81 76
76 75 78 69 75 77 85

Class B: 99 73 79 93 84 70 68 68 69 65 64 60 56 10 23 15 33
20 52 53 54 55 53 72

(a) Prepare a stem and leaf plot
(b) Which class has the greatest spread of observations?
(c) Which class had the better performing students?
(d) Which class had the poorer performing students?
(e) Which class did better overall?

Solution

(a)

Leaf	Stem	Leaf
	1	0 5
	2	0 3
5 0	3	3
8 7 6 3 0	4	
5 1	5	2 3 3 4 5 6
9 6 3 0	6	0 4 5 8 8 9
8 7 6 6 5 5	7	0 2 3 9
5 4 2 1 0	8	4
	9	3 9

Key 4 | 8 means 84   Key 3 | 9 means 39

(b) Class B has a greater spread with a range of 89; where as the range of A is 54.
(c) Some students in class B performed very well.
(d) Some students in class B performed poorly.
(e) Overall the performance of class A is better than that of class B

Example

Given the following stem and leaf plot.
Find the ringed values.

Stem	Leaf
1	3 4 5 6
2	0 1 2 3
3	4 5 6 7
4	0 0 2 3 3

If (a) 4 | 0 means 0.40

(b) 4 | 0 means 40

Solution

(a) 2 | 1 means 0.21 and 4 | 3 means 0.43

(b) 2 | 1 means 21 and 4 | 3 means 43

Example

The weekly wages, in thousands of dollars paid out by the company to an employee is given below.

375 484 548 623 735 828 946 483 668 642 781 857 587 645 792 620

Prepare a stem and leaf for the employee's wages.

Solution

Stem	Leaf
3	75
4	83 84
5	48 68 87
6	20 23 42 45
7	35 81 92
8	28 57
9	46

Key 7 | 35 means 735

Commas may be used to separate the leaves

1.6 Frequency Distributions

- A distribution is an arrangement of the observations in increasing or decreasing order of magnitude
- The frequency of a particular score is the number of times the score occurs in the data
- A frequency distribution is a tabular representation of the observations in ascending order of magnitude with their corresponding frequencies.

Example

Given the following data set.

5 6 5 4 7 8 5 4 8 6 6 5

Present the data set above in a frequency table.

Solution

Score (x)	Tally	frequency (f)
4	//	2
5	////	4
6	///	3
7	/	1
8	//	2
		12

Class Interval

- Class intervals are used to tabulate a larger data set.
- The frequency of a class interval is the number of observations which fall in the range specified by the interval

Standardized Frequencies

- Relative frequency = f/n_, where f is the frequency of a class interval and n is the number of observations
- Percentage frequency = f/n x 100
- Frequency density = $\frac{\text{frequency of class interval}}{\text{Width of class interval}}$

1.7 Histograms

A histogram is a graphical representation of a frequency distribution and is constructed in one of the following ways:

- The scores on the horizontal axis are plotted against the frequency or relative frequencies on the vertical axis in the form of rectangles.
- The class intervals on the horizontal axis are plotted against the frequency or relative frequency on the vertical axis in the form of rectangles.
- The class intervals on the horizontal axis are plotted against the frequency density on the vertical axis in the form of rectangles. This is used when the class intervals are not of equal width.

Note: No gaps are allowed between rectangles

Example

Fifteen pupils in class obtained the following scores.

2 3 1 5 3 3 2 3 4 4 5 5 2 3 1

(a) Using tally marks, present this information as a frequency distribution
(b) Construct a histogram
(c) Comment on the shape of the histogram

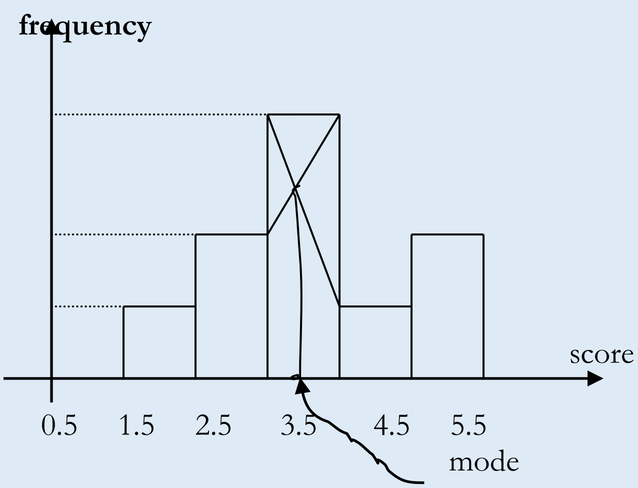
Solution

(a)

Score (x)	Tally	Frequency
1	//	2
2	///	3
3	~~////~~	5
4	//	2
5	///	3

(b)

Score	frequency
0.5 – 1.5	2
1.5 – 2.5	3
2.5 –3.5	5
3.5 – 4.5	2
4.5 – 5.5	3



(c) The histogram is almost symmetrical about the middle score. Thus the test is fair. The mode is 3

Note:

- Both the horizontal and vertical axes should be appropriately labeled
- If the observations are integers then each rectangle extends from one half below its observation (score) to one half above

Example

An actuary studying the causes of car accidents collected data over a period of one year. Along with some other variables he recorded the number of accidents according to age group . He compiled his results into a frequency distribution shown. Construct the histogram and super impose a frequency polygon. Estimate the mode using the histogram.

Age in years (x)	Number of accidents (f)
20 – 24	9
25 – 29	7
30 – 34	3
35 – 39	4
40 – 44	2
45 – 49	3
50 – 54	1
55 – 59	4
60 – 64	5
65 - 69	7

Solution

When constructing a histogram, use real limits

Age	No of accidents
19.5 – 24.5	9
24.5 – 29.5	7
29.5 – 34.5	3
34.5 – 39.5	4
39.5 – 44.5	2
44.5 – 49.5	3
49.5 – 54.5	1
54.5 – 59.5	4
59.5 – 64.5	5
64.5 – 69.5	7

When constructing a frequency polygon, join the middle points of the rectangles with line segments using a ruler and extend to the x – axis, alternatively class frequencies are plotted against class mid points

1.8 Cumulative Frequency Distribution

A cumulative frequency distribution is a tabular display of data showing how many scores lie below or above certain values, hence we talk of less than or more than cumulative frequency distributions.

- A less than cumulative distribution is obtained by adding each frequency to the sum of the preceding frequencies

- A more than cumulative distribution is obtained by subtracting each subsequent frequency from the previous sum of frequencies starting with the summation of frequencies.

Example

Using the following frequency distribution obtain:

(a) The less than cumulative frequency distribution

(b) more than cumulative frequency distribution

Score (x)	1	2	3	4	5	6	7	8	9
Frequency (f)	2	2	1	0	6	6	3	1	2

Solution

(a) Less than cumulative frequency distribution

Scores	< 0.5	<1.5	<2.5	<3.5	< 4.5	< 5.5	< 6.5	<7.5	< 8.5	< 9.5
Cum. Freq.	0	2	4	5	5	11	17	20	21	23

(b) More than cumulative frequency distribution

Scores	>0.5	>1.5	>2.5	>3.5	>4.5	>5.5	>6.5	>7.5	> 8.5	> 9.5
Comfrey.	23	21	19	18	18	12	6	3	2	0

1.9 Cumulative Frequency curve (O give)

The cumulative frequency curve is the graph of the cumulative frequency distribution. It is drawn either by:

i. Plotting the scores against the cumulative frequencies
ii. Plotting the scores against the percentage cumulative frequencies
iii. Plotting the scores against cumulative relative frequencies

Note: If a ruler is used to connect the points a frequency polygon is obtained.

The author recommends point three to be used . It makes it easier to estimate the percentiles once drawn. Cumulative relative frequencies are percentages given as a decimal e.g. 20% = 0.2

Uses of the cumulative frequency curves (ogive)

i. Determine how many observations lie below (above) a particular score.

ii. Read off directly the percentage of observations less (more) than any specified value, that is, it is used to estimate the percentiles e.g. the lower quartile, the median and the upper quartile

1.10 Percentiles

A percentile can be defined as a score below which lie a certain percentage of cases. The word percentile is derived from the word percent – percentage. A distribution can be divided into 100 equal parts called percentiles.

- The tenth percentile (P_{10}) is called a deciles
- The twentieth percentile (P_{20}) is called a second decile etc.
- The 25th percentile (P_{25}) is the lower quartile
- The 50th percentile (P_{50}) is the median or the 5th decile
- The 75th percentile (P_{75}) is the upper quartile

The percentiles are estimated using the ogive and are useful to describe the behaviour of the data set.

Example

In an accounting department 30 accounts were examined for errors. The information is compiled as follows;

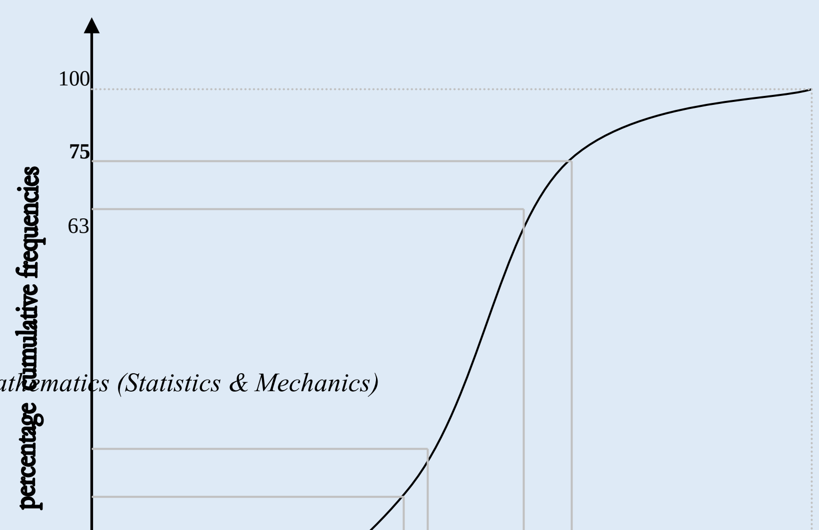
number of errors	1	2	3	4	5	6
number of accounts	4	5	8	6	4	3

(a) Set up a percentage cumulative frequency distribution
(b) Draw an ogive. Estimate the lower quartile, the median, the upper quartile, the fourth decile, the 63rd percentile and the semi – inter-quartile range.

Solution
(a)

Errors	< 0.5	< 1.5	< 2.5	< 3.5	< 4.5	< 5.5	< 6.5
Cum.freq.	0	4	9	17	23	27	30
% cum.freq.	0	13.3	30	56.7	76.7	90	100

(b)



50

40

25

0.5 1.5 2.5 3.5 4.5 5.5 6.5

number of errors

Reading off from the graph :

- the lower quartile (Q_1) = 2
- the median (Q_2) = 3
- the upper quartile (Q_3) = 4
- the fourth decile (D_4) = 2.9
- P_{63} = 3.9
- I.Q.R. = $Q_3 - Q_1$ = 2
- The semi inter quartile range is 1

Example

The following are the marks of 50 students marked out of 50.
18 14 28 25 20 32 22 17 5 30 40 31 7 23 26 20 30 11 28 25 19 39 15 24 19 27 23 23 9 28 37 18 24 20 21 25 17 37 12 38 22 30 28 16 16 29 8 22 23 39

(a) Taking class intervals of length 5, present this set of marks as a frequency table
(b) Prepare a stem and leaf plot
(c) Construct an ogive curve and estimate the quartiles
(d) Construct a histogram and estimate the mode. What is the disadvantage of using a histogram as away of representing data?

Solution

(a) The lowest mark = 5
The highest mark = 40

Marks (x)	Tally	f
5 – 9	////	4
10 – 14	///	3
15 – 19	~~////~~ ////	9
20 – 24	~~////~~ ~~////~~ ///	13
25 – 29		10

30 – 34	~~||||~~ ~~||||~~	5
35 – 39	~~||||~~	5
40 - 49	||||	1
	/	

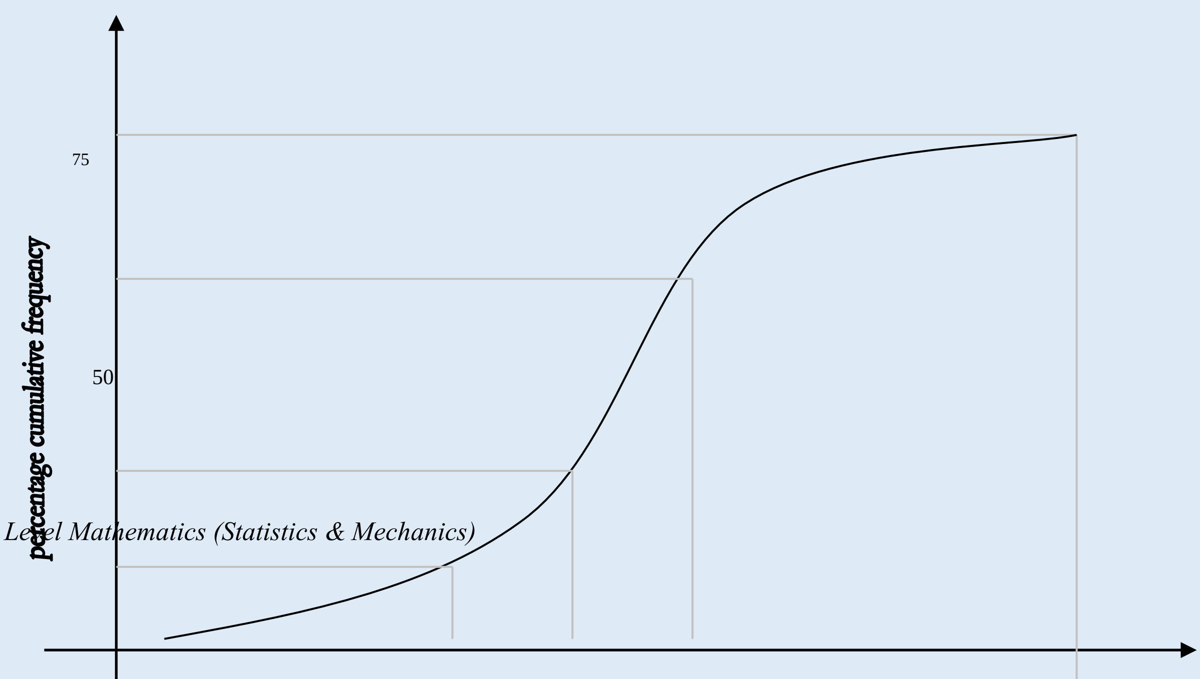
(b)

Stem	Leaf
0	5 7 8 9
1	1 2 4
1	5 6 6 7 7 8 8 9 9
2	0 0 0 1 2 2 2 3 3 3 3 4 4
2	5 5 5 6 7 8 8 8 8 9
3	0 0 0 1 2
3	7 7 8 9 9
4	0

key 2 | 2 means 22

(c)

marks	< 4.5	< 9.5	<14.5	< 19.5	< 24.5	< 29.5	< 34.5	<39.5	<44.5
c.f.	0	4	7	16	29	39	44	49	50
% c.f	0	8	14	32	58	78	88	98	100



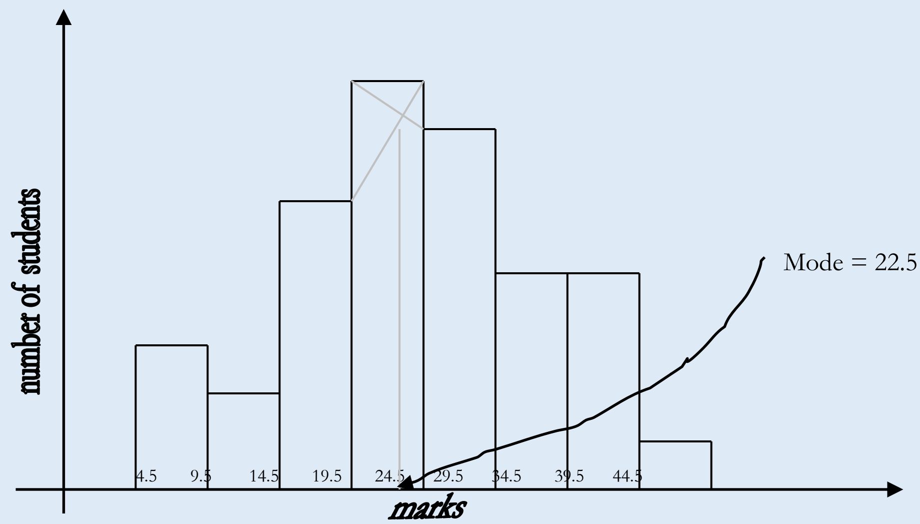
25

4.5 9.5 14.5 19.5 24.5 29.5 34.5 39.5 44.5

marks

- the lower quartile is 18
- the median is 23
- the upper quartile is 28.5

(c)



The disadvantage of a histogram in representing data is that, original data is lost through use of intervals No further "cooking" of data can be done

1.11 Measures of Central Tendency

There are many measures of central tendency (location). However in this course we will only discuss the three primary measures of central tendency, which are the arithmetic mean, the median and the mode. These statistics are generally used in every day life. They are also referred to as averages. Since they tend to summarise the entire data set.

1.12 The Mean $\bar{x}$ or U

This is the common average, that we are accustomed to and is denoted by ξ

$$\textbf{mean} = \frac{\textbf{sum of all measurements}}{\textbf{number of measurements}}$$

$$\bar{x} = \frac{1}{n}\sum_{i=1}^{n} X_i$$

The mean of ungrouped data

The formula is $\bar{x} = \frac{1}{n}\sum_{i=1}^{n} X_i$

Example

The following are marks obtained by 10 students in a test out of 10. Find the mean.

3 1 4 0 8 2 7 9 8 3

Solution

$$\bar{x} = \frac{1}{n}\sum_{i=1}^{n} X_i$$

n = 10

$\sum x = 3 + 1 + 4 + 0 + 8 + 2 + 7 + 9 + 8 + 3 = 45$

$$\bar{x} = \frac{45}{10}$$

$$\bar{x} = 4.5$$

It is general tedious to manually add large data sets. In such cases, a calculator is becomes handy. The author uses Sharp EL - 520L ADVANCED D. A.L. Twin Power. There are more advanced calculators than this one but this particular one will cater for all your statistical computing for this course. It is very important that you know how to use your calculator. All calculators come with manuals for help. A cell phone is more complicated to use than a calculator

The mean of grouped data

The mean of a frequency distribution is given by

$$\bar{x} = \frac{\sum fx}{\sum f}$$

Example

The marks obtained by 20 students in a multiple choice test are given as follows
3 1 6 0 4 4 1 5 2 3 3 3 1 4 3 3 4 2 3 3

Find the mean mark.

Solution

Mark (x)	Tally	Frequency (f)	fx
0	/	1	0
1	///	3	3
2	//	2	4
3	~~////~~ ///	8	24
4	////	4	16
5	/	1	5
6	/	1	6
	/		
		20	58

$$\bar{x} = \frac{\sum fx}{\sum f} = \frac{58}{20} = 2.9,\ \underline{\text{The mean mark is 2.9}}$$

The only draw back of the arithmetic mean is that it is sometimes an unreliable measure for location e.g. in a shoe shop, two customers bought size four, four bought size three, five bought size five and four bought size six. The mean size bought is 4.6. In this context this is a meaningless size otherwise the average size is five. The mean is also affected by extreme values; extreme values in this case means either too big or very small e.g. in a class of seven students, the following are their ages: 15 15 16 16 17 15 40. The mean age of this class is 19 years. But this does not reveal the true information about the group. The appropriate average for this group is 15.5

1.13 The Median

If the data set is arranged in ascending or descending order of magnitude, then the median is the middle score.

The median for ungrouped data

Given the following data sets:

a. 3 2 5 6 4
b. 4 7 5 1 2 0

Find the median

Solution

a. arrange the data in ascending order

2 3 4 5 6

n = 5 … odd number, then

median = $y_{\frac{1}{2}(n+1)} = y_{\frac{1}{2}(5+1)} = y_3 = 4$

The median is 4

b. 0 1 2 4 5 7

n = 6 ….. even number, then

$$\text{median} = \frac{1}{2}\left(y_{n/2} + y_{\frac{1}{2}(n+2)} \right)$$
$$= \frac{1}{2}(y_3 + y_4)$$
$$= \frac{1}{2}(2 + 4)$$
$$= 3$$

The median is 3

$$\text{Median} = \begin{cases} y_{\frac{1}{2}(n+1)}, & \text{for } n \text{ odd} \\ \frac{1}{2}\left(y_{n/2} + y_{\frac{1}{2}(n+2)} \right), & \text{for } n \text{ even} \end{cases}$$

1.14 The median for grouped data

If data is given in class intervals, the median can be read off the give . Also interpolation can be used. The formula for interpolation is:

$$\textbf{Median} = L_m + c_m \frac{\left[\frac{1}{2n} - F_{m-1}\right]}{f_m}$$

where, L_m is the lower limit of the median class
½ n is the position of the median
f_m is the frequency of the median class
F_{m-1} is the cumulative frequency of the class just before the

median class
c_m is the width of the class
c_m = upper real limit – lower real limit

Example
Given the following table.

x	3	4	5	6	7
f	4	3	2	3	1

Find the median

Solution
n = ∑f = 4 + 3 + 2 + 3 + 1 = 13 .. odd number , then
median = $y_{½(n+1)}$ = y_7 = 4, 4 occupies the seventh position
Example
Find the median weight of the following group of students.

Weight (kg)	50 – 54	55 – 59	60 – 64	65 – 69
Number of students	4	5	7	6

Solution
We may plot an ogive and read off the median from the curve. Let us use the interpolation formula

$$\textbf{Median} = L_m + c_m \frac{\left[\frac{1}{2n} - F_{m-1}\right]}{f_m}$$

½n = ½(4 + 5 + 7 + 6) = 11, the 11th position falls in the class 60 – 64, then 60 – 64 is the median class – converted to real limits it becomes 59.5 – 64.5, hence
L_m = 59.5, f_m = 7, F_{m-1} = 4 + 5 = 9 and c_m = 64.5 – 59.5 = 5
Substituting

$$\text{Median} = 59.5 + 5\,\frac{[11-9]}{7}$$

$$= 60.9$$

1.15 The Mode

The mode is the score, which occurs most often. A given data set could be a uni-modal, bi modal or multi modal or it does not have a mode altogether. If a grouped data is given in class intervals, the mode can be read off the histogram. Interpolation can also be used. The formula is:

Mode = L_m + $\underline{c_m\ [f_m - f_{m-1}]}$

$$[\, 2f_m - f_{m+1} - f_{m-1}]$$

Where, L_m is the lower limit of the modal class
f_m is the frequency of the modal class
f_{m+1} is the frequency of the class immediately above the modal class
f_{m-1} is the frequency of the class immediately below the modal class
c_m is the class width of the modal class

Example
Given the following data.

x	3	4	5	6
frequency	7	3	4	5

Find the mode.

Solution
The mode is 3. It has the highest frequency.

Example
The percentage marks of a thousand candidates in Zimsec examinations were grouped into class intervals of ten marks. The distribution of percentage marks is given in the following table.

mark	0 –9	10 – 19	20 - 29	30 - 39	40 - 49	50 - 59	60 - 69	70 – 79	80 - 89	90 - 99
Fre.	20	60	100	200	260	199	80	35	25	21

Find the mode

Solution
A histogram can be constructing, and the mode read off the diagram . Using interpolation

$$\textbf{Mode} = L_m + \frac{c_m\,[f_m - f_{m-1}]}{[\, 2f_m - f_{m+1} - f_{m-1}]}$$

The modal class is 40 – 49, which is 39.5 – 49.5, then :
$C_m = 49.5 - 39.5 = 10$
$F_m = 260$
$F_{m+1} = 199$
$F_{m-1} = 200$

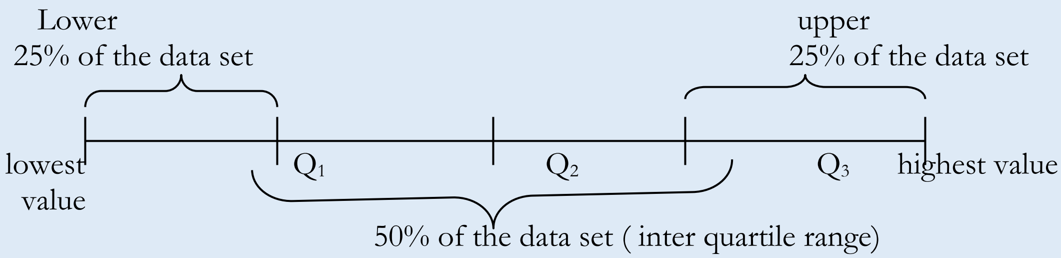
<u>Substituting</u>

$$\text{Mode} = 39.5 + \frac{10\,[260 - 200]}{[\,2(260) - 199 - 200]}$$

= 44.5

1.16 The Quartiles

These are primary measures of position and they are the lower quartile (Q_1) , the second quartile (Q_2), which is the median and the upper quartile (Q_3)



The Lower Quartile (Q_1)

If a data set is ordered in ascending or descending order, the lower quartile is the value Q_1 that has 25% of the observations below or equal to it. It can be noted from the diagram above that Q_1 is the median of the lower half.

The Upper Quartile (Q_3)

If a data set is ordered in the ascending or descending order, the upper quartile (Q3) is the value Q_3 that has 75% of the observations below or equal to it. It can be noted from the diagram above that Q_3 is the median of the upper half.

Example

Given the following data sets.

3 1 3 4 1 4 6 5

4 0 1 3 5 5 2

Find Q_2, Q_1 and Q_3

Solution

Method 1

1 1 3 3 4 4 5 6

$n = 8$

$Q_2 = ½ (y_4 + y_5) = ½ (3 + 4) = 3.5$

Taking the lower half: 1 1 3 3

$n = 4$

$Q_1 = ½ (y_2 + y_3)$

$= ½(1 + 3)$

$= 2$

Taking the upper half : 4 4 5 6

$n = 4$

$Q_3 = ½(y_2 + y_3)$

$= ½ (4 + 5)$

$= 4.5$

0 1 2 3 4 5 5
$n = 7$
$Q_2 = y_4 = 3$
Taking the lower half: 0 1 2
$Q_1 = 1$
Taking the upper half: 4 5 5
$Q_3 = 5$

Method 2

1 1 3 3 4 4 5 6
For Q_1: $k = 0.25$, $n = 8$, then $nk = 0.25 \times 8 = 2$ .. ..an integer, hence the position of Q_1 is $(2 + 0.5)^{th} = 2.5^{th}$
$Q_1 = ½ (1 + 3) = 2$
For Q_2 : $k = 0.5$, $n = 8$, then $nk = 0.5 \times 8 = 4$ ……an integer,
hence $i = 4 + 0.5 = 4.5^{th}$
$Q_2 = ½ (y_4 + y_5)$
$= ½ (3 + 4)$
$= 3.5$
For Q_3 : $k = 0.75$, $n = 8$, then $0.75 \times 8 = 6$ …. an integer,
$i = 6.5^{th}$
$Q_3 = ½ (y_6 + y_7)$

$= ½ (4 + 5)$
$= 4.5$

0 1 2 3 4 5 5
For Q_1: $k = 0.25$, $n = 7$, then $nk = 0.25 \times 7 = 1.75$ .not an integer, hence $i = 2$
$Q_1 = y_2$

For Q_2 : $k = 0.5$, $n = 7$, then $nk = 0.5 \times 7 = 3.5$ ..not an integer,
hence $i = 4$
$Q_2 = y_4 = 3$

For Q_3 : $k = 0.75$, $n = 7$, then $0.75 \times 7 = 5.25$ …not an integer,
$i = 6$, $Q_3 = y_6 = 5$
Example

Find the lower quartile, the median, the upper quartile and the inter quartile range of the following group of students.

Height (x)	150 – 154	155 – 159	160 – 164	165 – 169	170 –174	175 - 179
Frequency	5	10	15	20	8	2

Solution
A cumulative frequency curve can be plotted and the quartiles can then be read off the curve. Interpolation can be used. The formulae are:

$$Q_1 = L_q + \frac{c_q\left(\frac{1}{4}n - F_{q-1}\right)}{f_q}$$

$$Q_3 = L_q + \frac{c_q\left(\frac{3}{4}n - F_{q-1}\right)}{f_q}$$

where, c_q is the class width of the quartile class
f_q is the frequency of the quartile class
F_{q-1} is the cumulative frequency of the class up to but excluding the quartile class.

$n = \sum f = 5 + 10 + 15 + 20 + 8 + 2 = 60$
$\frac{1}{4}n = 15^{th}$; Q_1 falls in the interval 155 – 159, then $L_q = 154.5$,
$c_q = 159.5 - 154.5 = 5$, $f_q = 10$, $F_{q-1} = 5$, hence

$$Q_1 = 154.5 + \frac{5(15-5)}{10}$$
$$= 159.5$$

$\frac{1}{2}n = 30^{th}$: Q_2 falls in the interval 160 – 164, then $L_m = 159.5$, $c_m = 5$, $f_m = 15$,

$F_{m-1} = 15$, hence

$$\text{Median} = 159.5 + 5\frac{(30-15)}{15}$$
$$= 164.5$$

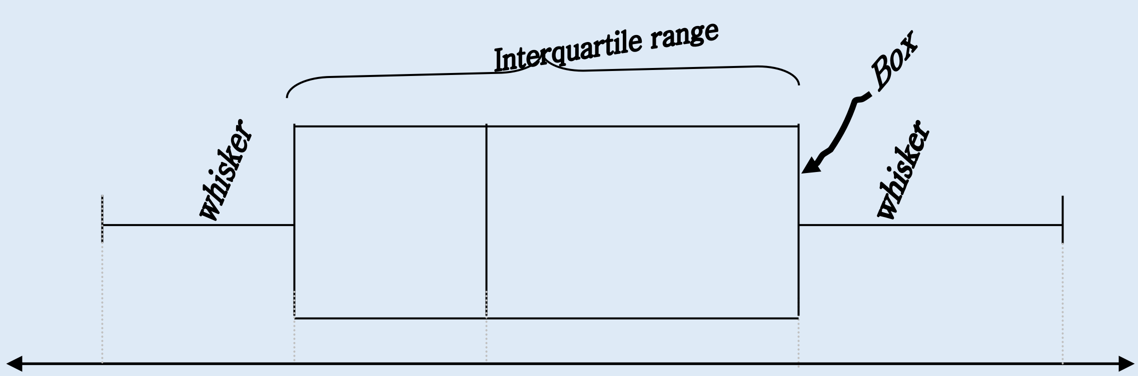
$\frac{3}{4}n = 45^{th}$: Q3 falls in the interval 165 – 169, then

$$Q3 = 164.5 + \frac{5(45-30)}{20}$$
$$= 168.25$$

The inter quartile range is $Q_3 - Q_1 = 168.25 - 159.5 = 9$

1.17 The Box and Whisker plot

A box plot is a device used to illustrate the range, median, quartiles and the inter quartile range of data. To plot a box plot you need a scaled line segment and the box plot can be plotted either horizontal or vertical.



Q1 Q2 Q3

lowest score

highest score

Range

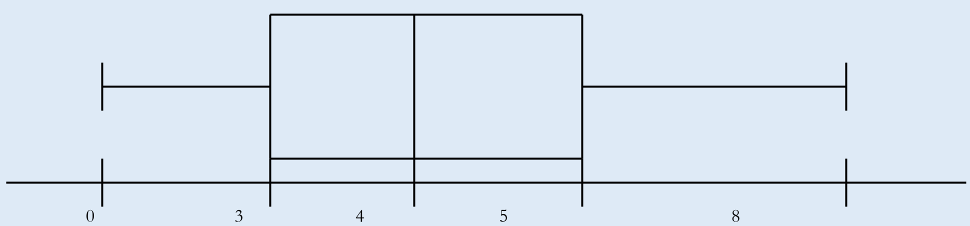
The inter quartile range is represented by the length of the box and the range is represented by the total length of the diagram.

1.18 Uses of Box plots

- Box plots can be used to compare two or more sets of data by drawing the box plots underneath each other or side by side
- Box plots can be used to determine skew-ness(shape of data) of the distribution
  (i) If $Q_3 - Q_2 > Q_2 - Q_1$, the distribution is positively skewed
  (ii) If $Q_3 - Q_2 < Q_2 - Q_1$, the distribution is negatively skewed
  (iii) If $Q_3 - Q_2 = Q_2 - Q_1$, the distribution is symmetric

Example

A computer was used to simulate the contents of 50 packs of cards. The results are shown in the box plot below.



number of cards

Use the information in the box plot to find the percentage of all the packs of cards that will contain the following number of cards:

i) Three cards or less
ii) four cards or less
iii) five cards or less

Solution

i) $P(X \leq 3) = 0.25$

ii) $P(X \leq 4) = 0.5$
iii) $P(X \leq 5) = 0.75$

1.19 Measures of Variation (Dispersion)

When interpreting a data set, it is vital to know its spread (dispersion) about a measure of location. This measure of spread quantifies how representative is the measure of location of a data set. For example in assessing the achievement of the students in a certain examination, we calculate the mean. But if the mean is not accompanied by a measure of spread, the assessment will be not accurate Common measures of dispersion are; the range, inter quartile range and the standard deviation.

The Range

The range is the difference between the largest value and the smallest value of the given set of data

R=max value- min value

Example

In a mathematics test, marls obtained by the students are as follows.

50 70 51 49 80 99

Find the range

Solution

Highest value = 99
Lowest value = 49
Range = 99 – 49 = 50

Example

Given the following frequency distribution. Find the range.

Mark	**frequency**
10 – 19	**4**
20 – 29	**5**
30 – 39	**2**
40 - 49	**7**

Solution

Range = 49.5 – 9.5 = 40

Example

Given the following distribution. Find the range.

Mark	Frequency
24 – 25	4
25 – 30	5

30 – 35	0
35 - 40	1

Solution

Range = 40 – 24 = 16

Clearly the range of marks is not a really good measure of the spread, as it depends only on the highest and lowest marks, and not on the spread of marks among the majority of candidates. However it is largely used in quality control, for small samples, because of its simplicity.

The inter quartile Range
It describes the spread of the middle half of the observations. It helps to quantify how representative is the median of the data set.
I.Q.R. = $Q_3 - Q_1$

1.20 The Variance

The variance of ungrouped data

Consider the deviations x_i - ξ, naturally if $x_i - \bar{x}$ is very small, x_i is very close to ξ, hence ξ may substitute x_i. If we square up $(x_i - \bar{x})$, i.e. $(x_i - \xi)^2$ and if x_i - ξ, is very small then $(x_i - \bar{x})^2$ is far smaller than $x_i - \bar{x}$, hence x_i are clustered around $\bar{x}$, similarly if $x_i - \bar{x}$

is very big, then $(x_i - \xi)^2$ is far bigger than x_i - ξ, in this case ξ is not representative of the data set. The objective is to find the average of $(x_i - \xi)^2$ i .e

$$\frac{\sum (x_i - \overline{x^2}}{n}$$

This new statistic is called a variance and is denoted by:

$$\frac{S^2 \sum (x_i - \overline{x^2}}{n}$$

Example
Calculate the variance of the following data set
3 0 4 3 5 1

Solution

$\bar{x} = \frac{\sum x}{n}$ $\quad\quad$ n = 6, $\sum x$ = 3 +0 + 4 + 3 + 5 + 1 =16

$\bar{x}$ = 1/6(16) = 8/3

x	$x_i - \bar{x}$,	$(x_i - \bar{x},)^2$
3	3 – 8/3 =1/3	$(1/3)^2$ = 1/9
0	- 8/3	64/9
4	4 - 8/3 = 4/3	4/3 = 16/9
3	3 – 8/3 = 1/3	$(1/3)^2$ = 1/9
5	5 – 8/3 = 7/3	$(7/3)^2$ = 49/9
1	1 – 8/3 = - 5/3	$(5/3)^2$ = 25/9
		156/9

S^2 = 1/6 (156/9) = 2.889 to 2 d.p

The convenient formula for calculating the variance is thus derived:

$$S^2 = 1/n \sum(x_i - \bar{x})^2$$
$$= 1/n \sum\{ x_i^2 - 2\bar{x} x_i + \bar{x}^2 \}$$
$$= 1/n \{ \sum x_i^2 - 2\bar{x} \sum x_i + \sum \bar{x}^2 \}$$
$$= 1/n\{ \sum x_i^2 - 2 \bar{x} n\xi + n\bar{x}^2\}$$
$$= 1/n \sum x_i^2 - \bar{x}^2 \text{ ,i.e.}$$
$$\mathbf{S^2= 1/n \sum x_i^2 - \bar{x}^2}$$

Referring to the previous example and using the above formula calculate S^2,

$\sum x^2 = 3^2 + 0^2 + 4^2 + 3^2 + 5^2 + 1^2 = 60$

$S^2 = 60/6 – (8/3)^2$
$= 10 – 64/9$
$= 2.889$

The variance of a grouped data

The formula for the variance of grouped data is given by:

$$S^2= 1/n \sum f x^2 - \bar{x}^2$$

where n = $\sum f$ and $\bar{x}$ = 1/n $\sum fx$

Example

The following heights are given to the nearest cm of 20 students

Height (cm)	Number of students
140 – 144	4
145 – 149	3
150 – 154	5
155- 159	8

Calculate the variance and the standard deviation

Solution

Height (h)	h_i	f_i	$f_i \times h_i$	h_i^2	$f_i \times h_i^2$
140 – 144	142	4	568	20164	80656
145 – 149	147	3	441	21609	64827
150 – 154	152	5	760	23104	115520
155- 159	157	8	1256	24649	197192

$\sum f = 20$, $\sum fh = 3025$, $\sum fh^2 = 458195$

$S^2 = 1/n \sum fx^2 - \bar{x}^2 = 458195/20 - (3025/20)^2 = 33.1875$

Standard deviation $= \sqrt{S^2} = S$, hence $S = \sqrt{33.1875} = 5.76$

1.21 Examination Type Questions

1. The marks of 500 candidates in Zimsec examinations were grouped into class intervals of 10 marks. The distribution is given in the table below.

Marks	frequency
0 – 9	25
10 – 19	37
20 – 29	60
30 – 39	65
40 – 49	81
50 – 59	80
60 – 69	61
70 – 79	55
80 – 89	28
90 - 99	8

i) Set up a cumulative frequency distribution

ii) Draw a cumulative frequency curve

iii) Find (a) the 25th percentile

(b) the 75th percentile
(c) the inter quartile range

iv) Estimate the median mark from the ogive

v) Draw a box plot

vi) Find the cut off mark necessary to:

(a) pass 80% of the candidates

(b) award a grade A to 10% of students

2. Twenty-four students in a random sample were asked to record their statistics marks. The marks are shown below.

67 76 85 42 93 48 93 46 52 63 70 72 44 66 87 78 47 66 50 72
82 56 58 44

(a) Construct a stem and leaf plot to represent the data set.

(b) Using a scale of I cm to 10%, draw a box and whisker plot to represent the data set, and comment on the symmetry of the distribution.

(c) Give one advantage of using

a. (i) a stem and leaf plot (ii) a box and whisker plot.

(d) Using the classes 40-44, 45-50,50-54 etc. Construct

(i) the frequency table (ii) the histogram

(iii) The frequency polygon

(e) estimate the mean.

(f) estimate the variance and the standard deviation

(g) Construct a lee than cumulative frequency table.

(h) draw an ogive and use it to estimate the median, the lower quartile, the upper quartile and the quartile range.

3. Below is a stem and leaf plot for the ages of people who went to a movie theatre.

stem	leaf
0	7 9
1	1
2	0 0 5
3	8
4	1

Key 3|1 means 31

(a) state one advantage of using a stem and leaf plot over a histogram.

(b) state the mode and the median.

(c) calculate the mean age of the people.

4. Pookie, Yandiswa's dog, has fleas. Her owner counts the number of fleas on Pookie each day for 25 days. The results are tabulated below.

Number of fleas (x)	0	1	2	3	4	5
Number of days Yandiswa had this number of fleas (f)	2	3	2	5	8	5

(a) find the proportion of days on which Yandiswa observed more than three fleas.
(b) find the mean number of fleas per day that Pookie had.

5.A marketing consultant observed 50 consecutive shoppers at a grocery store. One variable of interest was how much each spent in the store. The following data (in hundred thousands of dollars) gives the amount spent by each shopper.

37 2 63 20 14 33 8 12 6 9 64 30 6 10 11 19 20 23 26 20 15 40 15
38 19 20 33 34 43 61 20 52 18 11 13 16 17 21 45 29 27 39 12 31 32
39 14 13 14 28 18 36

(a) construct a stem and leaf plot
(b) Calculate: i) mean ii) mode iii) median iv) range v) lower quartile vii)upper quartile (vii) variance and viii) standard deviation of the shoppers.
(c) comment on the distribution of the data set.
(d) Group the data into classes 0-9, 10-19, 20-29 etc., until all the values have been accounted for.
(e) using the grouped data, obtained in part (d) construct an ogive and hence determine :
(i) the median (ii) the lower quartile (iii) the upper quartile
(iv) the inter-quartile range (v) estimate the mean

CHAPTER 2

PROBABILITY

OBJECTIVES
CALCULATE PROBABILITIES OF SIMPLE AND COMPOUND EVENTS
CALCULATE PROBABILITIES USING A TREE DIGRAM
CALCULATE CONDITIONAL PROBABILITIES

2.1 Introduction

Probability theory is fundamental to the area of statistical inference. Inferential statistics deals with generalising the behaviour of random variables from sample findings to the entire population. Probability theory is used to quantify the uncertainties involved in making these generalizations.

Definition.
A probability is the chance, or likelihood, of a particular outcome of a number of possible outcomes occurring at a given event.

2.2 Types of probabilities

There are two main types of probabilities:

1. s**ubjective** i.e., this type cannot be verified. It is based on guesses.

Example 1

What is the chance of you passing "A" level mathematics? It is difficult for you to know exactly what chances you have for passing "A" level mathematics

2. **Objective** i.e. this type can be known a prior, and can be calculated.

Example 2

Consider tossing a fair die once. The possible outcomes that can occur are 1, 2, 3, 4, 5, or 6.
Tossing a coin is an example of a statistical experiment, the outcomes are uncertain. The set of all possible outcomes of an experiment is called a sample space and is denoted by S and each individual outcome is called a sample point or simply a singleton event.
A set that contains more than one sample points is called an event and is denoted by a capital letter e.g. A, B, C, e.t.a.

$$P(A) = \frac{number.of\ .possible.outcomes}{total.outcomes}$$

A = event.

n (A) = number of possible outcomes in A of event A.
n (S) = total number of possible outcomes.
p (A) = probability of event A happening.

Example 3

If, S = { 1,2,3,4,5,6 }
A = {1,2 } B = { 4, 2, 6 } C = {1, 2,4,5} D = { 2, 3, 6 }

Find (i) p (A) (ii) p (B) (iii) p(C) (iv) p (D)

Solution.

(i) p (A) = n(A)/n(S)

$= \frac{2}{6}$

$= \frac{1}{3}$

(ii) p(B) = $\frac{n(B)}{n(S)}$

= 3/6

= ½

(iii) p(C) = n(C) /n(S)

$= \frac{4}{6}$

$= \frac{2}{3}$

(iv) p(D) = $\frac{n(D)}{n(S)}$

= 3/6

= ½

Note.

- P(S) = 1
- P(ϕ) = 0
- Probability is a measure from $0 \leq P(E) \leq 1$
  :$0 \rightarrow$ uncertain
  $1 \rightarrow$ certain

S and ϕ are called certain and impossible events respectively.

Example

Tossing a coin once. What is the probability of obtaining a head or a tail?

Solution.

S = { H ; T }

Hence p(S) = $\frac{1}{2} + \frac{1}{2} = 1 \textit{certain}$

p(S) = 1

Example

Tossing a die once, what is the probability of obtaining a number greater than 7?

Solution.

The event is impossible, hence , p(ϕ) = 0

Addition rule.

- P(A or B) = p(A) + p(B) – p(A ∩B).
- If A ∩ B = ϕ, then;
- P(A or B) = P(A) +P(B), A and B are said to be mutually exclusive events, they cannot happen at the same time.

Example

A candidate writes two examinations , Maths and Physics. The candidate assesses the chance of passing examination A at 0.30 , of passing examination B at 0.40 and of passing both at 0.10. What is the probability that the candidate passes examination Maths or Physics?

Solution.

P(A or B) = P(A) +P(B) –P(A ∩ B)
= 0.30 + 0.40 – 0.10
= 0.60

Example

A bag contains 3 green , 5 blue and 7 red balls . one ball is drawn at random from the bag. What is the probability that it is,

a) green? b) green or red? c) black?

Solution.

Total numbers of balls = 3 + 5 + 7 =15

S = { G,G, B,B,B,B,B, R,R,R,R,R,R,R }

G = green, B = blue R = red

hence, n(S) = 15

a) p(green) = p(G) = $\frac{2}{15}$

b) p(G or R) = p(G) + p(R)
$= \frac{2 + 7}{15}$
$= \frac{9}{15}$
$= \frac{3}{5}$

c) p(black) = 0

Example

An ordinary pack of playing cards contains 13 diamonds , 13 hearts, 13 clubs and 13 spades. There are 26 red cards and 26 black cards. The cards are numbered from one(ace) to eleven (11) and contain 4 As ,4Qs , and 4Ks. A card is selected at random , what is the probability that it is;

a) An ace? b) an ace and a red card?
c) A black card and a king (K)?

Solution.

Total number of cards = 52.

a) p(an ace) = $\frac{4}{52}$, ***there are 4 aces in a deck***.

$= \frac{1}{52}$

b) p(an ace and a red card) = p(a red ace), ***there are 2 red aces.***

$= \frac{2}{52}$

c) p(a black card or a king (K) =

= p(a black card) + p(a king (k) – p(a black card and a king(K))

$= \frac{26}{52} + \frac{4}{52} - \frac{2}{52}$

$= \frac{28}{52}$

$= \frac{7}{13}$

2.3 Multiplication law.

P(A and B) = P(A) x P(B), if A and B are independent events, i.e. they may happen at the same time.

Example

A biased coin is tossed twice. Let the probability of obtaining a head be 3/5. What is the probability that;

a) The first toss gives a head and the second toss gives a tail?
b) Both tosses give a head?

Solution.

a) p(H and T) = p(H) x p(T)

$$= \frac{3}{5} \times \frac{2}{5}$$

$$= \frac{6}{25}$$

b) p(H and H) = p(H) x p(H)

$$= \frac{3}{5} \times \frac{3}{5}$$

$$= \frac{9}{25}$$

2.4 Complementary events.

Let A be an event , hence, the event A^1 is called the complementary event of A.
Then, $A \cup A^1 = S$ and $A \cap A^1 = \phi$

Hence, A and A^1 are mutually exclusive events.

$P(A \text{ or } A^1) = p(A) + p(A^1)$
$= 1$

hence, $\mathbf{p(A) = 1 - p(A^1)}$

Example

The probability that a teacher arrives late for a session is 0.74. What is the probability of the teacher arriving on time for the session?

Solution.

The events ''arrive on time'' and ''late arrival'' are complementary.

hence, P(arrive on time) = 1 – P(late arrival)
= 1 – 0.74
= 0.26

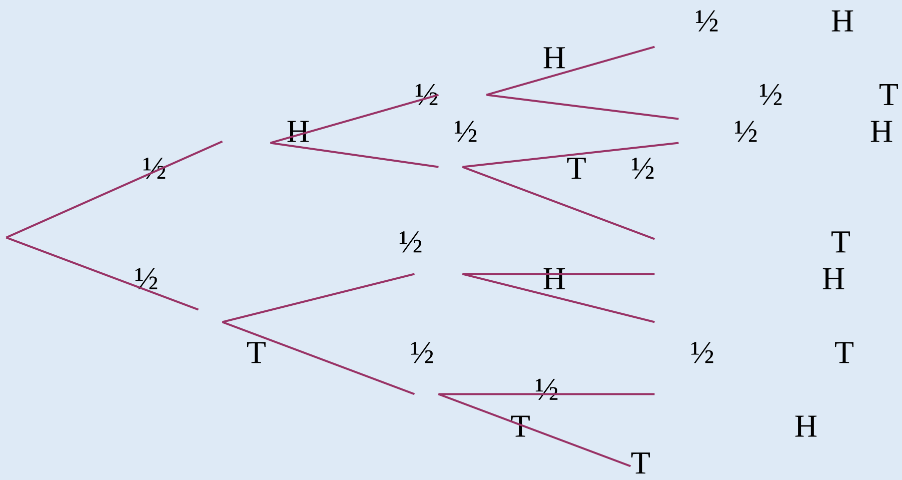
2.5 Probability tree diagrams

A probability tree diagram depicts events or sequences of events as branches of a tree. Each branch of the tree is labeled to indicate which event it represents. Along each branch, the probability of the event's occurrence is given.

Example

Tossing a coin three times. What is the probability of obtaining 2 heads?

Solution



Solution.

P(2 heads) = p(HTH) + p(HHT) + p(THH)

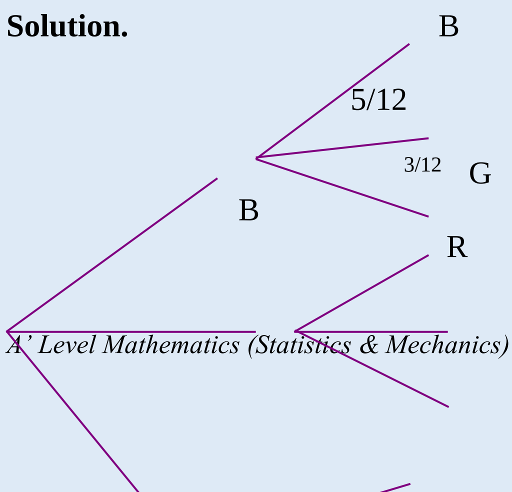
$= \frac{1}{8} + \frac{1}{8} + \frac{1}{8}$

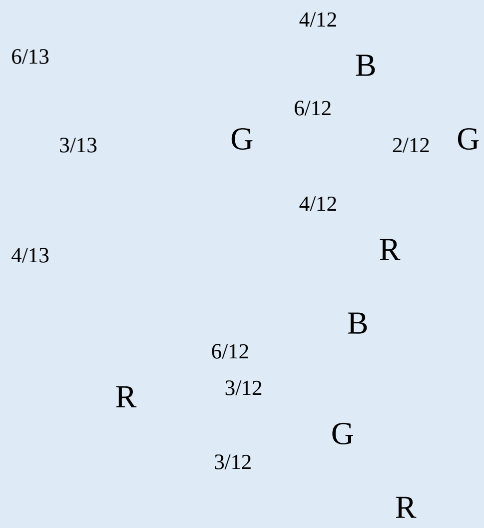
$= \frac{3}{8}$

Example

A bag contains six black balls, three green balls and four red balls. Two balls are selected at random without replacement. What is the probability of at least 1 black ball being selected

Solution.





$$BB = \frac{6}{13} \times \frac{5}{12} = \frac{5}{26}$$

$$BG = \frac{6}{13} \times \frac{3}{12} = \frac{3}{26}$$

$$BR = \frac{6}{13} \times \frac{4}{12} = \frac{2}{13}$$

$$GB = \frac{3}{13} \times \frac{4}{12} = \frac{1}{13}$$

$$GG = \frac{3}{13} \times \frac{2}{12} = \frac{1}{26}$$

$$GR = \frac{3}{13} \times \frac{4}{12} = \frac{1}{13}$$

$$RB = \frac{4}{13} \times \frac{6}{12} = \frac{2}{13}$$

$$RG = \frac{4}{13} \times \frac{3}{12} = \frac{1}{13}$$

$$RR = \frac{4}{13} \times \frac{3}{12} = \frac{1}{13}$$

P(at least one black ball) = p(RB) + p(GB) + p(BG) + p(BR) + p (BB)

= 4/13x 6/12 + 3/13 x 6/12 +6/13 x5/12 + 6/13 x4/12 +6/13 x3/12

= 2/13 +3/26 + 5/26 + 2/13 + 3/26

= 19/26

2.6 Conditional Probability.

Suppose we roll a fair die and we define event B: obtaining an odd number and event A : obtaining a number less than or equal to 3. We are interested in the probability that event B; obtaining an odd number, will occur given that event A: obtaining a number less than or equal to 3, has occurred or is certain to occur.

We call the probability of event B given event A, a conditional probability and we denote it by p(B/A).

hence, A = {1,2,3 } and B = { 1,3,5 }

S = { 1,2,3,4,5,6 }

$P(A) = \frac{3}{6}$; $p(B) = \frac{3}{6}$

$p(A \cap B) = \frac{2}{6}$

Since, A has occurred then the new sample space is A = { 1,2,3, } , then number of elements of B in A equals 2.

hence, $p(B|A) = \frac{n(A\cap B)}{n(A)}$

$$p(B|A) = \frac{2}{3}$$

but $p(B/A) = \frac{n(A\cap B)}{n(S)} \Big/ \frac{n(A)}{n(S)} = \frac{p(A\cap B)}{p(A)}$

i.e. $\mathbf{p(B|A) = \frac{p(A\cap B)}{p(A)}}$

Similarly, $\mathbf{p(B|A)} = \frac{p(A\cap B)}{P(B)}$.

If A and B are independent events

$P(A / B) = \frac{p(A\cap B)}{P(B)} = \frac{p(A) \cdot p(B)}{p(B)} = p(A)$

Also, $p(A\cap B) = p(A / B) \times p(B)$.

Example

75% of the graduates of a Maths program end up in management. About 25% of the graduates have both management jobs and the good public – speaking skills that such job usually requires, find, the conditional probability that graduates of this program will have good public – speaking skills given that they have management jobs.

Solution.

Define

A: = have good public – speaking skills.

B : = have management jobs

We now compute $p(A / B) = \frac{p(A \cap B)}{P(B)}$.

$P(B) = 0.75$; $p(A \cap B) = 0.25$

hence, $p(A/B) = \frac{0.25}{0.75}$

$= 1/3$

Example

The personal manager of a company classifies job applicants as qualified or unqualified for the job. The manager says that only 25% of the job applicants are qualified, and of those that are qualified, 20% list high school as their highest level of education but 30% of the qualified applicants list polytechnic and 50% list university. Of the unqualified 40% list high school, 40% polytechnic and only 20% list university.

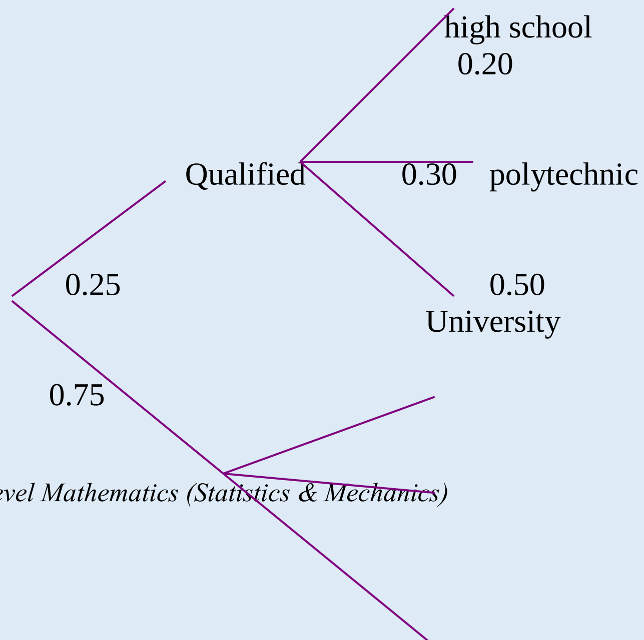
a) Draw a probability tree diagram.

b) Find the probability that an applicant is both qualified and a university graduate.

c) Find, the probability that an applicant comes from a polytechnic.

d) Find the probability, that the applicant is unqualified given that the applicant is a high school graduate.

Solution



Unqualified 0.40 high school

0.40 polytechnic

0.20

University

b) p(qualified and university) = 0.25 x 0.50
= 0.125

c) p(a polytechnic graduate) = p(qualified and poly) + p(unqualified and polytechnic)
= 0.25 x 0.30 + 0.75 x 0.40
= 0.075 + 0.30
= 0.375.

d) Define A : unqualified
B: high school graduate

$$P(A/B) = \frac{p(A \cap B)}{P(B)}$$

$$= \frac{p(\text{ unqualified and high school graduate})}{p(\text{ high school graduate})}$$

p(unqualified and high school) = 0.75 x 0.40
= 0.3

p(high school graduate) = 0.75 x 0.4 + 0.25 x 0.20
= 0.35

hence, $p(A / B) = \frac{0.3}{0.35}$
= 0.8571.

Example

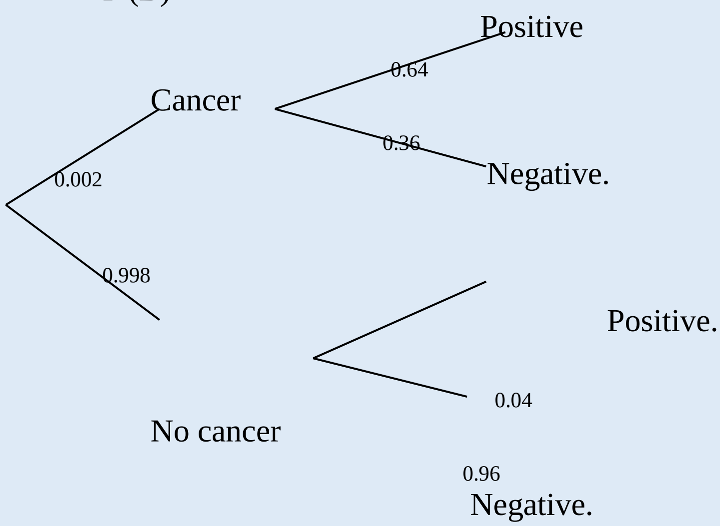
The annual incidence rate of cancer is some what rare and has been reported to be about 2 in 1000 people. One person is tested, the probability of being positive given that there is no cancer is 0.04, and the probability of negative given that there is cancer is 0.36. Find the probability that a person have cancer given that the person is positive.

Solution.

A: have cancer

B: is positive

$$P(A \mid B) = \frac{p(A \cap B)}{P(B)}$$



$$P(\text{cancer / positive}) = \frac{p(\text{cancer and positive})}{P(\text{positive})}$$

$$= \frac{0.002 \times 0.64}{0.002 \times 0.64 + 0.998 \times 0.04}$$

$$= \frac{0.00128}{0.0412}$$

$$= 0.031.$$

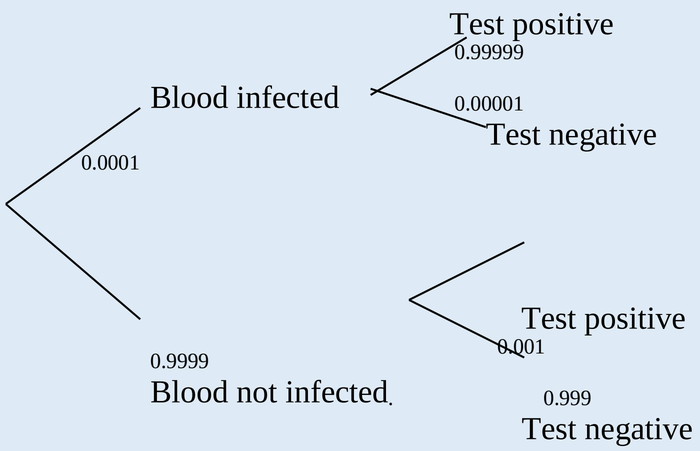
Example

Several hundreds of people were victimized by AIDS when they received transfusions of blood that contained the human immune deficiency virus that result in AIDS. At a particular time about 4 in 10 000 donor units of blood were infected with the AIDS virus. Routine testing of blood donations or antibodies of the AIDS virus began some time in previous years. Since testing was started, health care administrators have said that the public need not to worry about getting AIDS. From blood transfusions one desirable effect of screening potential blood donations for the AIDS virus has been that the proportion of infected donors has decreased. The proportion is quoted to be about 1 in 10 000 people. It has been estimated that about 1 in 100 000 current blood donations test negative, given that the blood is virus – positive. If about 1 in 1000 units of blood test positive when the blood does not have the virus then:

a)Find, the probability that the virus is infected given that the test is negative.

b) Find the probability that the blood is infected given that the test is positive.

Solution.



A: blood infected
B: test is negative.

$$P(A/B) = \frac{p(A \cap B)}{P(B)}$$

$P(A \cap B)$ = p(blood infected and test negative)
$= 0.0001 \times 0.00001$
$= 10^{-9}$

$p(B) = 10^{-9} + 0{,}9999 \times 0{,}999$
$= 10^{-9} + 0.9989\ 001$

$p(A/B) = 10^{-9} / (10^{-9} + 0.9989001)$
$= 0$

Example

Students in a class were given two mathematics problems to solve, the second of which was harder than the first. Within the class 5/6 of the students got the first correct and 7/12 got the second one correct. Of the students who got the first correct, 3/5 got the second one correct. Let A be the event that the student got the first problem correct and let B be the event that the student got the second one correct.

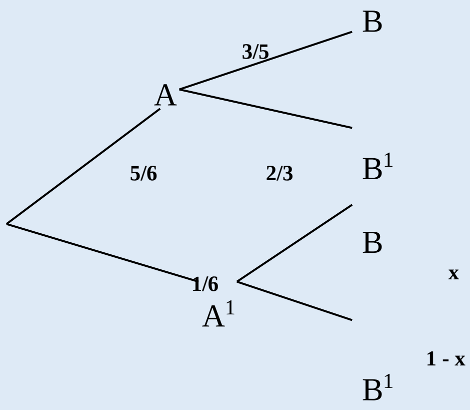
(i) Draw a tree diagram.

(ii) Find the probability that if a student was chosen at random from the class he or she got both answers correct.

(iii) Given that a student got the second problem right, find the probability that the first problem was solved correctly.

Solution

(i)



(ii) P (B) $= 5/6 \cdot 3/5 + 1/6x = 7/12$

$\therefore\ 1/2 + 1/6x = 7/12$

$x = 6(7/12 - 1/2)$

$x = 1/2$

Hence, P(A and B) $= 5/6 \times 3/5$

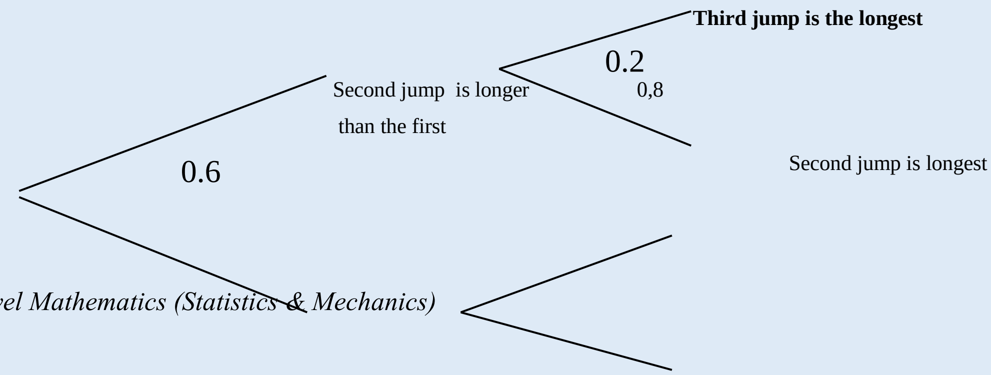
$= 1/2$

(iii) $P(A / B) = P(A \cap B) / P(B) = \dfrac{3/5 \times 5/6}{(3/5 \times 5/6 + \frac{1}{2} \times 1/6)}$

$= \quad 6/7$

Example

In a competition, a long – jumper takes three jumps. The probability that her second jump is longer than her first jump is 0.6. If her second jump is longer than her first then the probability that her third jump is the longest of the three is 0.2. If her first jump is longer than her second jump then the probability that her third jump is longest of the three is 0.3. The information concerning the results of her second and third jumps is shown in the following tree diagram.



0,4

0.3 Third jump is longest

0,7

First jump is longer than the second

First jump is longest

(i) Find the probability that, after her first jump, she improves with each subsequent jump.

(ii) Find the probability that she improves on her first jump.

(iii) Find the conditional probability that her second jump is longest, given that she improves on her first jump.

Solution

(i) p = 0.6 x 0.2 = 0.12

(ii) p = 0.6 x 0.2 + 0.6 x 0.8 + 0.4 x 0.3 = 0.72

(iii) We define A: second jump is longest. B: she improves on her first jump.

Hence, $P(A / B) = \dfrac{P(A \cap B)}{P(B)}$

$= \dfrac{0.6 \times 0.8}{0.72}$

$= 0.67$

2.7 Examination Type Questions

1. In a class of 30 pupils, 12 walk to school, 10 travel by bus and 2 travel by car. If 4 pupils are picked at random, obtain the probabilities that:

(a) they all travel by bus,

(b) they all travel by the same means

If two are picked at random from the class, find the probability that they travel by different means

2. A company has 3000 entries in its books which have to be audited. Of these 3000 entries 600 are larger than \$1M, and 30 of the 3000 contain errors. Only 12 of the entries over \$1M contain errors. An entry is chosen at random. What is the probability that the entry is:

(a) over \$1M;

(b) incorrect;

(c) incorrect and over \$1M;

(d) incorrect given that it is over \$1M;

(e) over \$1M, given that it is incorrect?

(f) Are the events {Incorrect} and {Over \$1M} independent?

3. Out of every summer days in a certain region, it is sunny and windy on 20 days, windy on 60 days, and sunny on 50 days. On a randomly selected summer day, what is :

(a) the conditional probability that it will be sunny, given that it is windy?
(b) the conditional probability that it will be windy, given that it is sunny?
(c) Are the two events independent?

4. A newspaper has fourteen pages, of which ten contain news, twelve contain advertisements and four contain advertisements only. If a page is selected at random, what is the conditional probability:

(a) that it contains advertisement, given that it contains news?
(b) that it contains news, given that it contains advertisements?

5 A computer company has three machines that produce a certain type of computer chip. Machine A produces 100 chips, machine B, produces 200 chips and machine C, 500 chips. Of the chips produced by machine A, 20% are found to defective, by machine B, 5% are found to be defective and by machine C, 10% are found to be defective.

a) construct a probability tree diagram.
b) find the probability that a chip is defective.
c) find the probability that a chip was produced by machine A given that it is defective.

6. A company has a machine that produces an important part in a production line. At the beginning of each day, the machine is set up correctly 90% of the time. If the machine is set up correctly, it will produce good parts 80% of the time. If it is set incorrectly, it will produce good parts 30% of the time. The company is considering a testing procedure.

a) what is the probability that the machine is set up and produced good parts?
b) If the machine is set up and produces a good part. What is the probability that it was set up correctly?

7. Three flower vendors X, Y and Z have equal chances of selling their flowers. X has 80 red and 20 white flowers, Y has 30 red and 40 white flowers and Z has 10 red and 6o white flowers. On Valentine’s day, Thandekile wants to buy a flower.

a) find the probability that she picks a red flower if she chooses a vendor at random.
b) given that she bought a red flower, find the probability that it came from Y.

8. Students in a class were given two tests A and B, to write. The second test, B, was harder than the first test, A. Within the class 0.6 of the students passed the first test and 0.5 passed the second test. Of the students who passed the first test, 0.3 passed the second test

a) draw a tree diagram.
b) find the probability that if a student was chosen at random from the class he or she passed both tests
c) given that a student passed the second test, find the probability that he or she passed the first test.

CHAPTER 3

DISCRETE RANDOM VARIABLES.

OBJECTIVES

- DEFINE A DISCRETE VARIABLE
- CONSTRUCT A P.D.F OF A GIVEN VARIABLES X
- PROBABILITIES USING A GIVEN OR CONSTRUCT p.d.f
- RECALL THE FORMULA FOR E(X) AND CALCULATE E(X)
- RECALL THE FORMULA FOR X^2 AND X AND CALCULATE X^2 AND X
- IDENTIFY THE BINOMIAL DISTRIBUTION
- CALCULATE PROBABILITIES , THE MEAN, THE VARIANCE, AND THE STANDARD DEVIATION OF A VARIABLE THAT FOLLOWS A BINOMIAL DISTRIBUTION
- IDENTIFY THE GEOMETRIC DISTRIBUTION
- CALCULATRE THAT FOLLOWS THE GEOMETRIC DISTRIBUTION

3.1 Definition

A variable X is said to be discrete if it takes integer values.

Example

If a coin is tossed twice and X is the number of heads obtained, hence, X take values 0,1 or 2 i.e. for X = 0; no heads obtained, the event is TT

For x = 1; 1 head is obtained, the event is HT or TH.

For x = 2; 2 heads are obtained, the event is HH.

Note the symbol X (capital letter) denotes the variable, while x (small letter) denotes specific values of X..

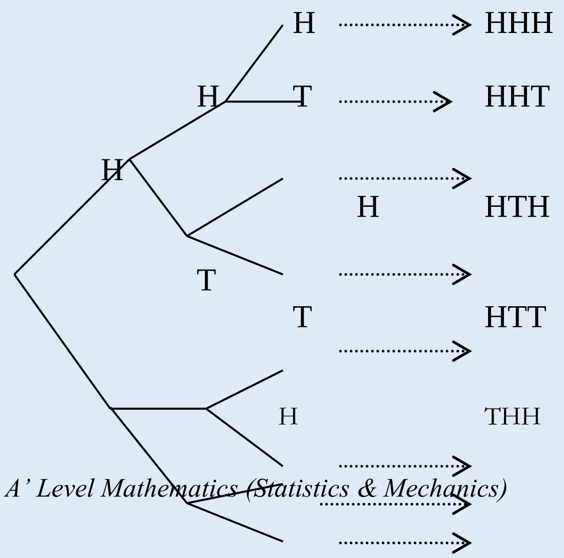
3.2 The probability distribution function (p.d.f) of X

The p.d.f of X is a rule that assigns probabilities to the values of X.

Example.

If a coin is cast thrice .Let X be the number of heads obtained. Construct the p.d.f. of X.

Solution.



T H

T THT
T H TTH

T TTT

X takes values 0, 1, 2, or 3.

$P(x = 0) = p(TTT) = \frac{1}{8}$

$p(x = 1) = p(HTT) + p(THT) + p(TTH) = \frac{1}{8} + \frac{1}{8} + \frac{1}{8} = \frac{3}{8}$

$p(x = 2) = p(HHT) + p(HTH) + p(THH) = \frac{3}{8}$

$p(x = 3) = p(HHH) = \frac{1}{8}$

The table represents the p.d.f of X.

X	0	1	2	3
P(X = x)	1/8	3/8	3/8	1/8

<u>Note</u>: the p.d.f. of X can be given in functional form: i.e.

$$P(X = x) = \begin{cases} 1/8 & \text{for } x = 0,3 \\ 3/8 & \text{for } x = 1,2 \end{cases}$$

$P(X = x) = p(x)$

$$\sum_{\text{all } x} p(X = x) = \sum_{\text{all } x} p(x) = 1$$

Hence, X is called a discrete random variable (d.r.v)

Example

Given that X is a d.r.v. find the value of a

X	-2	0	1	2
P(X = x)	1/5	a	3/15	2/5

Solution.

X is a d.r.v, hence, $\sum_{\text{All } x} p(x) = 1$

i.e. $\frac{1}{5} + a + \frac{3}{15} + \frac{2}{5} = 1$

$a = 1 - \frac{1}{5} - \frac{3}{15} - \frac{2}{5}$

a = 1/5

Example

X has the probability distribution function.

X	0	1	2
P(X = x)	1/7	2/7	4/7

Find $p(0 < x \leq 1)$

Solution.

$P(0 < X \leq 1) = P(X = 1)$
$= 2/7$

3.3 Expectation of X, E (X)

The mean value of a random variable X, also called the expected value, E(X) is defined by

$$\mu = E(X)$$

where,

$$E(X) = \sum_{\text{all } x} x p(x)$$

Example

If X is a discrete random variable. Find μ.

X	-4	3	5	7
P(X = x)	1/7	2/14	7/14	3/14

Solution.

$\mu = E(X)$

$E(X) = \sum_{\text{all } x} x\, p(x)$

$\sum x\, p(x) = -4(1/7) + 3(2/14) + 5(7/14) + 7(3/14)$

all x

$$= -\frac{4}{7} + \frac{6}{14} + \frac{35}{14} + \frac{21}{14}$$

$$= \frac{-8 + 6 + 35 + 21}{14}$$

$$= \frac{54}{14}$$

$$= \frac{27}{7}$$

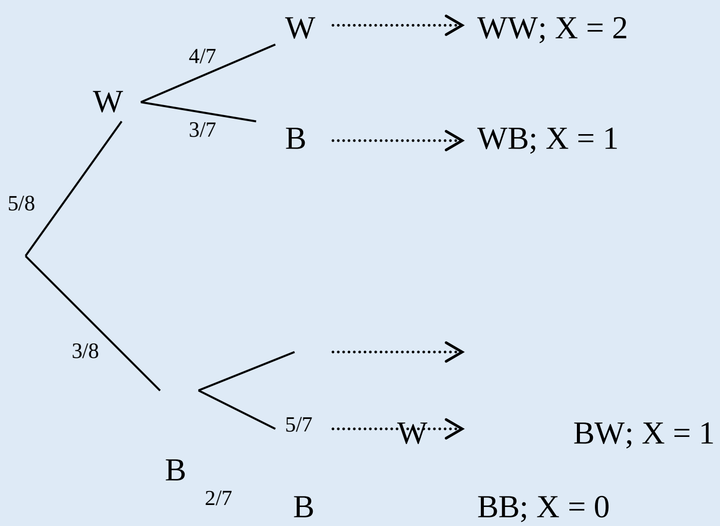
hence $\mu = \frac{27}{7}$

Example

A box contains 5 white and 3 black cricket balls. If two balls are drawn at random without replacement and X denotes the number of white balls.

a) Construct the p.d.f of X
b) Find E(X)

Solution.



$p(X = 0) = p(BB) = \frac{3}{8} \times \frac{2}{7} = \frac{3}{28}$

$p(X = 1) = p(BW) + p(WB) = \frac{5}{8} \times \frac{3}{7} + \frac{3}{8} \times \frac{5}{7}$

$= \frac{15 + 15}{56}$

$= \frac{30}{56}$

$= \frac{15}{28}$

$p(X = 2) = p(WW) = \frac{5}{8} \times \frac{4}{7} = \frac{10}{28}$

X	0	1	2
P(X = x)	3/28	15/28	10/28

b) $E(X) = \sum_{\text{all } x} x\, p(X = x)$

$= 0(3/28) + (15/28) + 2(10/28)$
$= 15/28 + 20/28$
$= \frac{35}{28}$

Let $Y = X^2$, where, X is a r.v, hence Y is also a r.v

hence, $\mathbf{E(Y) = \sum_{\text{all } x} X^2\, P(X = x)}$

i.e. $\mathbf{E(X^2) = \sum_{\text{all } x} x^2\, P(X = x)}$

Example

If X is d.r.v. Find $E(X^2)$

X	-1	0	2	3
P(X = x)	1/5	1/5	2/5	1/5

Solution.

$E(X^2) = \sum x^2\, p(x)$

$= (-1)^2 (1/5) + 0^2 (1/5) + 2^2 (2/5) + 3^2(1/5)$

$= \frac{1}{5} + 0 + \frac{8}{5} + \frac{9}{5}$

$= \frac{18}{5}$

3.4 The Variance and Standard Deviation.

The variance σ^2 of the .r.v X is defined as Var (X) = $E (X - \mu)^2$
where, μ = E (X) and σ^2 = Var (X)

hence $E (X - \mu)^2 = \sum_{\text{all } x} (x - \mu)^2\, p(x)$

On expanding we obtain $E(X^2) - \mu^2$

where $\mu^2 = E^2(X)$ and σ^2 = Var (x)

hence,

$$\sigma^2 = E(X^2) - \mu^2$$

$$\sigma = \sqrt{\sigma^2} \text{ i.e. } \sigma = \sqrt{\text{var } (x)}$$

Example

Compu- tech is a computer consultancy firm. The number of new clients that they have obtained each month has ranged from 0 to 6. The number of clients has the p.d.f. shown below.

No. of new clients	P(x)
0	0.05
1	0.10
2	0.15
3	0.35
4	0.20
5	0.10
6	0.05

a) Find the expected value.
b) The variance
c) The standard deviation

Solution

a) E (X) = 0(0.05) + 1(0.1) + 2(0.15) + 3 (0.35) + 4 (0.2) + 5(0.1) + 6(0.05)

$= 0.1 + 0.3 + 1.05 + 0.8 + 0.5 + 0.3$

$= 3.05$

b) $Var(X) = E(X^2) – E^2(X)$

$E(X^2) = 0^2(0,05) + 1^2(0,1) + 2^2(0,15) + 3^2(0,35) + 4^2(0,2) + 5^2(0,1) + 6^2(0,05)$

$= 0.1 + 0.6 + 3.15 + 3.2 + 2.5 + 1.8$

$= 11.35$

hence, $Var(X) = 11.35 - 3.05^2$

$= 2.0475$

c) $\sigma = \sqrt{var(x)}$

$= \sqrt{2.0475}$

$= 1.431$

Example

Let X be a d..r.v with the following distribution function:

X	-3	-1	0	2	4
P(X = x)	a	b	0.1	0.4	0.2

Given that E(X) = 1.2 and Var (X) = 4.36. Find the values of a and b.

Solution:

$E(X) = 1.2$

$E(X) = - 3a – b + 0 + 0.8 + 0.8$

i.e. $E(X) = -3a – b = 1.6$

hence, $-3a – b + 1.6 = 1.2$

$-3a – b = -0.4$

$\mathbf{3a + b = 0.4}$ **(1)**

$Var(X) = E(X^2) – E^2(X)$

$E(X^2) = 9a + b + 0 + 1.6 + 3.2$

$= 9a + b + 4.8$ (1)

hence, $4.36 = 9a + b + 4.8 – 1.2^2$

$4.36 = 9a + b + 4.8 – 1.44$

i.e. $9a + b = 4.36 + 1.44 – 4.8$

$\mathbf{9a + b = 1}$ **(2)**

Solving the simultaneous equations.

$3a = b = 0.4$

$(-)\ 9a + b = 1.0$

$-6a = -0.6$

$6a = 0.6$

$a = 0.1$

Use $\sum_{\text{all } x} p(x) = 1$ and E(X)/ or $\sum p(x) = 1$ and var.(x) to solve for a and b.

hence, $3(0.1) + b = 0.4$
$b = 0.4 - 0.3$
$b = 0.1$

Example

A computer is programmed to generate a random integer X in the set
{ 0, 1, 2, ... ,9}. The random variable Y is the remainder when X is divided by 4.
Give a table showing the probability distribution of Y. Show that the expectation of Y is 1.3 and find the standard deviation of Y.
Three successive observations of Y are made. Find the probability that each observation after the first is greater than the one before.

Solution

If 0 is divided by 4, the remainder is 0; If 1 is divided by 4, the remainder is 1; . . . . . ;if 8 is divided by 4, the remainder is 0; if 9 is divided by 4 the remainder is 1.
Y take values 0 1 2 3 0 1 2 3 0 1
$P(Y = 0) = 0.3$; $P(Y = 1) = 0.3$; $P(Y = 2) = 0.2$; $P(Y = 3) = 0.2$.

Y	0	1	2	3
P (Y = y)	0.3	0.3	0.2	0.2

$E(Y) = \sum y\, p(y) = 0 \times 0.3 + 1 \times 0.3 + 2 \times 0.2 + 3 \times 0.2$
$= 1.3$ (shown)
$E(Y^2) = \sum y^2 p(y) = 0^2 \times 0.3 + 1^2 \times 0.3 + 2^2 \times 0.2 + 3^2 \times 0.2$
$= 2.9$
$Var(Y) = E(Y^2) - E^2(Y)$
$= 2.9 - 1.3^2$
$= 1.21$, hence standard deviation $= \sqrt{1.21} = 1.1$
$p = p(0\ 1\ 2) + p(0\ 1\ 3) + p(0\ 2\ 3) + p(1\ 2\ 3)$
$= 0.3 \times 0.3 \times 0.2 + 0.3 \times 0.3 \times 0.2 + 0.3 \times 0.2 \times 0.2 + 0.3 \times 0.2 \times 0.2$
$= 0.06$

3.5 The Binomial Distribution.

Consider a d.r.v X with the following characteristics.
(i) There are n identical trials
(ii) The trials are independent.
(iii) There are two possible outcomes for each trial, success (S) or failure(F).
(iv) The probability of success on a single trial is equal to p and remains the same from trial to trial. The probability of failure is q where $q = 1-p$.
(v) The number of successes in n trials is denoted by x.

If all these characteristics are satisfied, we say X follows a binomial distribution with the parameters n and p. In symbols, X~ Bin (n, p)
hence, $\mathbf{P(X = x) = {}^nC_x\, p^x q^{n-x}}$
$\mathbf{x = 0,1,2,\ldots\ldots.n}$

Example

If $X \sim Bin(10;2/5)$

Find (a) $p(x = 1)$ (b) $p(1< x< 4)$ (c) $p(x \geq 1)$

Solution.

$X \sim Bin(10; 2/5)$

$P(X=x) = {}^{10}C_x\, (2/5)^x\, (3/5)^{10-x}$; $n = 10$, $p = 2/5$ hence, $q = 3/5$

a) $p(x = 1) = {}^{10}C_1\, (2/5)^1\, (3/5)^9$
$= 10 \times (2/5) \times (3/5)^9$
$= 0.04031$

(b) $p(1 < X < 4) = p(X = 2) + p(X = 3)$
$= {}^{10}C_2\, (2/5)^2\, (3/5)^8 + {}^{10}C_3\, (2/5)^3\, (3/5)^7$
$= 0.3359$

(c) $p(x \geq 1) = 1 - p(x = 0)$
$= 1 - {}^{10}C_0 \times (2/5)^0 \times (3/5)^{10}$
$= 1 - 0.6^{10}$
$= 0.9940$

Example.

In any box of 12 bananas, there are 4 bad bananas. A box is selected at random what is the probability of getting 6 good bananas?

Solution.

P(bad banana) = 4/12 = 1/3
P(good banana) = 2/3
Let X = number of good bananas
Hence, $X \sim Bin(12;2/3)$
$P(X = 6) = {}^{12}C_6 (2/3)^6\, (1/3)^6$
$= 924 \times (2/3)^6 \times (1/3)^6$
$= 0.1113$

If $X \sim Bin(n:p)$
$P(X = x) = {}^nC_x\, p^x\, q^{n-x}$, $x = 0,1\ldots n$
Then: $E(X) = npq$
$Var(X) = npq$ standard deviation= $\sqrt{npq}$

Example

If $X \sim Bin(20; ½)$

Find a) E(X) (b) Var (X)

Solution.

$X \sim Bin(20, ½)$,
$p = ½$ and $q = ½$, $n = 20$

a) $E(X) = np$
$= 20 \times ½$
$= 10$

(b) $Var(X) = npq$
$= 20 \times ½ \times ½$
$= 5$

Example

The probability of hitting a target with a missile is 0.45. How many shots must be fired in order to claim with 95% confidence that the target will be hit?

Solution.
Let X denote the number of hits,
hence, $X \sim Bin(n; 0.45)$. We find n such that $p(x \geq 1) \geq 95\%$

$P(X \geq 1) = 1 - p(x = 0)$

i.e. $1 - p(x = 0) \geq 0.95$

$p(x = 0) \leq 0.05$

${}^{n}C_0(0.45)^0 (0.55)^n \leq 0.05$

$0.55n \leq 1n\ 0.05$

$n \geq \frac{1n\ 0.05}{1n\ 0.55}$

$n \geq 5.010$

n at least 6 times.

Example

In an election in a certain country 60% of voters supported party X, while 40% supported party Y. If during the election a random sample of 15 voters is drawn.

a) What is the probability that all 15 voters would support party Y?
b) What is the probability that not one of the 15 voters would vote for party X?
c) What is the probability that a sample would have predicted the correct election result?
d) What is the expected number of people in the sample that would vote for party X?
d) What is the standard deviation of the number of people in the sample that would vote for party X?

Solution.

Each voter will either vote for party X (success) or party Y (failure). Let X be the number of people that vote for party X.

Hence X ~ Bin (15;0.6)

a) $p(x = 15) = {}^{15}C_{15}\, 0.6^{15}\, 0.4^0$
$= 0.0005$

b) $p(x = 0) = {}^{15}C_0\, 0.6^0\, (0.4)^{15}$
$= 0.000001073$

c) The sample would have predicted the correct election result had 40% of the sample voted for party Y, we therefore have

$40\% \times 15 = 6$

$\therefore \quad P(X = 6) = {}^{15}C_6\, (0.4)^6 (0.6)^9$
$= 0.2066$

d) $E(X) = np$
$= 15 \times 0.6$
$= 9$

e) $\sigma^2 = np(1 - p)$
$= 15 \times 0.6 \times 0.4$
$= 3.6$
$\sigma = \sqrt{3.6}$
$= 1.9$

3.6 The Geometric Distribution.

Consider a d.r.v X with the following characteristics:

a) x is the number of trials until a success is obtained.
b) In each trial, there are two mutually exclusive events ; success or a failure.
c) The trials are independent.
d) The probability of success, p is constant in each trial and $p + q = 1$

Hence, we say X follows a geometric distribution with a parameter p. In symbols,
$X \sim Geo(p)$ and $\mathbf{P(X = x) = q^{x-1}p \; ; \; x = 1,2,3,\ldots\ldots}$

Example

If $X \sim Geo(\frac{1}{4})$ find:

a) $p(x = 3)$
b) $p(x \geq 3)$
c) $p(x \leq 9)$

Solution

a) $P(X = x) = q^{x-1}p$

$p = \frac{1}{4}, \quad q = \frac{3}{4}$

$p(X = 3) = (\frac{3}{4})^{3-1}\frac{1}{4}$

$= (\frac{3}{4})^2(\frac{1}{4})$

$= 9/24$

(b) $p(x \geq 3) = \sum_{x=3}^{\infty} (\frac{3}{4})^{x-1}(\frac{1}{4})$

$= (1/3)\sum_{x=3}^{\infty} (\frac{3}{4})^x$

$= (1/3)\, x(0.421875/(1 - 0.75)$

$= 0.5625$

c) $p(x \leq 9) = p + qp + q^2p + q^3p + q^4p + q^5p + q^6p + q^7p + q^8p$

$= p(1 + q + \text{---------} + q^8)$

$= 1 - q^9$

$= 1 - (\frac{3}{4})^9$

$= 0.925$

If X~ Geo (p) then:

(i) $\mathbf{p(x \leq r) = 1 - q^r}$
(ii) $\mathbf{p(x > r) = q^r}$
(iii) $\mathbf{p(x > (a+b) / x > a) = q^b}$

Example

If $X \sim Geo(2/3)$. Find (a) $p(x \leq 4)$ (b) $p(x < 6)$ (c) $p(x > 3)$ (d) $p(x \geq 3)$ (e) $p(x > 7 / x > 3)$ (f) $p(x \geq 7 / x \geq 3)$

Solution.

a) $p(x \leq 4) = 1 - q^4$

$= 1 - (1/3)^4$

$= 0.9877$

b) $p(x < 6) = p(x \leq 5) = 1 - q^s$

$= 1 - (1/3)^5$

$= 0.9959$

c) $p(x > 3) = q^3$

$= (1/3)^3$

$= 0.0370$

d) $p(x \geq 3) = p(x > 2)$

$= q^2$

$= (1/3)^2$

$= 0.1111$

e) $p(x > 7 / x > 3) = q^{7-3}$

$= q^4$

$= (1/3)^4$

$= 0.0123$

f) $p(x \geq 7 / x \geq 3)$

$= \frac{p(x \geq 7 / x \geq 3)}{p(x \geq 3)}$

$= \frac{p(x \geq 7)}{p(x \geq 3)}$

$= \frac{p(x > 6)}{p(x > 2)}$

If X~ geo (p),
hence, $\mathbf{E(X) = \frac{1}{p}}$

$\mathbf{Var(X) = \frac{q}{p^2}}$

$= q^4 = (1/3)^4 = 0.0123$

Example

If X ~Geo (4/5)

Find ;

a) E(X) (b) Var (X)

c) Standard deviation of x

Solution.

X ~ geo (4/5) , p = 4/5 , q = 1/5.

a) $E(X) = 1/p$

$= ¼ /5$

$= 5/4$

$= 1.25$

b) $var(x) = \frac{q}{p^2}$

$= \frac{1/5}{(4/5)^2}$

$= \frac{1}{5} \times \frac{25}{16}$

$= 5/16$

$= 0.3125$

c) $\sigma = \sqrt{q/p^2}$

$= \sqrt{5/16}$

$= 0.559$

Example

If X ~ Geo (p), and E (X) = 2.

Find a) $p(x \leq 3)$ b) Var(X)

Solution

(a) E (X) = 2, hence, 1/p = 2, where, p = 1/2

Hence, $p(x \leq 3) = 1 - q^3$

$= 1 - (½)^3$

$= 1 - 1/8)$

= 7/8

b) $Var(X) = q/p^2$

$= 1/ 2 \times 4$

$= 2$

Example

A coin is loaded such that the probability of obtaining a head is 2/3. A coin is cast repeatedly until a head is obtained. Let X be the number of tosses until a head is obtained.

(i) State the p.d.f of x. (ii) Find p (x > 4).

(iii) Find E (X), i.e. expected number of tosses.

Solution.

(i) X ~ Geo (2/3)

p = 2/3, q = 1 – 2/3 = 1/3

(ii) $P(x > 3) = q^3$

$= (1/3)^3$

$= \frac{1}{27}$

(iii) $E(X) = \frac{1}{p}$

$= \frac{3}{2}$

$= 1.5$

3.7 Examination Type Questions

1. A bag contains five identical discs, two of which are marked with the letter A and three with the letter B. the discs are randomly drawn, one at a time without replacement , until both discs marked A are obtained. Show that the probability that 3 draws are required is 2/10. Given that X denotes the number of draws required to obtain both discs marked A, copy and complete the following table.

Value of X	2	3	4	5
probability of X		2/10		

Evaluate (i) E(X) (ii) $E(X^2)$ (iii) the variance of X

2. If X ~ Bin (n, 0.5) and P(X < 1) = 0.045. find n.

3. If the probability that it will rain on any given day in February is 0.4, calculate the probability that in a given week in February, it will rain on

(a) Exactly two days

(b) at least two days,

(c) more than half the days

(d) Exactly three days that is consecutive.

4. During February, the probability that it will rain on any given day is 0.55**.**

(a) Find the probability that the first rainy day in February is on the 6^{th}

(b) Given that it does not rain on the first 10 days, find the probability that it first rains on 14^{th} February.

(c) Find the probability that it does not rain before 8^{th} February.

5. If X ~ Geo(0.5), find (a) P(X > 4) (b) P(X < 6)

(c) P (X >7 /X > 4) (d) Var (X)

6 A fair coin is tossed and we denote by X the number of trials until a head appears for the first time. Construct the probability distribution function of X.

7. Find and sketch the graph of the probability distribution function of the random variable X equals sum of the two numbers obtained in throwing two fair dice.

8. A box contains five defective and three non-defective screws. Two screws are drawn at random without replacement. Let X be the random variable, the number of defective screws obtained. Construct the probability distribution of X. Find p (x >1).

CHAPTER 4

CONTINUOUS DISTRIBUTIONS

OBJECTIVES

- DEFINE A CONTINOUS VARIABLE
- CALCULATE PROBABILITIES USING A p.d.f. OF A CONTINOUS VARIABLE
- CALCULATE E(X) , X^2 AND X OF A CONTINOUS VARIABLE
- FIND THE CUMULATIVE DISTRIBUTION FUNCTION GIVEN p.d.f.
- DEFINE A UNIFORM DISTRIBUTION
- FIND THE p.d.f AND THE CUMULATIVE DISTRIBUTION OF A GIVEN UNIFORM DISTRIBUTION
- CALCULATE PROBABILITIES USING THE CUMULATIVE DISTRIBUTION

4.1 The probability density function (p.d.f.)

A continuous variable takes any value in an interval. The probability density function of x is given by f (x) hence,

(i)

$$\int_{-\infty}^{+\infty} f(x)dx = 1$$

(ii) f(x) ≥ 0, for all x.

Example

Given the function, defined as follows;

$$f(x) = \begin{cases} 0 \text{ for } x \le 0 \\ 2x \text{ for } 0 < x \le 1 \\ 0 \text{ for } x > 1 \end{cases}$$

Verify that f (x) is a probability density function.

Solution.

We check if (i) $f(x) \ge 0$

(ii) $\int_{-\infty}^{+\infty} f(x)dx = 1$

(i) $f(x) \ge 0$ as defined above

(ii)

$$\int_{-\infty}^{+\infty} f(x)dx = \int_{-\infty}^{0} f(x)dx + \int_{0}^{1} f(x)dx + \int_{1}^{+\infty} f(x)dx$$

$$= 0 + \int_0^1 2x\,dx + 0$$

$$= x^2 \Big|_0^1$$

$$= 1$$

Hence, f(x) is a probability density function

If X is a continuous distribution function then;

$$\mathbf{P(a < x \le b) = p(a \le x \le b) = p(a < x < b) = p(a \le x < b) = \int_a^b f(x)\,dx}$$

If $f(x) = \begin{cases} 0 \text{ for } x \le 0 \\ 2x \text{ for } 0 < x \le 1 \\ 0 \text{ for } x > 1 \end{cases}$

Calculate the probabilities:

(i) $p(0<x<0.5)$ (ii) $p(0.8 \le x \le 2)$

Solution.

(i) $p(0 < x < 0.5) = \int_0^{0.5} 2x\,dx$

P(a < x < b) is the area under the curve f(x) , above the x – axis and between the lines x= a and x = b.

$$= x^2 \Big|_0^{0.5}$$

$$= 0.25$$

(ii) $p(0.8 \leq x < 2) = \int_{0.8}^{1} 2x\, dx + \int_{1}^{+\infty} dx$

$$= x^2 \Big|_{0.8}^{1}$$

$$= 1 - 0.8^2$$

$$= 0.36$$

Example

If x is a random continuous variable with the following p.d.f.

$$f(x) = \begin{cases} k(1 - x^2) & \text{for } -1 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

Find a) the value of k

b) the mode.

Solution.

$$\int_{-\infty}^{+\infty} f(x)dx = 1; \textbf{ hence} \int_{-\infty}^{+\infty} f(x)dx = \int_{-\infty}^{0} f(x)dx + \int_{-1}^{1} k(1 - x^2)dx + \int_{1}^{+\infty} 0dx$$

$$= 0 + \int_{-1} k(1 - x^2)\, dx + 0$$

$$= k \int_{-1}^{1} (1 - x^2)\, dx$$

$$= k(x - x^3/3) \Big|_{-1}^{1}$$

$$= k[1 - 1/3 - (-1 + 1/3)]$$

$$= k[1 - 1/3 + 1 - 1/3$$

$$= \frac{4k}{3}$$

Hence, $\frac{4k}{3} = 1$

$\therefore$ k = 0.75

b) Let Mo be the mode of the p.d.f. Hence f(Mo) ≥ f(x) for all x. Since f(x) is a continuous function, we find M o by using the first derivative of f(x)

$f(x) = \frac{3}{4}(1 - x^2)$ for $-1 \le x \le 1$

Hence, $f^1(x) = \frac{3}{4}(-2x)$

$f^1(x) = 0$

$\frac{-3x}{2} = 0$

i.e. $x = 0$

$f^{11}(x) = -3/2 < 0$; at $x = 0$ we have the maximum value i.e. $x = 0$ is the mode of $f(x)$

Expectation, E(X)

We define the expected value of X as:

$$\mathbf{E(X)} = \int_{-\infty}^{+\infty} x\, f(x)\, dx \quad \text{i.e. } \mu = E(X)$$

Example

Find μ if,

$$f(x) = \begin{cases} e^{-x}\ ; x \ge 0 \\ 0 \text{ otherwise.} \end{cases}$$

Solution.

$\mu = E(X)$; hence, $E(X) = \int_{-\infty}^{+\infty} x\, f(x)dx$

$$= \int_0^{+\infty} xe^{-x}dx$$

Use integration by parts.

Let: $u = x$; $dv = e^{-x}dx$

$du = dx$; $v = -e^{-x}$, then

$$\int_0^{+\infty} xe^{-x}dx = -xe^{-x}\Big|_0^{+\infty} + \int_0^{+\infty} e^{-x}dx$$

$$= 0 + \int_0^{+\infty} e^{-x}dx$$

$$= -e^{-x}\Big|_0^{+\infty}$$

$$= 0 - (-1)$$

= 1

The variance, Var (X)

We define the variance of X as:

$$\mathbf{Var\ (X) = \int_{-\infty}^{+\infty} (x - \mu)^2 f(x)dx}$$

For calculation we use the formula

$$\mathbf{Var\ (X) = \int_{-\infty}^{+\infty} x^2 f(x) - E^2(X)}$$

where, E(X) = μ

Example

Find the variance and the standard deviation of x where:

$$f(x) = \begin{cases} \frac{1}{2}x & \text{for } 0 < x < 2 \\ 0 & \text{elsewhere} \end{cases}$$

Solution

$\text{Var}(x) = E(X^2) - E^2(X)$

$$E(X^2) = \int_0^2 x^2 \left(\tfrac{1}{2}x\right) dx$$

$$= \tfrac{1}{2} \int_0^2 x^3\, dx$$

$$= \tfrac{1}{8} x^4 \Big|_0^2$$

$$= \frac{16}{8}$$

$$= 2$$

$$E(X) = \int_0^2 x \left(\tfrac{1}{2} (x)\right) dx$$

$$= \tfrac{1}{2} \int_0^2 x^2\, dx$$

$$= \frac{x^3}{6} \Big|_0^2$$

$$= \frac{8}{6}$$

hence, $\text{var}(x) = 2 - (4/3)^2$

$$= 2 - \frac{16}{9}$$

$$= \frac{2}{9}$$

$\sigma = \sqrt{\text{var. }(x)}$

$= \sqrt{2/9}$

$= \frac{\sqrt{2}}{3}$

Example

Let X be a random continuous variable with the following density function.

$$f(x) = \begin{cases} 2e^{-2x} & \text{for } x \geq 0 \\ 0 & \text{elsewhere.} \end{cases}$$

Find, a) p(x>2). b) the mode. c) the median.
d) the lower quartile. e) the upper quartile.
f) the inter quartile range .

Solution.

a) $p(x > 2) = \int_{2}^{+\infty} 2\mathbf{e}^{-2x}\, dx$

$= -\mathbf{e}^{-2x} \Big|_{2}^{+\infty}$

$= [-(e^{-2(\infty)} - e^{-4})]$

$= -0 + e^{-4})$

$= e^{-4}$

$= 0.0183$

b) $f(x) = 2e^{-2x}$

$f^1(x) = -4e^{-2x}$

$f^1(x) = 0$

$-4e^{-2x} \neq 0$ for all values of x, but the mode occurs at x = 0 since 1 is the maximum value
of $f(x) = e^{-2x}$.

c**) Let m_e be the median , hence, $p(x \geq m_e) = p(x \leq m_e) = 0.5$**
i.e $p(x \leq me) = 0.5$

M e

$$\int_0 2e^{-2x}dx = ½$$

i.e. $-e^{-2x} \Big|_0^{Me} = ½$

$1 - e^{-2Me} = ½$

i.e. $e^{-2Me} = ½$ (***apply logarithms***)

$-2m_e = -1n\ 2$

$m_e = \frac{1n2}{2}$

$m_e = 0.3466$

b) Let Q_1 be the lower quartile, hence,

$P(x \le Q_1) = ¼$

i.e. $\int_0^{Q1} 2\ e^{-2x}dx = ¼$

$-e^{-2x} \Big|_0^{Q1} = ¼$

$1 - e^{-2Q1} = ¼$

$e^{-2Q1} = ¾$

$- 2Q_1 = 1n\ 0.75$

$Q_1 = \frac{1n\ 0.75}{-2}$

$Q_1 = 0.1438$

(c) Let Q_3 be the upper quartile hence, **$p(x \le Q_3) = 0.75$ or $p(x \ge Q_3) = 0.25$**

$$\int_0^{Q3} 2e^{-2x}\ dx = 0.75$$

$1 - e^{-2Q3} = 0.75$

$e^{-2Q3} = 0.75$

$e^{-2Q3} = 0.25$

$Q_3 = \frac{1n\ 0.25}{-2}$

$Q_3 = 0.6931$

f) I.Q.R $= Q_3 - Q_1$

$= 0.6931 - 0.1438$

$= 0.5493$

4.2 The Cumulative Distribution Function , F(x)

Let X be a random continuous variable with the following probability distribution function, f(x).

We define: $F(x) = p(X \le x)$

where, $p(X \le x) = \int_{-\infty}^{x} f(x)\,dx$

$$F(x) = \int_{-\infty}^{x} f(x)\,dx$$

Hence, $\mathbf{f(x) = F^1(x)}$

Example

Let X be r.v with the following p.d.f

$$f(x) = \begin{cases} 2x \text{ for } 0 < x \le 1 \\ 0 \text{ elsewhere} \end{cases}$$

Find the cumulative distribution function F(x) and sketch its graph.

Solution.

$F(x) = p(x \le x)$: i.e. $p(X \le x) = \int_{-\infty}^{x} f(x)dx$

<u>For $x \le 0$</u>

$f(x) = 0$, then; $F(x) = 0$

<u>For $x < 1$</u>

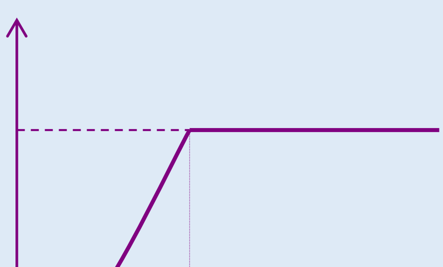
$F(x) = \int_{0}^{x} 2x\,dx = x^2$

<u>For $x > 1$</u>

$F(x) = \int_{-\infty}^{0} f(x) + \int_{0}^{1} f(x)dx + \int_{1}^{x} f(x)dx$

$F(x) = 0 + \int_{0}^{1} 2x\,dx + 0$

1



$= x^2 \Big|_0$ f(x)

$= 1$

1

hence, $F(x) = \begin{cases} 0 & \text{for } x \le 0 \\ x^2 & \text{for } 0 < x \le 1 \\ 1 & \text{for } x > 1 \end{cases}$

0 1 x

Example

Given the following cumulative distribution function

$$F(x) = \begin{cases} 1 - e^{-2x} & \text{for } x \ge 0 \\ 0 & \text{for } x < 0 \end{cases}$$

Find f(x)

Solution.

$f(x) = F^1(x)$

$$f(x) = \begin{cases} 2e^{-2x} & \text{for } x \ge 0 \\ 0 & \text{for } x < 0 \end{cases}$$

If X is a random variable, with f(x) its p.d.f and F(x) its cumulative distribution function, hence,

(i) $p(X \le x) = \int_{-\infty}^{x} f(x)dx$

(ii) $p(X > x) = 1 - \int_{-\infty}^{x} f(x)\,dx$

(iii) $p(a \le x \le b) = F(b) - F(a)$

(iv) $p(X = x) = 0$

(v) $p(x \le -\infty) = F(-\infty) = 0$

(vi) $p(x \le +\infty) = F(+\infty) = 1.$

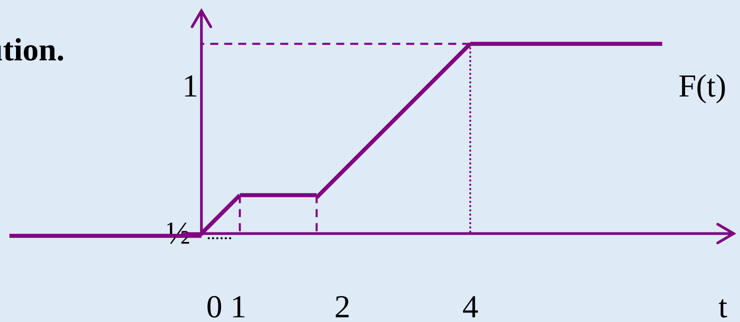
Example

Let T be the waiting time for a commuter bus by a student. The cumulative distribution function of t is given as follows:

$$F(t) = \begin{cases} 0 & \text{for } t \leq 0 \\ \tfrac{1}{2}t & \text{for } 0 \leq t \leq 1 \\ \tfrac{1}{2} & \text{for } 1 \leq t \leq 2 \\ \tfrac{1}{4}t & \text{for } 2 \leq t \leq 4 \\ 1 & \text{for } t \geq 4 \end{cases}$$

a) Sketch the graph of f(t)
b) Find the probability density function of t.
c) Find the probability that the waiting time would be ;
   (i) more than three minutes.
   (ii) between one and three minutes.

Solution.



b) f(t) = $F^1(t)$, finding $f^1(t)$ in each interval we obtain:

$$f(t) = \begin{cases} 0 & t < 0 \\ \tfrac{1}{2} & 0<t < 1 \\ 0 & 1 < t < 2 \\ \tfrac{1}{4} & 2 < t< 4 \\ 0 & t> 4 \end{cases}$$

At the points x = 0, 1 , 2 and 4. F^1 (x) does not exist.

$$\text{i.e} \quad f(t) = \begin{cases} \tfrac{1}{2} & 0 < t< 1 \\ \tfrac{1}{4} & 2 < t < 4 \\ 0 & \text{elsewhere} \end{cases}$$

c) (i) p(t.>3) = 1– F(3)
= 1 - ¼(3)
= 1 – ¾
= 0.25

(ii) p(1< t < 3) = F(3) – F (1)
= ¼ (3) - ½(1)
= ¾ - ½
= 0.25

Example

Given the cumulative distribution function of the random variable X:

$$F(x) = \begin{cases} ae^{x} \; ; & x \le 0 \\ \tfrac{1}{4} & 0 < x \le 1 \\ 1 - be^{-(x-1)} \; ; & x > 1 \end{cases}$$

a) Find the values of a and b
b) Find f(x) the p.d.f of x
c) (i) $p(x < 1)$ (ii) $p(x \ge 1)$ (iii) $p(-1 \le x \le 4)$

Solution.

a) $F(0) = \tfrac{1}{4}$ i.e. $ae^{0} = \tfrac{1}{4}$,

hence, $a = \tfrac{1}{4}$

$F(1) = \tfrac{1}{4}$
$1 - b\,e^{-0} = \tfrac{1}{4}$
$-b\,e^{-0} = -\tfrac{3}{4}$
$b = \tfrac{3}{4}$

$$\text{Hence, } F(x) = \begin{cases} \tfrac{1}{4} e^{x} \text{ for } & x \le 0 \\ \tfrac{1}{4} \text{ for } & 0 < x \le 1 \\ 1 - \tfrac{3}{4} e^{-(x-1)}, & x > 1 \end{cases}$$

b) $f(x) = F^{1}(x)$

i.e.

$$f(x) = \begin{cases} \tfrac{1}{4} e^{x}, & x \le 0 \\ \tfrac{3}{4} e^{-(x-1)} ; & x \ge 1 \\ 0 \text{ otherwise.} \end{cases}$$

c) (i) $p(x < 1) = F(1) - F(-\infty)$
$= \tfrac{1}{4} - 0$
$= 0.25$

(ii) $p(x \ge 1) = F(+\infty) - F(1)$
$= 1 - \tfrac{1}{4}$
$= 0.75$

(iii) $P(-1 \le x \le 4) = p(-1 \le x \le 0) + p(0 \le x \le 1) + p(1 \le x \le 4)$
$= F(0) - F(-1) + F(1) - F(0) + F(4) - F(1)$
$= \tfrac{1}{4} - \tfrac{1}{4} e^{-1} + \tfrac{1}{4} + \tfrac{1}{4} + 1 - \tfrac{3}{4} e^{-3} - \tfrac{1}{4}$
$= 0.87066$

4.3 The Uniform Distribution

A continuous random variable X is said to have a uniform distribution on the interval (a, b) if its density function is given by:

$$f(x) = \begin{cases} \dfrac{1}{b-a} & \text{for } a < x < b \\ 0 & \text{elsewhere.} \end{cases}$$

Hence, $E(X) = \dfrac{a+b}{2}$, $Var(X) = \dfrac{(b-a)^2}{12}$

Notation.

$X \sim U(a,b)$

$$F(x) = \begin{cases} 0 & \text{for } x \leq a \\ \dfrac{x-a}{b-a} & \text{for } a \leq x \leq b \\ 1 & \text{for } x \geq b \end{cases}$$

Example

Let X be uniformly distributed on the interval (0;1). Find;

(i) The cumulative distribution function F(x).

(ii) $p(x < ¾)$.

Solution.

(i)

$$f(x) = \begin{cases} 1 & 0 < x < 1 \\ 0 & \text{elsewhere.} \end{cases}$$

$$F(x) = \int_{-\infty}^{x} f(x)dx$$

$$F(x) = \begin{cases} 0 \ ; & x \leq 0 \\ x \ ; & 0 \leq x \leq 1 \\ 1 \ ; & x \geq 1 \end{cases}$$

(ii) $P(x < ¾) = F(¾) - F(-\infty)$

$= F(¾) - 0$

$= ¾$

Example
Suppose a point X is randomly selected from the interval (-a, a). Find the cumulative distribution function of X.

Solution.
$X \sim U(-a, a)$

$$F(x) = \begin{cases} 0 & x \leq -a \\ \dfrac{x+a}{2a} & -a \leq x \leq a \\ 1 & x \geq a \end{cases}$$

Example

The time, in minutes, taken by a student from her home to the train station has a uniform distribution in the interval 20 to 25. If the student leaves home at 7:05 am and the train departs at 7: 28 am, find the probability that the student will not catch the train.

Solution.

$X \sim U(20,25)$

$$f(x) = \begin{cases} 1/5 & \text{for } 20 \leq x \leq 25 \\ 0 & \text{otherwise.} \end{cases}$$

$$\begin{array}{r} 7.28. \\ -\ 7.05 \\ \hline 23 \text{ minutes}. \\ \hline \end{array}$$

Hence, we find

$P(x>23)$

i.e. $p(x>23) = \int_{23}^{25} 1/5 \, dx$

$$= x/5 \Big|_{23}^{25} = 0.40$$

4.4 Examination Type Questions

1. The continuous r.v. X has p.d.f. $f(x)$ where, $f(x) = kx^2$, for $0 \leq x \leq 2$

a) find the value of k

b) sketch the graph of $y = f(x)$

c) find the cumulative distribution function F(x)

d) find $P(X \geq 1)$

e) find $P(0.5 \leq x \leq 1.5)$

f) Find the mode and the median

2. The length X of an off- cut of wooden planking is a random variable which can take any value up to 0.5m. It is shown that the probability of the length being not more than x metres,
$0 \leq x \leq 0.5$ is equal to kx..

Determine:

a) the value of k

b) the p.d.f of X

c) the expected value of X

d) the standard deviation of X

3. The probability that a randomly chosen flight from Harare International Airport is delayed by more than x hours is $\frac{1}{100}(x - 10)^2$, for $x \in \Re$, $0 \leq x \leq 10$. No flights leave early, and none is delayed for more than 10hrs. The delay in hours, for a randomly chosen flight is denoted by X.

a) Find the median, m_e of X.

b) Find the cumulative distribution function, $F_x(x)$ of X and sketch the graph of $F_x(x)$

c) Find the probability density function, f (x) of X and sketch the graph of f(x)

d) Show that E (X) = 10/3

A random sample of two flights is taken. Find the probability that both flights are delayed by more than m_e hours, where, m_e is the median.

4. Given the following function.

$$F(x) = \begin{cases} 0 & x < 0 \\ 0.001\exp(-0.001x) & x \geq 0 \end{cases}$$

Show that f (x) is a probability density function

5. Given the following probability density function,

$$F(x) = \begin{cases} \frac{1}{2}\sin\pi x & 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

a) find the cumulative distribution function, F (x) and sketch its graph.

b) find the following probabilities

i) $P(X > 0.5)$ ii) $P(0.75 < X < 2)$

6. The cumulative distribution function of the random variable X is given below.

$$F(x) = \begin{cases} 0.25e^{x} & x \leq 0 \\ 0.25 & 0 < x \leq 1 \\ 1 - 0.75e^{-(x-1)} & x > 1 \end{cases}$$

a) find the probability density function of X
b) find E(X) and Var(X)
c) calculate the mode, the median, the lower quartile and the upper quartile.
d) determine;
   i) $P(X<1)$ ii) $P(X \geq 1)$ iii) $P(-1 \leq X \leq 4)$

7. A continuous random variable X takes values in the interval 0 to 4. It is given that $P(X > x) = c + dx^2$, for $0 \leq x \leq 4$

i) find the cumulative distribution function F(x) in terms of c and d.
ii) Find the values of the constants c and d
iii) find the probability density function, f (x)
iv) show that E(X) = 2.67
v) find the mode, the median, the lower quartile and the upper quartile
vi) determine $P(X < 0.12)$

8. A number X is randomly selected from the interval $(-\pi/2, \pi/2)$. Find the cumulative distribution function of X.

9. The length of the parts produced by an automatic machine has the following probability density function.

$$f(x) = \begin{cases} x/2 & 0 < x < 2 \\ 0 & \text{elsewhere} \end{cases}$$

A part is non-defective if its length is 1.2 ± 0.7.
If the parts produced are distributed in batches of size 10 and the buyer does not accept a batch if at least one part is defective. What is the probability that a batch selected at random is not accepted?

10. Let X be the angle between a fixed direction and the direction of the pointer of a spinner. If the spinner works properly , what is the probability distribution of X? Find $P(X > 1.2)$

11. Suppose that X has a uniform distribution on the interval $-1 \leq x \leq 1$. Find the density function f(x) and the cumulative distribution function F(x). Find $P(X \geq 0)$, $P(0 \leq x \leq 0.3)$ and $P(x = 0.5)$

CHAPTER 5

THE NORMAL DISTRIBUTION.

OBJECTIVES

- CALCULATE PROBABILITIES USING THE STANDARD NORMAL DISTRIBUTION
- STANDARDIZE A NORMAL DISTRIBUTION
- CALCULATE PROBABILITIES USING A STANDARDIZED NORMAL DISTRIBUTION
- CALCULATE PROBABILITIES USING A NORMAL APPROX TO THE BIOMIAL DISTRIBUTION
- APPLY CONTINOUS CORRECTIONS
- SOLVE PROBLEMS INVOLVING A NORMAL AND A BIOMIAL DISTRIBUTIONS

5.1 Introduction

The normal distribution is the most used distribution in every day life. It can be used as a limiting distribution. In industry, business e.t.c the normal distribution can be used to model many data sets e.g. the ages of the employees of a company. If these are not normally distributed, the company may decide to balance its staff, either by employing younger staff and retiring the older staff or simply by employing younger staff.

If x has a normal distribution, its probability density function is given by:

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\tfrac{1}{2}\left(\left(x - \mu\right) / \sigma\right)^2\right) \quad \text{for all real x.}$$

Notation.

$X \sim N(\mu, \sigma^2)$
hence, $E(X) = \mu$ and $\text{var}(x) = \sigma^2$

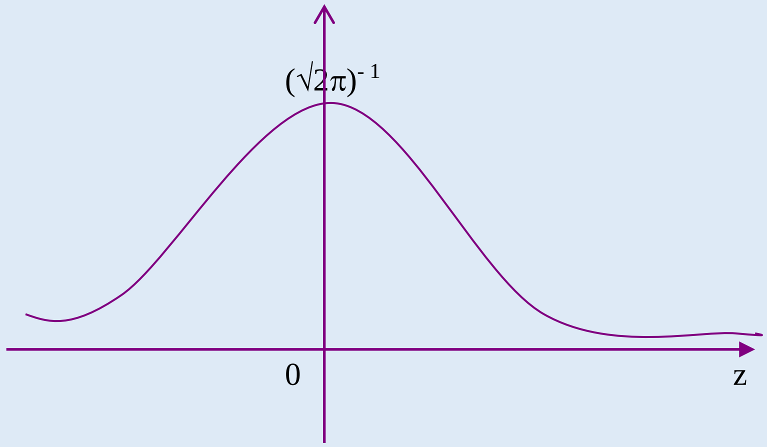
Observe that: **(i) $f(x) \geq 0$ for all real x**

(ii) $$\int_{-\infty}^{+\infty} f(x)\,dx = 1$$

(iii) μ can take any real values and $\sigma \geq 0$

From observation (ii) we note that for different values of μ and / or σ. We obtain different values of f(x).The normal distribution is a continuous distribution, to calculate probabilities we need to integrate the function f (x) in any given interval. At this level such integrals are not dealt with. To solve this problem, we consider the **standard** normal **distribution**, where **$\mu = 0$** and **$\sigma = 1$**. This is achieved by the change of variable $z = (x - \mu)/\sigma$. This process is called **standardisation.** The resultant standard normal distribution function is given by $f(z) = \frac{1}{\sqrt{2\pi}} \exp\left(-\tfrac{1}{2} z^2\right)$

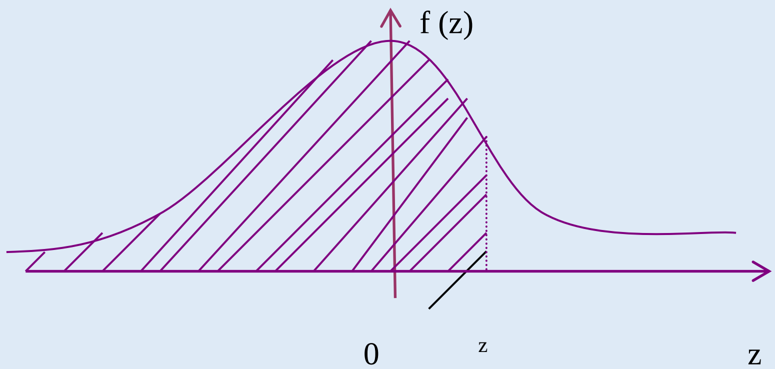
The integrals involving this function cannot be evaluated by the integral techniques studied so far. However, tables for the standard normal distribution are available. These make it easier to compute **any** probability for **any** normal distribution. The graph of the standard normal distribution function is given below.



(i) The graph is bell – shaped.
(ii) The graph is symmetric about the point $\mu = 0$.
(iii) The total area under the graph is 1.

5.2 Calculating probabilities

We use the tables of a standard normal distribution. These are distributed by ZIMSEC, for use in the examination [make sure you have one]. There are many different standard normal distribution tables. We will use the ones where the probability (area) given is that of P $(Z \leq z)$, (see the diagram below).

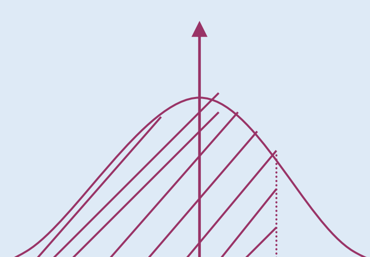


Since the graph is symmetrical with respect to the line $z = 0$, we only use the positive values of z.

Example
If $X \sim N(0;1)$. Find

a) $p(z < 1)$
b) $p(1 < z \leq 2)$
c) $p(z > 2)$
d) $p(z > -2)$
e) $p(-0.46 \leq z < -0.23)$
f) $p(-1 \leq z < 1)$
g) $p(|z| \leq 3.1$
h) $p(|z| > 2.5)$

Solution.
Always make a sketch.



P (z< 1) = 0.8413,

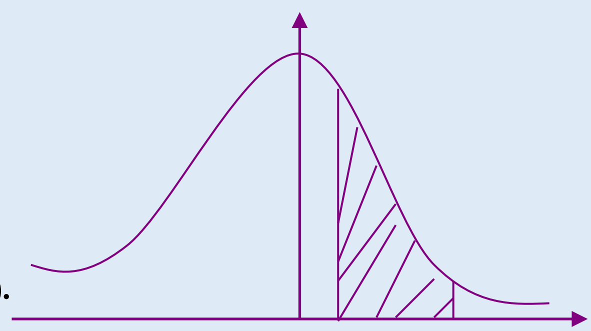
This is the value of the shaded area left of 1.

0 1

							Differences								
z	0.00	0.01	0.02	0.03	------------	0.9	1	2	3	4	5	6	7	8	9
0.0															
0.1															
0.2															
.															
.															
.															
1.0	0.8413														
.															
.															
2.0	0.9772														

Always add differences

b) p(1< z ≤ 2) = F(2) – F(1)
= 0.9772 – 0.8413
= 0.1359

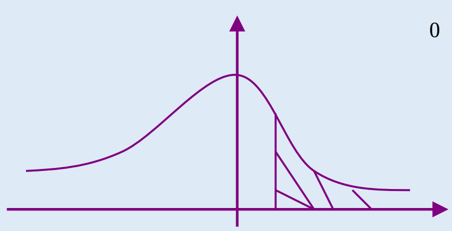


0 1 2

Note:
For the distribution function of z we use ϕ (z).

hence, **p (a < x < b) = ϕ (b) - ϕ (a)**

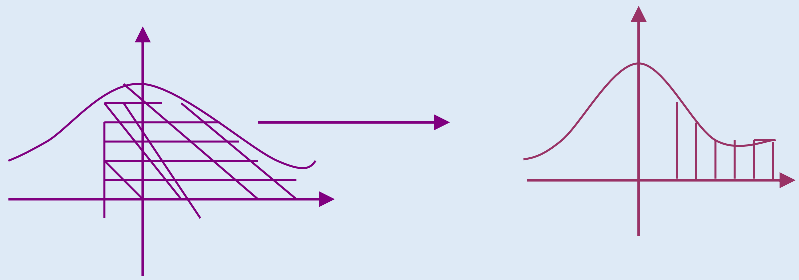
c) p(z > 2) = 1 – p(z≤ 2)
= 1 - ϕ (2)
= 1 – 0.9772
= 0.0228



0 1

hence, p (z > a) = 1 - ϕ (a)

d) p(z > -2) = p(z < 2) = ϕ (2)
= 0.9772.

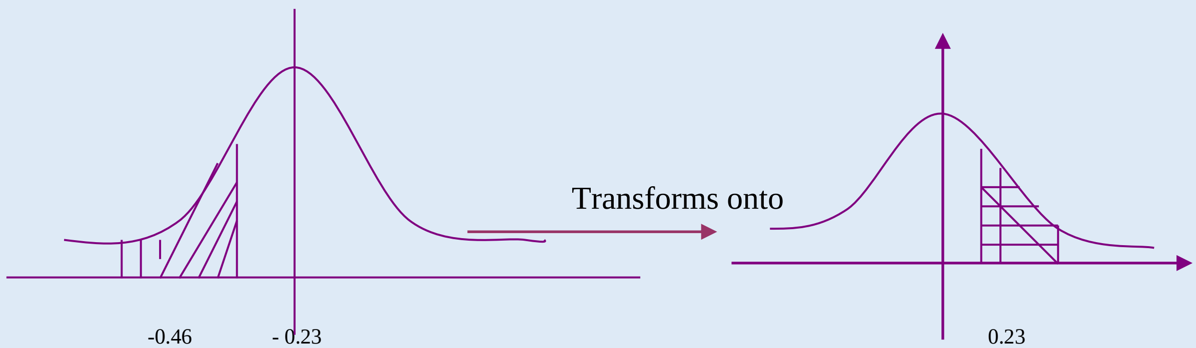


hence, **p (z > - a) = ϕ (a)**

- 2

2

(e)

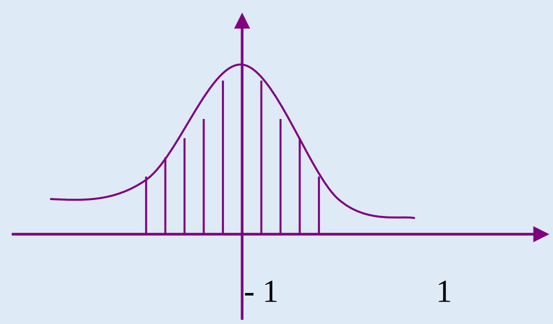


0.46

Hence, p(- 0.46 < z < - 0.23) = p(0.23 < z < 0.46)

i.e. p(-0.46 < z < - 0.23) = ϕ (0.46) - ϕ (0.23)
= 0.6772 – 0.59710
= 0.0862.

f) p(-1 ≤ z < 1) = p(z < 1) – p(z < -1)
= p(z < 1) – p(z>1)
= p(z < 1) – [1 – p(z<1)]
= 2p (z < 1) – 1
= 2 ϕ (1) – 1
= 2 × 0.8413 – 1
= 0.6926.



p (z < -a) = p (z > a) = 1 - ϕ (a)

p(|z| < a) = 2 ϕ (a) – 1.

g) p(|z| < 3.1| = 2 ϕ (3.1) – 1
= 2 x 0.9991 – 1

= 0. 9982

h) p) ($|z| > 2.5$)

= $p(z < -2.5) + p(z > 2.5)$

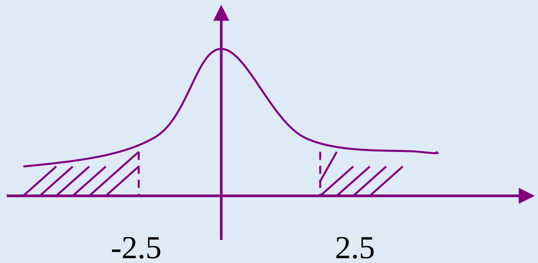
= $2p(z > 2.5)$

= $2[1 - p(z < 2.5)]$

= $2[1 - \phi(2.5)]$

= $2(1 - 0.9938)$

= 0.0124.



$p(|z| > a) = 2[1 - \phi(a)]$.

Example

If $X \sim N(0,1)$

Find a, if

(i) $p(z > a) = 0.9567$ (ii) $p(z < a) = 0.05$ (iii) $p(|z| < a) = 0.651$

Solution.

We are given the probabilities, hence we find the value of z.

(i) $p(z > a) = 0.9567$

hence, a is negative.

i.e. $p(z > a) = \phi(-a)$

Hence, $\phi(-a) = 0.9567$

$-a = Z_{0.9567}$

$a = -Z_{0.9567}$

$a = -1.713$

$[-1.710 + 0.003]$

Note. 0.9567 is nearer to 0.9564 which is less. If the probability cannot be found in the tables, always take the nearer ,but less probability and add the difference

(iii) $p(|z| < a|) = 0.651$

$2\phi(a) - 1 = 0.651$

$2\phi(a) = 1.651$

$\phi(a) = 0.855$

$a = Z_{0.855}$

$a = 1.06 + 0.001$

$a = 1.061$

Example

X ~ N (50 ; 49)

Find (i) p(x > 55) (ii) p(x ≤ 45) (iii) p(51 ≤ x ≤ 65)

Solution.

We standardize the distribution of x using the transformation: $z = \frac{x - \mu}{\sigma}$

(i) $\mu = 50$, $\sigma = \sqrt{49} = 7$, x = 55

hence, $z = (x - \mu) / \sigma$
$= (55 - 50) / 7$
$= 0.7143$

now, $p(x > 55) = p(z > 0.7143)$
$= 1 - \phi(0.7143)$
$= 1 - [\ 0.7611 + 0.0012\ ]$
$= 0.2377$

(ii) $x = 45 \Rightarrow z = (45 - 50) / 7$
$= -0.7143$

hence, $p(x \le 45) = p(z \le -0.7143)$
$= 1 - \phi(0.7143)$
$= 0.2377$

(iii) p(51 ≤ x ≤ 65 = p(51 ≤ z ≤ 65)
$= p(0.143 \le z \le 2.143)$
$= \phi(2.143) - \phi(0.143)$
$= 0.9839 - 0.5569$
= 0. 427

Example

A tea machine is regulated so that it dispenses an average of 15g of tea per cup. If the amount of tea is normally distributed with a standard deviation equal to 0.5g, then;

a) What is the probability that a cup will contain more than 16g?

b) What is the probability that a cup contains between 14.8 and 15.2g?

c) Below what value do we obtain from the smallest 15% of the drinks?

d) How many cups are expected to over flow, if 16.2g cups are used for the next 100 drinks.

Solution.

Let X be the mass of tea in any cup.

$X \sim N(15; 0.5^2)$

a) $p(x > 16) = p\left(z > \frac{16 - 15}{0.5}\right) = p(z > 2)$
$= 1 - \phi(2)$
$= 1 - 0.9772$
$= 0.0228$

b) $p(14.8 \le x \le 15.2) = p\left(\frac{14.8 - 15}{0.5} \le z \le \frac{15.2 - 15}{0.5}\right)$

$= p(- 0.4 \le z \le 0.4)$

$= p(|z| \le 0.4)$

$= 2\,\phi(0.4) - 1$

$= 2 \times 0.6554 - 1$

$= 0.3108$

c) $p(x \le a) = 0.15$

$p(z \le (a - 15) /0.5) = 0.15$

i.e. $1 - \phi((15 - a) / 0.5) = 0.15$

$\phi((15 - a) / 0.5) = 0.85$ **Note: change of sign from a – 15 to 15 - a**

$15 - a = 0.5 \times Z_{0.85}$

$-a = 0.5\,Z_{0.85} - 15$

$a = 15 - 0.5\,Z_{0.85}$

$a = 15 - 0.8 \times 1.036$

$a = 14.482$

∴ The smallest mount of tea dispensed is 14.48g.

d) $E(X) = np$; $n = 100$

$p = p(x > 16.2)$

$= p(z > 2.4)$

$= 1 - \phi(2.4)$

$= 1 - 0.9918$

$= 0.0082$

∴ $E(X) = 100 \times 0.0082$

$= 8.2$

$= 8$

Example

The annual rainfall in Bulawayo is known to be approximately a normally distributed variable with a mean of 70mm and standard deviation of 10mm a year .It is regarded as very wet for the city if the rainfall exceeds 85mm. The table below gives description for other annual rainfalls.

Rainfall for the year	Description of the year.
$x > 85$mm	Very wet
$65 \le x \le 85$	Moderate
$55 \le x < 65$	Fairly dry
$50 \le x \le 55$	Very dry
$x \le 50$	Drought year.

(a) State the probability that the rainfall in 2006 will exceed 70mm.

(b) What is the probability that 2006 will be a very wet year for the city?
(c) What is the probability that a particular year will have an annual rainfall that is fairly dry for the city?
(d) Assume, that the weather for any given year is independent of the weather for any other year, find:
   i) The probability that in a given four year period all four years will be fairly dry for the city.
   ii) The probability that in a given six year period, exactly four years will be fairly dry for the city.
(e) How many mm of annual rainfall is exceeded about 90% of the time.
(f) Show that the probability of a drought year is 0.0228. Assuming the independence of the weather of any other, how often would you expect a drought year to occur? Explain very briefly how you arrived at you answer.

Solution.

Let X represents the rainfall in mm

$X \sim N(70 : 100)$, hence, $\mu = 70$ and $\sigma = 10$.

a) $p(x > 70) = p(z > 0)$
$= 0.5$

b) $p(x > 85) = p(z > 1.5)$
$= 1 - \phi(1.5)$
$= 1 - 0.9332$
$= 0.0668$

(c) $p(55 \leq x < 65) = p\left(\frac{55 - 70}{10} \leq z < \frac{65 - 70}{10}\right)$
$= p(-1.5 \leq z < -0.5)$
$= p(0.5 < z \leq 1.5)$
$= \phi(1.5) - \phi(0.5)$
$= 0.9332 - 0.6915$
$= 0.2417$

d) (i) p (4 years fairly dry) $= {}^4C_4 (0.2417)^4 (1 - 0.2417)^0$
$= 0.2417^4$
$= 0.0034$

(ii) Let Y represent the number of years of fairly dry years, then :

$Y \sim Bin(6; 0.2417)$

$p(y = 4) = {}^6C_4 \times 0.2417^4 \times (0.7583)^2$

$= 0.0294$

e) Let a be the amount of rain fall then, $p(x > a) = 0.90$

$$p\left(z > \frac{a - 70}{10}\right) = 0.90$$

$$p\left(z < \frac{70 - a}{10}\right) = 0.90$$

Note change of sign.

$$\Phi\left(\frac{70 - a}{10}\right) = 0.9$$

$$\frac{70 - a}{10} = Z_{0.9}$$

$$-a = 10 \times Z_{0.9} - 70$$
$$a = 70 - 10 \times Z_{0.9}$$
$$a = 70 - 10 \times 1.282$$
$$a = 57.18\text{mm}$$

f) p(drought year)= $p(x < 50)$

$= p(z < -2)$

$= 1 - \phi(2)$

$= 1 - 0.9772$

$= 0.0228$

Let y be the number of years until a drought year occurs. Hence, Y~ Geo (0.0228).

and $E(Y) = \frac{1}{p}$

$= \frac{1}{0.0228}$

= 44 years. i.e. a drought year will occur after every 44 years

5.3 Normal Approximation to the Binomial Distribution.

Let X ~ Bin(n ;p) such that n is big, usually, $n \geq 30$, $np > 5$ and $nq > 5$, hence,

X~N (np; npq)

Continuity correction.

If X is a discrete variable, for X to become continuous we apply what we term as **continuous correction**. Either adding or subtracting 0.5 from the value of x achieves this.

Recall that if X is discrete $p(X = x) \neq 0$
But if X is continuous $p(X = x) = 0$

Example

Apply continuity correction to the following:

(i) $P(x > 40)$ (ii) $p(x \geq 40)$ (iii) $p(x = 40)$
(iv) $p(x < 40)$ (v) $P(x \leq 40)$

Solution.

(i) $p(x > 40) = p(x \geq 40.5)$ (ii) $p(x \geq 40) = p(x \geq 39.5)$
(iii) $p(x = 40) = p(39.5 \leq x < 40.5)$ (iv) $p(x < 40) = p(x < 39.5)$
(v) $P(x \leq 40) = p(x < 40.5)$

Example.

A die is tossed 120 times;
a) Find the probability that the face five appears 15 times or more.
b) Find the probability that the face five appears exactly 18 times.

Solution.
Assuming the die is fair;
$X \sim Bin(120; 1/6)$: $p(5) = 1/6$ and $p(\text{not } 5) = 5/6$
$E(X) = np$
$= 120 \times 1/6$
$= 20$, **hence use a normal approximation.**
$Var(x) = npq$
$= 120 \times 1/6 \times 5/6$
$= 50/3$

Remember to apply continuity correction

a) $p(x \geq 15) = p(x \geq 14.5)$
$= p\left(z \geq \frac{14.5 - 20}{\sqrt{(50/3)}}\right)$
$= p(z \geq -1.3472)$
$= \Phi(1.3472)$
$= 0.911$

Use the calculator to check $P(x \geq 15)$ using the Binomial Distribution what do you think about the approximation?

b) $p(X = 18) = p(17.5 \leq x < 18.5)$
$= P(-0.612 \leq z \leq -0.367)$
$= \Phi(0.612) - \Phi(0.367)$
$= 0.7298 - 0.6432$
$= 0.0866$

Use the calculator to check the approximation.

Example

In Zimbabwe it is estimated that the rate of HIV infection is 21.6%. Fifty expectant mothers were tested for HIV in July 2005. Find the probability that between 30 and 45 people, inclusive were found to be negative.

Solution.

Let X be the number of HIV negative people, hence, $X \sim Bin(50; 0.784)$

$$E(X) = 50 \times 0.784 = 39.2$$

$$Var(X) = npq = 50 \times 0.784 \times 0.216 = 8.4672$$

$$p(30 \leq x \leq 45) = p(29.5 \leq x \leq 45.5)$$
$$= p\left(\frac{29.5 - 39.2}{\sqrt{8.4672}} \leq z < \frac{45.5 - 39.2}{\sqrt{8.4672}}\right)$$
$$= p(-3.334 \leq z < 2.165)$$
$$= \Phi(2.165) + \Phi(3.334) - 1$$
$$= 0.9848 + 0.9986 - 1$$
$$= 0.9834$$

Example

The proportion of the population who is colour- blind is 8%.

(i) A random sample of 11 people is taken . Find the probability that at least two are colour – blind.

(ii) Using a suitable approximation, find the probability that at least 11 in a random sample of 200 people are colour- blind.

Solution

Let X be the number of people who are colour-blind, hence, $X \sim Bin(11\ 0.08)$.

(i) $P(X \geq 2) = 1 - P(X \leq 1) = 1 - P(X = 0) - P(X = 1)$
$$= 1 - {}^{11}C_0 0.08^0 \times 0.92^{11} - {}^{11}C_1\ 0.08 \times 0.92^{10}$$
$$= 1 - 0.92^{11} - 11 \times 0.08 \times 0.92^{10}$$
$$= 0.2181$$

(ii) $E(X) = 200 \times 0.08 = 16$ $\quad Var(X) = 200 \times 0.08 \times 0.92 = 14.72$

hence, $X \sim N(16; 14.72)$ $\quad P(X \geq 11) = P(X \geq 10.5)$
$$= P(z \geq -1.4335)$$
$$= \Phi(1.4335)$$
$$= 0.9242$$

5.4 Examination Type Questions

1. The marks in an examination were normally distributed with mean μ and standard deviation σ . 10% of the candidates had more than 70 marks and 20% had less than 35 marks. Find the values of μ and σ.

2. The masses of Bakers Inn bread is normally distributed with mean 760 grams and standard deviation of 5 grams. The nominal weight of a bread is 750 grams. What is the probability that a randomly chosen bread is under weight? If nine loaves are chosen at random what is the probability that at most two loaves are under weight?

3 A random variable X is such that $X \sim N(-6,16)$. Find the probability that (a) an item chosen at random will have a positive value, (b) out of nine items chosen at random, exactly three have a positive value.

4. It is estimated that 0.4 of the population of Bulawayo watched the match between Dynamos and Highlanders on television. If random samples of 200 people are interviewed, calculate the mean and variance of the number of people from these samples who watched the match on television. Find the probability that in a random sample of 200 people, more than 60 people watched the match on television.

5. If $X \sim N(12.3; 2.5^2)$, find $P(X < 17.3)$

6. In a warehouse of agricultural produce ,there are many watermelons. The mass X of a watermelon is normally distributed with a mean mass of 8kg and a standard deviation of 2kg.

a) Find the probability that a randomly selected melon has a mass between 7kg to 11kg.
b) find the value of h such that $P(X \leq h) = 0.3974$
c) Find the probability that in a sample of 10 watermelons ,5 weigh between 7 and 11 kg.

7. The potency (power) of a particular medicine is required to be in the range 47 to 56 units.

The potency of a batch of medicine is normally distributed, with a mean of 50 and a variance of 9.

a) Find the expected proportion of batches of this medicine which meet the potency requirement.

The potency of each batch must be checked and corrected if necessary. It costs \$50 to produce any particular batch. If the potency is too high, it costs an additional \$20 to correct it, but if the potency is too low ,it costs an additional \$25 to correct it. Let Y denote the total cost of producing a batch with the potency within the required limits.

b) Specify the possible values that Y can take and find the probability associated with each of these values.
c) Find the men value of Y i.e. the expected costs of a batch.

8. A chain is made from five links, which are selected at random from a population of links. Assume that the strengths of the links are normally distributed with mean 100 units and standard deviation 4 units.

a) Find the probability that a randomly selected link has strength of at least 96 units.
b) calculate the probability that all five links in a chain have strengths of at least 96 units.
c) Calculate the probability that at least one link in a chain has a strength of at least 96 units.
d) Let C denote the strengths of a randomly selected chain. Find the probability that C exceeds 93 units.

9. Sipho is a contestant in the long jump event at the Zimbabwean Champions. In a particular jump, Sipho jumps X metres. X is a normally distributed random variable with mean 7.3 and standard deviation 0.3.

a) For any jump, find the probability that Sipho jumps:
   i) more than 7.6 metres ii)less than 6.7 metres
   ii) between 6.7 and 7.6 metres

During the championships each competitor in the long jump event has five jumps**.**

b) For five jumps, find the probability, stated to 3s.f. that:
   i) Sipho's first two jumps are both less than 7.6metres and his last three jumps are all more than 7.6metres
   ii) at least four of Sipho's jumps are more than 7.6 metres
   iii) Sipho's final jump is more than 7.6metrs, given that at least four of his jumps are more 7.6metres
c) Between what two distances symmetrically placed about the mean would 68% of Sipho's jumps lie?
d) If 80% of Sipho's jumps are greater than h metres, what is the value of h?
e) During training for the championships, Sipho had 75 practice jumps. How many of these jumps would be expected to be more than 7.6metres?
   After his five Zimbabwean championship jumps, Sipho's best jump was 7.9 metres. He was leading the event with Zola's final jump still to come. In a particular jump, Zola jumps Ymetres, where Y is a normally distributed random variable with variance 0.25
f) If the probability that Zola's final jump is more than 7.9 metres is 0.25, find the mean Of distances that Zola jumps correct to 2 d.p.

10. Yandile has measured his heart rate, after morning run, over several weeks. He has found that his heart rate in beats per minute is normally distributed with mean value of 68 beats per minute and a standard deviation of 6 beats per minute. After a particular morning run, what is the probability that his heart rate will exceed 77 beats per minute?

CHAPTER 6

BASIC ASSUMPTIONS AND TERMINOLOGY

OBJECTIVES

- CALCULATE OF TWO OR MORE FORCES
- CALCULATE AN ANGLE BETWEEN TWO FORCES ACTING AT ANGLE
- RESOLVE A VECTOR ONTO TWO PERPENDICULAR COMPONENTS
- APPLY Lamis THEOREM
- RESOLVE FORCES USING Newton's laws of motion
- Calculate acceleration, velocity and DISTANCE OF A MOVING PARTICLE OVER SMOOTH AND ROUGH PLANES

6.1 Definition of terms

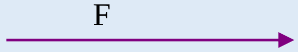
- A **particle** is a body whose dimensions are so small compared with other lengths involved that its position in space can be represented by a single point. For example a car can be represented as a particle
- A **bead** is a particle which is assumed to have a hole drilled through it so that it may be threaded onto a string or wire.

- A **light object** is one whose mass is so small compared with other masses being considered that the mass may be considered to be zero. A light string is one example.
- An **inextensible string** or **inelastic** is a string whose length remains the same whether motion is taking place or not.
- A **smooth surface** is one which offers so little frictional resistance to the motion of a body sliding across it that the friction may be ignored. a sheet of ice may be an example.
- A **smooth pulley** is one with no friction in its bearings.
- A **peg** is a pin or support from which a body may be hung or on which a body may rest. There is only one point of contact between the peg and body in either case

6.2 Forces

Forces are always represented as vectors, hence a force has a magnitude and a direction. Geometrical, it is represented by a line segment with an arrow.

Example $\vec{F}$

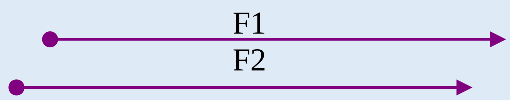


A particle is considered a dimensionless material. e.g. a car can be considered a particle.

The Resultant of forces acting on a particle.

The resultant is the sum of the two forces

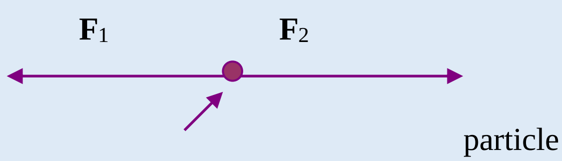
(i) Given two forces acting towards one direction.



$R = F_2 + F_1$

The resultant is the difference between the two forces and the particle is moving in the direction of the larger force in magnitude

(i) Given two forces acting towards different directions



If, $F_2 > F_1$

Then $\mathbf{R} = \mathbf{F}_2 - \mathbf{F}_1$

If, $F_1 = F_2$ in magnitude then, $\mathbf{R} = \mathbf{0}$, i.e. the system is in equilibrium.

Example
Find the resultant of the following forces:
(a) $\mathbf{F}_1 = 4i + 4j$ and $\mathbf{F}_2 = 6i + 6j$ (b) $\mathbf{F}_1 = 12j$ and $\mathbf{F}_2 = -5j$

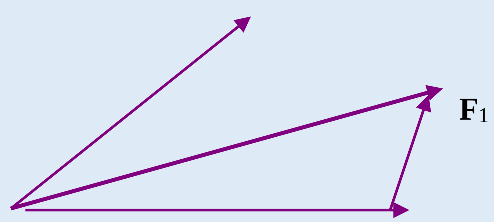
Solution
(a) These two forces are acting along the same line of action, hence,

$\mathbf{R} = \mathbf{F}_1 + \mathbf{F}_2$
$= 4\mathbf{i} + 4\mathbf{j} + 6\mathbf{i} + 6\mathbf{j}$
$= 10\mathbf{i} + 10\mathbf{j}$, hence, the equilibrant is $-10\mathbf{i} - 10\mathbf{j}$
$R = \sqrt{(10^2 + 10^2)}$
$= 10\sqrt{2}$

(b) $\mathbf{R} = 12\mathbf{j} + (-5\mathbf{j}) = 7\mathbf{j}$, hence, the equilibrant is $-7\mathbf{j}$, and $R = 7$. The particle is moving in the direction of $\mathbf{F}_1$

The resultant of two forces acting at an angle θ.

(a) Using the parallelogram rule



Method: translate the tail of F_1, to the head of F_2 at an angle of $(180 - \theta)^0$. The resultant is given by the magnitude of R: the diagonal of the parallelogram.

$\mathbf{F}_2$ $180 - \theta$

Hence using the cosine rule, we have

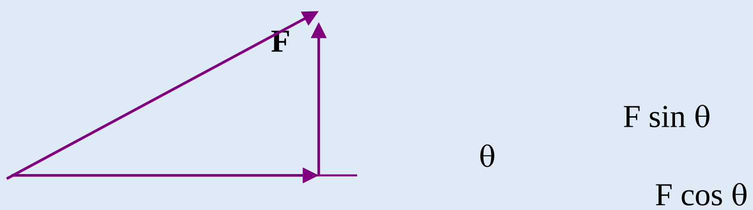
$R^2 = F_1^2 + F_2^2 - 2 F_1 \times F_2 \cos(180 - \theta)$
$R^2 = F_1^2 + F_2^2 - 2 F_1 \times F_2 \cos(180 - \theta)$
$R = \sqrt{(F_1^2 + F_2^2 - 2F \times F_2 \cos(180 - \theta))}$

Example
Given two forces $\mathbf{F}_1 = 4N$ and $\mathbf{F}_2 = 5N$, inclined at 60^0, find their resultant.
Solution
$R = \sqrt{(4^2 + 5^2 - 2 \times 4 \times 5 \times \cos 60^0)}$

$= \sqrt{(16 + 25 - 20)}$
$= \sqrt{21}$N

The other method called method of polygons can be used for any number of forces acting on the particle.
However we are not going to navigate through this technique., there are more elegant techniques to explore

Resolving a vector onto two perpendicular components.

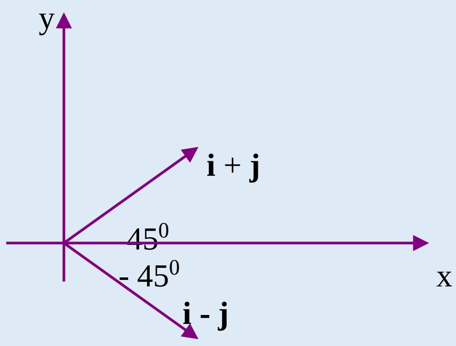


Hence, $\mathbf{F} = F\cos\theta\,\mathbf{i} + F\sin\theta\,\mathbf{j}$
$F\cos\theta$ is the horizontal component and $F\sin\theta$ is the vertical component.

b) **By resolving vectors.**
Example
Find the resultant of the following vectors.

$\mathbf{F}_1 = \mathbf{i} - \mathbf{j}$ and $\mathbf{F}_2 = \mathbf{i} + \mathbf{j}$

Solution.



<u>In the X – direction</u>

The component of F_1 is $X = F_1\cos(-45^0) + F_2\cos45^0$
$= \sqrt{(1+1)}\cos 45^0 + \sqrt{(1+1)}\cos 45^0$
$= \sqrt{2} \times \underline{\sqrt{2}} \quad + \sqrt{2} \times \underline{\sqrt{2}}$

2 2

= 2

Hence: $\mathbf{X} = (1 + 1)\,\mathbf{i} = 2\mathbf{i}$

<u>In the Y – direction</u>

Component $F_1 =$ $F_1 \sin(-45^0)$

Component of $F_2 =$ $F_2 \sin 45^0$

Hence: $\mathbf{Y} = (1 - 1)\,\mathbf{i} = 0\mathbf{i}$

Then $R^2 = X^2 + Y^2$

$R^2 = 2^2 + 0^2$

$R = \sqrt{4}$

$R = 2$

<u>$\mathbf{R} = 2\mathbf{i}$</u>

Example

Find the magnitude and the direction of the resultant of the following forces;
-i + 2j , 3i – 4j and –2i+5j.

Solution

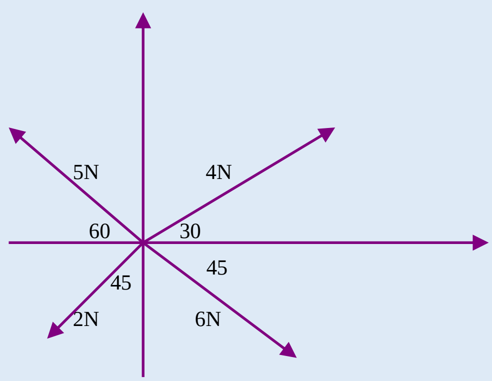
$X = -1 + 3 - 2$ $\qquad$ $Y = 2 - 4 + 5$

$= 0$ $\qquad$ $= 3$

hence, $\mathbf{R} = 3\mathbf{j}$ and $R = 3$. **R** acts in the positive direction of the y – axis

Example

Find the resultant and the direction of the following forces acting on a particle.



Solution

$X = -5\cos 60 - 2\cos 45 + 4\cos 30 + 6\cos 45$; $Y = 5\sin 60 + 4\sin 30 - 2\sin 45 - 6\sin 45$

$= -2.5 - \sqrt{2} + 2\sqrt{3} + 3\sqrt{2}$ $\qquad$ $= 2.5\sqrt{3} + 2 - \sqrt{2} - 3\sqrt{2}$

$= -2.5 + 2\sqrt{3} + 2\sqrt{2}$ $\qquad$ $= 2.5\sqrt{3} + 2 - 4\sqrt{2}$

$= 3.793\text{N}$ $\qquad$ $= 0.673\text{N}$

R = 3.793i + 0.673j

$\alpha = \tan^{-1}\left(\frac{0.673}{3,793}\right)$; $\alpha = 10^0$ to the horizontal

This technique can be applied to any situation involving more than two forces; more examples are given in the subsequent sections

The force that pulls all bodies towards the earth is called **the gravitational force** and its magnitude is called **the weight** and is given by **mg**, where **m** is the mass of the object and **g** is a constant**. g** = **9.81 m / s²** and is called the gravitational acceleration

The weight **always** acts vertically downwards

6.3 Lami' s Theorem

Used when three forces are acting at equilibrium

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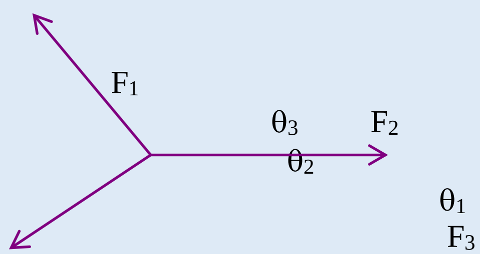
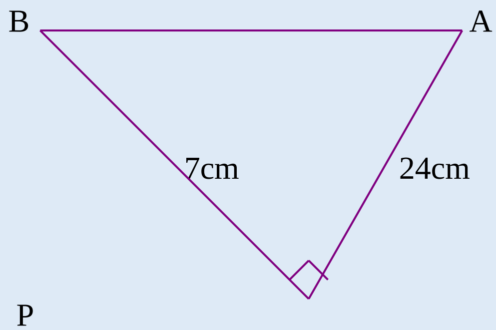


$F_1 / \sin\theta_1 = F_2 / \sin\theta_2 = F_3 / \sin\theta_3$

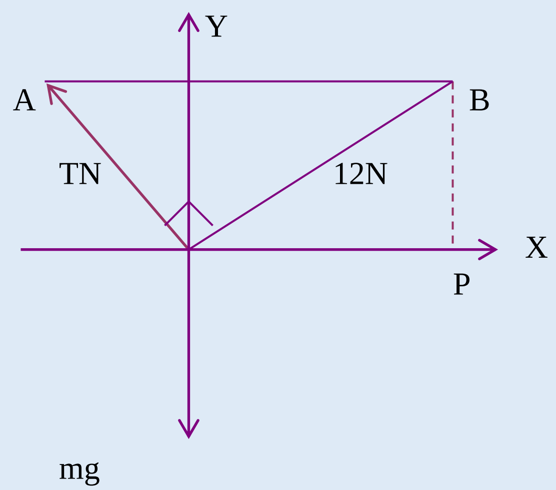
Example

The diagram shows a particle P of mass in kg attached to points A and B by two light inextensible strings, AP of length 24 cm and BP of length 7 cm.

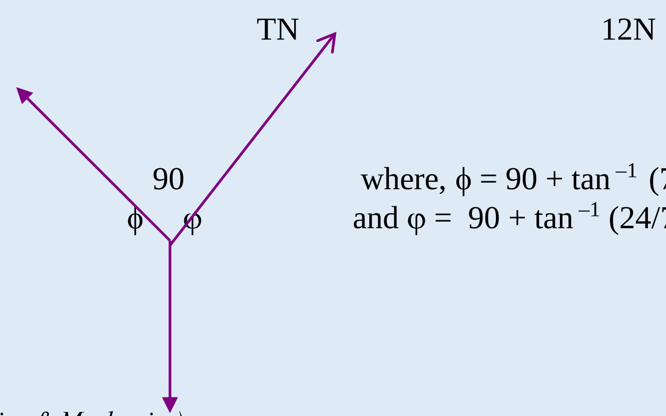


The points A and B are at the same horizontal level and P hangs freely with an angle APB equal to 90^0. Given that the tension in the string BP is 12N. Find
(i) the tension in the string AP?
(ii) the mass of P?

Solution.
The statement: "P hangs freely" means the system is in equilibrium. There are three forces acting on point P; the tension in the string AP, the tension in the string BP and the weight of particle P i.e. mg. Which is acting vertically downwards.



Since A and B are on the same horizontal level hence AB is parallel to PX
Angle B = angle BPX and angle A= angle APC
Hence :
B = $\tan^{-1}(24/7)$ A = $\tan^{-1}(7/24)$



where, $\phi = 90 + \tan^{-1}(7/24)$
and $\varphi = 90 + \tan^{-1}(24/7)$

mg

(i) $T/\sin(90 + \tan^{-1}(24/7)) = 12/\sin(90 + \tan^{-1}(7/24))$

$T = 12 \sin(90 + \tan^{-1}(24/7)) / \sin(90 + \tan^{-1}(7/24))$

T = 3.5N to 2 s. f.

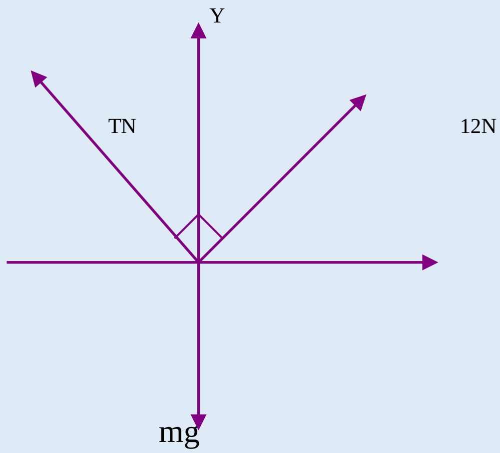
(ii) $\frac{mg}{\sin 90} = \frac{12}{\sin(90 + \tan^{-1}(7/24))}$

m = 1.25 kg to 2 s.f.

By solving vectors:

X = 0 since the system is in equilibrium. Hence resolving in the X – direction to obtain:

$X = 12\cos\theta - T\cos\theta$, hence, $12\cos\theta - T\cos\theta = 0$



ϕ θ

$\cos\theta = 7/25$

$\cos\phi = 24/25$

Hence: $12 \times 7/25 - T \times 24/25 = 0$

$T = \frac{12 \times 7/25}{24/25}$

T = 3.5N

(ii) **Also** Y = 0

$Y = mg – 12 \sin\theta - T \sin\phi$

$\sin\theta = 24/25, \sin\phi = 7 / 25$

Hence: $mg = 12 \times 24 /25 + 7/2 \times 7/25$

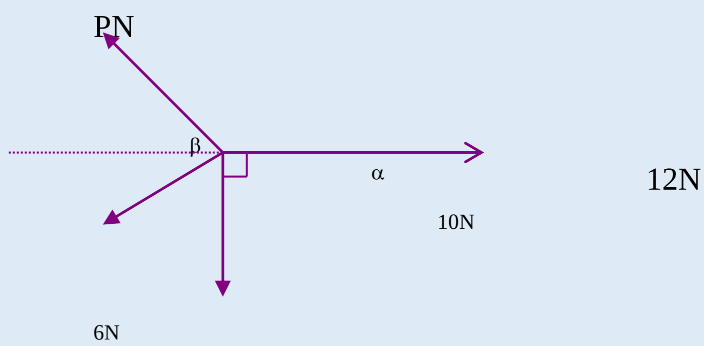
$mg = 288/25+49/50$

$mg = 12.5;$ $g= 9.81$

$m = 1.3kg.$

Example

The four coplanar forces shown in the diagram keep a particle in equilibrium. Given that $\sin \alpha = 4/5$, find the value of P and β



Solution

We cannot use Lami's theorem , since there are 4 forces in equilibrium not three. We will use the method of resolving vectors.

X –direction

$X = 0$

$X = 12 – 10\cos\alpha - P\cos\beta.$

Hence: $10\cos\alpha + P\cos\beta = 12$

$\sin\alpha = 4/5$ and $\cos \alpha = 3/5$

$P\cos \beta = 12 – 10 \times 3/5$

$P\cos \beta = 12 – 6$

$P\cos \beta = 6$ ……..(1)

Y -direction

$Y = 0$

$Y = 6 + 10\sin \alpha - P\sin \beta$

$0 = 6 + 10 \times 4/5 - P\sin \beta$

$P\sin \beta = 6 + 8$

$P\sin \beta = 14$ …….(2)

Divide (2) by (1) to obtain:

$P\sin\beta / P\cos\beta = 14/6,$ $\tan \beta = 7/3$

$\beta = \tan^{-1} (7/3)$

$\beta = 66.8^0$ to 1 d. p.

from (1) we have, $P = 6/\cos\beta$

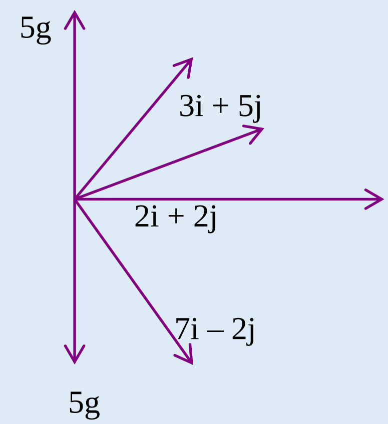
$P = 2\sqrt{58}N$

$P = 15.2N$ to 1 d. p.

Example

Three coplanar forces act upon a mass of 5 kg. The forces are represented by the vectors 7i – 2**j**, 3**i** + 5**j** and 2**i** + 2**j**. Calculate the resultant acceleration of the mass in magnitude and direction.

Solution:



$X = 7 + 2 + 3$

$= 12N$

$Y = 5 + 2 – 2 + 5g – 5g$

$Y = 5N$

$R = \sqrt{X^2 + Y^2}$

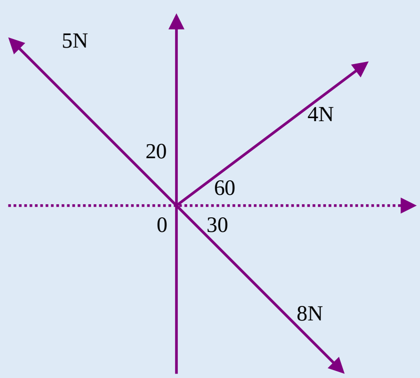
$R = \sqrt{12^2 + 5^2}$

$R = 13 m/s^2$

$\theta = \tan^{-1}(5/12)$ to the horizontal

Example

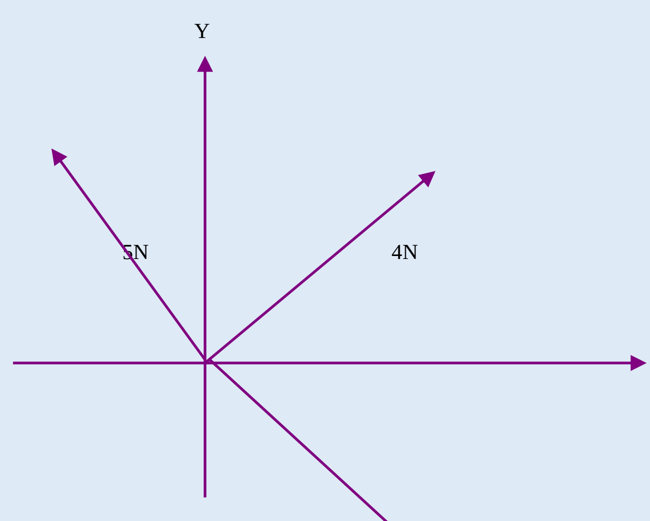
Find the magnitude of the resultant of the forces 4N, 5N and 8N shown in the diagram.



Solution:

$X = 4\cos 60^0 – 5\cos 70^0 + \cos 30^0$

$X = 7.218N$



$Y = 4\sin 60^0 – 8\sin 30^0 + 5\sin 70^0$
$Y = 4.16N.$

70 60
30

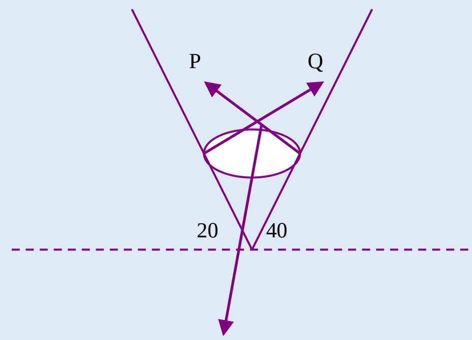
8N

$R = \sqrt{X^2 + Y^2}$
$= (4.16^2 + 7.2182)^{½}$
$= 8.3N$ to 2 sig. fig.

The direction of the resultant is
$\theta = \tan^{-1} (4. 163/7.218)$
$\theta = 29.97^0$
$\theta = 30^0$ to the horizontal i.e. the x – axis

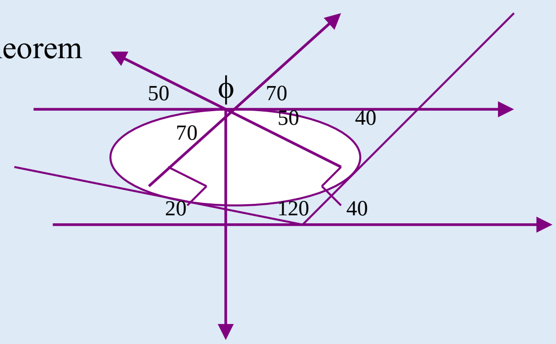
Example

The diagram shows a small ball, of mass 2kg resting in the horizontal groove between two smooth planes inclined at 20^0 and 40^0 to the horizontal.
Find the magnitudes of the contact forces P and Q



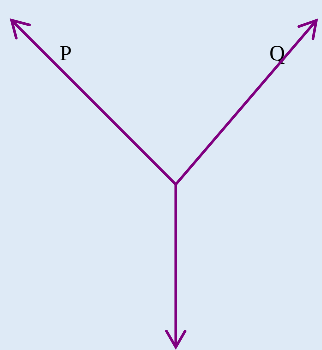
Solution:

Since there are three forces in equilibrium we use Lami's theorem
$2\phi + 2 \times 70 + 2 \times 50 = 360$
$2\phi = 120$
$\phi = 60^0$



Hence: 2g/sin60 = P/sin140 = Q/sin 140

where 2g/sin60 = P/sin140
P = 2gsin140/sin60
P = 7.7N



P = 7.7N to 2 s. f.

60

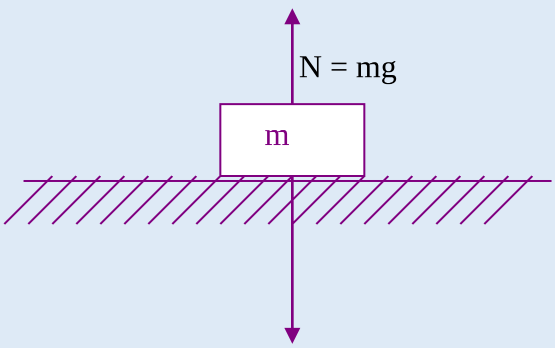
140° 160°

And Q = 2gsin140/sin60
Q = 15N

2g

6.4 Contact Forces

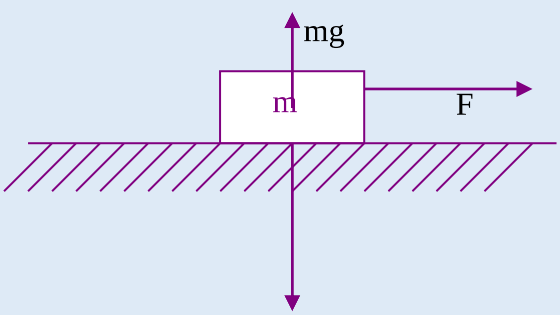
(a) For a body resting on a horizontal plane.



N is called the normal force and it acts vertically opposite the weight. It is equal in magnitude to the weight.

mg = weight.

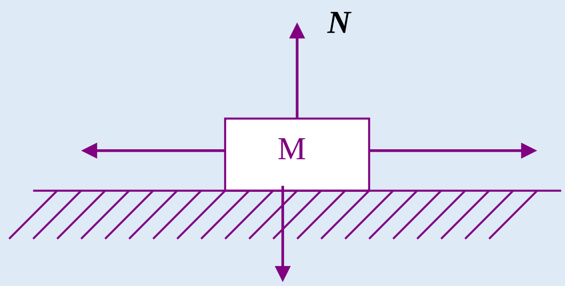
(b) For a body of mass m kg moving along a smooth horizontal plane.



The body is only moving in the direction of the force F, and F is the resultant force.

mg

(c) For a body moving along a rough horizontal plane*:*



Where F is the friction force and W is the weight

W = N F P

The friction force has a maximum value which depends on the normal reaction N, hence, friction force is proportional to the normal reaction. i.e.: W

$$F_{max} \propto N$$

Hence $$F_{max} = \mu N$$

Where μ is called the coefficient of friction.

If,

$F \leq F_{max} = \mu N$, then, P = F (no motion there is equilibrium)

$F = F_{max} = \mu$ N, then P = F (limiting equilibrium i.e. the particle is about to in the direction of the force P)

Otherwise

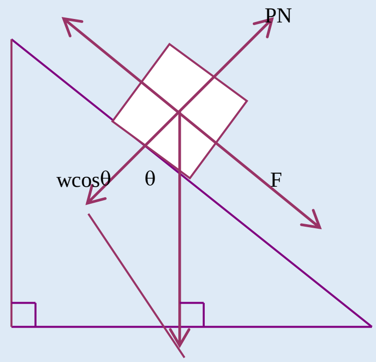
$P > \mu N$, then, $P > F$ (motion in direction of P i.e. the particle is moving in the direction 0f the force P).

Resolving on inclined planes.

W = mg (weight)

$F = \mu$ W cos θ

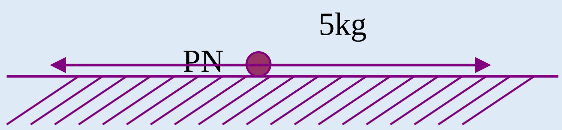
Frictional forces ALWAYS act in the opposite direction of a moving particle



wcosθ θ

w

Example



60N

The diagram shows a body of mass 5kg on a rough horizontal surface. The coefficient of friction between the body and the surface is 1/4. The body remains in equilibrium when acted on by force PN and 60N as shown. Find the ranges of values of P.

Solution:

$\mu = ¼$

$F_r = \mu N$

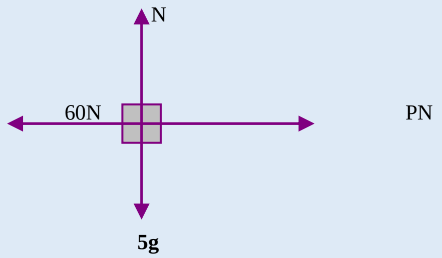
$F_r = 1/4 \times 5g$

$F_r = 1.25g$

If there is no motion, hence,

$F_r \leq \mu N$

$\mathbf{F_r \leq 1.25g}$



The movement is only in the X – direction, hence, we assume $P \leq 60N$, the body has a tendency to move in the direction of the 60N force hence:

$P + F = 60$ for equilibrium

Then; $\mathbf{P + \mu N \geq 60}$ **…….(1)** since $Fr \leq \mu N$

Similarly for $P \geq 60$, $60+F = P$, hence, $60 + \mu N \geq P$ **(2)**

From (1) $P \geq 60 - \mu N$ (3), hence combining **(2)** and **(3)** to obtain;

$$60 - \mu N \leq P \leq 60 + \mu N$$

$60 – 1.25g \leq P \leq 60 + 1.25g$, taking $g = 9.81$

$\mathbf{48 \leq P \leq 72}$ **to 2 s. f.**

Example

A body of mass M kg rests on a rough plane, which is inclined at θ to the horizontal. The coefficient of friction between the body and the plane is μ. Show that for a motion to take place down the plane $\tan \theta > \mu$

Solution:

For the motion to take place:

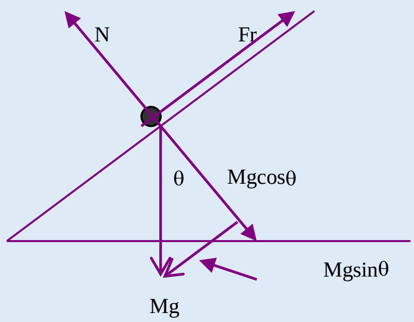
$Mg \sin \theta >$ frictional force

$Mg \sin \theta > \mu N$

$Mg \sin \theta > \mu Mg \cos \theta$

$Mg \sin \theta / Mg\cos\theta > \mu$

$\tan \theta > \mu$ (shown)



Example

"A particle is in limiting equilibrium" what do you understand by this statement?

A particle of mass 5 kg lies on a rough horizontal plane and is acted on by an upward force of P N at an angle of α to the horizontal, where $\cos \alpha = 0.8$. The coefficient of friction between the particle and the plane is 0.6. Find the value of P when the particle is in limiting equilibrium.

Solution:

(a) Limiting equilibrium if $F_r = \mu N$ i.e. if the particle is about to move

(b) $Y = 0$

$Y = P\sin\alpha - 5g + N$

$N = -P\sin\alpha + 5g$

$N = 5g - 0.6P$

Hence: $Fr = 0.6\,(5g - 3/5\,P)$

$X = 0$

$X = P\cos\alpha - F_r.$

$P\cos\alpha - (3g - 9/25P) = 0$

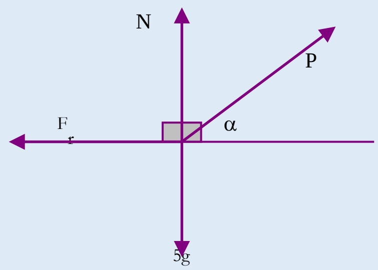
$0.8P = 3g - 0.36P$

$0.8\,P + 0.36P = 3g$

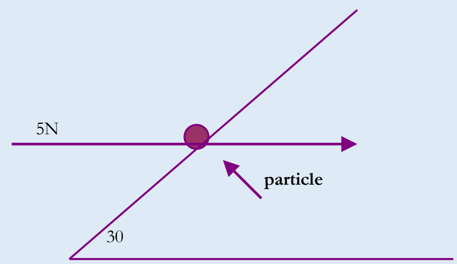
$1.16P = 3g$

$P = 3g/1.16$

$P = 25.37$ N



Example



A particle of mass 0.6kg is held in equilibrium on a rough plane, inclined at 30^0 to the horizontal, by means of a horizontal force of magnitude 5N. Given that the equilibrium is limiting, with the particle about to slip up a line of greatest slope of the plane, find the coefficient of friction between the particle and the plane

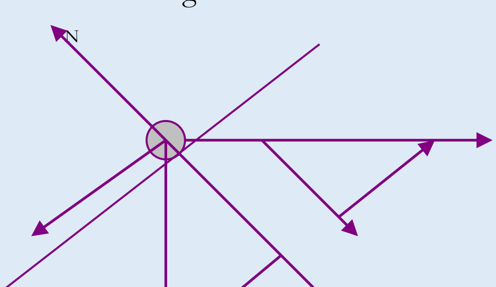
Solution

If the equilibrium is limiting, hence $F_r = \mu N$

The particle is about to slip upwards, hence the frictional force is acting downwards.

$X = 0$ and $Y = 0$. Hence, $N = 5\cos 60 + 0.6g\cos 30°$

$$\mathbf{N = 2.5 + 0.3\sqrt{3}g}$$



$F_r + 0.6g \sin 30 = 5 \sin 60$

$F_r = 5 \sin 60 - 0.6g \sin 30°$

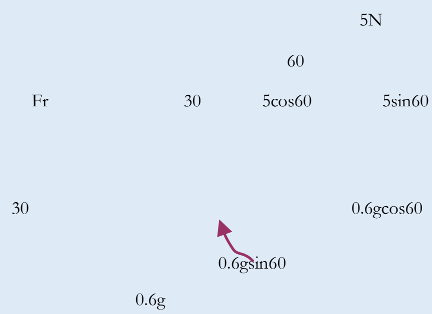
$F_r = 2.5\sqrt{3} - 0.3g$

$F_r = \mu N$

$2.5\sqrt{3} - 0.3g = \mu\ (5/2 + 0.3\sqrt{3}g)$

$\mu = (2.5\sqrt{3} - 0.3g) / (2.5 + 0.3\sqrt{3}g)$

$\mu = 0.183$



6.3 Kinematics

It refers to the movement of objects without considering the cause of the movement.

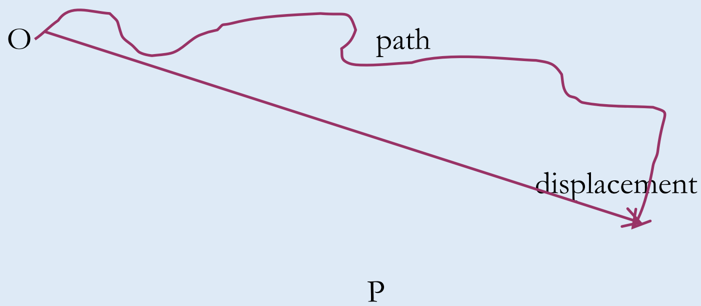
Rate of change.

"Rate of " simply means, "divided by time"

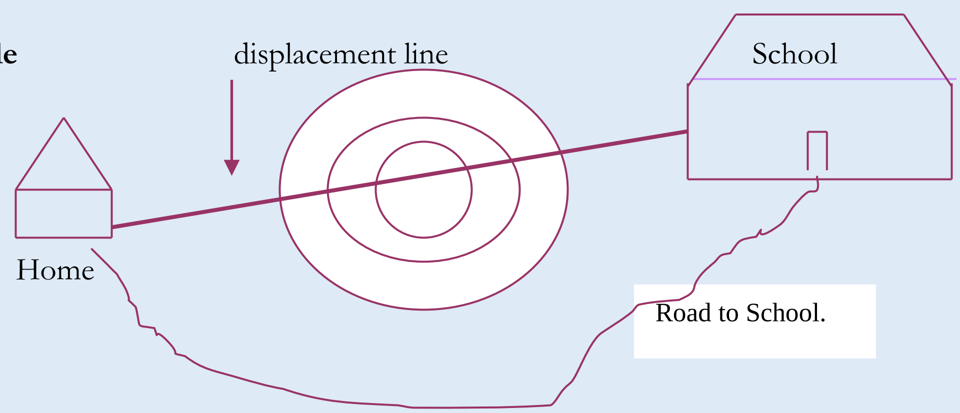
Displacement

Displacement is the straight line joining two end points. It is a vector.

Displacement from O to P



Example



Velocity

It is the rate at which the displacement changes with time, hence velocity is a vector.

Distance.

The length of the path actually followed. Usually it is a curve.

Speed.

Change of distance with time.

When traveling between to places, usually, you make a number of stops along the way. We then talk of the average speed.

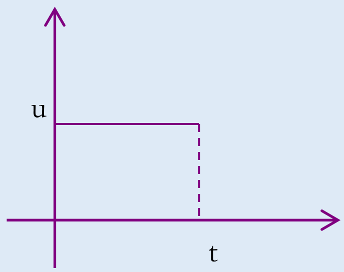
Average speed = Distance Traveled/Total Time Taken.

Similarly **Average velocity =** $\frac{\text{Total Displacement}}{\text{Time Taken.}}$

Acceleration = $\frac{\text{Change of Velocity}}{\text{Time Taken.}}$

Speed – Time Graphs.

If the vehicle moves with a constant speed, the speed time graph is as follows:



u = Speed = Constant.

The area of the rectangle is given by ut i.e. **s = ut**, where s is the distance travelled in time t, Now consider a case where the vehicle starts moving at a constant speed u and finish at a constant speed v. Hence

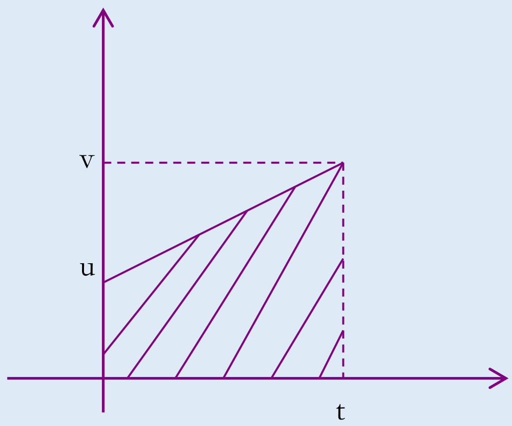
Average speed $= \frac{1}{2}(u + v)$

$\frac{1}{2}(u + v) = s/t$

$s = \frac{1}{2}t(u + v)$ (1)

Using the trapezium rule we have

$s = \frac{1}{2} t (u + v)$



Acceleration is the rate at which the velocity changes with time, i.e. the gradient of the velocity – time graph.

Hence: $a = (v - u)/t$

v = u + at

Sub: **(2)** in **(1)** to obtain: $s = \frac{1}{2}t(u + u + at)$

$s = \frac{1}{2}(2u + at^2)$

$s = ut + \frac{1}{2}at^2$

$$\mathbf{s = ut + \tfrac{1}{2}at^2}$$

From **(2)** $t = (v - u)/a$, sub (4) in (3) to obtain:

$s = u(v - u)/a + \frac{1}{2}a((v - u)/a)^2 = (v - u)[u/a + \frac{1}{2}(v - u)/a]$

$s = \frac{1}{2}(v - u)(u + v)/a$

$2as = v^2 - u^2$

$$\mathbf{v^2 = u^2 + 2as}$$

6.4 Falling bodies

The equations are similar except that $a = g$; and $g \approx 9.81$

If the body is freely falling, then a = g

If the body is vertically projected upwards then a = -g.

Example

A train travels a distance of 432m, starting and finishing at rest, in one minute. It first accelerates at $1/3 m/s^2$, then travels with a constant velocity and finally retards at $1 m/s^2$. Find the time taken to each of the three stages.

Solution.

$(v - 0)/(t_1 - 0) = 1/3$

$v/t_1 = 1/3$

$v = t_1/3$

i.e. $t_1 = 3v$

$(0 - v)/(60 - t_2) = -1$

$-v/(60 - t_2) = -1$

$-v = -60 + t_2$

$t_2 = 60 - v$

The area under the graph is equal to the distance

Hence: Area = 432

i.e. $\frac{1}{2}v(t_2 - t_1 + 60) = 432.$

$\frac{1}{2}v(60 - v - 3v + 60) = 432$

$\frac{1}{2}v(120 - 4v) = 432$

$60v - 2v^2 = 432$

$2v^2 - 60v + 432 = 0$

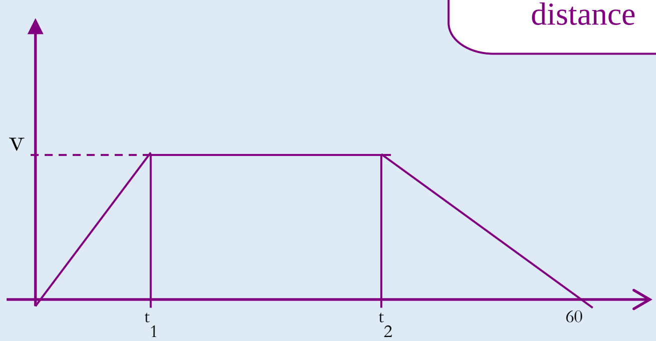
$v^2 - 30v + 216 = 0$

$v = \frac{1}{2}(30 \pm (900 - 864)^{\frac{1}{2}}$

$v = \frac{1}{2}(30 \pm 6)$

$v = 18$ or 12

v = 18m/s or 12m / s



<u>For v = 18m / s</u>

$t_1 = 3x18 = 545$

$t_2 = 60 - 18 = 425$

In this case $t_2 < t_1$

But $t_2 > t_1$: **contradiction!!**

<u>Reject v= 18m/s.</u>

<u>For v = 12m/s</u>

$t_1 = 3 \times 12 = 36$

$t_2 = 60 - 12 = 48$:These are admissible values

$\therefore t_1 = 36s$ for first stage.
$t_2 - t_1 = 48 - 36 = 12$ for the second stage
$60 - t_2 = 12s$ for the last stage.

Alternatively you could have assumed t_1: time for the first stage; t_2 time for the last stage.

Example
The brakes of a train are able to produce a retardation of $1.5m/s^2$. The train is approaching a station and is scheduled to stop at a platform. How far away from the station must the driver apply the brakes if the train is traveling at 100km / h?. If the brakes are applied 50m after this point at what speed will the train enter the station?

Solution.

$- 1.5m / s^2$; deceleration
$v = 0$: the train, comes to rest, the final speed is 0.
$u = 100km / h$: the initial speed of the train.
$s = ?$
Formula: $\mathbf{v^2 = u^2 + 2as}$
$u = 100km / h = 100 \times 1000 / 60^2 = 100000 / 3600 m / s$

We convert the initial speed to m / s, hence, substituting.

$$0^2 = (100 \times 1000)^2 / 3600 - 1.5 \times 2s$$
$$s = 1/3\ (100 \times 1000)^2 / 3600^2 \text{ ; hence, } s = \mathbf{257.2m}$$

If the brakes are applied 50m after this point, the distance to be traveled becomes:
257.2 – 50 = 207.2m, therefore:

$$v^2 = (100 \times 10000)^2 / 3600 - 2 \times 1.5 \times 207.2m$$
$$v^2 = 150.005$$
$$v = \sqrt{150.005}$$
$$\mathbf{v = 12.2m / s.}$$

Example

The skid marks, (black tyre marks on the road) left by a car which has been involved in an accident are found to be 35m long. Measurement show that on that stretch of the road the coefficient of friction $\mu = 0.65$. The acceleration of the car is calculated due to gravity i.e. $a = \mu g$. Find the speed of the vehicle at the start of the skid marks.
assume $g = 9.81m / s^2$.

Solution.
The sign of skid marks, indicate that the car was braking i.e. retarding:
The formula is: $\mathbf{v^2 = u^2 + 2aS}$

Hence: $s = 35m$ $\quad 0^2 = u^2 - 2\mu g \times 35$

$a = -\mu g = -0.65 \times 9.81$ $\quad 0 = u^2 - 2 \times 0.65 \times 9.81 \times 35$

$v = 0$ $\quad u^2 = 446.355$

$u = ?$ $\quad u = 21.13$m/s

The speed of the car is $\frac{21.13 \times 60 \times 60}{1000}$

= 76.1km / h

Example

A balloon rises vertically from rest on the ground with a constant acceleration 2m/s^2. A small stone is released from the balloon when it has risen to a height of 225m.

(i) Find the speed of the balloon when the stone is released?

(ii) Show that the time taken by the stone to reach the ground is approximately 6.8s.

Solution.

(i)If the stone is released from rest, hence

$a = 2$m/s^2

$u = 0$ the balloon started form rest

$v = 0$?

$s = 225$

Hence $v^2 = u^2 + 2as$

$v^2 = 0 + 2 \times 2 \times 225$

$v = \sqrt{900}$

v**= 30m / s.**

(ii) $s = 225$

$t = ?$

$a = g$

$u = 0$

$s = ut + ½at$

$225 = 0 \times t + ½ \times 9.81\ t^2$

$t^2 = 450/9.81$

$t = \pm 6.77$

$t = 6.8$s (shown)

Example

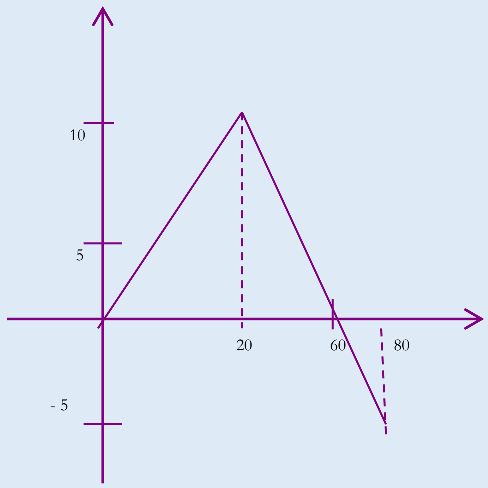
The diagram shows a velocity – time graph of a car during 80 seconds.

(i) What is the acceleration in the first 20 seconds?

(ii) Calculate the distance in the first 60 seconds?

(iii)Find the displacement of the particle during the 80 seconds?

(iv) Explain what happens from t = 60 to t = 80?



Solution.

(i) Acceleration is the gradient in a velocity- time graph..

$a = \frac{(v - u)}{}$

(ii) Distance covered in the first 60s is equal to the area under the graph from t = 0 to t = 60s t

a = $(10 - 0)/20$ $\quad\quad$ $s = 0.5 \times 60 \times 10$

a = $0.5 m/s^2$ $\quad\quad$ $s = 300m$

(iii) Displacement = area under graph from t = 0 to t = 60 plus area under the graph from t = 60 to t = 80.

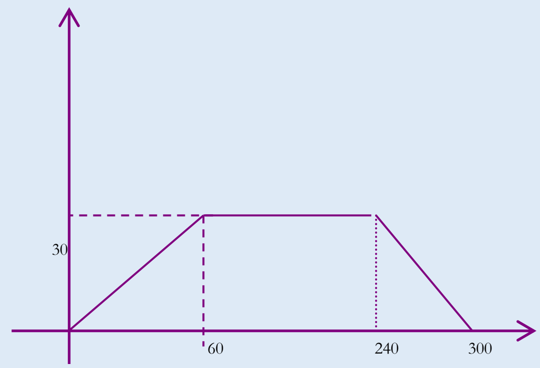
Hence: D = $300 - \frac{1}{2} \times 5 \times 20$
= $300 - 50$
= 250m.

The area under the graph is always negative

(iv) The car is going backwards.

Example

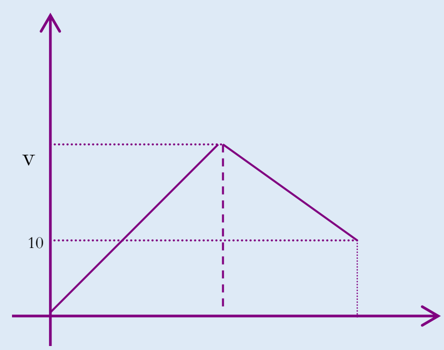
The train travels from a station P to the next station Q, arriving at Q exactly 5 mins after leaving P. The (t.v) graph for the train's journey is approximated by three straight-line segments, as shown in the figure .



(a)Write down the acceleration of the train during the first minute of its journey?

b) Find the distance from P to Q?

c) On one occasion, when the track is being repaired, the train is restricted to a maximum speed of 10m/sfor the 200m length of track lying mid- way between P and Q. The train always accelerates and decelerates at the rate shown in the figure. When not accelerating or decelerating or moving at the restricted speed of 10m/s the train travels at 30m/s. Sketch the (t,v) graph for the train's journey from P to Q, when speed restrictions is in force, and hence find how long the train takes to travel from P to Q on this occasion?



(ii)The figure shows the (t.v) graph for the train accelerating from rest up to a maximum speed of Vm/s and the decelerating to a speed of 10m/s. The acceleration and deceleration have the same value as shown in part (i) above; show that the distance traveled is $(2v^2 - 100)$ m. Determine whether the train in (ii) could be exceeding the normal speed of 30m/s when possible; make up the time lost due to the speed restriction when travelling from P to Q. Assume that the acceleration and deceleration must remain as before.

Solution:

(i) (a) Acceleration is equal to the, gradient hence :

a = $(v - u) / t$

a = $(30 - 0) / 60$

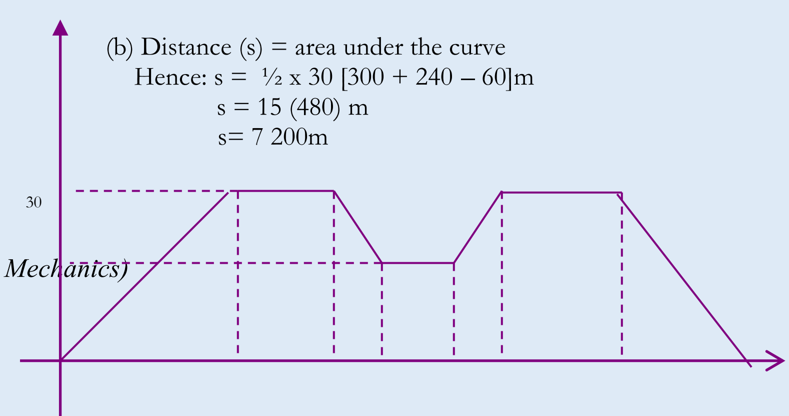
a = $0.5 m/s^2$.

(b) Distance (s) = area under the curve

Hence: s = $\frac{1}{2} \times 30\ [300 + 240 - 60]$m

s = 15 (480) m

s= 7 200m



We seek to find t_1, t_2, t_3, t_4 and t_6.

10

s_1 = distance covered from $t = 0$ to t_2

s_2 = distance covered from $t = t_2$ to t_3

60 t_1 t_2 t_3 t_4 t_5 t_6

s_3 = distance covered from t_3 to t_6.

$s_1 = s_3 = 2600m$

$s_2 = 2000m$

Total distance from P to Q: $s_1 + s_2 + s_3 = 7200m$.

$s_1 = \frac{1}{2} \times 30 \times 60 + (t_1 - 60) \times 30 + \frac{1}{2}(t_2 - t_1)(30 + 10)$

$2600 = 30^2 + 30(t_1 - 60) + 20(t_2 - t_1)$

$2600 = 900 + 30t_1 - 1800 + 20t_2 - 20t_1$.

$10t_1 + 20t_2 = 3500$

$t_1 + 2t_2 = 350$ **(1)**

$a = (v - u) / t$

$0.5 = (10 - 30) / (t_2 - t_1)$

$0.5 = -20 / (t_2 - t_1)$

$t_2 - t_1 = 40$

$-t_1 + t_2 = 40$ **(2)**

Solving the simultaneous equations to obtain:

$-t_1 + t_2 = 40$;

$\underline{t_1 + t_2 = 350}$

$3t_2 = 390$

$\underline{t_2 = 130s}$

$t_1 = 130 - 40 = 90s$

$\underline{t_1 = 90s}$

$s_2 = 2000$

$s_2 = 10(t_3 - t_2)$

$s_2 = (t_3 - 130)$

$\underline{t_3 = 330s}$

$a = (v - u) / t$

$0.5 = (30 - 10) / (t_4 - t_3)$

$\frac{1}{2} = 20 / (t_4 - t_3)$

$t_4 - t_3 = 40$

$t_4 = 40 + t_3$

$t_4 = 40 + 330$

$\underline{t_4 = 370s}$

$a = (v - u) / t$

$-0.5 = (0 - 30) / (t_6 - t_5)$

$t_6 - t_5 = 60$

but: $t_5 - t_4 = t_1 - 60$

$t_5 = t_4 + t_1 - 60$

$t_5 = 370 + 90 - 60$

$\underline{t_5 = 400s}$

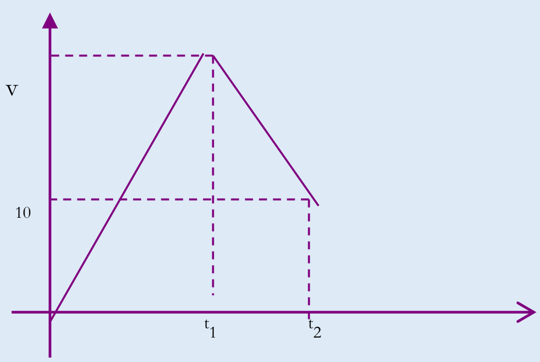
$t_6 = t_5 + 60$

$t_6 = 400 + 60$

$\underline{t_6 = 460s}$

<u>The total journey takes 460 s</u>

(iii) Distance (s) = Area under the graph.



$s = \frac{1}{2} v t_1 + \frac{1}{2}(t_2 - t_1)(v + 10)$

$s = \frac{1}{2} v t_1 + \frac{1}{2}(t_2 - t_1)(v + 10)$

$a = (v - u) / t$

$\frac{1}{2} = (v - 0) / t_1$

$\underline{\mathbf{t_1 = 2v.}}$

$-\frac{1}{2} = (10 - v) / (t_2 - t_1)$

$-(t_2 - t_1) = 20 - 2v$

$-t_2 + t_1 = 20 - 2v$

$t_2 = t_1 - 20 + 2v$

$t_2 = 2v - 20 + 2v$

$t_2 = 4v - 20$

$2v^2 - 100 = 2600$

$2v^2 = 2700$

$v^2 = 1350$

$\mathbf{v = 37m/s}$

$s = \frac{1}{2}(2v)v + \frac{1}{2}(2v - 20)(v + 10)$

$s = \frac{1}{2}(2v^2 + 2v^2 + 20v - 20v - 200)$

$s = \frac{1}{2}(4v^2 - 200)$

$\mathbf{s = (2v^2 - 100)\ m}$

Under the same conditions the speed could be increased to 37m/s. Hence time taken to cover the first stage is 37 / t =1/2
i.e. t = 74s

But before the speed restriction the time was t = 60s hence 74>60, no its not possible, to recover for the time lost of **160s = 460s – 300s.**

6.5 Examination Type Questions

1. A particle has an initial velocity of **i** + 2**j** and is accelerating in the direction **i** + **j**. If the magnitude of the acceleration is $5\sqrt{2}$, find the velocity vector and the speed of the particle after 2 seconds

2. A particle has position vector 2**i** + **j** initially and is moving with speed 10m/s in the direction 3**i** – 4**j**. Find its position vector when t = 3 and the distance it has traveled in those 3 seconds.
3. The brakes of a train can produce a retardation of 1.7m/s. If the train is traveling at 100km/hr and applies its brakes. What distance does it travel before stopping? If the driver applies the brakes 15m too late to stop at a station with what speed is the train traveling when it passes through that station?

4. A ball is thrown vertically upwards with a speed of 15m/s from a point 1 m above the ground. Find the speed with which it hits the floor. If it rebounds with a speed of which is half the speed which it hits the floor, find its greatest height after the first bounce.

5.A book falls from a window, which is 30m above the ground. Model the book as a particle and hence find the time taken to reach the ground. What assumptions have you made when adopting this model? What effect is this likely to have on your answer?

6 A train starts from rest at station a and moves with constant acceleration for 60s until it reaches a speed of 30m/s. It travels at this constant for T seconds and then decelerates uniformly for 1.2km, coming to rest at station b which is 14.1km from station A

a) sketch a speed – time graph for the journey
b) calculate the deceleration of the train
c) calculate the value of T
d) calculate the total time for the journey

7. A train stops at two stations P and Q, which are 2km apart. It accelerates uniformly from P at $1m/s^2$ for 15s and maintains a constant speed for a time before decelerating uniformly to rest at Q. If the deceleration is $0.5m/s^2$, find the time for which the train is traveling at a constant speed.

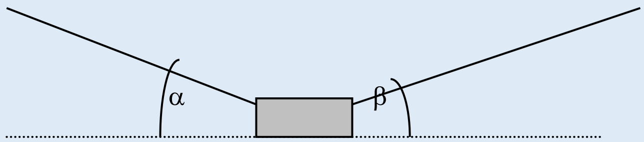
8.A car starts from rest at time t = 0 seconds and moves with a uniform acceleration of magnitude $2.3m/s^2$ along a straight horizontal road. After T seconds, when its speed is Vm/s, it immediately stops accelerating and maintains this steady speed until it hits a brick wall when it comes instantly to rest. The car has then traveled a distance of 776.25m in 30seconds.

a) sketch a speed – time graph to illustrate this information
b) write down an expression for V in terms of T
c) show that $T^2 - 60T + 675 = 0$
d) hence solve for T

9. Forces **P** and **Q** act on a particle at the origin O along the coordinate axes Ox, Oy respectively. Calculate the magnitude of the resultant force and the angle it makes with the x –axis when:

a) **P** =5N, **Q** = 2N
b) **P** = 7N, **Q** = 4N
c) **P** = 9N , **Q** = 40N

10. A particle of mass kg rests in equilibrium on a rough plane inclined at 30^0 to the horizontal. Find the normal contact force and the frictional force in terms of m and g.

11 

An electric light mass mkg in a workshop is held over a particular place by two strings attached to the light bulb holder. This is modelled by a particle suspended by two straight strings inclined at angles α and β above the horizontal. Find the tension in the strings when:

(a) $m = 0.5$, $\beta = \alpha = 60^0$

(b) $m = 0.5$, $\alpha = 60^0$, $\beta = 65^0$

CHAPTER 7

DYNAMICS

OBJECTIVES

- CALCULATE , ACCELERATION, VELOCITY AND DISTANCE USING THE LAWS OF MOTION
- CALCULATE THE COEFFICIENT OF MOTION
- SKETCH (T,X) AND T,V) GRAPHS
- RESOLVE FORCES ACTING ON A PULLEY

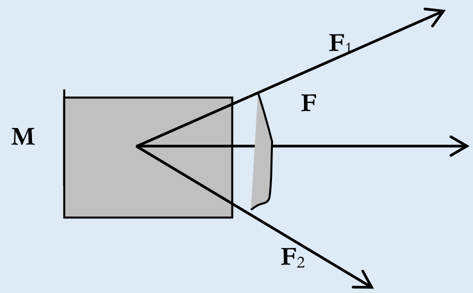
Dynamics is the study of forces on bodies whose motion is changing.

7.1 Newton's laws of motion.

1st Law.
Everybody continues in its state of rest or continues to move with uniform speed in a straight line unless it is acted upon by an external force.

The law stipulates that ***uniform motion*** and ***rest*** are dynamically equivalent. When forces balance, we have equilibrium of forces.

2nd Law.
The acceleration of a body is proportional to the resultant force and takes place in the direction of the resultant force.

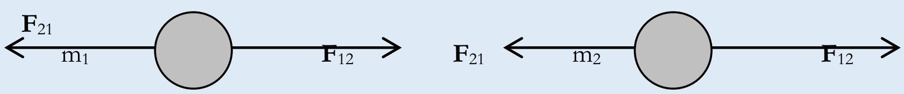


Hence: $\mathbf{F} = m\mathbf{a}$
Where **F** is the resultant force and **a** is the acceleration . This law is widely used in mechanics.

The second Newton's Law plays the following roles:

1. It provides an operational (usable) definition of forces. How to quantify the force.
2. It is an equation of motion when the force laws are known.

3rd Law
Action and reaction forces are equal and opposite.
When an object exerts a force on another object (an action), then the second object always responds with an equal force on the first object (the reaction) acting in the opposite direction.

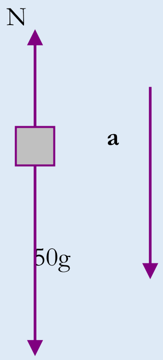


$$\mathbf{F}_{21} = -\mathbf{F}_{12}$$

Example

A woman of mass 50kg is standing on the floor of a lift, which is descending. By, making an appropriate diagram, or otherwise calculate the force which the floor of the lift exerts on the women's feet when the lift has an acceleration of 2m/s^z.

Solution

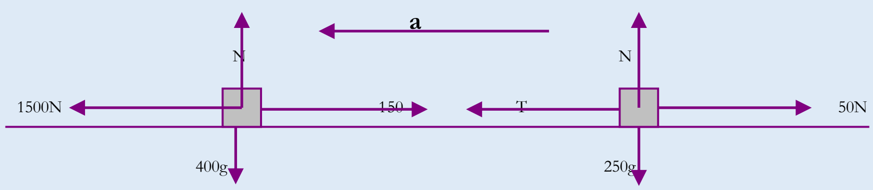


F = ma
F = 50g – N, since the lift is moving downwards.
m **= 50**
a = 2m/s^z
Hence:
50g – N = 50 x 2
N = 50 x 9 .81 – 100
N = 390.5N
N = 390 to 2 s. f.

Example

State Newton's 2nd Law.
A car of mass 400kg is pulling a caravan of mass 250kg along a horizontal road. There are constant resistances of 150N to the motion of the car and 50N to the motion of the caravan. Given that the tractive force of the car is 1500N.
Calculate
(i) the acceleration of the car and caravan.
(ii) the tension in the tow bar.

Solution:



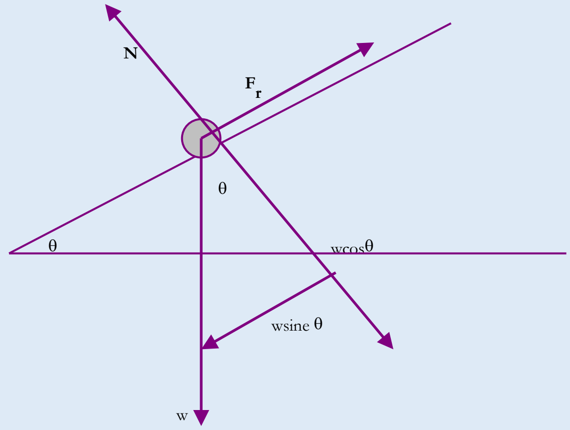
For the car:
1500 – 150 – T = 400a
1350 – T = 400a (1)

For the caravan:
T – 50 = 250a (2)
1350 – T = 400a
T – 50 = 250a
1300 = 650a
a = 1300/650
a= 2m/s^z

Hence: T = 50 + 250 x 2
T = 550N

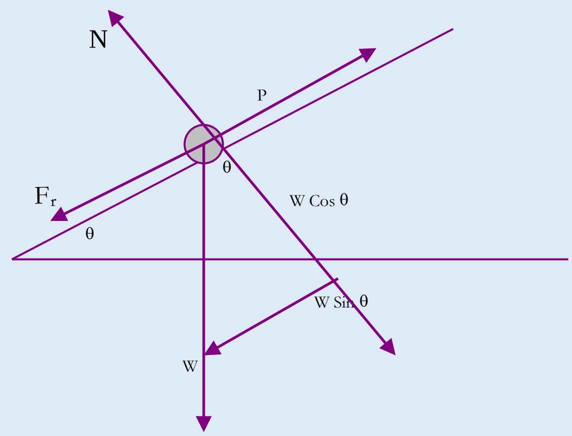
Motion on a rough inclined plane

Consider a body at rest under its own weight on a rough plane inclined at an angle θ to the horizontal. The forces acting upon the body are at equilibrium. The particle exerts a force on the plane, and by Newton's Law there is a reaction force on the particle. This reaction force can be resolved into a component ***normal*** to the plane and another component ***parallel*** to the plane. When the motion is about to take place, the friction force attains its maximum or limiting value.



W is the weight of the body
$N = w\cos\theta$, $F_r = w\text{sine}\ \theta$
F_r is the friction force
$F_r = w\sin\theta$
$F_r = \mu\, w\cos\theta$
$F/ F = w\sin\theta / \mu\, w\cos\theta$
$I = \tan\theta/\mu$
$\tan\theta = \mu$
Note: this happens when the system is in a Limiting equilibrium

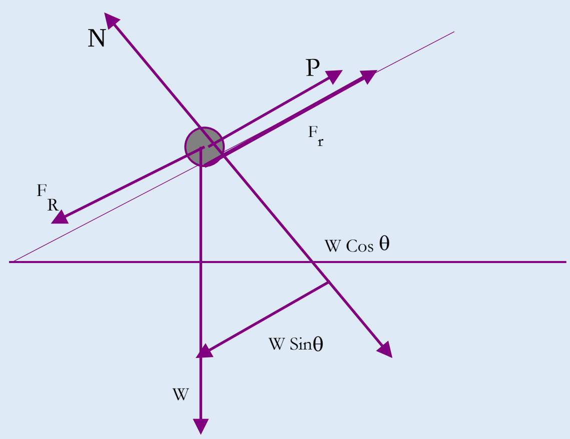
7.2 Motion up the slope



Since the body is moving up the slope, the friction force acts down wards, hence. applying Newton's 2nd Law of motion.

$F_R = ma$, where F_R is the resultant force
$F_R = P – F_r – W\sin\theta$
$P – F_r – W\sin\theta = ma$
$P - \mu N – W\sin\theta = ma$
$P - \mu W\cos\theta - W\sin\theta = ma$

7.3 Motion down the slope



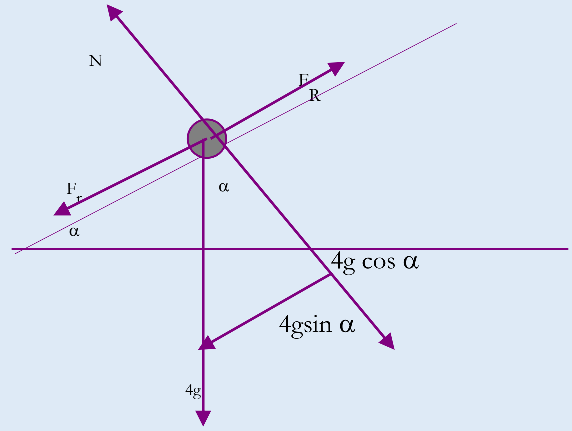
$W\sin\theta - P – F_r = F_R$
$W\sin\theta – P - \mu W\cos\theta = ma$

Example
A particle of mass 4kg is projected up a line of greatest slope of a rough plane inclined at an angle tan $\alpha = ¾$. Given that the speed of the projection is 10m/s and that the coefficient of friction between the particle and the plane is 1/5

Calculate

(i) The retardation of the particle.
(ii) The distance the particle moves up the plane before coming to instantaneous rest.

Solution:



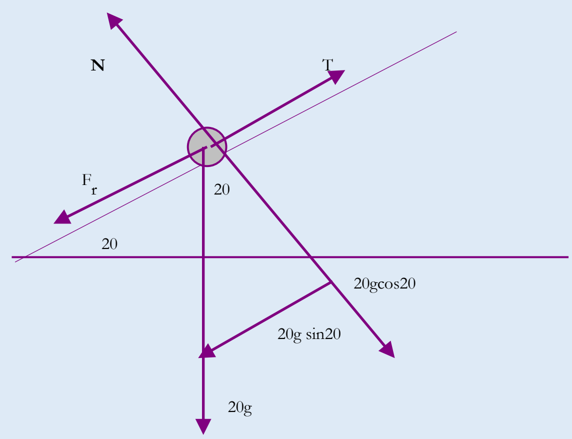
Since the particle is moving upwards.
The friction force F_r is acting against the movement.
If $\tan \alpha = \frac{3}{4}$, hence, $\sin\alpha = 3/5$ and $\cos\alpha = 4/5$
$F_r = \mu N$; $N = 4g \cos \alpha$
$F_r = 1/5 \times 4g \cos \alpha$
$F_r = 1/5 \times 4 \times 9.81 \times 4/5$
$F_r = 16/25 \times g$
$F_r = 6.2784N$
$F_R = ma$
$4g\sin \alpha + 16/25 + ma = 0$
$4g \times 3/5 + 16/25g = -ma$
$12\,g + 16/25g = -4a$
$a = \frac{1}{4} \times 76/25g$
$a = 19/25g$
$a = -7.4556$
The retardation is $7.5m/s^2$

(ii) $v = 0$ $\quad$ $v^2 = u^2 + 2a$
$a = -7.5$ $\quad$ $v = 100 - 2 \times 7.4556 \times 5$
$u = 10$ $\quad$ $s = 50/7.4556$
$s = 6.7m$

Example

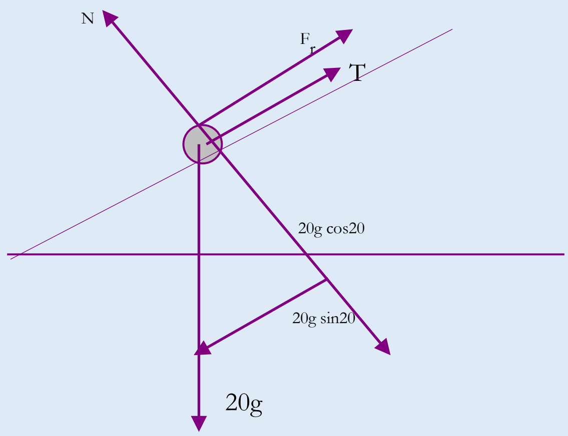
A man is holding a car battery of mass 20kg, at rest on a rough slope by pulling a rope attached to the battery. The slope is inclined to the horizontal at 20°, and the rope is parallel to a line of greatest slope. The coefficient of friction between the battery and the slope is 1/5. The tension in the rope is denoted by T.
(i) Find the maximum value of T.
(ii) Find the minimum value of T.

Solution



(i) $\mu = 1/5$
The system is at equilibrium, hence,
$F_R = ma = 0$
If the particle is about to move upwards, hence,
T acts against two forces, the friction force and
$20g \sin20°$, then T must be at a maximum.
Since, $F_R = 0$
i.e. $F_R = T - F_r - 20g\sin20$
$T = F_r + 20g \sin20^0$,
$F_r = \mu N$
$F_r = 1/5\ 20g\cos20_0$
$F_r = 4g\cos20^0$
$T = 4g \cos20° + 20g \sin20^0$
$T = 104N$

(b) The minimum value of T is attained when the battery is about to move downwards; hence, frictional force acts up the greatest slope.



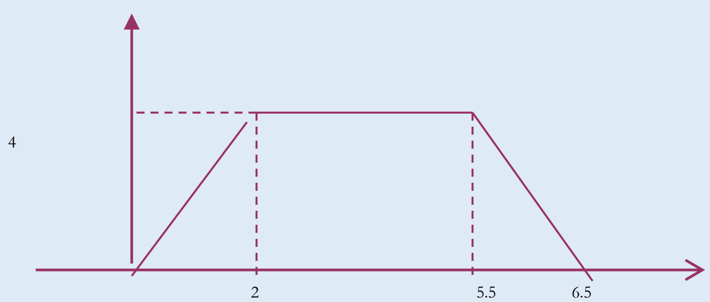
$F_R = 0$
$F_R = T + F_r - 20g\sin20$
i.e. $T = 20g \sin20 - F_r$
$T = 20g\sin 20 - 1/5 \times 20g \cos20$
$T = 20g \sin20 - 4g \cos20$
T = 30N

Example

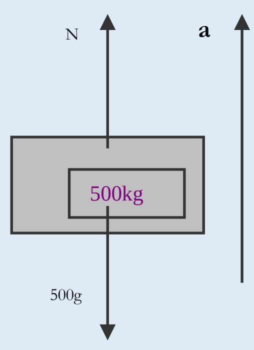
A lift travels vertically upwards from rest at a floor A to rest at floor B, which is 20m above A, in three stages as follows. At first the lift accelerates from rest at A at $2ms^{-2}$ for 2s. It then travels at a constant speed and finally it decelerates uniformly, coming to rest at B after a total time of 6.5s. Sketch the (t.v) graph for this motion, and find the magnitude of the constant acceleration. The mass of the lift and its contents is 500kg. Find the tension in the lift cable during the stage of the motion when the lift is accelerating upwards.

Solution

$a = (v - u) / t$
$2 = (v - 0) / 2$
$v = 4m/s$
s = Area under the curve
$s = ½ \times 2 \times 4 + 4(t_1 - 2) + ½ \times 4(6.5 - t_1)$
$s = 4 + 4t_1 - 8 + 13 - 2t_1$
i.e. $2t_1 + 9 = 20$
$t_1 = 5.5$
Hence: $a = (v - u) / t$
$a = (0 - 4) / (6.5 - 5.5)$
$a = -4m/s^2$



Since there is acceleration
$F_R = ma$
$F_R = N - 500g$, the lift is going upwards.
i.e. $N - 500g = 500a$
$N - 500g = 500 \times 2$
$N = 1000 + 500g$
$N = 5905N$
N **= 5900N to 2 s. f.**

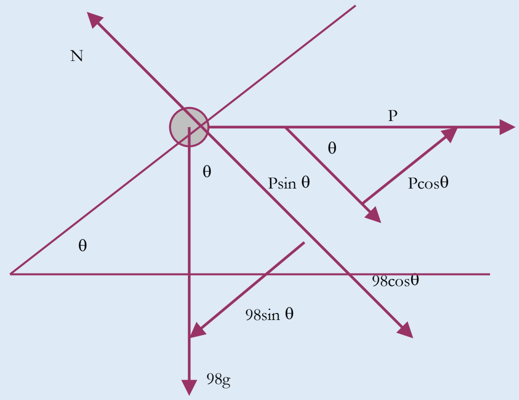


Example

A body of weight 98N is at rest on a rough inclined plane. A force P is applied horizontally. If $\mu = 0.35$, find the range of values of P over which the body will remain at rest.

Solution

We resolve P into two components, one parallel to the plane and the other perpendicular to the plane.
$P\cos\theta$ acts up the line of the greatest slope and $P\sin\theta$ acts perpendicular to the plane.



Case 1: the particle has a tendency of moving up the slope.

$F_R = P\cos\theta - 98\sin\theta - F_r$ ***BUT*** $F_r \leq \mu N$

$F_R = 0$

$P\cos\theta - 98\sin\theta - F_r = 0$

$P\cos\theta - 98\sin\theta - \mu N \leq 0$

$P\cos\theta - 0.35P\sin\theta \leq \mu(98\cos\theta + P\sin\theta) + 98\sin\theta$

$P\cos\theta - 0.35P\sin\theta \leq 34.3\cos\theta + 98\sin\theta$

$$P \leq \frac{34.3\cos\theta + 98\sin\theta}{\cos\theta - 0.35\sin\theta}$$

Case 2: The particle has a tendency of moving down the slope:

$F_R = -98\sin\theta + F_r + P\cos\theta$

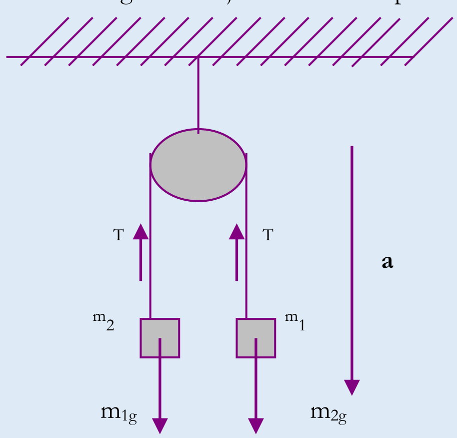
$P\cos\theta + F_r - 98\sin\theta = 0$: $F_r \leq \mu N$

$P\cos\theta + \mu N - 98\sin\theta \geq 0$

$$P \geq \frac{98\sin\theta - 34.3\cos\theta}{\cos\theta + 0.35\sin\theta}$$

Connected Particles

You have seen cranes and lift at the industry. e.g. heavy loads are loaded into the ships and big lorries by cranes. These cranes use the mechanism of connected bodies. We consider two particles of mass m_1 and m_2 where $m_1 > m_2$, which are connected by means of a light inextensible string of negligible mass. The string passes over a fixed frictionless pulley of negligible mass in such a way that the two particles can move up and down without rotating. The objective is to compute the acceleration and the tension in the string.



Since $m_2 > m_1$, we assume, the system is moving in the direction of m_2**g**.

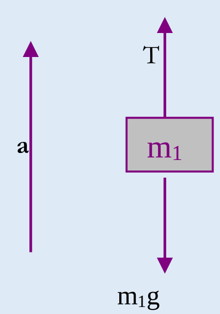
a is constant throughout the system. T is the same on either side of the pulley.

For a particle of mass m_1

$F = ma$

$F = T - m_1 g$

$T - m_1 g = m_1 a$ **(1).**

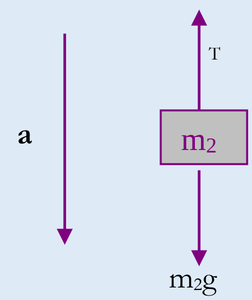


For particles of mass m_2

$F = ma$

$F = m_2g - T$

$m_2g - T = m_2a$ **(2)**



Solving the simultaneous equations.

$$T - m_1g = m_1a$$
$$-T + m_2g = m_2a$$

$$g(m_2 - m_1) = (m_1 + m_2)a$$

$$a = \frac{(m_2 - m_1)}{(m_1 + m_2)}g$$

$$T = m_1a + m_1g$$

$$T = m_1\frac{(m_2 - m_1)}{(m_1 + m_2)}g + m_2g$$

$$T = \frac{2m_2m_1g}{(m_1 + m_2)}$$

The force (pressure) exerted on the pulley; is $P = 2T$

$$P = \frac{4m_1m_2g}{(m_1 + m_2)}$$

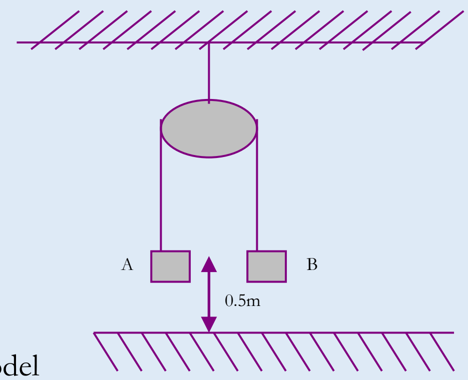
Example

Particles A of mass 0.5kg and B of mass 0.3kg are attached to the ends of a light inextensible string. The string passes over a fixed peg, and the system is released from rest with both points o f the string taunt and vertical and each particle 0.5m, above a fixed horizontal plane. (see diagram). Neglecting all resistances to motion, find:

(i) Acceleration of A and the tension in the string.

(ii) The time after release at which A hits the plane.

(b) The results in (a) are based on a mathematical model in which resistance to motion are neglected. Describe briefly one resisting force, other than air resistance, which would be present in a real system in which objects of unequal mass, hanging from a string passing over a fixed peg are in motion.



Solution

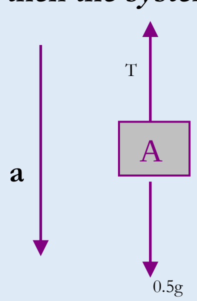
Since mass of A is bigger than mass of B, then the system is moving in the direction of A.

For particle A.

$F = ma$

$F = 0.5g - T$

$0.5g - T = 0.5a$ …….. **(1)**



For particle B.

F = ma

F = T – 0.3g

T – 0.3g = 0.3a…….. **(2)**

Hence, -T + 0.5g = 0.5a

T – 0.3g = 0.3a

0.2g = 0.8a

$a = \frac{0.2g}{0.8}$

a= g/ 4

a= 2.4525m/s^2.

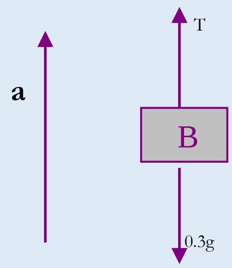
T = -0.5a + 0.5g

T = -0.5 x 2.4525 + 0.5 x 9.81

T = -1.22625 + 4.905

T = 3.67875

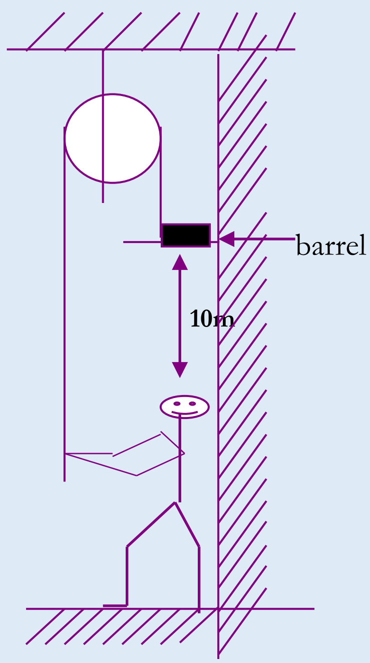
T= 3.7N



(b) Friction forces over the pulley system.

Example

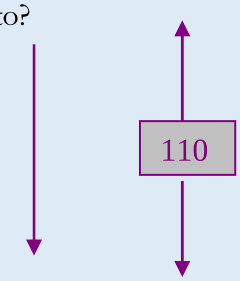
A man of mass 90kg is standing on the ground, holding one end of the rope. The rope passes over a pulley and has a barrel of bricks, of total mass 110kg, attached to the other end.
The barrel is temporarily supported by a plank (see diagram).Initially the distance between the man and the barrel is 10m. The rope is taut and the two parts of the rope are vertical. The plank breaks. The barrel and the man, who fails to let go off the rope, move towards each other.
The following are assumptions that are made
A: The rope is light
B: The rope is inextensible.
C: There is no air resistance.



With the above assumptions, together with the assumptions that the pulley is smooth, the man and the barrel modeled as particles move with constant accelerations of the same magnitude. Find the time after the pack breaks when the barrel hits the man

Sketch the (t.v) and the (t,x) graphs for the motion of the man up to the moment of impact. State which assumptions A, B and C are relevant to?

(a) The accelerations having the same magnitude
(b) The acceleration having the same constant magnitude.



Solution:

If the plank breaks, the load of mass 110kg moves downwards.

Hence, F = ma

F = 110g – T

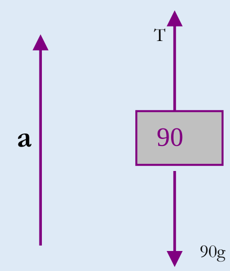
110g – T = 110a …….. **(1)**

The man goes upwards

Hence: F = ma

F = T – 90g

T – 90g = 90a …… **(2)**



T – 90g = 90a

<u>-T + 110g = 110a</u>

20g = 200a

a = 20g /200

a = g /10 m/s^2

The acceleration of the system is g /10m/s^2

Since the acceleration is the same for the man and the barrel of bricks , the distance covered by the man up wards is equals the distance covered by the barrel of bricks downwards, i.e. the distance covered by each is 5m. The impact will take place mid- way the distance of 10m and the initial speed is 0.

Hence: a = g /10m/s^2

u = 0

s = 5

t = ?

s = u t + ½ at^2

5 = 0 x t + ½ g /10 t^2

10 = g / 10 t^2

t^2 = 100 /g

t^2 = 10.1936

t = 3.19

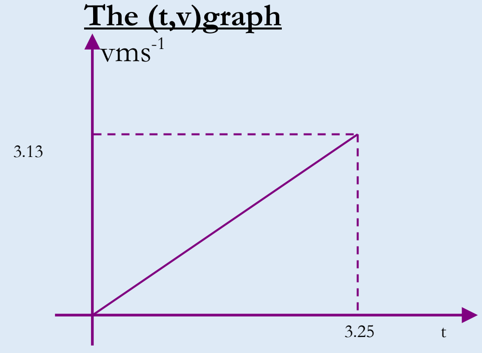
<u>t= 3.25 to 2 s. f.</u>

v = u + at

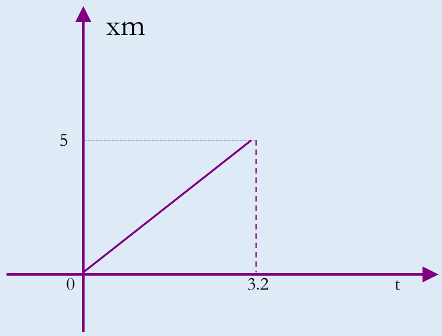
v = 3.2 x g /10

v **= 3.13m/.s**

<u>The (t,v)graph</u>



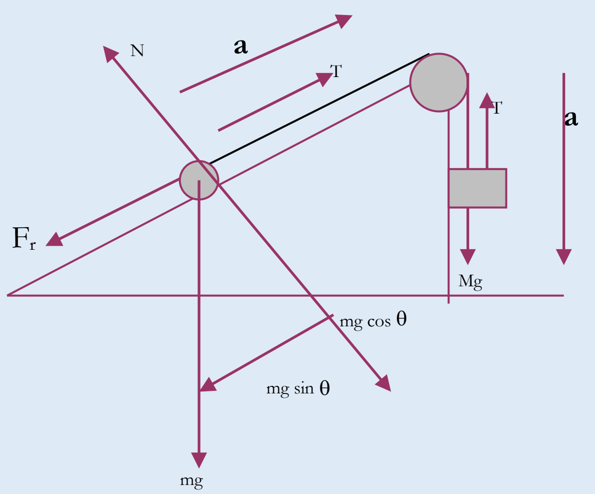
The (t,x) graph



(a) In order for the acceleration to have the same magnitude, the rope should be inextensible.
(b) For the acceleration to have the same magnitude and constant, the rope should be light and there should be no air resistance.

Example
Consider a particle of mass m kg which rests on the surface of a rough plane inclined at an angle θ to the horizontal. It is connected by a light inelastic string passing over a smooth pulley at the top of the plane to a particle of mass M (M>m) hanging freely. The coefficient of friction is μ. Find the acceleration of the system when it is released from rest and the force exerted by the string on the pulley.

Solution
If the system is released from rest the mass M moves a distance ym downwards. Since the string is in elastic, the mass m would move a distance ym up the plane.



<u>For particle M</u>
$Mg - T = Ma$

<u>For particle m</u>
$T - F_r - mg \sin\theta = ma$
$T - \mu N - mg\sin\theta = ma$
$T - \mu mg \cos\theta - mg \sin\theta = ma$
$Mg - T = Ma$
$\underline{T - \mu mg \cos\theta - mg \sin\theta = ma}$
$Mg - \mu mg \cos\theta - mg \sin\theta = (M + m)a$

$$a = \frac{Mg - mg(\mu \cos\theta + \sin\theta)}{(M + m)}$$

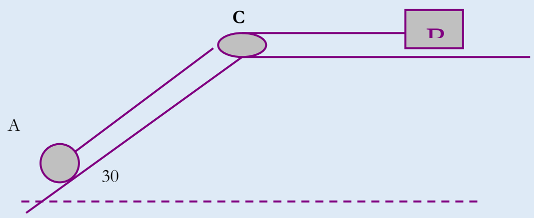
Hence: $$T = \frac{Mmg(\sin\theta + \mu\cos\theta + 1)}{(M + m)}$$

The force exerted by the string on the pulley is:

$$P = 2T = \frac{2Mmg\,(\sin\theta + \mu\cos\theta + 1)}{(M + m)}$$

Example

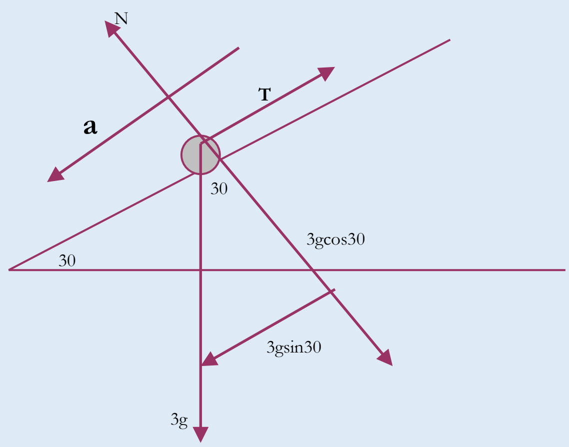
The diagram shows two particles, A of mass 3kg and B of mass 2kg, joined by a light in elastic string which passes over a smooth pulley at C. This system is held at rest with A on a smooth plane inclined at 30° to the horizontal surface. The coefficient of friction between B and the surface is 0.4. Show that, when the system is released from rest, the acceleration of each particle has a magnitude of $1.372 m/s^2$ and calculate the tension in the string.



Solution:

<u>For particle A</u>

3g Sin 30° - T = 3a …… **(1)**



<u>For particle B</u>

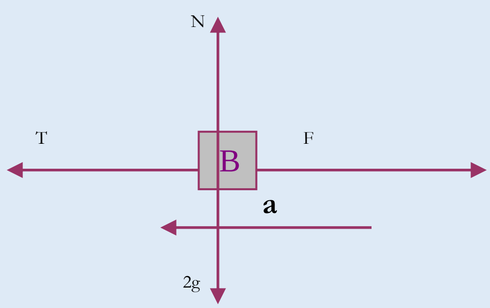
F = μ N

μ = 0.4

N = 2g

T – F = 2a

T – 0.4 x 2g = 2a. …. **(2)**



1.5 g – T = 3a

<u>T – 0.8g = 2a</u>

0.7g = 5a

a = 014g

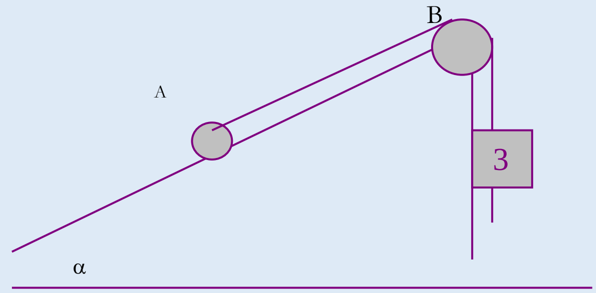
$a = 1.3734\ m/s^2$

T = 2a + 0.8g

T= 2 x 1.3734 + 0.8 x 9.81

T= 10.5948N

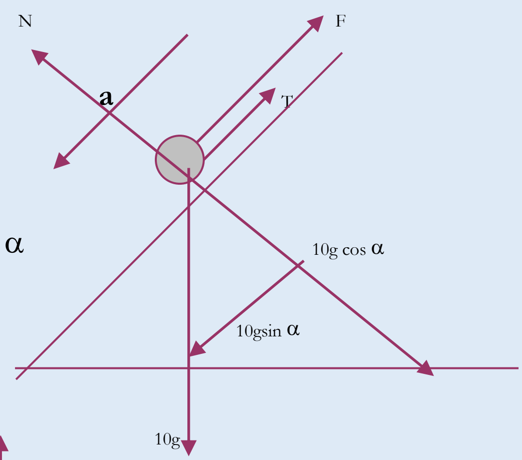
Example



The diagram shows two bodies A and B connected by a light inextensible string passing over a smooth fixed pulley.

The body A has a mass of 10kg and lies on a rough inclined plane at an angle α to the horizontal, where tanα= 4/3.
The body B has a mass of 3kg and hangs freely.
(i) A accelerates down the plane at $2m/s^2$. Calculate the tension in the string and show that the coefficient of friction between A and the plane is 0.4 to 1 decimal place.
(ii) Find the mass which when attached to B would just prevent A from sliding down the plane.

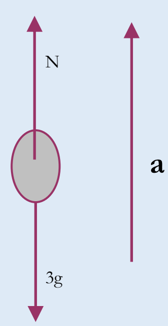
Solution:



<u>For particle A</u>
Since tanα = 4/3, then: cos α = 3 /5
and sin α = 4 / 5
10gsin α - F – T = 10a: F = μ N
Hence;

10gsin α - μ x10gcos α - T = 10a: N = 10g cos α
10g x 4 / 5_- μ x 10g x 3 / 5– T = 20
8g - 6μg – T = 20 **(1)**

<u>For particle B</u>



T – 3g = 3a
T = 3g + 3 x 2
T =6 + 3g
T = 35.43N
T = 35N to sig figures.

From (1)
8g - 6μg – 35.45 = 20
8g - 6μg = 55.43
6μg = 8g – 55.43
μ =(8g – 55.43) /(6g)
μ = 0.3916
μ = 0.4 to 1 d. p.

(ii)If A is prevented from sliding down the plane, hence the system is in equilibrium and $\mathbf{F_R = ma = 0}$

$F_R = 10g\sin\alpha - F - Mg$

$10g \sin \alpha - \mu N - Mg = 0$

$8g - 0.4 \times 10g \cos \alpha - Mg = 0$

$8g - 4g \times 3/5 - Mg = 0$

$Mg = 8g - 2.4$

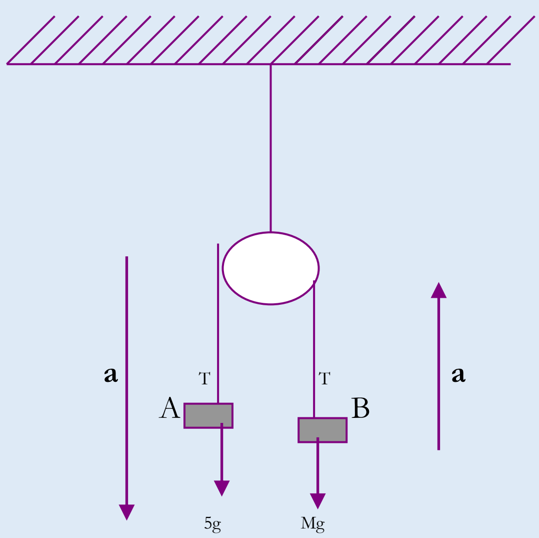
M = 5.6kg.

Example

Two particles A and B, of mass 5kg and Mkg. ($M<5$) respectively are connected by a light inextensible string passing over a smooth fixed pulley. Given that the particles accelerate at $0.2g\ m/s^2$ freely, calculate:

(i) The value of M

(ii) The force exerted by the string on the pulley during the motion.



Solution:

For particle A

$5g - T = 5a$ ….. **(1)**

For particle B

$T - Mg = Ma$ …….. **(2)**

Hence: from (1) : $T = 5g - 5a$

$T = 5(g - a)$

$T = 5(g - 0{,}2g)$

$T = 4g$

$T = 39.24N$

From (2)

$39.24 = Mg + Ma$

i.e. $M = 39.24\ (a + g)$

$M = 39.24 / 1.2g$

$M = 3.33kg$

The force exerted by the string on the pulley is 2T.

i.e. $P = 2T$

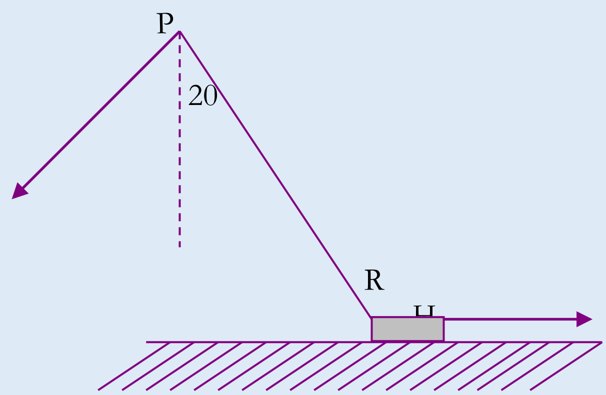
$P = 2 \times 39.24$

$P = 78.48N$

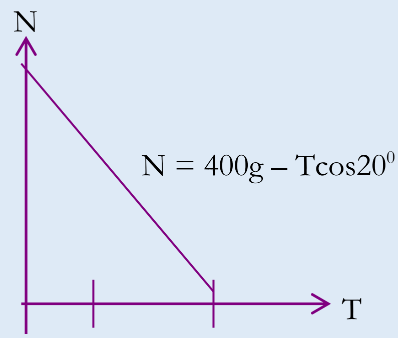
$P = 78.5N$

Example.

A stone slab, of mass 400kg, rests on a horizontal ground. It is lifted off the ground by a wire which has one end attached to a ring R fixed to the top of the slab. The wire passes over a fixed pulley P and a force X is applied to the other end. The wire R is inclined at an angle of 20° to the vertical. A rope is attached to R and pulled horizontally with a force H, as shown in the diagram above. The force applied to the wire and the rope are both gradually increased in such a way that the slab remains in equilibrium until it is on the point of lifting off the ground. Find the magnitudes of the two forces X and H at this instant, stating any one necessary assumption.



he wire and the rope are released and the rope is removed. The wire is again pulled, with RP inclined at 20° to the vertical. The tension in the wire is denoted by T, and the magnitudes of the frictional force exerted on the slab by the ground are F and N respectively. Given that the slab is in equilibrium, show that $N = 400g – T\cos 20^0$ and, Find an expression for F in terms of T.



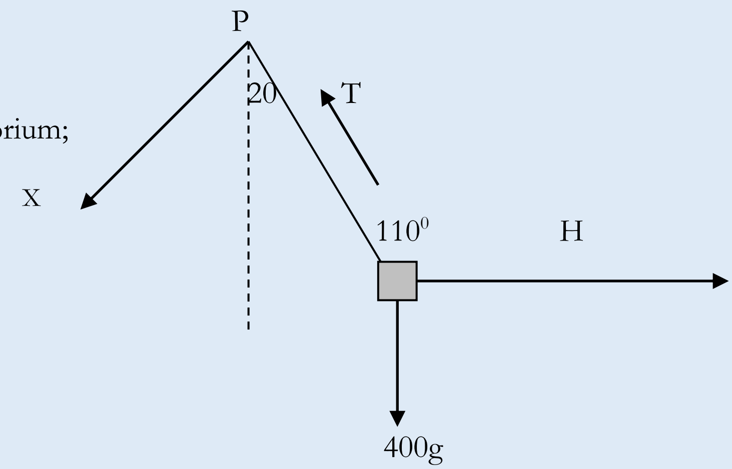
The figure shows a sketch of the graph of N against T. Copy the above figure, on your copy sketch.

(i) The graph of F against T.

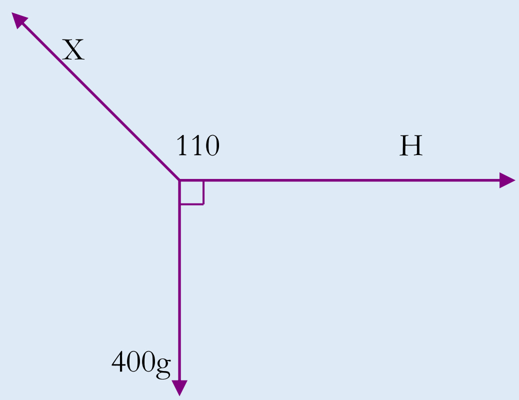
(ii) The graph of μ N against T, where μ is the coefficient of friction between the slab and the ground.

By considering the graphs, or otherwise, show that, as T is increased, the slab will slip before it is lifted off the ground, whatever the value of μ.

Solution:



The tension T = XN, since P is a pulley, and the forces H, T and W are in equilibrium; we use Lami's Theorem.



$$\frac{400g}{\sin 110} = \frac{X}{\sin 90}$$

i.e $X = \frac{400g}{\sin 110} \sin 90$

$X = 4175.8N$

$X = 4200N$ to 2 s.f.

$$\frac{H}{\sin 160} = \frac{400g}{\sin 110°}$$

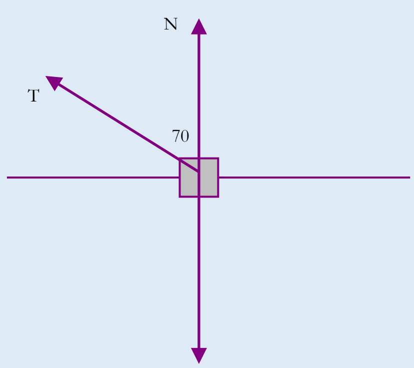
$$H = \frac{400g \sin 160°}{\sin 110°}$$

$H = 1428.219$

$H = 1400N$ to 2 s. f.

Assumption: the pulley is smooth for T to be equal to X.

Since the system is in equilibrium hence Y = 0 and X = 0



$Y = N + T\sin 70 - 400g$

$Y = N + T\cos 20 - 400g$; since, $\sin 70 = \cos 20$

$0 = N + T\cos 20 - 400g$

$N = 400g - T\cos 20°$ (shown)

$X = 0$

$X = T\cos 70 - F$

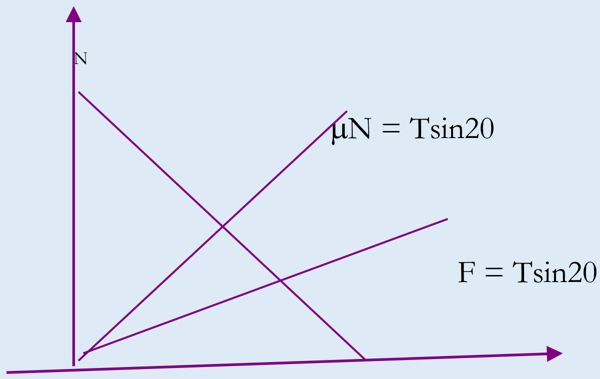
$X = T\sin 20° - F$, since, $\cos 70 = \sin 20$

$0 = T\sin 20° - F$

i.e. $F = T\sin 20°$

400g

If T increases N becomes very small .



T

7.4 Examination type questions

1.A stone slides in a straight line across a frozen pond. Given that the initial speed of the stone is 5m/s and that it slides 20m before coming to rest, calculate the coefficient of friction between the stone and the surface of the frozen pond.

2.A parcel of mass 5kg is released from rest on a rough ramp of inclination arcsin 3/5 and slides down the ramp. After 3 seconds the parcel has a speed of 4.9m/s. treating the parcel as a particle, find the coefficient of friction between the parcel and the ramp.

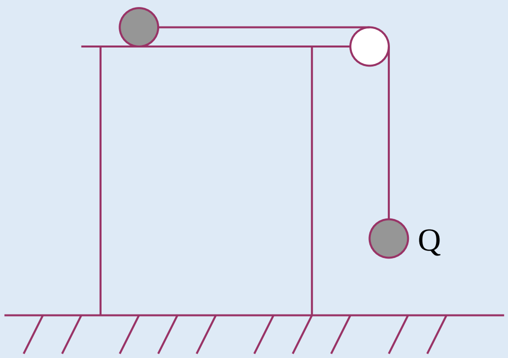
3.A block of mass 1.6kg is placed on a rough plane inclined at 450 to the horizontal. The coefficient of friction between the block and the plane is ¼. Model the block as a particle and hence find the acceleration of the block down the pane. Find the velocity of the block after 2 seconds, assuming that it starts from rest.

4.A car of mass 1000kg tows a caravan of mass 750kg along a horizontal road. The engine of the car exerts a forward force of 2.5KN. The resistance to the motion of the car and caravan are each k x their masses, where k is a constant. Given that the car accelerates at 1m/s, find the tension in the tow bar.

5. A car of mass 1000kg exerts a driving force of 2.2KN when pulling a caravan of mass 500kg along a horizontal road. The car and the caravan increase speed from rest to 4m/s while travelling 16m. Given that the resistances on the car and caravan are proportional to their masses, find these resistances and the tension in the tow bar

6.A particle of mass 0.5kg rests on a smooth inclined plane of angle arcsin 3/5. A light inextensible string is attached to this particle and passes over a smooth pulley at the top of the plane. A particle B of mass mkg hangs freely from the other end of the string. The system is released from rest with the string taut. Particle B descends 1m in 1second. Find the value of m and the tension in the string.

7.



The diagram shows a particle of mass 3kg lying on a smooth horizontal table top, which is 1.5m above the floor. A light inextensible string of lenght 1m connects P to a particle Q, also of mass 3kg, which hangs freely over the edge of the table. Initially P is held at rest at a point 0.5m from the pulley. When the system is released from rest find

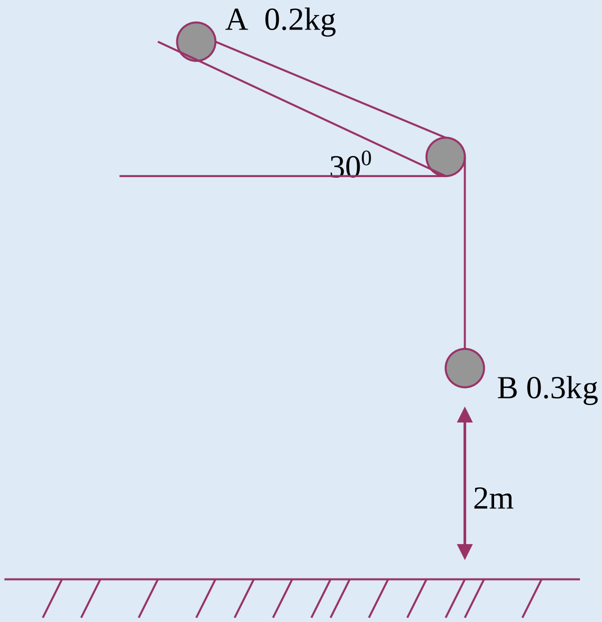
a) the speed of P when it reaches the pulley
b) the tension in the string
c) In this problem several mathematical models have been used. Identify three of these and describe the assumption, which have been made in using these models.

8.A particle P of mass 5kg rests on a rough inclined plane of angle arcsine 3/5. The coefficient of friction between P and the plane is 1/8. P is connected to a particle Q of mass 4kg by a light inextensible string, which lies along a line of greatest slope of the plane. Q is hanging freely 2m above a horizontal plane. The system is released from rest with the string taut. Assuming P does not reach the pulley find

a) the acceleration of the system
b) the time that elapses before Q hits the horizontal plane
c) the total distance P moves up the plane.

9 A particle A of mass 5kg rests on a rough plane incline at an angle 300 to the horizontal. A string attached to A lies along a line of greatest slope of the plane and passes over a smooth pulley at the top of the plane. A particle B of mass 6kg hangs vertical from the string 1m above a horizontal plane. The system is released from rest with the string taut. If B takes 2 seconds to reach the horizontal plane, find the coefficient of friction between A and the inclined plane. Find also the total distance that A moves up the plane, assuming A does not reach the pulley.

10



The diagram shows a particle A of mass 0.2kg held at rest on a smooth roof inclined at 30^0 to the horizontal. Particle A is attached to one end of a light inextensible string, which passes over smooth pulley at the edge of the roof. A particle B of mass 0.3kg is attached to the other end of the string and hangs freely at rest. Particle A is 3.8m from the pulley and particle B is 2m above the horizontal ground. The system is released from rest.

a) Find the acceleration of each particle while the string remains taut.

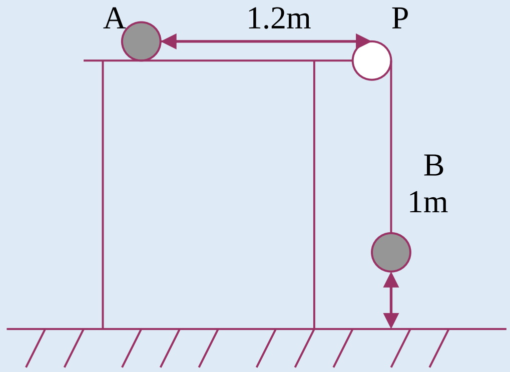
b) Find the speed with which B strikes the floor.

Assuming that B is brought to rest when it strikes the floor,

c) Find the speed with which A reaches the pulley.

In this problem several mathematical models have been used. Identify three of these and briefly describe the assumptions, which have been made in using these models.

11



A particle A of mass 3M lies on a rough horizontal table. The particle is attached to a light inextensible string which passes over a small smooth pulley P fixed at the edge of the table to the other end of the string is attached a particle B of mass 2M which hangs freely. AP is perpendicular to the edge of the table and A, P and B are in the same vertical plane. The system is released from rest with the string taut. When A is 1.2m from the edge of the table and B is 1m above the floor, as shown in the diagram. The particle B reaches the floor after 2 seconds, and does not rebound.

a) find the acceleration of A during the first 2 seconds

b) Find the coefficient of friction between particle A and the table

a) Determine by calculation whether A reaches P

CHAPTER 8

PROJECTILES.

OBJECTIVES

- DEDUCE THE CARTESIAN EQUATION OF A PROJECTICE
- CALCULATE THE POSITION VECTOR AT ANY GIVEN INSTANCE
- CALCULATE THE HIGHEST POINT
- CALCULATE THE RANGE
- CALCULATE THE TIME OF FLIGHT
- CALCULATE THE ANGLE OF PROJECTION
- CALCULATE SPEED OF PROJECTION

Cricket is a popular sport in Zimbabwe at the present moment. Think of Tatenda Taibu, Heath Streak e.t.c. These guys live on cricket. Had you ever thought why Heath Streak at one time, was the best batsman in Zimbabwe? To hit a six, you need to hit a cricket ball hard enough and project it at an angle so that it goes over the boundary without being caught. If it is caught before it "flies" over the boundary, you are **out**. Obvious air resistance will affect its movement and also the force of gravity will always "pull" the cricket ball downwards. In this case, the cricket ball is considered a projectile. Other examples of projectiles are a golf ball, a stone thrown to hit a bird on a tree branch, satellites rockets and missiles (Tom hawk / scud)

In order to model the path taken by projectiles we make the following assumptions:

- The projectile may be modelled by a particle.
- The acceleration due to gravity is constant
- The motion of the projectile takes place in a vertical plane
- The only force acting on the ball is its weight

❖ The first assumption ignores the size and the shape of the projectile

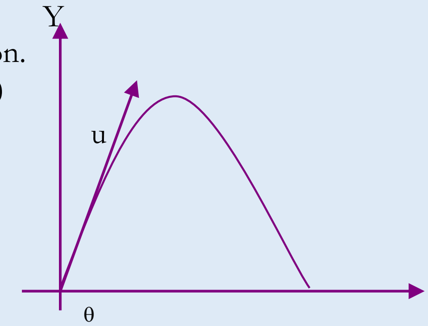
❖ The second assumption ignores the variation of gravity with position on the Earth's surface and height above sea level.

❖ The third assumption ignores any sideways movement due to crosswinds or spin.

❖ The fourth assumption ignores the effect of any air resistance

Assuming that the projectile is launched from the ground with a given speed, say u m/s at a particular angle θ, $0 < \theta < 90$, we need to find:

(a) The position vector at any given instance.
(b) The highest point the particle reaches.
(c) The range of the particle.
(d) The time of flight.



The projectile moves in the Y – direction and X – direction.
We resolve the initial speed u, to obtain: $u_y = u \sin \theta$
$u_x = u \cos\theta$
Hence: the position of the projectile when $t = 0$ is $u = \cos \theta\, i + u \sin \theta\, j$

Movement in the x – direction.

$v_x = u_x + at$
$s = u_x t + \frac{1}{2} at^2$
where, u_x is the initial velocity in the x – direction.

Use the motion equations to derive the projectile equations

No force is acting in the X – direction, hence: $\mathbf{a = 0}$
$v_x = u \cos \theta$
$s = u\, t \cos\theta$
i.e. $x = u\, t \cos\theta$

Movement in Y – direction.

$v_y = u_y + at$
$y = u_y t + \frac{1}{2} at^2$
In this case $a = -g$

Hence: $v_y = u \sin\theta - gt$
$y = u\, t\sin\theta - \frac{1}{2} g\, t^2$

$v_x = u\cos\theta$
$v_y = u\sin\theta - gt$
$x = ut\cos\theta$
$y = ut\sin\theta - \frac{1}{2}gt^2$
$v = \sqrt{(v^2_x + v^2_y)}$

The value of $y = ut\sin\theta - \frac{1}{2} gt^2$ gives the height of the projectile at time t.

8.1 The maximum height.

$y = ut\sin\theta - \frac{1}{2} gt^2$
y depends on t; hence to find the maximum height we differentiate y with respect to t, and equate to zero, i.e. $y^1 = u \sin \theta - gt$

$y^1 = 0$

$u \sin \theta = gt$, hence, $\mathbf{t = \dfrac{u \sin \theta}{g}}$

$y^{11} = -g < 0$

Hence there is global maximum at **$t = \frac{u \sin \theta}{g}$**

Hence: $y_{max} = u \left(\frac{u \sin \theta}{g}\right) - \frac{1}{2} g \left(\frac{u \sin \theta}{g^2}\right)^2$

$y_{max} = \frac{u^2 \sin^2 \theta}{g} - \frac{1}{2} \frac{u^2 \sin^2 \theta}{g}$

$y_{max} = \frac{u^2 \sin^2 \theta}{2g}$

Time of flight

The projectiles start at y = 0 and will end at y = 0.

Hence: $y = 0$

$ut \sin \theta - \frac{1}{2} gt^2 = 0$

$t (u \sin \theta - \frac{1}{2}gt) = 0$

$t = 0$ or $u \sin \theta - \frac{1}{2}gt = 0$

$t = 0$ or $t = \frac{2u\sin \theta}{g}$

$t = \frac{2u\sin\theta}{g}$

8.2 The Range of the particle.

$x = ut\cos \theta$

$x = u \left(\frac{2u \sin \theta}{g}\right)\cos \theta$

$X = \frac{2u^2 \sin \theta \cos \theta}{g}$ i.e **$X = \frac{u^2 \sin 2\theta}{g}$**

Maximum Range

$x = \frac{u^2 \sin 2\theta}{g}$, hence, x is a function in θ.

$x^1 (\theta) = \frac{2u^2 \cos 2\theta}{g}$

$x^1 (\theta) = 0$

$\frac{2u^2}{g} \cos 2 \theta = 0$ i.e $\cos 2\theta = 0$ hence, $2\theta = 90°$, where, $\theta = 45°$

The maximum range occurs when the projectile is projected at 45°

Hence: $x_{max} = \frac{u^2 \sin 90}{g}$

$x_{max} = \frac{u^2}{g}$

8.3 The Cartesian equation of the trajectory of a projectile.

$x = ut\cos \theta$

$y = ut \sin \theta - \frac{1}{2} gt^2$

We eliminate t: i.e $t = \frac{x}{u \cos \theta}$, substituting:

$y = \frac{u x\sin \theta}{u \cos \theta} - \frac{\frac{1}{2}g\, x^2}{(u \cos \theta)^2}$

$y = x\tan \theta - \frac{g x^2}{2u^2 \cos^2 \theta}$

$y = x \tan \theta - \frac{gx^2}{2u^2} (\sec^2 \theta)$

This is a parabola opening downwards. To find the turning point and the gradient at any point of this curve calculus may be used

$$y = x \tan\theta - \frac{gx^2}{2u^2}(1 + \tan^2\theta)$$

Example.

A particle is projected from a point O on horizontal ground with a speed of 40m/s at an angle of 30° to the horizontal. Given that 0.7s, after projection, the particle is at a point P calculate:

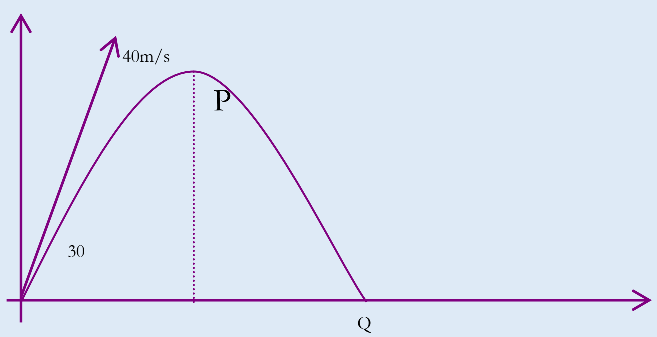
(i) The height of P above the ground.

(ii) The particle reaches the ground again at the point Q. Calculate:

(a) The time of flight.

(b) The distance OQ.

Solution:



(i) $y = u_y t - \frac{1}{2}gt^2$

$u_y = 40 \sin 30^0$

$\mathbf{u_y = 20m/s}$

Hence: $y = 20t - \frac{1}{2}gt^2$

For $t = 0.7$

$y = 20 \times 0.7 - \frac{1}{2}g \times (0.7)^2$, $g = 9.81$

y **= 11.6m**

(ii) Time of flight is found by solving the quadratic equation.

$y = 0$

$20t - \frac{1}{2}gt^2 = 0$

$t = 0$ or $t = 40 / g$

$t = 40 / 9.81$

$t = 4.15$

$x = 40t \cos\theta$

$x = 40 \times 40 / g\cos 30°$

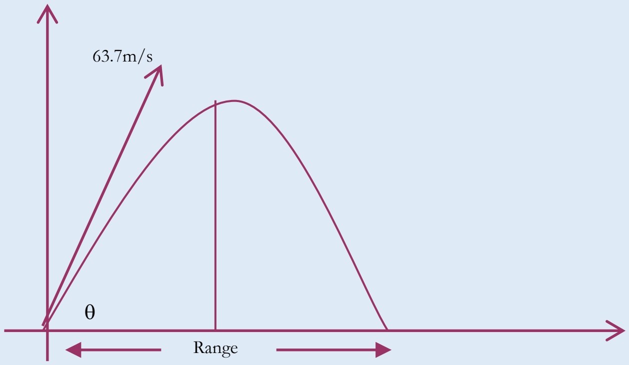
$x = \frac{1600 \times \sqrt{3}}{9.81}$

$x = 141$ m.

Example.

A particle P is projected from a point O on a horizontal plane with speed 63,7m/s at angle θ to the horizontal, where $\tan\theta = 5/12$. Find:

(i) The time taken for P to return to the plane
(ii) The maximum height attained by P
(iii) The range of P.



Solution:

$\tan\theta = 5/12$, hence , $\cos\theta = 12/13$ and $\sin\theta = 5/13$

(i) we calculate the time of flight

$y = u_y t - \frac{1}{2}gt^2$

$u_y = u\sin\theta$

$u = 63.7$

$y = 0$

$ut\sin\theta - \frac{1}{2}gt^2 = 0$

$t(u\sin\theta - \frac{1}{2}gt) = 0$

$t = 0$ or $t = \frac{2u\sin\theta}{g}$

hence, $t = \frac{2 \times 63.7 \times 5/13}{9.81}$

$t = 4.99s$ i.e. $t = 5s$

(ii) $y = ut\sin\theta - \frac{1}{2}gt^2$

$y^1 = u\sin\theta - gt$

$y^1 = 0$

$u\sin\theta - gt = 0$

$t = \frac{u\sin\theta}{g}$; time to reach the max. height.

$y_{max} = \frac{u(\sin\theta)}{g}\sin\theta - \frac{\frac{1}{2}g\,u(\sin\theta)^2}{g}$

$y_{max} = \frac{u^2\sin^2\theta}{2g}$

$y_{max} = \frac{63.7 \times (5/13)^2}{2 \times 9.81}$

$y_{max} = 30.59$ m

$y_{max} = 31$m .

Hence: $R = u\left(\frac{2u\sin\theta}{g}\right)\cos\theta$

$R = \frac{2u^2\sin\theta\cos\theta}{g}$

$R = 2 \times 63.7^2 \times \frac{5}{13} \times \frac{12}{13} / 9.81$

$R = 293.700.$

R = 294m to 3 s. f.

Example

a) A tennis ball is projected with a horizontal velocity of 40 m/s and moves freely under gravity. After moving for 3 seconds:

i) state the horizontal component of the velocity.

ii) find the vertical component of the velocity.

iii) find the speed and direction of the tennis ball.

b) A particle is projected from a point O, h metres above the horizontal. The velocity of projection is 50m/s at an angle of $\sin^{-1}(7/25)$ above the horizontal. The particle strikes the ground at a point 240m away from a point on the ground directly below O. Find:

i) the time for which the particle is in the air.

ii) the value of h.

Solution

a (i) the horizontal component is 40 m/s.

ii) the vertical component is $v_y = - gt$

$$v_y = - 9.81 \times 3 = - 29.43\text{m/s}$$

iii) $V = \sqrt{(29.43^2 + 40^2)} = 66.8\text{m/s}$:$\tan\alpha = - 29.43/ 40$ i.e. $\alpha = - \tan^{-1}(29.43/40)$

$\alpha = - 36.4^0$

b i) $x = ut\cos \alpha = 50\text{x}5\text{x}24/25 = 5\text{s}$

ii) At $t = 5$, $y = - h$, hence, $- h = ut\sin\alpha - ½ g t^2$

$-h = 50 \times 5 \times 7/25 – ½ \times 9.81 \times 5^2$

$h = 52.625\text{m}$

8.4 Examination Type Questions

1A stone throne upwards from the top of a vertical cliff 56m high falls into the sea 4 seconds later, 32m from the foot of the cliff. Find the speed and direction of projection. A second stone is thrown at the same time, in the same vertical plane, at the same speed and at the same angle to the horizontal, but downwards. Find how long it will take to reach the sea and the distance between points of entry of the stones into the water. (Take g to be 10m/s)

2. Two particles are projected simultaneously from points A and B on level ground and a distance 150m apart. The first particle is projected vertically upwards from A with an initial velocity of *u* m/s and the second particle is projected from B towards A with an angle of projection α. If the particles collide when they are both at their greatest height above the level of AB, prove that $\tan \alpha = u^2/(150g)$

3. The particle is projected so that it just clears the wall. Find the initial velocity of the particle, if

a) the wall is 5m high

b) the floor of the wall is 30m horizontally from the point of projection

c) the particle is moving at angle of arctan ½ to the downward vertical as it passes over the wall.

4.A particle is projected from a point O with an initial velocity of 60m/s, at an angle 45^0 to the horizontal. At the same instant a second particle is projected in the opposite direction with initial speed 50m/s from a point level with O and 90m from 0. Find the angle of projection of the second particle if they collide and the time at which this occurs

5.A particle is projected from a point O with an initial velocity of 21m/s at an angle of arctan4/3 to the horizontal and one second later another particle is projected from O with an initial velocity of 31m/s at an angle arctan ¾ to the horizontal. Prove that the particles collide and find when this occurs. Find also the direction in which each particle is moving when they collide take g = 9.81

6. An arrow which has an initial speed of 50m/s is aimed at a target which is a strip, 5m at a distance 90m from the point of projection. Find the values between which the angle of projection must lie so that the arrow hits the target.

7. Two projectiles are fired simultaneously from points P and Q on horizontal ground and collide head on when traveling horizontally. The first projectile is fired with speed vm/s at an angle of elevation α and the second is fired with speed v/2m/s at an angle of elevation β. If $PQ = v^2 \sin\beta / (2g)$, show that:

a) $2\sin\alpha = \sin\beta$

b) $2 = 2\cos\alpha + \cos\beta$

Hence show that $\cos\alpha = 7/8$

8. Two boys stand on horizontal ground at a distance apart. One throws a ball from a height of 2h with velocity v and the other catches it at height h. If θ is the inclination above the horizontal at which the first boy throws the ball, show that:

$g\, a^2\tan^2\theta - 2v^2 a\tan\theta + ga^2 - 2v^2h = 0$, when $a = 2\sqrt{2h}$ and $v2 = 2gh$, calculate

i) the value of θ

ii) the greatest height attained by the ball above the ground, in terms of h.

9.Two particles are projected with the same speed from the same point. The angles of projections are 2α and α and a time T elapses between the instants of projection. If the particles collide in flight, find the speed of projection in terms of T and α. If the collision occurs when one of the particles is at its greatest height, show that α is given by

$$5\cos^4\alpha - \cos^2\alpha - = 0$$